Exam 2

Quantum Mechanics

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Problem 1.

Solution.

Some of the states have the same energy, so we will need to use degenerate perturbation theory. Specifically, the subspaces spanned by $\mathcal{A} = \{\left|0^{(0)}\right\rangle, \left|1^{(0)}\right\rangle\}$ and $\mathcal{B} = \{\left|2^{(0)}\right\rangle, \left|4^{(0)}\right\rangle\}$ have a degeneracy while the lone ket $\left|3^{(0)}\right\rangle$ is nondegenerate. We assume that a perturbed ket $\alpha \in \mathcal{A}$ can be written as a linear combination of the unperturbed kets:

$$|\alpha\rangle = \sum_{n \in \mathcal{A}} \langle n|\alpha\rangle |n\rangle$$

The first order correction is given by

$$V|\alpha\rangle = \sum_{n \in \mathcal{A}} \langle n|\alpha\rangle (H - H_0)|n\rangle = \Delta_\alpha^{(1)} |\alpha\rangle$$

We therefore need to find the eigenvectors and eigenvalues of the matrix

$$|V_{\mathcal{A}} - \Delta_\alpha I| = \det \begin{pmatrix} 2\cos\theta - \Delta_\alpha & 2\sin\theta e^{-i\phi} \\ 2\sin\theta e^{i\phi} & -2\cos\theta - \Delta_\alpha \end{pmatrix} = 0$$

which is easy to solve, and we get the first order shifts $\Delta_\alpha^{(1)} = \pm 2$. It is the same process for the $\mathcal{B}$ subspace

$$|V_{\mathcal{B}} - \Delta_\beta I| = \det \begin{pmatrix} 4\cos\theta - \Delta_\beta & 4\sin\theta e^{-i\phi} \\ 4\sin\theta e^{i\phi} & -4\cos\theta - \Delta_\beta \end{pmatrix} = 0$$

It is pretty much the same matrix, so $\Delta_\beta^{(1)} = \pm 4$. For the first order correction to the nondegnerate ket $\left|3^{(0)}\right\rangle$, we use nondegenerate perturbation theory to first order

$$\begin{aligned}\Delta_3^{(1)} &= \lambda V_{33} + \lambda^2 \sum_{j\neq 3} \frac{|V_{j3}|^2}{E_3^{(0)} - E_j^{(0)}} \\ &= \lambda + \lambda^2 \left(-\frac{3}{\epsilon} - \frac{3}{\epsilon}\right) \\ &= \lambda - \frac{6\lambda^2}{\epsilon} \approx \lambda\end{aligned}$$

To get the corrections to the ground state eigenvector, we can again use nondegenerate perturbation theory

$$\begin{aligned}\left|3^{(1)}\right\rangle &= \lambda \left|3^{(0)}\right\rangle + \lambda^2 \sum_{j\neq 3} \left|j^{(0)}\right\rangle \frac{V_{j3}}{E_3^{(0)} - E_j^{(0)}} \\ &= \lambda \left|3^{(0)}\right\rangle + \frac{\lambda^2}{\epsilon}\left(\left|2^{(0)}\right\rangle + \left|4^{(0)}\right\rangle\right)\end{aligned}$$

In the limit $\lambda \to 0$, the eigenvectors are the "good" linear combinations. To find them we need to find the eigenvectors of the submatrices $V_{\mathcal{A}}$ and $V_{\mathcal{B}}$. ■

Problem 2.

$$\begin{aligned}|\alpha_\pm\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right) \\ |\beta_\pm\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle \pm i\,|1\rangle\right)\end{aligned}$$

