Homework 2

Quantum Mechanics
February 3, 2023 C Seitz

Problem 1. *2.2*

Solution.
The matrix representation of A is

$$A = \begin{pmatrix} \langle 0| A |0\rangle & \langle 0| A |1\rangle \\ \langle 1| A |0\rangle & \langle 1| A |1\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In the output basis

$$A = \begin{pmatrix} \langle 0| A |0\rangle & \langle 0| A |1\rangle \\ \langle 1| A |0\rangle & \langle 1| A |1\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We can choose a different basis, say $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

In this basis A takes the form:

$$A' = UA = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

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Problem 2. *2.9*

Solution.

$$\begin{aligned} \sigma_z &= |1\rangle \langle 1| - |0\rangle \langle 0| \\ \sigma_x &= |1\rangle \langle 0| + |0\rangle \langle 1| \\ \sigma_y &= i\,|0\rangle \langle 1| - i\,|1\rangle \langle 0| \end{aligned}$$

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Problem 3. *2.12*

Solution. A matrix is diagonalizable if and only if the algebraic multiplicity equals the geometric multiplicity of each eigenvalue. It is easy to show that the characteristic equation here is $(1-\lambda)^2 = 0$ which only has one solution. ■

Problem 4. *2.17*

Solution.

If H is normal, it must be diagonalizable and has the eigendecomposition

$$H = U\Lambda U^\dagger$$

where U is some unitary matrix. The conjugate transpose is

$$H^\dagger = U^\dagger \Lambda^\dagger U$$

If $H = H^\dagger$, and Λ is diagonal, then

$$U^\dagger \Lambda^\dagger U = U\Lambda U^\dagger$$

which means $\Lambda = \Lambda^\dagger$ i.e. the eigenvalues are real. Furthermore, if Λ is diagonal and purely real, then clearly $H = H^\dagger$. ■

Problem 5. *2.18*

Solution. For a unitary matrix $U^\dagger U = I$, so for an eigenvector $|\alpha\rangle$,

$$\langle\alpha| U^\dagger U |\alpha\rangle = \langle\alpha| I |\alpha\rangle = 1$$

and $\langle\alpha| U^\dagger U |\alpha\rangle = \lambda^*\lambda$, so $\lambda^*\lambda = 1$. ■

Problem 6. *2.24*

Solution. ■

Problem 7. *Grahm-Schmidt*

Solution. It suffices to show that the following matrix has nonzero determinant:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

And it is straightforward to show that

$$\det(A) = 1$$

Therefore these vectors are indeed linearly independent, but not orthogonal. We can make them orthogonal using the Graham-Schmidt procedure. Let $|0\rangle, |1\rangle, |2\rangle$ be our non-orthogonal basis vectors.

$$|0'\rangle = |0\rangle$$

For the second basis vector we have

$$\begin{aligned} |1'\rangle &\propto |1\rangle - \langle 0'|1\rangle |0\rangle \\ &= |1\rangle - |0\rangle \end{aligned}$$

and with the appropriate normalization we get

$$|1'\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

For the third basis vector we have

$$\begin{aligned} |2'\rangle &\propto |2\rangle - \langle 0|2\rangle |0\rangle - \langle 1'|2\rangle |1'\rangle \\ &= |2\rangle \end{aligned}$$

and with the appropriate normalization we get

$$|2'\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

The physical interpretation of this relative to the standard basis for $\mathbb{R}^3$ is that we rotated the standard basis $\phi = -\pi/4$ and $\theta = \pi/4$.

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Problem 8. *Normal matrix parameterization*

Solution.

$$\begin{aligned}(a \cdot \sigma)(a^* \cdot \sigma) &= (a^* \cdot \sigma)(a \cdot \sigma) \\ &\Rightarrow a \cdot a^* + i(a \times a^*) \cdot \sigma = a^* \cdot a + i(a^* \times a) \cdot \sigma \\ &\Rightarrow (a \times a^*) \cdot \sigma = (a^* \times a) \cdot \sigma \\ &\Rightarrow (a \times a^*) = (a^* \times a)\end{aligned}$$

which occurs when $a = a^*$ and the vector is strictly real. If this is satistied, the operator is normal, and has a spectral decomposition:

$$\begin{aligned}A &= \lambda_1 |\lambda_1\rangle \langle\lambda_1| + \lambda_2 |\lambda_2\rangle \langle\lambda_2| \\ &= \lambda_1 P_1 + \lambda_2 P_2\end{aligned}$$

