Homework 3

Quantum Mechanics
February 19, 2023 C Seitz

Problem 1. *Problem 2.48*

Solution.
The polar decomposition of a matrix is $A = UJ = KU$ where U is a unitary operator and J, K are positive operators that satisfy $J = \sqrt{A^\dagger A}$ and $K = \sqrt{AA^\dagger}$. If P is itself a positive matrix we can immediately say that its polar decomposition is $P = IP = PI$. If the matrix to decompose is unitary then of course $UU^\dagger = U^\dagger U = I$ so its decomposition is itself. If the matrix to decompose is Hermitian, then $H = H^\dagger$ and

$$J = K = \sqrt{H^2} = \sqrt{\sum_i \lambda_i^2 \left|i\right\rangle \left\langle i\right|} = \sum_i \left|\lambda_i\right| \left|i\right\rangle \left\langle i\right|$$

and its polar decomposition is therefore $U \sum_i \left|\lambda_i\right| \left|i\right\rangle \left\langle i\right|$ or $\sum_i \left|\lambda_i\right| \left|i\right\rangle \left\langle i\right| U$.

I will give some examples which demonstrate these properties, but are not very computationally intensive. The examples for positive matrices are the 2 x 2 matrix

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \quad B = A = \begin{pmatrix} 5 & 1 & 1 & 1 \end{pmatrix}$$

An example of a 2 x 2 Hermitian matrix is the Hamiltonian

$$H = -\gamma \sigma_z B_0$$

The matrix J is

$$J = K = \gamma B_0 \left(\left|+\right\rangle \left\langle +\right| + \left|-\right\rangle \left\langle -\right|\right) = \gamma B_0 I$$

Clearly for both right and left polar decompositions we have $U = \sigma_z$. A 2 x 2 unitary example can be seen from the time-evolution corresponding to this Hamiltonian

$$\begin{aligned} U &= e^{-iHt} \\ &= e^{i\omega t} \left|+\right\rangle \left\langle+\right| + e^{-i\omega t} \left|-\right\rangle \left\langle-\right| \\ &= \begin{pmatrix} e^{i\omega t} & 0 \\ 0 & e^{-i\omega t} \end{pmatrix} \end{aligned}$$

where $\omega = -\gamma B_0$. Obviously $J = \sqrt{U^\dagger U} = I$. A 4 x 4 example is:

$$H = \gamma \left(\sigma_{1z} \otimes \sigma_{2z}\right)$$

Written out explicitly, it is

$$H = \gamma \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix J is

$$\begin{aligned} J = K = \gamma(&\left|++\right\rangle \left\langle++\right| + \left|+-\right\rangle \left\langle+-\right| \\ &+ \left|-+\right\rangle \left\langle-+\right| + \left|--\right\rangle \left\langle--\right|) = \gamma I \end{aligned}$$

and in this case we have that $U = \sigma_{1z} \otimes \sigma_{2z}$, which can by easily checked to be unitary. We can again give the unitary example by considering time evolution

$$\begin{aligned} U &= e^{-iHt} \\ &= \begin{pmatrix} e^{i\gamma t} & 0 & 0 & 0 \\ 0 & e^{-i\gamma t} & 0 & 0 \\ 0 & 0 & e^{-i\gamma t} & 0 \\ 0 & 0 & 0 & e^{i\gamma t} \end{pmatrix} \end{aligned}$$

and again we can see that $\sqrt{U^\dagger U} = I$.

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Problem 2. *Problem 2.49*

Solution.

The polar decomposition is $A = UJ$. The spectral decomposition of J is

$$J = \sqrt{\sum_i \lambda_i \lambda_i^* |i\rangle \langle i|} = \sum_i |\lambda_i| |i\rangle \langle i|$$

For the unitary matrix U, we have

$$U = \sum_j \lambda_j |j\rangle \langle j|$$

Therefore the product UJ reads

$$\begin{aligned} UJ &= \left(\sum_j \lambda_j |j\rangle \langle j| \right) \left(\sum_i |\lambda_i| |i\rangle \langle i| \right) \\ &= \sum_{ij} |\lambda_i| \lambda_j |i\rangle \langle i|j\rangle \langle j| \end{aligned}$$

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Problem 3. *Problem 2.50*

Solution.

