Bell's Inequality

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Bell's inequality (classical)

Tsirelson's inequality (quantum)

Expectation of QS and RS

$$
\begin{aligned}
\langle QS \rangle &= \langle \psi | \, QS \, | \psi \rangle \\
&= -\frac{1}{2^{3/2}} \left(\langle 01| - \langle 10| \right) \left(Z_1 Z_2 + Z_1 X_2 \right) \left(|01\rangle - |10\rangle \right) \\
&= -\frac{1}{2^{3/2}} \left(\langle 01| \, Z_1 Z_2 \, |01\rangle + \langle 10| \, Z_1 Z_2 \, |10\rangle \right) \\
&= \frac{2}{2^{3/2}} = \frac{1}{\sqrt{2}}
\end{aligned}
$$

$$
\begin{aligned}
\langle RS \rangle &= \langle \psi | \, RS \, | \psi \rangle \\
&= -\frac{1}{2^{3/2}} \left(\langle 01| - \langle 10| \right) \left(X_1 Z_2 + X_1 X_2 \right) \left(|01\rangle - |10\rangle \right) \\
&= \frac{1}{2^{3/2}} \left(\langle 01| \, X_1 X_2 \, |10\rangle + \langle 10| \, X_1 X_2 \, |01\rangle \right) \\
&= \frac{2}{2^{3/2}} = \frac{1}{\sqrt{2}}
\end{aligned}
$$

Expectation of QT and RT

$$\begin{aligned}
\langle QT \rangle &= \langle \psi | \, QT \, | \psi \rangle \\
&= \frac{1}{2^{3/2}} \left(\langle 01 | - \langle 10 | \right) \left(Z_1 Z_2 - Z_1 X_2 \right) \left(|01\rangle - |10\rangle \right) \\
&= -\frac{1}{2^{3/2}} \left(\langle 01 | \, Z_1 Z_2 \, |01\rangle + \langle 10 | \, Z_1 Z_2 \, |10\rangle \right) \\
&= \frac{2}{2^{3/2}} = \frac{1}{\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
\langle RT \rangle &= \langle \psi | \, RT \, | \psi \rangle \\
&= -\frac{1}{2^{3/2}} \left(\langle 01 | - \langle 10 | \right) \left(X_1 Z_2 + X_1 X_2 \right) \left(|01\rangle - |10\rangle \right) \\
&= \frac{1}{2^{3/2}} \left(\langle 01 | \, X_1 X_2 \, |10\rangle + \langle 10 | \, X_1 X_2 \, |01\rangle \right) \\
&= \frac{2}{2^{3/2}} = \frac{1}{\sqrt{2}}
\end{aligned}$$

Why do Bell states saturate the bound?

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho)$$