Homework 6

Quantum Mechanics
October 28th, 2022 C Seitz

Problem 1. *Problem 3.12 from Sakurai*

Solution.

In general the ensemble average of an operator $[A]$ is defined as

$$[A] = \mathrm{Tr}(\rho A)$$

where $\hat{\rho} = \sum_i w_i \rho_i$ and $\rho_i = |\alpha_i\rangle \langle\alpha_i|$

$$\begin{aligned}\hat{\rho} &= a\,|+\rangle\,\langle+| + (1-a)\,|-;y\rangle\,\langle-;y| \\ &= \begin{pmatrix} \frac{1-a}{2} + a & \frac{1}{2}i(1-a) \\ -\frac{1}{2}i(1-a) & \frac{1-a}{2} \end{pmatrix}\end{aligned}$$

$$\begin{aligned}[][S_x] &= \mathrm{Tr}(\hat{\rho} S_x) \\ &= \frac{\hbar}{2}\mathrm{Tr}\left(\begin{pmatrix} \frac{1}{2}i(1-a) & \frac{1-a}{2} + a \\ \frac{1-a}{2} & -\frac{1}{2}i(1-a) \end{pmatrix}\right) = 0\end{aligned}$$

$$\begin{aligned}[][S_y] &= \mathrm{Tr}(\hat{\rho} S_x) \\ &= \frac{\hbar}{2}\mathrm{Tr}\left(\begin{pmatrix} \frac{a-1}{2} & -i\left(\frac{1-a}{2} + a\right) \\ \frac{1}{2}i(1-a) & \frac{a-1}{2} \end{pmatrix}\right) = \frac{\hbar}{2}(a-1)\end{aligned}$$

$$\begin{aligned}[][S_z] &= \mathrm{Tr}(\hat{\rho} S_z) \\ &= \frac{\hbar}{2}\mathrm{Tr}\left(\begin{pmatrix} \frac{1-a}{2} + a & -\frac{1}{2}i(1-a) \\ -\frac{1}{2}i(1-a) & \frac{a-1}{2} \end{pmatrix}\right) = \frac{\hbar}{2}a\end{aligned}$$

This is what we would expect for the pure ensemble cases $a = 1$ and $a = 0$.

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Problem 2. *Problem 3.13 from Sakurai*

Solution.

The state vector in the S_z basis has the form

$$|\alpha\rangle = c_+ |+\rangle + c_- |-\rangle$$

First note that

$$\langle S_z \rangle = |c_+|^2 - |c_-|^2 \quad |c_+|^2 + |c_-|^2 = 1$$

Together, these equations tell us the magnitude of each complex component.

$$|c_+|^2 = \frac{\langle S_z \rangle + 1}{2} \quad |c_-|^2 = \frac{1 - \langle S_z \rangle}{2}$$

$$\begin{aligned}
\langle S_x \rangle &= \langle \alpha| (|+\rangle \langle -| + |-\rangle \langle +|)(c_+ |+\rangle + c_- |-\rangle) \\
&= \langle \alpha| (c_- |+\rangle + c_+ |-\rangle) \\
&= (c_+^* \langle +| + c_-^* \langle -|)(c_- |+\rangle + c_+ |-\rangle) \\
&= c_+^* c_- + c_-^* c_+ \\
&= |c_+||c_-|(e^{i(\theta-\phi)} + e^{i(\phi-\theta)}) \\
&= 2|c_+||c_-| \cos(\theta - \phi)
\end{aligned}$$

$$\begin{aligned}
\langle S_y \rangle &= \langle \alpha| ((i |+\rangle \langle -| - i |-\rangle \langle +|)(c_+ |+\rangle + c_- |-\rangle) \\
&= i \langle \alpha| (c_- |+\rangle - c_+ |-\rangle) \\
&= i(c_+^* \langle +| + c_-^* \langle -|)(c_- |+\rangle - c_+ |-\rangle) \\
&= c_+^* c_- - c_-^* c_+ \\
&= |c_+||c_-|(e^{i(\theta-\phi)} - e^{i(\phi-\theta)}) \\
&= 2i|c_+||c_-| \sin(\theta - \phi)
\end{aligned}$$

So $\langle S_x \rangle$ gives us the phase difference of c_+ and c_-. Then the sign of $\langle S_y \rangle$ tells us whether θ or ϕ is larger, since sine is odd. This is all we can hope to extract from the expectation values, since multiplying by a global phase $e^{i\delta} |\alpha\rangle$ has no effect on the expectation values.

