Homework 2

Quantum Mechanics

August 29th, 2022 Clayton Seitz

Problem 1. *Problem 1.12 from Sakurai*

Solution.

If we choose the representation such that $|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $|2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ then we can use the definition of the outer product to show that

$$H = a\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$

The energy eigenvalues are then found by

$$\begin{aligned}\det(H-\lambda I) &= \det\begin{pmatrix}a-\lambda & a\\ a & -a-\lambda\end{pmatrix}\\ &= (a-\lambda)(-a-\lambda)-a^2\\ &= \lambda^2-2a^2 = 0\end{aligned}$$

therefore $E_\pm = \pm\sqrt{2a}$. The + eigenvector $|\psi_1\rangle$ is given by the system

$$\begin{aligned}(\psi_1^1+\psi_1^2) &= \sqrt{\frac{2}{a}}\psi_1^1\\ (\psi_1^1-\psi_1^2) &= \sqrt{\frac{2}{a}}\psi_1^2\end{aligned}$$

The − eigenvector $|\psi_2\rangle$ is given by the system

$$\begin{aligned}(\psi_2^1+\psi_2^2) &= -\sqrt{\frac{2}{a}}\psi_2^1\\ (\psi_2^1-\psi_2^2) &= -\sqrt{\frac{2}{a}}\psi_2^2\end{aligned}$$

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Problem 2. *Problem 1.13 from Sakurai*

Solution.

Writing H out in matrix form gives

$$\begin{aligned} H &= H_{11}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + H_{12}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + H_{22}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} H_{11}+H_{12}+H_{22}+1 & H_{11}-H_{12}-H_{22}+1 \\ H_{11}-H_{12}+H_{22}-1 & H_{11}+H_{12}-H_{22}-1 \end{pmatrix} \end{aligned}$$

$$\det(H-\lambda I) = \det\begin{pmatrix} H_{11}+H_{12}+H_{22}+1-\lambda & H_{11}-H_{12}-H_{22}+1 \\ H_{11}-H_{12}+H_{22}-1 & H_{11}+H_{12}-H_{22}-1-\lambda \end{pmatrix}$$

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Problem 3. *Problem 1.15 from Sakurai*

Solution. After the first measurement along $+\hat{z}$, all of our atoms are prepared in the $|+\rangle$ state in the S_z basis. At the next apparatus oriented along $\hat{n}$, more atoms will be filtered out since $|+\rangle$ is not an eigenket of the $\mathbf{S}\cdot\hat{n}$ operator. Recall that $|+\rangle_n$ is

$$|+\rangle_n = \cos\frac{\beta}{2}|+\rangle + \sin\frac{\beta}{2}|-\rangle$$

The probability the state $|+\rangle$ survives is given by the inner product

$$\begin{aligned} |\langle +|+\rangle_n|^2 &= |\langle +|\cos\frac{\beta}{2}|+\rangle + \langle +|\sin\frac{\beta}{2}|-\rangle|^2 \\ &= \cos^2\frac{\beta}{2} \end{aligned}$$

After this, all atoms are in the $|+\rangle_n$ state. We then filter the atoms one more time with an apparatus along $-\hat{z}$. The fraction that survive this one is given by

$$
\begin{aligned}
|\langle -|+\rangle_n|^2 &= |\langle -|\cos\frac{\beta}{2}|+\rangle + \langle -|\sin\frac{\beta}{2}|-\rangle|^2 \\
&= \sin^2\frac{\beta}{2}
\end{aligned}
$$

Therefore the fraction output is $\cos^2\frac{\beta}{2}\sin^2\frac{\beta}{2}$. We can maximize this function by setting $\beta = \pi/2$

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Problem 4. *Problem 1.16 from Sakurai*

Solution.

We have the observable

$$
O = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
$$

$$
\begin{aligned}
\det(O - \lambda I) &= \det\begin{pmatrix} -\lambda & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda \end{pmatrix} \\
&= -\lambda\left(\lambda^2 - \frac{1}{2}\right) - \frac{1}{\sqrt{2}}\left(-\frac{\lambda}{\sqrt{2}}\right) \\
&= -\lambda^3 + \lambda = 0
\end{aligned}
$$

Clearly our eigenvalues are $\lambda = \pm 1$

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Problem 5. *Problem 1.23 from Sakurai*

Solution.

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Problem 6. *Problem 1.24 from Sakurai*

Solution. For the ground state, the position space wavefunction $|\psi\rangle$ is a solution to the eigenvalue equation

$$
\begin{aligned}
H|\psi\rangle &= \left[\frac{\boldsymbol{p}^2}{2m} + \boldsymbol{V}(x)\right]|\psi\rangle \\
&= -\frac{\hbar^2}{2m}\frac{\partial^2|\psi\rangle}{\partial x^2} + V(x)|\psi\rangle \\
&= E|\psi\rangle
\end{aligned}
$$

We set the boundary conditions $\psi(0) = 0$ and $\psi(a) = 0$ since the wavefunction must vanish at the two walls. Since $V(x) = 0$ inside the well, Schrodinger's equation reduces to

$$
H|\psi\rangle = -\frac{\hbar^2}{2m}\frac{\partial^2|\psi\rangle}{\partial x^2} = E|\psi\rangle
$$

This equation has the general solution

$$
|\psi\rangle = A\exp(ikx) + B\exp(-ikx)
$$

Given our boundary condition $\psi(a) = 0$, the wavelength must satisfy $ka = n\pi$ which means that $k = \frac{n\pi}{a}$ for integer $n > 0$, which gives us the solution

$$
|\psi\rangle = A\sin\left(\frac{n\pi x}{a}\right)
$$

It is straightforward to show that

$$
\langle\psi|\psi\rangle = \frac{2}{a}\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)dx = 1
$$

Giving the eigenkets

$$
|\psi\rangle = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)
$$

the variance in position when $n = 1$ is

$$
\begin{aligned}
\langle(\Delta x)^2\rangle &= \langle x^2\rangle - \langle x\rangle^2 \\
&= \langle\psi| x^2 |\psi\rangle - (\langle\psi| x |\psi\rangle)^2 \\
&= x^2\langle\psi|\psi\rangle - (x\langle\psi|\psi\rangle)^2 \\
&= \frac{2}{a}\int_0^a x^2\sin^2\left(\frac{\pi x}{a}\right) - \left(\frac{2}{a}\int_0^a x\sin^2\left(\frac{\pi x}{a}\right)\right)^2
\end{aligned}
$$

We can immediately write the value of $(\langle x \rangle)^2$ based on the symmetry of the wavefunction

$$\left(\int_0^a x \sin^2\left(\frac{\pi x}{a}\right) dx \right)^2 = \frac{a^2}{4}$$

The term $\langle x^2 \rangle$ is given by the integral

$$\int_0^a x^2 \sin^2(\alpha x)\, dx$$

$$\begin{aligned}
\langle (\Delta p)^2 \rangle &= \langle p^2 \rangle - \langle p \rangle^2 \\
&= \langle \psi | \, p^2 \, | \psi \rangle - (\langle \psi | \, p \, | \psi \rangle)^2 \\
&= \langle \psi | \, \hbar^2 \frac{\partial^2}{\partial x^2} \, | \psi \rangle - \left(\langle \psi | - i\hbar \frac{\partial}{\partial x} \, | \psi \rangle \right)^2 \\
&= -c\alpha^2\hbar^2 \int_0^a \sin(\alpha x) dx
\end{aligned}$$

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