

# Homework 3

Quantum Mechanics

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## Problem 1. Problem 2.48

### Solution.

The polar decomposition of a matrix is  $A = UJ = KU$  where  $U$  is a unitary operator and  $J, K$  are positive operators that satisfy  $J = \sqrt{A^\dagger A}$  and  $K = \sqrt{AA^\dagger}$ . If  $P$  is itself a positive matrix we can immediately say that its polar decomposition is  $P = IP = PI$ . If the matrix to decompose is unitary then of course  $UU^\dagger = U^\dagger U = I$  so its decomposition is itself. If the matrix to decompose is Hermitian, then  $H = H^\dagger$  and

$$J = K = \sqrt{H^2} = \sqrt{\sum_i \lambda_i^2 |i\rangle \langle i|} = \sum_i |\lambda_i| |i\rangle \langle i|$$

and its polar decomposition is therefore  $U \sum_i |\lambda_i| |i\rangle \langle i|$  or  $\sum_i |\lambda_i| |i\rangle \langle i| U$ . ■

## Problem 2. Problem 2.49

### Solution.

The polar decomposition is  $A = UJ$ . The spectral decomposition of  $J$  is

$$J = \sqrt{\sum_i \lambda_i \lambda_i^* |i\rangle \langle i|} = \sum_i |\lambda_i| |i\rangle \langle i|$$

For the unitary matrix  $U$ , we have

$$U = \sum_j \lambda_j |j\rangle \langle j|$$

Therefore the product  $UJ$  reads

$$\begin{aligned}
 UJ &= \left( \sum_j \lambda_j |j\rangle \langle j| \right) \left( \sum_i |\lambda_i| |i\rangle \langle i| \right) \\
 &= \sum_{ij} |\lambda_i| \lambda_j |i\rangle \langle i|j\rangle \langle j|
 \end{aligned}$$

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**Problem 3.** *Problem 2.50*

**Solution.**

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

First, consider

$$A^\dagger A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad AA^\dagger = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$J = \sqrt{A^\dagger A} =$$

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**Problem 4.** *Problem 2.51*

**Solution.** The Hadamard gate  $H$  is unitary if  $H^\dagger = H^{-1}$ . It is easy to see that

$$H^\dagger = H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

It's inverse is

$$H^{-1} = -\frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = H$$

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**Problem 5.** *Problem 2.52*

**Solution.**

$$H^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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**Problem 6.** *Problem 2.53*

**Solution.** Writing out the characteristic equation gives that the eigenvalues are  $\lambda = \pm\sqrt{2}$ .

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**Problem 7.** *Problem 2.54*

**Solution.** Since the two operators commute, they are simultaneously diagonalizable. Consider the following spectral decompositions

$$A = \sum_n a_n |n\rangle \langle n|$$

$$B = \sum_n b_n |n\rangle \langle n|$$

Therefore, it must be true that

$$A + B = \sum_n (a_n + b_n) |n\rangle \langle n|$$

Now these matrices are Hermitian so their eigenvectors are orthogonal, and the product of matrix exponentials is just

$$\begin{aligned} \exp(A) \exp(B) &= \left( \sum_n \exp(a_n) |n\rangle \langle n| \right) \left( \sum_m \exp(b_m) |m\rangle \langle m| \right) \\ &= \sum_{m,n} \delta_{mn} \exp(a_n) \exp(b_m) |n\rangle \langle m| \\ &= \sum_n \exp(a_n) \exp(b_n) |n\rangle \langle n| \\ &= \sum_n \exp(a_n + b_n) |n\rangle \langle n| \\ &= \exp(A + B) \end{aligned}$$

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**Problem 8.** *Problem 2.55***Solution.**

$$\begin{aligned}
UU^\dagger &= \exp\left(\frac{-iH(t_2 - t_1)}{\hbar}\right) \exp\left(\frac{iH(t_2 - t_1)}{\hbar}\right) \\
&= \left(\sum_n \exp\left(\frac{-iE_n(t_2 - t_1)}{\hbar}\right) |n\rangle \langle n|\right) \left(\sum_m \exp\left(\frac{iE_m(t_2 - t_1)}{\hbar}\right) |m\rangle \langle m|\right) \\
&= \sum_{m,n} \delta_{mn} |n\rangle \langle m| \\
&= \sum_n |n\rangle \langle n| = I
\end{aligned}$$

where  $H$  is a Hermitian operator. ■

**Problem 9.** *Problem 2.56***Solution.**

$U$  is unitary so its eigenvalues  $u_n$  have unit norm, which means

$$K = -i \log(U) = -i \sum_n \log(u_n) |n\rangle \langle n| = \sum_n \theta |n\rangle \langle n|$$

