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# Uncertainty-Aware Localization Microscopy by Variational Diffusion

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## Abstract

1 Fast extraction of physically relevant information from images using deep neu-  
2 ral networks has led to significant advances in fluorescence microscopy and its  
3 application to the study of biological systems. For example, the application of  
4 deep networks for kernel density (KD) estimation in single molecule localization  
5 microscopy (SMLM) has accelerated super-resolution imaging of densely-labeled  
6 structures in the cell. However, simple and interpretable uncertainty quantification  
7 is lacking in these applications, and remains a necessary modeling component in  
8 high-risk research. We propose a generative modeling framework for KD estima-  
9 tion in SMLM based on variational diffusion. This approach allows us to probe the  
10 structure of the posterior on KD estimates, creating an additional avenue toward  
11 quality control. We demonstrate that data augmentation with traditional SMLM  
12 architectures followed by a diffusion process permits simultaneous high-fidelity  
13 super-resolution with uncertainty estimation of regressed KDEs.

## 14 1 Introduction

15 Deep models have attracted tremendous attention from researchers in the natural sciences, with several  
16 foundational applications arising in microscopy [Weigert et al., 2018, Falk et al., 2019]. Recently,  
17 the application of deep image translation in single-molecule localization microscopy (SMLM) has  
18 received considerable interest. SMLM techniques are a mainstay of fluorescence microscopy, which  
19 localize “blinking” fluorescent molecules to produce a pointillist representation of biomolecules in  
20 the cell at diffraction-unlimited precision [Rust et al., 2006, Betzig et al., 2006]. Recently, the use of  
21 deep models to perform localization has been proposed as an alternative to traditional localization  
22 algorithms, in order to increase imaging speed and labeling density. In previous applications of deep  
23 models to localization microscopy, super-resolution images have been recovered from a sparse set of  
24 localizations with conditional generative adversarial networks [Ouyang et al., 2018] or localization  
25 itself can be performed using traditional convolutional networks [Nehme et al., 2020, Speiser et al.,  
26 2021]. Here, we train a deep model to perform localization indirectly by predicting kernel density  
27 (KD) estimates of a population of fluorescent molecules.

28 Kernel density estimation in SMLM is necessarily performed using a single low-resolution image,  
29 and thus common measures of model performance are based on localization errors computed over  
30 ensembles of simulated images. Unfortunately, this choice precludes computation of uncertainty at  
31 test time under a fixed model. Bayesian probability theory is therefore an attractive alternative, which  
32 offers us mathematically grounded tools to reason about uncertainty.

33 Here, we address aleatoric uncertainty in KD estimation, by modeling a posterior on high-resolution  
34 KD estimates conditioned on a low-resolution image. Our approach is based on a type of score based  
35 generative model [Song et al., 2021], referred to as a denoising diffusion probabilistic model (DDPM)  
36 in the literature [Ho et al., 2020, Song et al., 2021]. We find that this technique is complementary



Figure 1: Generative model of single molecule localization microscopy images. Low resolution images  $\mathbf{x}$  are generated from coordinates  $\theta$  by integration of the optical transfer function  $O$  and sampling from the likelihood (1):  $\mathbf{x} \sim p(\mathbf{x}|\theta) = \prod_k p(\mathbf{x}_k|\theta)$ . A kernel density estimate  $\mathbf{y}$  is inferred from  $\mathbf{x}$

to relevant existing approaches to uncertainty estimation, which would primarily address epistemic sources of uncertainty, using techniques such as ensembling [Lakshminarayanan et al., 2017] or Monte Carlo dropout [Gal et al., 2022].

Our approach is inspired by recent variational perspectives on diffusion [Dirmeier et al., 2023, Ribeiro et al., 2024, Kingma et al., 2021, 2023]. A primary advantage of this approach is that it provides a mechanism for scalable variational inference, which can be trained using a variational bound written in terms of the signal-to-noise ratio of the diffused data, and a simple noise estimation loss. Indeed, recent efforts have shown that the variational bound can be reparameterized to give several more conventional diffusion losses [Kingma et al., 2021, 2023, Ribeiro et al., 2024].

In the remainder of this paper, we first introduce the likelihood of low-resolution images in localization microscopy, and show uncertainty quantification in a rudimentary example scenario. Then, we introduce KD estimation as an alternative to direct localization using low-resolution images, followed by demonstration of our variational diffusion model for measuring uncertainty KD estimation at scale.

## 2 Background

### 2.1 Image Likelihood and Localization Error

The central objective of SMLM is to infer a set of molecular coordinates  $\theta = (\theta_u, \theta_v)$  from measured low resolution images  $\mathbf{x}$ . The likelihood on a particular pixel  $p(\mathbf{x}_k|\theta)$  is taken to be a convolution of Poisson and Gaussian distributions, due to shot noise  $p(s_k) = \text{Poisson}(\omega_k)$  and sensor readout noise  $p(\zeta_k) = \mathcal{N}(o_k, w_k^2)$

$$p(\mathbf{x}_k|\theta) = A \sum_{q=0}^{\infty} \frac{1}{q!} e^{-\omega_k} \omega_k^q \frac{1}{\sqrt{2\pi} w_k} e^{-\frac{(\mathbf{x}_k - g_k q - o_k)^2}{2w_k^2}} \approx \text{Poisson}(\omega'_k) \quad (1)$$

where  $A$  is some normalization constant. For the sake of generality, we include a per-pixel gain factor  $g_k$ , which is often unity. Sampling from  $p(\mathbf{x}_k|\theta)$  is trivial; however, for computation of a lower bound on uncertainty in  $\theta$ , the summation in (1) can be difficult to work with. Therefore, we choose to use a Poisson approximation for simplification, valid under a range of experimental conditions [Huang et al., 2013]. After subtraction of a known offset  $o_k$  of the pixel array, which can be easily measured, we have  $\omega'_k = \omega_k + w_k^2$ . The expectation of the Poisson process  $\omega_k$  at each pixel of the image must then be computed from the optical impulse response  $O(u, v)$ , which is taken to be an isotropic Gaussian in two-dimensions. Consider an idealized scenario where an isolated fluorescent molecule exists in the image plane. For a particular pixel  $k$  of width  $\delta$  centered at  $\xi_k = (u_k, v_k)$ , we define, along the first image dimension

$$\Delta E_u := \int_{u_k - \delta/2}^{u_k + \delta/2} O(u; \theta_u) du = \frac{1}{2} \left( \text{erf} \left( \frac{u_k + \frac{\delta}{2} - \theta_u}{\sqrt{2}\sigma_{\mathbf{x}}} \right) - \text{erf} \left( \frac{u_k - \frac{\delta}{2} - \theta_u}{\sqrt{2}\sigma_{\mathbf{x}}} \right) \right) \quad (2)$$