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

First, consider

$$A^\dagger A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad AA^\dagger = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Let

$$J = \sqrt{j_1} |j_1\rangle \langle j_1| + \sqrt{j_2} |j_2\rangle \langle j_2|$$

where, for example, $|j_1\rangle$ is the first eigenvector of $A^\dagger A$. According to Mathematica, $|j_1\rangle = \frac{1+\sqrt{5}}{2}|0\rangle + |1\rangle$, and $|j_2\rangle = \frac{1-\sqrt{5}}{2}|0\rangle + |1\rangle$, in the standard basis. The eigenvalues are $j_1 = \frac{3+\sqrt{5}}{2}$ and $j_2 = \frac{3-\sqrt{5}}{2}$. We can then write normalized eigenvectors as

$$|j_1\rangle = \frac{1}{\sqrt{1+(1+\sqrt{5})^2}}\Big((1+\sqrt{5})|0\rangle + 2|1\rangle\Big)$$

$$|j_2\rangle = \frac{1}{\sqrt{1+(1-\sqrt{5})^2}}\Big((1-\sqrt{5})|0\rangle + 2|1\rangle\Big)$$

Putting it all together we find that

$$J = \frac{1}{\sqrt{5}}\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$$

The matrix U is then found by solving $U = AJ^{-1}$. Mathematica says:

$$U = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

The matrix K is found in a similar fashion:

$$K = AU^{-1} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

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Problem 4. *Problem 2.51*

Solution. The Hadamard gate H is unitary if $H^\dagger = H^{-1}$. It is easy to see that

$$H^\dagger = H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

It's inverse is

$$H^{-1} = -\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = H$$

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Problem 5. *Problem 2.52*

Solution.
$$H^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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Problem 6. *Problem 2.53*

Solution. Writing out the characteristic equation gives that the eigenvalues are $\lambda = \pm\sqrt{2}$. ■

Problem 7. *Problem 2.54*

Solution. Since the two operators commute, they are simultaneously diagonalizable. Consider the following spectral decompositions

$$A = \sum_n a_n |n\rangle \langle n|$$

$$B = \sum_n b_n |n\rangle \langle n|$$

Therefore, it must be true that

$$A + B = \sum_n (a_n + b_n) |n\rangle \langle n|$$

Now these matrices are Hermitian so their eigenvectors are orthogonal, and the product of matrix exponentials is just

$$\begin{aligned}
\exp(A)\exp(B) &= \left(\sum_n \exp(a_n) |n\rangle \langle n|\right) \left(\sum_m \exp(b_m) |m\rangle \langle m|\right) \\
&= \sum_{m,n} \delta_{mn} \exp(a_n) \exp(b_m) |n\rangle \langle m| \\
&= \sum_n \exp(a_n) \exp(b_n) |n\rangle \langle n| \\
&= \sum_n \exp(a_n + b_n) |n\rangle \langle n| \\
&= \exp(A + B)
\end{aligned}$$

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Problem 8. *Problem 2.55*

Solution.

$$
\begin{aligned}
UU^\dagger &= \exp\left(\frac{-iH(t_2-t_1)}{\hbar}\right)\exp\left(\frac{iH(t_2-t_1)}{\hbar}\right) \\
&= \left(\sum_n \exp\left(\frac{-iE_n(t_2-t_1)}{\hbar}\right)|n\rangle\langle n|\right)\left(\sum_m \exp\left(\frac{iE_m(t_2-t_1)}{\hbar}\right)|m\rangle\langle m|\right) \\
&= \sum_{m,n}\delta_{mn}|n\rangle\langle m| \\
&= \sum_n |n\rangle\langle n| = I
\end{aligned}
$$

where H is a Hermitian operator.

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Problem 9. *Problem 2.56*

Solution.

U is unitary so its eigenvalues u_n have unit norm, which means

$$K = -i\log(U) = -i\sum_n \log(u_n)|n\rangle\langle n| = \sum_n \theta|n\rangle\langle n|$$

since

$$\log(u_n) = \log\big(|u_n|e^{i\theta}\big) = \log(|u_n|) + i\theta = i\theta$$

Therefore, $K = K^\dagger$ since $\theta \in \mathbb{R}$.

