The Abelian Hidden Subgroup Problem

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Introduction

Dimension of n-qubit Hilbert space $N = 2^n$

The quantum fourier transform (QFT) transforms a quantum state $|\psi\rangle \to |\phi\rangle$ via the transformation of basis states:

$$\mathrm{QFT}\,|j\rangle = \frac{1}{2^{n/2}} \sum_{k=1}^{2^n} e^{2\pi ijk/2^n} \,|k\rangle$$

Equivalently, on the state $|\psi\rangle = \sum_j \psi_j \,|j\rangle$ reads

$$\mathrm{QFT}\,|\psi\rangle = |\phi\rangle = \frac{1}{2^{n/2}} \sum_{j=1}^{2^n} \psi_j \left(\sum_{k=1}^{2^n} e^{2\pi ijk/2^n} \,|k\rangle \right)$$

which turns out to be a unitary transformation

Product representation of the QFT

Computational basis ket $|j\rangle = |j_1 j_2 ... j_n\rangle$

Fourier basis ket $|k\rangle = |k_1 k_2 ... k_n\rangle$

Converting k to binary: $k = \sum_l k_l 2^l$

Also, note that $|k\rangle = |k_1 k_2 ... k_n\rangle = \bigotimes_{l=1}^{n} |k_l\rangle$

Product representation of the QFT

$$
\begin{aligned}
\mathrm{QFT}\,|j\rangle &= \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} e^{2\pi i j k/2^n} |k\rangle \\
&= \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} e^{2\pi i j \sum_l k_l 2^{-l}} \bigotimes_{l=1}^{n} |k_l\rangle \\
&= \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} \bigotimes_{l=1}^{n} e^{2\pi i j k_l 2^{-l}} |k_l\rangle \\
&= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \sum_{k_l=0}^{1} e^{2\pi i j k_l 2^{-l}} |k_l\rangle \\
&= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left(|0\rangle + e^{2\pi i j 2^{-l}} |1\rangle \right)
\end{aligned}
$$

Phase estimation of a unitary operator

An important module in many quantum algorithms that uses QFT

Consider an eigenvector $|u\rangle$ of a Unitary operator U. Its eigenvalue can be written as $u = e^{2\pi i\theta}$

$$U|u\rangle = u|u\rangle = e^{2\pi i\theta}|u\rangle$$

The Hidden Subgroup Problem

Let G be a group and X a finite set and $f : G \to X$ a function that *hides* a subgroup $H \leq G$. The problem is to determine a generating set for H

Simon's problem. Given a 2-1 function $f : \{0,1\}^n \to \{0,1\}^n$ such that there is a secret string $s \in \{0,1\}^n$ where $f(x) = f(y)$ if and only if $x \oplus y = s$. Equivalently $f(x) = f(y) = f(x \oplus y)$ which gives the periodicity of f

The function f is a black box. Clasically you would solve the problem by drawing pairs x, y and checking if $f(x) = f(y)$. If they match, you can obviously retrieve $s = x \oplus y$

Clasically the problem scales as $\mathcal{O}(2^{n/2})$ but we Simon designed a quantum algorithm that scales as $\mathcal{O}(n)$.

The standard solution to the HSP

The first register in Simon's algorithm is a uniform superposition over all possible input strings x. The second register are ancillary bits that will store $f(x)$. We assume we have some oracle function U_f which will compute and store $f(x)$ in the ancillary bits

$$|\psi\rangle = H^{\otimes n}|0^n\rangle = \frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} |x\rangle$$

As in the standard solution, the oracle function then does

$$O_f(|\psi\rangle |0^m\rangle) = \frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} |x\rangle |f(x)\rangle$$

Then we measure the second register which collapses the system to a superposition of the two inputs that map to our measured output $|f(a)\rangle$

$$\frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} |x\rangle |f(x)\rangle \to (|a\rangle + |a \oplus s\rangle) \otimes |f(a)\rangle$$

The standard solution to the HSP

Essentially when we measure the second register we end up with an equal superposition of x and $x \oplus s$. But how do we use that superposition to actually find s?