The expected value at each pixel is then  $\omega_k \propto \Delta E_u(u_k, \theta_u, \sigma_{\mathbf{x}}) \Delta E_v(v_k, \theta_v, \sigma_{\mathbf{x}})$ . Using this, sampling from the convolution distribution in (1) can be carried out by  $\mathbf{x}_k = s_k + \zeta_k$  for

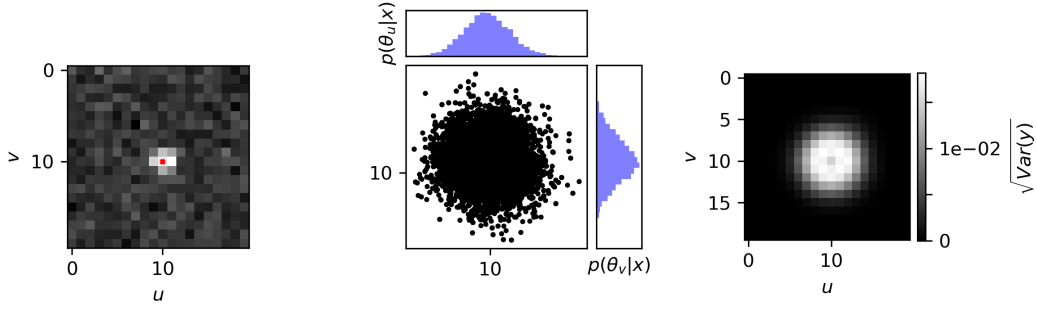


Figure 2: Estimation of the marginal variances  $\sqrt{\text{Var}(\mathbf{y}_k)}$  for an isolated fluorescent emitter. MCMC sampling is carried out using the low-resolution image (left) to estimate the posterior  $p(\theta|\mathbf{x})$  (middle) which is sampled  $\theta \sim p(\theta|\mathbf{x})$  and combined with (3) to estimate marginal variances (right)

69  $s_k \sim \text{Poisson}(\omega_k)$ ,  $\eta_k \sim \mathcal{N}(o_k, w_k^2)$ . The complete generative process is depicted in (Figure 1). In  
70 this simple case, an estimate of the posterior  $p(\theta|\mathbf{x})$  can be found by Markov Chain Monte Carlo  
71 (MCMC) sampling as shown in (Figure 2). Details of the MCMC algorithm used here are outlined in  
72 the Appendix.

## 73 2.2 Gaussian kernel density estimation

74 Direct optimization of the likelihood in (1) from observations  $\mathbf{x}$  alone is challenging when fluorescent  
75 emitters are dense within the field of view and fluorescent signals significantly overlap. However,  
76 convolutional neural networks (CNNs) have recently proven to be powerful tools fluorescence  
77 microscopy to extract parameters describing fluorescent emitters such as color, emitter orientation,  
78  $z$ -coordinate, and background signal Zhang et al. [2018], Kim et al. [2019], Zelger et al. [2018]. For  
79 localization tasks, CNNs typically employ upsampling layers to reconstruct Bernoulli probabilities of  
80 emitter occupancy [Speiser et al., 2021] or KD estimates with higher resolution than experimental  
81 measurements [Nehme et al., 2020]. We choose to use KD estimates in our model, denoted by  $\mathbf{y}$ ,  
82 which are latent in the low-resolution data  $\mathbf{x}$ . KDEs are the most common data structure used in  
83 SMLM, and can be easily generated from molecular coordinates, alongside observations  $\mathbf{x}$ .

84 Similar to the generative process on low resolution images  $\mathbf{x}$ , we generate KDEs  $\mathbf{y}$  by repurposing  
85 the generative model (1) on an unsampled image without noise. In other words, we cast Gaussian KD  
86 estimation as a noiseless image generation process on the domain of  $\mathbf{y}$ . Under a fixed configuration  
87 of  $N$  particles  $\theta$ , the value of a non-normalized KDE pixel  $\mathbf{y}_k$  is given by

$$\mathbf{y}_k(\theta) = \sum_{n=1}^N \Delta E_u(u_k, \theta_u, \sigma_{\mathbf{y}}) \Delta E_v(v_k, \theta_v, \sigma_{\mathbf{y}}) \quad (3)$$

88 where the hyperparameter  $\sigma_{\mathbf{y}}$  is a Gaussian kernel width.

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90 We now consider more realistic datasets  $(\mathbf{x}_i, \mathbf{y}_{0,i}, \hat{\mathbf{y}}_i)_{i=1}^N$  of observed images  $\mathbf{x}_i$  true KD images  $\mathbf{y}_{0,i}$ ,  
91 and augmented low-resolution inputs  $\hat{\mathbf{y}}_i = \phi(\mathbf{x}_i)$ , where  $\phi$  is a CNN. Observations  $\mathbf{x}_i$  are simulated  
92 under the convolution distribution (1) and KDEs are generated by (4).

