The Fokker-Planck Equation

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1 The Fokker-Planck Equation

1.1 Kramers-Moyal Expansion

Consider the following Ito stochastic differential equation

$$d\vec{x} = F(\vec{x}, t) + G(\vec{x}, t)dW$$

The SDE given above corresponds to the Kramers-Moyal expansion (KME) of a transition density $T(x', t'|x, t)$ see (Risken 1989) for a full derivation.

$$\frac{\partial P(x,t)}{\partial t} = \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial x}\right)^n [M_n(x,t)P(x,t)] \tag{1}$$

where M_n is the nth moment of the transition density. In the diffusion approximation, the KME becomes the Fokker-Planck equation (FPE) (Risken 1989). For the sake of demonstration, consider the univariate case with random variable x and the form of $T(x', t'|x, t)$ is a Gaussian with mean $\mu(t)$ and variance $\sigma^2(t)$. In this scenario, the FPE applies because $M_n = 0$ for all $n > 2$. Given that $M_1(x, t) = \mu(t)$ (drift) and $M_2(x, t) = \sigma^2(t)$ (diffusion), the FPE reads

$$\frac{\partial P(x,t)}{\partial t} = \left(-\frac{\partial}{\partial x} M^{(1)}(t) + \frac{1}{2}\frac{\partial^2}{\partial x^2} M^{(2)}(t)\right) P(x,t) \tag{2}$$

We can additionally define the term in parentheses as a differential operator acting on $P(x, t)$

$$\hat{\mathcal{L}}_{FP} = \left(-\frac{\partial}{\partial x} M^{(1)}(t) + \frac{1}{2}\frac{\partial^2}{\partial x^2} M^{(2)}(t)\right) \tag{3}$$

It is common to additionally define the probability current $J(x, t)$ as

$$J(x,t) = \left(M^{(1)}(t) - \frac{1}{2}\frac{\partial}{\partial x} M^{(2)}(t) \right) P(x,t) \tag{4}$$

This definition provides some useful intuition. The value of $J(x,t)$ is the net probability flux into the interval between x and $x + dx$ at at time t. This also allows us to write the FPE as a continuity equation

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x} \tag{5}$$

1.2 The Heat Equation

The well-known heat equation is a special case of the FPE where $M^{(1)}(t) = 0$ and $M^{(2)}(t) = \sigma^2$

$$\frac{\partial P(x,t)}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 P(x,t)}{\partial x^2} \tag{6}$$

The FPE is a parabolic partial differential equation which are difficult to solve directly. We will show a first example of solving the FPE on the above heat equation by Fourier transformation

$$\frac{\partial}{\partial t} \int P(x,t) e^{i\omega t} dx = \frac{\sigma^2}{2} \int \frac{\partial^2 P(x,t)}{\partial x^2} e^{i\omega t} dx \tag{7}$$

We know that $\mathcal{F}[\partial_x f] = -i\omega \mathcal{F}[f]$ and $\mathcal{F}[\partial_x^2 f] = \omega^2 \mathcal{F}[f]$ which allows us to write the heat equation as a first order equation

$$\frac{\partial \tilde{P}(\omega,t)}{\partial t} = -\frac{\sigma^2 \omega^2}{2} \tilde{P}(\omega,t) \tag{8}$$

which suggests the solution $c \cdot \exp\left(-\alpha\omega^2 t\right)$. We can find the solution in the spatial domain by inverse Fourier transformation

1.3 The Multivariate Case

If we now generalize the above equation to a case where we are faced with many variables $\boldsymbol{x} = (x_1, x_2, ..., x_n)$. The continuity equation becomes

$$\frac{\partial P(\vec{x},t)}{\partial t} = -\vec{\nabla} \cdot J(\vec{x},t) \tag{9}$$

where the multivariate probability current now has the interpretation of the net flux into or out of a volume dx^n centered around $\boldsymbol{x}$. If we consider each dimension,

$$J(x_i,t) = \left(M_i^{(1)}(t) - \sum_j \frac{\partial}{\partial x_j} M_{ij}^{(2)}(t) \right) P(\vec{x},t) \tag{10}$$

The full Fokker-Planck equation then reads

$$\frac{\partial P(\vec{x},t)}{\partial t} = \vec{\nabla} \cdot J(\vec{x},t) \tag{11}$$

$$= \sum_{i=1}^{N} \left(-\frac{\partial}{\partial x_i} M_i^{(1)}(t) + \sum_{j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} M_{ij}^{(2)}(t) \right) P(\vec{x},t) \tag{12}$$

It proves quite useful in this form because we can see that the Fokker-Planck equation represents a differentiation operator acting on the distribution $P(\vec{x},t)$

$$\hat{\mathcal{L}}_{FP} = \sum_{i=1}^{N} \left(-\frac{\partial}{\partial x_i} M_i^{(1)}(t) + \sum_{j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} M_{ij}^{(2)}(t) \right) \tag{13}$$

1.4 Ornstein-Uhlenbeck Process

If the transition density is Gaussian then the density is fully specified by the first two moments $M^{(1)}(t) = \vec{\mu}(t)$ and $M^{(2)}(\vec{x},t) = \Sigma(t)$. The moments can also be functions of $\vec{x}$. Both of these possibilities are evident in the Ornstein-Uhlenbeck (OU) process. Let the drift vector be a linear function of the state $\vec{x}$ and the diffusion matrix the square of the Gaussian covariances

$$M^{(1)}(t) = \Gamma \vec{x} \qquad M^{(2)}(t) = 2D$$

with $D = \Sigma\Sigma^T$ which is assumed to be independent of time.

$$\hat{\mathcal{L}}_{FP} = \sum_{i=1}^{N} \left(-\frac{\partial}{\partial x_i} \Gamma \vec{x} + \sum_{j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} D \right) \tag{14}$$