Homework 5

Quantum Mechanics
October 17th, 2022 C Seitz

Problem 1. *Problem 3.10 from Sakurai*

Solution.

$$\begin{aligned}\exp(i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})\theta) &= \begin{pmatrix}\cos\theta + in_z\sin\theta & (-in_x+n_y)\sin\theta \\ (in_x-n_y)\sin\theta & \cos\theta - in_z\sin\theta\end{pmatrix} \\ &= \begin{pmatrix}e^{-(i\alpha+\gamma)/2}\cos\frac{\beta}{2} & -e^{-(i\alpha-\gamma)/2}\sin\frac{\beta}{2} \\ e^{-(i\alpha-\gamma)/2}\sin\frac{\beta}{2} & e^{(i\alpha+\gamma)/2}\cos\frac{\beta}{2}\end{pmatrix}\end{aligned}$$

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Equating the trace of these matrices gives

$$2\cos\theta = 2\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\frac{\beta}{2}$$

So $\theta = \cos^{-1}(\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\frac{\beta}{2})$

Problem 2. *Problem 3.20 from Sakurai*

Solution.
Recall that

$$J_\pm = J_x \pm iJ_y$$

and thus $J_x = (J_+ + J_-)/2$ and $J_y = \frac{J_+ - J_-}{2i}$. We know that the matrix elements of $J_\pm$ are

$$\langle j', m'|\, J_\pm \,|j, m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\hbar\delta_{jj'}\delta_{m,m'+1}$$

where j is our usual shorthand for $\hbar^2 j(j+1)$ (the eigenvalue of J^2) and m is short for $m\hbar$ (the eigenvalue of J_z). For a spin-1 system, $j = 1$ and $m = -1, 0, 1$ which gives the eigenkets $|1,-1\rangle, |1,0\rangle, |1,-1\rangle$

$$J_+ = \begin{pmatrix} 0 & \sqrt{2}\hbar & 0 \\ 0 & 0 & \sqrt{2}\hbar \\ 0 & 0 & 0 \end{pmatrix} \quad J_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2}\hbar & 0 & 0 \\ 0 & \sqrt{2}\hbar & 0 \end{pmatrix}$$

$$J_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad J_y = \frac{\hbar}{\sqrt{2}i} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

We can use Mathematica to find the eigenvectors of these two matrices

$$|J_x;+\rangle = \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix} \quad |J_x;0\rangle = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \quad |J_x;-1\rangle = \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix}$$

$$|J_y;+\rangle = \begin{pmatrix} -1/2 \\ -i\sqrt{2} \\ 1/2 \end{pmatrix} \quad |J_y;0\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \quad |J_y;-1\rangle = \begin{pmatrix} -1/2 \\ i\sqrt{2} \\ 1/2 \end{pmatrix}$$

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Problem 3. *Problem 3.22 from Sakurai*

Solution.

We are asked to derive

$$\langle x| L_z |\alpha\rangle = -i\hbar \frac{\partial}{\partial \phi} \langle x|\alpha\rangle$$

$$\begin{aligned} \langle x| L_z |\alpha\rangle &= \langle x| (xp_y - yp_x) |\alpha\rangle \\ &= i\hbar \langle x| y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} |\alpha\rangle \end{aligned}$$

We need to write these partial derivatives in spherical coordinates to complete the proof. I just looked up the coordinate transformation and made the substitution

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\begin{aligned}
\langle x| L_z |\alpha\rangle &= i\hbar \langle x| y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} |\alpha\rangle \\
&= i\hbar \left(y\frac{\partial}{\partial x} \langle x|\alpha\rangle - x\frac{\partial}{\partial y} \langle x|\alpha\rangle \right) \\
&= ir\hbar \sin\phi \sin\theta \left(\sin\theta\cos\phi \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \right) \\
&- ir\hbar \cos\phi \sin\theta \left(\sin\theta\sin\phi \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \right) \\
&= ir\hbar \sin\phi\sin\theta \left(-\frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\theta} \right) - ir\hbar\cos\phi\sin\theta \left(\frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \right) \\
&= i\hbar \left(-\sin^2\phi\frac{\partial}{\partial\phi} \right) - i\hbar \left(\cos^2\phi \frac{\partial}{\partial\phi} \right) \\
&= -i\hbar\frac{\partial}{\partial\phi}
\end{aligned}
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Problem 4. *Problem 3.23 from Sakurai*

Solution.

