Homework 3

Quantum Mechanics

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Problem 1. *Problem 2.1 from Sakurai*

Solution. The Heisenberg equation of motion reads

$$\frac{dA}{dt} = \frac{1}{i\hbar}[A, H]$$

For the spin precession problem, we have the Hamiltonian

$$H = -\left(\frac{eB}{mc}\right) S_z = \omega S_z$$

For $A = S_x, S_y, S_z$, the time evolution is given by

$$\begin{aligned}
\frac{dS_x}{dt} &= \frac{\omega}{i\hbar}[S_x, S_z] = -\omega S_y \\
\frac{dS_y}{dt} &= \frac{\omega}{i\hbar}[S_y, S_z] = \omega S_x \\
\frac{dS_z}{dt} &= \frac{\omega}{i\hbar}[S_z, S_z] = 0
\end{aligned}$$

The above system has a straightforward solution:

$$\begin{aligned}
S_x(t) &= S_x(0)\cos(\omega t) \\
S_y(t) &= S_y(0)\sin(\omega t) \\
S_z(t) &= S_z(0)
\end{aligned}$$

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Problem 2. *Problem 2.3 from Sakurai*

Solution. We are given that $\vec{B} = B\hat{z}$ and that we are in the eigenstate $|\psi(0)\rangle = |\boldsymbol{S} \cdot \hat{\boldsymbol{n}}\rangle_+$, which reads

$$\begin{aligned} |\psi(0)\rangle &= \psi_+ |+\rangle + \psi_- |-\rangle \\ &= \cos\frac{\beta}{2} |+\rangle + \sin\frac{\beta}{2} |-\rangle \end{aligned}$$

where we have set $\alpha = 0$ since the ket is in the x-z plane. This state will evolve according to a Hamiltonian

$$H = -\left(\frac{eB}{m_e c}\right) S_z$$

Let $\omega = |e|B/m_e c$ giving $H = \omega S_z$. We have the energies

$$E_\pm = \mp\frac{e\hbar B}{2m_e c} = \mp\hbar\omega$$

$$\begin{aligned} |\psi(t)\rangle &= \psi_+(0) \exp\left(\frac{-iE_+ t}{\hbar}\right) |+\rangle + \psi_-(0) \exp\left(\frac{-iE_- t}{\hbar}\right) |-\rangle \\ &= \cos\frac{\beta}{2} \exp\left(\frac{-i\omega t}{2}\right) |+\rangle + \sin\frac{\beta}{2} \exp\left(\frac{i\omega t}{2}\right) |-\rangle \end{aligned}$$

In general, the probability of measuring $|+\rangle_x = \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} |-\rangle$ is given by the inner product

$$\begin{aligned} |\langle S_x; +|\psi; t\rangle|^2 = &\left| \left(\frac{1}{\sqrt{2}} \langle +| + \frac{1}{\sqrt{2}} \langle -|\right) \cdot \right. \\ &\left. \left(\psi_+ \exp\left(\frac{-i\omega t}{2}\right) |+\rangle + \psi_- \exp\left(\frac{i\omega t}{2}\right) |-\rangle\right) \right|^2 \\ &= \left| \frac{1}{\sqrt{2}} \cos\frac{\beta}{2} \exp\left(\frac{-i\omega t}{2}\right) + \frac{1}{\sqrt{2}} \sin\frac{\beta}{2} \exp\left(\frac{i\omega t}{2}\right) \right|^2 \end{aligned}$$

Using the half-angle identity for $\sin\theta$ and some straightforward arithmetic gives

$$|\langle S_x;+|\psi;t\rangle|^2 = \frac{1+\sin\beta\cos\omega t}{2}$$

For the time-dependence of $\langle S_x\rangle$, we have

$$\begin{aligned}
\langle S_x\rangle(t) &= \langle\psi;t|\, S_x\,|\psi;t\rangle \\
&= \left(\psi_+ \exp\left(\frac{i\omega t}{2}\right)\langle +| + \psi_- \exp\left(\frac{-i\omega t}{2}\right)\langle -|\right) \\
&\cdot \frac{\hbar}{2}\left(\psi_+ \exp\left(-\frac{i\omega t}{2}\right)|-\rangle + \psi_- \exp\left(\frac{i\omega t}{2}\right)|+\rangle\right)
\end{aligned}$$

Subsituting ψ_+ and ψ_- with the same values as above, we get

$$\langle S_x\rangle(t) = \frac{\hbar}{2}\sin\beta\cos\omega t$$

When $\beta = \pi/2$ the probability oscillates between 0 and 1 with frequency ω and when $\beta = 0$ then the probability is always 1/2, as expected. The expectation value also makes sense because when $\beta = 0$, we can get $\pm\hbar/2$ with equal probability, giving zero on average. When $\beta = \pi/2$ the expectation value oscillates between $\hbar/2$ and $-\hbar/2$.