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Problem 10. *Problem 2.57*

Solution.

$$L_l|\alpha\rangle = \frac{\ell|l\rangle}{|\ell|}$$

$$M_m\frac{\ell|l\rangle}{|\ell|} = \frac{m\ell}{|m||\ell|}|m\rangle$$

which is equivalent to

$$
\begin{aligned}
N_{m\ell} |\alpha\rangle &= M_m L_\ell |\alpha\rangle \\
&= \frac{|m\rangle \langle m|\ell\rangle \langle \ell|}{|m||\ell|} |\alpha\rangle \\
&= \frac{\ell |m\rangle \langle m|}{|m||\ell|} |\ell\rangle \\
&= \frac{m\ell}{|m||\ell|} |m\rangle
\end{aligned}
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Problem 11. *Problem 2.58*

Solution.

Since the system is in an eigenstate of M with eigenvalue m, the average will be m

$$\langle M\rangle = \langle m| M |m\rangle = \langle m| m |m\rangle = m$$

The variance must then be zero

$$
\begin{aligned}
(\Delta M)^2 &= \langle M^2\rangle - \langle M\rangle^2 \\
&= m^2 - m^2 = 0
\end{aligned}
$$

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Problem 12. *Problem 2.59*

Solution.

$$\langle 0| X |0\rangle = \langle 0|1\rangle = 0$$

$$
\begin{aligned}
(\Delta X)^2 &= \langle X^2\rangle - \langle X\rangle^2 \\
&= \langle X^2\rangle \\
&= \langle 0| X^2 |0\rangle \\
&= 1
\end{aligned}
$$

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Problem 13. *Problem 2.60*

Solution.

$$\begin{aligned}\vec{v}\cdot\sigma &= \begin{pmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z \end{pmatrix} \\ &= v_z\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right) + (v_1 - iv_2)\,|0\rangle\langle 1| + (v_1 + iv_2)\,|1\rangle\langle 0|\end{aligned}$$

from the outer product representations of $\sigma_x, \sigma_y, \sigma_z$. The corresponding characteristic equation is

$$\lambda^2 - (v_z^2 + v_y^2 + v_x^2) = 0$$

If $\vec{v}$ is normalized then $\lambda = \pm 1$. We now show that the projectors onto the respective eigenspaces are $P_\pm = (I \pm \vec{v}\cdot\sigma)/2$. Let $|\pm\rangle$ be the eigenvectors of $\vec{v}\cdot\sigma$ with eigenvalues ± 1, respectively.

$$\begin{aligned}P_+ &= |+\rangle\langle +| \\ &= \frac{|+\rangle\langle +| + |-\rangle\langle -| + |+\rangle\langle +| - |-\rangle\langle -|}{2} \\ &= \frac{I + \vec{v}\cdot\sigma}{2}\end{aligned}$$

since by spectral decomposition we know that $\vec{v}\cdot\sigma = |+\rangle\langle +| - |-\rangle\langle -|$. Of course, we also have that

$$\begin{aligned}P_- &= |-\rangle\langle -| \\ &= \frac{|+\rangle\langle +| + |-\rangle\langle -| - |+\rangle\langle +| + |-\rangle\langle -|}{2} \\ &= \frac{I - \vec{v}\cdot\sigma}{2}\end{aligned}$$

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Problem 14. *Problem 2.61*

Solution.

Let $|0\rangle$ and $|1\rangle$ be the eigenvectors of σ_z.

$$
\begin{aligned}
p(+) &= |c_+|^2 \\
&= \langle 0| P_+ |0\rangle \\
&= \langle 0| \frac{I + \vec{v} \cdot \sigma}{2} |0\rangle \\
&= \frac{1}{2} \left(1 + \langle 0| \left(v_3 \left(|0\rangle \langle 0| - |1\rangle \langle 1|\right) + (v_1 - iv_2) |0\rangle \langle 1| + (v_1 + iv_2) |1\rangle \langle 0|\right) |0\rangle\right) \\
&= \frac{1}{2} (1 + v_3)
\end{aligned}
$$

The state of the system must be then in the eigenvector $|+\rangle$ of $\vec{v} \cdot \sigma$ with eigenvalue $+1$. This can be conveniently obtained by applying the measurement operator P_+, which was obtained in the last problem. Consider,

$$
\begin{aligned}
P_+ |0\rangle &= \frac{I + \vec{v} \cdot \sigma}{2} |0\rangle \\
&= \frac{1}{2} \left((1 + v_3) |0\rangle + (v_1 + iv_2) |1\rangle\right)
\end{aligned}
$$

Then applying the appropriate normalization, we get

$$
|+\rangle = \frac{1}{2\sqrt{(1 + v_3)/2}} \left((1 + v_3) |0\rangle + (v_1 + iv_2) |1\rangle\right)
$$

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