Homework 6

Quantum Mechanics

October 28th, 2022 C Seitz

Problem 1. *Problem 3.12 from Sakurai*

Solution.

In general the ensemble average of an operator $[A]$ is defined as

$$[A] = \sum_i w_i \langle \alpha_i | A | \alpha_i \rangle$$

where $\sum_i w_i = 1$

$$\begin{aligned}
[\sigma_x] &= a \langle + | \sigma_x | + \rangle + (1-a) \langle -; y | \sigma_x | -; y \rangle \\
&= a \langle + | \left(| + \rangle \langle - | + | - \rangle \langle + | \right) | + \rangle + (1-a) \langle -; y | \left(| + \rangle \langle - | + | - \rangle \langle + | \right) | -; y \rangle \\
&= 0
\end{aligned}$$

$$\begin{aligned}
[\sigma_y] &= a \langle + | \sigma_y | + \rangle + (1-a) \langle -; y | \sigma_y | -; y \rangle \\
&= ai \langle + | \left(| + \rangle \langle - | - | - \rangle \langle + | \right) | + \rangle + i(1-a) \langle -; y | \left(| + \rangle \langle - | - | - \rangle \langle + | \right) | -; y \rangle \\
&= i(1-a) \langle -; y | \left(-\frac{i}{\sqrt{2}} | + \rangle - \frac{1}{\sqrt{2}} | - \rangle \right) \\
&= -i(1-a) \langle -; y | +; y \rangle = 0
\end{aligned}$$

$$\begin{aligned}
[\sigma_z] &= a \langle + | \left(| - \rangle \langle - | - | + \rangle \langle + | \right) | + \rangle + i(1-a) \langle -; y | \left(| - \rangle \langle - | - | + \rangle \langle + | \right) | -; y \rangle \\
&= -a + i(1-a) \langle -; y | \left(-\frac{i}{\sqrt{2}} | + \rangle - \frac{1}{\sqrt{2}} | - \rangle \right)
\end{aligned}$$

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Problem 2. *Problem 3.13 from Sakurai*

Solution.

The state vector in the S_z basis has the form

$$|\alpha\rangle = c_+ |+\rangle + c_- |-\rangle$$

First note that

$$\langle S_z\rangle = |c_+|^2 - |c_-|^2 \quad |c_+|^2 + |c_-|^2 = 1$$

Together, these equations tell us the magnitude of each complex component.

$$|c_+|^2 = \frac{\langle S_z\rangle + 1}{2} \quad |c_-|^2 = \frac{1 - \langle S_z\rangle}{2}$$

$$\begin{aligned}
\langle S_x\rangle &= \langle\alpha| (|+\rangle \langle -| + |-\rangle \langle +|)(c_+ |+\rangle + c_- |-\rangle) \\
&= \langle\alpha| (c_- |+\rangle + c_+ |-\rangle) \\
&= (c_+^* \langle +| + c_-^* \langle -|)(c_- |+\rangle + c_+ |-\rangle) \\
&= c_+^* c_- + c_-^* c_+ \\
&= |c_+||c_-|(e^{i(\theta-\phi)} + e^{i(\phi-\theta)}) \\
&= 2|c_+||c_-| \cos(\theta - \phi)
\end{aligned}$$

$$\begin{aligned}
\langle S_y\rangle &= \langle\alpha| ((i |+\rangle \langle -| - i |-\rangle \langle +|)(c_+ |+\rangle + c_- |-\rangle) \\
&= i \langle\alpha| (c_- |+\rangle - c_+ |-\rangle) \\
&= i(c_+^* \langle +| + c_-^* \langle -|)(c_- |+\rangle - c_+ |-\rangle) \\
&= c_+^* c_- - c_-^* c_+ \\
&= |c_+||c_-|(e^{i(\theta-\phi)} - e^{i(\phi-\theta)}) \\
&= 2i|c_+||c_-| \sin(\theta - \phi)
\end{aligned}$$

So $\langle S_x\rangle$ gives us the phase difference of c_+ and c_-. Then the sign of $\langle S_y\rangle$ tells us whether θ or ϕ is larger, since sine is odd. This is all we can hope to extract from the expectation values, since multiplying by a global phase $e^{i\delta} |\alpha\rangle$ has no effect on the expectation values.

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Problem 3. *Problem 3.14 from Sakurai*

Solution.

$$\rho = \sum_i w_i |\psi_i\rangle \langle\psi_i| \\ = \frac{1}{3}\left(|\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta| + |2\rangle\langle 2|\right)$$

We can write this out explicitly in the subspace spanned by $|0,1,2\rangle$

$$|\alpha\rangle\langle\alpha| = \frac{1}{2}\begin{pmatrix}1 & 1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} \quad |\beta\rangle\langle\beta| = \frac{1}{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 1\\ 0 & 1 & 1\end{pmatrix} \quad |2\rangle\langle 2| = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\rho = \frac{1}{6}\begin{pmatrix}1 & 1 & 0\\ 1 & 2 & 1\\ 0 & 1 & 2\end{pmatrix}$$

Now recall that $H = \hbar\omega(N + \frac{1}{2})$ which is

$$H = \left(\frac{\hbar\omega}{2}\right)\mathbb{I}_{3\times 3} + \begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}$$

$$[H] = \mathrm{Tr}(\rho H) = \hbar\omega\mathrm{Tr}(\rho N + \rho/2) = \hbar\omega\left(\mathrm{Tr}(\rho N) + \mathrm{Tr}(\rho/2)\right) = \frac{17\hbar\omega}{12}$$

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Problem 4. *Problem 3.15 from Sakurai*

Solution.

$$\rho(t_0) = \sum_i w_i |\psi_i; t_0\rangle \langle\psi_i; t_0|$$

In the Schrodinger picture, the coefficients of the state vectors evolve. Therefore,

$$
\begin{aligned}
\rho(t) &= \sum_i w_i \, \mathcal{U}(t,t_0) \left|\psi_i;t_0\right\rangle \left\langle\psi_i;t_0\right| \mathcal{U}^\dagger(t,t_0) \\
&= \mathcal{U}(t,t_0) \left(\sum_i w_i \left|\psi_i;t_0\right\rangle \left\langle\psi_i;t_0\right| \right) \mathcal{U}^\dagger(t,t_0) \\
&= \mathcal{U}(t,t_0)\rho(t_0)\mathcal{U}^\dagger(t,t_0)
\end{aligned}
$$

For a pure ensemble in state $|\psi_i\rangle$, the density matrix is

$$
\rho(t_0) = \left|\psi_i;t_0\right\rangle \left\langle\psi_i;t_0\right|
$$

At a later time, the density matrix is

$$
\begin{aligned}
\rho(t) &= \mathcal{U}(t,t_0)\rho(t_0)\mathcal{U}^\dagger(t,t_0) \\
&= \mathcal{U}(t,t_0) \left|\psi_i;t_0\right\rangle \left\langle\psi_i;t_0\right| \mathcal{U}^\dagger(t,t_0) \\
&= \left|\psi_i;t_0\right\rangle \left\langle\psi_i;t_0\right|
\end{aligned}
$$

since the eigenkets are stationary ■

Problem 5. *Problem 3.16 from Sakurai*

Solution. ■

Problem 6. *Problem 3.40 from Sakurai*

Solution. ■