Homework 7

Quantum Mechanics
November 4, 2022 C Seitz

Problem 1. *5.1*

Solution.
We are concerned here with the new ground state ket $|0\rangle$ and the new ground state energy shift Δ_0 in the presence of perturbation $V = bx$.

$$|0\rangle = \left|0^0\right\rangle + \sum_{j\neq 0} \left|j^0\right\rangle \frac{V_{j0}}{E_0^0 - E_j^0} + ...$$

$$\Delta_0 = V_{00} + \sum_{j\neq 0} \frac{|V_{j0}|^2}{E_0^0 - E_j^0} + ...$$

$$V_{nk} = b\left\langle i^0\right| x \left|j^0\right\rangle = b\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{j}\delta_{i,j-1} + \sqrt{j+1}\delta_{i,j+1}\right)$$

The lowest nonvanishing order is then V_{01}. Therefore

$$\Delta_0 = -\frac{b^2\hbar}{2m\omega}\frac{1}{\hbar\omega} = -\frac{b^2}{2m\omega^2}$$

To solve it exactly, notice that the potential is of the form

$$V_1(x) = ax^2 + bx$$

The new potential shifts to the left by $b/2$, has a new minimum at $-b/2a$. So it is really just the original problem, we just have to make a change of coordinates and shift the equilibrium point down by $-b/2m\omega^2$. Therefore,

$$\Delta = -\frac{b}{2a} = -\frac{b}{2m\omega^2}$$

which is exactly what we got with perturbation theory.

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Problem 2. *5.2*

Solution.

In general, the first order shift in the energy levels i is

$$\Delta_i = V_{ii} = \left\langle i^0 \right| V \left| i^0 \right\rangle$$

Furthermore, the perturbation Hamiltonian is

$$V = \frac{V_0 x}{L}$$

So we just need to identify the matrix elements along the diagonal of this matrix:

$$\begin{aligned} V_{ii} = \left\langle i^0 \right| V \left| i^0 \right\rangle &= \frac{V_0}{L} \left\langle i^0 \right| x \left| i^0 \right\rangle \\ &= \frac{V_0}{L} \frac{2}{L} \int_0^L x \sin^2 \left(\frac{n\pi x}{L} \right) dx \\ &= \frac{V_0}{L} \end{aligned}$$

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Problem 3. *5.5*

Solution.

Up to order λ^2, we have

$$\begin{aligned} |i\rangle = \left| i^0 \right\rangle &+ \lambda \sum_{j \neq i} \frac{V_{ij}}{E_i^0 - E_j^0} \left| j^0 \right\rangle \\ &+ \lambda^2 \left(\sum_{j \neq i} \sum_{l \neq i} \frac{V_{jl} V_{li} \left| j^0 \right\rangle}{(E_i^0 - E_j^0)(E_i^0 - E_l^0)} - \sum_{j \neq i} \frac{V_{ii} V_{ji} \left| j^0 \right\rangle}{(E_i^0 - E_j^0)^2}) \right) \end{aligned}$$

Now recall that we chose the normalization $\langle i^0 | i \rangle = 1$ (which can be seen from the above equation), so what we have to calculate is $|\langle i^0 | i \rangle|^2 / |\langle i | i \rangle|^2$

$$\begin{aligned}
\langle i|i\rangle &= 1 + \lambda^2 \left(\sum_{j \neq i} \frac{V_{ij}^*}{E_i^0 - E_j^0} \langle j^0| \right) \left(\sum_{j \neq i} \frac{V_{ij}}{E_i^0 - E_j^0} |j^0\rangle \right) \\
&= 1 + \lambda^2 \sum_{j \neq i} \frac{V_{ij}^* V_{ij}}{\left(E_i^0 - E_j^0\right)^2}
\end{aligned}$$

So the probability is just

$$|\langle i^0 | i \rangle|^2 / |\langle i | i \rangle|^2 = \left(1 + 2\lambda^2 \sum_{j \neq i} \frac{V_{ij}^* V_{ij}}{\left(E_i^0 - E_j^0\right)^2} \right)^{-1} + \mathcal{O}(\lambda^3)$$

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Problem 4. *5.7*

Solution.

