Homework 7

Quantum Mechanics
November 4, 2022 C Seitz

Problem 1. *5.1*

Solution.
We are concerned here with the new ground state ket $|0\rangle$ and the new ground state energy shift Δ_0 in the presence of perturbation $V = bx$.

$$|0\rangle = \left|0^0\right\rangle + \sum_{j\neq 0} \left|j^0\right\rangle \frac{V_{j0}}{E_0^0 - E_j^0} + ...$$

$$\Delta_0 = V_{00} + \sum_{j\neq 0} \frac{|V_{j0}|^2}{E_0^0 - E_j^0} + ...$$

$$V_{nk} = b\left\langle i^0\right| x \left|j^0\right\rangle = b\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{j}\delta_{i,j-1} + \sqrt{j+1}\delta_{i,j+1}\right)$$

The lowest nonvanishing order is then V_{01}. Therefore

$$\Delta_0 = -\frac{b^2\hbar}{2m\omega}\frac{1}{\hbar\omega} = -\frac{b^2}{2m\omega^2}$$

To solve it exactly, notice that the potential is of the form

$$V_1(x) = ax^2 + bx$$

The new potential shifts to the left by $b/2$, has a new minimum at $-b/2a$. So it is really just the original problem, we just have to make a change of coordinates and shift the equilibrium point down by $-b/2m\omega^2$. Therefore,

$$\Delta = -\frac{b}{2a} = -\frac{b}{2m\omega^2}$$

which is exactly what we got with perturbation theory.

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Problem 2. *5.2*

Solution.

In general, the first order shift in the energy levels i is

$$\Delta_i = V_{ii} = \left\langle i^0 \right| V \left| i^0 \right\rangle$$

Furthermore, the perturbation Hamiltonian is

$$V = \frac{V_0 x}{L}$$

So we just need to identify the matrix elements along the diagonal of this matrix:

$$\begin{aligned} V_{ii} = \left\langle i^0 \right| V \left| i^0 \right\rangle &= \frac{V_0}{L} \left\langle i^0 \right| x \left| i^0 \right\rangle \\ &= \frac{V_0}{L}\frac{2}{L}\int_0^L x \sin^2\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{V_0}{L} \end{aligned}$$

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Problem 3. *5.5*

Solution.

Up to order λ^2, we have

$$\begin{aligned} |i\rangle = \left|i^0\right\rangle &+ \lambda \sum_{j\neq i} \frac{V_{ij}}{E_i^0 - E_j^0} \left|j^0\right\rangle \\ &+ \lambda^2 \left(\sum_{j\neq i}\sum_{l\neq i} \frac{V_{jl}V_{li}\left|j^0\right\rangle}{(E_i^0 - E_j^0)(E_i^0 - E_l^0)} - \sum_{j\neq i} \frac{V_{ii}V_{ji}\left|j^0\right\rangle}{(E_i^0 - E_j^0)^2}) \right) \end{aligned}$$

Now recall that we chose the normalization $\left\langle i^0 \middle| i \right\rangle = 1$ (which can be seen from the above equation), so what we have to calculate is $|\left\langle i^0 \middle| i \right\rangle|^2 / |\left\langle i | i \right\rangle|^2$

$$\begin{aligned}
\langle i|i\rangle &= 1 + \lambda^2 \left(\sum_{j \neq i} \frac{V_{ij}^*}{E_i^0 - E_j^0} \left\langle j^0 \right| \right) \left(\sum_{j \neq i} \frac{V_{ij}}{E_i^0 - E_j^0} \left| j^0 \right\rangle \right) \\
&= 1 + \lambda^2 \sum_{j \neq i} \frac{V_{ij}^* V_{ij}}{\left(E_i^0 - E_j^0\right)^2}
\end{aligned}$$

So the probability is just

$$|\left\langle i^0 \middle| i \right\rangle|^2 / |\left\langle i | i \right\rangle|^2 = \left(1 + 2\lambda^2 \sum_{j \neq i} \frac{V_{ij}^* V_{ij}}{\left(E_i^0 - E_j^0\right)^2} \right)^{-1} + \mathcal{O}(\lambda^3)$$

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Problem 4. *5.7*

Solution.

