Training a dynamical system by using multivariate information

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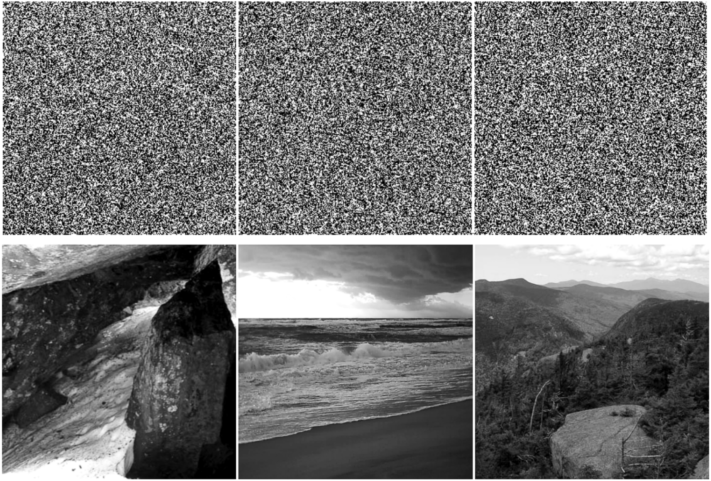
Introduction

Neuroethology argues that neural networks evolve according to the stimuli to which they are exposed



Channel coding for neural networks

Networks of neurons can be viewed as a communication channel
Except this communication channel *learns* the transformation F based on the statistical structure of its input X. Visual cortex has learned an encoding for visual scenes (that perhaps maximizes information)

Leaky integrate and fire neurons

A realistic LIF model might look like

$$\tau_m \frac{d\mathrm{V}[I]}{dt} = (\mathrm{V}[I] - E) \sum_j \mathrm{W}^0[I, j] + (\mathrm{V}[I] - E_{in}) \sum_k \mathrm{W}^1[I, k])$$

Instead, we ignore changes in the voltage of the postsynaptic neuron due to subthreshold voltages of the presynaptic neuron and let matrices W learn the input-output voltage relationship

$$V[j, t{+}1] = \alpha V[j, t{+}1] + \sum_{i \neq j} W^0_{ij} z[i, t] + \sum_i W^1_{ij} x[i, t{+}1] - z[j, t] v_{th}$$

where $z = H(v - v_{th})$

RNN Gradients

Say we have a model $\Phi = (W^0, W^1)$ and want to use gradient descent to train a network to have a target rate or a target branching parameter. The rate and its associated loss for a single unit is

$$r(t) = \frac{1}{\Delta t}\int_t^{t+\Delta t} d\tau \langle \rho(\tau) \rangle \qquad \mathcal{L} = \alpha(r - r_0)^2$$

We would like the standard update

$$\Delta W_{ij} = -\eta \frac{\partial \mathcal{L}}{\partial W_{ij}}$$

But it is intractable to compute $\frac{\partial \mathcal{L}}{\partial W_{ij}}$ since $\rho(t)$ depends on other neurons through space and time.

Factorizing loss gradients for BPTT

BPTT involves unrolling an RNN into a large feedforward network where each layer is a time step.

$$\frac{\partial \mathcal{L}}{\partial W_{ij}^t} = \frac{\partial \mathcal{L}}{\partial h_j^t} \frac{\partial h_j^t}{\partial W_{ij}^t}$$

and the total gradient is a sum over the layers (time)

$$\frac{\partial \mathcal{L}}{\partial W_{ij}^t} = \sum_t \frac{\partial \mathcal{L}}{\partial h_j^t} \frac{\partial h_j^t}{\partial W_{ij}^t}$$