We can write this Hamiltonian as

$$H_0 = H_x + H_y$$

Given some state $|n, m\rangle$, where H_x acts on $|n\rangle$ and H_y acts on $|m\rangle$.

$$\begin{aligned}
H_0 |n, m\rangle &= (H_x + H_y) |n, m\rangle \\
&= (E_x^n + E_y^m) |n, m\rangle \\
&= \hbar\omega \left(n + \frac{1}{2} + m + \frac{1}{2} \right) \\
&= \hbar\omega (n + m + 1)
\end{aligned}$$

So the energies of the three lowest states are $\hbar\omega, 2\hbar\omega, 3\hbar\omega$. There is a degeneracy - two unique $|n, m\rangle$ have the same eigenvalue for the first excited state. For example $|0, 1\rangle$ and $|1, 0\rangle$ have the same energy. Now, we are given the perturbation

$$V = \delta m \omega^2 xy$$

and we need to find a general matrix representation in the unperturbed $|n, m\rangle$ basis. I will use a shorthand:

$$\begin{aligned}
V_{ij} &= \delta m\omega^2 \langle i| xy |j\rangle \\
&= \delta m\omega^2 \langle i_x| x |j_x\rangle \langle i_y| y |j_y\rangle \\
&= \delta \frac{\hbar\omega}{2} \left(\sqrt{j_x}\delta_{i_x,j_x-1} + \sqrt{j_x+1}\delta_{i_x,j_x+1}\right) \left(\sqrt{j_y}\delta_{i_y,j_y-1} + \sqrt{j_y+1}\delta_{i_y,j_y+1}\right)
\end{aligned}$$

First, we consider the ground state. To first order,

$$\Delta_i = V_{ii} = 0$$

This can be seen from the above equation, where the matrix elements of V_{ij} are always zero when $i = j$. For the first exicted state, there is a degeneracy, so we need to diagonalize V in the $|1, 0\rangle$, $|0, 1\rangle$ subspace. The matrix is

$$\begin{aligned}
V' &= \begin{pmatrix} \langle 0,1| V |0,1\rangle & \langle 0,1| V |1,0\rangle \\ \langle 1,0| V |0,1\rangle & \langle 1,0| V |1,0\rangle \end{pmatrix} \\
&= \delta \frac{\hbar\omega}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\end{aligned}$$

which is a familiar matrix, so we can immediately write

$$\Delta_1^1 = \pm\delta\frac{\hbar\omega}{2}$$

and the eigenvectors are $\frac{1}{\sqrt{2}}\left(|0,1\rangle \pm |1,0\rangle\right)$

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Problem 5. *5.12a*

Solution.

We need to find the eigenvectors of V in the degenerate subspace. If they are all independent, then the three-fold degeneracy is lifted. I will suppress the quantum numbers n and m_s for brevity

$$V = \begin{pmatrix} \langle 1,-1| V |1,-1\rangle & \langle 1,-1| V |1,0\rangle & \langle 1,-1| V |1,1\rangle \\ \langle 1,0| V |1,-1\rangle & \langle 1,0| V |1,0\rangle & \langle 1,0| V |1,1\rangle \\ \langle 1,1| V |1,-1\rangle & \langle 1,1| V |1,0\rangle & \langle 1,1| V |1,1\rangle \end{pmatrix}$$

