Applied and Numerical Harmonic Analysis

Gabor Analysis and Algorithms

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Theory and Applications

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Foreword

In his paper *Theory of Communication* [Gab46], D. Gabor proposed the use of a family of functions obtained from one Gaussian by time- and frequency-shifts. Each of these is well concentrated in time and frequency; together they are meant to constitute a complete collection of building blocks into which more complicated time-depending functions can be decomposed. The application to communication proposed by Gabor was to send the coefficients of the decomposition into this family of a signal, rather than the signal itself. This remained a proposal—as far as I know there were no serious attempts to implement it for communication purposes in practice, and in fact, at the critical time-frequency density proposed originally, there is a mathematical obstruction; as was understood later, the family of shifted and modulated Gaussians spans the space of square integrable functions [BBGK71, Per71] (it even has one function to spare [BGZ75] ...) but it does not constitute what we now call a *frame*, leading to numerical instabilities. The Balian-Low theorem (about which the reader can find more in some of the contributions in this book) and its extensions showed that a similar mishap occurs if the Gaussian is replaced by any other function that is "reasonably" smooth and localized. One is thus led naturally to considering a higher time-frequency density.

Interestingly, the same time-frequency lattice of functions was also proposed in an entirely different context by von Neumann [vN55], and became subsequently known as the von Neumann lattice, and lived an essential parallel life among quantum physicists (witness [BBGK71, Per71, BGZ75]). In addition, there is also a very clear connection to the short-time Fourier transform or windowed Fourier transform, used extensively in electrical engineering. Here too, the need to go to *overcritical sampling*, corresponding to the higher time-frequency density mentioned above, was discovered, independently.

Of course, in order to be useful practically, a transform must not only have good mathematical properties; it must also go hand-in-hand with efficient discrete algorithms, and for the Gabor transform these were developed extensively in the last decade.

Yet, despite this long history, and a lot of work by mathematicians, physicists and engineers alike, there are still many interesting and useful aspects of the Gabor transform to be explored and exploited. This book is an illustration of the continuing vigor of research on the Gabor transform, with mathematical developments, as well as approaches to numerical algorithms and a variety of applications.

Ingrid Daubechies
Princeton, New Jersey

Preface

This is the first book devoted to the subject of *Gabor analysis*. Since Dennis Gabor's fundamental paper of 1946, half a century has passed, but only in the last 10–15 years Gabor expansions have gained popularity in the signal processing community and under mathematicians. A number of basic questions has been put on firm mathematical grounds, and on the practical side efficient algorithms for numerical implementations have been developed, not to mention the variety of applications presented over the years.

The editors have asked a team of authors to cover the wide range of problems and methods coming together in Gabor analysis, and to give readers a survey of the present state of the field. We believe that the field has reached a first stage of maturity, which suggests summarizing existing results, trying to unify terminology and prepare ground for further investigations.

In this sense we also anticipate hope that this book will become a widely used general reference, and that it will motivate further research in the field and stimulate communication between mathematicians, engineers, and other scientists. We also hope to demonstrate through this book that Gabor analysis is not just "the unimportant, old-fashioned uncle" of the wavelet transform, but a fascinating field of mathematics and signal analysis, still offering high potential for further applications.

The book is addressing a broad audience, such as mathematicians looking for interesting problems with relevance to signal processing, as well as the engineering community or computer scientists, who care for efficient algorithms, and to applied scientists looking for powerful methods of signal analysis. The book is also supposed to provide rich material for graduate seminars and courses on Gabor analysis.

Due to the diversity of topics covered in the different chapters, the required background to fully appreciate their content varies. Because this book is written by scientists from different fields, most readers will find it appropriate to start with the topic of their main interest, and collect relevant (mostly mathematical) background by following the cross connections to other chapters. The book is sufficiently self-contained in order to allow a reader with a general background in signal analysis (or alternatively just the corresponding concepts of mathematical analysis) to profit from it and to use it as a guide toward a better understanding of Gabor analysis.

We encourage all readers, in particular those who would like to send some constructive criticism, to contact the editors by email or through the *Gabor Digest*, located at `http://tyche.mat.univie.ac.at`. This forum may be used by the readers of this book to find out about comments on the articles, updates, hints to recent publications, and further valuable information.

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We would like to thank the authors for their excellent contributions and their willingness to follow our editorial suggestions in order to give the book a more uniform appearance. We gratefully acknowledge stimulating discussions with Karlheinz Gröchenig and generous assistance of Johann "Niki" Lutz, Norbert Kaiblinger, Peter Prinz, Mitch Rauth, and Georg Zimmermann from the Numerical Harmonic Analysis Group (NUHAG) at our Department. In particular we are greatly indebted to Werner Kozek (also from the NUHAG team) for his constant support and valuable suggestions. We are grateful to Lauren Lavery and Wayne Yuhasz of Birkhäuser for their friendly help, and to John Benedetto, the editor of the Applied and Numerical Harmonic Analysis series, who encouraged us to edit this book. Finally we want to thank Katharina and Gabriela for their understanding and support during the period of preparation for this book.

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Introduction

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Thomas Strohmer

> *"It is probably true quite generally that in the history of human thinking the most fruitful developments frequently take place at those points where two different lines of thought meet. Hence, if they actually meet, that is, if they are at least so much related to each other that a real interaction can take place, then one may hope, that new and interesting developments may follow."*
>
> **Werner Heisenberg**

In order to analyze and describe complicated phenomena, mathematicians, engineers and physicists like to represent them as a superposition of simple, well-understood objects. A significant part of research has gone into the development of methods to find such representations. These methods have become important in many areas of our scientific and technological activity. They are used for instance in telecommunications, medical imaging, geophysics, and engineering. An important aspect of many of these representations is the chance to extract relevant information from a signal or the underlying process, which is actually present but hidden in its complex representation. For example, one may apply linear transformations with the aim that the information can be read off more easily from the new representation of the signal. Such transformations are used for many diverse tasks such as analysis and diagnostics, compression and coding, transmission and reconstruction.

Over many years the Fourier transform was the main tools in applied mathematics and signal processing for these purposes. But due to the large diversity of problems with which science is confronted on a regular basis, it is clear that there does not exist a single universal method which is well adapted to all those problem simultaneously. Nowadays there are many efficient analysis tools at our disposal. In this book we concentrate on methods which can be summarized under the name *Gabor analysis*, an area of research which is both theoretically appealing and successfully used in applications.

In the sequel we give a brief overview on several streams of development, leading up to what is nowadays called Gabor analysis. For the reader who

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is not familiar with some of the symbols and tools that are used throughout the book, we have included a section entitled *Mathematical notation and basic tools* after the *Reader's guide.*

From Fourier expansions to Gabor expansions

Motivated by the study of heat diffusion, Fourier asserted that an arbitrary function f in $[0,1)$ could be represented by a trigonometric series

$$f(t) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n t}$$

where

$$\hat{f}(n) = \int_0^1 f(t) e^{-2\pi i n t} dt \,.$$

A good part of mathematical analysis developed since then was devoted to the attempt to make Fourier's statement precise. Despite the delicate problems of convergence, Fourier series are a powerful and widely used tool in mathematics, engineering, physics, and other areas. The existence of the Fast Fourier Transform has extended this use enormously in the past thirty years. Fourier expansions are not only useful to study single functions or function spaces, they can also be applied to study operators between function spaces. It is a well-known fact that the trigonometric basis $\{e^{2\pi i n t}, n \in \mathbb{Z}\}$ diagonalizes translation invariant operators on the interval $[0,1)$, identified with the torus. However the Fourier system is not adapted to represent local information in time of a function or an operator, since the representation functions themselves are not at all localized in time, we have $|e^{2\pi i n t}| = 1$ for all n and t. A local perturbation of $f(t)$ may result in a perturbation of all expansion coefficients $\hat{f}(\omega)$. Roughly speaking the same remarks apply to the Fourier transform. The Fourier transform is an ideal tool to study stationary signals and processes (where the properties are statistically invariant over time). However many physical processes and signals are nonstationary, they evolve with time, such as speech or music.

Let us take for instance a short segment of Mozart's Magic Flute (say thirty seconds and the corresponding number of samples, as they are stored on a CD). If we represent this piece of music as a function of time, we may be able to perceive the transition from one note to the next, but we get little insight about which notes are in play. On the other hand the Fourier representation may give us a clear indication about the prevailing notes in terms of the corresponding frequencies, but information about the moment of emission and duration of the notes is masked in the phases. Although both representations are mathematically correct, but one does not have to be a member of the Vienna Philharmonic Orchestra to find neither of them very satisfying. According to our hearing sensations we would intuitively prefer a representation which is local both in time and frequency, like music notation, which tells the musician which note to play at a given moment. Additionally such a local time-frequency representation should be discrete, so that it is better adapted to applications.

Dennis Gabor had similar considerations in mind, when he introduced in 1946 in his "Theory of Communication" a method to represent a one-dimensional signal in two dimensions, with time and frequency as coordinates [Gab46]. Gabor's research in communication theory was driven by the question how to represent locally as good as possible by a finite number of data the information of a signal which is given a priori through uncountably many function values $f(t)$. He was strongly influenced by developments in quantum mechanics, in particular by Heisenberg's *uncertainty principle* and by the fundamental results of Nyquist [Nyq24] and Hartley [Har28] on the limits for the transmission of information over a channel.

Gabor proposed to expand a function f into a series of elementary functions, which are constructed from a single building block by translation and modulation (i.e. translation in the frequency domain). More precisely he suggested to represent f by the series

$$f(t) = \sum_{n,m\in\mathbb{Z}} c_{m,n} g_{m,n}(t)\,. \tag{1}$$

where the elementary functions $g_{m,n}$ are given by

$$g_{m,n}(t) = g(t-na)e^{2\pi imbt}, \quad m,n \in \mathbb{Z} \tag{2}$$

for a fixed function g and *time-frequency shift parameters* $a, b > 0$. A typical set of Gabor elementary functions is illustrated in Figure 1.

We could also say that the $g_{m,n}$ in (2) are obtained by shifting g along a *lattice* $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ in the *time-frequency plane*. If g and its Fourier transform $\hat{g}$ are essentially localized at the origin, then $g_{m,n}$ is essentially localized at (na, mb) in the time-frequency plane. Hence each such elementary function $g_{m,n}$ essentially occupies a certain area ("logon") in the time-frequency plane. Each of the expansion coefficients $c_{m,n}$, associated to a certain area of the time-frequency plane via $g_{m,n}$, represents one *quantum of information.* For properly chosen shift parameters a, b the $g_{m,n}$ cover the time-frequency plane, as demonstrated in Figure 2.

Gabor proposed to use the Gauss function and its translations and modulations with shift parameters $ab = 1$ as elementary signals, since they "assure the best utilization of the information area in the sense that they possess the smallest product of effective duration by effective width" [Gab46]. Recall that the *uncertainty principle inequality* [Ben94] states that for all

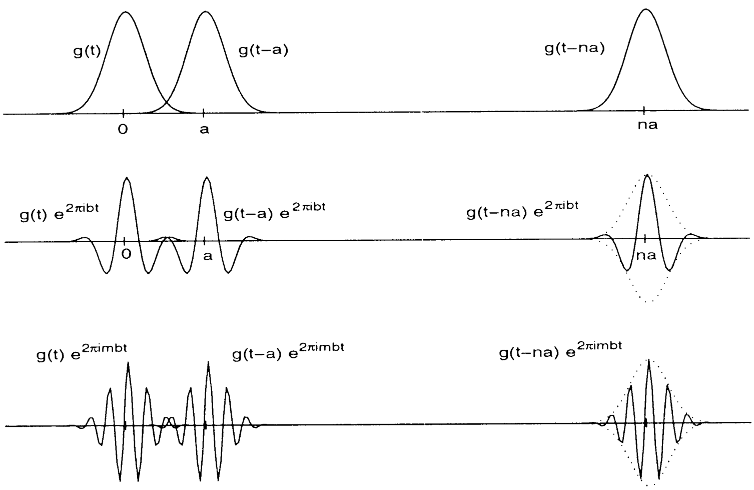


FIGURE 1. Gabor's elementary functions $g_{m,n}(t) = g(t-na)e^{2\pi imbt}$ are shifted and modulated copies of a single building block g (a and b denoting the time shift and the frequency shift parameter, respectively). Each $g_{m,n}$ has an envelope of the shape of g (only the real part of the functions $g_{m,n}$ is shown).

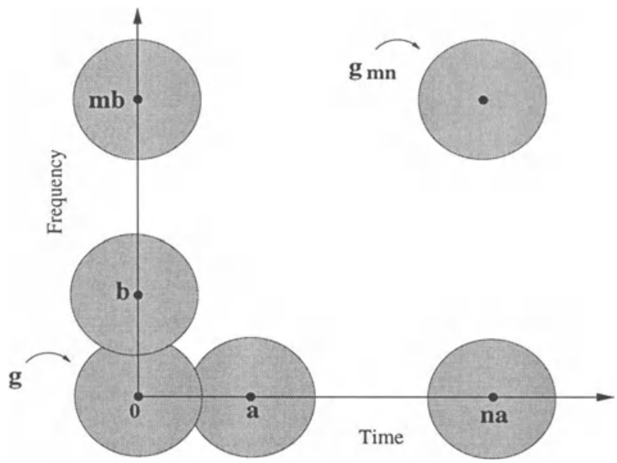


FIGURE 2. If g is localized at the origin in the time-frequency plane, then $g_{m,n}$ is localized at the point (na, mb). For appropriate lattice constants a, b the $g_{m,n}$ cover the whole time-frequency plane.

functions $f \in \boldsymbol{L}^2(\mathbb{R})$ and for all points (t_0, ω_0) in the time-frequency plane

$$\|f\|_2^2 \leq 4\pi \|(t - t_0)f(t)\|_2 \|(\omega - \omega_0)\hat{f}(\omega)\|_2$$

where equality is achieved only by functions of the form

$$g(t) = Ce^{2\pi i t \omega_0} e^{-s(t-t_0)^2}, \quad C \in \mathbb{C},\ s > 0, \tag{3}$$

i.e., by modulated and translated Gaussians. The Fourier transform of the Gauss function is of the same analytic form, its sharpness is reciprocal.

It is obvious that time series and Fourier series are limiting cases of Gabor's series expansion. The first one may be obtained by letting $s \to 0$ in (3), in which case the $g_{m,n}$ approximate the delta distribution δ, in the second case, the $g_{m,n}$ become ordinary sine and cosine waves for $s \to \infty$.

The idea to represent a function f in terms of the time-frequency shifts of a single atom g did not only originate in communication theory but somewhat 15 years earlier also in quantum mechanics. In an attempt to expand general functions (quantum mechanical states) with respect to states with minimal uncertainty, John von Neumann [vN55] introduced in 1932 a set of *coherent states* on a lattice with lattice constants $ab = h$ in the *phase space* with position and momentum as coordinates (here h is the *Plack constant*). These states, associated with the *Weil–Heisenberg group* are in principle the same used by Gabor. Therefore the system $\{g_{m,n}\}$ is also called *Weyl-Heisenberg system*, and the time-frequency lattice with lattice constants $ab = 1$ is also referred to as *von Neumann lattice.* We recommend the book of Klauder and Skagerstam for an excellent review on coherent states [KS85].

Only two years after Gabor's paper, Shannon published "A Mathematical Theory of Communication" [Sha48]. It should be emphasized that the temporal coincidence is not the only connection between Gabor theory and Shannon's principles of information theory. Both, Shannon and Gabor, tried to "cover" the time–frequency plane with a set of functions, transmission signals for digital communication in Shannon's case and building blocks for natural signals in Gabor's case. While Gabor explicitly suggested the Gaussian function and Weyl–Heisenberg structure, Shannon only emphasized the relevance of orthonormal bases without explicitly suggesting a signal set design. Yet, the determination of a *critical density* (referred to as degrees of freedom per time and bandwidth in Shannon's work) was one of the key mathematical prerequisites for Shannon's famous Capacity Theorem. In summary, both Gabor and Shannon worked about the same time on communication engineering problems related to Heisenberg uncertainty and phase space density, where at that time only very few mathematicians, most prominently von Neumann, had touched upon their basics. Note, however, that Shannon's work certainly had a greater impact on the engineering community than the work of Gabor.

Two questions arise immediately with an expansion of the form (6):

- Can any $f \in \boldsymbol{L}^2(\mathbb{R})$ be written as superposition of such $g_{m,n}$?
- How can the expansion coefficients $c_{m,n}$ in (6) be computed?

Gabor gave an iterative method to estimate the $c_{m,n}$, which was analyzed in [GP92]. However an analytic method to compute the expansion coefficients was not known until Bastiaans published a solution in 1980 [Bas80a].

Representations of the form (6) belong to the general framework of *atomic decompositions.* The goal is to find simple elements – the atoms – of a function space and the "assembly rule", that allows the reconstruction of all the elements of the function space using these atoms. Thus in our context the building block g is also called "Gabor atom". See [CR80] for atomic decompositions in entire function spaces and [FG89a, FG89b, FG92a] for atomic decompositions in conjunction with Gabor analysis.

While Gabor was awarded the Nobel Prize in Physics in 1971 for the conception of holography, his paper on "Theory of Communication" went almost unnoticed until the early 80's, when the work of Bastiaans and Janssen refreshed the interest of mathematicians and engineers in Gabor analysis. The connection to wavelet theory[1] and the increasing interest of scientists in signal analysis and frame theory was then very much influenced by the work of I. Daubechies [DGM86, Dau90, Dau92]. But before we proceed to the 80's let us go back to the 30's and 40's and follow the development of Gabor theory from the signal analysis point of view.

Local time-frequency analysis and short time Fourier transform

Time-frequency analysis plays a central role in signal analysis. Already long ago it has been recognized that a global Fourier transform of a long time signal is of little practical value to analyze the frequency spectrum of a signal. High frequency bursts for instance cannot be read off easily from $\hat{f}$. Transient signals, which are evolving in time in an unpredictable way (like a speech signal or an EEG signal) necessitate the notion of frequency analysis that is local in time.

In 1932, Wigner derived a distribution over the phase space in quantum mechanics [Wig32]. It is a well-known fact that the *Wigner distribution* of an $\boldsymbol{L}^2$-function f is the *Weyl symbol* of the orthogonal projection operator onto f [Fol89]. Some 15 years later, Ville, searching for an "instantaneous spectrum" – influenced by the work of Gabor – introduced the same

[1] In the wavelet literature the functions $g_{m,n}$ are often referred to as *Gabor wavelets.*

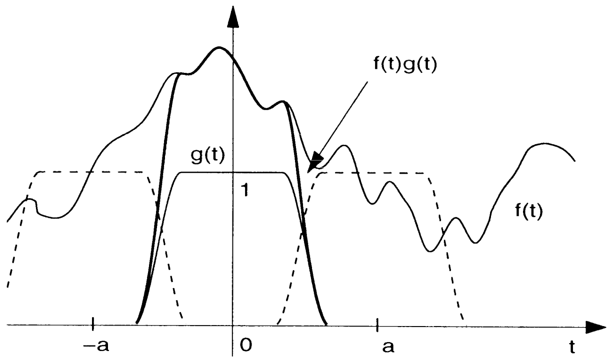


FIGURE 3. For fixed t_0 the short time Fourier transform of a function $f(t)$ describes the local spectral content of $f(t)$ near t_0, as a function of ω. It is defined as the Fourier transform of $f(t)g(t-t_0)$, where $g(t)$ is a (often compactly supported) window function, localized around the origin. Moving the center of the window g along the real line, allows to obtain "snapshots" of the time-frequency behavior of f. We depict a collection of such shifted windows, with $t_0 = -a, 0, a$.

transform in signal analysis [Vil48]. Unfortunately the non-linearity of the Wigner distribution causes many interference phenomena, which makes it less attractive for many practical purposes [Coh95].

A different approach to obtain a local time-frequency analysis (suggested by various scientists, among them Ville), is to cut the signal first into slices, followed by doing a Fourier analysis on these slices. But the functions obtained by this crude segmentation are not periodic, which will be reflected in large Fourier coefficients at high frequencies, since the Fourier transform will interpret this jump at the boundaries as a discontinuity or an abrupt variation of the signal. To avoid these artifacts, the concept of windowing has been introduced. Instead of localizing f by means of a rectangle function, one uses a smooth window-function for the segmentation, which is close to 1 near the origin and decays towards zero at the edges. Popular windows which have been proposed for this purpose are associated with the names Hamming, Hanning, Bartlett, or Kaiser. If the window is in $\boldsymbol{C}^{\infty}$ (i.e. infinitely differentiable) one finds that for any $\boldsymbol{C}^{\infty}$-function f the localized Fourier coefficients show at least polynomial decay in the frequency direction.

The resulting local time-frequency analysis procedure is referred to as (continuous) *short time Fourier transform* or *windowed Fourier transform* [AR77]. It is schematically represented in Figure 3. In mathematical notation, the short time Fourier transform (STFT) of an arbitrary function $f \in \boldsymbol{L}^2(\mathbb{R})$ with respect to a given (often compactly supported) window g

is defined as

$$\mathcal{V}_g f(t,\omega) = \int_{\mathbb{R}} f(s)\overline{g(s-t)}e^{-2\pi i\omega s}ds\,.$$

The function f can be recovered from its STFT via the inversion formula

$$f(t) = \frac{1}{\|g\|_{L^2}^2} \iint_{\mathbb{R}\times\mathbb{R}} \mathcal{V}_g f(s,\omega) g(t-s) e^{2\pi i\omega t} dt d\omega\,.$$

It is possible to derive the inversion formula (the integral is understood in the mean square sense) from the following formula, which itself can be seen as an immediate consequence of *Moyal's formula.* In particular it implies that for a normalized window g satisfying $\|g\|_2 = 1$ the mapping $f \mapsto \mathcal{V}_g f$ is an isometric embedding from $\boldsymbol{L}^2(\mathbb{R})$ into $\boldsymbol{L}^2(\mathbb{R}^{2d})$

$$\|\mathcal{V}_g f\|_{\boldsymbol{L}^2(\mathbb{R}\times\mathbb{R})} = \|g\|_{\boldsymbol{L}^2(\mathbb{R})}\|f\|_{\boldsymbol{L}^2(\mathbb{R})}\,.$$

The STFT and the *spectrogram* $|\mathcal{V}_g f(t,\omega)|^2$ have become standard tools in signal analysis. However the STFT has also its disadvantages, such as the limit in its time-frequency resolution capability, which is due to the uncertainty principle. Low frequencies can be hardly depicted with short windows, whereas short pulses can only poorly be localized in time with long windows, see also Figure 4 for an illustration of this fact. These limitations in the resolution were one of the reasons for the invention of wavelet theory. (A recent approach to overcome this drawback is to use multiple analysis windows (see e.g., [SS94, ZZ97a] and Chapter 12).

