**TTIC 31230 Fundamentals of Deep Learning**

**Problems for GANs.**

**Problem 1. Conditional GANs** In a conditional GAN we model a conditional distribution $\text{Pop}(y|x)$ defined by a population distribution on pairs $\langle x, y\rangle$. For conditional GANs we consider the probability distribution over triples $\langle x, y, i\rangle$ defined by

$$\begin{aligned}\tilde{P}_\Phi(i=1) &= 1/2\\ \tilde{P}_\Phi(y|x, i=1) &= \text{pop}(y|x)\\ \tilde{P}_\Phi(y|x, i=-1) &= p_\Phi(y|x)\end{aligned}$$

(a) Write the conditional GAN adversarial objective function for this problem in terms of $\tilde{P}(x, y, i)$, $P_\Phi(y|x)$ and $P_\Psi(i|y, x)$.

**Solution**:

$$\Phi^* = \operatorname*{argmax}_\Phi \min_\Psi E_{x,y,i\sim\tilde{P}(x,y,i)} \; -\ln P_\Psi(i|x, y)$$

**Problem 2. GAN instability**

Consider the following adversarial objective where $x$ and $y$ are scalars (real numbers).

$$\max_x \min_y xy$$

(a) Write the differential equation for gradient flow of this adversarial objective.

**Solution**:

$$\begin{aligned}\frac{dx}{dt} &= y\\ \frac{dy}{dt} &= -x\end{aligned}$$

(b) Give a general solution to your differential equation. (Hint: It goes in a circle). You solution should have parameters allowing for any given initial value of $x$ and $y$.

**Solution**:

$$x = r_0 \sin(t + \Theta_0)$$

$$y = r_0 \cos(t + \Theta_0)$$

**Problem 3. Contrastive GANs.**

A GAN can be built with a "contrastive" discriminator. Rather than estimate the probability that $y$ is from the population, the discriminator must select which of $y_1, \ldots, y_N$ is from the population.

More formally, for $N \geq 2$ let $\tilde{P}_\Phi^{(N)}$ be the distribution on tuples $\langle i, y_1, \ldots, y_N \rangle$ defined by drawing one "positive" from Pop and $N-1$ IID negatives from $P_\Phi$; then inserting the positive at a random position among the negatives; and returning $(i, y_1, \ldots, y_N)$ where $i$ is the index of the positive.

$$\Phi^* = \operatorname*{argmax}_\Phi \min_\Psi \; E_{(i,y_1,\ldots,y_{N+1}) \sim \tilde{P}_\Phi^{(N)}} \; -\ln p_\Psi(i|y_1, \ldots, y_{N+1}) \quad (1)$$

Restate the above definition of $\tilde{P}_\Phi^{(N)}$ and the GAN adversarial objective for the case of conditional constrastive GANs.

**Solution**:

$$\Phi^* = \operatorname*{argmax}_\Phi \min_\Psi \; E_{(i,y_1,\ldots,y_{N+1},x) \sim \tilde{P}_\Phi^{(N)}} \ln -P_\Psi(i|y_1, \ldots, y_{N+1}, x)$$