### 93 3.1 Problem Statement

94 Kernel density estimates produced by the traditional deep architectures for localization microscopy  
95 produce strong results, but lack uncertainty quantification. Unfortunately, the posterior  $p(\theta|\mathbf{x})$  has  
96 no known analytical form and can be difficult to compute at test time, since (i) molecules cannot  
97 be easily resolved and therefore  $\theta$  is of unknown dimension and (ii)  $\theta$  can be high dimensional and

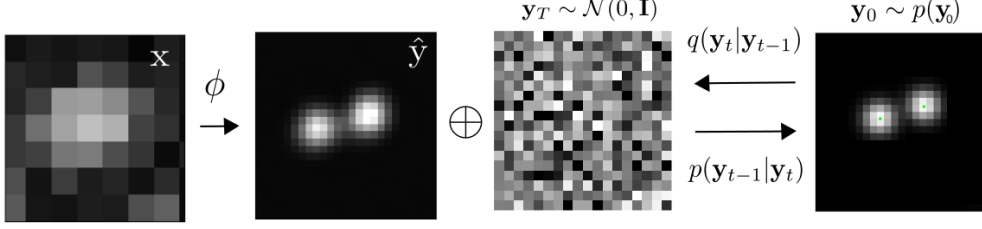


Figure 3: Conditional diffusion model for sampling kernel density estimates. An initial KDE estimate  $\hat{\mathbf{y}}$  is found by the CNN  $\phi$ , which is used as conditional input in the DDPM

efficient exploration of the parameter space is challenging. The central goal of this paper is to instead model a conditional distribution on the latent  $\mathbf{y}$ :  $p(\mathbf{y}|\mathbf{x})$ , where  $\mathbf{y}$  is of known dimensionality. We choose to model  $p(\mathbf{y}|\mathbf{x})$  with a diffusion model, given that the distribution  $p(\mathbf{y}|\mathbf{x})$  is expensive to compute, even if  $p(\theta|\mathbf{x})$  were known.

Recent advances in generative modeling, particularly diffusion models [Sohl-Dickstein et al., 2015, Ho et al., 2020, Song et al., 2021] present a unique opportunity to integrate uncertainty awareness into the localization microscopy toolkit. However, sampling from diffusion models can be computationally expensive, given that generation amounts to solving a complex stochastic differential equation, effectively mapping a simple base distribution to the complex data distribution. The solution of such equations requires numerical integration with very small step sizes, resulting in thousands of neural network evaluations [Saharia et al., 2021, Vahdat et al., 2021]. For conditional generation tasks in high-risk applications, generation complexity is further exacerbated by the need for the highest level of detail in generated samples. Therefore, we propose that sampling is preceded by an augmentation network  $\phi$ , which in essence generates an initial estimate to guide the diffusion process. Reasoning for this choice in our application is two-fold:

**Synthesis Speed.** By training the augmentation network  $\phi$  to obtain an approximate estimate of  $\mathbf{y}_0$ , we can reduce the number of iterations, since the diffusion model only needs to model the remaining mismatch, resulting in a less complex model from which sampling becomes easier. Speed is critical in SMLM applications, which can produce thousands of images in a single experiment.

**Sample Fidelity.** Since Langevin dynamics will often be initialized in low-density regions of the data distribution, inaccurate score estimation in these regions will negatively affect the sampling process. Moreover, mixing can be difficult because of the need of traversing low density regions to transition between modes of the distribution [Song et al., 2019].

### 3.2 Variational Diffusion

Diffusion models [Sohl-Dickstein et al., 2015, Ho et al., 2020, Song et al., 2021] are a class of generative models originally inspired by nonequilibrium statistical physics, which slowly destroy structure in a data distribution via a fixed Markov chain referred to as the *forward process*. In the present context, we leverage the variational interpretation of this model class [Kingma et al., 2021, 2023] to approximate the posterior  $p(\mathbf{y}|\mathbf{x})$ .

**Diffusion Model.** We use a forward process which gradually adds Gaussian noise to the latent  $\mathbf{y}_0$  in discrete time, according to a variance schedule  $\beta_t$ :

$$q(\mathbf{y}_T|\mathbf{y}_0) = \prod_{t=1}^T q(\mathbf{y}_t|\mathbf{y}_{t-1}) \quad q(\mathbf{y}_t|\mathbf{y}_{t-1}) = \mathcal{N}\left(\sqrt{1-\beta_t}\mathbf{y}_{t-1}, \beta_t\mathbf{I}\right) \quad (4)$$

An important property of the forward process is that it admits sampling  $\mathbf{y}_t$  at an arbitrary timestep  $t$  in closed form [Ho et al., 2020]. Using the notation  $\alpha_t := 1 - \beta_t$  and  $\gamma_t := \prod_{s=1}^t \alpha_s$ , we have  $q(\mathbf{y}_t|\mathbf{y}_0) = \mathcal{N}(\sqrt{\gamma_t}\mathbf{y}_0, (1 - \gamma_t)\mathbf{I})$  or  $\mathbf{y}_t = \sqrt{\gamma_t}\mathbf{y}_0 + \sqrt{1 - \gamma_t}\epsilon$  for  $\epsilon \sim \mathcal{N}(0, \mathbf{I})$ . The signal to noise ratio (SNR) as defined in [Kingma et al., 2023], at a time step  $t$  reads  $\text{SNR}_t = \gamma_t/(1 - \gamma_t)$ .