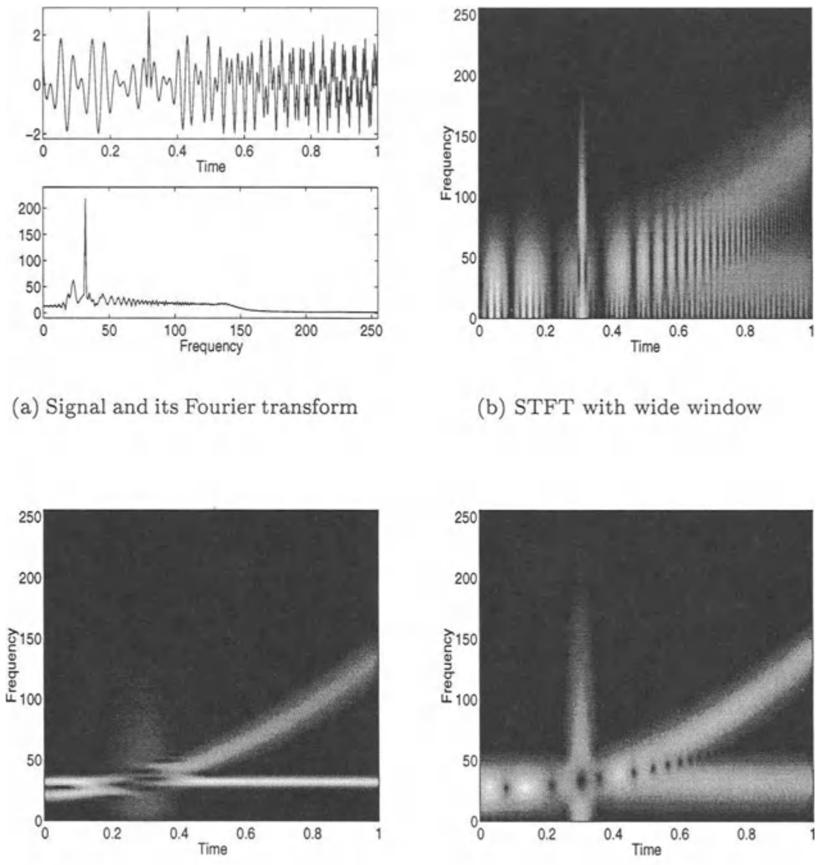
Another disadvantage for many practical purposes is the high redundancy of the STFT. This fact suggests to ask, if we can reduce this redundancy by sampling $\mathcal{V}_g f(t,\omega)$. The natural discretization for t,ω is $t = na, \omega = mb$ where $a,b > 0$ are fixed, and n,m range over $\mathbb{Z}$, i.e., to sample $\mathcal{V}_g f$ over a time-frequence lattice of the form $a\mathbb{Z} \times b\mathbb{Z}$.

Large values of a,b give a coarse discretization, whereas small values of a,b lead to a dense sampled STFT.

Using the operator notation T_t and M_ω for translation and modulation, respectively, we can express the STFT of f with respect to a given window g as

$$\mathcal{V}_g f(t,\omega) = \int_{\mathbb{R}} f(s)\overline{g(s-t)}e^{-2\pi i\omega s}ds = \langle f, T_t M_\omega g\rangle\,.$$

Hence the sampled STFT of a function f can also be interpreted as the set of inner products of f with the members of the family $\{g_{m,n}\} = \{T_{na}M_{mb}g\}$ with discrete labels in the lattice $a\mathbb{Z}\times b\mathbb{Z}$. It is obvious that the members of this family are constructed in the same way as the representation functions



(a) Signal and its Fourier transform

(b) STFT with wide window

(c) STFT with narrow window

(d) STFT with medium window

FIGURE 4. A signal, its Fourier transform and short time Fourier transforms with windows of different duration. (a) The signal itself consists of a constant sine wave (with 35 Hz), a quadratic chirp (starting at time 0 with 25 Hz and ending after one second at 140 Hz) and a short pulse (appearing after 0.3 sec.). (b) Using a wide window for the STFT leads to good frequency resolution. The constant frequency term can be clearly seen, also the quadratic chirp. However the short pulse is hardly visible. (c) Using a narrow window gives good time resolution, clearly localizing the short pulse at 0.3 sec., but the information about the constant harmonic gets very unsharp. (d) In this situation a window of medium width yields a satisfactory resolution both in time and frequency.

$g_{m,n}$ in Gabor's series expansion. Thus the sampled STFT is also referred to as *Gabor transform*.

It is intuitively appealing that the STFT can be seen as limit of the Gabor transform for $(a, b) \to (0, 0)$ (in a suitable mathematical sense, cf. Chapter 3).

Two questions arise immediately with the discretization of the STFT

- Do the discrete STFT coefficients $\langle f, g_{m,n} \rangle$ completely characterize f (i.e., does $\langle f_1, g_{m,n} \rangle = \langle f_2, g_{m,n} \rangle$ for all m, n imply that $f_1 = f_2$)?
- A stronger formulation is: Can we reconstruct f in a numerically stable way from the $\langle f, g_{m,n} \rangle$?

Recall, that in connection with the Gabor expansion of a function we have asked

- Can any function in $\boldsymbol{L}^2(\mathbb{R})$ be written as superposition of the elementary building blocks?
- How can we compute the expansion coefficients $c_{m,n}$ in the series $f = \sum c_{m,n} g_{m,n}$?

It turns out that the question of recovering f from the samples (at lattice points) of its STFT with respect to the window g is actually dual to the problem of finding coefficients for the Gabor expansion of f with atom g, using the same lattice to generate the time-frequency shifts of g. Both problems can be successfully and mathematically rigorously attacked using the concept of frames and surprisingly for both questions the same "dual" Gabor atom has to be used.

Fundamental properties of Gabor frames

The theory of frames is due to Duffin and Schaeffer [DS52], and was introduced in the context of nonharmonic Fourier series only six years after Gabor published his paper. Despite this fact one has to say that frame analysis has become popular much later in sampling theory, time-frequency analysis and wavelet theory, ignited by the papers [DGM86] and [Dau90]. Expository treatments of frames can be found in [You80], [Dau92], and [BW94]. Based on the results on coorbit spaces in [FG89a], Gröchenig [Grö91] suggested a very general concept of *Banach frames*, which applies to series expansions in Banach spaces which do not have to be Hilbert spaces in a natural way. See Chapters 3 and 5 for more details on Banach frames.

A system $\{g_{m,n}\} = \{T_{na} M_{mb} g\}$ is a *Gabor frame* or *Weyl–Heisenberg* *frame* for $\boldsymbol{L}^2(\mathbb{R})$, if there exist two constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{m,n\in\mathbb{Z}} |\langle f, g_{m,n}\rangle|^2 \leq B\|f\|^2 \tag{4}$$

holds for all $f \in \boldsymbol{L}^2(\mathbb{R})$. For a Gabor frame $\{g_{m,n}\}$ the *analysis mapping* (also called Gabor transform) T_g, given by

$$T_g : f \to \{\langle f, g_{m,n}\rangle\}_{m,n} \tag{5}$$

and its adjoint, the *synthesis mapping* (also called Gabor expansion) T_g^*, given by

$$\{c_{m,n}\} \to \sum_{m,n\in\mathbb{Z}} c_{m,n}\, g_{m,n} \qquad \{c_{m,n}\} \in \ell^2(\mathbb{Z}) \tag{6}$$

are bounded linear operators. The *Gabor frame operator* S_g is defined by $S_g = T_g^* T_g$. Explicitly,

$$S_g f = \sum_{m,n\in\mathbb{Z}} \langle f, g_{m,n}\rangle\, g_{m,n}\,. \tag{7}$$

(We will often drop the subscript and denote the Gabor frame operator simply by S).

If $\{g_{m,n}\}$ constitutes a Gabor frame for $\boldsymbol{L}^2(\mathbb{R})$, any function $f \in \boldsymbol{L}^2(\mathbb{R})$ can be written as

$$f = \sum_{m,n} \langle f, g_{m,n}\rangle\, \gamma_{m,n} = \sum_{m,n} \langle f, \gamma_{m,n}\rangle\, g_{m,n} \tag{8}$$

where $\gamma_{m,n}$ are the elements of the dual frame, given by $\gamma_{m,n} = S^{-1} g_{m,n}$. Equation (8) provides a constructive answer how to recover f from its Gabor transform $\{\langle f, g_{m,n}\rangle\}_{m,n\in\mathbb{Z}}$ for given analysis window g and how to compute the coefficients in the series expansion $f = \sum c_{m,n} g_{m,n}$ for given atom g. The key is the corresponding dual frame $\{S^{-1} g_{m,n}\}_{m,n\in\mathbb{Z}}$.

More results on Gabor frames can be found for example in [Dau92, Wal92, FG92a, BW94, Kai94b] and in the work of Janssen, and Ron and Shen. Gabor frames for other spaces than $\boldsymbol{L}^2$ have been studied in [Dau90, BW94, BHW95, FG92b, FG96]. An interesting representation of the Gabor frame operator has been derived by Walnut [Wal92].

A detailed analysis of Gabor frames brings forward some features that are basic for a further understanding of Gabor analysis. Most of these features are not shared by other frames such as wavelet frames.

Commutation relations of the Gabor frame operator

One can easily check (see also [Dau90]) that the Gabor frame operator commutes with translations by a and modulations by b, i.e.,

$$ST_a = T_a S\,, \quad SM_b = M_b S\,. \tag{9}$$

It follows that S^{-1} also commutes with T_a and M_b, so that

$$\gamma_{m,n} = S^{-1} g_{m,n} = S^{-1} T_{na} M_{mb} g = T_{na} M_{mb} S^{-1} g = T_{na} M_{mb} \gamma\,.$$

Therefore the elements of the dual Gabor frame $\{\gamma_{m,n}\}$ are generated by a single function γ, analogously to the $g_{m,n}$. This observation bears important computational advantages. To compute the dual system $\{\gamma_{m,n}\}$ one computes the (canonically) *dual atom* $\gamma = S^{-1}g$ and derives all other elements $\gamma_{m,n}$ of the dual frame by translations and modulations.

Clearly, since the elements of a frame are in general linear dependent, there are many choices for the coefficients $c_{m,n}$ and even different choices of γ are possible. However, speaking with the words of I. Daubechies, the coefficients determined by the dual frame, are the most economical ones, in the sense that they have minimal ℓ^2-norm among all possible sets of coefficients, and at the same time γ is the L^2-function with minimal norm for which (8) is valid.

Critical sampling, oversampling, and the Balian–Low Theorem

We have mentioned earlier that Gabor suggested to use the Gaussian function as atom g, since it minimizes the uncertainty principle inequality. Recalling that the $g_{m,n} = T_{na}M_{mb}g$ are the coherent states associated to the Weyl-Heisenberg group in quantum mechanics (see e.g. [KS85, Per86]), we remind that this choice corresponds to the *canonical coherent states.* Although coherent states play an important role in many branches of theoretical and mathematical physics and pure mathematics. there is no chapter in this book devoted to coherent states and related topics, such as von Neumann algebras and Lie groups. This "omission" is due to the limit of space, which makes it impossible to present these important topics in the particularity they would deserve, and due to the fact that they are covered in numerous books.

Exploiting the link between Gaussian coherent states and the Bargmann space of entire functions it was proved in 1971 by Peremolov [Per71] and independently by Bargmann et al. [BBGK71] that the canonical coherent states $g_{m,n}$ are complete in $L^2(\mathbb{R})$ if and only if $ab \le 1$. Bacry, Grossmann and Zak [BGZ75] showed in 1975 that if $ab = 1$, then

$$\inf_{f \in L^2(\mathbb{R})} \sum_{m,n} |\langle f, g_{m,n}\rangle|^2 = 0 \tag{10}$$

although the $g_{m,n}$ are complete in $\boldsymbol{L}^2(\mathbb{R})$. Formula (10) implies that for Gabor's original choice of the Gaussian and $ab = 1$, the set $\{g_{m,n}\}$ is not a frame for $\boldsymbol{L}^2(\mathbb{R})$. Thus there is no numerically stable algorithm to reconstruct f from the $\langle f, g_{m,n}\rangle$.

Bastiaans [Bas80b, Bas81] was the first who has published an analytic solution to compute the Gabor expansion coefficients for the case $a = b = 1$ and g equal the Gaussian. He constructed a function γ, such that

$$f = \sum_{m,n} \langle f, g_{m,n}\rangle \gamma_{m,n} \tag{11}$$

with $\gamma_{m,n} = T_{na}M_{mb}\gamma$. Note however that (11) does not even converge in a weak $\boldsymbol{L}^2$-sense, in fact γ is not in $\boldsymbol{L}^2$, as was pointed out by Janssen [Jan82], cf. also [DJ93]. Janssen [Jan81] showed that convergence holds only in the sense of distributions.

Using entire function methods, Lyubarskii [Lyu92] and independently Seip and Wallsten [SW92a] showed that for the Gaussian g the family $\{g_{m,n}\}$ is a frame whenever $ab < 1$. According to Janssen the dual function γ is then even a Schwartz function.

As a corollary of deep results on C^*-algebras by Rieffel [Rie81] it was proved that the set $\{g_{m,n}\}$ is incomplete in $\boldsymbol{L}^2(\mathbb{R})$ for *any* $g \in \boldsymbol{L}^2(\mathbb{R})$, if $ab > 1$. This fact can be seen as a Nyquist criterion for Gabor systems. The non-constructive proof makes use of the properties of the von Neumann algebras, generated by the operators $T_{na}M_{mb}$. Daubechies [Dau90] derived this result for the special case of rational ab. In [Jan94b] Janssen showed that the $g_{m,n}$ cannot establish a frame for any $g \in \boldsymbol{L}^2(\mathbb{R})$, if $ab > 1$ without any restriction on ab. One year earlier Landau proved the weaker result that $\{g_{m,n}\}$ cannot be a frame for $\boldsymbol{L}^2(\mathbb{R})$ if $ab > 1$ and both g and $\hat{g}$ satisfy certain decay conditions [Lan93]. On the other hand his result includes the case of irregular Gabor systems, where the sampling set is not necessarily a lattice in $\mathbb{R} \times \mathbb{R}$.

All these results remind on the role of the Nyquist density for sampling and reconstruction of bandlimited functions in Shannon's Sampling Theorem. Hence it is natural to classify Gabor systems according to the corresponding sampling density of the time-frequency lattice:

- *oversampling* – $ab < 1$: Frames with excellent time-frequency localization properties exist (a particular example are frames with Gaussian g and appropriate oversampling rate).
- *critical sampling* – $ab = 1$: Frames and orthonormal bases are possible, but – as we will see below – without good time-frequency localization;
- *undersampling* – $ab > 1$: In this case any Gabor family will be incomplete, in the sense that the closed linear span is a proper subspace of $\boldsymbol{L}^2(\mathbb{R})$, in particular one cannot have a frame for $\boldsymbol{L}^2(\mathbb{R})$.

The case $ab = 1$ is also distinguished among all others by the fact, that the time-frequency shift operators, which are used to build the coherent frame, commute with each other (without non-trivial factor).

Clearly there exist many choices for g, so that $\{g_{m,n}\}$ is a frame or even an orthonormal basis (ONB) for $\boldsymbol{L}^2(\mathbb{R})$. Two well-known examples of functions for which the family $\{T_{na}M_{mb}g\}$ constitutes an ONB are the rectangle function (which is 1 for $0 \leq t \leq 1$ and zero else), and the sinc-function $g(t) = \sin \pi t / \pi t$. However in the first case $\int \omega^2 |\hat{g}(\omega)|^2 = \infty$, in the second case $\int t^2 |g(t)|^2 = \infty$. Thus these choices lead to systems with bad localization properties in either time or frequency. Even if we drop the orthogonality requirement, we cannot construct Riesz bases with good time-frequency localization properties for the limit case $ab = 1$. This is the contents of the celebrated Balian-Low Theorem [Bal81, Low85], which describes one of the key facts in Gabor analysis:

Balian–Low Theorem: *If the $g_{m,n}$ constitute a Riesz basis for $\boldsymbol{L}^2(\mathbb{R})$, then*

$$\int_{-\infty}^{+\infty} |g(t)|^2 t^2 \int_{-\infty}^{+\infty} |g(\omega)|^2 \omega^2 = \infty$$

See Chapter 2 for an expository treatment of the Balian–Low Theorem and its consequences.

According to Gabor's heuristics the integer lattice in the time-frequency plane was chosen to make the choice of coefficients "as unique as possible" (unfortunately one cannot have strict uniqueness since there are bounded sequences which represent the zero-function in a non-trivial, but only distributional way). In focusing his attention on this uniqueness problem he apparently overlooked that the use of well-localized building blocks to obtain an expansion $f = \sum_{m,n} c_{m,n} g_{m,n}$ does not imply that the computation of the coefficient $c_{m,n}$ can be carried out by a "local" procedure, using only information localized around the point (an, bm) in the time-frequency plane. The problem of lack of time-frequency locality of the Gabor coefficients is not only severe in the critical case, but becomes more and more serious as one uses a sequence of lattices which are close-to-critical sampling. This fact becomes clear by observing that the corresponding dual functions lose their time-frequency localization, see also Figure 5.

It appears that Gabor families having some (modest) redundancy, which allows to have a pair of dual Gabor atoms (g, γ) where each function of this dual pair is well localized in time and frequency (cf. also Figure 5), are more appropriate as a tool in Gabor's original sense. Clearly under such premises one has to give up the uniqueness of coefficients and even the uniqueness of γ. The choice $\gamma = S^{-1}g$ is in some sense canonical and – as we have seen above – appropriate Gabor coefficients can be easily determined as samples of the STFT with window γ.

Is there no way to obtain an orthonormal basis for $\boldsymbol{L}^2(\mathbb{R})$ with good time-frequency properties based on Gabor's approach?

Wilson observed that for the study of the kinetic operator in quantum mechanics, one does not need basis functions that distinguish between positive and negative frequencies of the same order [Wil87, SRWW87]. Musicians probably would also agree to such a relaxation of requirements. Thus we are looking for complete orthonormal systems which are essential of Weyl-Heisenberg type, but allowing to have linear combinations of $g_{t,s}$ with $g_{t,-s}$. It turns out that by this seemingly small modification a family of orthonormal bases for $\boldsymbol{L}^2(R)$ can be constructed, the so-called *Wilson bases*, avoiding the Balian–Low phenomenon. Wilson's suggestion was turned into a construction by Daubechies, Jaffard and Journé [DJJ91], who gave a recipe how to obtain such an orthonormal Wilson basis from a tight Gabor frame for $a = 1/2$ and $b = 1$. See also Chapter2 and the research articles [Aus94, FGW92, Tac96].

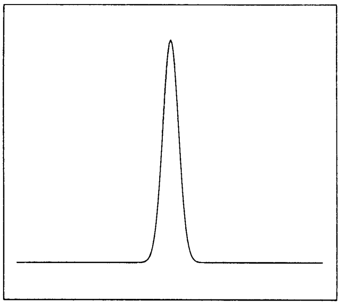
A general construction that includes many examples of Wilson bases as well as wavelet bases are the *local trigonometric bases* of Coifman and Meyer [CM91b]. Similar constructions were proposed by Malvar in [Mal90b] in a signal processing context. For more details, and especially the connection with Wilson bases, see [Aus94] and [AWW92].

In 1993 Mallat and Zhang [MZ93] introduced the *matching pursuit* for representing highly non-stationary signals. The idea is to expand a given signal in a small number of atoms, by selecting those atoms from a given *time-frequency dictionary*, which match best the different structures included in a signal. The analysis functions of this dictionary are constructed by translations, dilations (like in the wavelet expansion) and modulations (like in the Gabor expansion) from a single atom.

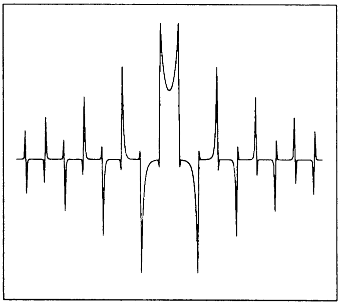
Clearly there is a trade-off between using a huge library of atoms in order to construct an "optimal" representations of a signal (tied to a given quality criterion) and the computational costs to find the representation functions out of this library. Attempts to handle this trade-off comprise multi-wavelets and multi-window Gabor schemes (see [SS94, ZZ97a] and Chapter 12 in this volume). A more detailed discussion of this dynamic field of research is beyond the scope of this introduction.

Wexler–Raz duality condition

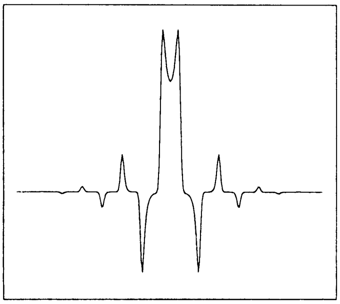
Ignoring fine mathematical details in our explanation (cf. Chapter 3 or Chapter 1 for details) one can say that a system of the form $\gamma_{m,n} = T_{na}M_{mb}\gamma$ is dual to $g_{m,n}$ if $f \in \boldsymbol{L}^2(\mathbb{R})$ has an $\boldsymbol{L}^2$-convergent represen-



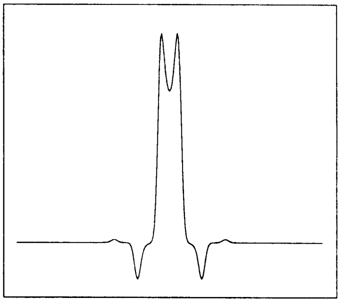
(a) Gabor function g



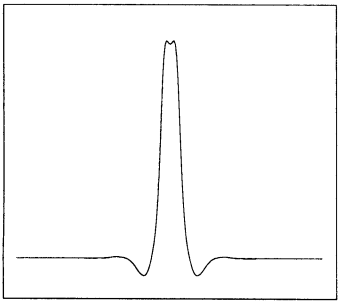
(b) Dual function for critical sampling



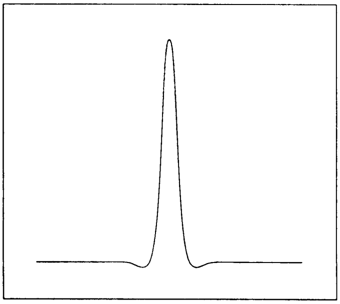
(c) Dual function for oversampling rate 1.06



(d) Dual function for oversampling rate 1.25



(e) Dual function for oversampling rate 1.67



(f) Dual function for oversampling rate 2.4

FIGURE 5. Dual Gabor functions for different oversampling rates. The dual window approaches Bastiaan's dual function for critical sampling and approximates the given Gabor window g with increasing oversampling rate.

tation of the form

$$f = \sum_{m,n} \langle f, \gamma_{m,n} \rangle \, g_{m,n} \, .$$

Any function γ generating a dual system $\gamma_{m,n}$ is called a *dual function* for g with respect to the given time-frequency lattice. Among all dual functions the *canonical dual* obtained via $\gamma = S^{-1}g$ has least energy. If we want to distinguish it from other duals we denote it by $^\circ\gamma$.

