

# Homework 4

Quantum Mechanics

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CLAYTON SEITZ

**Problem 1.** *Problem 2.14 from Sakurai*

**Solution.**

We are given that the state vector is

$$|\alpha; t_0\rangle = \exp\left(\frac{-ipa}{\hbar}\right) |0\rangle$$

The Heisenberg equation of motion for  $x(t)$  reads

$$x(t) = x(0) \cos(\omega t) + \frac{p(0)}{m} \sin(\omega t)$$

Therefore

$$\begin{aligned} \langle x \rangle &= \langle \alpha; t_0 | x(t) | \alpha; t_0 \rangle \\ &= \langle \alpha; t_0 | \left( x(0) \cos(\omega t) + \frac{p(0)}{m} \sin(\omega t) \right) | \alpha; t_0 \rangle \\ &= \langle 0 | \exp\left(\frac{ipa}{\hbar}\right) \left( x(0) \cos(\omega t) + \frac{p(0)}{m} \sin(\omega t) \right) \exp\left(\frac{-ipa}{\hbar}\right) | 0 \rangle \\ &= \langle 0 | \exp\left(\frac{ipa}{\hbar}\right) x(0) \exp\left(-\frac{ipa}{\hbar}\right) | 0 \rangle \cos(\omega t) \\ &\quad + \frac{1}{m} \langle 0 | \exp\left(\frac{ipa}{\hbar}\right) p(0) \exp\left(-\frac{ipa}{\hbar}\right) | 0 \rangle \sin(\omega t) \end{aligned}$$

We can simplify this last expression by using the Baker-Hausdorff lemma for arbitrary operators  $G$  and  $A$

$$\exp(iG\lambda)A\exp(-iG\lambda) = A + i\lambda[G, A] + \dots$$

$$\begin{aligned}
\langle x \rangle &= \langle 0 | \exp \left( \frac{ipa}{\hbar} \right) x(0) \exp \left( -\frac{ipa}{\hbar} \right) | 0 \rangle \cos(\omega t) \\
&\quad + \frac{1}{m} \langle 0 | \exp \left( \frac{ipa}{\hbar} \right) p(0) \exp \left( -\frac{ipa}{\hbar} \right) | 0 \rangle \sin(\omega t) \\
&= \langle 0 | \left( x + \frac{ia}{\hbar} [p, x] \right) | 0 \rangle \cos(\omega t) \\
&\quad + \frac{1}{m} \langle 0 | \left( p + \frac{ia}{\hbar} [p, p] \right) | 0 \rangle \sin(\omega t) \\
&= (\langle 0 | x | 0 \rangle + a) \cos(\omega t) = a \cos(\omega t)
\end{aligned}$$

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**Problem 2.** *Problem 2.15 from Sakurai*

**Solution.**

First note that the state is being translated  $x \rightarrow x + a$

$$|\alpha; t_0\rangle = \exp \left( \frac{-ipa}{\hbar} \right) |0\rangle = \mathcal{J}(a) |0\rangle$$

where  $\mathcal{J}$  is the translation operator. Using that

$$\langle x | 0 \rangle = \pi^{-1/4} x_0^{1/2} \exp \left( -\frac{1}{2} \left( \frac{x}{x_0} \right)^2 \right)$$

we expect to be able to show

$$\begin{aligned}
\langle x | \exp \left( \frac{-ipa}{\hbar} \right) | 0 \rangle &= \langle x + a | 0 \rangle \\
&= \pi^{-1/4} x_0^{1/2} \exp \left( -\frac{1}{2} \left( \frac{x + a}{x_0} \right)^2 \right)
\end{aligned}$$

where  $x_0 = \sqrt{\frac{\hbar}{m\omega}}$ . The probability that  $|\alpha\rangle$  is measured to be in the state  $|0\rangle$  is

$$\langle \alpha | 0 \rangle = \pi^{-1/2} x_0 \int \exp \left( -\frac{1}{2} \left( \frac{x + a}{x_0} \right)^2 \right) \exp \left( -\frac{1}{2} \left( \frac{x}{x_0} \right)^2 \right) dx$$

which is an integral of a product of Gaussians. ■

**Problem 3.** *Problem 2.16 from Sakurai*

**Solution.**

We will assume the form of the annihilation and creation operators

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left( x + \frac{ip}{m\omega} \right)$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left( x - \frac{ip}{m\omega} \right)$$