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Problem 3. *Problem 2.9 from Sakurai*

Solution.

We were given the wavefunction

$$\langle x|\alpha\rangle = A(x-a)^2(x+a)^2\exp(ikx)$$

To find the normalization constant A, we evaluate

$$\begin{aligned}
A &= \left(\int_{-a}^{+a}\langle\alpha|x\rangle\langle x|\alpha\rangle\, dx\right)^{-1/2} \\
&= \left(\int_{-a}^{+a}(x-a)^4(x+a)^4 dx\right)^{-1/2} \\
&= \frac{\sqrt{315}}{16a^{9/2}}
\end{aligned}$$

according to Mathematica. The expectation value $\langle x \rangle$ is found by integrating

$$
\begin{aligned}
\langle x \rangle &= \int_{-a}^{a} x \langle \alpha | x \rangle \langle x | \alpha \rangle \, dx \\
&= \int_{-a}^{+a} x \langle \alpha | x \rangle \langle x | \alpha \rangle \, dx \\
&= \int_{-a}^{+a} x (x-a)^4 (x+a)^4 dx \\
&= 0
\end{aligned}
$$

since the function is odd. This makes physical sense, given the overall symmetry of the wavefunction. Notice that the complex exponential vanishes due to the complex conjugation. The expectation value $\langle x^2 \rangle$ is found by integrating

$$
\langle x^2 \rangle = \int_{-a}^{+a} x^2 (x-a)^4 (x+a)^4 dx
$$

This we do not expect to be zero, and indeed it is not:

Integrate $\left[\boldsymbol{x\text{^}2(x-a)^4(x+a)^4, \{x, -a, a\}}\right]$

$\frac{256a^{11}}{3465}$

and multiplying by A^2 gives

$$
\langle x^2 \rangle = \left(\frac{\sqrt{315}}{16a^{9/2}} \right)^2 \frac{256a^{11}}{3465} = \frac{a^2}{11}
$$

The expectation value $\langle p \rangle$ is found similarly

$$
\begin{aligned}
\langle p \rangle &= \int_{-a}^{+a} \langle \alpha | x \rangle \frac{\hbar}{i} \frac{\partial}{\partial x} \langle x | \alpha \rangle \, dx \\
&= A^2 \int_{-a}^{+a} (x-a)^2 (x+a)^2 \exp(-ikx) \frac{\hbar}{i} \frac{\partial}{\partial x} (x-a)^2 (x+a)^2 \exp(ikx)
\end{aligned}
$$

Leaving out the factor of $\hbar$ from p gives

$$\boldsymbol{f = D\left[(x-a)^2(x+a)^2\mathrm{Exp}[Ikx],\{x,1\}\right]}$$

$$\textbf{Integrate}\left[\boldsymbol{(x-a)^2(x+a)^2\mathrm{Exp}[-Ikx]*-I*f,\{x,-a,a\}}\right]$$

$$2e^{ikx}(-a+x)^2(a+x)+2e^{ikx}(-a+x)(a+x)^2+ie^{ikx}k(-a+x)^2(a+x)^2$$

$\frac{256a^9k}{315}$

Bringing in the factor of $\hbar$ and multiplying by A^2 gives a familiar expression

$$\langle p\rangle = \hbar k$$

The expectation value $\langle p^2\rangle$ is found similarly

$$\begin{aligned}\langle p^2\rangle &= -\int_{-a}^{+a}\langle\alpha|x\rangle\,\hbar^2\frac{\partial^2}{\partial x^2}\langle x|\alpha\rangle\,dx\\ &= -A^2\int_{-a}^{+a}(x-a)^2(x+a)^2\exp(-ikx)\hbar^2\frac{\partial^2}{\partial x^2}(x-a)^2(x+a)^2\exp(ikx)\end{aligned}$$

$$\boldsymbol{f = -D\left[(x-a)^2(x+a)^2\mathrm{Exp}[Ikx],\{x,2\}\right]}$$

$$\textbf{Integrate}\left[\boldsymbol{(x-a)^2(x+a)^2\mathrm{Exp}[-Ikx]*f,\{x,-a,a\}}\right]$$

$$-2e^{ikx}(-a+x)^2-4(a+x)\left(2e^{ikx}(-a+x)+ie^{ikx}k(-a+x)^2\right)-$$
$$(a+x)^2\left(2e^{ikx}+4ie^{ikx}k(-a+x)-e^{ikx}k^2(-a+x)^2\right)$$