Solution. We are after the ensemble expectation values $[x^n]$ and $[p^n]$. In general,

$$[A] = \sum_n w_n \langle\alpha_n| A |\alpha_n\rangle$$

First, we should show

$$
\begin{aligned}
(a+a^\dagger)^n |0\rangle &= \sum_{k=0}^{n} \binom{n}{k} a^k (a^\dagger)^{n-k} |0\rangle \\
&= \sum_{k=0}^{n} \binom{n}{k} a^k \sqrt{(n-k)!}\, |n-k\rangle \\
&= \sum_{k=0}^{n} \binom{n}{k} \frac{(n-k)!}{\sqrt{(n-2k)!}} |n-2k\rangle
\end{aligned}
$$

$$
\begin{aligned}
(a+a^\dagger)^n |1\rangle &= \sum_{k=0}^{n} \binom{n}{k} a^k (a^\dagger)^{n-k} |1\rangle \\
&= \sum_{k=0}^{n} \binom{n}{k} a^k \sqrt{(n-k+1)!}\, |n-k+1\rangle \\
&= \sum_{k=0}^{n} \binom{n}{k} \frac{(n-k+1)!}{\sqrt{(n-2k+1)!}} |n-2k+1\rangle
\end{aligned}
$$

For $(a-a^\dagger)^n$ we just replace $(a^\dagger)^{n-k} \to (-1)^{n-k}(a^\dagger)^{n-k}$ in the sum.

$$
\left(a+a^\dagger\right)^n |\alpha_+\rangle = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{(n-k)!}{\sqrt{(n-2k)!}} |n-2k\rangle + \frac{(n-k+1)!}{\sqrt{(n-2k+1)!}} |n-2k+1\rangle \right)
$$

Hitting this with a bra $\langle 0|$ will select terms of the sum that satisfy $k = n/2$ (which only occurs for even n) or $k = (n+1)/2$ (which occurs for odd n). On the other hand, hitting this with a bra $\langle 1|$ will select terms of the sum that satisfy $k = (n-1)/2$ (odd n) or $k = n/2$ (even n).

First, consider even n,

$$
\begin{aligned}
\langle 0| \left(a+a^\dagger\right)^n |\alpha_+\rangle &= \binom{n}{n/2} \left(\frac{n}{2}\right)! \\
\langle 1| \left(a+a^\dagger\right)^n |\alpha_+\rangle &= \binom{n}{n/2} \left(\frac{n}{2}\right)!
\end{aligned}
$$

For odd n,

$$\begin{aligned}
\langle 0| \left(a + a^\dagger\right)^n |\alpha_+\rangle &= \binom{n}{(n+1)/2} \left(\frac{(n-1)}{2}\right)! \\
\langle 1| \left(a + a^\dagger\right)^n |\alpha_+\rangle &= \binom{n}{(n+1)/2} \left(\frac{(n+3)}{2}\right)!
\end{aligned}$$

I will refer to these implicitly from here on.

$$\begin{aligned}
[x^n] &= \langle \alpha_+| x^n |\alpha_+\rangle \\
&= \frac{1}{2}\left(\frac{\hbar}{2m\omega}\right)^{\frac{n}{2}} \left(\langle 0| \left(a + a^\dagger\right)^n |\alpha_+\rangle + \langle 1| \left(a + a^\dagger\right)^n |\alpha_+\rangle\right)
\end{aligned}$$

For the $|\beta_+\rangle$ ensemble, the only difference is the factor of i in front of the $|1\rangle$ ket, but the result is the same as the $|\alpha_+\rangle$ ensemble

$$\begin{aligned}
[x^n] &= \langle \beta_+| x^n |\beta_+\rangle \\
&= \frac{1}{2}\left(\frac{\hbar}{2m\omega}\right)^{\frac{n}{2}} \left(\langle 0| \left(a + a^\dagger\right)^n |\beta_+\rangle + \langle 1| \left(a + a^\dagger\right)^n |\beta_+\rangle\right)
\end{aligned}$$

For the mixture of $|\alpha_+\rangle$ and $|\alpha_-\rangle$

$$[x^n] = \frac{1}{2}\left(\langle \alpha_+| x^n |\alpha_+\rangle + \langle \alpha_-| x^n |\alpha_-\rangle\right)$$

To find the second term we need the odd and even cases again,

$$\begin{aligned}
\langle 0| \left(a + a^\dagger\right)^n |\alpha_-\rangle &= \binom{n}{n/2} \left(\frac{n}{2}\right)! \\
\langle 1| \left(a + a^\dagger\right)^n |\alpha_-\rangle &= \binom{n}{n/2} \left(\frac{n}{2}\right)!
\end{aligned}$$

For odd n,

$$\begin{aligned}\langle 0| \left(a + a^\dagger\right)^n |\alpha_-\rangle &= -\binom{n}{(n+1)/2}\left(\frac{(n-1)}{2}\right)! \\ \langle 1| \left(a + a^\dagger\right)^n |\alpha_-\rangle &= -\binom{n}{(n+1)/2}\left(\frac{(n+3)}{2}\right)!\end{aligned}$$

■

Problem 3.

