Homework 1

Quantum Mechanics
August 29th, 2022 CLAYTON SEITZ

Problem 1. *For the spin 1/2 state $|+\rangle_x$, evaluate both sides of the inequality*

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}|\langle[A,B]\rangle|^2$$

for the operators $A = S_x$ and $B = S_y$, and show that the inequality is satisfied. Repeat for the operators $A = S_z$ and $B = S_y$

Solution.

Let $A = S_x$ and $B = S_y$. The variance $\langle(\Delta S_x)^2\rangle$ in state $|+\rangle_x$ must be zero since $|+\rangle_x$ is an eigenvector of S_x

$$\langle(\Delta S_x)^2\rangle = \langle S_x^2\rangle - \langle S_x\rangle^2 = 0$$

Therefore, the LHS of the above inequality is zero. The commutator $[S_x, S_y] = i\hbar S_z$ and

$$\langle S_z\rangle = \langle +|_x \, S_z \, |+\rangle_x = 0$$

Clearly the inequality is satisfied since both sides are zero. Now let $A = S_z$ and $B = S_y$. Since the state is prepared in $|+\rangle_x$, the variances $\langle(\Delta S_x)^2\rangle$ and $\langle(\Delta S_x)^2\rangle$ must be 1/4 (this is just a fair coin toss).
The commutator $[S_z, S_y] = -i\hbar S_x$ and $\langle S_x\rangle = \frac{\hbar}{2}$. The inequality then reads

$$\frac{1}{16} \geq \frac{\hbar^2}{16}$$

which is satisfied given that $\hbar \approx 10^{-34}$ J $\cdot$ s

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Problem 2. *Suppose a 2×2 matrix X (not necessarily Hermitian, nor unitary) is written as*

Solution.

$$\begin{aligned}\mathrm{Tr}(X) &= \mathrm{Tr}(a_0) + \mathrm{Tr}\left(\sum_k a_k\sigma_k\right)\\ &= 2a_0\end{aligned}$$

$$\begin{aligned}\mathrm{Tr}(\sigma_k X) &= \mathrm{Tr}\left(\sigma_k a_0 + \sigma_k \sum_j a_j\sigma_j\right)\\ &= \mathrm{Tr}\left(\sigma_k a_0 + \sum_j a_j\sigma_k\sigma_j\right)\\ &= \mathrm{Tr}\left(\sum_j a_j\sigma_k\sigma_j\right)\end{aligned}$$

We can write out the equation $X = a_0 + \sigma \cdot a$ explicitly

$$X = \begin{pmatrix} a_0 + a_3 & a_1 - ia_3 \\ a_1 + ia_2 & a_0 - a_3 \end{pmatrix}$$

Thus we have four equations involving X_{ij}'s and a_k for $k = (1, 2, 3)$. We can manipulate those four equations to show that

$$\begin{aligned}a_0 &= \frac{X_{11} + X_{22}}{2}\\ a_1 &= \frac{X_{12} + X_{21}}{2}\\ a_2 &= \frac{X_{21} - X_{12}}{2}\\ a_3 &= \frac{X_{11} - X_{22}}{2}\end{aligned}$$

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Problem 3.

Solution.

$$\sigma \cdot a' = \exp\left(\frac{i\sigma \cdot \hat{n}\phi}{2}\right) \sigma \cdot a \exp\left(-\frac{i\sigma \cdot \hat{n}\phi}{2}\right)$$

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Problem 4.

Solution.

$$A(|i\rangle + |j\rangle) = i\,|i\rangle + j\,|j\rangle$$

If we have degenerate eigenvalues i.e., $i = j$ then

$$A(|i\rangle + |j\rangle) = i(|i\rangle + |j\rangle)$$

and $|i\rangle + |j\rangle$ is also an eigenvector of A

$\blacksquare$

Problem 5.

Solution.

$$\begin{aligned} \mathbf{S} \cdot \hat{n} &= \sin\beta\cos\alpha\ S_x + \sin\beta\sin\alpha\ S_y + \cos\beta\ S_z \\ &= \begin{pmatrix} \cos\beta & \sin\beta\exp(-i\alpha) \\ \sin\beta\exp(i\alpha) & -\cos\beta \end{pmatrix} \end{aligned}$$

Let a and b represent the components of the eigenket $|+\rangle$ of this operator. We then need to solve the following system for the components a and b

$$\begin{aligned} a(\cos\beta - e) + b(\sin\beta\exp(-i\alpha) - e) = 0 \\ a(\sin\beta\exp(i\alpha) - e) + b(-\cos\beta - e) = 0 \end{aligned}$$

where e represents the eigenvalue.

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Problem 6.

Solution. $\blacksquare$