Homework 8

Quantum Mechanics
November 9, 2022 C Seitz

Problem 1. *5.27*

Solution.

$$\frac{\langle\tilde{0}|\, H\, |\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle} \geq E_0$$

The denominator is easy to compute

$$2\int_{-\infty}^{0} e^{\beta x} dx = \frac{1}{\beta}$$

The numerator

$$\begin{aligned}
\langle\tilde{0}|\, H\, |\tilde{0}\rangle &= \int_{-\infty}^{\infty} \psi^*(x) H \psi(x) dx \\
&= \int_{-\infty}^{0} \psi^*(x) H \psi(x) dx + \int_{0}^{\infty} \psi^*(x) H \psi(x) dx
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{0} \psi^*(x) H \psi(x) dx &= \int_{-\infty}^{0} -e^{\beta x} \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} e^{\beta x} + \frac{1}{2} m\omega^2 x^2 e^{2\beta x} dx \\
&= \int_{-\infty}^{0} e^{2\beta x} \left(\frac{1}{2} m\omega^2 x^2 - \frac{\hbar^2 \beta^2}{2m} \right) dx \\
&= \Big|_{-\infty}^{0} \frac{1}{2} m\omega^2 \frac{e^{2\beta x}(1 - 2\beta x + 2\beta^2 x^2)}{4\beta^3} - e^{2\beta x} \frac{\hbar^2 \beta}{4m} \\
&= \frac{1}{2} m\omega^2 \frac{1}{4\beta^3} - \frac{\hbar^2 \beta}{4m}
\end{aligned}$$

$$
\begin{aligned}
\int_0^\infty \psi^*(x) H \psi(x) dx &= \int_0^\infty -e^{-\beta x}\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}e^{-\beta x} + \frac{1}{2}m\omega^2 x^2 e^{-2\beta x} dx \\
&= \int_0^\infty e^{-2\beta x}\left(\frac{1}{2}m\omega^2 x^2 - \frac{\hbar^2\beta^2}{2m}\right) dx \\
&= \Big|_0^\infty \frac{1}{2}m\omega^2 \frac{e^{-2\beta x}(1 + 2\beta x + 2\beta^2 x^2)}{4\beta^3} - e^{-2\beta x}\frac{\hbar^2 \beta}{4m} \\
&= \frac{1}{2}m\omega^2 \frac{1}{4\beta^3} - \frac{\hbar^2\beta}{4m}
\end{aligned}
$$

$$
\bar{H} = \frac{\langle\tilde{0}|\, H \,|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle} = \frac{m\omega^2}{4\beta^2} - \frac{\hbar^2\beta^2}{2m}
$$

$$
\frac{d\bar{H}}{d\beta} = -\frac{m\omega^2}{4\beta} - \frac{\hbar^2\beta}{m} = 0
$$

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Problem 2. *5.29*

Solution.

We have the full time-dependent Hamiltonian

$$
H(t) = H_0 + F_0 x \cos\omega t
$$

We need to find $|\psi(t)\rangle$, which amounts to finding the expansion coefficients $c_n(t)$. In the interaction picture, we have that

$$
i\hbar \dot{c_n}(t) = \sum_m V_{nm} e^{i\omega_{nm} t} c_m(t)
$$

for $\omega_{nm} = (E_n - E_m)/\hbar$.

$$
\begin{aligned}
V_{nm} &= F_0 \cos\omega t \,\langle n|\, x \,|m\rangle \\
&= \frac{F_0\cos\omega t}{2}\sqrt{\frac{\hbar}{2m\omega_0}}\left(\sqrt{n+1}\delta_{m,n-1} + \sqrt{n}\delta_{m,n+1}\right)
\end{aligned}
$$

But the initial condition says that $|\psi(0)\rangle = |0\rangle$, so $n = 0$ and the only term of the summation that survives has $m = 1$. Therefore,

$$\begin{aligned} i\hbar\dot{c_1}(t) &= V_{10}e^{i\omega_0 t}c_0(t) \\ &= \frac{F_0\cos\omega t}{2}\sqrt{\frac{\hbar}{2m\omega_0}}e^{i\omega_0 t}c_0(t) \end{aligned}$$

Solving for $c_1(t)$,

$$c_1(t) = \frac{F_0}{2}\sqrt{\frac{\hbar}{2m\omega_0}}\int_0^t e^{i\omega_0 t}\cos\omega t dt$$

The expectation value $\langle x\rangle$ is just

$$\begin{aligned} \langle x\rangle &= \langle\psi(t)|\, x\,|\psi(t)\rangle \\ &= (\langle 0|\, c_0 + \langle 1|\, c_1(t))\,\frac{a+a^\dagger}{2}\,(c_0\,|0\rangle + c_1(t)\,|1\rangle) \end{aligned}$$

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Problem 3. *5.30*

Solution. ■

Problem 4. *5.32*

Solution. ■

Problem 5. *5.35*

Solution. ■

Problem 6. *5.36*

Solution. ■