Homework 7

Quantum Mechanics

November 3, 2022   C Seitz

Problem 1. *5.1*

Solution.

We are concerned here with the new ground state ket $|0\rangle$ in the presence of H_1 and the new ground state energy shift Δ_0.

$$|0\rangle = |0^0\rangle + \sum_{k \neq 0} |k^0\rangle \frac{V_{k0}}{E_0^0 - E_k^0} + \dots$$

$$\Delta_0 = V_{00} + \sum_{k \neq 0} \frac{|V_{k0}|^2}{E_0^0 - E_k^0} + \dots$$

$$V_{nk} = b \langle n^0 | x | k^0 \rangle = b \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{k} \delta_{n,k-1} + \sqrt{k+1} \delta_{n,k+1} \right)$$

The lowest nonvanishing order is then V_{01}. Therefore

$$\Delta_0 = -\frac{b^2 \hbar}{2m\omega} \frac{1}{\hbar\omega} = -\frac{b^2}{2m\omega^2}$$

To solve it exactly, notice that the potential is of the form

$$V_1(x) = ax^2 + bx$$

The new potential shifts to the left by $b/2$, has a new minimum at $-b/2a$, and the gradient has changed:

$$V'(x) = 2ax \rightarrow V_1'(x) = 2ax + b$$

This change in the gradient will not change the energy differences w.r.t. the original problem (why?) so we have really just shifted the equilibrium point down by $-b/2m\omega^2$.

$$\Delta = -\frac{b}{2a} = -\frac{b}{2m\omega^2}$$

which is exactly what we got with perturbation theory.

∎

Problem 2. *5.2*

Solution. The perturbation Hamiltonian is

$$H_1 = \frac{Vx}{L}$$

$$\begin{aligned}
V_{nk} = \left\langle n^0 \right| H_1 \left| k^0 \right\rangle &= \frac{V}{L} \left\langle n^0 \right| x \left| k^0 \right\rangle \\
&= \frac{V}{L} \int_0^L x \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{k\pi x}{L}\right) dx
\end{aligned}$$

$$\Delta_0 = V_{00} + \sum_{k\neq 0} \frac{|V_{k0}|^2}{E_0^0 - E_k^0} + \dots$$

∎

Problem 3. *5.5*

Solution. What we are asked to find is $\left| \left\langle n^0 | n \right\rangle \right|^2$. Up to order λ^2, we have

$$\begin{aligned}
|n\rangle = \left| n^0 \right\rangle + \lambda \sum_{k\neq n} \left| k^0 \right\rangle \frac{V_{kn}}{E_n^0 - E_k^0} + \lambda^2 \Big(\sum_{k\neq n} \sum_{k\neq l} \frac{V_{kl}V_{ln}\left| k^0 \right\rangle}{(E_n^0 - E_k^0)(E_n^0 - E_l^0)} \\
- \sum_{k\neq n} \frac{V_{nn}V_{kn}\left| k^0 \right\rangle}{(E_n^0 - E_k^0)^2} \Big)
\end{aligned}$$

Therefore,

$$\langle n^0 | n \rangle = 1 + \lambda^2 \sum_{k \neq n} \sum_{k \neq l} \frac{V_{kl} V_{ln}}{(E_n^0 - E_k^0)(E_n^0 - E_l^0)} \delta_{nk}$$

∎

Problem 4. *5.7*

Solution. ∎

Problem 5. *5.12a*

Solution. ∎

Problem 6. *5.24*

Solution. ∎