Homework 4

Quantum Mechanics

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Problem 1. *Problem 2.14 from Sakurai*

Solution.

We are given that the state vector is

$$|\alpha; t_0\rangle = \exp\left(\frac{-ipa}{\hbar}\right)|0\rangle$$

The Heisenberg equation of motion for $x(t)$ reads

$$x(t) = x(0)\cos(\omega t) + \frac{p(0)}{m}\sin(\omega t)$$

Therefore

$$\begin{aligned}
\langle x\rangle &= \langle \alpha; t_0|\, x(t)\, |\alpha; t_0\rangle \\
&= \langle \alpha; t_0| \left(x(0)\cos(\omega t) + \frac{p(0)}{m}\sin(\omega t)\right)|\alpha; t_0\rangle \\
&= \langle 0| \exp\left(\frac{ipa}{\hbar}\right)\left(x(0)\cos(\omega t) + \frac{p(0)}{m}\sin(\omega t)\right)\exp\left(\frac{-ipa}{\hbar}\right)|0\rangle \\
&= \langle 0| \exp\left(\frac{ipa}{\hbar}\right) x(0) \exp\left(-\frac{ipa}{\hbar}\right)|0\rangle \cos(\omega t) \\
&+ \frac{1}{m}\langle 0| \exp\left(\frac{ipa}{\hbar}\right) p(0) \exp\left(\frac{-ipa}{\hbar}\right)|0\rangle \sin(\omega t)
\end{aligned}$$

We can simplify this last expression by using the Baker-Haussdorf lemma for arbitrary operators G and A

$$\exp(iG\lambda)A\exp(-iG\lambda) = A + i\lambda[G, A] + ...$$

$$
\begin{aligned}
\langle x\rangle &= \langle 0|\exp\left(\frac{ipa}{\hbar}\right)x(0)\exp\left(-\frac{ipa}{\hbar}\right)|0\rangle\cos(\omega t)\\
&+\frac{1}{m}\langle 0|\exp\left(\frac{ipa}{\hbar}\right)p(0)\exp\left(\frac{-ipa}{\hbar}\right)|0\rangle\sin(\omega t)\\
&= \langle 0|\left(x+\frac{ia}{\hbar}[p,x]\right)|0\rangle\cos(\omega t)\\
&+\frac{1}{m}\langle 0|\left(p+\frac{ia}{\hbar}[p,p]\right)|0\rangle\sin(\omega t)\\
&= (\langle 0|\,x\,|0\rangle + a)\cos(\omega t) = a\cos(\omega t)
\end{aligned}
$$

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Problem 2. *Problem 2.15 from Sakurai*

Solution.

We are given the state

$$|\alpha; t_0\rangle = \exp\left(\frac{-ipa}{\hbar}\right)|0\rangle$$

Using that

$$\langle x|0\rangle = \pi^{-1/4}x_0^{-1/2}\exp\left(-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right)$$

we expect to be able to show

$$
\begin{aligned}
\langle x|\exp\left(\frac{-ipa}{\hbar}\right)|0\rangle &= \langle x-a|0\rangle\\
&= \pi^{-1/4}x_0^{-1/2}\exp\left(-\frac{1}{2}\left(\frac{x-a}{x_0}\right)^2\right)
\end{aligned}
$$

where $x_0 = \sqrt{\frac{\hbar}{m\omega}}$. The probability that $|\alpha\rangle$ is measured to be in the state $|0\rangle$ is given by the inner product

$$
\begin{aligned}
\langle\alpha|0\rangle &= \pi^{-1/2}x_0^{-1}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\left(\frac{x-a}{x_0}\right)^2\right)\exp\left(-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right)dx \\
&= \pi^{-1/2}x_0^{-1}\int_{-\infty}^{\infty}\exp\left(-\frac{1}{2}\frac{(x-a)^2+x^2}{x_0^2}\right)dx
\end{aligned}
$$

The numerator can be alternatively written as

$$
\begin{aligned}
(x-a)^2+x^2 &= 2x^2-2ax+a^2 \\
&= 2(x-a/2)^2+a^2/2
\end{aligned}
$$

so the integral becomes

$$
\begin{aligned}
\langle\alpha|0\rangle &= \pi^{-1/2}x_0^{-1}e^{-a^2/4x_0^2}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-a/2)^2}{x_0^2}\right)dx \\
&= e^{-a^2/4x_0^2}
\end{aligned}
$$

and the probability is then $e^{-a^2/2x_0^2}$

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Problem 3. *Problem 2.16 from Sakurai*

Solution.