since

$$\log(u_n) = \log(|u_n| e^{i\theta}) = \log(|u_n|) + i\theta = i\theta$$

Therefore,  $K = K^\dagger$  since  $\theta \in \mathbb{R}$ . ■

**Problem 10.** *Problem 2.57***Solution.**

$$L_l |\alpha\rangle = \frac{\ell |l\rangle}{|\ell|}$$

$$M_m \frac{\ell |l\rangle}{|\ell|} = \frac{m\ell}{|m||\ell|} |m\rangle$$

which is equivalent to

$$\begin{aligned}
 N_{m\ell} |\alpha\rangle &= M_m L_\ell |\alpha\rangle \\
 &= \frac{|m\rangle \langle m|\ell\rangle \langle \ell|}{|m||\ell|} |\alpha\rangle \\
 &= \frac{\ell |m\rangle \langle m|}{|m||\ell|} |\ell\rangle \\
 &= \frac{m\ell}{|m||\ell|} |m\rangle
 \end{aligned}$$

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**Problem 11.** *Problem 2.58*

**Solution.**

Since the system is in an eigenstate of  $M$  with eigenvalue  $m$ , the average will be  $m$

$$\langle M \rangle = \langle m| M |m\rangle = \langle m| m |m\rangle = m$$

The variance must then be zero

$$\begin{aligned}
 (\Delta M)^2 &= \langle M^2 \rangle - \langle M \rangle^2 \\
 &= m^2 - m^2 = 0
 \end{aligned}$$

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**Problem 12.** *Problem 2.59*

**Solution.**

$$\langle 0| X |0\rangle = \langle 0|1\rangle = 0$$

$$\begin{aligned}
 (\Delta X)^2 &= \langle X^2 \rangle - \langle X \rangle^2 \\
 &= \langle X^2 \rangle \\
 &= \langle 0| X^2 |0\rangle \\
 &= 1
 \end{aligned}$$

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**Problem 13.** *Problem 2.60***Solution.**

$$\begin{aligned}
\vec{v} \cdot \sigma &= \begin{pmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z \end{pmatrix} \\
&= v_z (|0\rangle \langle 0| - |1\rangle \langle 1|) + (v_1 - iv_2) |0\rangle \langle 1| + (v_1 + iv_2) |1\rangle \langle 0|
\end{aligned}$$

from the outer product representations of  $\sigma_x, \sigma_y, \sigma_z$ . The corresponding characteristic equation is

$$\lambda^2 - (v_z^2 + v_y^2 + v_x^2) = 0$$

If  $\vec{v}$  is normalized then  $\lambda = \pm 1$ . We now show that the projectors onto the respective eigenspaces are  $P_{\pm} = (I \pm \vec{v} \cdot \sigma) / 2$ . Let  $|\pm\rangle$  be the eigenvectors of  $\vec{v} \cdot \sigma$  with eigenvalues  $\pm 1$ , respectively.

$$\begin{aligned}
P_+ &= |+\rangle \langle +| \\
&= \frac{|+\rangle \langle +| + |-\rangle \langle -| + |+\rangle \langle +| - |-\rangle \langle -|}{2} \\
&= \frac{I + \vec{v} \cdot \sigma}{2}
\end{aligned}$$

since by spectral decomposition we know that  $\vec{v} \cdot \sigma = |+\rangle \langle +| - |-\rangle \langle -|$ . Of course, we also have that

$$\begin{aligned}
P_- &= |-\rangle \langle -| \\
&= \frac{|+\rangle \langle +| + |-\rangle \langle -| - |+\rangle \langle +| + |-\rangle \langle -|}{2} \\
&= \frac{I - \vec{v} \cdot \sigma}{2}
\end{aligned}$$

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**Problem 14.** *Problem 2.61***Solution.**

Let  $|0\rangle$  and  $|1\rangle$  be the eigenvectors of  $\sigma_z$ .

$$\begin{aligned}
p(+) &= \langle 0 | P_+ | 0 \rangle \\
&= \langle 0 | \frac{I + \vec{v} \cdot \sigma}{2} | 0 \rangle \\
&= \frac{1}{2} (1 + \langle 0 | (v_3 (|0\rangle \langle 0| - |1\rangle \langle 1|) + (v_1 - iv_2) |0\rangle \langle 1| + (v_1 + iv_2) |1\rangle \langle 0|) | 0 \rangle) \\
&= \frac{1}{2} (1 + v_3)
\end{aligned}$$

The state of the system must be then in the eigenvector  $|+\rangle$  of  $\vec{v} \cdot \sigma$  with eigenvalue  $+1$ . This can be conveniently obtained by applying the measurement operator  $P_+$ , which was obtained in the last problem:

$$\begin{aligned}
P_+ |0\rangle &= \frac{I + \vec{v} \cdot \sigma}{2} |0\rangle \\
&= \frac{1}{2} (|0\rangle + (v_3 + v_1 + iv_2) |1\rangle)
\end{aligned}$$

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