$\frac{256a^7}{105}+\frac{256a^9k^2}{315}$

Bringing in a factor of $\hbar^2$ and multiplying by A^2 gives

$$\langle p^2\rangle = \frac{3\hbar^2}{a^2}+\hbar^2k^2$$

The variance $\langle(\Delta x)^2\rangle$ is just

$$\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2 = \frac{a^2}{11}$$

The variance $\langle(\Delta p)^2\rangle$ is just

$$\begin{aligned}\langle(\Delta p)^2\rangle &= \langle p^2\rangle - \langle p\rangle^2 \\ &= \frac{3\hbar^2}{a^2}\end{aligned}$$

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Problem 4. *Problem 2.10 from Sakurai*

Solution. Let $|\psi\rangle = \alpha\,|a'\rangle + \beta\,|a''\rangle$ be an eigenvector of the Hamiltonian. Note that this must be real for the eigenvalue to be real. That means that

$$\begin{aligned}H\,|\psi\rangle &= \left(|a'\rangle\,\delta\,\langle a''| + |a''\rangle\,\delta\,\langle a'|\right)\left(\alpha\,|a'\rangle + \beta\,|a''\rangle\right) \\ &= \delta\left(\alpha\,|a''\rangle + \beta\,|a'\rangle\right)\end{aligned}$$

Therefore $\alpha = \beta = \frac{1}{\sqrt{2}}$ or $\alpha = \frac{1}{\sqrt{2}}$ and $\beta = -\frac{1}{\sqrt{2}}$. Giving eigenvalues $\pm\delta$. To get the time evolution of the state, we need to express these in the basis of H. Just based on inspection of the the two bases, we can tell that

$$\begin{aligned}|a'\rangle &= \frac{1}{\sqrt{2}}\left(|\psi_1\rangle - |\psi_2\rangle\right) \\ |a''\rangle &= \frac{1}{\sqrt{2}}\left(|\psi_1\rangle + |\psi_2\rangle\right)\end{aligned}$$

and, since the Hamiltonian is time-independent, a state prepared in $|a'\rangle$ will evolve according to

$$|\alpha(t)\rangle = \frac{1}{\sqrt{2}}\exp\left(\frac{-i\delta t}{\hbar}\right)|\psi_1\rangle - \frac{1}{\sqrt{2}}\exp\left(\frac{i\delta t}{\hbar}\right)|\psi_2\rangle$$

The probability of finding the system in the state $|a''\rangle$ at a later time is

$$
\begin{aligned}
|\langle a''|\alpha(t)\rangle|^2 &= \Bigg| \frac{1}{\sqrt{2}} \left(\langle \psi_1 | + \langle \psi_2 | \right) \\
&\cdot \left(\frac{1}{\sqrt{2}} \exp\left(\frac{-i\delta t}{\hbar} \right) |\psi_1\rangle - \frac{1}{\sqrt{2}} \exp\left(\frac{i\delta t}{\hbar} \right) |\psi_2\rangle \right) \Bigg|^2 \\
&= \frac{1}{4} \sin^2 \frac{\delta t}{\hbar}
\end{aligned}
$$

This could describe a system in which the eigenvectors of the Hamiltonian are simultaneous with the eigenvectors of S_x, however the states $|a'\rangle$ and $|a''\rangle$ are expressed in the S_z basis.

