Bell's Inequality

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Basically I want to relate the expansion coefficients of the two-qubit pure state to the degree of entanglement using entanglement entropy. Then, I want to demonstrate that entangled states can violate Bell's inequality (but I'm not sure if this is exactly correct or under what conditions). Finally, I want to show maximally entangled states that saturate the Tsirelson bound.

CHSH Inequality

Define 4 spin operators along arbitrary directions
$Q = \vec{q} \cdot \sigma, R = \vec{r} \cdot \sigma, S = \vec{s} \cdot \sigma, T = \vec{t} \cdot \sigma.$

Alice: $Q, R = \pm 1$ Bob: $S, T = \pm 1$

Combination of correlations between Alice and Bobs measurements are bounded according to the CHSH inequality

$$E(Q \otimes S) + E(R \otimes S) + E(R \otimes T) - E(Q \otimes T) \leq 2$$

Exceeding this upper bound violates locality

Calculating expectations

Let $\vec{q} = (0,0,1), \vec{r} = (1,0,0), \vec{s} = (-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}), \vec{t} = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$
Take an arbitrary state

$$|\psi\rangle = \alpha\,|00\rangle + \beta\,|01\rangle + \gamma\,|10\rangle + \delta\,|11\rangle$$

$$\begin{aligned}
\langle AB\rangle &= \langle\psi|A\otimes B|\psi\rangle \\
&= (\alpha^*\langle 00| + \beta^*\langle 01| + \gamma^*\langle 10| + \delta^*\langle 11|)\otimes(\vec{a}\cdot\vec{\sigma})\otimes(\vec{b}\cdot\vec{\sigma})\otimes(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle \\
&\quad\times(\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle) \\
&= \alpha^*\beta(\vec{a}\cdot\vec{\sigma})01(\vec{b}\cdot\vec{\sigma})10 + \alpha^*\gamma(\vec{a}\cdot\vec{\sigma})00(\vec{b}\cdot\vec{\sigma})00 \\
&\quad+\beta^*\delta(\vec{a}\cdot\vec{\sigma})11(\vec{b}\cdot\vec{\sigma})01 + \gamma^*\delta(\vec{a}\cdot\vec{\sigma})10(\vec{b}\cdot\vec{\sigma})11
\end{aligned}$$

where $(\vec{a}\cdot\vec{\sigma})_{ij}$ denotes the ij-th element of the matrix $\vec{a}\cdot\vec{\sigma}$.

Entanglement entropy of a two-qubit system

Suppose the system is in a pure state

$$|\psi\rangle = \alpha\,|00\rangle + \beta\,|01\rangle + \gamma\,|10\rangle + \delta\,|11\rangle$$

$$\rho_{AB} = |\psi\rangle\,\langle\psi| =$$

$$\rho_{A} = \mathrm{Tr}_{B}(\rho_{AB}) =$$