We will assume the form of the annihilation and creation operators

$$
\begin{aligned}
a &= \sqrt{\frac{m\omega}{2\hbar}}\left(x+\frac{ip}{m\omega}\right) \\
a^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}\left(x-\frac{ip}{m\omega}\right)
\end{aligned}
$$

Adding these equations gives and rearranging we can express x as

$$
x = \sqrt{\frac{\hbar}{2m\omega}}\left(a+a^\dagger\right)
$$

$$
\begin{aligned}
\langle m| x |n\rangle &= \langle m| \sqrt{\frac{\hbar}{2m\omega}} \left(a + a^\dagger\right) |n\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}} \left(\langle m| a |n\rangle + \langle m| a^\dagger |n\rangle\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n}\delta_{m,n-1} + \sqrt{n+1}\delta_{m,n+1}\right)
\end{aligned}
$$

Subtracting the creation operator from the annihalation operator allows us to write the momentum operator as

$$
p = i\sqrt{\frac{m\hbar\omega}{2}} \left(a^\dagger - a\right)
$$

$$
\begin{aligned}
\langle m| p |n\rangle &= \langle m| \left(i\sqrt{\frac{m\hbar\omega}{2}} \left(a^\dagger - a\right)\right) |n\rangle \\
&= \left(i\sqrt{\frac{m\hbar\omega}{2}} \left(\langle m| a^\dagger |n\rangle - \langle m| a |n\rangle\right)\right) \\
&= i\sqrt{\frac{m\hbar\omega}{2}} \left(\sqrt{n+1}\delta_{m,n+1} - \sqrt{n}\delta_{m,n-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
\langle m| \{x,p\} |n\rangle &= \langle m| xp |n\rangle + \langle m| px |n\rangle \\
&= \frac{i\hbar}{2} \langle m| \left((a^\dagger)^2 - a^2\right) |n\rangle + \frac{i\hbar}{2} \langle m| \left((a^\dagger)^2 + a^\dagger a - a a^\dagger - a^2\right) |n\rangle \\
&= \frac{i\hbar}{2}(\sqrt{n+1}\sqrt{n+2}\delta_{m,n+2} + \sqrt{n}\sqrt{n-1}\delta_{m,n-2})
\end{aligned}
$$

since only the cross terms will survive.

$$
\begin{aligned}
\langle m| x^2 |n\rangle &= \frac{\hbar}{2m\omega} \langle m| \left(a^2 + a a^\dagger + a^\dagger a + (a^\dagger)^2\right) |n\rangle \\
&= \frac{\hbar}{2m\omega} \left((2n+1)\delta_{mn} + \sqrt{m}\sqrt{n+1}\delta_{m-1,n+1} + \sqrt{m+1}\sqrt{n}\delta_{m+1,n-1}\right)
\end{aligned}
$$

$$\begin{aligned}\langle m| p^2 |n\rangle &= -\frac{m\hbar\omega}{2} \langle m| \left(a^2 - aa^\dagger - a^\dagger a + (a^\dagger)^2\right) |n\rangle \\ &= -\frac{m\hbar\omega}{2} \left((2n+1)\delta_{mn} - \sqrt{m}\sqrt{n+1}\delta_{m-1,n+1} - \sqrt{m+1}\sqrt{n}\delta_{m+1,n-1}\right)\end{aligned}$$

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To validate the virial theorem we write,

$$\frac{1}{m}\langle p^2\rangle = m\omega^2\langle x^2\rangle$$

From the above, we can see that this is satisfied for an energy eigenstate

$$\begin{aligned}\frac{1}{m}\langle p^2\rangle &= \frac{\hbar\omega}{2}(2n+1) \\ m\omega^2\langle x^2\rangle &= m\omega^2\frac{\hbar}{2m\omega}(2n+1) = \frac{\hbar\omega}{2}(2n+1)\end{aligned}$$

Problem 4. *Problem 2.28 from Sakurai*

Solution.

First of all, the solution is not trivial since x does not commute with the Hamiltonian since $[x, p^2] \neq 0$. At $t = t_0$ we are in the position eigenstate

$$\langle x|\alpha; t_0\rangle = \delta\left(x - \frac{L}{2}\right)$$

Since this is the infinite square well, we have the following energy eigenstates, in the position representation

$$\langle x|\alpha\rangle = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

Of course $|\alpha; t_0\rangle$ is not an eigenstate of H, so this state will measurably evolve in time. The state $|\alpha; t_0\rangle$ in the energy basis is

$$\begin{aligned}|\beta\rangle &= \sum_n |\epsilon_n\rangle \langle\epsilon_n|\alpha; t_0\rangle \\ &= \sqrt{\frac{2}{L}}\sum_n \sin\left(\frac{n\pi}{2}\right)|\epsilon_n\rangle\end{aligned}$$