It is immediately obvious that the elements along the diagonal must be zero, by symmetry. Also, the matrix is Hermitian (which can be verified by inspecting the spherical harmonics $Y_l^m(\theta, \phi)$. Therefore, we just need to find $\langle 1, -1| V |1, 0\rangle$, $\langle 1, -1| V |1, 1\rangle$ and $\langle 1, 0| V |1, 1\rangle$. Write,

$$
\begin{aligned}
\langle 1, -1| x^2 |1, 0\rangle &= \int r^2 \sin^2\theta \cos^2\phi Y_1^{-1}(\theta, \phi) Y_1^0(\theta, \phi) r dr d\theta d\phi \\
&\propto \int r^2 \sin^2\theta \cos^2\phi \sin\theta e^{i\phi} \cos\theta r dr d\theta d\phi = 0
\end{aligned}
$$

due to the θ integral. It is similar for $\langle y^2\rangle$ - the integral over θ is again zero and $\langle 1, -1| V |1, 0\rangle = 0$. Moving on,

$$
\begin{aligned}
\langle 1, 0| x^2 |1, 1\rangle &= \int r^2 \sin^2\theta \cos^2\phi Y_1^0(\theta, \phi) Y_1^1(\theta, \phi) r dr d\theta d\phi \\
&\propto \int r^2 \sin^2\theta \cos^2\phi \sin\theta e^{-i\phi} \cos\theta r dr d\theta d\phi = 0
\end{aligned}
$$

So it is basically the same as before and $\langle 1, 0| V |1, 1\rangle = 0$. Finally,

$$
\begin{aligned}
\langle 1, -1| x^2 |1, 1\rangle &= \int r^2 \sin^2\theta \cos^2\phi Y_1^{-1}(\theta, \phi) Y_1^1(\theta, \phi) r dr d\theta d\phi \\
&\propto \int r^3 \sin^4\theta \cos^2\phi dr d\theta d\phi
\end{aligned}
$$

$$
\begin{aligned}
\langle 1, -1| y^2 |1, 1\rangle &= \int r^2 \sin^2\theta \cos^2\phi Y_1^{-1}(\theta, \phi) Y_1^1(\theta, \phi) r dr d\theta d\phi \\
&\propto \int r^3 \sin^2\theta \cos^4\phi dr d\theta d\phi
\end{aligned}
$$

We are not asked to work this integral out in detail, but it is nonzero and we will ultimately get a matrix like

$$
V = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}
$$

since α is real. Finally,

$$\det(V-\lambda I)=\det\begin{pmatrix}-\lambda & 0 & \alpha\\ 0 & -\lambda & 0\\ \alpha & 0 & -\lambda\end{pmatrix}=-\lambda^3+\lambda\alpha^2=0$$

which has three solutions $\lambda=0,\pm\alpha$. So the degeneracy is broken.

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Problem 6. *5.24*

Solution.

$$V=\frac{A}{2\hbar^2}\left(\boldsymbol{J}^2-\boldsymbol{L}^2-\boldsymbol{S}^2\right)+\frac{B}{\hbar}\left(J_z+S_z\right)$$

We have two states for $l=0$ of form $|lsjm\rangle$

$$\left|0,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle,\left|0,\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle$$

and six states for $l=1$

$$\left|1,\frac{1}{2},\frac{3}{2},\frac{3}{2}\right\rangle,\left|1,\frac{1}{2},\frac{3}{2},-\frac{3}{2}\right\rangle,\left|1,\frac{1}{2},\frac{1}{2},\frac{1}{2}\right\rangle,\left|1,\frac{1}{2},\frac{3}{2},\frac{1}{2}\right\rangle,\left|1,\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right\rangle,\left|1,\frac{1}{2},\frac{3}{2},-\frac{1}{2}\right\rangle$$

For an arbitrary state $|lsjm\rangle$

$$\begin{aligned}\langle lsjm|\,V\,|lsjm\rangle &= \langle lsjm|\,\frac{A}{2\hbar^2}\left(\boldsymbol{J}^2-\boldsymbol{L}^2-\boldsymbol{S}^2\right)+\frac{B}{\hbar}\left(J_z+S_z\right)|lsjm\rangle\\ &=\frac{A}{2}\left(j(j+1)+l(l+1)+s(s+1)\right)+B(m+s)\end{aligned}$$

Suppose we group together the $l=0$ states into a 2x2 matrix

$$V_1=\begin{pmatrix}\frac{3A}{4}+B & 0\\ 0 & \frac{3A}{4}\end{pmatrix}$$

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