Adding these equations gives and rearranging we can express  $x$  as

$$x = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger)$$

$$\begin{aligned} \langle m | x | n \rangle &= \langle m | \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger) | n \rangle \\ &= \sqrt{\frac{\hbar}{2m\omega}} (\langle m | a | n \rangle + \langle m | a^\dagger | n \rangle) \\ &= \sqrt{\frac{\hbar}{2m\omega}} (\sqrt{n}\delta_{m,n-1} + \sqrt{n+1}\delta_{m,n+1}) \end{aligned}$$

Subtracting the creation operator from the annihilation operator allows us to write the momentum operator as

$$p = i\sqrt{\frac{m\hbar\omega}{2}} (a^\dagger - a)$$

$$\begin{aligned} \langle m | p | n \rangle &= \langle m | \left( i\sqrt{\frac{m\hbar\omega}{2}} (a^\dagger - a) \right) | n \rangle \\ &= \left( i\sqrt{\frac{m\hbar\omega}{2}} (\langle m | a^\dagger | n \rangle - \langle m | a | n \rangle) \right) \\ &= i\sqrt{\frac{m\hbar\omega}{2}} (\sqrt{n+1}\delta_{m,n+1} - \sqrt{n}\delta_{m,n-1}) \end{aligned}$$

$$\begin{aligned}
\langle m | \{x, p\} | n \rangle &= \langle m | xp | n \rangle + \langle m | px | n \rangle \\
&= \frac{i\hbar}{2} \langle m | ((a^\dagger)^2 - a^2) | n \rangle + \frac{i\hbar}{2} \langle m | ((a^\dagger)^2 + a^\dagger a - aa^\dagger - a^2) | n \rangle \\
&= \frac{i\hbar}{2} (\sqrt{n+1}\sqrt{n+2}\delta_{m,n+2} + \sqrt{n}\sqrt{n-1}\delta_{m,n-2})
\end{aligned}$$

since only the cross terms will survive.

$$\begin{aligned}
\langle m | x^2 | n \rangle &= \frac{\hbar}{2m\omega} \langle m | (a^2 + aa^\dagger + a^\dagger a + (a^\dagger)^2) | n \rangle \\
&= \frac{\hbar}{2m\omega} \left( (2n+1)\delta_{mn} + \sqrt{m}\sqrt{n+1}\delta_{m-1,n+1} + \sqrt{m+1}\sqrt{n}\delta_{m+1,n-1} \right) \\
\langle m | p^2 | n \rangle &= -\frac{m\hbar\omega}{2} \langle m | (a^2 - aa^\dagger - a^\dagger a + (a^\dagger)^2) | n \rangle \\
&= -\frac{m\hbar\omega}{2} \left( (2n+1)\delta_{mn} - \sqrt{m}\sqrt{n+1}\delta_{m-1,n+1} - \sqrt{m+1}\sqrt{n}\delta_{m+1,n-1} \right)
\end{aligned}$$

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To validate the virial theorem we write,

$$\frac{1}{m} \langle p^2 \rangle = m\omega^2 \langle x^2 \rangle$$

From the above, we can see that this is satisfied for an energy eigenstate

$$\begin{aligned}
\frac{1}{m} \langle p^2 \rangle &= \frac{\hbar\omega}{2} (2n+1) \\
m\omega^2 \langle x^2 \rangle &= m\omega^2 \frac{\hbar}{2m\omega} (2n+1) = \frac{\hbar\omega}{2} (2n+1)
\end{aligned}$$

**Problem 4.** *Problem 2.28 from Sakurai*

**Solution.**

First of all, the solution is not trivial since  $x$  does not commute with the Hamiltonian since  $[x, p^2] \neq 0$ . At  $t = t_0$  we are in the position eigenstate

$$\langle x | \alpha; t_0 \rangle = \delta \left( x - \frac{L}{2} \right)$$

Since this is the infinite square well, we have the following energy eigenstates, in the position representation

$$\langle x|\alpha\rangle = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

Of course  $|\alpha; t_0\rangle$  is not an eigenstate of  $H$ , so this state will measurably evolve in time. The state  $|\alpha; t_0\rangle$  in the energy basis is

$$\begin{aligned} |\beta\rangle &= \sum_n |\epsilon_n\rangle \langle \epsilon_n|\alpha; t_0\rangle \\ &= \sqrt{\frac{2}{L}} \sum_n \sin\left(\frac{n\pi}{2}\right) |\epsilon_n\rangle \end{aligned}$$