134 The usual procedure is then to learn a parametric representation of the *reverse process*, and therefore  
 135 generate samples of the latent  $\mathbf{y}_0$  from  $p(\mathbf{y}_0|\mathbf{x})$ . Formally,  $p_\psi(\mathbf{y}_0|\mathbf{x}) = \int p_\psi(\mathbf{y}_{0:T}|\mathbf{x})d\mathbf{y}_{1:T}$  where  
 136  $\mathbf{y}_t$  is a latent representation with the same dimensionality of the data and  $p_\psi(\mathbf{y}_{0:T}|\mathbf{x})$  is a Markov  
 137 process, starting from a noise sample  $p_\psi(\mathbf{y}_T) = \mathcal{N}(0, I)$ . Writing this Markov process gives

$$p_\psi(\mathbf{y}_{0:T}|\mathbf{x}) = p_\psi(\mathbf{y}_T) \prod_{t=1}^T p_\psi(\mathbf{y}_{t-1}|\mathbf{y}_t, \mathbf{x}) \quad p_\psi(\mathbf{y}_{t-1}|\mathbf{y}_t, \mathbf{x}) = \mathcal{N}(\mu_\psi(\mathbf{y}_t, \gamma_t), \beta_t I) \quad (5)$$

138 where we reuse the variance schedule of the forward process [Ho et al., 2020]. From (5) it can be  
 139 seen that the learnable parameter in the reverse process is the expectation of the transition  $\mu_\psi$ , where  
 140  $\psi$  is a neural network.

141 Learning the reverse process can be approached by either regressing noise  $\epsilon$  from the forward process,  
 142 or the true latent  $\mathbf{y}_0$ , as there is a deterministic relationship between them. We adopt the former for  
 143 consistency with other work, and define  $\psi$  as a neural denoising function which regresses the noise  $\epsilon$   
 144 from a noisy  $\mathbf{y}_t$ . A relation between the noise estimate  $\epsilon_\psi$  and  $\mu_\psi$  is given in the Appendix, which  
 145 gives an intuition for sampling. The proposed sampling scheme is depicted in (Figure 3).

146 **Variational Objective.** Following [Kingma et al., 2021], we interpret the reverse process as a  
 147 hierarchical generative model that samples a sequence of latents  $\mathbf{y}_t$ , with time running backward.  
 148 Training of the model is achieved through the variational bound

$$-\log p(\mathbf{y}_0) \leq -\mathbb{E}_{q(\mathbf{y}_{1:T}|\mathbf{y}_0)} \log \left( \frac{p_\psi(\mathbf{y}_{1:T}, \mathbf{y}_0)}{q(\mathbf{y}_{1:T}|\mathbf{y}_0)} \right) \quad (6)$$

$$= D_{KL}(q(\mathbf{y}_T|\mathbf{y}_0)||p(\mathbf{y}_T)) + \mathbb{E}_{q(\mathbf{y}_1|\mathbf{y}_0)} \log p(\mathbf{y}_0|\mathbf{y}_1) + \mathcal{L}_\psi \quad (7)$$

149 where we have omitted conditioning on the low-resolution  $\mathbf{x}$  to simplify the notation. Note that, this  
 150 is similar to a hierarchical VAE, but in a diffusion model  $q(\mathbf{y}_{1:T}|\mathbf{y}_0)$  is fixed by the forward process  
 151 rather than learned. The so-called diffusion loss  $\mathcal{L}_\psi$  is shown in Appendix A, and is the term of  
 152 interest as the first two terms do not contribute meaningfully to the loss [Ho et al., 2020]. Furthermore,  
 153 it has become standard to use simplified forms of  $\mathcal{L}_\psi$ , such as a noise estimation loss, as this has  
 154 shown superior performance. Importantly,  $\mathcal{L}_\psi$  is simply a reweighted variant of a family of diffusion  
 155 objectives [Kingma et al., 2021, 2023]. We use the following Monte Carlo estimate of  $\mathcal{L}_\psi$ , which  
 156 demonstrates that the variational bound can be written in terms of the common noise-estimation loss

$$\mathcal{L}_\psi = \mathbb{E}_{\epsilon \sim \mathcal{N}(0, I), t \sim U(1, T)} \left[ \left( \frac{\text{SNR}_{t-1}}{\text{SNR}_t} - 1 \right) \|\epsilon - \epsilon_\psi\|_2^2 \right] \quad (8)$$

157 A full derivation of this objective is outlined in the Appendix. Note that  $\text{SNR}_t$  is monotonically  
 158 decreasing with  $t$ , and thus  $\frac{\text{SNR}_{t-1}}{\text{SNR}_t} = \frac{\gamma_{t-1}}{\gamma_t} \frac{1-\gamma_t}{1-\gamma_{t-1}} \geq 1$ , ensuring  $\mathcal{L}_\psi \geq 0$ . In this paper, we choose  
 159 to use a uniformly weighted loss and leave the exploration of the weighted loss to future work.

## 160 4 Experiments

161 All training data consists of low-resolution  $20 \times 20$  images, setting  $\sigma_{\mathbf{x}} = 0.92$  in units of low-resolution  
 162 pixels, for consistency with common experimental conditions with a 60x magnification objective  
 163 lens and numerical aperture (NA) of 1.4. We multiply  $\omega_k$  by a constant  $i_0 = 200$  for experiments for  
 164 consistency with typical fluorophore emission rates. All KDEs have dimension  $80 \times 80$ , are scaled  
 165 between  $[0, 1]$ , and are generated using  $\sigma_{\mathbf{y}} = 3.0$  pixels in the upsampled image. For a typical CMOS  
 166 camera, this results in KDE pixels with lateral dimension of  $\approx 27\text{nm}$ . Initial coordinates  $\theta$  were  
 167 drawn uniformly over a two-dimensional disc with a radius of 7 low-resolution pixels.

