Bell's Inequality

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Basically I want to relate the expansion coefficients of the two-qubit pure state to the degree of entanglement using entanglement entropy. Then, I want to demonstrate that entangled states can violate Bell's inequality (but I'm not sure if this is exactly correct or under what conditions). Finally, I want to show maximally entangled states that saturate the Tsirelson bound.

CHSH Inequality

Define 4 spin operators along arbitrary directions
$Q = \vec{q} \cdot \sigma, R = \vec{r} \cdot \sigma, S = \vec{s} \cdot \sigma, T = \vec{t} \cdot \sigma.$

Alice: Q, R Bob: S, T

Combination of correlations between Alice and Bobs measurements are bounded according to the CHSH inequality

$$E(Q \otimes S) + E(R \otimes S) + E(R \otimes T) - E(Q \otimes T) \leq 2$$

Let $\vec{q} = (0,0,1), \vec{r} = (1,0,0), \vec{s} = (-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}), \vec{t} = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$

Calculating expectations

$$\vec{q}\cdot\sigma\otimes\vec{s}\cdot\sigma = \begin{pmatrix} \vec{s}\cdot\sigma & 0 \\ 0 & -\vec{s}\cdot\sigma \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\vec{r}\cdot\sigma\otimes\vec{s}\cdot\sigma = \begin{pmatrix} 0 & \vec{s}\cdot\sigma \\ \vec{s}\cdot\sigma & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 1 \\ -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}$$

$$\vec{r}\cdot\sigma\otimes\vec{t}\cdot\sigma = \begin{pmatrix} 0 & \vec{t}\cdot\sigma \\ \vec{t}\cdot\sigma & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix}$$

Calculating expectations

$$\vec{q}\cdot\sigma\otimes\vec{t}\cdot\sigma = \begin{pmatrix} \vec{t}\cdot\sigma & 0 \\ 0 & -\vec{t}\cdot\sigma \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\langle\vec{q}\cdot\sigma\otimes\vec{s}\cdot\sigma\rangle = \frac{1}{\sqrt{2}}(-\alpha^*(\alpha+\beta)+\beta^*(\beta-\alpha)+\gamma^*(\gamma+\delta)+\delta^*(\gamma-\delta))$$

$$\langle\vec{r}\cdot\sigma\otimes\vec{s}\cdot\sigma\rangle = \frac{1}{\sqrt{2}}(-\alpha^*(\gamma+\delta)+\beta^*(\delta-\gamma)-\gamma^*(\alpha+\beta)+\delta^*(\beta-\alpha))$$

$$\langle\vec{r}\cdot\sigma\otimes\vec{t}\cdot\sigma\rangle = \frac{1}{\sqrt{2}}(\alpha^*(\gamma-\delta)-\beta^*(\delta+\gamma)+\gamma^*(\alpha-\beta)-\delta^*(\beta+\alpha))$$

$$\langle\vec{q}\cdot\sigma\otimes\vec{t}\cdot\sigma\rangle = \frac{1}{\sqrt{2}}(\alpha^*(\alpha-\beta)-\beta^*(\beta+\alpha)+\gamma^*(\delta-\delta)+\delta^*(\gamma+\delta))$$

Entanglement entropy of a two-qubit system

Assume ρ_{AB} is a pure state but not necessarily separable.

$$\rho_{AB} = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* & \alpha\gamma^* & \alpha\delta^* \\ \beta\alpha^* & |\beta|^2 & \beta\gamma^* & \beta\delta^* \\ \gamma\alpha^* & \gamma\beta^* & |\gamma|^2 & \gamma\delta^* \\ \delta\alpha^* & \delta\beta^* & \delta\gamma^* & |\delta|^2 \end{pmatrix}$$