Solution. In this two particle system, we have $j_1 = 1$ and $j_2 = 2$. We are told the z component of the individual angular momenta $m_1 = -1$ and $m_2 = 2$. So we use the state kets $|j_1, j_2; m_1, m_2\rangle$ where m_1 and m_2 follow the usual rules. So the state is $|1, 2; -1, 2\rangle$.

$$\begin{aligned}\langle J^2\rangle &= \langle 1, 2; -1, 2| J^2 |1, 2; -1, 2\rangle \\ &= \langle 1, 2; -1, 2| \left(J_1^2 + J_2^2 + 2J_{1z}J_{2z}\right) |1, 2; -1, 2\rangle \\ &= j_1(j_1+1)\hbar^2 + j_2(j_2+1)\hbar^2 - 4\hbar^2 \\ &= 2\hbar^2 + 6\hbar^2 - 4\hbar^2 \\ &= 4\hbar^2\end{aligned}$$

We cannot directly compute the expectation values of J_x, J_y, J_z in this basis, because $|j_1, j_2, m_1, m_2\rangle$ are not eigenkets of J_x, J_y, J_z. But can change basis:

$$|j_1, j_2, m_1, m_2\rangle = \sum_{m_1, m_2} |j_1, j_2; jm\rangle \langle j_1, j_2; jm|j_1, j_2; m_1, m_2\rangle$$

We need to determine $\langle j_1, j_2; jm|j_1, j_2; m_1, m_2\rangle$ which are the Clebsch-Gordon coefficients.

Noting that $m = m_1 + m_2 = 1$, we find

$$|1, 2; -1, 2\rangle = \sqrt{\frac{3}{5}}\,|1, 2; 11\rangle + \sqrt{\frac{1}{3}}\,|1, 2; 21\rangle + \sqrt{\frac{1}{15}}\,|1, 2; 31\rangle$$

The expectation value $\langle J_z\rangle$ is then

$m = 1$

m_1, m_2 \ j	3	2	1
2, −1	$\sqrt{\frac{1}{15}}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{3}{5}}$
1, 0	$\sqrt{\frac{8}{15}}$	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{3}{10}}$
0, 1	$\sqrt{\frac{2}{5}}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{10}}$

Figure 1: Clebsch-Gordon coefficients for $j_1 = 1$, $j_2 = 2$, $m = 1$

$$
\begin{aligned}
\langle J_z \rangle &= \langle 1,2;-1,2| \, J_z \, |1,2;-1,2\rangle \\
&= \left(\sqrt{\frac{3}{5}} \langle 1,2;11| + \sqrt{\frac{1}{3}} \langle 1,2;21| + \sqrt{\frac{1}{15}} \langle 1,2;31| \right) J_z \\
&\left(\sqrt{\frac{3}{5}} |1,2;11\rangle + \sqrt{\frac{1}{3}} |1,2;21\rangle + \sqrt{\frac{1}{15}} |1,2;31\rangle \right) \\
&= \hbar \left(\frac{3}{5} + \frac{1}{3} + \frac{1}{15} \right) = \hbar
\end{aligned}
$$

as we should expect. For J_x, J_y,

$$
\begin{aligned}
\langle J_x \rangle &= \langle 1,2;-1,2| \, J_x \, |1,2;-1,2\rangle \\
&= \left(\sqrt{\frac{3}{5}} \langle 1,2;11| + \sqrt{\frac{1}{3}} \langle 1,2;21| + \sqrt{\frac{1}{15}} \langle 1,2;31| \right) \\
&\frac{1}{2} \left(J_+ + J_- \right) \left(\sqrt{\frac{3}{5}} |1,2;11\rangle + \sqrt{\frac{1}{3}} |1,2;21\rangle + \sqrt{\frac{1}{15}} |1,2;31\rangle \right) \\
&= 0
\end{aligned}
$$

and $\langle J_y \rangle = 0$ since neither J_+ nor J_- connects two $|j_1, j_2; jm\rangle$ states.