168 **Localization RMSE.** In order to verify the initial predictions made by the augmentation model  
 169  $\phi$ , we simulated a dataset  $(\mathbf{x}_i, \mathbf{y}_{0,i}, \hat{\mathbf{y}}_i)_{i=1}^N$  with  $N = 1000$ . Objects in the KDE  $\hat{\mathbf{y}}_i$  are detected  
 170 using the Laplacian of Gaussian (LoG) detection algorithm [Kong et al., 2013], which permits  
 171 more direct comparison of model predictions to the Cramer-Rao lower bound on localization error,  
 172 compared to other image similarity measures. Localization is carried out from scale-space maxima

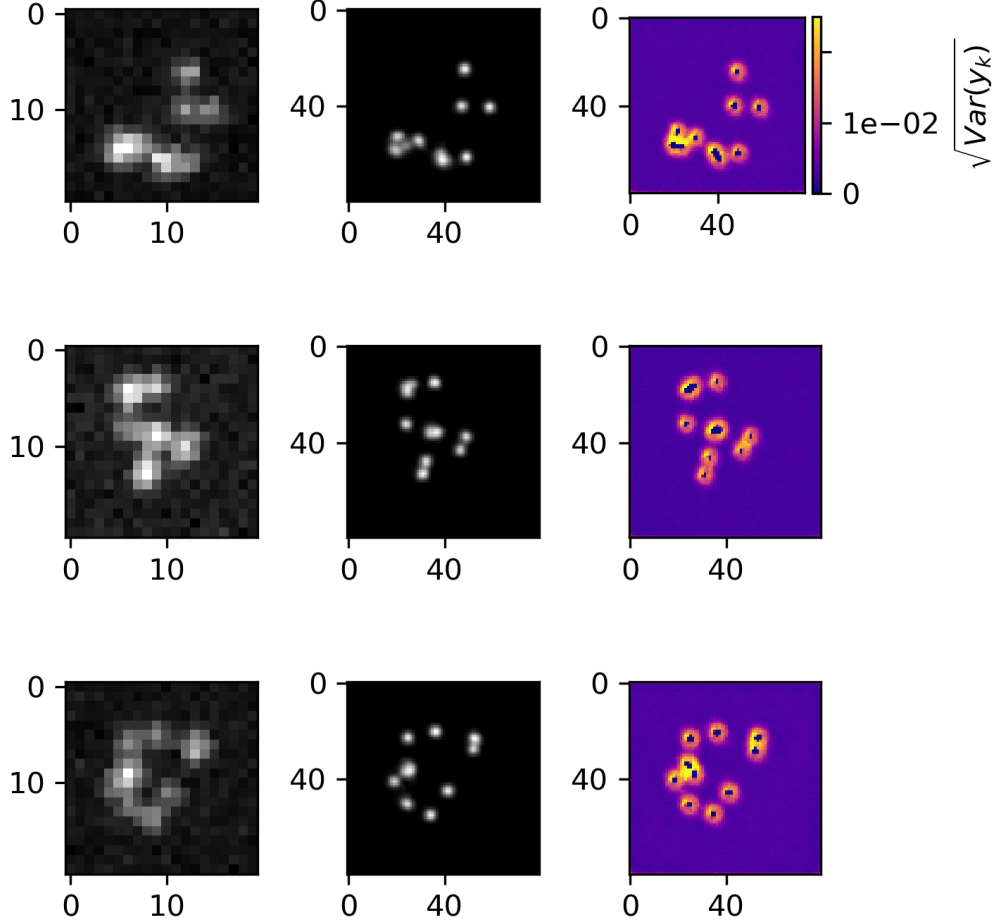


Figure 4: Non cherry-picked estimation of marginal variances. A low-resolution image  $\mathbf{x}$  (left column) is transformed by  $\phi$  to produce a KDE estimate  $\hat{\mathbf{y}}$  (middle column) and a DDPM  $\psi$  computes a map of marginal variances (right column)

173 directly in LoG, as opposed to fitting a model function to KDEs. A particular LoG localization in  
 174 the KDE is paired to the nearest ground truth localization and is unpaired if a localization is not  
 175 within 5 KDE pixels of any ground truth localization. In addition to localization error, we measure a  
 176 precision  $P = TP / (TP + FP) = 1.0$  and recall  $R = TP / (TP + FN) = 0.85$ , where TP denotes  
 177 true positive localizations, FP denotes false positive localizations, and FN denotes false negative  
 178 localizations.

179 **Variational Diffusion.** We set  $T = 100$  for all experiments and treat forward process variances  $\beta_t$   
 180 as hyperparameters, with a linear schedule from  $\beta_0 = 10^{-4}$  to  $\beta_T = 10^{-2}$ . These constants were  
 181 chosen to be small relative to ground truth KDEs, which are scaled to  $[-1, 1]$ , ensuring that forward  
 182 process distribution  $\mathbf{y}_T \sim q(\mathbf{y}_T | \mathbf{y}_0)$  approximately matches the reverse process  $\mathbf{y}_T \sim \mathcal{N}(0, I)$  at  
 183  $t = T$ . Example KD estimates from low-resolution images and the marginal variances obtained from  
 184 sampling  $N = 100$  samples from  $p_\psi(\mathbf{y}_0 | \mathbf{x})$  are shown in (Figure 4).

## 5 Conclusion

We proposed a variational diffusion model for uncertainty-aware localization microscopy. Our approach builds on recent advancements in conditional diffusion models, to model the posterior distribution on high-resolution KD estimates from low-resolution inputs. This tractable posterior distribution is constructed by first augmenting low resolution inputs to a KD estimate using the DeepSTORM architecture with minor modifications [Nehme et al., 2020]. Conditioning a diffusion model on this initial estimate permits sampling with relatively fewer samples than most existing diffusion models in similar applications, thereby making computation of marginal variances possible. Our approach made three core contributions: (i) we derived a relationship between the posterior on kernel density estimates with the posterior on molecular locations, and (ii) we demonstrated that a diffusion model can model a distribution on KDEs with qualitatively similar marginal variances expected from predictions made using MCMC. By using a recently discovered relationship of the variational lower bound to a traditional noise-estimation objective, we can confidently approximate the true posterior.