Wexler and Raz have obtained an elegant formulation of the duality condition of the systems $\{g_{m,n}\}$ and $\{\gamma_{m,n}\}$. For the case of Gabor analysis over $\mathbb{R}$ their basic result translates into the following characterization of duality with respect to given lattic econstants a, b

$$\langle \gamma, T_{n/b} M_{m/a} g \rangle = ab \; \delta_{n,0} \, \delta_{m,0} \, , \qquad m, n \in \mathbb{Z} \, ,$$

i.e., if and only if the two Weyl-Heisenberg systems generated from g and γ respectively, with respect to the adjoint lattice constant $1/b, 1/a$, are biorthogonal to each other.

There are different ways to understand this *Wexler–Raz condition* , one of them being a beautiful representation of the Gabor frame operator due to Janssen, who has shown in [Jan95b] that it can be written as a series of time-frequency shifts along the adjoint lattice with parameters $1/b, 1/a$ [Jan95b]. In Chapter 1 a connection to what Janssen calls the *fundamental identity of Gabor analysis* is established. This representation has also been derived at about the same time by Daubechies, H. Landau and E. Landau in [DLL95] and by Ron and Shen in [RS95c]. It should be noted that the approaches used in these three references are quite different.

Ron and Shen [RS95c] have observed that there is an important duality between between oversampling and undersampling lattices (in a multidimensional setting). An $\boldsymbol{L}^2$-function g generates a frame for given a, b if and only if it generates a Riesz basis (for its closed linear span) for the lattice constants $1/b, 1/a$.

This survey would be incomplete without mentioning a major tool in the analysis of Gabor systems, namely the *Zak-transform.* Actually, this transform (in the mathematical community known as the *Weil–Brezin transform* [Fol89]) has been introduced by Gelfand [Gel50] (1950), and was rediscovered by A. Weil and independently by J. Zak (Zak himself called it the kq-transform). A review of the Zak transform and its use in signal analysis can be found in [Jan88]. The Zak transform is in fact highly efficient for the case of integer oversampling (i.e., $1/ab \in \mathbb{N}$), because in this case it diagonalizes the frame operator. In a generalized form it can also be used to study Gabor expansions for rational oversampling. Recently other techniques (like the Kohn–Nirenberg correspondence or, for numerical purposes, the concept of unitary matrix factorization) indexunitary factorization have turned out to be powerful tools for the analysis of rationally oversampled Gabor frames.

Group theory as unifying language

Most often Gabor theory is investigated for functions on $\mathbb{R}$, i.e., the continuous, non-periodic and one-dimensional case of Gabor theory is discussed. Only in the last years alternative settings have been considered. Gabor expansions for discrete signals can be seen as part of Gabor theory over $\mathbb{Z}$, cf. [AC96, Li94a, Jan94a, ML94, ZZ93a], whereas numerical implementations can only work with finite signals. Since these are naturally identified with discrete and periodic signals, Gabor theory over finite cyclic groups (e.g., see [FCS95, Orr93a, Orr93b, Pri96b, QC93, RN96, WR90]) is the appropriate model for this situation. In image processing Gabor theory over $\mathbb{R}^2$ or $\mathbb{R}^d$ (as already discussed in [RS95c]) and in its discrete version over $\mathbb{Z}^d$ (or just products of finite abelian groups) has to be used [AZG91, Dau88b, Li94c].

The essential ingredients of Gabor theory are the commutative (=abelian) group of translations in combination with another commutative group, the so-called *dual* group of modulation operators. Hence it is possible to extend Gabor theory to the general setting of locally compact abelian (LCA) groups $\mathcal{G}$, which includes all setting discussed above. Through the Haar measure one has a natural $\boldsymbol{L}^2$-space on $\mathcal{G}$, and and the existence of sufficiently many "pure frequencies", called the *characters* of $\mathcal{G}$, is assured. The theory and the formal computations on all these groups are then the same, and their derivation becomes somewhat repetitive, often with unnecessary notation problems. As K. Gröchenig says in Chapter 6: "In a sense, present-day Gabor theory resembles the state of abelian harmonic analysis in the 1930's before it was discovered that all but a few theorems on Fourier integrals and Fourier series could be formulated for general LCA groups".

In the same way as for standard Fourier analysis the setting of LCA groups is the natural framework for a description of general results in Gabor theory. Some of the chapters of this book point in this direction, and further publications in this direction are in preparation.

Thinking of practical people like engineers we expect some natural scepticism against such "abstract nonsense". Nevertheless we express our expectation, that such a unified approach to Gabor analysis will be found useful, as it brings out most clearly the basic features of Gabor analysis, avoiding a repetitive rewriting of the same mathematical arguments, just with more and more indices or a somewhat different notation. From this point of view it also becomes more natural to see the typically used time-frequency lattices of the form $a\mathbb{Z}^d \times b\mathbb{Z}^d$ as quite special, since they depend on the choice of a coordinate system. Instead, it becomes more natural to make use of non-separable time-frequency lattices Λ of the form $\Lambda = M\mathbb{Z}^{2d}$, for some non-singular $2d \times 2d$-matrix M. Such non-separable lattices are discussed in [FCS95, FPK96, Pri96a, RS96, ZZ97a]. In applications where the orientation of the analysis windows is important, such as in texture analysis or pattern recognition, it may be advantageous to use non-separable analysis functions, cf. [Li94c, FPK96]) and Chapters 12 and 13.

Independently from the possibility for generalization of the domain, even for the case $\mathcal{G} = \mathbb{R}$ the structure of the group of all unitary operators (on $\boldsymbol{L}^2(\mathbb{R})$) generated by the time-frequency shift operators establishes a connection to representation theory of locally compact groups (resp. Lie groups). Indeed, the study of time-frequency shifts is intimately related to the study of the so-called Schrödinger representation of the reduced Heisenberg group (see e.g. the book of Schempp [Sch86]). Since this is a three-dimensional group, with the torus as third component (besides the time and the frequency parameter), the repeated appearance of exponential terms of the form e^{iax} often has a very natural explanation from the group theoretical point of view [Fol89, HR63]. This viewpoint also allows to draw interesting conclusions with respect to Gabor theory, only to mention results on the atomic decomposition of modulation spaces [FG89a].

Algorithms and Applications

In the last years a considerable part of research in Gabor analysis has been dedicated to the development of numerical algorithms. One of the most important problem from a numerical viewpoint is an efficient computation of a dual atom γ for given atom g and a given lattice parameters a, b. For a long time no fast and stable algorithms were known for this purpose, it was even believed that computationally inexpensive methods to not exist, but in the last years a number of efficient methods have been derived, see Chapter 8 and [QC96] for references.

There are many other numerical aspects that become important, when we are trying to realize Gabor methods in applications. Many of these aspects are discussed in Chapters 8–14, where the reader can also find further pointers to the relevant literature. Several journals in the field such as *The Journal of Fourier Analysis and Applications*, *Applied and Computational Harmonic Analysis*, and the IEEE journals publish regularly papers on Gabor analysis and their applications.

Gabor systems are applied in numerous engineering applications, many of them without obvious connection to the traditional field of time–frequency analysis for deterministic signals. Detailed information (including many references to further work) about the use of Gabor systems in such diverse fields as image analysis, object recognition, optics, filter banks, or signal detection can be found in the corresponding chapters of this book. In the sequel we quickly review some other applications (and apologize that they are not discussed in the particularity they deserve).

Any countable set of test functions $\{f_n\}$ in a Hilbert space conveys a linear mapping between function spaces and sequence spaces. In one direction scalar products $\langle f, f_n \rangle$ are taken (analysis mapping), and in the other direction the members $c = \{c_n\}$ from the sequence space are used as coefficient sequences in a series of the form $\sum_n c_n f_n$ (synthesis mapping). In our concrete context the analysis mapping is given by the Gabor transform (5) (i.e., the sampled short time Fourier transform) and the synthesis mapping is given by the Gabor expansion (6). In principle there exists two basic setups for the use of Gabor systems, which pervade most applications:

- The overall system acts on the sequence space $\boldsymbol{\ell}^2(\mathcal{G})$ (where $\mathcal{G}$ is some LCA group) by (i) Gabor synthesis, (ii) (desired or undesired and mostly linear but probably nonlinear) modification, (iii) Gabor analysis. This setup underlies e.g. the so–called multicarrier modulation schemes in digital communication, see Fig. 6, but also applies to system identification and radar tracking procedures.
- The overall system acts on the function space $\boldsymbol{L}^2(\mathcal{G})$ by (i) Gabor analysis, (ii) (desired or undesired and mostly nonlinear but also linear) modification, (iii) Gabor synthesis. Typical tasks where one encounters this setup (illustrated in Fig. 7) include signal enhancement, denoising or image compression.

Speech signal analysis:
Speech signals are one of the classical applications of linear time–frequency representations. In fact, the analysis of speech signals was the driving force that led to the invention of (a nondigital filter bank realization of) the *spectrogram* [KDL46, PKG47]. The advent of the FFT [CT65] made the STFT up to now to the standard tool of speech analysts (specialists are able to read simple sentences from proper spectrogram plots) [Opp70, Ril89]. For speech processing applications such as speech coding [MQ92, Nol93] or speech recognition [Fla92, RW93] the redundancy of the STFT is no longer desirable and is avoided by sampling the STFT, leading to the Gabor transform. Note that for such applications Gabor analysis serves only as a first stage decorrelating transform which is followed by fairly complicated typically nonlinear (pattern recognition or source coding) algorithms [Rab90].

Time–Varying Spectral Estimation:
Time–invariant spectral estimation, a standard procedure for time series analysis, is of fundamental relevance in most areas of science and technology. The basic principle is to model the time series as a realization of a

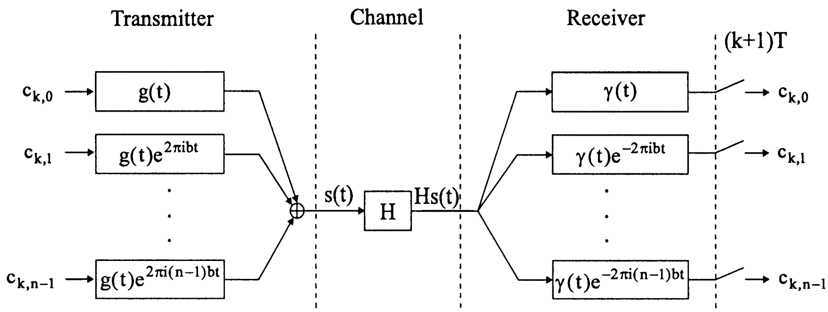


FIGURE 6. Digital communication systems transmit sequences of binary data over a continuous time physical channel. The transmitter sends a signal, formed by linear combination of the $g_{m,n}$, weighted by the binary coefficients $c_{m,n}$ (=Gabor synthesis). The receiver recovers the coefficients by computing inner products with the corresponding analysis functions $\gamma_{m,n}$ (=Gabor analysis). The editors wishes to thank P. Prinz for producing this figure and Figure 7.

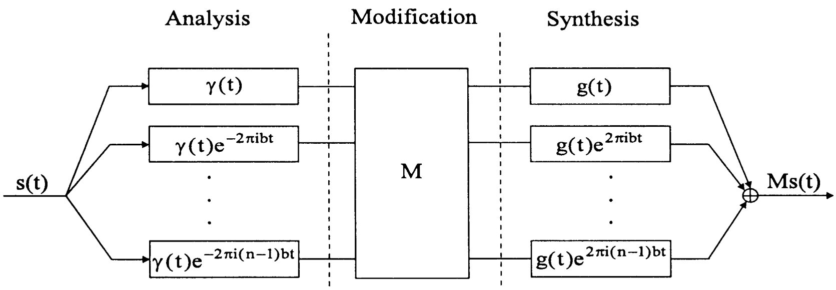


FIGURE 7. Schematic representation of Gabor setup for signal denoising or compression. The coefficients (computed by Gabor transform) are subject to linear or nonlinear (such as thresholding or source coding) modifications. The modified (e.g., denoised or decompressed) signal is obtained by Gabor expansion.

stationary random process whose power spectrum can be characterized by a typically lower dimensional data set (uniformly spaced frequency samples in a nonparametric setup or, e.g., poles and zeros of an ARMA model). There exists a large variety of robust and efficient procedures for power spectrum estimation [Sch91, Gar88, Por94]. The natural generalization of the stationary framework to nonstationary environments leads to a longstanding theoretical problem: What is the proper definition of a time–varying power spectrum of a nonstationary process? Classical definitions are Priestley's evolutionary spectrum [Pri65] and the Wigner–Ville spectrum which are closely related to the operator symbols of Weyl and Kohn–Nirenberg (discussed in Chapter 7). Whether or not one trusts either definition it is clear that a time–varying power spectrum is (i) time–frequency parametrized and (ii) a quadratic form of the data, hence the magnitude–squared Gabor coefficients establish the basis of any nonparametric estimation procedure [MF85, Rie93]. The inevitable bias/variance tradeoff can be handled by using a sparser sampling lattice and employing multiple windows as originally proposed by Thompson for the stationary case [Tho82]. Note however, that a recent approach of Mallat et al. discards the lattice structure [MPZ96] achieving increased adaptivity by the choice of tree–structured bases.

Representation and Identification of Linear Systems:
The theory of linear time–invariant (LTI) systems and in particular the symbolic calculus of transfer functions is a standard tool in all areas of mechanical and electrical engineering. Strict translation invariance is however almost always a pragmatical modeling assumption which establishes a more or less accurate approximation to the true physical system. Hence it is a problem of longstanding interest to generalize the transfer function concept from LTI to linear time–varying (LTV) systems. Such a time–varying transfer function was suggested by Zadeh in 1950 [Zad50]. It is formally equivalent to the Weyl–Heisenberg operator symbol of Kohn and Nirenberg [KN65]. Pseudodifferential operators are the classical way to establish a symbol classification that keeps some of the conceptual power which the Fourier transform has for LTI systems. Recently Gabor frames have turned out to be a useful tool to the analysis of pseudodifferential operators [Tac94, HRT, GH97]. A less known concept to impose smoothness restrictions on the Weyl–Heisenberg operator symbol is the so-called underspread/overspread classification in electrical engineering (see also Chapter 10 in this volume).

Digital Communication:
Digital communication systems transmit sequences of binary data over a continuous time physical channel. An ideal physical channel is bandlimited without in–band distortions. Under this idealized assumption, digital communication systems can be implemented by selecting a Gabor–structured

orthonormal system, transmitting a linear combination of the elementary signals, weighted by the binary coefficients (Gabor synthesis) and the receiver recovers the coefficients by computing inner products with the known basis functions (matched filter receiver = Gabor analysis), see Fig.6. However, in wireless communication systems which is one of the challenging research areas, the physical channel is subject to severe linear distortions and hundreds of users communicate over the same frequency band at the same time. Traditional OFDM (orthogonal frequency division multiplex) systems can be interpreted as orthonormal Gabor systems with critical sampling $ab = 1$ and come therefore with the well-known bad time-frequency localization properties of the building blocks. Since completeness is not a concern here, recent works suggest the use of a coarser grid $ab > 1$, together with good TF–localized atoms [San96, KM97] to obtain more robustness.

Image representation and biological vision:
Gabor functions were successfully applied to model the response of simple cells in the *visual cortex* [Dau80, Mar80]. A survey on this topic can be found in [NTC96]. In our notation, each pair of adjacent cells in the visual cortex represents the real and imaginary part of one coefficient $c_{m,n}$ corresponding to $g_{m,n}$. Clearly the Gabor model cannot capture the variety and complexity of the visual system, but it seems to be a key in further understanding of biological vision.

Among the people who paved the way for the use of Gabor analysis in pattern recognition and computer vision one certainly has to mention Zeevi, M. Porat, and their coworkers [ZP84, ZP88, PZ89, ZG92] and Daugman [Dau87, Dau88b, Dau93]. Motivated by biological findings, Daugman [Dau88b] and Zeevi and Porat [PZ88] proposed the use of Gabor functions for image processing applications, such as image analysis and image compression. Ebrahimi, Kunt and coworkers used Gabor expansions for still image and video compression [EK91]. Many modern transform based data compression methods try to combine the advantages of Gabor systems with the advantages of wavelets.

Since techniques from signal processing are of increasing importance in medical diagnostics, we mention a few applications of Gabor analysis in medical signal processing. The Gabor transform has been used for the analysis of brain function, such as for detection of epileptic seizures in EEG signals, see e.g., [BKQQ$^+$97], or for the study of sleep spindles (see Chapter 11 in [QC96]). The role of the Heisenberg group in magnetic resonance imaging has been recently analyzed by Schempp, see for example [Sch95].

Another important application, where Gabor systems and the Heisenberg group play a significant role is radar, see for instance [Sch84b, Sch84a, Bag90, AGW95] and Chapter 10 in the book of Qian and Chen [QC96].

Reader's guide

Depending on the setup in which the main questions are formulated one may see Gabor analysis as a particular chapter (or circle of problems) within the following fields:

- If the convergence of Gabor series for certain classes of functions or distributions or continuity properties of the frame operator are discussed, one may regard Gabor analysis as a chapter of *functional analysis* or *operator theory.*
- Drawing the attention on the Heisenberg group, which is fundamental in analyzing the Gabor system $\{T_{na}M_{mb}g\}$ of modulations and translations and exploiting this link to study Gabor theory with group theoretical methods, shows this field as a branch of *abstract harmonic analysis.*
- If the focus is on the development of efficient and stable algorithms, we are dealing with a problem of *numerical mathematics.* For computer implementations we have to deal with a finite model. The general problem of expanding "finite" signals, such as vectors of finite length or digital images of format $N \times M$, is essentially a chapter of *linear algebra.*
- If Gabor expansions are used for certain applications in communication theory or time-frequency signal analysis, it may be seen as a chapter of *signal processing.*
- When we use the properties of the Gabor transform to extract local features or to study textures in images, we are dealing with a classical topic of *pattern recognition* or *image analysis.*
- If we apply Weyl-Heisenberg frames to the analysis of coherent states, (e.g., see a recent paper of Zak [Zak96]), we study a classical field in *quantum physics.*

The chapters of this book reflect this diversity and also the interrelations between these fields. Although all chapters are more or less self-contained, there are many cross-connections between the chapters. We have tried to use a consistent notation throughout the book to aid the reader in discovering and following these cross-connections.

In a nutshell the organization of the chapters may be characterized in the following way. In the first half of the book the emphasis is on theoretical questions, mainly concentrating on mathematical problems, whereas in the chapters of the second half the focus is on applications and numerical algorithms. In the sequel we give a brief overview of the chapters.

The first chapter by A.J.E.M. Janssen gives a detailed survey on the condition of duality for Gabor frames for the continuous-time and the discrete-time case. Many results are derived for the general framework of translation invariant systems or filter banks. Besides the formulation of the duality condition in various domains and formulas for the frame bounds, the author derives a representation of the Gabor frame operator as a series of time-frequency shift operators. This representation is intimately related to the "fundamental identity of Gabor analysis", which is obtained by means of classical Fourier methods.

Chapter 2 by John J. Benedetto, Christopher Heil, and David F. Walnut, presents a tutorial on the famous Balian–Low Theorem (BLT) and on related topics such as Wilson bases and the uncertainty principle. As a variation on the theme, a so-called Amalgam BLT is derived, the difference to the standard BLT is illustrated by examples. Major tools in this chapter are the Zak transform and the notion of frames. Furthermore the authors give completeness conditions for irregular Gabor systems $\{g_{p,q}\}_{(p,q)\in\Lambda}$, based on the density of Λ, where Λ is a discrete subset (but not necessarily a lattice) of $\mathbb{R} \times \widehat{\mathbb{R}}$.

The space $\boldsymbol{S}_0$ can be characterized as the smallest time-frequency homogeneous Banach space of (continuous) functions. Exploiting the properties of $\boldsymbol{S}_0$, Hans G. Feichtinger and Georg Zimmermann demonstrate in Chapter 3 that this space is particularly useful for Gabor analysis. Thus they present an extended version of the fundamental identity of Gabor analysis; the continuous dependence of the canonical dual Gabor atom on the given Gabor window and on the lattice; continuity of thresholding and masking operators from signal processing; and an approximate Balian–Low Theorem stating that for close-to-critical lattices, the dual Gabor atoms progressively lose their time-frequency localization.

In Chapter 4, Richard Rochberg and Kazuya Tachizawa use Gabor frames for the derivation of sufficient conditions on the Weyl symbol, to ensure that the corresponding pseudodifferential operators belong to the Schatten–von Neumann classes $S_{p,q}$. Furthermore they give estimates for the size of eigenfunctions of pseudodifferential operators. This chapter also shows nicely how localized systems can be applied to obtain an approximate diagonalization of operators. In particular, the authors use local trigonometric bases to approximately diagonalize elliptic pseudo-differential operators.

Chapter 5 by O.Christensen is devoted to the problem of perturbation of frames, with an emphasis on the consequences for Gabor frames. Based on functional analytic arguments it is explained that both small changes of the lattice points of the time-frequency lattice or of the Gabor atom lead to coherent families which are still frames for $\boldsymbol{L}^2(\mathbb{R})$. Furthermore, it is described how the frame bounds change with the perturbation parameters.