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Problem 3. *Problem 3.14 from Sakurai*

Solution.

$$\begin{aligned}\hat{\rho} &= \sum_i w_i \left|\psi_i\right\rangle \left\langle\psi_i\right| \\ &= \frac{1}{3}\left(\left|\alpha\right\rangle\left\langle\alpha\right| + \left|\beta\right\rangle\left\langle\beta\right| + \left|2\right\rangle\left\langle 2\right|\right)\end{aligned}$$

We can write this out explicitly in the subspace spanned by $\left|0,1,2\right\rangle$

$$\left|\alpha\right\rangle\left\langle\alpha\right| = \frac{1}{2}\begin{pmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} \quad \left|\beta\right\rangle\left\langle\beta\right| = \frac{1}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & 1\end{pmatrix} \quad \left|2\right\rangle\left\langle 2\right| = \frac{1}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 2\end{pmatrix}$$

$$\hat{\rho} = \frac{1}{6}\begin{pmatrix}1 & 1 & 0\\ 1 & 2 & 1\\ 0 & 1 & 3\end{pmatrix}$$

Now recall that $H = \hbar\omega(N + \frac{1}{2})$ which reads

$$H = \hbar\omega\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix} + \left(\frac{\hbar\omega}{2}\right)\mathbb{I}_{3\times 3}$$

$$\begin{aligned}[H] = \mathrm{Tr}(\rho H) &= \hbar\omega\mathrm{Tr}(\rho N + \rho/2) \\ &= \hbar\omega\left(\mathrm{Tr}(\rho N) + \mathrm{Tr}(\rho/2)\right) \\ &= \frac{11}{6}\hbar\omega\end{aligned}$$

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Problem 4. *Problem 3.15 from Sakurai*

Solution.

$$\rho(t_0) = \sum_i w_i \left|\psi_i; t_0\right\rangle \left\langle\psi_i; t_0\right|$$

In the Schrodinger picture, the coefficients of the state vectors evolve. Therefore,

$$\begin{aligned}\rho(t) &= \sum_i w_i\, \mathcal{U}(t,t_0)\, |\psi_i; t_0\rangle \langle\psi_i; t_0| \mathcal{U}^\dagger(t,t_0) \\ &= \mathcal{U}(t,t_0) \left(\sum_i w_i\, |\psi_i; t_0\rangle \langle\psi_i; t_0| \right) \mathcal{U}^\dagger(t,t_0) \\ &= \mathcal{U}(t,t_0)\rho(t_0)\mathcal{U}^\dagger(t,t_0)\end{aligned}$$

For a pure ensemble in state $|\psi_i\rangle$, the density matrix is

$$\rho(t_0) = |\psi_i; t_0\rangle \langle\psi_i; t_0|$$

At a later time, the density matrix is

$$\begin{aligned}\rho(t) &= \mathcal{U}(t,t_0)\rho(t_0)\mathcal{U}^\dagger(t,t_0) \\ &= \mathcal{U}(t,t_0)\, |\psi_i; t_0\rangle \langle\psi_i; t_0| \mathcal{U}^\dagger(t,t_0) \\ &= |\psi_i; t_0\rangle \langle\psi_i; t_0|\end{aligned}$$

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Problem 5. *Problem 3.16 from Sakurai*

Solution. A 3x3 matrix has 9 parameters, but we only need ■

Problem 6. *Problem 3.40 from Sakurai*

Solution. The singlet state is

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|{+-}\rangle + |{-+}\rangle)$$

If B doesn't make a measurement, B will have no effect on A's measurement. So the probability for A to obtain $s_{1z} = \hbar/2$ is of course $1/2$. The probability that A measures $s_{1x} = \hbar/2$ in this state is also $1/2$. This is because obtaining $s_{1x} = \hbar/2$ is equiprobable for the two states in the singlet superposition.

If observer B has determined that $s_{2z} = \hbar/2$, then observer A must observe $s_{1z} = -\hbar/2$ since the measurement made by B collapses $|\psi\rangle$ to $|{-+}\rangle$.

Furthermore, if observer B has measured $s_{2z} = \hbar/2$, then particle 1 must be in the $|+\rangle$ state (as stated before) which means $s_{1x} = \pm\hbar/2$ with equal probability.

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