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Problem 9. *2.26*

Solution. Writing out $|\psi\rangle^{\otimes 2}$ explicitly, we have

$$|\psi\rangle^{\otimes 2} = \frac{1}{2} (|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

or in terms of tensor products we have

$$|\psi\rangle^{\otimes 2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Writing out $|\psi\rangle^{\otimes 3}$ explicitly, we have

$$|\psi\rangle^{\otimes 3} = \frac{1}{2^{3/2}} \left(|000\rangle + |001\rangle + |100\rangle + |010\rangle + |101\rangle + |111\rangle + |110\rangle + |011\rangle\right)$$

or in terms of tensor products we have

$$|\psi\rangle^{\otimes 3} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2^{3/2}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

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Problem 10. *2.27*

Solution.

$$X \otimes Z = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$I \otimes X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$X \otimes I = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Clearly from this last result, the tensor product does not necessarily commute.

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Problem 11. *2.33*

Solution. Consider the outer product representation of the Hadamard gate

$$\begin{aligned} H &= \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right) \\ &= \frac{1}{\sqrt{2}}\sum_{x,y}(-1)^{x\cdot y}|x\rangle\langle y| \end{aligned}$$

To see how this generalizes, consider:

$$\begin{aligned} |0\rangle\langle 0| \otimes |0\rangle\langle 0| &= (|0\rangle \otimes |0\rangle)(\langle 0| \otimes \langle 0|) \\ &= |00\rangle \otimes \langle 00| \end{aligned}$$

This is generally true of the tensor product, so multiplying sums like the one above n times will give a similar expression for vectors $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$H^{\otimes n} = \frac{1}{\sqrt{2}}\sum_{\boldsymbol{x},\boldsymbol{y}}(-1)^{\boldsymbol{x}\cdot\boldsymbol{y}}|\boldsymbol{x}\rangle\langle\boldsymbol{y}|$$

For $n = 2$, we just need to evaluate this sum for all possible binary strings of length 2 and add them up according to this scheme, which gives

$$H^{\otimes 2} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

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Problem 12. *2.34*

Solution. We can compute these functions of matrices if we have its spectral decomposition. For an arbitrary matrix A, with spectral decomposition, the square root of A is simply

$$\sqrt{A} = \sum_n \sqrt{\lambda_n}\,|\lambda_n\rangle\langle\lambda_n|$$

and its logarithm is

$$\log A = \sum_n \log \lambda_n \left|\lambda_n\right\rangle \left\langle\lambda_n\right|$$

Therefore we start by finding the eigendecomposition of the matrix. The characteristic equation is

$$\lambda^2 - 8\lambda + 7 = 0$$

so the two eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 7$. The eigenvectors are:

$$\left|\lambda_1\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}, \left|\lambda_2\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

Thus for its square root we get

$$\begin{aligned}\sqrt{A} &= \sum_n \sqrt{\lambda_n} \left|\lambda_n\right\rangle \left\langle\lambda_n\right| \\ &= \left|\lambda_1\right\rangle \left\langle\lambda_1\right| + \sqrt{7} \left|\lambda_2\right\rangle \left\langle\lambda_2\right| \\ &= \frac{1}{2}\begin{pmatrix}1+\sqrt{7} & -1+\sqrt{7}\\ -1+\sqrt{7} & 1+\sqrt{7}\end{pmatrix}\end{aligned}$$

and the logarithm is

$$\begin{aligned}\log A &= \log 7 \left|\lambda_2\right\rangle \left\langle\lambda_2\right| \\ &= \frac{\log 7}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\end{aligned}$$

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Problem 13. *2.35*

Solution.
First recall that $(\vec{a}\cdot\vec{\sigma})^k$ is identity for even k and is just $(\vec{a}\cdot\vec{\sigma})$ for odd k.

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\begin{aligned}
\exp(i\theta\vec{a}\cdot\vec{\sigma}) &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(i\theta\vec{a}\cdot\vec{\sigma}\right)^k \\
&= \sum_{k=0}^{\infty}\frac{(i\theta)^k}{k!}\left(\vec{a}\cdot\vec{\sigma}\right)^k \\
&= \sum_{k\ odd}^{\infty}\frac{(-1)^k\theta^k}{k!}\left(\vec{a}\cdot\vec{\sigma}\right)^k + i\sum_{k\ even}^{\infty}\frac{(-1)^k(\theta)^k}{k!}\left(\vec{a}\cdot\vec{\sigma}\right)^k \\
&= \cos\theta I + i\sin\theta(\vec{a}\cdot\vec{\sigma})
\end{aligned}
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Problem 14. *2.39*

Solution.

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