Most results in the chapters described above are derived for functions defined on $\mathbb{R}, \mathbb{R}^d$, and $\mathbb{Z}$. In Chapters 6 and 7, the focus is on the more general setting of (elementary) locally compact abelian groups. This group theoretical setting allows a unifying approach to Gabor analysis, without repeating proofs for different settings (such as $\mathbb{R}, \mathbb{R}^d, \mathbb{Z}, \mathbb{Z}_N$). In Chapter 6 Karlheinz Gröchenig uses Liebs' inequalities for the short time Fourier transform to express an uncertainty principle for the setting of LCA groups. Using the Zak transform, introduced on LCA groups already by A. Weil, the author analyzes Gabor frames for the cases of critical sampling and integer oversampling. In this context it turns out that the known version of the Balian–Low Theorem does not hold for discrete and compact groups. A notion of density is defined and necessary density conditions for lattices are derived to generate Gabor frames.

In Chapter 7 by Hans G. Feichtinger and Werner Kozek, various new aspects come into play. The duality conditions are extended to general discrete subgroups Λ (with compact quotient), allowing a unified formulation for continuous-time, discrete-time, multi-dimensional signals, including the case of separable and non-separable lattices and atoms. The Gabor frame operator is interpreted as a time-frequency periodization (with respect to Λ) of the one-dimensional projection onto the vector space generated by the atom g. The case of multi-window Gabor analysis is obtained as a simple modification, replacing the rank-1 operator $K : f \to \langle f, g \rangle g$ by some finite rank operator. In order to derive these results, the concept of Gelfand triples, the spreading function and a generalized Kohn–Nirenberg correspondence turn out to be powerful tools.

One of the challenging numerical problems in Gabor analysis is the development of efficient methods for the inversion of the frame operator in order to compute the dual window. In Chapter 8, Thomas Strohmer presents a unifying approach to derive fast numerical algorithms, based on unitary factorizations of the Gabor frame operator. Using classical tools from linear algebra the author shows that different algorithms for the computation of the dual window correspond to different factorizations of the frame operator. Based on simple number theoretic conditions new structural properties of the frame operator for the finite setting are derived. The chapter concludes with a discussion of the conjugate gradient method, demonstrating the efficiency of certain preconditioners by numerical experiments.

Oversampled filter banks offer increased design freedom and noise immunity as compared to critically sampled filter banks (FB). Since these advantages come at the cost of greater computational complexity, those oversampled FBs are of particular interest, which can be realized in practice with little computational effort. In Chapter 9, Helmut Bölcskei and Franz Hlawatsch study oversampled DFT FBs and oversampled cosine modulated FBs which allow efficient implementations. Exploiting the link of the Gabor frame operator with the filter bank polyphase matrix, they present a frame-theoretic analysis of filter banks. Conditions for perfect reconstruction are given and the increased design freedom is illustrated by numerical simulations.

In many applications the choice of the analysis window g and the lattice parameters a, b follows traditional rules of thumb. In Chapter 10, Werner Kozek presents a theoretical framework for the adaptation of continuous ("fully oversampled") and discrete Gabor frames to underspread operators in the sense of approximate diagonalization. Underspread operators correspond to slowly time-varying systems with finite memory or to quasistationary random processes. The optimization criteria for the adaptation are formulated in terms of the ambiguity function of g and the spreading function of the operator. Numerical experiments demonstrate the practical performance of the proposed method.

Recently, the problem of signal detection based on linear time-frequency representations has received considerable attention. In Chapter 11 Ariela Zeira and Benjamin Friedlander formulate the problem of detecting a Gabor transient as problem of detecting a subspace signal in background noise. Using global ratio likelihood tests, they derive matched subspace detectors and discuss their sensitivity in terms of mismatch of the model parameters. Based on the sensitivity analysis they develop robust matched subspace detectors and analyze their performance.

In Chapter 12 Yehoshua Y. Zeevi, Meir Zibulski and Moshe Porat present a detailed analysis of multi-window Gabor expansions. They extend the Balian–Low Theorem to the case of multi-windows and discuss also the case of non-separable time-frequency lattices. Applications to image processing and computer vision are presented with regard to texture images, and considered in the context of two typical tasks: image representation by partial information and pattern recognition. In both cases the results indicate that the multi-window approach allows to overcome drawbacks of the single window approach.

The subject of Chapter 13 is the application of Gabor functions to object recognition. Jezekiel Ben-Arie and Zhiqian Wang show that 3-D pose invariant object recognition can be achieved by a specific multi-window configuration of Gabor functions. By matching the extracted local spectral signature against a set of iconic models using multi-dimensional indexing in the frequency domain, the authors are able to identify objects under varying conditions with high reliability, as shown by numerical experiments.

The final chapter, written by Martin J. Bastiaans, presents applications in the field of optics. Special attention is paid to Gaussian windows, which are related to the classical concept of Gaussian light beams in optics. The propagation of an optical signal in terms of its Gabor coefficients is investigated and the relation of the case of critical sampling to the degrees of freedom of an optical signal is discussed. The author demonstrates how a product of Zak transforms establishes the basis of a coherent-optical setup which can be used to generate the Gabor transform.

A comprehensive bibliography completes the book.

Mathematical notation and basic tools

The purpose of this section is to fix some notation and conventions that are used throughout the book, as well as to present a few analytic tools, that will appear frequently. This section is intended as a reference rather than a systematic text. The editors and contributors made some effort to use a consistent notation throughout the book. In the few cases where different symbols are used (such as i and j to denote the imaginary unit) there should not be a possibility of confusion.

Spaces and operators

We use the standard symbols $\mathbb{R}, \mathbb{C}, \mathbb{Z}$ and $\mathbb{N}$ to denote the set of real numbers, complex numbers, integers and natural numbers, respectively. The torus is given by $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ and this is thus naturally isomorphic to the range of the group homomorphism $x \mapsto e^{2\pi i x}$ in $\mathbb{C}$.

The Lebesgue space $\boldsymbol{L}^p(\mathbb{R})$ consists of all measurable functions f such that $\|f\|_p < \infty$ for $1 \le p < \infty$ where

$$\|f\|_p = \left(\int_{-\infty}^{+\infty} |f(x)|^p \, dx \right)^{1/p} \tag{12}$$

and $\|f\|_\infty = \operatorname{ess\,sup} |f(x)|$, where $\operatorname{ess\,sup} h(x) = \inf\{\lambda \in \mathbb{R} : h(x) \le \lambda \text{a.e.}\}$. Here integration is understood in the sense of Lebesgue and therefore often integration limits are not given explicitly. It is well known that $\boldsymbol{L}^p(\mathbb{R})$ is a Banach space with norm $\|\cdot\|_p$ for $1 \le p \le \infty$ in the sense that *Cauchy sequences* are convergent.

Furthermore $\boldsymbol{L}^2(\mathbb{R})$ is a Hilbert space, i.e. a complete space with inner product $\langle f, g \rangle = \int f(x)\overline{g(x)}\, dx$, where $\overline{g(x)}$ is the complex conjugate of $g(x)$. Another frequently used Hilbert space is $\boldsymbol{\ell}^2(\mathbb{Z}) = \{c_n \in \mathbb{C} : \sum_{n\in\mathbb{Z}} |c_n|^2 < \infty\}$ with inner product $\langle u, v \rangle = \sum_{n\in\mathbb{Z}} u_n \overline{v_n}$. The *Cauchy-Schwarz inequality* states that $|\langle u, v \rangle| \le \|u\|_2 \|v\|_2$.

We will occasionally encounter the following spaces: $\boldsymbol{C}$ is the space of continuous functions on $\mathbb{R}$ and $\boldsymbol{C}_0$ consists of all $f \in \boldsymbol{C}$ which are vanishing at infinity. $\boldsymbol{S}(\mathbb{R})$ is the *Schwartz class* of infinitely differentiable functions which are rapidly decreasing at infinity, and $\boldsymbol{S}'(\mathbb{R})$ is its topological dual, the space of *tempered distributions*. $\boldsymbol{S}$ is dense in $\boldsymbol{L}^p$ for $p < \infty$.

The *Fourier transform* of $f \in \boldsymbol{L}^1(\mathbb{R})$ is given pointwise as

$$(\mathcal{F}f)(\xi) = \hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x) e^{-2\pi i x \xi} \, dx \, . \tag{13}$$

The *Lemma of Riemann–Lebesgue* states that if $f \in \boldsymbol{L}^1(\mathbb{R})$ then $\hat{f}$ is a uniformly continuous function with $\lim \hat{f}(\omega) = 0$ for $|\omega| \to 0$. Recall that a function h is *uniformly continuous* if for all $\varepsilon > 0$ there is a $\delta > 0$ such that $|h(x) - h(y)| \leq \varepsilon$ whenever $|x - y| \leq \delta$.

If $\hat{f} \in \boldsymbol{L}^1(\mathbb{R})$ the *inversion of the Fourier transform* is given by

$$f(x) = \int_{-\infty}^{+\infty} \hat{f}(\xi) e^{2\pi i \xi x}\, d\xi\,. \tag{14}$$

For $f, g \in \boldsymbol{L}^2(\mathbb{R}) \cap \boldsymbol{L}^1(\mathbb{R})$ we have the unitarity relation $\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle$ with the *Plancherel formula* $\|f\|_2 = \|\hat{f}\|_2$ as a special case. Since $\boldsymbol{L}^2(\mathbb{R}) \cap \boldsymbol{L}^1(\mathbb{R})$ is a dense subspace of $\boldsymbol{L}^2(\mathbb{R})$ this implies that the Fourier transform may be extended via a canonical limiting process to a unitary mapping (actually automorphism) of $\boldsymbol{L}^2(\mathbb{R})$ onto itself, cf. [Kat68].

Due to the normalization of the Fourier transform the *Poisson Summation Formula* becomes completely symmetric. It is valid at least for $f \in \boldsymbol{S}(\mathbb{R})$ and reads

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\,. \tag{15}$$

If f, g are complex-valued functions defined on $\mathbb{R}$, their *convolution* $f * g$ is the function $(f * g)(x) = \int f(x - y) g(y)\, dy$ provided that the integral exists (almost everywhere).

If K is an invertible operator or matrix, then K^{-1} is its *inverse.* The *pseudo-inverse* [CM91a] of an operator K is denoted by K^+, K^* is the *adjoint operator* of K (i.e., the conjugate transpose if K is a matrix).

Given a function f we define the following operators:

$$\text{Translation:} \quad T_a f(x) = f(x - a) \quad \text{for } a \in \mathbb{R} \tag{16}$$

$$\text{Modulation:} \quad M_b f(x) = e^{2\pi i x b} f(x) \quad \text{for } b \in \mathbb{R} \tag{17}$$

Each of these is a unitary operator from $L^2(\mathbb{R})$ onto itself, and we have

$$(T_a f)^\wedge = M_{-a}\hat{f}\,, \quad (M_a f)^\wedge = T_a \hat{f}\,, \quad M_b T_a = e^{2\pi i a b} T_b M_a\,. \tag{18}$$

For a general window $g \in \boldsymbol{L}^2(\mathbb{R})$ the *short time Fourier transform* of a function $f \in \boldsymbol{L}^2(\mathbb{R})$ is defined by [Fol89]

$$\mathcal{V}_g f(x, \xi) = \int f(z) \overline{g(z - x)} e^{-2\pi i \xi z}\, dz = \langle f, T_x M_\xi g\rangle\,. \tag{19}$$

It is a bounded, continuous and square integrable function.

The *Zak transform* of $f \in \boldsymbol{L}^2(\mathbb{R})$ is formally defined by [Fol89]

$$\mathcal{Z} f(x, \xi) = \sum_{k\in\mathbb{Z}} f(x + k) e^{2\pi i k \xi}\,. \tag{20}$$

Bases and frames

Let $\mathbb{H}$ be a separable Hilbert space (e.g. $\boldsymbol{L}^2(\mathbb{R})$) equipped with inner product $\langle f, g\rangle$ and norm $\|f\| = \sqrt{\langle f, f\rangle}$. A sequence $\{e_n\}_{n\in\mathbb{Z}}$ is a (Schauder) *basis* for $\mathbb{H}$ if for all $f \in \mathbb{H}$ there exist *unique* scalars c_n dependent on f such that

$$f = \sum c_n e_n \,. \tag{21}$$

A basis is called *unconditional* if the series expansion is convergent to f in the norm independently of the specific order of the sequence $\{e_n\}$. A basis is *bounded* if $0 < \inf \|e_n\| \leq \sup \|e_n\| < \infty$.

Separability of $\mathbb{H}$ is equivalent to the existence of a countable *orthonormal* family of vectors $\{e_n\} \subseteq \mathbb{H}$, i.e. a sequence satisfying[2]

$$\langle e_n, e_m\rangle = \delta_{n,m} \tag{22}$$

which is also complete in the sense that every $f \in \mathbb{H}$ can be approximated by finite linear combinations of elements from $\{e_n\}$. In a Hilbert space $\mathbb{H}$ completeness of a family $\{g_i\}$ can equivalently be described by the property that $\langle f, g_i\rangle = 0$ for all $i \in I$ implies $f = 0$. Especially in the context of Gabor families the reader should be aware that this does not always imply that a series expansion of arbitrary elements $f \in \mathbb{H}$ is possible. Indeed, Gabor's original family is an instance of this situation. Only for the case of finite dimensional spaces it is true, that every element which can be approximated in the norm by finite linear combinations can also be written as a series (actually as a finite sum).

The most important property of a complete orthonormal basis (as opposed to non-orthogonal bases or general frames) is the simplicity of a series expansion. Every $f \in \mathbb{H}$ can be expressed as

$$f = \sum_n \langle f, e_n\rangle e_n \,. \tag{23}$$

and square summability of coefficients is guaranteed by the *Parseval's formula* for orthonormal bases, which reads $\|f\|^2 = \sum_n |\langle f, e_n\rangle|^2$ for all $u \in \mathbb{H}$.

For a so-called *Riesz basis* $\{f_n\}$ one can always find a *biorthogonal family* $\{\tilde{f}_n\}$, satisfying $\langle f_k, \tilde{f}_n\rangle = \delta_{k,n}$, which generates $\boldsymbol{\ell}^2$-coefficients $c_n = \langle f, \tilde{f}_n\rangle$ for each $f \in \mathbb{H}$. The projection onto the closed linear span of the $\{f_n\}$ is of the form $f \mapsto \sum_n \langle f, \tilde{f}_n\rangle f_n$. Usually a Riesz basis for a closed subspace of a (separable) Hilbert space is characterized as a sequence $\{f_n\}$ for which one has two positive constants $C, D > 0$ such that $C\|c\|^2 \leq \|\sum_n c_n f_n\|^2 \leq D\|c\|^2$ for each sequence $c = \{c_n\} \in \boldsymbol{\ell}^2$.

[2] Here $\delta_{n,m}$ denotes the *Kronecker delta* with the usual meaning: $\delta_{n,m} = 1$ if $n = m$, and 0 if $n \neq m$.

Every Riesz basis is a *linear independent* system, in the sense that a finite linear combination $\sum_{k=1}^{r} c_r f_{n_k} = 0$ if and only if $c_r = 0$ for $1 \leq k \leq r$. However, the converse is not true. Actually, any Weyl-Heisenberg family obtained from the Gauss function is linear independent in $\boldsymbol{L}^2(\mathbb{R})$, but there are many ways to represent the zero function using a well convergent series with non-zero coefficients.

Frames are a generalization of (orthonormal) bases in a different direction [DS52, You80]. A sequence $\{f_n\}$ in $\mathbb{H}$ is a *frame* for $\mathbb{H}$ if there exist constants A, $B > 0$ such that

$$A\|f\|^2 \leq \sum |\langle f, f_n\rangle|^2 \leq B\|f\|^2 \qquad \forall f \in \mathbb{H}. \tag{24}$$

Valid constants A, B are called *frame bounds.* The frame is *tight* if $A = B$, and it is *exact* if it is no longer a frame when any one of its elements is removed. A sequence that satisfies the upper frame bound (and which may or may not satisfy the lower frame bound) is called a *Bessel sequence.*

Any Riesz basis for $\mathbb{H}$ is a frame, and conversely a frame is a Riesz basis if and only if it is exact. A Riesz basis $\{f_n\}$ for $\mathbb{H}$ can be also characterized by the fact that there exists a continuous, invertible, linear mapping T from $\mathbb{H}$ onto itself such that $\{Tf_n\}$ forms an orthonormal basis for $\mathbb{H}$.

Similar to a Riesz basis for $\mathbb{H}$, every frame provides a series representation of arbitrary elements $f \in \mathbb{H}$, i.e. there exists a sequence of square summable coefficients c_n so that $f = \sum c_n f_n$. If the frame is not exact then these expansion coefficients will not be unique. However, there is always at least one computable, canonical choice. This can be obtained by making use of the *frame operator*

$$Sf = \sum_n \langle f, f_n\rangle f_n, \tag{25}$$

which is a continuous, invertible and linear mapping from $\mathbb{H}$ onto itself.

The sequence $\{\tilde{f}_n\}$ with $\tilde{f}_n = S^{-1} f_n$ forms another frame for $\mathbb{H}$, called the *dual frame* of $\{f_n\}$. With A, B being frame bounds for S the numbers $1/B$ and $1/A$ are frame bounds for the dual frame, since the frame operator for the dual frame is just S^{-1}. Using the dual frame we obtain the series representation for all $f \in \mathbb{H}$

$$f = \sum_n \langle f, \tilde{f}_n\rangle f_n = \sum_n \langle f, f_n\rangle \tilde{f}_n\,. \tag{26}$$

with $c_n = \langle f, \tilde{f}_n\rangle$ being the minimal norm sequence which can be used for a representation of f. Furthermore the summations in (26) converge unconditionally, i.e., independently of the order of summation.

1

The duality condition for Weyl-Heisenberg frames

A.J.E.M. Janssen

ABSTRACT – We present formulations of the condition of duality for Weyl-Heisenberg systems in the time domain, the frequency domain, the time-frequency domain, and, for rational time-frequency sampling factors, the Zak transform domain, both for the one-dimensional time-continuous case and the one-dimensional time-discrete case. Many of the results we obtain are presented in the more general framework of shift-invariant systems or filter banks, and we establish, for instance, relations with the polyphase matrix approach from filter bank theory. The formulation of the duality condition in various domains is notably useful for the design of perfect reconstructing shift-invariant Weyl-Heisenberg analysis and synthesis systems under restrictions of the constituent filter responses which may be stated in any of the domains just mentioned. In all considered domains we present formulas for frame operators and frame bounds, and we compute and characterize minimal dual systems.

1.1 Introduction

Let us start by explaining what we mean by a shift-invariant system, both for the time-continuous and the time-discrete case. Such a system consists of a collection of functions or sequences g_{nm} of the form

$$g_{nm} = g_m(\cdot - n\Delta) \ , \ (n,m) \in \mathbb{Z} \times \mathcal{I} \ , \tag{1.1.1}$$

where $g_m \in L^2(\mathbb{R})$ or $l^2(\mathbb{Z})$, $\Delta = a > 0$ or $\Delta = N \in \mathbb{N}$, and $\mathcal{I} = \mathbb{Z}$ or $\{0, ..., M-1\}$ with $M \in \mathbb{N}$. The case with $g_m \in L^2(\mathbb{R})$, $\Delta > 0$, $\mathcal{I} = \mathbb{Z}$ is referred to as the time-continuous case while the case with $g_m \in l^2(\mathbb{Z})$, $\Delta \in \mathbb{N}$, $\mathcal{I} = \{0, ..., M-1\}$ is referred to as the time-discrete case. We do not consider systems that consist of functions or sequences g_m that are periodic.

We are interested in finding dual systems

$$\gamma_{nm} = \gamma_m(\cdot - n\Delta) \ , \ (n,m) \in \mathbb{Z} \times \mathcal{I} \ , \tag{1.1.2}$$

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by which we mean that any $f \in L^2(\mathbb{R})$ or $l^2(\mathbb{Z})$ has an $L^2(\mathbb{R})$-or $l^2(\mathbb{Z})$-convergent representation

$$f = \sum_{n,m} \langle f, \gamma_{nm} \rangle \, g_{nm} \; . \tag{1.1.3}$$

For this to be meaningful we require the involved systems to have a finite frame upper bound: the system (1.1.1) has a finite frame upper bound when there is a $B_g < \infty$ such that

$$\sum_{n,m} |\langle f, g_{nm} \rangle|^2 \leq B_g \, \|f\|^2 \; , \; f \in L^2(\mathbb{R}) \text{ or } l^2(\mathbb{Z}) \; , \tag{1.1.4}$$

and any $B_g < \infty$ such that (1.1.4) holds is called a frame upper bound.

When the system (1.1.1) has a finite frame upper bound B_g, one can define an analysis operator T_g and a synthesis operator T_g^* by

$$T_g : f \in L^2(\mathbb{R}) \text{ or } l^2(\mathbb{Z}) \to T_g \, f = (\langle f, g_{nm} \rangle)_{n,m} \; , \tag{1.1.5}$$

and

$$T_g^* : \underline{\alpha} \in l^2(\mathbb{Z} \times \mathcal{I}) \to T_g^* \underline{\alpha} = \sum_{n,m} \alpha_{nm} \, g_{nm} \in L^2(\mathbb{R}) \text{ or } l^2(\mathbb{Z}) \; , \tag{1.1.6}$$

respectively. These T_g and T_g^* are bounded linear operators with operator norm $\leq \; B_g^{1/2}$. Observe that T_g and T_g^* as defined by (1.1.5) and (1.1.6) are indeed adjoint operators with respect to the inner product $\langle \; , \; \rangle$. When the system (1.1.2) has a finite frame upper bound as well, the duality condition (1.1.3) can be expressed as

$$T_g^* \, T_\gamma = I \; , \tag{1.1.7}$$

where I denotes the identity operator of $L^2(\mathbb{R})$ or $l^2(\mathbb{Z})$.

