Homework 4

Quantum Mechanics
Sept 22nd, 2022 Clayton Seitz

Problem 1. *Problem 2.14 from Sakurai*

Solution.

We are given that the state vector is

$$|\alpha\rangle = \exp\left(\frac{-ipa}{\hbar}\right)|0\rangle$$

The Heisenberg equation of motion reads

$$\frac{dx}{dt} = \frac{1}{i\hbar}[x, H] = 0$$

Therefore $x = x_0$ for all $t \geq t_0$

$$\begin{aligned}
\langle x\rangle &= \int x_0 \langle x|\alpha\rangle \langle\alpha|x\rangle\, dx \\
&= \int x \exp\left(\frac{-ipa}{\hbar}\right)\langle x|0\rangle \exp\left(\frac{ipa}{\hbar}\right)\langle 0|x\rangle\, dx \\
&= \int x_0 \,|\langle x|0\rangle|^2\, dx \\
&= \int x_0 \,|\langle x|0\rangle|^2\, dx
\end{aligned}$$

We could write out $\langle x|0\rangle$, its complex conjugate, and do the integral. Instead recall the general expression for the matrix element of x

$$\langle n'|\, x\, |n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\delta_{n',n-1} + \sqrt{n+1}\delta_{n',n+1}\right)$$

which is zero when $n = n'$ which means that $\langle x \rangle = 0$

$\blacksquare$

Problem 2. *Problem 2.15 from Sakurai*

Solution. We were given the state

$$|\alpha\rangle = \exp\left(\frac{-ipa}{\hbar}\right)|0\rangle$$

$$\langle x|\alpha\rangle = \pi^{-1/4} x_0^{1/2} \exp\left(\frac{-ipa}{\hbar}\right) \exp\left(-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right)$$

where $x_0 = \sqrt{\frac{\hbar}{m\omega}}$. The Hamiltonian operator $\hat{H}$ is independent of time so we have the unitary time evolution operator

$$\mathcal{U}(t) = \exp\left(-\frac{i\hat{H}t}{\hbar}\right)$$

Assuming $|\alpha\rangle$ is expressed in the energy basis, this can be alternatively be written as the power series

$$\mathcal{U}(t) = \sum_{n=0}^{\infty} \frac{\hat{H}^n}{n!} \to \mathcal{U}(t)\,|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\hat{H}^n}{n!}\,|\alpha\rangle$$

$$\sum_{n=0}^{\infty} \frac{\alpha^n}{n!}\,|\alpha\rangle = \sum_{n} \exp\left(\frac{-i\alpha_n t}{\hbar}\right)|\alpha_n\rangle$$

The probability that $|\alpha\rangle$ is measured to be in the state $|0\rangle$ is

$$\langle 0|\alpha\rangle\,\langle\alpha|0\rangle = \exp\left(\frac{-ipa}{\hbar}\right)\langle 0|0\rangle \exp\left(\frac{ipa}{\hbar}\right)\langle 0|0\rangle = 1$$

This probability does not change for $t > 0$. This is clear when we look at the state

$$|\alpha; t\rangle = \exp\left(-\frac{iE_0 t}{\hbar}\right) \exp\left(\frac{-ipa}{\hbar}\right) |0\rangle$$

The second exponential is just a complex number and is time independent. The first exponential is just a phase, which is not measurable directly. In other words, when we hit this state with the dual ket $\langle 0|$, the phase goes away and we are left with a time-independent probability density.

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Problem 3. *Problem 2.16 from Sakurai*

Solution.

