Exam 1

Quantum Mechanics
October 14th, 2022 C Seitz

Problem 1.

Solution. Both operators are Hermitian, so their eigenvalues are real. The eigenvectors of operator A are

$$|a_1\rangle = \frac{1}{2}\begin{pmatrix} -1 \\ i\sqrt{2} \\ 1 \end{pmatrix} \quad |a_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad |a_3\rangle = \frac{1}{2}\begin{pmatrix} -1 \\ -i\sqrt{2} \\ 1 \end{pmatrix}$$

with eigenvalues $3\lambda, 2\lambda, \lambda$, in that order. Notice that $|a_1\rangle$ and $|a_3\rangle$ are not orthogonal in this basis. The eigenvectors of operator B are

$$|b_1\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \quad |b_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad |b_3\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

with eigenvalues $5\lambda, 3\lambda, \lambda$, in that order. If the physicist sends particles in state $|1\rangle$ and we measure A, the probability we observe the particle to be in the state $|a_1\rangle$, $|a_2\rangle$, and $|a_3\rangle$ can be found by using the expressions for eigenkets of A above. These probabilities are

$$|\langle a_1|1\rangle|^2 = \frac{1}{4}$$
$$|\langle a_2|1\rangle|^2 = \frac{1}{2}$$
$$|\langle a_3|1\rangle|^2 = \frac{1}{4}$$

The values of A that correspond with each beam are given by the eigenvalues of A written above. The relative intensities are just given by the relative probabilities. After the coffee spill, measuring A filters out only the beam corresponding to state $|a_2\rangle$. Therefore, these are the only particles that enter the apparatus measuring B. So when we measure B, the number of beams we get depends on the inner products

$$
\begin{aligned}
|\langle b_1|a_2\rangle|^2 &= \frac{1}{2} \\
|\langle b_2|a_2\rangle|^2 &= 0 \\
|\langle b_3|a_2\rangle|^2 &= \frac{1}{2}
\end{aligned}
$$

so there are only two beams. The intensities are equal for these two beams. The values of B that correspond with these two beams are b_1 and b_3. ■

Problem 2.

Solution. We are given the commutation relations between the operators C and D and the Hamiltonian, which suggests we should use the Heisenberg picture. The Heisenberg equations of motion are

$$
\begin{aligned}
\frac{dC}{dt} &= -\frac{1}{i\hbar}[H, C] = \alpha D - \beta C \\
\frac{dD}{dt} &= -\frac{1}{i\hbar}[H, D] = -\alpha C - \beta D
\end{aligned}
$$

When $\beta = 0$, the system becomes

$$
\begin{aligned}
\frac{dC}{dt} &= -\frac{1}{i\hbar}[H, C] = \alpha D \\
\frac{dD}{dt} &= -\frac{1}{i\hbar}[H, D] = -\alpha C
\end{aligned}
$$

which has the solution

$$
\begin{aligned}
C(t) &= -c_0 \cos(\alpha t) \\
D(t) &= d_0 \sin(\alpha t)
\end{aligned}
$$

$$
\begin{aligned}
\langle C(t)\rangle &= \langle\alpha|\, C(t)\, |\alpha\rangle \\
&= -\langle\alpha|\, c_0 \cos(\alpha t)\, |\alpha\rangle \\
&= -c_0 \cos(\alpha t)
\end{aligned}
$$

since $|\alpha\rangle$ is presumed to be normalized.

$$\begin{aligned}\langle D(t)\rangle &= \langle\alpha| D(t) |\alpha\rangle \\ &= -\langle\alpha| d_0 \sin(\alpha t) |\alpha\rangle \\ &= d_0 \sin(\alpha t)\end{aligned}$$

since $|\alpha\rangle$ is again presumed to be normalized. The constants α and β must then relate to the angular frequency of $C(t)$ and $D(t)$. Of course, when $\beta = 0$, only α determines the angular frequency of $C(t)$ and $D(t)$.

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Problem 3.

Solution.
We are given the wavefunction

$$\psi(r,\phi) = Ae^{-br^2}$$

We know that probability current is related to the gradient of the phase of the wavefunction. Regardless of what A is (purely real, imaginary, or complex) it is a constant. The exponential is real if b is real, so the phase of $\psi(r,\phi)$ is the same for all (r,ϕ) and therefore the probability current is zero everywhere. For the wavefunction

$$\psi(r,\phi) = Ae^{-br^2}e^{-im\phi}$$

The phase of the wavefunction is clearly dependent on ϕ, so there is a probability current.

$$\rho(r,\phi) = |A|^2 e^{-2br^2}$$

We can always write the wavefunction in the form:

$$\begin{aligned}\psi(r,\phi) &= \sqrt{\rho(r,\phi)}\exp\left(\frac{iS(r,\phi)}{\hbar}\right) \\ &= Ae^{-br^2}e^{-im\phi}\end{aligned}$$

so $-im\phi = iS/\hbar$. Expanding S gives

$$S(r,\phi) = -m\hbar\phi$$

The probability flux is related to $S(r,\phi)$ by

$$\begin{aligned}
\boldsymbol{j}(r,\phi) &= \frac{\rho(r,\phi)\nabla S}{m} \\
&= \frac{\rho}{m}\left(\frac{\partial S}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial S}{\partial \phi}\hat{\phi}\right) \\
&= -\frac{\hbar\rho}{r}\hat{\phi}
\end{aligned}$$

In the limit $r \to \infty$, we can see that

$$\lim_{r\to\infty} \boldsymbol{j}(r,\phi) = \lim_{r\to\infty} \; -\frac{\hbar\rho}{r}\hat{\phi} = 0$$

which makes sense if the particle is localized to some region of space. However, $\rho(r,\phi)$ is static because

$$\nabla \cdot \boldsymbol{j} = \frac{j_r}{r} + \frac{\partial j_r}{\partial r}\frac{1}{r} + \frac{\partial j_\phi}{\partial \phi} = 0$$

This situation might correspond to a particle with spin.

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