When the system (1.1.1) has a finite frame upper bound B_g, the frame operator S_g is defined by $S_g = T_g^* \, T_g$. Explicitly,

$$S_g : f \in L^2(\mathbb{R}) \text{ or } l^2(\mathbb{Z}) \to S_g \, f = \sum_{n,m} \langle f, g_{nm} \rangle g_{nm} \in L^2(\mathbb{R}) \text{ or } l^2(\mathbb{Z}) \; , \tag{1.1.8}$$

and there holds

$$S_g \leq B_g \, I \; . \tag{1.1.9}$$

When there is a $A_g > 0$ such that

$$\sum_{n,m} |\langle f, g_{nm} \rangle|^2 \geq A_g \, \|f\|^2 \; , \; f \in L^2(\mathbb{R}) \text{ or } l^2(\mathbb{Z}) \; , \tag{1.1.10}$$

or, equivalently, S_g is invertible with

$$S_g \geq A_g\, I \ , \tag{1.1.11}$$

we say that the system (1.1.1) has a positive lower frame bound, and any $A_g > 0$ such that (1.1.10) holds is called a lower frame bound. In the case that the system (1.1.1) has both a finite frame upper bound and a positive frame lower bound, we say that (1.1.1) is a frame. Then a dual system is given by

$$^\circ\gamma_{nm} = (S_g^{-1}\, g_m)(\cdot - n\Delta), (n,m) \in \mathbb{Z} \times \mathcal{I} \ , \tag{1.1.12}$$

and this system is also a frame. It is essential to note here that the frame operator S_g commutes with all relevant time-shift operators. More generally, when the systems (1.1.1) and (1.1.2) have finite frame upper bounds and (1.1.3) holds for all f, both systems are a frame. The dual system (1.1.12) is special in the following sense: for any $f \in L^2(\mathbb{R})$ or $l^2(\mathbb{Z})$ and any $\underline{\alpha} \in l^2(\mathbb{Z} \times \mathcal{I})$ with

$$f = \sum_{n,m} \alpha_{nm}\, g_{nm} \tag{1.1.13}$$

there holds

$$\sum_{n,m} |\langle f, {}^\circ\gamma_{nm}\rangle|^2 \leq \sum_{n,m} |\alpha_{nm}|^2 \tag{1.1.14}$$

with equality if and only if $\langle f, {}^\circ\gamma_{nm}\rangle = \alpha_{nm}$ for all $(n,m) \in \mathbb{Z} \times \mathcal{I}$. One checks, furthermore, easily that $S_g\, S_{^\circ\gamma} = I$, so that $S_{^\circ\gamma}$ is the inverse frame operator. We refer to this dual system as the minimal dual system. For generalities about frames and shift-invariant systems we refer to [Dau90], Sec. 3.2, [Dau92] Sec. II, and [RS95b], Sec. 1.3.

A particular example of a shift-invariant system arises when we take

$$g_m(t) = e^{2\pi imbt}\, g(t) \ , \ t \in \mathbb{R}, \ m \in \mathbb{Z} \ , \tag{1.1.15}$$

with $b > 0$ and $g \in L^2(\mathbb{R})$ (time-continuous case), or

$$g_m(j) = e^{2\pi imj/M}\, g(j) \ , \ j \in \mathbb{Z} \ , \ m = 0, ..., M-1 \ , \tag{1.1.16}$$

with $M \in \mathbb{N}$ and $g \in l^2(\mathbb{Z})$ (time-discrete case). It is customary here to ignore the phase factors in g_{nm}, γ_{nm} when studying duality questions, since these vanish anyway at the right-hand side of (1.1.3). Hence one has for the time-continuous case

$$g_{nm}(t) = e^{2\pi imbt}\, g(t-na) \text{ rather than } e^{2\pi imb(t-na)}\, g(t-na) \tag{1.1.17}$$

for $(n,m) \in \mathbb{Z} \times \mathbb{Z}$, etc. We refer to the systems so obtained as (time-continuous or time-discrete) Weyl-Heisenberg systems, and the g in (1.1.15) or (1.1.16) is sometimes called a prototype function or sequence. Since the frame operator S_g commutes now with all relevant time-shift and frequency-shift operators, it turns out that when g_{nm}, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, is a Weyl-Heisenberg frame, then so is the minimal dual system ${}^\circ\gamma_{nm}$, $(n,m) \in \mathbb{Z} \times \mathcal{I}$.

In the next sections we work out the condition of duality for two shift-invariant systems, and in particular two Weyl-Heisenberg systems, in various domains. Specifically, we present for a shift-invariant system g_{nm}, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, a necessary and sufficient condition, in terms of the Fourier transforms of the g_m, to be a frame. Also, we present a necessary and sufficient condition for two shift-invariant systems g_{nm}, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, and γ_{nm}, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, both having a finite frame upper bound, to be dual. Furthermore, we give a representation result for frame operators in the Fourier domain, we present a method to compute the minimal dual system and we characterize it in the Fourier domain. By specialization to Weyl-Heisenberg systems, we obtain formulations of the conditions of having a finite frame upper bound, duality, etc., both in the frequency domain and in the time domain. A third domain where these conditions and computation methods are worked out is the time-frequency domain. This consists of a detailed elaboration of the celebrated Wexler–Raz condition [WR90] of biorthogonality of two Weyl-Heisenberg systems. Finally, the conditions and computation methods for Weyl-Heisenberg systems are worked out in a fourth domain, viz. the Zak transform domain; for time-continuous Weyl-Heisenberg systems it is required here that the product $\Delta \cdot b$ of the two shift parameters, see (1.1.1) and (1.1.15), is rational.

There are some good reasons for working out the duality condition in various domains. First of all there is the issue of computational advantages in calculating (minimal) dual systems which can be easier in one domain than in the other. For instance, as we shall see in Subsec. 1.4.3, the frame operator S_g (time-continuous case) has the representation

$$S_g = \frac{1}{ab} \sum_{k,l} \langle g, U_{kl}\, g\rangle\, U_{kl} \tag{1.1.18}$$

in the time-frequency domain, where U_{kl} is the time-frequency shift operator

$$(U_{kl}\, f)(t) = e^{2\pi i l t/a}\, f(t - k/b)\ ,\ t \in \mathbb{R}\ , \tag{1.1.19}$$

defined for $f \in L^2(\mathbb{R})$. It can be expected, certainly when $ab \ll 1$, that the representation (1.1.18) is more efficient than the "direct" representation (1.1.8) with g_{nm} given by (1.1.17). Hence computation and inversion of the frame operator is more easily done by using (1.1.18) than by using (1.1.8). Also, the advantages can be of a more algebraic nature, in the sense that the matrices and operators one has to calculate with are sparse or structured.

A second reason to consider duality in different domains arises when one wants to use (1.1.3) as an efficient way to represent signals f by means of the expansion coefficient $\langle f, \gamma_{nm} \rangle$. In Subsec. 1.6.2 we shall present a number of constraints, as they may arise in certain data-storage and -transmission applications, to which such a representation method may be subjected. (Note that for applications of this type, it is almost pointless to want to calculate dual systems at high speed: it is the signal processing involved in using these systems that should be fast.) Now the constraints just mentioned are of several types. There are hard ones, such as the condition of duality, and there are soft ones, such as the condition of good frequency discrimination or smoothness of the constituent filter impulse responses. Also, the constraints may have been formulated in various domains (notably, but not exclusively, in the time domain and in the frequency domain). A possible strategy to design filter impulse responses satisfying some or many of the constraints could be to introduce a cost functional that incorporates soft constraints, and to minimize this functional under the condition that hard constraints are satisfied. Since one may have to switch here iteratively from one domain to another, it would be very convenient when the hard constraints were formulated in any of the domains in which the soft constraints are formulated. As to the duality constraint we have been successful in this respect.

Most of the results presented here are proved somewhere in the literature, quite often in a form that is accessible for most of the readers. This is perhaps not so for some of the results for the time-discrete case. However, usually the proofs of the latter results consist of slight adaptations of the proofs for the corresponding time-continuous results. The reason for nevertheless including the results (but not their proofs) for the time-discrete case is that the formulation of these results does not always follow straightforwardly from their time-continuous counterparts. For these reasons most of the proofs are omitted (with appropriate references, however, whenever possible). An exception has been made for the material presented in Sec. 1.2 on shift-invariant systems, that was also covered for the most part by Ron and Shen in [RS95b], and for the proof of the fundamental identity in Sec. refs:4 on which the Wexler-Raz approach is based.

1.2 Time-continuous shift-invariant systems

In this section we consider time-continuous shift-invariant systems, i.e. we take $\mathcal{I} = \mathbb{Z}$, $\Delta = a > 0$, and we have $g_m, \gamma_m \in L^2(\mathbb{R})$ in (1.1.1), (1.1.2). We shall present, in the frequency domain, equivalent conditions for a system g_{nm}, $(n,m) \in \mathbb{Z}^2$, to have a finite frame upper bound and to be a frame, an equivalent condition for two systems g_{nm}, $(n,m) \in \mathbb{Z}^2$, and γ_{nm}, $(n,m) \in \mathbb{Z}^2$, to be dual, a characterization of the minimal dual system, and a resulting algorithm to compute the minimal dual system. Furthermore, we give a representation result for the frame operator in the frequency domain, and we present a link with polyphase operators. Some of the results in this section can be found in the recent paper [RS95b] by Ron and Shen. However, their presentation is rather different from ours, and we have not been able to find the results on frame operator representation and on computation of minimal dual systems, so we have chosen to give all proofs here.

We first need some preparation. We consider $L^2(\mathbb{R})$ with the inner product norm $\|f\| = \langle f,f\rangle^{1/2}$, where

$$\langle f,h\rangle = \int_{-\infty}^{\infty} f(t)\, h^*(t)\, dt \ , \quad f,h \in L^2(\mathbb{R}) \ . \tag{1.2.1}$$

Also, we let T_x be the translation operator

$$(T_x\, f)(t) = f(t+x) \ , \quad t \in \mathbb{R}, \ \ f \in L^2(\mathbb{R}) \ , \tag{1.2.2}$$

where $x \in \mathbb{R}$ (since we use T_x only in Prop. 2.1 below, there is no confusion possible with the analysis operator T_g of (1.1.5)). Finally, we denote for $h \in L^2(\mathbb{R})$ by $\hat{h} = \mathcal{F}h$ the Fourier transform

$$\hat{h}(\nu) = (\mathcal{F}h)(\nu) = \int_{-\infty}^{\infty} e^{-2\pi i\nu t}\, h(t)\, dt \ , \quad \text{a.e. } \nu \in \mathbb{R} \ . \tag{1.2.3}$$

Proposition 1.2.1 *Assume that the systems $g_{nm} = g_m(\cdot - na)$, $(n,m) \in \mathbb{Z}^2$, and $\gamma_{nm} = \gamma_m(\cdot - na)$, $(n,m) \in \mathbb{Z}^2$, have finite frame upper bounds B_g, B_γ (the systems do not need to be related by duality). Then*

$$\sum_m |\hat{g}_m(\nu)|^2 \le a\, B_g \ , \ \sum_m |\hat{\gamma}_m(\nu)|^2 \le a\, B_\gamma \ , \quad \text{a.e. } \nu \in \mathbb{R} \ . \tag{1.2.4}$$

When $f,h \in L^2(\mathbb{R})$, the function

$$\rho(f,h)(x) = \sum_{n,m} \langle T_x\, f, g_{nm}\rangle \langle \gamma_{nm}, T_x\, h\rangle \tag{1.2.5}$$

is continuous and periodic in x with period a, and has the Fourier series

$$\rho(f,h)(x) \sim \sum_k c_k\, e^{-2\pi ikx/a} \ , \tag{1.2.6}$$

$$c_k = \frac{1}{a} \int_{-\infty}^{\infty} \hat{f}(\nu)\, \hat{h}^*(\nu + k/a) \sum_m \hat{g}_m^*(\nu)\, \hat{\gamma}_m(\nu + k/a)\, d\nu \ , \quad k \in \mathbb{Z} \ . \tag{1.2.7}$$

Proof: First assume that $f, h \in \mathcal{S}$ (Schwarz space of smooth and rapidly decaying functions). By absolute and bounded convergence of the series in (1.2.5) we have

$$c_k = \frac{1}{a} \int_0^a e^{2\pi ikx/a} \sum_{n,m} \langle f, g_m(\cdot - x - na)\rangle \langle h, \gamma_m(\cdot - x - na)\rangle^* \, dx =$$

$$= \frac{1}{a} \sum_m \int_{-\infty}^{\infty} e^{2\pi ikx/a} \langle f, g_m(\cdot - x)\rangle \langle h, \gamma_m(\cdot - x)\rangle^* \, dx \; . \tag{1.2.8}$$

Now for $m \in \mathbb{Z}$, $\mu \in \mathbb{R}$ we have

$$\int_{-\infty}^{\infty} e^{2\pi i\mu x} \langle f, g_m(\cdot - x)\rangle \langle h, \gamma_m(\cdot - x)\rangle^* \, dx = \tag{1.2.9}$$

$$= \int_{-\infty}^{\infty} e^{2\pi i\mu x} \left(\int_{-\infty}^{\infty} e^{2\pi i\nu x} \hat{f}(\nu)\, \hat{g}_m^*(\nu)\, d\nu \right) \cdot$$

$$\cdot \left(\int_{-\infty}^{\infty} e^{2\pi i\nu x} \hat{h}(\nu)\, \hat{\gamma}_m^*(\nu)\, d\nu \right)^* dx =$$

$$= \int_{-\infty}^{\infty} \hat{f}(\nu)\, \hat{g}_m^*(\nu) \left(\hat{h}(\nu + \mu)\, \hat{\gamma}_m^*(\nu + \mu) \right)^* d\nu \; ,$$

where we have applied Parseval's theorem to the functions $\hat{f} \cdot \hat{g}_m^* \in L^2(\mathbb{R})$ and $\hat{h}(\cdot + \mu)\, \hat{\gamma}_m^*(\cdot + \mu) \in L^2(\mathbb{R})$. Accordingly,

$$c_k = \frac{1}{a} \sum_m \int_{-\infty}^{\infty} \hat{f}(\nu)\, \hat{h}^*(\nu + k/a)\, \hat{g}_m^*(\nu)\, \hat{\gamma}_m(\nu + k/a)\, d\nu \; . \tag{1.2.10}$$

Now take $f = h$, $g_m = \gamma_m$, $k = 0$ in (1.2.10). Then

$$\frac{1}{a} \int_{-\infty}^{\infty} |\hat{f}(\nu)|^2 \sum_m |\hat{g}_m(\nu)|^2 \, d\nu = c_0 = \tag{1.2.11}$$

$$= \frac{1}{a} \int_0^a \sum_{n,m} |\langle T_x\, f, g_{nm}\rangle|^2 \, dx \le B_g \, ||f||^2 = B_g \int_{-\infty}^{\infty} |\hat{f}(\nu)|^2 \, d\nu \; .$$

From this the first (and, similarly, the second) inequality in (1.2.4) follows. Next, we interchange the sum and the integral in (1.2.10), and (1.2.7) follows.

For general $f, h \in L^2(\mathbb{R})$ we get (1.2.7) easily from a density argument (observe that the right-hand side integral in (1.2.7) converges absolutely). Finally, the continuity of $\rho(f,h)$ can be shown by using the frame upper bound conditions on the two systems g_{nm}, $(n,m) \in \mathbb{Z}^2$, and γ_{nm}, $(n,m) \in \mathbb{Z}^2$, together with the fact that $\|T_x f - f\| \to 0$ as $x \to 0$ when $f \in L^2(\mathbb{R})$. The proof is complete. ❑

Theorem 1.2.2 *Assume that the systems $g_{nm}, (n,m) \in \mathbb{Z}^2$, and γ_{nm}, $(n,m) \in \mathbb{Z}^2$, have finite frame upper bounds. Then the systems are dual in the sense that*

$$\langle f, h\rangle = \sum_{n,m} \langle f, g_{nm}\rangle \langle \gamma_{nm}, h\rangle \ , \ \ f, h \in L^2(\mathbb{R}) \ , \tag{1.2.12}$$

if and only if

$$\varphi_k(\nu) := \sum_m \hat{g}^*_m(\nu - k/a)\, \hat{\gamma}_m(\nu) = a\, \delta_{ko} \ , \ \text{a.e. } \nu \in \mathbb{R}, \ \ k \in \mathbb{Z} \ . \tag{1.2.13}$$

Proof: Assume that the two systems are dual, and let $f, h \in L^2(\mathbb{R})$. Then, see Prop. 2.1,

$$\rho(f,h)(x) = \langle f, h\rangle \ , \ \ x \in \mathbb{R} \ . \tag{1.2.14}$$

Hence by (1.2.7)

$$\int_{-\infty}^{\infty} \hat{f}(\nu - k/a)\, \hat{h}^*(\nu)\, \varphi_k(\nu)\, d\nu = a\, \delta_{ko} \int_{-\infty}^{\infty} \hat{f}(\nu)\, \hat{h}^*(\nu)\, d\nu \ , \ \ k \in \mathbb{Z} \ , \tag{1.2.15}$$

and (1.2.13) easily follows.

Conversely, assume that (1.2.13) holds. Then by (1.2.6)–(1.2.7) for $f, h \in L^2(\mathbb{R})$

$$\rho(f,h)(x) \sim \langle f, h\rangle \ . \tag{1.2.16}$$

Since both sides of (1.2.16) are continuous functions of x, they agree for all $x \in \mathbb{R}$. Taking $x = 0$ we get (1.2.12), and the proof is complete. ❑

To proceed we need some preparation and a definition. We let

$$H_g(\nu) := (\hat{g}_m(\nu - k/a))_{k\in\mathbb{Z}, m\in\mathbb{Z}} \ , \ \text{a.e. } \nu \in \mathbb{R} \ . \tag{1.2.17}$$

Proposition 1.2.3 *Assume that the system g_{nm}, $(n,m) \in \mathbb{Z}^2$, has a finite frame upper bound B_g. Then $H_g(\nu)$ of (1.2.17) defines for a.e. $\nu \in \mathbb{R}$ a* *bounded linear operator of $l^2(\mathbb{Z})$ with operator norm $\leq (a\,B_g)^{1/2}$. Explicitly, we have for a.e. $\nu \in \mathbb{R}$*

$$\sum_k \left| \sum_m \hat{g}_m(\nu - k/a)\,\beta_m \right|^2 \leq a\,B_g\,||\underline{\beta}||^2 \ , \underline{\beta} \in l^2(\mathbb{Z}) \ , \tag{1.2.18}$$

where $||\underline{\beta}|| = (\sum_m |\beta_m|^2)^{1/2}$ is the norm of $\underline{\beta} \in l^2(\mathbb{Z})$.

Proof: Let $\alpha_{nm} \neq 0$ for only finitely many n, m, and let

$$\alpha_m(\nu) = \sum_n \alpha_{nm}\, e^{-2\pi i n a \nu} \ , \ \nu \in \mathbb{R} \ . \tag{1.2.19}$$

Also let J be an interval of length $1/a$. Then we have

$$\int_J \sum_k \left|\sum_m \alpha_m(\nu)\,\hat{g}_m(\nu - k/a)\right|^2 d\nu = \int_{-\infty}^{\infty} \left|\sum_m \alpha_m(\nu)\,\hat{g}_m(\nu)\right|^2 d\nu$$
$$= \int_{-\infty}^{\infty} \left|\sum_{n,m} \alpha_{nm}\, e^{-2\pi i n a \nu}\,\hat{g}_m(\nu)\right|^2 d\nu = \left\|\sum_{n,m} \alpha_{nm}\, g_{nm}\right\|^2, \tag{1.2.20}$$

where we have used $1/a$-periodicity of the α_m and Parseval's theorem. Now, see below (1.1.6), the far right-hand side of (1.2.2)e is bounded by $B_g\,||\underline{\alpha}||^2$, and since

$$||\underline{\alpha}||^2 = \sum_{n,m} |\alpha_{nm}|^2 = a \int_J \sum_m |\alpha_m(\nu)|^2\, d\nu \ , \tag{1.2.21}$$

we get

$$\int_J \sum_k \left|\sum_m \alpha_m(\nu)\,\hat{g}_m(\nu - k/a)\right|^2 d\nu \leq a\,B_g \int_J \sum_m |\alpha_m(\nu)|^2\, d\nu \ . \tag{1.2.22}$$

Next fix $\underline{\beta} \in l^2(\mathbb{Z})$ with $\beta_m \neq 0$ for only finitely many $m \in \mathbb{Z}$, and choose in (1.2.19)

$$\alpha_m(\nu) = \beta_m\,\varphi(\nu) \ ; \ \varphi(\nu) = \sum_n \varphi_n\, e^{-2\pi i n a \nu} \tag{1.2.23}$$

with $\varphi_n \neq 0$ for only finitely many $n \in \mathbb{Z}$. Then (1.2.22) gives

$$\int_J |\varphi(\nu)|^2 \sum_k \left|\sum_m \hat{g}_m(\nu - k/a)\,\beta_m\right|^2 d\nu \leq a\,B_g\,||\underline{\beta}||^2 \int_J |\varphi(\nu)|^2\, d\nu \ . \tag{1.2.24}$$

By varying φ over all allowed $\frac{1}{a}$-periodic functions, we see that

$$\sum_k \Big|\sum_m \hat{g}_m(\nu - k/a)\,\beta_m\Big|^2 \leq a\,B_g\,||\underline{\beta}||^2 \text{ , a.e. } \nu \in J, \tag{1.2.25}$$

where the null set involved in (1.2.25) may depend on $\underline{\beta}$.