Deriving e-prop from BPTT

Consider the first term above. The hidden state is computed by some function $h_j^t = F(z_j^t, h_j^{t-1}, W)$. Backpropagating through time is then

$$\frac{\partial \mathcal{L}}{\partial h_j^t} = \frac{\partial \mathcal{L}}{\partial z_j^t}\frac{\partial z_j^t}{\partial h_j^t} + \frac{\partial \mathcal{L}}{\partial h_j^{t+1}}\frac{\partial h_j^{t+1}}{\partial h_j^t}$$

which must be expressed recursively

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial h_j^t} &= \frac{\partial \mathcal{L}}{\partial z_j^t}\frac{\partial z_j^t}{\partial h_j^t} + \left(\frac{\partial \mathcal{L}}{\partial z_j^{t+1}}\frac{\partial z_j^{t+1}}{\partial h_j^{t+1}} + (\ldots)\frac{\partial h_j^{t+2}}{\partial h_j^{t+1}}\right)\frac{\partial h_j^{t+1}}{\partial h_j^t} \\
&= L_j^t \frac{\partial z_j^t}{\partial h_j^t} + \left(L_j^{t+1}\frac{\partial z_j^{t+1}}{\partial h_j^{t+1}} + (\ldots)\frac{\partial h_j^{t+2}}{\partial h_j^{t+1}}\right)\frac{\partial h_j^{t+1}}{\partial h_j^t} \\
&= L_j^t \frac{\partial z_j^t}{\partial h_j^t} + \left(L_j^{t+1}\frac{\partial z_j^{t+1}}{\partial h_j^{t+1}} + (\ldots)\frac{\partial h_j^{t+2}}{\partial h_j^{t+1}}\right)\frac{\partial h_j^{t+1}}{\partial h_j^t}
\end{aligned}$$

Deriving e-prop from BPTT

Plugging into the original factorization gives

$$\frac{\partial \mathcal{L}}{\partial W_{ij}} = \left(\sum_t L_j^t \frac{\partial z_j^t}{\partial h_j^t} + \left(L_j^{t+1} \frac{\partial z_j^{t+1}}{\partial h_j^{t+1}} + (\ldots) \frac{\partial h_j^{t+2}}{\partial h_j^{t+1}} \right) \frac{\partial h_j^{t+1}}{\partial h_j^t} \right) \frac{\partial h_j^{t'}}{\partial W_{ij}}$$

You can then collect terms that are multiplied L_j^t

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial W_{ij}} &= \sum_t L_j^t \frac{\partial z_j^t}{\partial h_j^t} \left(\sum_{t' \le t} \left(\prod_{t'} \frac{\partial h_j^{t'+1}}{\partial h_j^{t'}} \right) \frac{\partial h_j^{t'}}{\partial W_{ij}} \right) \\ &= \sum_t L_j^t \frac{\partial z_j^t}{\partial h_j^t} \epsilon_{\mathrm{ij}}^{\mathrm{t}} = \sum_t L_j^t e_{ij}^t \end{aligned}$$

The eligibility vector for LIF neurons

We would like to know the eligibility is per synapse ϵ_{ij}^t for a network of RNN neurons.
Gradients can be computed using e-prop if we use a pseudo-derivative $\psi_j^t = \frac{\partial z_j^t}{\partial v_j^t}$ and use the fact that the eligibility vector ϵ is just a low passed filter of z.

$$\Delta W_{ij} = -\eta \sum_t \frac{\partial \mathcal{L}}{\partial z_j^t} \psi_j^t \mathcal{F}_\alpha(z_i^t)$$

Constraining the global firing rate distribution

We can define a constraint on the variance of the global firing rate (which simultaneously constrains the mean)

$$\mathcal{L} = \beta(\sigma - \sigma_r)^2 \qquad \sigma = \frac{1}{T}\sum_t (r - \mu_r)^2$$

where we constrain branching by constraining the variance s of the global firing rate where branching $\to 1$ as $s \to 0$.

$$L_j^t = \frac{\partial \mathcal{L}}{\partial z_j^t} = \frac{\partial \mathcal{L}}{\partial \sigma}\frac{\partial \sigma}{\partial n}\frac{\partial n}{\partial z_j^t} = \pm\beta(\sigma - \sigma_r)\cdot(r - \mu_r)$$

Think push-pull. Some variation is necessary for refractoriness.

Receptive fields of neurons in a low-rate network

Multivariate information theory

Adaptation of the transfer function

How do neuron transfer functions adapt to stimuli in an unsupervised manner?

Adaptation defines an energy function over phase space

Generalization bounds

What is the distance of a code defined by a particular energy function E

The energy function defines a dynamical system

The energy function is a generative model

Application to natural image statistics