From this, we can show the probability of measuring the particle in energy eigenstate  $|\epsilon_n\rangle$

$$\begin{aligned} \langle \epsilon_m|\beta\rangle &= \sqrt{\frac{2}{L}} \sum_n \sin\left(\frac{n\pi}{2}\right) \langle \epsilon_m|\epsilon_n\rangle \\ &= \sqrt{\frac{2}{L}} \sum_n \sin\left(\frac{n\pi}{2}\right) \delta_{mn} \\ &= \sqrt{\frac{2}{L}} \sin\left(\frac{m\pi}{2}\right) \end{aligned}$$

and therefore  $|\langle \epsilon_m|\beta\rangle|^2 = \frac{2}{L} \sin^2\left(\frac{m\pi}{2}\right)$ . The relative probabilities with respect to the ground state are then given by

$$r_{m+1} = \sin^2\left(\frac{(m+1)\pi}{2}\right) \csc^2\left(\frac{m\pi}{2}\right)$$

Since we know a representation for  $|\alpha; t_0\rangle$  in the energy basis, we can determine the time evolution of the wavefunction  $\langle x|\alpha\rangle$

$$\begin{aligned} |\alpha; t\rangle &= \mathcal{U}(t) |\beta\rangle \\ &= \sqrt{\frac{2}{L}} \sum_n \exp\left(\frac{-i\epsilon_n t}{\hbar}\right) \sin\left(\frac{n\pi}{2}\right) |\epsilon_n\rangle \end{aligned}$$

which has the position representation (wave function)

$$\begin{aligned}
\langle x|\alpha; t\rangle &= \psi(x, t) \\
&= \sqrt{\frac{2}{L}} \sum_n \exp\left(\frac{-i\epsilon_n t}{\hbar}\right) \sin\left(\frac{n\pi}{2}\right) \langle x|\epsilon_n\rangle \\
&= \sqrt{\frac{2}{L}} \sum_n \exp\left(\frac{-i\epsilon_n t}{\hbar}\right) \sin\left(\frac{n\pi}{2}\right) \psi_n(x)
\end{aligned}$$

where  $\psi_n(x)$  are the energy eigenstates given above. ■

**Problem 5.** *Problem 2.29 from Sakurai*

**Solution.**

For a delta-potential, Schrodinger's equation reads

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} - \nu_0 \delta(x) \psi(x) = E \psi(x)$$

We then solve Schrodinger's equation in two regions. Let the first region be  $x < 0$

$$\frac{d^2\psi_I}{dx^2} = -\frac{2mE}{\hbar^2} \psi_I(x) = \kappa_0^2 \psi_I(x)$$

for  $\kappa_0 = \sqrt{-2mE}/\hbar$ . If  $E < 0$ , then the general solution is

$$\psi_I(x) = A \exp(\kappa_0 x) + B \exp(-\kappa_0 x)$$

We choose  $B = 0$  on physical grounds. Now for the second region  $x > 0$ , we have a similar situation, but this time we choose an exponential decay

$$\psi_{II}(x) = A \exp(-\kappa_0 x)$$

We have chosen the constant to be the same as the first region, to preserve continuity of  $\psi(x)$ .

$$\psi(x) = \begin{cases} A \exp(\kappa_0 x) & x < 0 \\ A \exp(-\kappa_0 x) & x \geq 0 \end{cases}$$

The constant  $A$  is found by enforcing the normalization condition:

$$2A^2 \int_{-\infty}^0 \exp(2\kappa_0 x) dx = 1$$

and it is straightforward to show that  $A = \sqrt{\kappa_0}$

The energy is  $E = -\hbar^2 \kappa_0^2 / 2m$ . The usual trick for relating this to the strength of the potential  $\nu_0$  is to integrate Schrodingers equation

$$- \int_{-\epsilon}^{+\epsilon} \frac{\hbar^2}{2m} \frac{d\psi^2}{dx^2} dx - \int_{-\epsilon}^{+\epsilon} \nu_0 \delta(x) \psi(x) dx = 0$$

as  $\epsilon \rightarrow 0$ . Ignoring the normalization, because it will cancel

$$\left| \frac{d\psi}{dx} \right|_{-\epsilon}^{+\epsilon} = -2\kappa_0 = -\frac{2m\nu_0}{\hbar^2} \psi(0)$$

So we find that  $\kappa_0 = \nu_0 m / \hbar^2$  so  $E = -m\nu_0^2 / 2\hbar^2$ . This is unique so we just have one bound state. There are of course unbound states when  $E > 0$ , for which the solutions would be complex exponentials. ■