Now let V be a dense, countable set in $l^2(\mathbb{Z})$ of $\underline{\beta}$'s with $\beta_m \neq 0$ for only finitely many $m \in \mathbb{Z}$, and let $N_1 \subset J$ be a null set outside of which

$$\sum_k \Big|\sum_m \hat{g}_m(\nu - k/a)\,\beta_m\Big|^2 \leq a\,B_g\,||\underline{\beta}||^2 \ , \ \underline{\beta} \in V \ . \tag{1.2.26}$$

Also, let $N_2 \subset J$ be a null set outside of which

$$\sum_m |\hat{g}_m(\nu - k/a)|^2 \leq a\,B_g \ , \ k \in \mathbb{Z} \ , \tag{1.2.27}$$

see Prop. 2.1. Then take $\underline{\beta} \in l^2(\mathbb{Z})$, and let $\underline{\beta}^{(M)} \in V$ such that $||\underline{\beta}^{(M)} - \underline{\beta}|| \to 0$ as $M \to \infty$. When $\nu \notin N_1 \cup N_2$, we have by Fatou's lemma

$$\begin{aligned}\sum_k \Big|\sum_m \hat{g}_m(\nu - k/a)\,\beta_m\Big|^2 &= \sum_k \liminf_{M\to\infty} \Big|\sum_m \hat{g}_m(\nu - k/a)\,\beta_m^{(M)}\Big|^2 \\ &\leq \liminf_{M\to\infty} \sum_k \Big|\sum_m \hat{g}_m(\nu - k/a)\,\beta_m^{(M)}\Big|^2 \leq a\,B_g\,||\underline{\beta}||^2 \ .\end{aligned} \tag{1.2.28}$$

We have shown now that for any interval J of length $1/a$ there is a null set N $(= N_1 \cup N_2)$ such that

$$\sum_k \Big|\sum_m \hat{g}_m(\nu - k/a)\,\beta_m\Big|^2 \leq a\,B_g\,||\underline{\beta}||^2 \ , \ \underline{\beta} \in l^2(\mathbb{Z})\,,\ \nu \in J \setminus N \ . \tag{1.2.29}$$

This completes the proof. ❑

We are now ready to give a representation result in the frequency domain of the frame operator

$$S_g\,f = \sum_{n,m} \langle f, g_{nm}\rangle\, g_{nm} \ , \ f \in L^2(\mathbb{R}) \ . \tag{1.2.30}$$

For the case of Weyl-Heisenberg frames the formulas (1.2.31)–(1.2.32) below are also known as the Walnut representation of the frame operator, see [Wal92], Prop. 2.4.

Theorem 1.2.4 *Assume that the system g_{nm}, $(n,m) \in \mathbb{Z}^2$, has a finite frame upper bound B_g, and let $f \in L^2(\mathbb{R})$. Then we have*

$$\widehat{S_g\, f}(\nu) = \frac{1}{a} \sum_k d_k(\nu)\, \hat{f}(\nu - k/a) \tag{1.2.31}$$

with absolute convergence for a.e. $\nu \in \mathbb{R}$, where, see (1.2.17),

$$d_k(\nu) = \big(H_g(\nu)\, H_g^*(\nu)\big)_{ok} = \sum_m \hat{g}_m(\nu)\, \hat{g}_m^*(\nu - k/a)\ ,\ k \in \mathbb{Z}\ . \tag{1.2.32}$$

Proof: We have by Props. 2.1 and 2.3

$$\begin{aligned}\sum_k |d_k(\nu)|^2 &= \sum_k \Big|\sum_m \hat{g}_m(\nu)\, \hat{g}_m^*(\nu - k/a)\Big|^2 \le \\ &\le a\, B_g \sum_m |\hat{g}_m(\nu)|^2 \le (a\, B_g)^2\end{aligned} \tag{1.2.33}$$

for a.e. $\nu \in \mathbb{R}$. Since $\hat{f} \in L^2(\mathbb{R})$, so that $\sum_k |\hat{f}(\nu - k/a)|^2 < \infty$ for a.e. $\nu \in \mathbb{R}$, we see that the right-hand side of (1.2.31) is a.e. well-defined as an absolutely convergent series. Also, by the Cauchy-Schwarz inequality, the right-hand side of (1.2.31) represents an $L^2_{\text{loc}}(\mathbb{R})$-function.

Now let $h \in \mathcal{S}$. We shall show that

$$\langle S_g\, f, h\rangle = \frac{1}{a} \int_{-\infty}^{\infty} \Big(\sum_k d_k(\nu)\, \hat{f}(\nu - k/a)\Big)\, \hat{h}^*(\nu)\, d\nu\ . \tag{1.2.34}$$

From this and the above, the result follows by density of $\mathcal{S}$ in $L^2(\mathbb{R})$ and Parseval's theorem.

To show (1.2.34) we observe that by (1.2.33) and the Cauchy-Schwarz inequality

$$\begin{aligned}&\sum_k \int_{-\infty}^{\infty} |d_k(\nu)|\, |\hat{f}(\nu - k/a)|\, |\hat{h}(\nu)|\, d\nu \le \\ &\le a\, B_g \int_{-\infty}^{\infty} \Big(\sum_k |\hat{f}(\nu - k/a)|^2\Big)^{\frac{1}{2}}\, |h(\nu)|\, d\nu\ .\end{aligned} \tag{1.2.35}$$

The right-hand side of (1.2.35) is easily seen to be finite, whence the function $\rho(f,h)$ of Prop. 2.1 with $\gamma_m = g_m$, $m \in \mathbb{Z}$, has an absolutely convergent

Fourier series. Therefore

$$\langle S_g\, f, h\rangle = \rho(f,h)(0) = \sum_k c_k = \tag{1.2.36}$$

$$= \sum_k \frac{1}{a} \int_{-\infty}^{\infty} \hat{f}(\nu)\, \hat{h}^*(\nu + k/a) \sum_m \hat{g}_m^*(\nu)\, \hat{g}_m(\nu + k/a)\, d\nu$$

$$= \sum_k \frac{1}{a} \int_{-\infty}^{\infty} d_k(\nu)\, \hat{f}(\nu - k/a)\, \hat{h}^*(\nu)\, d\nu \ .$$

By (1.2.35) the series and the integral in the last member of (1.2.36) may be interchanged, and we arrive at (1.2.34). This completes the proof. ❑

We next present a basic result, also to be found in [RS95b], on frame bounds in terms of the operators $H_g(\nu)$ of (1.2.17).

Theorem 1.2.5 *Let $g_m \in L^2(\mathbb{R})$, $m \in \mathbb{Z}$, and let $A \geq 0$, $B < \infty$. Then*

$$A\,\|f\|^2 \leq \sum_{n,m} |\langle f, g_{nm}\rangle|^2 \leq B\,\|f\|^2 \ , \ f \in L^2(\mathbb{R}) \tag{1.2.37}$$

$$\Leftrightarrow$$

$$a\,A\,I \leq H_g(\nu)\, H_g^*(\nu) \leq a\,B\,I \ , \ \text{a.e. } \nu \in \mathbb{R} \ ,$$

where I denotes the identity operator of $l^2(\mathbb{Z})$.

Proof: When at least one of the members of (1.2.37) holds, we have that $H_g(\nu)\, H_g^*(\nu) \leq a\,B\,I$ by Prop. 2.3 since $H_g(\nu)$ and $H_g^*(\nu)$ have the same operator norm. Next let $f \in \mathcal{S}$ with $\hat{f}$ compactly supported, and let $m \in \mathbb{Z}$. There holds the $L^2([0, a^{-1}])$-convergent Fourier expansion

$$\sum_k \hat{g}_m(\nu - k/a)\, \hat{f}^*(\nu - k/a) \sim \sum_n c_{nm}\, e^{2\pi i n a \nu} \tag{1.2.38}$$

with

$$c_{nm} = a\langle g_{nm}, f\rangle \ , \ n \in \mathbb{Z} \ . \tag{1.2.39}$$

Thus, when J is an interval of length a^{-1},

$$\int_J \Big|\sum_k \hat{g}_m(\nu - k/a)\, \hat{f}^*(\nu - k/a)\Big|^2 d\nu = a \sum_n |\langle f, g_{nm}\rangle|^2 \ . \tag{1.2.40}$$

Denote for $\nu \in \mathbb{R}$

$$\underline{\hat{f}}(\nu) = \Big(\hat{f}(\nu - k/a)\Big)_{k\in\mathbb{Z}} \ . \tag{1.2.41}$$

Observe that for any interval J of length a^{-1} we have

$$\|f\|^2 = \int_J \|\underline{\hat{f}}(\nu)\|^2 \, d\nu \,. \tag{1.2.42}$$

Then, when at least one of the members of (1.2.37) holds, we see from (1.2.40) that for any interval J of length a^{-1}

$$\begin{aligned}\int_J \|H_g^*(\nu)\underline{\hat{f}}(\nu)\|^2 d\nu &= \sum_m \int_J \Big|\sum_k \hat{g}_m(\nu - k/a)\hat{f}^*(\nu - k/a)\Big|^2 d\nu \\ &= a \sum_{n,m} |\langle f, g_{nm}\rangle|^2 \,.\end{aligned} \tag{1.2.43}$$

$\Rightarrow$. By what has been said above we only have to show that $\|H_g^*(\nu)\,\underline{\beta}\|^2 \geq a\,A\,\|\underline{\beta}\|^2$ for $\underline{\beta} \in l^2(\mathbb{Z})$. Let $\hat{\varphi} \in \mathcal{S}$ be supported by an interval J of length a^{-1}, and let $\underline{\beta} \in l^2(\mathbb{Z})$ with $\beta_k \neq 0$ for only finitely many $k \in \mathbb{Z}$. For $\nu \in \mathbb{R}$ define $\hat{f}(\nu) := \beta_k\, \hat{\varphi}(\nu + k/a)$, where $k \in \mathbb{Z}$ is such that $\nu + k/a \in J$. This f satisfies

$$\underline{\hat{f}}(\nu) = \hat{\varphi}(\nu)\,\underline{\beta} \,, \quad \nu \in J \,. \tag{1.2.44}$$

Therefore we get from (1.2.43), the first member of (1.2.37) and (1.2.42) that

$$\begin{aligned}&\int_J |\hat{\varphi}(\nu)|^2 \, \|H_g^*(\nu)\,\underline{\beta}\|^2 \, d\nu = a \sum_{n,m} |\langle f, g_{nm}\rangle|^2 \geq a\,A\,\|f\|^2 \\ &= a\,A \int_J \|\underline{\hat{f}}(\nu)\|^2 \, d\nu = a\,A\,\|\underline{\beta}\|^2 \int_J |\hat{\varphi}(\nu)|^2 \, d\nu \,.\end{aligned} \tag{1.2.45}$$

By varying $\hat{\varphi}$ over all elements of $\mathcal{S}$ supported by J we get

$$\|H_g^*(\nu)\,\underline{\beta}\|^2 \geq a\,A\,\|\underline{\beta}\|^2 \,, \text{ a.e. } \nu \in J \,, \tag{1.2.46}$$

where the null set involved in (1.2.46) may depend on $\underline{\beta}$.

Now let V be a countable dense set in $l^2(\mathbb{Z})$ of $\underline{\beta}$'s such that $\beta_k \neq 0$ for only finitely many $k \in \mathbb{Z}$, and let $N_1 \subset J$ be a null set such that

$$\|H_g^*(\nu)\,\underline{\beta}\|^2 \geq a\,A\,\|\underline{\beta}\|^2 \,, \ \underline{\beta} \in V \,, \ \nu \in J \setminus N_1 \,. \tag{1.2.47}$$

Also, let $N_2 \subset J$ be a null set such that

$$\|H_g^*(\nu)\,\underline{\beta}\|^2 \leq a\,B\,\|\underline{\beta}\|^2 \,, \ \underline{\beta} \in l^2(\mathbb{Z}) \,, \ \nu \in J \setminus N_2 \,. \tag{1.2.48}$$

When now $\nu \in J \setminus (N_1 \cup N_2)$ and $\underline{\beta} \in l^2(\mathbb{Z})$, we choose $\underline{\beta}^{(M)} \in V$ with $||\underline{\beta}^{(M)} - \underline{\beta}|| \to 0$ as $M \to \infty$, and we conclude from (1.2.47) and (1.2.48) that

$$\begin{aligned} ||H_g^*(\nu)\,\underline{\beta}||^2 &= \lim_{M\to\infty} ||H_g^*(\nu)\,\underline{\beta}^{(M)}||^2 \geq a\,A \lim_{M\to\infty} ||\underline{\beta}^{(M)}||^2 \\ &= a\,A\,||\underline{\beta}||^2 \,. \end{aligned} \tag{1.2.49}$$

Then the proof of $\Rightarrow$ is easily completed.
$\Leftarrow$. Let $f \in \mathcal{S}$ with $\hat{f}$ boundedly supported and let J be an interval of length a^{-1}. Then (1.2.42) and (1.2.43) show that the quantity $a\sum_{n,m}|\langle f, g_{nm}\rangle|^2$ lies between $a\,A\,||f||^2$ and $a\,B\,||f||^2$. From this we get the first member of (1.2.37) and the proof is complete. ❑

We next present a result on the computation of minimal dual systems and a characterization of these systems in terms of generalized inverses.

Theorem 1.2.6 *Assume that the system g_{nm}, $(n,m) \in \mathbb{Z}^2$, is a frame, and denote by $\underline{c}(\nu) \in l^2(\mathbb{Z})$ for a.e. $\nu \in \mathbb{R}$ the least-norm solution $\underline{c} = (c_m)_{m\in\mathbb{Z}}$ of the linear system*

$$\sum_m \hat{g}_m(\nu - k/a)\, c_m = a\,\delta_{ko} \ , \ \ k \in \mathbb{Z} \,. \tag{1.2.50}$$

Then there holds

$$^\circ\hat{\gamma}_m(\nu) = c_m^*(\nu) \ , \ \ m \in \mathbb{Z} \, , \ \text{a.e. } \nu \in \mathbb{R} \,. \tag{1.2.51}$$

Proof: When $\underline{e} = (\delta_{ko})_{k\in\mathbb{Z}}$, the least-norm solution

$$\underline{c}(\nu) = a\,H_g^*(\nu)\,\left(H_g(\nu)\,H_g^*(\nu)\right)^{-1}\,\underline{e} \tag{1.2.52}$$

of $H_g(\nu)\,\underline{c} = a\,\underline{e}$ is explicitly given by

$$c_m(\nu) = a\sum_k \left(\left(H_g(\nu)\,H_g^*(\nu)\right)^{-1}\right)_{ko} \hat{g}_m^*(\nu - k/a) \ ,$$
$$m \in \mathbb{Z} \, , \ \text{a.e. } \nu \in \mathbb{R} \,. \tag{1.2.53}$$

On the other hand by Theorem 2.4

$$\hat{g}_m(\nu) = \widehat{S_g\,{}^\circ\gamma_m}(\nu) = \frac{1}{a}\sum_k \left(H_g(\nu)\,H_g^*(\nu)\right)_{ok}\, {}^\circ\hat{\gamma}_m(\nu - k/a) \ ,$$
$$m \in \mathbb{Z} \, , \ \text{a.e. } \nu \in \mathbb{R}. \tag{1.2.54}$$

Now using the fact that for all $l, k \in \mathbb{Z}$

$$\left(H_g(\nu - l/a)\,H_g^*(\nu - l/a)\right)_{ok} = \left(H_g(\nu)\,H_g^*(\nu)\right)_{l,l+k} \ , \ \text{a.e. } \nu \in \mathbb{R} \ , \tag{1.2.55}$$

we get for all $l \in \mathbb{Z}$

$$\hat{g}_m(\nu - l/a) = \frac{1}{a} \sum_k \left(H_g(\nu)\, H_g^*(\nu)\right)_{lk} {}^\circ\hat{\gamma}_m(\nu - k/a) \ , \quad m \in \mathbb{Z} \ , \text{ a.e. } \nu \in \mathbb{R} \ . \tag{1.2.56}$$

Letting for $m \in \mathbb{Z}$

$$\underline{\hat{g}}_m(\nu) = (\hat{g}_m(\nu - l/a))_{l\in\mathbb{Z}} \ , \ {}^\circ\underline{\hat{\gamma}}_m(\nu) = ({}^\circ\hat{\gamma}_m(\nu - k/a))_{k\in\mathbb{Z}} \ , \quad \text{a.e. } \nu \in \mathbb{R} \ , \tag{1.2.57}$$

we thus have

$$\underline{\hat{g}}_m(\nu) = \frac{1}{a}\, H_g(\nu)\, H_g^*(\nu)\, {}^\circ\underline{\hat{\gamma}}_m(\nu) \ , \ m \in \mathbb{Z} \ , \text{ a.e. } \nu \in \mathbb{R} \ . \tag{1.2.58}$$

That is

$${}^\circ\underline{\hat{\gamma}}_m(\nu) = a \left(H_g(\nu)\, H_g^*(\nu)\right)^{-1} \underline{\hat{g}}_m(\nu) \ , \quad m \in \mathbb{Z} \ , \text{ a.e. } \nu \in \mathbb{R} \ . \tag{1.2.59}$$

When we write out (1.2.59) for the 0^{th} coordinate, we get

$${}^\circ\hat{\gamma}_m(\nu) = a \sum_k \left(\left(H_g(\nu)\, H_g^*(\nu)\right)^{-1}\right)_{ok} \hat{g}_m(\nu - k/a) \ , \quad m \in \mathbb{Z} \ , \text{ a.e. } \nu \in \mathbb{R} \ . \tag{1.2.60}$$

The right-hand side of (1.2.60) equals $c_m^*(\nu)$ by (1.2.52) (observe that $H_g(\nu)\, H_g^*(\nu)$ is Hermitean), and the proof is complete. ❑

Theorem 1.2.7 *Assume that the systems $g_{nm}, (n,m) \in \mathbb{Z}^2$, and γ_{nm}, $(n,m) \in \mathbb{Z}^2$, have finite frame upper bounds. Then they are dual if and only if*

$$H_g(\nu)\, H_\gamma^*(\nu) = a\, I \ , \ \text{a.e. } \nu \in \mathbb{R} \ . \tag{1.2.61}$$

Moreover,

$$H_{\circ\gamma}^*(\nu) = a\, H_g^*(\nu) \ \left(H_g(\nu)\, H_g^*(\nu)\right)^{-1} \ , \ \text{a.e. } \nu \in \mathbb{R} \ , \tag{1.2.62}$$

so that

$$\frac{1}{a}\, H_{\circ\gamma}(\nu)\, H_{\circ\gamma}^*(\nu) = \left(\frac{1}{a}\, H_g(\nu)\, H_g^*(\nu)\right)^{-1} \ , \ \text{a.e. } \nu \in \mathbb{R} \ . \tag{1.2.63}$$

Proof: We can write (1.2.61) as

$$\sum_m \hat{g}_m(\nu - k/a)\, \hat{\gamma}^*_m(\nu - l/a) = a\, \delta_{kl}\ ,\ \ k, l \in \mathbb{Z}\ ,\ \text{a.e. } \nu \in \mathbb{R}\ . \tag{1.2.64}$$

This is just (1.2.13) applied with $\nu - l/a$ instead of ν.

To show (1.2.62) we just note that (1.2.59) implies that

$$H_{\circ\gamma}(\nu) = a\left(H_g(\nu)\, H^*_g(\nu)\right)^{-1} H_g(\nu)\ ,\ \text{a.e. } \nu \in \mathbb{R}\ , \tag{1.2.65}$$

and (1.2.62) follows upon conjugation. Finally, (1.2.63) is a direct consequence of (1.2.62), and the proof is complete.
❑

As a consequence of (1.2.63) and Theorem 2.4 we see that the inverse frame operator $S_g^{-1} = S_{\circ\gamma}$ corresponding to the minimal dual system has the representation

$$\widehat{S_{\circ\gamma}\, f}(\nu) = \sum_k \left(\frac{1}{a}\, H_g(\nu)\, H^*_g(\nu)\right)^{-1}_{ok} \hat{f}(\nu - k/a)\ ,\ \text{a.e. } \nu \in \mathbb{R}\ , \tag{1.2.66}$$

when $f \in L^2(\mathbb{R})$.

We conclude this section by presenting a relation with what might be called polyphase operators because of their analogy with the polyphase matrices for the filter banks of Subsec. 1.6.2. Define for $l \in \mathbb{Z}$ the operator $P(l)$ by

$$P(l) : w \in L^2\left([0, a)\right) \to \left(\int_0^a g_m(t - la)\, w(t)\, dt\right)_{m\in\mathbb{Z}} \in l^2(\mathbb{Z})\ , \tag{1.2.67}$$

and put

$$P_g(e^{2\pi i\nu}) = \sum_{l=-\infty}^{\infty} P(l)\, e^{2\pi i l\nu}\ ,\ \text{a.e. } \nu \in \mathbb{R}\ . \tag{1.2.68}$$

Thus for $w \in L^2\left([0, a)\right)$

$$P_g(e^{2\pi i\nu})\, w = \left(\int_0^1 (\mathcal{Z}_a\, g_m)(s, \nu)\, w_a(s)\, ds\right)_{m\in\mathbb{Z}}\ . \tag{1.2.69}$$

Here $\mathcal{Z}_a$ is a Zak transform, defined for $h \in L^2(\mathbb{R})$ by

$$(\mathcal{Z}_a\, h)(s, \nu) = a^{1/2} \sum_{l=-\infty}^{\infty} h\left(a(s - l)\right)\, e^{2\pi i l\nu}\ ,\ \text{a.e. } s, \nu \in \mathbb{R}\ , \tag{1.2.70}$$

see Sec. 1.5, and $w_a(s) = a^{1/2}\, w(as)$. Now P_g and H_g are related to one another according to

$$H_g^T(\nu) = a^{1/2}\, P_g(e^{2\pi i a\nu})\, D(\nu)\, \mathcal{F}_a^{-1} \ , \tag{1.2.71}$$

with $\mathcal{F}_a^{-1}$ and $D(\nu)$ the unitary operators defined by

$$\mathcal{F}_a^{-1} : \underline{x} \in l^2(\mathbb{Z}) \to a^{-1/2} \sum_{k=-\infty}^{\infty} e^{2\pi i k t/a}\, x_k \in L^2\left([0,a)\right) \ , \tag{1.2.72}$$

$$D(\nu) : w \in L^2\left([0,a)\right) \to e^{-2\pi i\nu t}\, w(t) \in L^2\left([0,a)\right) \ . \tag{1.2.73}$$

1.3 Weyl-Heisenberg systems as shift-invariant systems

In this section we apply the results of Sec. refs:2 to shift-invariant systems of the Weyl-Heisenberg type in which we have

$$g_m(t) = e^{2\pi i m b t}\, g(t) \ , \ t \in \mathbb{R}, \ m \in \mathbb{Z} \ , \tag{1.3.1}$$

with $g \in L^2(\mathbb{R})$ and $b > 0$ (the factor $\exp(-2\pi i n m a b)$ is therefore included in $g_{nm}(t)$, see (1.1.17), but does not play a significant role).