We can write this Hamiltonian as

$$H_0 = H_x + H_y$$

Given some state $|n, m\rangle$, where H_x acts on $|n\rangle$ and H_y acts on $|m\rangle$.

$$\begin{aligned}
H_0 |n, m\rangle &= (H_x + H_y) |n, m\rangle \\
&= (E_x^n + E_y^m) |n, m\rangle \\
&= \hbar\omega \left(n + \frac{1}{2} + m + \frac{1}{2} \right) \\
&= \hbar\omega(n + m + 1)
\end{aligned}$$

So the energies of the three lowest states are $\hbar\omega, 2\hbar\omega, 3\hbar\omega$. There is a degeneracy - two unique $|n, m\rangle$ have the same eigenvalue for the first excited state. For example $|0, 1\rangle$ and $|1, 0\rangle$ have the same energy. Now, we are given the perturbation

$$V = \delta m \omega^2 xy$$

and we need to find a general matrix representation in the unperturbed $|n, m\rangle$ basis. I will use a shorthand:

$$
\begin{aligned}
V_{ij} &= \delta m\omega^2 \langle i| xy |j\rangle \\
&= \delta m\omega^2 \langle i_x| x |j_x\rangle \langle i_y| y |j_y\rangle \\
&= \delta \frac{\hbar\omega}{2} \left(\sqrt{j_x}\delta_{i_x,j_x-1} + \sqrt{j_x+1}\delta_{i_x,j_x+1}\right) \left(\sqrt{j_y}\delta_{i_y,j_y-1} + \sqrt{j_y+1}\delta_{i_y,j_y+1}\right)
\end{aligned}
$$

First, we consider the ground state. To first order,

$$
\Delta_i = V_{ii} = 0
$$

This can be seen from the above equation, where the matrix elements of V_{ij} are always zero when $i = j$. For the first exicted state, there is a degeneracy, so we need to diagonalize V in the $|1, 0\rangle$, $|0, 1\rangle$ subspace. The matrix is

$$
\begin{aligned}
V' &= \begin{pmatrix} \langle 0, 1| V |0, 1\rangle & \langle 0, 1| V |1, 0\rangle \\ \langle 1, 0| V |0, 1\rangle & \langle 1, 0| V |1, 0\rangle \end{pmatrix} \\
&= \delta \frac{\hbar\omega}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\end{aligned}
$$

which is a familiar matrix, so we can immediately write

$$
\Delta_1^1 = \pm\delta \frac{\hbar\omega}{2}
$$

and the eigenvectors are $\frac{1}{\sqrt{2}} (|0, 1\rangle \pm |1, 0\rangle)$

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Problem 5. *5.12a*

Solution.

We need to find the eigenvectors of V in the degenerate subspace. If they are all independent, then the three-fold degeneracy is lifted. I will suppress the quantum numbers n and m_s for brevity

$$
V = \begin{pmatrix} \langle 1, -1| V |1, -1\rangle & \langle 1, -1| V |1, 0\rangle & \langle 1, -1| V |1, 1\rangle \\ \langle 1, 0| V |1, -1\rangle & \langle 1, 0| V |1, 0\rangle & \langle 1, 0| V |1, 1\rangle \\ \langle 1, 1| V |1, -1\rangle & \langle 1, 1| V |1, 0\rangle & \langle 1, 1| V |1, 1\rangle \end{pmatrix}
$$

It is immediately obvious that the elements along the diagonal must be zero, by symmetry. Also, the matrix is Hermitian (which can be verified by inspecting the spherical harmonics $Y_l^m(\theta, \phi)$. Therefore, we just need to find $\langle 1,-1| V |1,0\rangle$, $\langle 1,-1| V |1,1\rangle$ and $\langle 1,0| V |1,1\rangle$. Write,

$$\begin{aligned} \langle 1,-1| x^2 |1,0\rangle &= \int r^2 \sin^2\theta \cos^2\phi Y_1^{-1}(\theta,\phi) Y_1^0(\theta,\phi) r dr d\theta d\phi \\ &\propto \int r^2 \sin^2\theta \cos^2\phi \sin\theta e^{i\phi} \cos\theta r dr d\theta d\phi = 0 \end{aligned}$$

due to the θ integral. It is similar for $\langle y^2\rangle$ - the integral over θ is again zero and $\langle 1,-1| V |1,0\rangle = 0$. Moving on,

$$\begin{aligned} \langle 1,0| x^2 |1,1\rangle &= \int r^2 \sin^2\theta \cos^2\phi Y_1^{0}(\theta,\phi) Y_1^1(\theta,\phi) r dr d\theta d\phi \\ &\propto \int r^2 \sin^2\theta \cos^2\phi \sin\theta e^{-i\phi} \cos\theta r dr d\theta d\phi = 0 \end{aligned}$$

So it is basically the same as before and $\langle 1,0| V |1,1\rangle = 0$. Finally,

$$\begin{aligned} \langle 1,-1| x^2 |1,1\rangle &= \int r^2 \sin^2\theta \cos^2\phi Y_1^{-1}(\theta,\phi) Y_1^1(\theta,\phi) r dr d\theta d\phi \\ &\propto \int r^3 \sin^4\theta \cos^2\phi dr d\theta d\phi \end{aligned}$$

$$\begin{aligned} \langle 1,-1| y^2 |1,1\rangle &= \int r^2 \sin^2\theta \cos^2\phi Y_1^{-1}(\theta,\phi) Y_1^1(\theta,\phi) r dr d\theta d\phi \\ &\propto \int r^3 \sin^2\theta \cos^4\phi dr d\theta d\phi \end{aligned}$$