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Problem 5. *Problem 2.12 from Sakurai*

Solution. The state is prepared in

$$
|\alpha; t = 0\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{\exp(i\delta)}{\sqrt{2}} |1\rangle
$$

In general, the energies of $|n\rangle$ are $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$. Therefore, the time dependence of the state can be evalulated as

$$
\begin{aligned}
|\alpha; t\rangle &= \exp\left(-\frac{iHt}{\hbar} \right) |\alpha; t\rangle \\
&= \frac{1}{\sqrt{2}} \exp \frac{-i\omega t}{2} |0\rangle + \frac{1}{\sqrt{2}} \exp(i\delta) \exp \frac{-3i\omega t}{2} |1\rangle
\end{aligned}
$$

$$
\langle x|\alpha; t\rangle = \frac{1}{\sqrt{2}} \exp \frac{-i\omega t}{2} \langle x|0\rangle + \frac{1}{\sqrt{2}} \exp(i\delta) \exp \frac{-3i\omega t}{2} \langle x|1\rangle
$$

and we know in general that the position representation of $|n\rangle$ i.e., $\langle x|n\rangle$ are

$$
\langle x|n\rangle = \psi_n(x) = \frac{1}{\sqrt{2^n\, n!}} \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}} x \right)
$$

$$
\begin{aligned}
\langle x\rangle(t) &= \langle\alpha;t|\,x\,|\alpha;t\rangle \\
&= \left(\frac{1}{\sqrt{2}}\exp\frac{i\omega t}{2}\langle 0| + \frac{1}{\sqrt{2}}\exp(-i\delta)\exp\frac{3i\omega t}{2}\langle 1|\right) \\
&\quad x\left(\frac{1}{\sqrt{2}}\exp\frac{-i\omega t}{2}|0\rangle + \frac{1}{\sqrt{2}}\exp(i\delta)\exp\frac{-3i\omega t}{2}|1\rangle\right) \\
&= \frac{1}{2}\langle 0|\,x\,|0\rangle + \frac{1}{2}\langle 1|\,x\,|1\rangle \\
&\quad + \frac{1}{2}\exp(i\delta)\exp(-i\omega t)\langle 0|\,x\,|1\rangle + \frac{1}{2}\exp(-i\delta)\exp(i\omega t)\langle 1|\,x\,|0\rangle
\end{aligned}
$$

Now recall the general expression for the matrix element of x

$$
\langle n'|\,x\,|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\delta_{n',n-1} + \sqrt{n+1}\delta_{n',n+1}\right)
$$

which means that the above expression simplifies to

$$
\begin{aligned}
\langle x\rangle(t) &= \frac{1}{2}\exp(i\delta)\exp(-i\omega t)\langle 0|\,x\,|1\rangle + \frac{1}{2}\exp(-i\delta)\exp(i\omega t)\langle 1|\,x\,|0\rangle \\
&= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(\exp(i\delta)\exp(-i\omega t) + \exp(-i\delta)\exp(i\omega t)\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\cos(\delta - \omega t) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t - \delta)
\end{aligned}
$$

For momentum, we can just replace the operator x with p in the expressions above:

$$
\begin{aligned}
\langle p\rangle(t) &= \langle\alpha;t|\,x\,|\alpha;t\rangle \\
&= \left(\frac{1}{\sqrt{2}}\exp\frac{i\omega t}{2}\langle 0| + \frac{1}{\sqrt{2}}\exp(-i\delta)\exp\frac{3i\omega t}{2}\langle 1|\right) \\
&\quad \hat{p}\left(\frac{1}{\sqrt{2}}\exp\frac{-i\omega t}{2}|0\rangle + \frac{1}{\sqrt{2}}\exp(i\delta)\exp\frac{-3i\omega t}{2}|1\rangle\right) \\
&= \frac{1}{2}\langle 0|\,p\,|0\rangle + \frac{1}{2}\langle 1|\,p\,|1\rangle \\
&\quad + \frac{1}{2}\exp(i\delta)\exp(-i\omega t)\langle 0|\,p\,|1\rangle + \frac{1}{2}\exp(-i\delta)\exp(i\omega t)\langle 1|\,p\,|0\rangle
\end{aligned}
$$

and we have another general expression for the matrix element of p

$$\langle n'| p |n\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(-\sqrt{n}\delta_{n',n-1} + \sqrt{n+1}\delta_{n',n+1}\right)$$

which again means that the above expression simplifies to

$$\begin{aligned}\langle p\rangle(t) &= \frac{1}{2}\exp(i\delta)\exp(-i\omega t)\langle 0| p |1\rangle + \frac{1}{2}\exp(-i\delta)\exp(i\omega t)\langle 1| p |0\rangle \\ &= \frac{i}{2}\sqrt{\frac{m\hbar\omega}{2}}\left(-\exp(i\delta)\exp(-i\omega t) + \exp(-i\delta)\exp(i\omega t)\right) \\ &= -\sqrt{\frac{m\hbar\omega}{2}}\sin(\omega t - \delta)\end{aligned}$$