1.3.1 Frequency-domain results

Since

$$\mathcal{F}\left(e^{2\pi i m b t}\, h(t)\right)(\nu) = \hat{h}(\nu - mb) \ , \ \text{a.e. } \nu \in \mathbb{R}, \ m \in \mathbb{Z} \ , \tag{1.3.2}$$

when $h \in L^2(\mathbb{R})$, we see that two Weyl-Heisenberg systems g_{nm}, $(n,m) \in \mathbb{Z}^2$, and γ_{nm}, $(n,m) \in \mathbb{Z}^2$, both having a finite frame upper bound, are dual if and only if

$$\sum_m \hat{g}^*(\nu - mb - k/a)\, \hat{\gamma}(\nu - mb) = a\, \delta_{ko} \ , \ \text{a.e. } \nu \in \mathbb{R}, \ k \in \mathbb{Z} \ . \tag{1.3.3}$$

Observing that the left-hand side of (1.3.3) is periodic in ν with period b, one can show, by computing Fourier coefficients, that the two systems (when both have a finite frame upper bound) are dual if and only if

$$\langle \gamma, U_{kl}\, g \rangle = ab\, \delta_{ko}\, \delta_{lo} \ , \ k, l \in \mathbb{Z} \ , \tag{1.3.4}$$

where

$$(U_{kl}\, h)(t) = e^{2\pi i l t/a}\, h(t-k/b)\ ,\ \text{a.e. } t\in\mathbb{R},\ (k,l)\in\mathbb{Z}^2\ , \tag{1.3.5}$$

for $h \in L^2(\mathbb{R})$. Formula (1.3.4) is the well-known Wexler-Raz biorthogonality condition for dual Weyl-Heisenberg systems, see [WR90, Jan94b], Prop. A, [Jan95b], (1.12), [DLL95], Sec. 3, [RS95c], Subsec. 3.3.

An important consequence of (1.3.4) is the following result: the Weyl-Heisenberg system g_{nm}, $(n,m)\in\mathbb{Z}^2$, can be a frame only if $ab \le 1$. Indeed, when the system is a frame, then it follows from minimality, see (1.1.14), that $|(g, {}^\circ\gamma)| \le 1$ by comparing the two expansions

$$g = \sum_{n,m} \langle g, {}^\circ\gamma_{nm}\rangle\, g_{nm} = \sum_{n,m} \delta_{no}\,\delta_{mo}\, g_{nm}\ . \tag{1.3.6}$$

And then (1.3.4) with $l = k = 0$ shows that $ab \le 1$. We note that this argument, which was presented in [Jan94b], end of Sec. 1, has been extended in [Janar] to more general shift-invariant systems g_{nm}, $(n,m)\in\mathbb{Z}^2$, where the g_m, $m\in\mathbb{Z}$, have certain frequency localization properties.

As a consequence of Theorem 2.4 we have for the Weyl-Heisenberg system g_{nm}, $(n,m)\in\mathbb{Z}^2$, the frame operator representation

$$\widehat{S_g\, f}(\nu) = \frac{1}{a}\sum_k d_k(\nu)\,\hat f(\nu-k/a) \tag{1.3.7}$$

when $f\in L^2(\mathbb{R})$ with absolute convergence for a.e. $\nu\in\mathbb{R}$. Here

$$d_k(\nu) = \sum_m \hat g(\nu-mb)\,\hat g^*(\nu-mb-k/a)\ . \tag{1.3.8}$$

We observe that the convergence of the right-hand side of (1.3.7) is in $L^2(\mathbb{R})$-sense when there is a $c>0$ such that

$$\sum_k \operatorname*{ess\,sup}_{\nu\in[0,c)} |\hat g(\nu-kc)| < \infty \tag{1.3.9}$$

(Wiener amalgam space, see [Wal92], Sections 1 and 2). The representation (1.3.7) is also known as the Walnut representation of frame operators.

As a consequence of Theorem 2.5 we have that the Weyl-Heisenberg system g_{nm}, $(n,m)\in\mathbb{Z}^2$, is a frame with frame bounds A, B if and only if

$$a\,A\,I \le H_g(\nu)\,H_g^*(\nu) \le a\,B\,I\ ,\ \text{a.e. } \nu\in\mathbb{R}\ , \tag{1.3.10}$$

with

$$H_g(\nu) = (\hat g(\nu-mb-k/a))_{k\in\mathbb{Z},m\in\mathbb{Z}}\ ,\ \text{a.e. } \nu\in\mathbb{R}\ . \tag{1.3.11}$$

The minimal dual system ${}^{\circ}\gamma_{nm}$, $(n,m) \in \mathbb{Z}^2$, can be computed according to Theorem 2.6 as

$$ {}^{\circ}\hat{\gamma}(\nu) = a \sum_k \left(\left(H_g(\nu)\, H_g^*(\nu) \right)^{-1} \right)_{ok} \hat{g}(\nu - k/a) \ , \ \text{a.e. } \nu \in \mathbb{R} \ . \quad (1.3.12) $$

Moreover, Theorem 2.6 shows that for any other dual system γ_{nm}, $(n,m) \in \mathbb{Z}^2$, there holds for a.e. $\nu \in \mathbb{R}$

$$ \sum_{m=-\infty}^{\infty} |{}^{\circ}\hat{\gamma}(\nu - mb)|^2 \leq \sum_{m=-\infty}^{\infty} |\hat{\gamma}(\nu - mb)|^2 \ , \quad (1.3.13) $$

with equality if and only if ${}^{\circ}\hat{\gamma}(\nu - mb) = \hat{\gamma}(\nu - mb)$ for $m \in \mathbb{Z}$. By integration of (1.3.13) over $\nu \in [0, b)$ and using Parseval's theorem we see that

$$ \|{}^{\circ}\gamma\|^2 \leq \|\gamma\|^2 \quad (1.3.14) $$

with equality if and only if ${}^{\circ}\gamma = \gamma$ a.e. Hence the minimal dual ${}^{\circ}\gamma$ has the least energy among all duals γ.

We conclude this subsection by considering the case $ab = 1$ in some more detail (this case is called the critically sampled case since a Weyl-Heisenberg system can only be a frame when $ab \leq 1$). In that case the operator $H_g(\nu)$ in (1.3.11) is Toeplitz, and so is $H_g(\nu)\, H_g^*(\nu)$ in (1.3.10). There holds

$$ \left(H_g(\nu)\, H_g^*(\nu) \right)_{kl} = \sum_{m=-\infty}^{\infty} \hat{g}(\nu - (m+k)\, b)\, \hat{g}^*(\nu - (m+l)\, b) \ , $$

$$ \text{a.e. } \nu \in \mathbb{R} \ . \quad (1.3.15) $$

It follows from the theory of Toeplitz operators of $l^2(\mathbb{Z})$ that the spectrum of $H_g(\nu)\, H_g^*(\nu)$ is contained in the interval $[m(\nu), M(\nu)]$ with

$$ m(\nu) = \operatorname*{ess\,inf}_{\vartheta} F_g(\vartheta, \nu), M(\nu) = \operatorname*{ess\,sup}_{\vartheta} F_g(\vartheta, \nu), \quad (1.3.16) $$

where

$$ F_g(\vartheta, \nu) = \sum_{k=-\infty}^{\infty} \left(H_g(\nu)\, H_g^*(\nu) \right)_{ko} e^{2\pi i k \vartheta} = \quad (1.3.17) $$

$$ = \left| \sum_{m=-\infty}^{\infty} \hat{g}(\nu - mb)\, e^{2\pi i m \vartheta} \right|^2 \ , \ \text{a.e. } \vartheta \in \mathbb{R} \ . $$

Here we need $\nu \in \mathbb{R}$ such that $\sum_m |\hat{g}(\nu - mb)|^2 < \infty$ (this holds for a.e. $\nu \in \mathbb{R}$). When there are $m > 0$, $M < \infty$ such that

$$ m \leq m(\nu) \leq M(\nu) \leq M \ , \ \text{a.e. } \nu \in \mathbb{R} \ , \quad (1.3.18) $$

we have that g_{nm}, $(n,m) \in \mathbb{Z}^2$, is a frame, and $H_g(\nu)$ is invertible for a.e. $\nu \in \mathbb{R}$, so that there is only one dual γ.

1.3.2 Time-domain results

The results of Sec. 1.3.1 have consequences in the time-domain as well since

$$\mathcal{F}\left[e^{2\pi imb(t-na)}\, g(t-na)\right](\nu) = e^{-2\pi ina\nu}\, \hat{g}(\nu - mb)\ . \tag{1.3.19}$$

Thus (ignoring factors $\exp(2\pi inmab)$) when g_{nm}, $(n,m) \in \mathbb{Z}^2$, is a Weyl-Heisenberg system with shift parameters a, b, then $\hat{g}_{nm}$, $(n,m) \in \mathbb{Z}^2$, is such a system with shift parameters b, a. In particular, we have that the systems g_{nm}, $(n,m) \in \mathbb{Z}^2$, and γ_{nm}, $(n,m) \in \mathbb{Z}^2$, are dual (for the shift parameters a, b) if and only if

$$\sum_n g(t + na - l/b)\, \gamma^*(t + na) = b\, \delta_{lo}\ ,\ \text{a.e. } t \in \mathbb{R},\ l \in \mathbb{Z}\ . \tag{1.3.20}$$

And the minimal dual ${}^\circ\gamma$ can be computed as

$${}^\circ\gamma(t) = b \sum_l \left(M_g(t)\, M_g^*(t)\right)_{ol}^{-1} g(t - l/a)\ ,\ \text{a.e. } t \in \mathbb{R} \tag{1.3.21}$$

where

$$M_g(t) = (g(t + na - l/b))_{l\in\mathbb{Z},n\in\mathbb{Z}}\ . \tag{1.3.22}$$

Let us consider the problem of computing dual γ's in the case that g has bounded support. We ask ourselves whether it is possible to have a dual γ with bounded support as well. In the case of critical sampling ($ab = 1$) this is only possible when $M_g(t)$ (a Toeplitz operator) is a diagonal matrix for a.e. $t \in \mathbb{R}$. Indeed we have then that $M_g(t)$ and $M_\gamma^*(t)$ are each other's inverse, and banded Toeplitz operators of $l^2(\mathbb{Z})$ have unbanded inverses, except in the case of diagonal matrices. When, however, $ab < 1$ the situation is different.

Consider the case that $b^{-1} = qa$ where $q \in \mathbb{N}$, $q \geq 2$, and that $g \in L^2(\mathbb{R})$ is supported by an interval $[0, ra]$ with $r \in \mathbb{N}$. We try to satisfy (1.3.20) with a γ supported by $[ua, va]$ where $u \leq 0$, $v \geq r$. Observe that we need to satisfy (1.3.20) for $0 \leq t < a$ only by periodicity of the left-hand side. Now for $t \in [0, a)$ we have

$$g(t + na - lqa)\, \gamma^*(t + na) = 0\ ,\ n \in \mathbb{Z}\ , \tag{1.3.23}$$

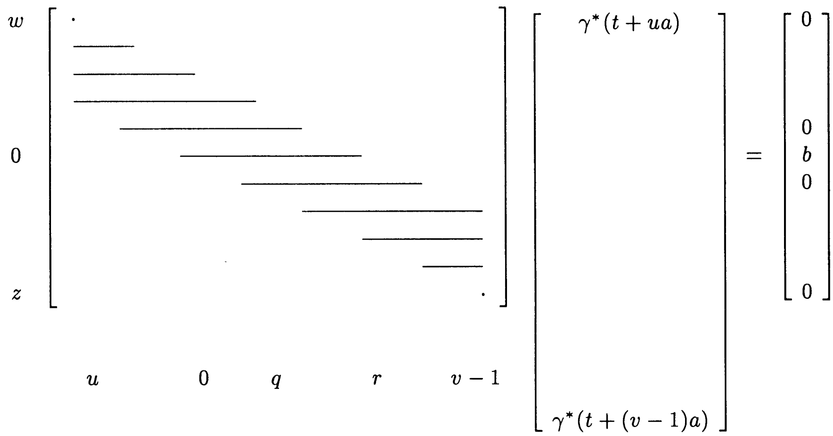
when $lq < u - r + 1$ or $lq > v - 1$, whence (1.3.20) holds for these l, and the condition (1.3.20) reduces for $t \in [0, a)$ to a linear system of

$$z - w + 1\ ;\ z = \left\lfloor \frac{v-1}{q} \right\rfloor,\ w = -\left\lfloor \frac{r-1-u}{q} \right\rfloor \tag{1.3.24}$$

equations in the $v-u$ unknowns $\gamma(t+na)$, $n=u,...,v-1$. Here we have denoted $\lfloor x \rfloor$ = largest integer $\leq x$. When the number in (1.3.24) $\leq v-u$, so that the coefficient matrix

$$M_g(t\,;\,u,v) = (g(t+na-lqa))_{l,u-r+1\leq lq\leq v-1,n=u,\dots,v-1} \tag{1.3.25}$$

has at least as many columns as rows, this system has a fair chance to be solvable. Schematically, the situation may be depicted as follows, where the row index l runs from top to bottom and the column index n runs from left to right:



Assuming (1.3.25) has full row rank for all $t\in[0,a)$, least-norm solutions ${}^\circ\gamma^{uv}$ are then found by letting

$$({}^\circ\gamma^{uv}(t+na))^*_{n=u,\dots,v-1} = b\,M_g^+(t\,;\,u,v)\,\underline{e}^{uv}\ ,t\in[0,a)\ , \tag{1.3.26}$$

with $M_g^+(t\,;\,u,v)$ the generalized inverse of $M_g(t\,;\,u,v)$ and $b\,\underline{e}^{uv}$ the right-hand side vector in the above scheme. Observe that for all dual γ^{uv} supported by $[ua,va]$ we have

$$\sum_{n=u}^{v-1} |{}^\circ\gamma^{uv}(t+na)|^2 \leq \sum_{n=u}^{v-1} |\gamma^{uv}(t+na)|^2\ ,\ t\in[0,a)\ , \tag{1.3.27}$$

whence $||{}^\circ\gamma^{uv}||^2 \leq ||\gamma^{uv}||^2$ by integration.

Consider the following example. Let $a=1$, $q=2$ (so that $b=1/2$), and let

$$g(t) = \begin{cases} \frac{1}{6}\sqrt{3}\,(1-\cos\frac{1}{2}\pi t) & 0\leq t\leq 4, \\ 0 & \text{otherwise,} \end{cases} \tag{1.3.28}$$

so that g is the familiar raised cosine window. For $M = 1, 2, \ldots$, denote

$$_M{}^\circ\gamma = {}^\circ\gamma^{-2M+2,2M+2} \tag{1.3.29}$$

with ${}^\circ\gamma^{uv}$ given by (1.3.26). We have displayed $_M{}^\circ\gamma$ for $M = 1, 2, 3, 4$ in Figure 1.3.1(a)-(e).

It now happens that in the present case one can determine the non-support restricted minimal ${}^\circ\gamma$ of (1.3.21) analytically, see [Jan96], Sec. 3. This yields

$$ {}^\circ\gamma(t+n) = \frac{3}{2\sqrt{2}} \sum_{k=\lfloor \frac{1}{2}n \rfloor - 1}^{\lfloor \frac{1}{2}n \rfloor} \left(\frac{-1}{3+2\sqrt{2}}\right)^k g(t+n-2k) \ , \qquad t \in [0,1) \, , \ n \in \mathbb{Z} \ , \tag{1.3.30}$$

and this ${}^\circ\gamma$ has been displayed in Fig. 1.3.1(e). Apparently $_M{}^\circ\gamma \to {}^\circ\gamma$ as $M \to \infty$.

1.4 Weyl-Heisenberg systems in the time-frequency domain

As in Sec. 1.3 we let $a > 0$, $b > 0$, and we consider Weyl-Heisenberg systems derived from a $g \in L^2(\mathbb{R})$. We use in this section the notation

$$g_{nm}(t) = g_{na,mb}(t) = e^{2\pi imbt} \, g(t - na) \ , \ t \in \mathbb{R} \ , \tag{1.4.1}$$

where

$$h_{x,y}(t) = e^{2\pi iyt} \, h(t - x) \ , \ t \in \mathbb{R} \ , \tag{1.4.2}$$

for $h \in L^2(\mathbb{R})$, $x \in \mathbb{R}$, $y \in \mathbb{R}$. We also consider the time-frequency shift operators U_{kl} defined for $k \in \mathbb{Z}, l \in \mathbb{Z}$ by

$$U_{kl} \, h = h_{k/b,l/a} \ , \ h \in L^2(\mathbb{R}) \ . \tag{1.4.3}$$

The proofs of the main results of this section can be found in [Jan95b, DLL95, RS95c], where it is noted that the approaches used in these references are quite different (indeed, [Jan95b, DLL95, RS95c] have been written independently and more or less simultaneously). We shall follow the approach in [Jan95b] which is based on what we call the Fundamental Identity below, also see [DLL95], Sec. 3. It is the only thing we prove in this section; its proof can be assembled from [Jan94b], proof of Prop. A and [Jan95b], Props. 2.3 and 2.4. The technique we use in this proof is due to Tolimieri and Orr, see [TO95], Sec. 2.

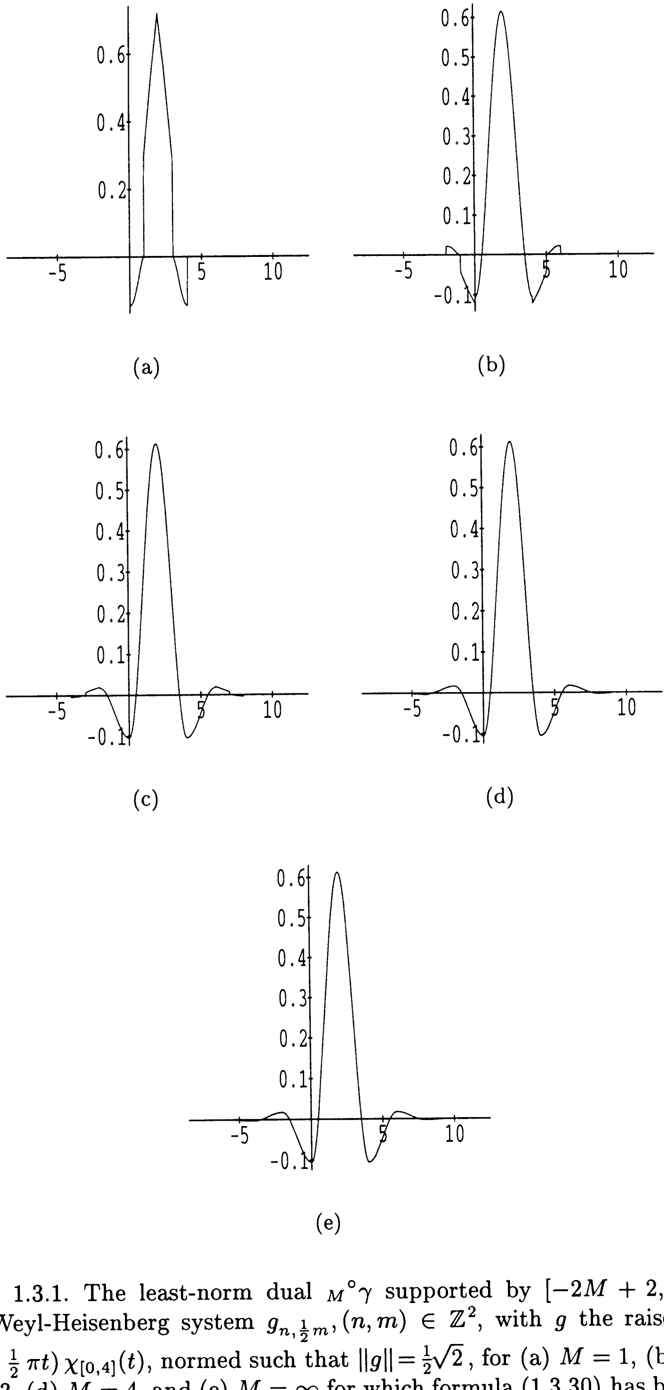


FIGURE 1.3.1. The least-norm dual ${}_M{}^\circ\gamma$ supported by $[-2M + 2, 2M + 2]$ for the Weyl-Heisenberg system $g_{n,\frac{1}{2}m}$, $(n, m) \in \mathbb{Z}^2$, with g the raised cosine $c(1 - \cos \frac{1}{2}\pi t)\,\chi_{[0,4]}(t)$, normed such that $\|g\| = \frac{1}{2}\sqrt{2}$, for (a) $M = 1$, (b) $M = 2$, (c) $M = 3$, (d) $M = 4$, and (e) $M = \infty$ for which formula (1.3.30) has been used. The author wishes to thank his colleague M. Maes for producing the figures.

1.4.1 Fundamental Identity

Let $f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)} \in L^2(\mathbb{R})$, and assume that at least one of $f^{(1)}$, $f^{(2)}$ and at least one of $f^{(3)}$, $f^{(4)}$ generates a Weyl-Heisenberg system (for the parameters a, b) with a finite frame upper bound. Also assume that

$$\sum_{k,l} |\langle f^{(3)}, f^{(2)}_{k/b,l/a}\rangle| \, |\langle f^{(1)}_{k/b,l/a}, f^{(4)}\rangle| < \infty \ . \tag{1.4.4}$$

Then

$$\sum_{n,m} \langle f^{(1)}, f^{(2)}_{na,mb}\rangle\langle f^{(3)}_{na,mb}, f^{(4)}\rangle = \frac{1}{ab} \sum_{k,l} \langle f^{(3)}, f^{(2)}_{k/b,l/a}\rangle\langle f^{(1)}_{k/b,l/a}, f^{(4)}\rangle \ . \tag{1.4.5}$$

Proof: Consider the function

$$H(x,y) = \sum_{n,m} \langle f^{(1)}_{x-na,y-mb}, f^{(2)}\rangle\langle f^{(3)}, f^{(4)}_{x-na,y-mb}\rangle \tag{1.4.6}$$

which is periodic in x and y with periods a and b, respectively. This H is continuous which follows from the finite frame upper bound assumption and the fact that

$$||f_{x,y} - f_{t,s}|| \to 0 \ , \ (x,y) \to (t,s) \ , \tag{1.4.7}$$

for $f \in L^2(\mathbb{R})$. Here it is also useful to note that

$$\begin{aligned} \langle f_{x-na,y-mb}, h\rangle &= \langle f_{x,y}, h_{na,mb}\rangle \, e^{-2\pi i yna + 2\pi i mnab} \\ &= \langle f_{-na,-mb}, h_{-x,-y}\rangle \, e^{-2\pi i mbx + 2\pi i xy} \end{aligned} \tag{1.4.8}$$

for $f, h \in L^2(\mathbb{R})$, $x, y \in \mathbb{R}$, and $n, m \in \mathbb{Z}$. Therefore, H has the Fourier series expansion

$$H(x,y) \sim \frac{1}{ab} \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} c_{uv} \, e^{-2\pi i ux/a - 2\pi i vy/b} \ , \tag{1.4.9}$$

where

$$c_{uv} = \int_0^a \int_0^b H(x,y) \, e^{2\pi i ux/a + 2\pi i vy/b} \, dx \, dy \ . \tag{1.4.10}$$

We shall show below that

$$c_{uv} = \langle f^{(1)}_{v/b,-u/a}, f^{(4)}\rangle\langle f^{(3)}, f^{(2)}_{v/b,-u/a}\rangle \ , \ u, v \in \mathbb{Z} \ . \tag{1.4.11}$$

From this and the assumption (1.4.4), we see that the continuous function H coincides everywhere with its absolutely convergent Fourier series, and then taking $x = y = 0$ in (1.4.9) we get the result.