We will assume the form of the annihilation and creation operators

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right)$$
$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{ip}{m\omega}\right)$$

Adding these equations gives and rearranging we can express x as

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)$$

$$\begin{aligned}\langle m| x |n\rangle &= \langle m| \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)|n\rangle \\ &= \sqrt{\frac{\hbar}{2m\omega}}\left(\langle m| a |n\rangle + \langle m| a^\dagger |n\rangle\right) \\ &= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\delta_{m,n-1} + \sqrt{n+1}\delta_{m,n+1}\right)\end{aligned}$$

Subtracting the creation operator from the annihalation operator allows us to write the momentum operator as

$$p = i\sqrt{\frac{m\hbar\omega}{2}}\left(a^\dagger - a\right)$$

$$
\begin{aligned}
\langle m| p |n\rangle &= \langle m| \left(i\sqrt{\frac{m\hbar\omega}{2}} \left(a^\dagger - a\right)\right) |n\rangle \\
&= \left(i\sqrt{\frac{m\hbar\omega}{2}} \left(\langle m| a^\dagger |n\rangle - \langle m| a |n\rangle\right)\right) \\
&= i\sqrt{\frac{m\hbar\omega}{2}} \left(\sqrt{n+1}\delta_{m,n+1} - \sqrt{n}\delta_{m,n-1}\right)
\end{aligned}
$$

$$
\begin{aligned}
\langle m| \{x, p\} |n\rangle &= \langle m| xp |n\rangle + \langle m| px |n\rangle \\
&= \frac{i\hbar}{2} \langle m| \left((a^\dagger)^2 - a^2\right) |n\rangle + \frac{i\hbar}{2} \langle m| \left((a^\dagger)^2 + a^\dagger a - a a^\dagger - a^2\right) |n\rangle \\
&= \frac{i\hbar}{2} \left(\sqrt{n+1}\sqrt{n+2}\delta_{m,n+2} - \sqrt{n}\sqrt{n-1}\delta_{m,n-2}\right) \\
&+ \frac{i\hbar}{2} (\sqrt{n+1}\sqrt{n+2}\delta_{m,n+2} + \sqrt{n}\sqrt{n-1}\delta_{m,n-2})
\end{aligned}
$$

$$
\langle m| x^2 |n\rangle = \frac{\hbar}{2m\omega} \langle m| \left(a^2 + a a^\dagger + a^\dagger a + (a^\dagger)^2\right) |n\rangle
$$

$$
\langle m| p^2 |n\rangle = -\frac{m\hbar\omega}{2} \langle m| \left((a^\dagger)^2 + a^\dagger a - a a^\dagger - a^2\right) |n\rangle
$$

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Problem 4. *Problem 2.28 from Sakurai*

Solution.

First of all, the solution is not trivial since x does not commute with the Hamiltonian since $[x, p^2] \neq 0$. We are told that

$$
\langle x|\alpha; t_0\rangle = \delta\left(x - \frac{L}{2}\right)
$$

Even though $|\alpha; t_0\rangle$ is an eigenstate of x, we are not in an eigenstate of H. This is just the infinite square well, which has energy eigenstates

$$\langle x|\alpha\rangle = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

Let $\alpha = \pi/L$, the probability of finding a particle in the eigenstate corresponding to n can be written as

$$\begin{aligned} |\langle x|\alpha\rangle|^2 &= \sin^2(n\alpha x + m\alpha x) \\ &= \frac{1-\cos(2n\alpha x + 2m\alpha x)}{2} \end{aligned}$$

Since $|\alpha\rangle$ is not an eigenstate of H, then the state will evolve in time. We evolve $|\alpha\rangle$ by changing to the energy basis, evolving in time, and changing back to the position representation.

$$\begin{aligned} \mathbb{I}\,|\psi\rangle &= \int_0^L dx\,|x\rangle\langle x| \left(\sum_n c_n \ \exp\left(-\frac{i\epsilon_n t}{\hbar}\right)|\epsilon_n\rangle\right) \\ &= \sum_n c_n(0) \ \exp\left(-\frac{i\epsilon_n t}{\hbar}\right) \int_0^L dx\,|x\rangle\langle x|\epsilon_n\rangle \\ &= \sum_n c_n(0) \ \exp\left(-\frac{i\epsilon_n t}{\hbar}\right) \psi_n(x) \end{aligned}$$

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Problem 5. *Problem 2.29 from Sakurai*

Solution. ■

Problem 6. *Problem 2.32 from Sakurai*

Solution. ■