

Homework 10

Quantum Mechanics

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Problem 1. 4.7

Solution.

The wave function in three dimensions for a free particle ($V = 0$), is

$$\begin{aligned}\psi(\mathbf{x}, t) &= u(\mathbf{x})e^{-iE_n t/\hbar} \\ \psi^*(\mathbf{x}, -t) &= u^*(\mathbf{x})e^{-iE_n t/\hbar}\end{aligned}$$

where $u(\mathbf{x}) = e^{i\vec{p}\cdot\vec{k}}$. Note that the phase remains unchanged under complex conjugation and time reversal. Now if we reverse the direction of momentum i.e. $|p\rangle \rightarrow |p'\rangle$ for $\vec{p} \cdot \vec{p}' = -1$,

$$\psi'(\mathbf{x}, t) = u'(\mathbf{x})e^{-iE_n t/\hbar}$$

Notice that $u'(\mathbf{x}) = e^{-i\vec{p}'\cdot\vec{k}} = u^*(\mathbf{x})$. Therefore $\psi'(\mathbf{x}, t) = \psi^*(\mathbf{x}, -t)$

$$\chi_+(\hat{n}) = \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{i\gamma} \end{pmatrix}$$

It is also known that

$$\chi_-(\hat{n}) = \begin{pmatrix} -\sin \frac{\theta}{2} e^{-i\gamma} \\ \cos \frac{\theta}{2} \end{pmatrix}$$

So we just need to show that the given transformation gives this result:

$$-i\sigma_2\chi^*(\hat{n}) = -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \cos \frac{\theta}{2} \\ \sin \frac{\theta}{2} e^{-i\gamma} \end{pmatrix} = \begin{pmatrix} -\sin \frac{\theta}{2} e^{-i\gamma} \\ \cos \frac{\theta}{2} \end{pmatrix}$$

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Problem 2. 4.8

Solution.

First note that

$$H\Theta|n\rangle = \Theta H|n\rangle = E_n\Theta|n\rangle$$

so $|n\rangle$ and $\Theta|n\rangle$ have the same energy. If the states are nondegenerate then $|n\rangle$ and $\Theta|n\rangle$ represent the same state. Their wavefunctions are then the same:

$$\langle x'|n\rangle = \langle n|x'\rangle^*$$

which occurs if they are real, or have a phase difference independent of x . For this reason the wavefunction $\psi = e^{ip\cdot x/\hbar}$ does not violate time reversal invariance, because it is degenerate with $e^{ip\cdot x/\hbar}$.

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Problem 3. 4.9

Solution.

$$\begin{aligned} \Theta|\alpha\rangle &= \int d^3\mathbf{p} \Theta|\mathbf{p}\rangle \langle\mathbf{p}|\alpha\rangle^* \\ &= \int d^3\mathbf{p} |-\mathbf{p}\rangle \langle\mathbf{p}|\alpha\rangle^* \\ &= \int d^3\mathbf{p} |\mathbf{p}\rangle \langle-\mathbf{p}|\alpha\rangle^* \\ &= \phi^*(-p) \end{aligned}$$

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Problem 4. 4.10

Solution.

We first try to prove that

$$\Theta |j, m\rangle = e^{i\delta} (-1)^m |j, -m\rangle$$

Consider acting on $|j, m\rangle$ with J_z :

$$J_z \Theta |j, m\rangle = -\Theta J_z |j, m\rangle = -m (\Theta |j, m\rangle)$$

so clearly $\Theta |j, m\rangle$ behaves like $|j, -m\rangle$. The $e^{i\delta}$ part is there because we can always include an arbitrary phase.

$$\begin{aligned} \Theta \mathcal{D}(R) \Theta^{-1} &= \Theta \left(1 - i \frac{\mathbf{J} \cdot \hat{n}}{\hbar} \right) \Theta^{-1} \\ &= \left(1 + i \frac{\Theta \mathbf{J} \Theta^{-1} \cdot \hat{n}}{\hbar} \right) \\ &= \left(1 - i \frac{\mathbf{J} \cdot \hat{n}}{\hbar} \right) = \mathcal{D}(R) \end{aligned}$$

so the time reversed state is just $\mathcal{D}(R) \Theta |j, m\rangle$. ■

Problem 5. 4.11**Solution.**

Since the Hamiltonian is time reversal invariant, then energy eigenkets transform as $\Theta |\alpha\rangle = e^{i\delta} |\alpha\rangle$,

$$\begin{aligned} \langle \mathbf{L} \rangle &= \langle \alpha | \mathbf{L} | \alpha \rangle \\ &= e^{i\delta} e^{-i\delta} \langle \alpha | \Theta \mathbf{L} \Theta^{-1} | \alpha \rangle \\ &= -\langle \alpha | \mathbf{L} | \alpha \rangle \end{aligned}$$

This is only satisfied when $\langle \mathbf{L} \rangle = 0$. We know that when the Hamiltonian is invariant under time-reversal, the eigenkets must be real.

$$\begin{aligned}
\psi(r, \theta, \phi) &= \sum_{l,m} F_{lm}(r) Y_l^m(\theta, \phi) \\
&\rightarrow e^{i\delta} \sum_{l,m} F_{lm}^*(r) (Y_l^m(\theta, \phi))^* \\
&= e^{i\delta} \sum_{l,m} F_{lm}^*(r) (-1)^m Y_l^m(\theta, \phi)
\end{aligned}$$

which means that $F_{lm}(r) = e^{-i\delta} (-1)^m F_{lm}^*(r)$

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Problem 6. 4.12

Solution.

We were given the Hamiltonian:

$$H = AS_z^2 + B(S_x^2 - S_y^2)$$

This Hamiltonian must be invariant under time reversal because these are all scalar values which are invariant e.g, $\Theta S_x^2 \Theta^{-1} = S_x^2 \Theta \Theta^{-1} = S_x^2$ and A and B are real. The explicit matrix representation is

$$H = \hbar^2 \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix}$$

which according to Mathematica has the following eigenvectors

$$\begin{aligned}
|E_1\rangle &= (|+1\rangle + |-1\rangle)/\sqrt{2} \\
|E_{-1}\rangle &= (|+1\rangle - |-1\rangle)/\sqrt{2} \\
|E_0\rangle &= |0\rangle
\end{aligned}$$

with eigenvalues $\hbar^2(A+B)$, $\hbar^2(A-B)$, 0 respectively. These eigenvectors transform in the following way under time reversal:

$$\begin{aligned}
\Theta |E_1\rangle &= (\Theta | +1\rangle + \Theta | -1\rangle)/\sqrt{2} = -(| +1\rangle + | -1\rangle)/\sqrt{2} \\
\Theta |E_{-1}\rangle &= (\Theta | +1\rangle - \Theta | -1\rangle)/\sqrt{2} = -(| +1\rangle - | -1\rangle)/\sqrt{2} \\
\Theta |E_0\rangle &= (-1)^0 |E_0\rangle = |E_0\rangle
\end{aligned}$$

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