## 6 Broader Impact

The development of a method for uncertainty estimation in super-resolution imaging, as proposed here, holds implications beyond its immediate application in SMLM. By leveraging diffusion models for uncertainty estimation, this approach not only enhances the reliability of super-resolution image reconstructions but also extends its utility to a diverse array of domains. The incorporation of a guided diffusion process facilitates efficient reconstruction while maintaining interpretation of the underlying uncertainty. Importantly, the principles underlying this method resonate across various fields, suggesting its potential applicability in domains beyond microscopy. For instance, the extension of similar techniques to general image processing tasks highlights the potential to address uncertainty in a wide range of applications in bioimaging or medical imaging. Moreover, the utilization of diffusion models for uncertainty estimation aligns with a broader trend in leveraging probabilistic frameworks for enhancing deep learning applications, with implications extending to fields such as natural language processing, computer vision, and autonomous systems. By bridging these interdisciplinary boundaries, this method not only addresses a critical need in localization microscopy but also contributes to the advancement of uncertainty-aware deep learning methodologies.

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## 271 A Appendix

### 272 A.1 Sampling

273 Sampling from the reverse process  $p_\psi(\mathbf{y}_{t-1}|\mathbf{y}_t, \mathbf{x})$  is achieved by estimation of the noise  $\epsilon_\psi$  from  $\mathbf{y}_t$   
 274 by the denoising model  $\psi$ , and therefore estimation of  $\mathbf{y}_0$

$$\hat{\mathbf{y}}_0 = \frac{1}{\sqrt{\gamma_t}}(\mathbf{y}_t - \sqrt{1 - \gamma_t}\epsilon_\psi) \quad (9)$$

275 followed by sampling from the forward process  $\mathbf{y}_{t-1} \sim q(\mathbf{y}_{t-1}|\hat{\mathbf{y}}_0) = \mathcal{N}(\sqrt{\gamma_{t-1}}, (1 - \gamma_{t-1})I)$ .

### 276 A.2 Derivation of the variational bound

277 We now derive the so-called diffusion loss  $\mathcal{L}_\psi$ , written in (8) in the main text. Similar derivations can  
 278 be found in [Kingma et al., 2021, Ribeiro et al., 2024], and we include it here only for completeness

$$\begin{aligned} -\log p(\mathbf{y}_0) &\leq -\mathbb{E}_{q(\mathbf{y}_{1:T}|\mathbf{y}_0)} \log \frac{p(\mathbf{y}_{0:T})}{q(\mathbf{y}_{1:T}|\mathbf{y}_0)} \\ &= -\mathbb{E}_{q(\mathbf{y}_{1:T}|\mathbf{y}_0)} \log \frac{p(\mathbf{y}_T)p(\mathbf{y}_0|\mathbf{y}_1) \prod_{t=2}^T p(\mathbf{y}_{t-1}|\mathbf{y}_t)}{q(\mathbf{y}_T|\mathbf{y}_0) \prod_{t=2}^T q(\mathbf{y}_{t-1}|\mathbf{y}_t, \mathbf{y}_0)} \\ &= -\mathbb{E}_{q(\mathbf{y}_{1:T}|\mathbf{y}_0)} \left[ p(\mathbf{y}_0|\mathbf{y}_1) + \log \frac{p(\mathbf{y}_T)}{q(\mathbf{y}_T|\mathbf{y}_0)} + \sum_{t=2}^T \log \frac{p(\mathbf{y}_{t-1}|\mathbf{y}_t)}{q(\mathbf{y}_{t-1}|\mathbf{y}_t, \mathbf{y}_0)} \right] \\ &= -\mathbb{E}_{q(\mathbf{y}_{1:T}|\mathbf{y}_0)} [p(\mathbf{y}_0|\mathbf{y}_1)] + D_{KL}(q(\mathbf{y}_T|\mathbf{y}_0)||p(\mathbf{y}_T)) \\ &\quad + \sum_{t=2}^T \mathbb{E}_{q(\mathbf{y}_t|\mathbf{y}_0)} D_{KL}(q(\mathbf{y}_{t-1}|\mathbf{y}_t, \mathbf{y}_0)||p(\mathbf{y}_{t-1}|\mathbf{y}_t)) \end{aligned}$$

279 As before, we omit conditioning on  $\mathbf{x}$  to simplify the notation. The first term is typically ignored,  
 280 as it does not contribute meaningfully to the loss [Ribeiro et al., 2024]. Furthermore, the second  
 281 term is approximately zero by construction. Therefore we are left with the last term, called the  
 282 diffusion loss  $\mathcal{L}_\psi$ . The KL-divergence of  $q$  and  $p$  is between two Gaussians with identical variances  
 283  $\sigma^2 = \frac{(1-\gamma_{t-1})(1-\alpha_t)}{1-\gamma_t}$ , and expectations

$$\mu = \frac{\sqrt{\gamma_{t-1}}(1-\alpha_t)}{1-\gamma_t}\mathbf{y}_0 + \frac{\sqrt{\alpha_t}(1-\gamma_{t-1})}{1-\gamma_t}\mathbf{y}_t \quad \mu_\psi = \frac{\sqrt{\gamma_{t-1}}(1-\alpha_t)}{1-\gamma_t}\hat{\mathbf{y}}_0 + \frac{\sqrt{\alpha_t}(1-\gamma_{t-1})}{1-\gamma_t}\mathbf{y}_t$$

284 for a fixed noise schedule [Saharia et al., 2021]. Therefore, we have

$$\begin{aligned} D_{KL}(q(\mathbf{y}_{t-1}|\mathbf{y}_t, \mathbf{y}_0)||p(\mathbf{y}_{t-1}|\mathbf{y}_t)) &= \frac{1}{2\sigma^2} \|\mu - \mu_\psi\|_2^2 \\ &= \frac{1}{2} \frac{\gamma_{t-1}(1-\alpha_t)}{(1-\gamma_{t-1})(1-\gamma_t)} \|\mathbf{y}_0 - \hat{\mathbf{y}}_0\|_2^2 \\ &= \frac{1}{2} \frac{\gamma_{t-1}((1-\gamma_t) - \alpha_t(1-\gamma_{t-1}))}{(1-\gamma_{t-1})(1-\gamma_t)} \|\mathbf{y}_0 - \hat{\mathbf{y}}_0\|_2^2 \\ &= \frac{1}{2} \frac{\gamma_{t-1}((1-\gamma_t) - \frac{\gamma_t}{\gamma_{t-1}}(1-\gamma_{t-1}))}{(1-\gamma_{t-1})(1-\gamma_t)} \|\mathbf{y}_0 - \hat{\mathbf{y}}_0\|_2^2 \\ &= \frac{1}{2} \left( \frac{\gamma_{t-1}}{1-\gamma_{t-1}} - \frac{\gamma_t}{1-\gamma_t} \right) \|\mathbf{y}_0 - \hat{\mathbf{y}}_0\|_2^2 \\ &= \frac{1}{2} (\text{SNR}_{t-1} - \text{SNR}_t) \|\mathbf{y}_0 - \hat{\mathbf{y}}_0\|_2^2 \end{aligned}$$

