Homework 6

Quantum Mechanics

October 28th, 2022 C Seitz

Problem 1. *Problem 3.12 from Sakurai*

Solution.

In general the ensemble average of an operator [A] is defined as

$$[A] = \mathrm{Tr}(\rho A)$$

where $\hat{\rho} = \sum_i w_i \rho_i$ and $\rho_i = |\alpha_i\rangle \langle\alpha_i|$

$$\begin{aligned}
\hat{\rho} &= a\,|+\rangle\langle+| + (1-a)\,|-;y\rangle\langle-;y| \\
&= \frac{1}{2}\begin{pmatrix} 2a & 0 \\ 0 & 0 \end{pmatrix} + \frac{1-a}{2}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} a+1 & -i(1-a) \\ i(1-a) & a-1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
[\sigma_x] &= \mathrm{Tr}(\hat{\rho}\sigma_x) \\
&= \mathrm{Tr}\left(\frac{1}{2}\begin{pmatrix} i(1-a) & a-1 \\ a+1 & -i(1-a) \end{pmatrix}\right) = 0
\end{aligned}$$

$$\begin{aligned}
[\sigma_y] &= \mathrm{Tr}(\hat{\rho}\sigma_x) \\
&= \mathrm{Tr}\left(\frac{1}{2}\begin{pmatrix} -(1-a) & i(a-1) \\ i(a+1) & (1-a) \end{pmatrix}\right) = 0
\end{aligned}$$

$$\begin{aligned}
[\sigma_z] &= \mathrm{Tr}(\hat{\rho}\sigma_z) \\
&= \mathrm{Tr}\left(\frac{1}{2}\begin{pmatrix} a+1 & -i(1-a) \\ i(1-a) & 1-a \end{pmatrix}\right) = 1
\end{aligned}$$

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Problem 2. *Problem 3.13 from Sakurai*

Solution.

The state vector in the S_z basis has the form

$$|\alpha\rangle = c_+ |+\rangle + c_- |-\rangle$$

First note that

$$\langle S_z \rangle = |c_+|^2 - |c_-|^2 \quad |c_+|^2 + |c_-|^2 = 1$$

Together, these equations tell us the magnitude of each complex component.

$$|c_+|^2 = \frac{\langle S_z \rangle + 1}{2} \quad |c_-|^2 = \frac{1 - \langle S_z \rangle}{2}$$

$$\begin{aligned}
\langle S_x \rangle &= \langle \alpha| (|+\rangle \langle -| + |-\rangle \langle +|)(c_+ |+\rangle + c_- |-\rangle) \\
&= \langle \alpha| (c_- |+\rangle + c_+ |-\rangle) \\
&= (c_+^* \langle +| + c_-^* \langle -|)(c_- |+\rangle + c_+ |-\rangle) \\
&= c_+^* c_- + c_-^* c_+ \\
&= |c_+||c_-|(e^{i(\theta-\phi)} + e^{i(\phi-\theta)}) \\
&= 2|c_+||c_-| \cos(\theta - \phi)
\end{aligned}$$

$$\begin{aligned}
\langle S_y \rangle &= \langle \alpha| ((i |+\rangle \langle -| - i |-\rangle \langle +|)(c_+ |+\rangle + c_- |-\rangle) \\
&= i \langle \alpha| (c_- |+\rangle - c_+ |-\rangle) \\
&= i(c_+^* \langle +| + c_-^* \langle -|)(c_- |+\rangle - c_+ |-\rangle) \\
&= c_+^* c_- - c_-^* c_+ \\
&= |c_+||c_-|(e^{i(\theta-\phi)} - e^{i(\phi-\theta)}) \\
&= 2i|c_+||c_-| \sin(\theta - \phi)
\end{aligned}$$

So $\langle S_x \rangle$ gives us the phase difference of c_+ and c_-. Then the sign of $\langle S_y \rangle$ tells us whether θ or ϕ is larger, since sine is odd. This is all we can hope to extract from the expectation values, since multiplying by a global phase $e^{i\delta} |\alpha\rangle$ has no effect on the expectation values.

