Homework 4

Quantum Mechanics

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Problem 1. *Problem 2.14 from Sakurai*

Solution.

We are given that the state vector is

$$|\alpha\rangle = \exp\left(\frac{-ipa}{\hbar}\right)|0\rangle$$

The Heisenberg equation of motion reads

$$\frac{dx}{dt} = \frac{1}{i\hbar}[x, H] = 0$$

Therefore $x = x_0$ for all $t \geq t_0$

$$\begin{aligned}
\langle x\rangle &= \int x_0 \langle x|\alpha\rangle \langle\alpha|x\rangle\, dx \\
&= \int x \exp\left(\frac{-ipa}{\hbar}\right)\langle x|0\rangle \exp\left(\frac{ipa}{\hbar}\right)\langle 0|x\rangle\, dx \\
&= \int x_0 \mid \langle x|0\rangle \mid^2 dx \\
&= \int x_0 \mid \langle x|0\rangle \mid^2 dx
\end{aligned}$$

We could write out $\langle x|0\rangle$, its complex conjugate, and do the integral. Instead recall the general expression for the matrix element of x

$$\langle n'|\, x\, |n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\delta_{n',n-1} + \sqrt{n+1}\delta_{n',n+1}\right)$$

which is zero when $n = n'$ which means that $\langle x \rangle = 0$

$\blacksquare$

Problem 2. *Problem 2.15 from Sakurai*

Solution. We were given the state

$$|\alpha\rangle = \exp\left(\frac{-ipa}{\hbar}\right)|0\rangle$$

$$\langle x|\alpha\rangle = \pi^{-1/4}x_0^{1/2}\exp\left(\frac{-ipa}{\hbar}\right)\exp\left(-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right)$$

where $x_0 = \sqrt{\frac{\hbar}{m\omega}}$. The Hamiltonian operator $\hat{H}$ is independent of time so we have the unitary time evolution operator

$$\mathcal{U}(t) = \exp\left(-\frac{i\hat{H}t}{\hbar}\right)$$

Assuming $|\alpha\rangle$ is expressed in the energy basis, this can be alternatively be written as the power series

$$\mathcal{U}(t) = \sum_{n=0}^{\infty}\frac{\hat{H}^n}{n!} \to \mathcal{U}(t)|\alpha\rangle = \sum_{n=0}^{\infty}\frac{\hat{H}^n}{n!}|\alpha\rangle$$

$$\sum_{n=0}^{\infty}\frac{\alpha^n}{n!}|\alpha\rangle = \sum_{n}\exp\left(\frac{-i\alpha_n t}{\hbar}\right)|\alpha_n\rangle$$

The probability that $|\alpha\rangle$ is measured to be in the state $|0\rangle$ is

$$\langle 0|\alpha\rangle\langle\alpha|0\rangle = \exp\left(\frac{-ipa}{\hbar}\right)\langle 0|0\rangle\exp\left(\frac{ipa}{\hbar}\right)\langle 0|0\rangle = 1$$

This probability does not change for $t > 0$. This is clear when we look at the state

$$|\alpha; t\rangle = \exp\left(-\frac{iE_0 t}{\hbar}\right) \exp\left(\frac{-ipa}{\hbar}\right) |0\rangle$$

The second exponential is just a complex number and is time independent. The first exponential is just a phase, which is not measurable directly. In other words, when we hit this state with the dual ket $\langle 0|$, the phase goes away and we are left with a time-independent probability density.

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Problem 3. *Problem 2.16 from Sakurai*

Solution. ■

Problem 4. *Problem 2.28 from Sakurai*

Solution. ■

Problem 5. *Problem 2.29 from Sakurai*

Solution. ■

Problem 6. *Problem 2.32 from Sakurai*

Solution. ■