From this, we can show the probability of measuring the particle in energy eigenstate $|\epsilon_n\rangle$

$$
\begin{aligned}
\langle \epsilon_m | \beta \rangle &= \sqrt{\frac{2}{L}} \sum_n \sin\left(\frac{n\pi}{2}\right) \langle \epsilon_m | \epsilon_n \rangle \\
&= \sqrt{\frac{2}{L}} \sum_n \sin\left(\frac{n\pi}{2}\right) \delta_{mn} \\
&= \sqrt{\frac{2}{L}} \sin\left(\frac{m\pi}{2}\right)
\end{aligned}
$$

and therefore $|\langle \epsilon_m | \beta \rangle|^2 = \sqrt{\frac{2}{L}} \sin^2\left(\frac{m\pi}{2}\right)$. The relative probabilities with respect to the ground state are then given by

$$
r_{m+1} = \sin^2\left(\frac{(m+1)\pi}{2}\right) \csc^2\left(\frac{m\pi}{2}\right)
$$

Since we know a representation for $|\alpha; t_0\rangle$ in the energy basis, we can determine the time evolution of the wavefunction $\langle x | \alpha \rangle$

$$
\begin{aligned}
|\alpha; t\rangle &= \mathcal{U}(t) |\beta\rangle \\
&= \sqrt{\frac{2}{L}} \sum_n \exp\left(\frac{-i\epsilon_n t}{\hbar}\right) \sin\left(\frac{n\pi}{2}\right) |\epsilon_n\rangle
\end{aligned}
$$

which has the position representation (wave function)

$$
\begin{aligned}
\langle x | \alpha; t \rangle &= \psi(x, t) \\
&= \sqrt{\frac{2}{L}} \sum_n \exp\left(\frac{-i\epsilon_n t}{\hbar}\right) \sin\left(\frac{n\pi}{2}\right) \langle x | \epsilon_n \rangle \\
&= \sqrt{\frac{2}{L}} \sum_n \exp\left(\frac{-i\epsilon_n t}{\hbar}\right) \sin\left(\frac{n\pi}{2}\right) \psi_n(x)
\end{aligned}
$$

where $\psi_n(x)$ are the energy eigenstates given above.

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Problem 5. *Problem 2.29 from Sakurai*

Solution.

For a delta-potential, Schrodinger's equation reads

$$-\frac{\hbar^2}{2m}\frac{d\psi^2}{dx^2} - \nu_0\delta(x)\psi(x) = E\psi(x)$$

We then solve Schrodingers equation in two regions. Let the first region be $x < 0$

$$\frac{d\psi_I^2}{dx^2} = -\frac{2mE}{\hbar^2}\psi_I(x) = \kappa_0^2\psi_I(x)$$

for $\kappa_0 = \sqrt{-2mE}/\hbar$. If $E < 0$, then the general solution is

$$\psi_I(x) = A\exp(\kappa_0 x) + B\exp(-\kappa_0 x)$$

We choose $B = 0$ on physical grounds. Now for the second region $x > 0$, we have a similar situation, but this time we choose an exponential decay

$$\psi_{II}(x) = A\exp(-\kappa_0 x)$$

We have chosen the constant to be the same as the first region, to preserve continuity of $\psi(x)$.

$$\psi(x) = \begin{cases} A\exp(\kappa_0 x) & x < 0 \\ A\exp(-\kappa_0 x) & x \geq 0 \end{cases}$$

The constant A is found by enforcing the normalization condition:

$$2A^2\int_{-\infty}^{0}\exp(2\kappa_0 x)dx = 1$$

and it is straightforward to show that $A = \sqrt{\kappa_0}$

The energy is $E = -\hbar^2\kappa_0^2/2m$. The usual trick for relating this to the strength of the potential ν_0 is to integrate Schrodingers equation

$$-\int_{-\epsilon}^{+\epsilon} \frac{\hbar^2}{2m}\frac{d\psi^2}{dx^2}dx - \int_{-\epsilon}^{+\epsilon} \nu_0\delta(x)\psi(x)dx = 0$$

as $\epsilon \to 0$. Ignoring the normalization, because it will cancel

$$\Big|_{-\epsilon}^{+\epsilon}\frac{d\psi}{dx} = -2\kappa_0 = -\frac{2m\nu_0}{\hbar^2}\psi(0)$$

So we find that $\kappa_0 = \nu_0 m/\hbar^2$ so $E = -m\nu_0^2/2\hbar^2$. This is unique so we just have one bound state. There are of course unbound states when $E > 0$, for which the solutions would be complex exponentials.