285 Reparameterizing the loss in terms of the noise, using  $\|\mathbf{y}_0 - \hat{\mathbf{y}}_0\|_2^2 = \frac{1-\gamma_t}{\gamma_t} \|\epsilon_0 - \epsilon_\psi\|_2^2$  [Ribeiro et al.,  
286 2024], we arrive at

$$\mathcal{L}_\psi = \frac{1}{2} \sum_{t=2}^T \mathbb{E}_{q(\mathbf{y}_t|\mathbf{y}_0)} \left( \frac{\text{SNR}_{t-1}}{\text{SNR}_t} - 1 \right) \|\epsilon - \epsilon_\psi\|_2^2$$

287 Using a Monte Carlo estimate of  $\mathcal{L}_\psi$  [Kingma et al., 2023] which optimizes random terms of the  
288 summation to avoid calculating all terms simultaneously, we arrive at the objective written in the  
289 main text (8)

$$\mathcal{L}_\psi = \mathbb{E}_{\epsilon \sim \mathcal{N}(0, I), t \sim U(1, T)} \left[ \left( \frac{\text{SNR}_{t-1}}{\text{SNR}_t} - 1 \right) \|\epsilon - \epsilon_\psi\|_2^2 \right]$$

### 290 A.3 Optical impulse response

291 It is common to describe the optical impulse response of a microscope as a two-dimensional isotropic  
292 Gaussian [Zhang et al., 2007]. This is an approximation to the more rigorous diffraction models given  
293 by [Richards and Wolf, 1959, Gibson and Lanni, 1989]. Over a continuous domain, the impulse  
294 response reads

$$O(u, v) = \frac{1}{2\pi\sigma_x^2} e^{-\frac{(u-\theta_u)^2 + (v-\theta_v)^2}{2\sigma_x^2}}$$

295 The above expression can be interpreted as a probability distribution over locations where a photon  
296 can be detected. Therefore, for discrete detectors, we discretize this expression by integrating over  
297 pixels. The number of photon arrivals will follow Poisson statistics, with expected value

$$\omega_k = i_0 \left( \int_{u_k - \delta/2}^{u_k + \delta/2} O(u; \theta_u) du \right) \left( \int_{v_k - \delta/2}^{v_k + \delta/2} O(v; \theta_v) dv \right)$$

298 The scalar quantity  $i_0$  represents the amplitude of the signal, which is proportional the quantum  
299 efficiency of a pixel  $\eta$ , the duration of exposure,  $\Delta$ , and the number of photons emitted by a fluorescent  
300 molecule  $N_0$ . With no loss of generality,  $\Delta = \eta = 1$  and there is a single free parameter  $N_0$ . A  
301 simple change of variables  $u' = u - \theta_u$  and  $v' = v - \theta_v$  gives

$$\omega_k = i_0 \left( \int_{u_k - \delta/2 - \theta_u}^{u_k + \delta/2 - \theta_u} O(u) du \right) \left( \int_{v_k - \delta/2 - \theta_v}^{v_k + \delta/2 - \theta_v} O(v) dv \right)$$

302 One of these terms can be written as

$$\begin{aligned} \int_{u_k - \delta/2 - \theta_u}^{u_k + \delta/2 - \theta_u} O(u) du &= \int_0^{u_k + \delta/2 - \theta_u} O(u) du - \int_0^{u_k - \delta/2 - \theta_u} O(u) du \\ &= \frac{1}{2} \left( \text{erf} \left( \frac{u_k + \frac{\delta}{2} - \theta_u}{\sqrt{2}\sigma_x} \right) - \text{erf} \left( \frac{u_k - \frac{\delta}{2} - \theta_u}{\sqrt{2}\sigma_x} \right) \right) \end{aligned}$$

303 where we have used the common definition  $\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$ .

### 304 A.4 Metropolis-Hastings MCMC

305 To obtain numerical estimates of  $p(\theta|\mathbf{x}) \propto p(\mathbf{x}|\theta)p(\theta)$  and therefore  $p(\mathbf{y}|\mathbf{x})$ , for the isolated fluo-  
306 rescent molecule as shown in (Figure 2), we used Metropolis-Hastings Markov Chain Monte Carlo

(MCMC) to estimate the posterior on coordinates. Under the Poisson approximation in (1), the model negative log-likelihood is

$$\ell(\mathbf{x}|\theta) = -\log \prod_k \frac{e^{-(\omega'_k)} (\omega'_k)^{n_k}}{n_k!} = \sum_k \log n_k! + \omega'_k - n_k \log (\omega'_k) \quad (10)$$

where  $n_k$  is the observed number events at a pixel. MCMC is asymptotically exact, which is not guaranteed by variational methods which may rely on a Laplace approximation around the MLE. We choose a uniform prior  $p(\theta)$ , and Metropolis-Hastings is run for  $10^4$  iterations, the first  $10^3$  iterations are discarded as burn-in. A proposal  $\theta' = \theta + \Delta\theta$  was generated with  $\Delta\theta \sim \mathcal{N}(0, \sigma^2 I)$  where  $\sigma^2 = 0.05$ . The acceptance probability is

$$\alpha = e^{\beta(\ell(\theta) - \ell(\theta'))}$$

We choose  $\beta = 0.2$  to achieve a target acceptance rate of 0.5.

