Bell's Inequality

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CHSH Inequality

Alice: Q, R Bob: S, T

Classical observables described by the joint distribution $P(Q, R, S, T)$. Combination of correlations between Alice and Bobs measurements are bounded according to the CHSH inequality

$$|E(QS) + E(RS) + E(RT) - E(QT)| \leq 2$$

Tsirelson's Inequality

Quantum mechanics breaks the CHSH inequality

For the quantum version, define 4 spin operators along arbitrary directions
$Q = \vec{q} \cdot \sigma, R = \vec{r} \cdot \sigma, S = \vec{s} \cdot \sigma, T = \vec{t} \cdot \sigma$.
The book uses $\vec{q} = (0, 0, 1), \vec{r} = (1, 0, 0), \vec{s} = (-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}), \vec{t} = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$

One can prove that (and I do this partially on the next slide)

$$|\langle Q \otimes S \rangle + \langle R \otimes S \rangle + \langle R \otimes T \rangle - \langle Q \otimes T \rangle| \leq 2\sqrt{2}$$

Clearly this exceeds the CHSH bound by a factor of $\sqrt{2}$.

The Tsirelson bound

I will start from the solution to Problem 2.3 in the book:

$$(Q \otimes S + R \otimes S + R \otimes T - Q \otimes T)^2 = 4I + [Q, R] \otimes [S, T]$$

Jensen's inequality:

$$\begin{aligned}\langle (Q \otimes S + R \otimes S + R \otimes T - Q \otimes T) \rangle^2 &\leq \langle (Q \otimes S + R \otimes S + R \otimes T - Q \otimes T)^2 \rangle \\ &= \langle 4I + [Q, R] \otimes [S, T] \rangle \\ &= 4 + \langle [Q, R] \otimes [S, T] \rangle\end{aligned}$$

This gives us a very general bound on the LHS for possible $Q, R, S, T, |\psi\rangle$; however, we may still want to know how $\langle (Q \otimes S + R \otimes S + R \otimes T - Q \otimes T) \rangle$ depends on the $|\psi\rangle$ we pick.

Assumptions and objectives

As in the book, Assume $\vec{q}, \vec{r}$ and $\vec{s}, \vec{t}$ are orthogonal and all $\vec{q}, \vec{r}, \vec{s}, \vec{t}$ live in a plane (say x-z)

Now draw a set of pure states $|\psi\rangle$ possibly with different degrees of entanglement

Question: How does the correlation function vary across states with different degrees of entanglement?

The Tsirelson bound

Using that $(\sigma \cdot \vec{a})(\sigma \cdot \vec{b}) = \vec{a} \cdot \vec{b} + i\sigma \cdot (\vec{a} \times \vec{b})$

$$[A, B] = i\sigma \cdot (\vec{a} \times \vec{b} - \vec{b} \times \vec{a}) = 2i\sigma \cdot (\vec{a} \times \vec{b})$$

Let $\vec{n} = \vec{q} \times \vec{r}$ and $\vec{m} = \vec{s} \times \vec{t}$. Per our constraints, we could have $\vec{n} = \vec{m} = \hat{y}$.

$$\begin{aligned}
\langle [Q, R] \otimes [S, T] \rangle &= -4 \langle \psi | \sigma \cdot \vec{n} \otimes \sigma \cdot \vec{m} | \psi \rangle \\
&= -4 \langle \psi | \sigma \cdot \hat{y} \otimes \sigma \cdot \hat{y} | \psi \rangle \\
&= -4 \langle \psi | \sigma_{1y} \otimes \sigma_{2y} | \psi \rangle
\end{aligned}$$

$$\langle (Q \otimes S + R \otimes S + R \otimes T - Q \otimes T) \rangle \leq 2\sqrt{1 - \langle \psi | \sigma_{1y} \otimes \sigma_{2y} | \psi \rangle}$$

