Bell's Inequality

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CHSH and Tsirelson's Inequalities

Alice: Q, R Bob: S, T

Classical observables distributed according to $P(Q, R, S, T)$. Combination of correlations between Alice and Bobs measurements are bounded according to the CHSH inequality

$$|E(QS) + E(RS) + E(RT) - E(QT)| \leq 2$$

For the quantum version, define 4 spin operators along arbitrary directions $Q = \vec{q} \cdot \sigma, R = \vec{r} \cdot \sigma, S = \vec{s} \cdot \sigma, T = \vec{t} \cdot \sigma$.

$$|\langle Q \otimes S \rangle + \langle R \otimes S \rangle + \langle R \otimes T \rangle - \langle Q \otimes T \rangle| \leq 2\sqrt{2}$$

The Tsirelson bound

Solution to Problem 2.3 in the book:

$$(Q \otimes S + R \otimes S + R \otimes T - Q \otimes T)^2 = 4I + [Q, R] \otimes [S, T]$$

Jensen's inequality:

$$\begin{aligned}
\langle (Q \otimes S + R \otimes S + R \otimes T - Q \otimes T) \rangle^2 &\leq \langle (Q \otimes S + R \otimes S + R \otimes T - Q \otimes T)^2 \rangle \\
&= \langle 4I + [Q, R] \otimes [S, T] \rangle \\
&= 4 + \langle [Q, R] \otimes [S, T] \rangle
\end{aligned}$$

The Tsirelson bound

Using that $(\sigma \cdot \vec{a})(\sigma \cdot \vec{b}) = \vec{a} \cdot \vec{b} + i\sigma \cdot (\vec{a} \times \vec{b})$

$$[A, B] = i\sigma \cdot (\vec{a} \times \vec{b} - \vec{b} \times \vec{a}) = 2i\sigma \cdot (\vec{a} \times \vec{b})$$

Let $\vec{n} = \vec{q} \times \vec{r}$ and $\vec{m} = \vec{s} \times \vec{t}$.

$$\langle [Q, R] \otimes [S, T] \rangle = -4 \langle \psi | \, \sigma \cdot \vec{n} \otimes \sigma \cdot \vec{m} \, | \psi \rangle \leq 4$$

Inserting this and taking the square root of both sides of the expression on the last slide then gives,

$$\langle (Q \otimes S + R \otimes S + R \otimes T - Q \otimes T) \rangle \leq 2\sqrt{2}$$

Question: How does the LHS scale with entanglement entropy?

The book uses $\vec{q}=(0,0,1), \vec{r}=(1,0,0), \vec{s}=(-\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}), \vec{t}=(-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}})$, which gives

$$\vec{r}\cdot\sigma\otimes\vec{s}\cdot\sigma=\begin{pmatrix}0 & \vec{s}\cdot\sigma\\ \vec{s}\cdot\sigma & 0\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0 & -1 & -1\\ 0 & 0 & -1 & 1\\ -1 & -1 & 0 & 0\\ -1 & 1 & 0 & 0\end{pmatrix}$$

$$\vec{r}\cdot\sigma\otimes\vec{t}\cdot\sigma=\begin{pmatrix}0 & \vec{t}\cdot\sigma\\ \vec{t}\cdot\sigma & 0\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & 0 & 1 & -1\\ 0 & 0 & -1 & -1\\ 1 & -1 & 0 & 0\\ -1 & -1 & 0 & 0\end{pmatrix}$$

$$\vec{q}\cdot\sigma\otimes\vec{t}\cdot\sigma=\begin{pmatrix}\vec{t}\cdot\sigma & 0\\ 0 & -\vec{t}\cdot\sigma\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1 & 0 & 0\\ -1 & -1 & 0 & 0\\ 0 & 0 & -1 & 1\\ 0 & 0 & 1 & 1\end{pmatrix}$$