We can write the wavefunction given in spherical coordinates

$$\psi(\boldsymbol{x}) = \langle x|\alpha\rangle = r\left(\cos\phi\sin\theta + \sin\phi\sin\theta + \cos\theta\right) f(r)$$

If this is an eigenfunction of L^2, then we should be able to write it in terms of the spherical harmonics $Y_l^m(\theta,\phi)$. We can, and it is

$$\psi(\boldsymbol{x}) = \langle x|\alpha\rangle = \sqrt{\frac{8\pi}{3}} \left(\frac{Y_1^{-1} + Y_1^1}{2} + \frac{Y_1^{-1} - Y_1^1}{2i} + \frac{3}{\sqrt{2}} Y_1^0 \right) r f(r)$$

So it must be an eigenfunction of L^2. Recall that the spherical harmonics form an orthonormal basis, so

$$\langle l, m|l, m'\rangle = \int (Y_l^m)^* Y_l^{m'} d\boldsymbol{x} = \delta_{m,m'}$$

The probability amplitudes are then just

$$\langle 1,-1|\alpha\rangle = \int (Y_1^{-1})^* \psi(\boldsymbol{x}) d\boldsymbol{x} = \sqrt{\frac{8\pi}{3}} \left(\frac{1}{2} + \frac{1}{2i}\right) rf(r)$$
$$\langle 1,0|\alpha\rangle = \int (Y_1^{0})^* \psi(\boldsymbol{x}) d\boldsymbol{x} = \sqrt{\frac{8\pi}{3}} \frac{3}{\sqrt{2}} rf(r)$$
$$\langle 1,1|\alpha\rangle = \int (Y_1^{1})^* \psi(\boldsymbol{x}) d\boldsymbol{x} = \sqrt{\frac{8\pi}{3}} \left(\frac{1}{2} - \frac{1}{2i}\right) rf(r)$$

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Problem 5. *Problem 3.24 from Sakurai*

Solution.

$$\langle l,m| L_x |l,m\rangle = \frac{1}{2} \langle l,m| (L_+ + L_-) |l,m\rangle = 0$$
$$\langle l,m| L_y |l,m\rangle = \frac{1}{2i} \langle l,m| (L_+ - L_-) |l,m\rangle = 0$$

$$\begin{aligned}
\langle l,m| L_x^2 |l,m\rangle &= \langle l,m| L_y^2 |l,m\rangle \\
&= \frac{1}{4} \langle l,m| \left(L_+^2 + L_+L_- + L_-L_+ + L_-^2\right) |l,m\rangle \\
&= \frac{1}{4} \langle l,m| (L_+L_- + L_-L_+) |l,m\rangle \\
&= \frac{1}{4} \left(\hbar^2 l(l+1) - m^2\hbar^2\right) + \frac{1}{4} \left(\hbar^2 l(l+1) - m^2\hbar^2\right) \\
&= \frac{1}{2} \left(\hbar^2 l(l+1) - m^2\hbar^2\right)
\end{aligned}$$

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Problem 6. *Problem 3.38 from Sakurai*

Solution.

We are asked to write J_y when $j = 1$. This was already done in Problem 2 above. The result was

$$J_y = \frac{\hbar}{\sqrt{2}i}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

The problem suggests that we should think about the matrix exponential $\exp(-iJ_y\beta/\hbar)$, which is of course

$$\exp(-iJ_y\beta/\hbar) = 1 - (iJ_y\beta/\hbar) + (iJ_y\beta/\hbar)^2 - (iJ_y\beta/\hbar)^3 + ...$$

Now notice that

$$J_y^2 = \frac{-\hbar^2}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} = \frac{-\hbar^2}{2}\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

and if we multiply this by J_y we see that, when $j = 1$, we have the property that $J_y^3 = -\hbar^2 J_y$. Therefore

$$\begin{aligned}
\exp(-iJ_y\beta/\hbar) &= 1 - (iJ_y\beta/\hbar) + (iJ_y\beta/\hbar)^2 - (iJ_y\beta/\hbar)^3 + (iJ_y\beta/\hbar)^4 + ... \\
&= 1 - (iJ_y\beta/\hbar) + J_y^2\,(i\beta/\hbar)^2/2! - J_y^3\,(i\beta/\hbar)^3/3! + J_y^4\,(i\beta/\hbar)^4/4! + ... \\
&= 1 - (iJ_y\beta/\hbar) + J_y^2\,(i\beta/\hbar)^2/2! + \hbar^2 J_y\,(i\beta/\hbar)^3/3! - \hbar^2 J_y^2\,(i\beta/\hbar)^4/4! + ... \\
&= 1 - i\frac{J_y}{\hbar}\sum_{n=0}^{\infty}(-1)^n\beta^{2n+1}/(2n+1)! - \frac{J_y^2}{\hbar^2}\sum_{m=1}^{\infty}(-1)^m\beta^{2m}/(2m)! \\
&= 1 - i\frac{J_y}{\hbar}\sin\beta + \frac{J_y^2}{\hbar^2}(\cos\beta - 1)
\end{aligned}$$

For the third part we just see that

$$\begin{aligned}
\exp(-iJ_y\beta/\hbar) &= 1 - i\frac{J_y}{\hbar}\sin\beta - \frac{J_y^2}{\hbar^2}(1-\cos\beta) \\
&= 1 - \frac{\sin\beta}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} + \frac{1-\cos\beta}{2}\begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1+\cos\beta}{2} & \frac{\sin\beta}{\sqrt{2}} & \frac{1-\cos\beta}{2} \\ -\frac{\sin\beta}{\sqrt{2}} & \cos\beta & \frac{\sin\beta}{\sqrt{2}} \\ \frac{1-\cos\beta}{2} & -\frac{\sin\beta}{\sqrt{2}} & \frac{1+\cos\beta}{2} \end{pmatrix}
\end{aligned}$$

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