Now, if we measure the total angular momentum and obtain the largest possible value, then we are in the state $|1,2;31\rangle$ in the $|j_1, j_2; jm\rangle$ basis. However, to compute J_{1z} and J_{2z} we need to transform this back to the $|j_1, j_2, m_1, m_2\rangle$ basis. Looking up the coefficients, we get

$$|1,2;31\rangle = \left(\sqrt{\frac{1}{15}}\,|1,2;-12\rangle + \sqrt{\frac{2}{5}}\,|1,2;10\rangle + \sqrt{\frac{8}{15}}\,|1,2;01\rangle\right)$$

$$\begin{aligned}
\langle J_{1z}\rangle &= \langle 1,2;31|\,J_{1z}\,|1,2;31\rangle \\
&= \left(\sqrt{\frac{1}{15}}\,\langle 1,2;-12| + \sqrt{\frac{2}{5}}\,\langle 1,2;10| + \sqrt{\frac{8}{15}}\,\langle 1,2;01|\right) \\
J_{1z}&\left(\sqrt{\frac{1}{15}}\,|1,2;-12\rangle + \sqrt{\frac{2}{5}}\,|1,2;10\rangle + \sqrt{\frac{8}{15}}\,|1,2;01\rangle\right) \\
&= -\hbar\frac{1}{15} + \hbar\frac{2}{5} = \frac{\hbar}{3}
\end{aligned}$$

$$\begin{aligned}
\langle J_{2z}\rangle &= \langle 1,2;31|\,J_{2z}\,|1,2;31\rangle \\
&= \left(\sqrt{\frac{1}{15}}\,\langle 1,2;-12| + \sqrt{\frac{2}{5}}\,\langle 1,2;10| + \sqrt{\frac{8}{15}}\,\langle 1,2;01|\right) \\
J_{2z}&\left(\sqrt{\frac{1}{15}}\,|1,2;-12\rangle + \sqrt{\frac{2}{5}}\,|1,2;10\rangle + \sqrt{\frac{8}{15}}\,|1,2;01\rangle\right) \\
&= 2\hbar\frac{1}{15} + \hbar\frac{8}{15} = \frac{2\hbar}{3}
\end{aligned}$$

The probability that J_{1z} and J_{2z} never change from their original values is given by

$$|\,\langle 1,2;-12|1,2;31\rangle\,|^2 = 1/15$$

If we instead measure the smallest possible value, we are in state $|1,2;11\rangle$. The third particle being added has $j_3 = 1$ and $m_3 = -1$. We can consider the first two particles as a single composite particle in state $|jm\rangle = |11\rangle$. We now have two particles with $j = 1$. Taking $m_1 = 1$ and $m_2 = -1$ and reading off the table above, the probabilities are

$$\mathbf{Pr}(j) = \begin{cases} \frac{1}{3}, j = 0 \\ \frac{1}{2}, j = 1 \\ \frac{1}{6}, j = 2 \end{cases}$$

$m = 0$

m_1, m_2 \ j	2	1	0
1, −1	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$
0, 0	$\sqrt{\frac{2}{3}}$	0	$-\sqrt{\frac{1}{3}}$
−1, 1	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$

Figure 2: Clebsch-Gordon coefficients for $j_1 = 1$, $j_2 = 1$ and $m = 0$

Finally, the expectation value of J^2 for this three particle system is

$$\begin{aligned}
\langle J^2 \rangle &= \langle 1,1;1,-1|\, J^2 \,|1,1;1,-1\rangle \\
&= \langle 1,1;1,-1|\left(J_1^2 + J_2^2 + 2J_{1z}J_{2z}\right)|1,1;1,-1\rangle \\
&= j_1(j_1+1)\hbar^2 + j_2(j_2+1)\hbar^2 - 2\hbar^2 \\
&= 2\hbar^2
\end{aligned}$$

$\blacksquare$