Quantifying entanglement: partial traces

$$
\begin{aligned}
\mathrm{Tr}_A(\rho_{AB}) &= \sum_{ijkl} \rho_{ij}^{kl} \mathrm{Tr}_A(|i\rangle\langle k|) \otimes |j\rangle\langle l| \\
&= \sum_i \left(\sum_{jl} \rho_{ij}^{il} |j\rangle\langle l| \right) \\
&= (\rho_{00}^{00} + \rho_{10}^{10}) |0\rangle\langle 0| + (\rho_{00}^{01} + \rho_{10}^{11}) |0\rangle\langle 1| + (\rho_{01}^{00} + \rho_{11}^{10}) |1\rangle\langle 0| + (\rho_{01}^{01} + \rho_{11}^{11}) |1\rangle\langle 1|
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Tr}_B(\rho_{AB}) &= \sum_{ijkl} \rho_{ij}^{kl} |i\rangle\langle k| \otimes \mathrm{Tr}_B(|j\rangle\langle l|) \\
&= \sum_j \left(\sum_{ik} \rho_{ij}^{kj} |i\rangle\langle k| \right) \\
&= (\rho_{00}^{00} + \rho_{01}^{01}) |0\rangle\langle 0| + (\rho_{00}^{10} + \rho_{01}^{11}) |0\rangle\langle 1| + (\rho_{10}^{00} + \rho_{11}^{01}) |1\rangle\langle 0| + (\rho_{10}^{10} + \rho_{11}^{11}) |1\rangle\langle 1|
\end{aligned}
$$

Reduced density matrices for an arbitrary state

$$\mathrm{Tr}_A(\rho_{AB}) = \begin{pmatrix} \rho_{00}^{00} + \rho_{10}^{10} & \rho_{00}^{01} + \rho_{10}^{11} \\ \rho_{01}^{00} + \rho_{11}^{10} & \rho_{01}^{01} + \rho_{11}^{11} \end{pmatrix} = \begin{pmatrix} |\alpha|^2 + |\gamma|^2 & \alpha\beta^* + \gamma\delta^* \\ \beta\alpha^* + \delta\gamma^* & |\beta|^2 + |\delta|^2 \end{pmatrix}$$

$$\mathrm{Tr}_B(\rho_{AB}) = \begin{pmatrix} \rho_{00}^{00} + \rho_{01}^{01} & \rho_{00}^{10} + \rho_{01}^{11} \\ \rho_{10}^{00} + \rho_{11}^{01} & \rho_{10}^{10} + \rho_{11}^{11} \end{pmatrix} = \begin{pmatrix} |\alpha|^2 + |\beta|^2 & \alpha\gamma^* + \beta\delta^* \\ \gamma\alpha^* + \delta\beta^* & |\gamma|^2 + |\delta|^2 \end{pmatrix}$$

Definition of entanglement entropy

The entanglement entropy of a bipartite system is the Von Neumann entropy of either reduced density matrix (it doesn't matter which one we choose)

$$S(\rho_A) = -\mathrm{Tr}(\rho_A \log \rho_A) = -\sum_x \lambda_x \log \lambda_x$$

for eigenvalues λ_x of ρ_A. This tells us: do the reduced states ρ_A and ρ_B contain all the *information* in ρ_{AB}?

Entanglement entropy and Tsirelson's bound

Sanity check:

Draw random reals $a, b, c, d, e, f, g, h \sim U([0,1]^8)$

Construct $|\psi\rangle = (a + ib)|00\rangle + (c + id)|01\rangle + (e + if)|10\rangle + (g + ih)|11\rangle$

Normalize $|\psi\rangle \to \frac{|\psi\rangle}{\sum_n |c_n|^2}$

Compute $S(\rho_A)$ and $2\sqrt{1 - \langle\psi|\,\sigma_{1y} \otimes \sigma_{2y}\,|\psi\rangle}$

Entanglement entropy and Tsirelson's bound