Homework 7

Quantum Mechanics
November 3, 2022 C Seitz

Problem 1. *5.1*

Solution.
We are concerned here with the new ground state ket $|0\rangle$ in the presence of H_1 and the new ground state energy shift Δ_0.

$$|0\rangle = \left|0^0\right\rangle + \sum_{k\neq 0}\left|k^0\right\rangle \frac{V_{k0}}{E_0^0 - E_k^0} + ...$$

$$\Delta_0 = V_{00} + \sum_{k\neq 0}\frac{|V_{k0}|^2}{E_0^0 - E_k^0} + ...$$

$$V_{nk} = b\left\langle n^0\right| x \left|k^0\right\rangle = b\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{k}\delta_{n,k-1} + \sqrt{k+1}\delta_{n,k+1}\right)$$

The lowest nonvanishing order is then V_{01}. Therefore

$$\Delta_0 = -\frac{b^2\hbar}{2m\omega}\frac{1}{\hbar\omega} = -\frac{b^2}{2m\omega^2}$$

To solve it exactly, notice that the potential is of the form

$$V_1(x) = ax^2 + bx$$

The new potential shifts to the left by $b/2$, has a new minimum at $-b/2a$, and the gradient has changed:

$$V'(x) = 2ax \rightarrow V_1'(x) = 2ax + b$$

This change in the gradient will not change the energy differences w.r.t. the original problem (why?) so we have really just shifted the equilibrium point down by $-b/2m\omega^2$.

$$\Delta = -\frac{b}{2a} = -\frac{b}{2m\omega^2}$$

which is exactly what we got with perturbation theory.

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Problem 2. *5.2*

Solution. The perturbation Hamiltonian is

$$H_1 = \frac{Vx}{L}$$

$$\begin{aligned} V_{nk} = \left\langle n^0 \right| H_1 \left| k^0 \right\rangle &= \frac{V}{L} \left\langle n^0 \right| x \left| k^0 \right\rangle \\ &= \frac{V}{L} \int_0^L x \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{k\pi x}{L}\right) dx \end{aligned}$$

$$\Delta_0 = V_{00} + \sum_{k \neq 0} \frac{|V_{k0}|^2}{E_0^0 - E_k^0} + ...$$

■

Problem 3. *5.5*

Solution. What we are asked to find is $|\left\langle n^0 \middle| n \right\rangle|^2 / |\left\langle n^0 \middle| n \right\rangle|^2$. Up to order λ^2, we have

$$\begin{aligned} |n\rangle = \left| n^0 \right\rangle + \lambda \sum_{k \neq n} \left| k^0 \right\rangle \frac{V_{kn}}{E_n^0 - E_k^0} + \lambda^2 \Big(\sum_{k \neq n} \sum_{k \neq l} \frac{V_{kl} V_{ln} \left| k^0 \right\rangle}{(E_n^0 - E_k^0)(E_n^0 - E_l^0)} \\ - \sum_{k \neq n} \frac{V_{nn} V_{kn} \left| k^0 \right\rangle}{(E_n^0 - E_k^0)^2} \Big) \end{aligned}$$

Therefore,

$$\langle n^0 | n \rangle = 1 + \lambda^2 \sum_{k \neq n} \sum_{k \neq l} \frac{V_{kl} V_{ln}}{(E_n^0 - E_k^0)(E_n^0 - E_l^0)} \delta_{nk}$$

$\blacksquare$

Problem 4. *5.7*

Solution.

We can write this Hamiltonian as

$$H_0 = H_x + H_y$$

Given some state $|n, m\rangle$, where H_x acts on $|n\rangle$ and H_y acts on $|m\rangle$.

$$\begin{aligned} H_0 |n, m\rangle &= (H_x + H_y) |n, m\rangle \\ &= (E_x^n + E_y^m) |n, m\rangle \\ &= \hbar\omega \left(n + \frac{1}{2} + m + \frac{1}{2} \right) \\ &= \hbar\omega(n + m + 1) \end{aligned}$$

So the energies of the three lowest states are $\hbar\omega, 2\hbar\omega, 3\hbar\omega$. There is a degeneracy - two unique $|n, m\rangle$ have the same eigenvalue for the first excited state. For example $|0, 1\rangle$ and $|1, 0\rangle$ have the same energy. Now, the perturbation Hamiltonian is

$$H_0 = H_x + H_y + \delta m\omega^2 xy$$

First, to zeroth order, the energy eigenkets will not change because the diagonal of V will always be zero. This should be clear from the following calculations. For the ground state, recall

$$V_{nk} = \langle n^0 | \, x \, | k^0 \rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{k} \delta_{n,k-1} + \sqrt{k+1} \delta_{n,k+1} \right)$$

Therefore,

$$
\begin{aligned}
V_{0k} &= \delta m\omega^2 \langle 0,0|\, xy\, |n,m\rangle \\
&= \frac{\hbar}{2m\omega}\delta m\omega^2 \langle n,m|1,1\rangle
\end{aligned}
$$

and $\langle 0,0|1,1\rangle = 0$ so V_{00} is zero (I have just used that x can be written in terms of a_x and $a_x^\dagger$ and the same for y). To first order, $V_{(n,m),0}$ must all be zero due to orthogonality. So there is no first order shift for the ground state. For the first excited state,

$$
\begin{aligned}
V_{1k} &= \delta m\omega^2 \langle 0,1|\, xy\, |n,m\rangle \\
&= \frac{\hbar}{2m\omega}\delta m\omega^2 \sqrt{2}
\end{aligned}
$$

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\Delta_1 = \frac{V_{1k}^2}{2\hbar\omega} + \frac{V_{1k}^2}{\hbar\omega}
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Problem 5. *5.12a*

Solution. ■

Problem 6. *5.24*

Solution. ■