**Problem 6.** *Problem 2.32 from Sakurai*

**Solution.** Let us define

$$\begin{aligned} \psi_I &= A \exp(\alpha x) \\ \psi_{II} &= B \exp(ikx) + C \exp(-ikx) \\ \psi_{III} &= D \exp(-\alpha x) \end{aligned}$$

Here  $\alpha, k$  are constants. We can enforce continuity in the wavefunction itself at  $x = -a$  and  $x = +a$

$$\begin{aligned} A \exp(-\alpha a) &= B \exp(-ika) + C \exp(ika) \\ D \exp(-\alpha a) &= B \exp(ika) + C \exp(-ika) \end{aligned}$$

And we can also enforce continuity in the first-order derivative at these points

$$\begin{aligned} \alpha A \exp(-\alpha a) &= ikB \exp(-ika) - ikC \exp(ika) \\ -\alpha D \exp(-\alpha a) &= ikB \exp(ika) - ikC \exp(-ika) \end{aligned}$$

This system of four equations can be written in matrix form

$$\begin{pmatrix} e^{-\alpha a} & e^{-ika} & e^{ika} & 0 \\ 0 & e^{ika} & e^{-ika} & e^{-\alpha a} \\ \alpha e^{-\alpha a} & ike^{-ika} & ike^{ika} & 0 \\ 0 & ike^{ika} & -ike^{-ika} & -\alpha e^{-\alpha a} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = 0$$

According to Mathematica, the determinant is

$$\mathcal{D} = \exp(-2a(ik + \alpha)) \left( -\exp(4iak)(k - i\alpha)^2 + (k + i\alpha)^2 \right)$$

If the determinant is zero, then a solution exists. The determinant  $\mathcal{D}$  is zero when

$$\exp(-2iak)(k + i\alpha)^2 = \exp(2iak)(k - i\alpha)^2$$

Notice that we have just distributed the  $\exp(-2ika)$  from the prefactor. If we let  $z = \exp(-iak)(k + i\alpha)$  then the above equation just reads  $z^2 = (z^*)^2$  or  $z = \pm z^*$ .

Considering the purely real solution first, we make the substitutions

$$\begin{aligned} \exp(-iak) &\rightarrow \frac{\sqrt{k^2 + \alpha^2}}{k + i\alpha} \\ \exp(iak) &\rightarrow \frac{\sqrt{k^2 + \alpha^2}}{k - i\alpha} \end{aligned}$$



$$\begin{pmatrix} e^{-\alpha a} & \frac{-i\sqrt{k^2+\alpha^2}}{k+i\alpha} & \frac{\sqrt{k^2+\alpha^2}}{k-i\alpha} & 0 \\ 0 & \frac{\sqrt{k^2+\alpha^2}}{k-i\alpha} & \frac{-i\sqrt{k^2+\alpha^2}}{k+i\alpha} & e^{-\alpha a} \\ \alpha e^{-\alpha a} & ik\frac{-i\sqrt{k^2+\alpha^2}}{k+i\alpha} & ik\frac{\sqrt{k^2+\alpha^2}}{k-i\alpha} & 0 \\ 0 & ik\frac{\sqrt{k^2+\alpha^2}}{k-i\alpha} & -ik\frac{-i\sqrt{k^2+\alpha^2}}{k+i\alpha} & -\alpha e^{-\alpha a} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = 0$$

Now considering the purely imaginary solution, we make the substitutions

$$\begin{aligned} \exp(-iak) &\rightarrow \frac{i\sqrt{k^2+\alpha^2}}{k+i\alpha} \\ \exp(iak) &\rightarrow \frac{-i\sqrt{k^2+\alpha^2}}{k-i\alpha} \end{aligned}$$

$$\begin{pmatrix} e^{-\alpha a} & \frac{i\sqrt{k^2+\alpha^2}}{k+i\alpha} & \frac{-i\sqrt{k^2+\alpha^2}}{k-i\alpha} & 0 \\ 0 & \frac{-i\sqrt{k^2+\alpha^2}}{k-i\alpha} & \frac{i\sqrt{k^2+\alpha^2}}{k+i\alpha} & e^{-\alpha a} \\ \alpha e^{-\alpha a} & ik\frac{i\sqrt{k^2+\alpha^2}}{k+i\alpha} & ik\frac{-i\sqrt{k^2+\alpha^2}}{k-i\alpha} & 0 \\ 0 & ik\frac{-i\sqrt{k^2+\alpha^2}}{k-i\alpha} & -ik\frac{i\sqrt{k^2+\alpha^2}}{k+i\alpha} & -\alpha e^{-\alpha a} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = 0$$

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