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Problem 6. *Problem 2.32 from Sakurai*

Solution. Let us define

$$\begin{aligned}
\psi_I &= A\exp(\alpha x)\\
\psi_{II} &= B\exp(ikx) + C\exp(-ikx)\\
\psi_{III} &= D\exp(-\alpha x)
\end{aligned}$$

Here α, k are constants. We can enforce continuity in the wavefunction itself at $x = -a$ and $x = +a$

$$\begin{aligned}
A\exp(-\alpha a) &= B\exp(-ika) + C\exp(ika)\\
D\exp(-\alpha a) &= B\exp(ika) + C\exp(-ika)
\end{aligned}$$

And we can also enforce continuity in the first-order derivative at these points

$$\begin{aligned}
\alpha A\exp(-\alpha a) &= ikB\exp(-ika) - ikC\exp(ika)\\
-\alpha D\exp(-\alpha a) &= ikB\exp(ika) - ikC\exp(-ika)
\end{aligned}$$

This system of four equations can be written in matrix form

$$\begin{pmatrix} e^{-\alpha a} & e^{-ika} & e^{ika} & 0 \\ 0 & e^{ika} & e^{-ika} & e^{-\alpha a} \\ \alpha e^{-\alpha a} & ike^{-ika} & ike^{ika} & 0 \\ 0 & ike^{ika} & -ike^{-ika} & -\alpha e^{-\alpha a} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = 0$$

According to Mathematica, the determinant is

$$\mathcal{D} = \exp(-2a(ik+\alpha))\left(-\exp(4iak)(k-i\alpha)^2 + (k+i\alpha)^2\right)$$

If the determinant is zero, then a solution exists. The determinant $\mathcal{D}$ is zero when

$$\exp(-2iak)(k+i\alpha)^2 = \exp(2iak)(k-i\alpha)^2$$

Notice that we have just distributed the $\exp(-2ika)$ from the prefactor. If we let $z = \exp(-iak)(k+i\alpha)$ then the above equation just reads $z^2 = (z^*)^2$ or $z = \pm z^*$.

Considering the purely real solution first, we make the substitutions

$$\exp(-iak) \rightarrow \frac{\sqrt{k^2+\alpha^2}}{k+i\alpha}$$
$$\exp(iak) \rightarrow \frac{\sqrt{k^2+\alpha^2}}{k-i\alpha}$$

which gives a new matrix

$$\begin{pmatrix} e^{-\alpha a} & \frac{-i\sqrt{k^2+\alpha^2}}{k+i\alpha} & \frac{\sqrt{k^2+\alpha^2}}{k-i\alpha} & 0 \\ 0 & \frac{\sqrt{k^2+\alpha^2}}{k-i\alpha} & \frac{-i\sqrt{k^2+\alpha^2}}{k+i\alpha} & e^{-\alpha a} \\ \alpha e^{-\alpha a} & ik\frac{-i\sqrt{k^2+\alpha^2}}{k+i\alpha} & ik\frac{\sqrt{k^2+\alpha^2}}{k-i\alpha} & 0 \\ 0 & ik\frac{\sqrt{k^2+\alpha^2}}{k-i\alpha} & -ik\frac{-i\sqrt{k^2+\alpha^2}}{k+i\alpha} & -\alpha e^{-\alpha a} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = 0$$

$$\psi_I(x) = -\frac{Ae^{-a\alpha}\sqrt{\alpha^2+k^2}\sin(kx)}{k}$$

$$\psi_{II}(x) = -Ae^{-\alpha x}$$

Now considering the purely imaginary solution, we make the substitutions

$$\exp(-iak) \to \frac{i\sqrt{k^2+\alpha^2}}{k+i\alpha}$$
$$\exp(iak) \to \frac{-i\sqrt{k^2+\alpha^2}}{k-i\alpha}$$

which again gives a new matrix

$$\begin{pmatrix} e^{-\alpha a} & \frac{i\sqrt{k^2+\alpha^2}}{k+i\alpha} & \frac{-i\sqrt{k^2+\alpha^2}}{k-i\alpha} & 0 \\ 0 & \frac{-i\sqrt{k^2+\alpha^2}}{k-i\alpha} & \frac{i\sqrt{k^2+\alpha^2}}{k+i\alpha} & e^{-\alpha a} \\ \alpha e^{-\alpha a} & ik\frac{i\sqrt{k^2+\alpha^2}}{k+i\alpha} & ik\frac{-i\sqrt{k^2+\alpha^2}}{k-i\alpha} & 0 \\ 0 & ik\frac{-i\sqrt{k^2+\alpha^2}}{k-i\alpha} & -ik\frac{i\sqrt{k^2+\alpha^2}}{k+i\alpha} & -\alpha e^{-\alpha a} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = 0$$

In this case the solution reads

$$\psi_I(x) = \frac{Ae^{-a\alpha}\sqrt{\alpha^2+k^2}\cos(kx)}{k}$$

$$\psi_{II}(x) = Ae^{-\alpha x}$$

$$\blacksquare$$