$$\vec{q}\cdot\sigma\otimes\vec{s}\cdot\sigma = \begin{pmatrix} \vec{s}\cdot\sigma & 0 \\ 0 & -\vec{s}\cdot\sigma \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\langle \vec{q}\cdot\sigma\otimes\vec{s}\cdot\sigma\rangle = \frac{1}{\sqrt{2}}\left(-\alpha^*(\alpha+\beta)+\beta^*(\beta-\alpha)+\gamma^*(\gamma+\delta)+\delta^*(\gamma-\delta)\right)$$
$$\langle \vec{r}\cdot\sigma\otimes\vec{s}\cdot\sigma\rangle = \frac{1}{\sqrt{2}}\left(-\alpha^*(\gamma+\delta)+\beta^*(\delta-\gamma)-\gamma^*(\alpha+\beta)+\delta^*(\beta-\alpha)\right)$$
$$\langle \vec{r}\cdot\sigma\otimes\vec{t}\cdot\sigma\rangle = \frac{1}{\sqrt{2}}\left(\alpha^*(\gamma-\delta)-\beta^*(\delta+\gamma)+\gamma^*(\alpha-\beta)-\delta^*(\beta+\alpha)\right)$$
$$\langle \vec{q}\cdot\sigma\otimes\vec{t}\cdot\sigma\rangle = \frac{1}{\sqrt{2}}\left(\alpha^*(\alpha-\beta)-\beta^*(\beta+\alpha)+\gamma^*(\delta-\delta)+\delta^*(\gamma+\delta)\right)$$

Entanglement entropy: partial traces

$$
\begin{aligned}
\mathrm{Tr}_A(\rho_{AB}) &= \sum_{ijkl} \rho_{ij}^{kl} \mathrm{Tr}_A(|i\rangle\langle k|) \otimes |j\rangle\langle l| \\
&= \sum_i \left(\sum_{jl} \rho_{ij}^{il} |j\rangle\langle l| \right) \\
&= (\rho_{00}^{00} + \rho_{10}^{10})|0\rangle\langle 0| + (\rho_{00}^{01} + \rho_{10}^{11})|0\rangle\langle 1| + (\rho_{01}^{00} + \rho_{11}^{10})|1\rangle\langle 0| + (\rho_{01}^{01} + \rho_{11}^{11})|1\rangle\langle 1|
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Tr}_B(\rho_{AB}) &= \sum_{ijkl} \rho_{ij}^{kl} |i\rangle\langle k| \otimes \mathrm{Tr}_B(|j\rangle\langle l|) \\
&= \sum_j \left(\sum_{ik} \rho_{ij}^{kj} |i\rangle\langle k| \right) \\
&= (\rho_{00}^{00} + \rho_{01}^{01})|0\rangle\langle 0| + (\rho_{00}^{10} + \rho_{01}^{11})|0\rangle\langle 1| + (\rho_{10}^{00} + \rho_{11}^{01})|1\rangle\langle 0| + (\rho_{10}^{10} + \rho_{11}^{11})|1\rangle\langle 1|
\end{aligned}
$$

Reduced density matrices for an arbitrary two-qubit state

$$\rho_B = \mathrm{Tr}_A(\rho_{AB}) = \begin{pmatrix} \rho^{00}_{00} + \rho^{10}_{10} & \rho^{01}_{00} + \rho^{11}_{10} \\ \rho^{00}_{01} + \rho^{10}_{11} & \rho^{01}_{01} + \rho^{11}_{11} \end{pmatrix} = \begin{pmatrix} |\alpha|^2 + |\gamma|^2 & \alpha\beta^* + \gamma\delta^* \\ \beta\alpha^* + \delta\gamma^* & |\beta|^2 + |\delta|^2 \end{pmatrix}$$

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) = \begin{pmatrix} \rho^{00}_{00} + \rho^{01}_{01} & \rho^{10}_{00} + \rho^{11}_{01} \\ \rho^{00}_{10} + \rho^{01}_{11} & \rho^{10}_{10} + \rho^{11}_{11} \end{pmatrix} = \begin{pmatrix} |\alpha|^2 + |\beta|^2 & \alpha\gamma^* + \beta\delta^* \\ \gamma\alpha^* + \delta\beta^* & |\gamma|^2 + |\delta|^2 \end{pmatrix}$$

The entanglement entropy of a bipartite system is the Von Neumann entropy of either reduced density matrix

$$S(\rho_A) = -\mathrm{Tr}(\rho_A \log \rho_A) = -\sum_x \lambda_x \log \lambda_x$$

for eigenvalues λ_x of ρ_A.

Relating the correlation function to entanglement

Algorithm:

Draw random reals $a, b, c, d, e, f, g, h \sim U([0,1]^8)$

Construct $|\psi\rangle = (a + ib)|00\rangle + (c + id)|01\rangle + (e + if)|10\rangle + (g + ih)|11\rangle$

Normalize $|\psi\rangle \rightarrow \frac{|\psi\rangle}{\sum_n |c_n|^2}$

Compute $S(\rho_A)$ and $\langle\psi| (Q \otimes S + R \otimes S + R \otimes T - Q \otimes T) |\psi\rangle$

Scatter plot