Gaussian Graphical Model Demo

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Outline

One-dimensional case

Let's first consider a toy example and work our way to the Gaussian graphical model. The standard definition of the 1D Gaussian is

$$P(x|\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{1}$$

We'd like to maximize the log-likelihood $\mathcal{L}_\theta$ of the parameters $\theta = (\mu, \sigma)$ i.e.

$$\theta^* = \underset{\theta}{\mathrm{argmin}} - \log P(\theta|X)$$

Bayesian Inference

We can use Bayesian inference to estimate the optimal parameters θ given a sample of data drawn from $P(x)$.

$$P(\theta|x) = \frac{P(x|\theta)P(\theta)}{\int_\theta P(x|\theta)P(\theta)d\theta} \tag{2}$$

In MAP estimation, we try to maximize the numerator as a function of θ. In MLE we assume a uniform prior and try to maximize $P(x|\theta) = \prod_{i=1}^{N} P(x_i|\theta)$.

Those are standard methods (which work nicely in this simple case) but let's try and use MCMC instead

Metropolis MCMC

Let $\tilde{P}(\theta|x) = P(x|\theta)P(\theta)$

In the Metropolis algorithm, we randomly choose starting parameter values μ_0 and σ_0. We will define two proposal distributions $T_\mu(\mu'|\mu) = \mathcal{N}(\mu, \sigma_\mu^2)$ and $T_\sigma(\sigma'|\sigma) = \mathcal{N}(\sigma, \sigma_\sigma^2)$

Iterate:

- Draw $\mu' \sim T_\mu(\mu'|\mu)$, $\sigma' \sim T_\sigma(\sigma'|\sigma)$
- Compute $a_\mu = \min\left(1, \frac{P(\mu)}{P(\mu')}\right)$, $a_\sigma = \min\left(1, \frac{P(\sigma)}{P(\sigma')}\right)$
- Accept μ' w.p. a_μ and σ' w.p. a_σ

MCMC Run: 1D Gaussian Parameters

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