In the Heisenberg picture, we have the Heisenberg equations of motion

$$\begin{aligned}\frac{dp}{dt} &= -m\omega^2 x \\ \frac{dx}{dt} &= \frac{p}{m}\end{aligned}$$

It is has been shown in the text how to uncouple these in terms of the ladder operators and solve the system for the time dependent operators $x(t)$ and $p(t)$

$$\begin{aligned}x(t) &= x(0)\cos(\omega t) + \left(\frac{p(0)}{m\omega}\right)\sin(\omega t) \\ p(t) &= -m\omega x(0)\sin(\omega t) + p(0)\cos(\omega t)\end{aligned}$$

To get $\langle x\rangle(t)$ we have

$$\begin{aligned}\langle x\rangle(t) &= \langle\alpha| x(0)\cos(\omega t) + \left(\frac{p(0)}{m\omega}\right)\sin(\omega t)|\alpha\rangle \\ &= \langle\alpha| x(0)|\alpha\rangle\cos(\omega t) + \langle\alpha| p(0)|\alpha\rangle\frac{\sin(\omega t)}{m\omega}\end{aligned}$$

$$\begin{aligned}
\langle\alpha| x(0) |\alpha\rangle &= \frac{1}{2}\langle 0| x(0) |0\rangle + \frac{1}{2}\langle 1| x(0) |1\rangle \\
&+ \frac{1}{2}\exp(i\delta)\langle 0| x(0) |1\rangle + \frac{1}{2}\exp(i\delta)\langle 1| x(0) |0\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}}\exp(i\delta)
\end{aligned}$$

The factor of one-half disappeared in the $\langle\alpha| x(0) |\alpha\rangle$ term since $\langle n'| x |n\rangle$ is real and therefore equal to its complex conjugate

$$\begin{aligned}
\langle x\rangle(t) &= \sqrt{\frac{\hbar}{2m\omega}}\exp(i\delta)\cos(\omega t) + \frac{i}{2}\exp(i\delta)\sqrt{\frac{m\hbar\omega}{2}}\frac{\sin(\omega t)}{m\omega} \\
&= \sqrt{\frac{\hbar}{2m\omega}}\exp(i\delta)\exp(i\omega t) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\exp(i(\omega t + \delta))
\end{aligned}$$

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Problem 6. *Problem 2.13 from Sakurai*

Solution.

The Heisenberg equations of motion are

$$\begin{aligned}
\frac{dp}{dt} &= F \\
\frac{dx}{dt} &= \frac{p_0 + Ft}{m}
\end{aligned}$$

The solution is simply

$$\begin{aligned}
p(t) &= p(0) + Ft \\
x(t) &= x(0) + \frac{p(0)}{m}t + \frac{1}{2}\frac{Ft^2}{m}
\end{aligned}$$

$$\begin{aligned}
\langle x\rangle(t) &= \langle\alpha| \left(x(0) + \frac{p_0}{m}t + \frac{1}{2}\frac{Ft^2}{m}\right) |\alpha\rangle \\
&= x_0 + \frac{p_0 t}{m} + \frac{Ft^2}{2m}
\end{aligned}$$

and for $\langle p \rangle$ we have

$$\begin{aligned} \langle p \rangle(t) &= \langle \alpha | \, p(0) + Ft \, | \alpha \rangle \\ &= p_0 + Ft \end{aligned}$$

In the Schrodinger picture, this Hamiltonian is not a constant

$$H(t) = \frac{p(t)^2}{2m} + V(x,t) = \frac{p(t)^2}{2m} + Fx(t)$$

although, using the canonical commutation relations, we can see that $H(t_0)$ commutes with $H(t)$. So we should be able to define

$$\mathcal{U}(t, t_0) = \exp\left(-\frac{i}{\hbar} \int_{t_0}^{t} H(t) dt \right)$$

such that $|\alpha; t\rangle = \mathcal{U}(t, t_0) \, |\alpha; t_0\rangle$. If we evaluated that expression, then we could write $|\alpha; t\rangle$ explicitly.

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