## A.5 Cramer-Rao Lower Bound

Reliable inference of  $\theta$  from  $\mathbf{x}$  in general requires performance metrics for model selection. We use the Fisher information as an information theoretic criteria to assess the quality of the data augmentation model  $\phi$  tested here, with respect to the root mean squared error (RMSE) of our predictions of  $\theta$  [Chao et al., 2016]. The Poisson log-likelihood  $\ell(\mathbf{x}|\theta)$  is also convenient for computing the Fisher information matrix [Smith et al., 2010] and thus the Cramer-Rao lower bound, which bounds the variance of a statistical estimator of  $\theta$ , from below i.e.,  $\text{var}(\hat{\theta}) \geq I^{-1}(\theta)$ . The Fisher information is straightforward to compute under the Poisson log-likelihood in (1). In general, the Fisher information is given by the expression

$$I_{ij}(\theta) = \mathbb{E}_{\theta} \left( \frac{\partial \ell}{\partial \theta_i} \frac{\partial \ell}{\partial \theta_j} \right) \quad (11)$$

For an arbitrary parameter, we find that, for a Poisson log-likelihood  $\ell$

$$\begin{aligned} \frac{\partial \ell}{\partial \theta_i} &= \frac{\partial}{\partial \theta_i} \sum_k \log n_k! + \omega'_k - n_k \log (\omega'_k) \\ &= \sum_k \frac{\partial \omega'_k}{\partial \theta_i} \left( \frac{\omega'_k - n_k}{\omega'_k} \right) \end{aligned}$$

Using this result, we can compute the Fisher information matrix  $I(\theta)$

$$I_{ij}(\theta) = \mathbb{E}_{\theta} \left( \sum_k \frac{\partial \omega'_k}{\partial \theta_i} \frac{\partial \omega'_k}{\partial \theta_j} \left( \frac{\omega'_k - n_k}{\omega'_k} \right)^2 \right) = \sum_k \frac{1}{\omega'_k} \frac{\partial \omega'_k}{\partial \theta_i} \frac{\partial \omega'_k}{\partial \theta_j}$$

A fundamental lower bound on the variance in our estimates of  $\theta$  then is found from its inverse:  $\text{CRLB} = I^{-1}(\theta)$ . This result is used to show in (Figure 5), that the data augmentation model  $\phi$  efficiently estimates molecular coordinates under the experimental conditions tested here.

## A.6 Neural Networks $\psi, \phi$

**DeepSTORM CNN  $\phi$ .** The DeepSTORM CNN, for 3D localization, can be viewed as a deep kernel density estimator, reconstructing kernel density estimates  $\mathbf{y}$  from low-resolution inputs  $\mathbf{x}$ . We utilize a simplified form of the original architecture [Nehme et al., 2020] for 2D localization, which we denote  $\phi$  in this paper, which consists of three main modules: a multi-scale context aggregation module, an upsampling module, and a prediction module. For context aggregation, the architecture utilizes dilated convolutions to increase the receptive field of each layer. The upsampling module is then

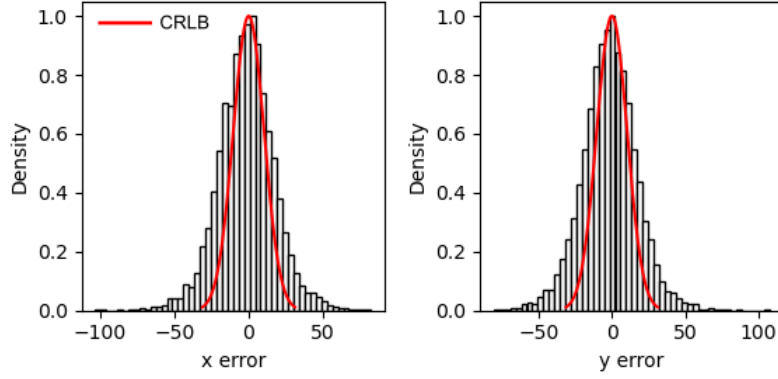


Figure 5: Localization errors of the trained model  $\phi$ . The Cramer-Rao lower bound is shown in red, computing by taking the diagonal elements of  $I^{-1}(\theta)$ .

336 composed of two consecutive 2x resize-convolutions, computed by nearest-neighbor interpolation,  
 337 to increase the lateral resolution by a factor of 4. Additional details regarding this architecture can  
 338 be found in the original paper Nehme et al. [2020]. The terminal prediction module contains three  
 339 additional convolutional blocks for refinement of the upsampled image, followed by an element-wise  
 340 HardTanh. The architecture is trained using the objective  $\mathcal{L}_\phi = \frac{1}{N} \sum_{n=1}^N (\mathbf{y}_{0,n} - \hat{\mathbf{y}}_n)^2$ .

341 **DDPM**  $\psi$ . To represent the reverse process, we used a DDPM architecture originally proposed in  
 342 [Saharia et al., 2021]. We chose the U-Net backbone to have channel multipliers  $[1, 2, 4, 8, 8]$  in  
 343 the downsampling and upsampling paths of the architecture. In this architecture, parameters are  
 344 shared across time, which is specified to the network using the Transformer sinusoidal position  
 345 embedding, and uses self-attention at the  $16 \times 16$  feature map resolution. To condition the model on  
 346 the input  $\hat{\mathbf{y}}$ , we concatenate the  $\hat{\mathbf{y}}$  estimated by DeepSTORM along the channel dimension, which  
 347 are scaled to  $[0, 1]$ , with  $\mathbf{y}_T \sim \mathcal{N}(0, I)$ . Others have experimented with more sophisticated methods  
 348 of conditioning, but found that the simple concatenation yielded similar generation quality [Saharia  
 349 et al., 2021].