We are not asked to work this integral out in detail, but it is nonzero and we will ultimately get a matrix like

$$V = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}$$

since α is real. Finally,

$$\det(V-\lambda I)=\det\begin{pmatrix}-\lambda & 0 & \alpha\\ 0 & -\lambda & 0\\ \alpha & 0 & -\lambda\end{pmatrix}=-\lambda^3+\lambda\alpha^2=0$$

which has three solutions $\lambda=0,\pm\alpha$. So the degeneracy is broken.

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Problem 6. *5.24*

Solution.

We have seen before that we can either express the state in the basis of $\boldsymbol{J}^2,\boldsymbol{L}^2,\boldsymbol{S}^2,J_z$ or $\boldsymbol{L}^2,\boldsymbol{S}^2,L_z,S_z$. In the former, our perturbation Hamiltonian V reads

$$V=\frac{A}{2\hbar^2}\left(\boldsymbol{J}^2-\boldsymbol{L}^2-\boldsymbol{S}^2\right)+\frac{B}{\hbar}\left(J_z+S_z\right)$$

The only hiccup of this representation is that S_z is not diagonal in the $|lsjm\rangle$ basis, so we will have to specify those matrix elements explicitly. For an arbitrary state $|lsjm\rangle$

$$\begin{aligned}\langle lsjm|\,V\,|lsjm\rangle &= \langle lsjm|\,\frac{A}{2\hbar^2}\left(\boldsymbol{J}^2-\boldsymbol{L}^2-\boldsymbol{S}^2\right)+\frac{B}{\hbar}\left(J_z+S_z\right)|lsjm\rangle\\ &=\frac{A}{2}\left(j(j+1)-l(l+1)-s(s+1)\right)+Bm+B\,\langle lsjm|\,S_z\,|lsjm\rangle\end{aligned}$$

We can write these states alternatively as

$$\begin{aligned}\left|j=l\pm\frac{1}{2},m\right\rangle &= \pm\sqrt{\frac{l\pm m+\frac{1}{2}}{2l+1}}\left|m_l=m-\frac{1}{2},m_s=\frac{1}{2}\right\rangle\\ &\quad+\sqrt{\frac{l\mp m+\frac{1}{2}}{2l+1}}\left|m_l=m+\frac{1}{2},m_s=-\frac{1}{2}\right\rangle\end{aligned}$$

which means that

$$\left\langle j = l \pm \frac{1}{2}, m \right| S_z \left| j = l \pm \frac{1}{2}, m \right\rangle = \frac{\hbar}{2}(|c_+|^2 - |c_-|^2)$$
$$= \frac{\hbar}{2}\frac{1}{2l+1}\left(\left(l \pm m + \frac{1}{2}\right) - \left(l \mp m + \frac{1}{2}\right)\right)$$
$$= \pm\frac{m\hbar}{2l+1}$$

So we can write down the general matrix element as

$$\langle lsjm| V |lsjm\rangle = \langle lsjm| \frac{A}{2\hbar^2}\left(\boldsymbol{J}^2 - \boldsymbol{L}^2 - \boldsymbol{S}^2\right) + \frac{B}{\hbar}\left(J_z + S_z\right)|lsjm\rangle$$
$$= \frac{A}{2}\left(j(j+1) - l(l+1) - s(s+1)\right) + Bm \pm B\frac{m}{2l+1}$$
$$= \frac{A}{2}\left(j(j+1) - l(l+1) - s(s+1)\right) + Bm\left(1 \pm \frac{1}{2l+1}\right)$$

Now, let's write out the 8 states. We have two states for $l = 0$ of form $|lsjm\rangle$

$$\left|0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\rangle, \left|0, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle$$

There are also six states for $l = 1$

$$\left|1, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}\right\rangle, \left|1, \frac{1}{2}, \frac{3}{2}, -\frac{3}{2}\right\rangle, \left|1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\rangle, \left|1, \frac{1}{2}, \frac{3}{2}, \frac{1}{2}\right\rangle, \left|1, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right\rangle, \left|1, \frac{1}{2}, \frac{3}{2}, -\frac{1}{2}\right\rangle$$

We are given the hint that two matrices will not be diagonal. We can spot Suppose we group together the $l = 0$ states into a 2x2 matrix

$$V_1 = \begin{pmatrix} \frac{3A}{4} + B & 0 \\ 0 & \frac{3A}{4} \end{pmatrix}$$

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