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Problem 3. *Problem 3.14 from Sakurai*

Solution.

$$\begin{aligned}\rho &= \sum_i w_i \left|\psi_i\right\rangle \left\langle\psi_i\right| \\ &= \frac{1}{3}\left(\left|\alpha\right\rangle\left\langle\alpha\right| + \left|\beta\right\rangle\left\langle\beta\right| + \left|2\right\rangle\left\langle 2\right|\right)\end{aligned}$$

We can write this out explicitly in the subspace spanned by $|0,1,2\rangle$

$$\left|\alpha\right\rangle\left\langle\alpha\right| = \frac{1}{2}\begin{pmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} \quad \left|\beta\right\rangle\left\langle\beta\right| = \frac{1}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & 1\end{pmatrix} \quad \left|2\right\rangle\left\langle 2\right| = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\rho = \frac{1}{6}\begin{pmatrix}1 & 1 & 0\\ 1 & 2 & 1\\ 0 & 1 & 2\end{pmatrix}$$

Now recall that $H = \hbar\omega(N + \frac{1}{2})$ which is

$$H = \left(\frac{\hbar\omega}{2}\right)\mathbb{I}_{3\times 3} + \frac{\hbar\omega}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}$$

$$[H] = \mathrm{Tr}(\rho H) = \frac{\hbar\omega}{2}\mathrm{Tr}(\rho N + \rho) = \frac{\hbar\omega}{2}\left(\mathrm{Tr}(\rho N) + \mathrm{Tr}(\rho)\right)$$

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Problem 4. *Problem 3.15 from Sakurai*

Solution.

$$\rho(t_0) = \sum_i w_i \left|\psi_i; t_0\right\rangle \left\langle\psi_i; t_0\right|$$

In the Schrodinger picture, the coefficients of the state vectors evolve. Therefore,

$$
\begin{aligned}
\rho(t) &= \sum_i w_i\, \mathcal{U}(t,t_0) \left|\psi_i; t_0\right\rangle \left\langle \psi_i; t_0\right| \mathcal{U}^\dagger(t,t_0) \\
&= \mathcal{U}(t,t_0) \left(\sum_i w_i \left|\psi_i; t_0\right\rangle \left\langle \psi_i; t_0\right| \right) \mathcal{U}^\dagger(t,t_0) \\
&= \mathcal{U}(t,t_0)\rho(t_0)\mathcal{U}^\dagger(t,t_0)
\end{aligned}
$$

For a pure ensemble in state $|\psi_i\rangle$, the density matrix is

$$\rho(t_0) = \left|\psi_i; t_0\right\rangle \left\langle \psi_i; t_0\right|$$

At a later time, the density matrix is

$$
\begin{aligned}
\rho(t) &= \mathcal{U}(t,t_0)\rho(t_0)\mathcal{U}^\dagger(t,t_0) \\
&= \mathcal{U}(t,t_0) \left|\psi_i; t_0\right\rangle \left\langle \psi_i; t_0\right| \mathcal{U}^\dagger(t,t_0) \\
&= \left|\psi_i; t_0\right\rangle \left\langle \psi_i; t_0\right|
\end{aligned}
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Problem 5. *Problem 3.16 from Sakurai*

Solution. A 3x3 matrix has 9 parameters, but we only need ■

Problem 6. *Problem 3.40 from Sakurai*

Solution. The singlet state is

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|{+}{-}\rangle + |{-}{+}\rangle)$$

The probability is 1/2 to obtain $s_{1z} = \hbar/2$. The probability is 1/2 to obtain $s_{1x} = \hbar/2$ since obtaining $s_{1x} = \hbar/2$ is equiprobable for the two states in the singlet superposition. If observer B has determined that $s_{2z} = \hbar/2$, then observer A must observe $s_{1z} = -\hbar/2$ since the measurement made by B collapses $|\psi\rangle$ to $|{-}{+}\rangle$. Furthermore, if observer B has measured $s_{2z} = \hbar/2$, then particle 1 must be in the $|+\rangle$ state which means $s_{1x} = \pm\hbar/2$ with equal probability.

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