Homework 5

Quantum Mechanics

October 17th, 2022 C Seitz

Problem 1. *Problem 3.10 from Sakurai*

Solution.

$$\begin{aligned}\exp(i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})\theta) &= \begin{pmatrix}\cos\theta + in_z\sin\theta & (-in_x + n_y)\sin\theta \\ (in_x - n_y)\sin\theta & \cos\theta - in_z\sin\theta\end{pmatrix} \\ &= \begin{pmatrix}e^{-(i\alpha+\gamma)/2}\cos\frac{\beta}{2} & -e^{-(i\alpha-\gamma)/2}\sin\frac{\beta}{2} \\ e^{-(i\alpha-\gamma)/2}\sin\frac{\beta}{2} & e^{(i\alpha+\gamma)/2}\cos\frac{\beta}{2}\end{pmatrix}\end{aligned}$$

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Equating the trace of these matrices gives

$$2\cos\theta = 2\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\frac{\beta}{2}$$

So $\theta = \cos^{-1}(\cos\left(\frac{\alpha+\gamma}{2}\right)\cos\frac{\beta}{2})$

Problem 2. *Problem 3.20 from Sakurai*

Solution.

Recall that

$$J_{\pm} = J_x \pm iJ_y$$

and thus $J_x = (J_+ + J_-)/2$ and $J_y = \frac{J_+ - J_-}{2i}$. We know that the matrix elements of $J_{\pm}$ are

$$\langle j', m'|\, J_{\pm}\, |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)}\hbar\delta_{jj'}\delta_{m,m'+1}$$

where j is our usual shorthand for $\hbar^2 j(j+1)$ (the eigenvalue of J^2) and m is short for $m\hbar$ (the eigenvalue of J_z). For a spin-1 system, $j = 1$ and $m = -1, 0, 1$ which gives the eigenkets $|1,-1\rangle, |1,0\rangle, |1,-1\rangle$

$$J_+ = \begin{pmatrix} 0 & \sqrt{2}\hbar & 0 \\ 0 & 0 & \sqrt{2}\hbar \\ 0 & 0 & 0 \end{pmatrix} \quad J_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2}\hbar & 0 & 0 \\ 0 & \sqrt{2}\hbar & 0 \end{pmatrix}$$

$$J_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad J_y = \frac{\hbar}{\sqrt{2}i} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

We can use Mathematica to find the eigenvectors of these two matrices

$$|J_x;+\rangle = \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix} \quad |J_x;0\rangle = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \quad |J_x;-1\rangle = \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix}$$

$$|J_y;+\rangle = \begin{pmatrix} -1/2 \\ -i\sqrt{2} \\ 1/2 \end{pmatrix} \quad |J_y;0\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \quad |J_y;-1\rangle = \begin{pmatrix} -1/2 \\ i\sqrt{2} \\ 1/2 \end{pmatrix}$$

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Problem 3. *Problem 3.22 from Sakurai*

Solution.

We are asked to derive

$$\langle x| L_z |\alpha\rangle = -i\hbar \frac{\partial}{\partial \phi} \langle x|\alpha\rangle$$

$$\begin{aligned}
\langle x| L_z |\alpha\rangle &= \langle x| (xp_y - yp_x) |\alpha\rangle \\
&= -\langle x| i\hbar(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}) |\alpha\rangle \\
&= \left(r\cos\phi\sin\theta\left(\sin\phi\sin\theta\frac{\partial}{\partial r}\right) - r\sin\phi\sin\theta\left(\cos\phi\sin\theta\frac{\partial}{\partial r}\right)\right)\langle x|\alpha\rangle \\
&+ \left(r\cos\phi\sin\theta\left(\frac{1}{r}\cos\theta\sin\phi\right) - r\sin\phi\sin\theta\left(\frac{1}{r}\sin\theta\sin\phi\right)\right)\langle x|\alpha\rangle \\
&+ \left(r\cos\phi\sin\theta\left(-\frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\theta}\right) - r\sin\phi\sin\theta\left(\cos\phi\sin\theta\frac{\partial}{\partial r}\right)\right)\langle x|\alpha\rangle
\end{aligned}$$

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Problem 4. *Problem 3.23 from Sakurai*

Solution.

We can write the wavefunction given in spherical coordinates

$$\psi(\boldsymbol{x}) = \langle x|\alpha\rangle = r\left(\cos\phi\sin\theta + \sin\phi\sin\theta + \cos\theta\right) f(r)$$

If this is an eigenfunction of L^2, then we should be able to write it in terms of the spherical harmonics $Y_l^m(\theta,\phi)$. We can show that

$$\psi(\boldsymbol{x}) = \langle x|\alpha\rangle = \sqrt{\frac{8\pi}{3}}\left(\frac{Y_1^{-1}+Y_1^1}{2} + \frac{Y_1^{-1}-Y_1^1}{2i} + \frac{3}{\sqrt{2}}Y_1^0\right) rf(r)$$

So it must be an eigenfunction of L^2. The probability amplitudes are

$$\langle 1,-1|\alpha\rangle = \sqrt{\frac{8\pi}{3}}\left(\frac{1}{2}+\frac{1}{2i}\right)\langle 1,1|\alpha\rangle \qquad = \sqrt{\frac{8\pi}{3}}\left(\frac{1}{2}+\frac{1}{2i}\right)$$

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Problem 5. *Problem 3.24 from Sakurai*

Solution.

$$\begin{aligned}
\langle l,m|\, L_x \,|l,m\rangle &= \frac{1}{2}\langle l,m|\,(L_+ + L_-)\,|l,m\rangle = 0\\
\langle l,m|\, L_y \,|l,m\rangle &= \frac{1}{2i}\langle l,m|\,(L_+ - L_-)\,|l,m\rangle = 0
\end{aligned}$$

$$\begin{aligned}
\langle l,m|\, L_x^2 \,|l,m\rangle &= \frac{1}{4}\langle l,m|\left(L_+^2 + L_+L_- + L_-L_+ + L_-^2\right)|l,m\rangle\\
&= \frac{1}{4}\langle l,m|\,(L_+L_- + L_-L_+)\,|l,m\rangle\\
&= \frac{1}{4}\left(\hbar^2 l(l+1) - m^2\hbar^2\right) + \frac{1}{4}\left(\hbar^2 l(l+1) - m^2\hbar^2\right)\\
&= \frac{1}{2}\left(\hbar^2 l(l+1) - m^2\hbar^2\right)
\end{aligned}$$

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Problem 6. *Problem 3.38 from Sakurai*

Solution. ■