We now show (1.4.11). By inserting (1.4.6) into (1.4.10) and rearranging sums and integrals (which is allowed by absolute and dominated convergence of the series in (1.4.6)) we get

$$c_{uv} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle f^{(1)}_{x,y}, f^{(2)} \rangle \langle f^{(3)}, f^{(4)}_{x,y} \rangle \, e^{2\pi i u x/a + 2\pi i v y/b} \, dx\, dy \tag{1.4.12}$$

with absolute convergence at the right-hand side. There holds

$$\langle f^{(1)}_{x,y}, f^{(2)} \rangle \, e^{2\pi i u x/a + 2\pi i v y/b} = \langle (f^{(1)}_{v/b,-u/a})_{x,y}, f^{(2)}_{v/b,-u/a} \rangle \tag{1.4.13}$$

for $x, y \in \mathbb{R}$ and $u, v \in \mathbb{Z}$. And now using the formula

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle k^{(1)}_{x,y}, k^{(2)} \rangle \langle k^{(3)}, k^{(4)}_{x,y} \rangle \, dx\, dy = \langle k^{(1)}, k^{(4)} \rangle \langle k^{(3)}, k^{(2)} \rangle \ , \tag{1.4.14}$$

valid for $k^{(1)}, k^{(2)}, k^{(3)}, k^{(4)} \in L^2(\mathbb{R})$, see [Fol89], Section 1.4, the formula (1.4.11) follows. This completes the proof.
❑

1.4.2 Wexler-Raz biorthogonality condition for duality

Assume that the two Weyl-Heisenberg systems $g_{na,mb}$, $(n, m) \in \mathbb{Z}^2$, and $\gamma_{na,mb}$, $(n, m) \in \mathbb{Z}^2$, have finite frame upper bounds. Then the systems are dual if and only if

$$\langle \gamma, g_{k/b,l/a} \rangle = ab\, \delta_{ko}\, \delta_{lo} \ , \quad k, l \in \mathbb{Z} \ . \tag{1.4.15}$$

A version of this result was first presented by Wexler and Raz [WR90] in 1990, and it has been applied since then extensively by many authors (among which Wexler and Raz themselves), see e.g. [Jan95b, FR94, QCL92, QC93, QC94]. A proof of the above result based on the Fundamental Identity in 1.4.1 can be found in [Jan94b]; also see Subsec. 1.3.1.

The condition (1.4.15) can be written as

$$U_g \, \gamma = \underline{\sigma} \ ; \ \underline{\sigma} = ab(\delta_{ko}\, \delta_{lo})_{k,l \in \mathbb{Z}} \ , \tag{1.4.16}$$

where U_g is the linear mapping of $L^2(\mathbb{R})$ defined by

$$U_g \, f = (\langle f, g_{k/b,l/a} \rangle)_{k,l \in \mathbb{Z}} \ , \quad f \in L^2(\mathbb{R}) \ . \tag{1.4.17}$$

Assuming the mapping U_g to be bounded and right-invertible (equivalently, $U_g U_g^*$ is bounded and invertible), one can compute a minimal energy $\gamma = {}^{oo}\gamma$ according to

$$ {}^{oo}\gamma = U_g^*(U_g U_g^*)^{-1} \underline{\sigma} \ , \tag{1.4.18} $$

or, more explicitly, as (see (1.4.3))

$$ {}^{oo}\gamma = ab \sum_{k,l} \left((U_g U_g^*)^{-1}\right)_{kl;oo} U_{kl}\, g \ . \tag{1.4.19} $$

Here we have used that

$$ U_g^* \underline{\alpha} = \sum_{k,l} \alpha_{kl}\, U_{kl}\, g \ , \ \underline{\alpha} \in l^2(\mathbb{Z}^2) \ . \tag{1.4.20} $$

Now one can ask whether the minimal dual ${}^{o}\gamma = S_g^{-1}\, g$ coincides with ${}^{oo}\gamma$ of (1.4.18). This is indeed so as we see from (1.3.14). A different approach to this result was presented in [DLL95], Sec. 4 and in [DLL95], Sec. 5. Yet another proof of the identity ${}^{o}\gamma = {}^{oo}\gamma$ as given in [Jan95b], Prop. 3.3 is based on the following result for frame bounds.

Theorem 1.4.1 *For any $A \geq 0$, $B < \infty$ there holds*

$$ A\,\|f\|^2 \leq \sum_{n,m} |\langle f, g_{na,mb}\rangle|^2 \leq B\,\|f\|^2 \ , \ f \in L^2(\mathbb{R}) \ , \tag{1.4.21} $$

$$ \Leftrightarrow $$

$$ A\, I \leq \frac{1}{ab} U_g U_g^* \leq B\, I \ , $$

where I is now the identity operator of $l^2(\mathbb{Z}^2)$.

We note that Theorem 4.1 holds for all $a, b > 0$, the most interesting case being $ab \leq 1$: for $ab > 1$ we must have $A = 0$. Observe also the following. Since U_g and U_g^* have the same operator norm, we have

$$ \frac{1}{ab} U_g U_g^* \leq B\, I \Leftrightarrow \forall_{f \in L^2(\mathbb{R})} \left[\sum_{k,l} |\langle f, g_{k/b,l/a}\rangle|^2 \leq ab\, B\, \|f\|^2 \right] . \tag{1.4.22} $$

Hence the system $g_{na,mb}$, $(n,m) \in \mathbb{Z}^2$, has the frame upper bound B_g if and only if the system $g_{k/b,l/a}$, $(k,l) \in \mathbb{Z}^2$, has the frame upper bound $ab\, B_g$. It is also seen from (1.4.22) that Theorem 4.1 is trivial for $ab = 1$.

The approach in (1.4.18) for computing a (minimal) dual γ can be generalized as follows. When X is a bounded positive definite linear operator of $L^2(\mathbb{R})$, the dual $\gamma = {}^{o}_{X}\gamma$ with minimal value of $\langle X\gamma, \gamma\rangle$ is given by

$$ {}^{o}_{X}\gamma = X^{-1}\, U_g^*(U_g\, X^{-1}\, U_g^*)^{-1}\, \underline{\sigma} \ . \tag{1.4.23} $$

This approach has been further generalized in [DLL95], Sec. 7, with worked out examples and explicit results, to include certain unbounded operators X as well.

1.4.3 Frame operator representation

Assume that the system $g_{na,mb}$, $(n,m) \in \mathbb{Z}^2$, has a finite frame upper bound. Then the frame operator S_g has the representation

$$S_g = \frac{1}{ab} \sum_{k,l} c_{kl}\, U_{kl} \ , \tag{1.4.24}$$

where

$$c_{kl} = (U_g\, U_g^*)_{kl;oo} = \langle g, g_{k/b,l/a} \rangle \ , \ k,l \in \mathbb{Z} \ , \tag{1.4.25}$$

in the sense that for any $f, h \in L^2(\mathbb{R})$ such that

$$\sum_{k,l} |\langle U_{kl}\, f, h \rangle|^2 < \infty \tag{1.4.26}$$

there holds

$$\langle S_g\, f, h \rangle = \frac{1}{ab} \sum_{k,l} c_{kl} \langle U_{kl}\, f, h \rangle \ . \tag{1.4.27}$$

In the case that g satisfies Tolimieri and Orr's condition A , see [TO95] Sec. 3,

$$E := \sum_{k,l} |\langle g, g_{k/b,l/a} \rangle| < \infty \ , \tag{1.4.28}$$

the system $g_{na,mb}$, $(n,m) \in \mathbb{Z}^2$, has the finite frame upper bound E/ab, and the representation (1.4.24) is unconditional.

For the proofs of these results we refer to [Jan95b], Props. 2.6 and 2.8. We observe that (1.4.24) is particularly useful when ab is small, for then the c_{kl} must be expected to decay rapidly with $k^2 + l^2 \to \infty$, so that only a few terms of (1.4.24) should be enough for accurate approximation of S_g. We also note that (1.4.24) implies that

$$\frac{ab}{\|g\|^2}\, S_g \to I \ , \ (a,b) \to (0,0) \ , \tag{1.4.29}$$

when g is sufficiently well-behaved. In particular, for such g, the system $g_{na,mb}$, $(n,m) \in \mathbb{Z}^2$, is a frame when ab is sufficiently small.

The set of operators that have the form of the right-hand side of (1.4.24) has been studied in [Jan95b], Secs. 4, 5, with particular attention for the issue that ${}^\circ\gamma$ inherits smoothness and decay properties of g when $g_{na,mb}$, $(n,m) \in \mathbb{Z}^2$, is a frame. A very interesting connection with Von Neumann algebra theory has been worked out in [DLL95], Sec. 6.

1.4.4 Characterization of minimum dual

The two Weyl-Heisenberg systems $g_{na,mb}$, $(n,m) \in \mathbb{Z}^2$, and $\gamma_{na,mb}$, $(n,m) \in \mathbb{Z}^2$, are dual if and only if

$$\frac{1}{ab}\, U_g\, U_\gamma^* = I \; . \tag{1.4.30}$$

Furthermore, we have that

$$\frac{1}{ab}\, U_{{}^\circ\gamma}^* = U_g^*(U_g\, U_g^*)^{-1} \tag{1.4.31}$$

when $g_{na,mb}$, $(n,m) \in \mathbb{Z}^2$, is a frame, so that in particular

$$\frac{1}{ab}\, U_{{}^\circ\gamma}\, U_{{}^\circ\gamma}^* = \left(\frac{1}{ab}\, U_g\, U_g^*\right)^{-1} . \tag{1.4.32}$$

The proof of this result is in [Jan95b], proof of Prop. 3.4. As a consequence of (1.4.32) we have that (when $g_{na,mb}$, $(n,m) \in \mathbb{Z}^2$) is a frame) the inverse frame operator $S_g^{-1} = S_{{}^\circ\gamma}$ has the representation

$$S_{{}^\circ\gamma} = \sum_{k,l} \left(\left(\frac{1}{ab}\, U_g\, U_g^*\right)^{-1}\right)_{kl;oo} U_{kl} \tag{1.4.33}$$

with the same precautions as in (1.4.24)–(1.4.28).

1.4.5 Connection between $H_g(\nu)$ and U_g

Let Φ denote the Hilbert space $L^2([0,b) \times \mathbb{Z})$, so that

$$\varphi \in \Phi \Leftrightarrow \underline{\varphi}(\nu) \in l^2(\mathbb{Z}) \, , \; \nu \in [0,b) \; \& \tag{1.4.34}$$

$$||\varphi||^2 := \int_0^b \sum_k |\varphi_k(\nu)|^2 \, d\nu < \infty \; .$$

Now let $g_{na,mb}$, $(n,m) \in \mathbb{Z}^2$, be a Weyl-Heisenberg system with finite frame upper bound, and define a linear operator $\mathcal{H}_g$ of Φ by

$$(\mathcal{H}_g\, \varphi)(\nu) = H_g^T(\nu)\, \underline{\varphi}(\nu) \; , \; \text{a.e. } \nu \in [0,b) \; , \tag{1.4.35}$$

for $\varphi \in \Phi$. Then there holds

$$U_g^* = \sqrt{b}\, \mathcal{F}^{-1}\, R\, \mathcal{H}_g\, Q \; , \tag{1.4.36}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform of $L^2(\mathbb{R})$, and R and Q are unitary operators, defined by

$$Q : \underline{\alpha} \in l^2(\mathbb{Z}^2) \to \left(\frac{1}{\sqrt{b}} \sum_{k=-\infty}^{\infty} \alpha_{kl} \, e^{2\pi ikl/ab} \, e^{-2\pi ik\nu/b} \right)_{l \in \mathbb{Z}} \in \Phi \ , \quad (1.4.37)$$

and

$$R : \varphi \in \Phi \to \varphi_{-\lfloor \nu/b \rfloor}(\nu - \lfloor \nu/b \rfloor \, b) \in L^2(\mathbb{R}) \ , \quad (1.4.38)$$

respectively. The relation (1.4.36) between the operators $H_g(\nu)$, $0 \leq \nu < b$, and U_g through unitary operators shows, for instance, that Theorem 2.5 for Weyl-Heisenberg systems, see (1.3.10)–(1.3.11), and Theorem 4.1 are really the same.

1.5 Rational Weyl-Heisenberg systems in the Zak transform domain

We consider in this section Weyl-Heisenberg systems $g_{na,mb}$, $(n, m) \in \mathbb{Z}^2$, for the special case that ab is rational, $ab = p/q$ with $p, q \in \mathbb{N}$ and $\mathrm{GCD}(p, q) = 1$.

Let $\lambda > 0$. We define for $h \in L^2(\mathbb{R})$ the Zak transform $\mathcal{Z}_\lambda \, h$ by

$$(\mathcal{Z}_\lambda \, h)(t, \nu) = \lambda^{1/2} \sum_{k=-\infty}^{\infty} h\,(\lambda(t-k)) \; e^{2\pi ik\nu} \ , \quad (1.5.1)$$

where the right-hand side has to be interpreted in an $L^2_{\mathrm{loc}}(\mathbb{R}^2)$-sense. Of the many properties of the Zak transform, see [Jan88], we mention

$$\int_{-\infty}^{\infty} f(t) \, h^*(t) \, dt = \int_0^1 \int_0^1 (\mathcal{Z}_\lambda \, f)(t, \nu)(\mathcal{Z}_\lambda \, h)^*(t, \nu) \, dt \, d\nu \ , \quad (1.5.2)$$

$$\lambda^{1/2} \, f(\lambda t) = \int_0^1 (\mathcal{Z}_\lambda \, f)(t, \nu) \, d\nu \ , \text{a.e. } t \in \mathbb{R} \ , \quad (1.5.3)$$

$$(\mathcal{Z}_\lambda \, f)(t+1, \nu) = e^{2\pi i\nu} (\mathcal{Z}_\lambda \, f)(t, \nu) \ , \text{a.e. } t, \nu \in \mathbb{R} \ , \quad (1.5.4)$$

$$(\mathcal{Z}_\lambda \, f)(t, \nu+1) = (\mathcal{Z}_\lambda \, f)(t, \nu) \ , \text{a.e. } t, \nu \in \mathbb{R} \ , \quad (1.5.5)$$

$$(\mathcal{Z}_\lambda \, \hat{f})(t, \nu) = e^{2\pi i\nu t} (\mathcal{Z}_{1/\lambda} \, f)(-\nu, t) \ , \text{a.e. } t, \nu \in \mathbb{R} \ , \quad (1.5.6)$$

where $f, h \in L^2(\mathbb{R})$. Moreover, for any $Z \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$ such that

$$Z(t+1, \nu) = e^{2\pi i\nu} \, Z(t, \nu), Z(t, \nu+1) = Z(t, \nu) \ , \quad \text{a.e. } t, \nu \in \mathbb{R} \ , \quad (1.5.7)$$

there is a unique $f \in L^2(\mathbb{R})$ such that $Z = \mathcal{Z}_\lambda f$. We note here that when Z in (1.5.7) is, in addition, continuous, we have $Z(t,\nu) = 0$ for some $(t,\nu) \in [0,1)^2$.

The usefulness of the Zak transform for studying frame bound questions and computation of (minimal) duals for the case that ab is rational has been demonstrated notably by Zibulski and Zeevi, see [ZZ92b, ZZ93b, ZZ97b]; also see [Dau92], p. 978 and 981, [FZ95b, Jan95c, BG]. We shall make the choice $\lambda = b^{-1}$, and suppress the subscript λ in $\mathcal{Z}_\lambda$ so that

$$(\mathcal{Z}\, h)(t,\nu) = b^{-1/2} \sum_{k=-\infty}^{\infty} h\Big(\frac{t-k}{b}\Big)\, e^{2\pi i k\nu}, \text{ a.e. } t,\nu \in \mathbb{R}\ , \tag{1.5.8}$$

for $h \in L^2(\mathbb{R})$. At the end of this section we shall indicate what the results given below would look like when the choice $\lambda = a$ instead of $\lambda = b^{-1}$ were made in $\mathcal{Z}_\lambda$.

We set for $f, h \in L^2(\mathbb{R})$ and a.e. $t,\nu \in \mathbb{R}$

$$\Phi^f(t,\nu) = p^{-1/2}\Big((\mathcal{Z}f)\Big(t - l\frac{p}{q}, \nu + \frac{k}{p}\Big)\Big)_{k=0,\dots,p-1\,;\,l=0,\dots,q-1}\ , \tag{1.5.9}$$

$$A^{fh}(t,\nu) = \Big(A^{fh}_{kr}(t,\nu)\Big)_{k,r=0,\dots,p-1} = \Phi^f(t,\nu)\,\big(\Phi^h(t,\nu)\big)^*\ . \tag{1.5.10}$$

Observe that by (1.5.2), (1.5.4) and (1.5.5) there holds

$$\int_0^{q^{-1}} \int_0^{p^{-1}} \sum_{k=0}^{p-1} \sum_{l=0}^{q-1} \Phi^f_{kl}(t,\nu)\,\big(\Phi^h_{kl}(t,\nu)\big)^*\, dt\, d\nu = \frac{1}{p}\,\langle f,h\rangle\ . \tag{1.5.11}$$

Here we also note that (since $\mathrm{GCD}(p,q) = 1$) by (1.5.4)

$$(\mathcal{Z}f)\Big(t - l\frac{p}{q}, \nu + \frac{k}{p}\Big) = e^{2\pi i\varphi}(\mathcal{Z}f)\Big(t - \frac{\pi(l)}{q}, \nu + \frac{k}{p}\Big)\ , \tag{1.5.12}$$

where the permutation π of $\{0,\dots,q-1\}$ and $\varphi = \varphi(k,l,\nu) \in \mathbb{R}$ are both independent of f. For later reference we write (1.5.11) alternatively as

$$\frac{1}{p}\,\langle f,h\rangle = \sum_{l=0}^{q-1} \int_0^{q^{-1}} \int_0^{p^{-1}} \langle \Phi^f_{.l}(t,\nu), \Phi^h_{.l}(t,\nu)\rangle\, dt\, d\nu\ , \tag{1.5.13}$$

where $\Phi^f_{.l}(t,\nu), \Phi^h_{.l}(t,\nu) \in \mathbb{C}^p$ are the l^{th} columns of $\Phi^f(t,\nu)$, $\Phi^h(t,\nu)$, and the inner product in the right-hand side integral of (1.5.13) is the inner product of $\mathbb{C}^p$.

1.5.1 Frame operator representation

Assume that the system $g_{na,mb}$, $(n,m) \in \mathbb{Z}^2$, has a finite frame upper bound, and let $f \in L^2(\mathbb{R})$. Then there holds

$$\Phi^{S_g f}(t,\nu) = A^{gg}(t,\nu)\,\Phi^f(t,\nu)\ ,\ \text{a.e. } t,\nu \in \mathbb{R}\ , \tag{1.5.14}$$

and

$$(S_g\, f, f) = \frac{1}{p}\sum_{l=0}^{q-1}\int_0^{q^{-1}}\int_0^{p^{-1}} \langle A^{gg}(t,\nu)\,\Phi^f_{.l}(t,\nu), \Phi^f_{.l}(t,\nu)\rangle\, dt\, d\nu\ . \tag{1.5.15}$$

These formulas follow from (10) and (13) in [ZZ93b] after some further manipulations.

1.5.2 Frame bounds

For any $A \geq 0$, $B < \infty$ there holds

$$A\,||f||^2 \leq \sum_{n,m} |\langle f, g_{na,mb}\rangle|^2 \leq B\,||f||^2\ ,\ f \in L^2(\mathbb{R}) \tag{1.5.16}$$

$$\Leftrightarrow$$

$$A\,I_{p\times p} \leq A^{gg}(t,\nu) \leq B\,I_{p\times p}\ ,\ t,\nu \in \mathbb{R}\ .$$

This is shown in [ZZ93b], and it also readily follows from (1.5.13) and (1.5.15). Note that for checking the right-hand side member of (1.5.16) one can restrict oneselves to $t \in [0, q^{-1})$, $\nu \in [0, p^{-1})$. Explicitly,

$$\begin{aligned} A^{gg}(t+q^{-1},\nu) =& F\,A^{gg}(t,\nu)\,F^{-1}\ , \\ A^{gg}(t,\nu+p^{-1}) =& J\,A^{gg}(t,\nu)\,J^{-1} \end{aligned} \tag{1.5.17}$$

with

$$F = \text{diag}(\exp(-2\pi i m_0 k/p),\ k = 0,...,p-1)\ , \tag{1.5.18}$$

where $m_0 \in \mathbb{Z}$ depends only on p, q, and J is the permutation matrix corresponding to the permutation $0 \to 1 \to \ldots p-1 \to 0$.

1.5.3 Fourier series expansion of matrix elements

Let $f, h \in L^2(\mathbb{R})$. Then $A^{fh}_{kr}(t,\nu)$ is p/q-periodic in t and 1-periodic in ν with Fourier series

$$A^{fh}_{kr}(t,\nu) \sim \sum_{n,m} d_{nmkr}\, e^{-2\pi i n t q/p}\, e^{-2\pi i m \nu}\ , \tag{1.5.19}$$

where

$$d_{nmkr} = \begin{cases} \frac{q}{p}\, e^{-2\pi i m r/p} \langle f, h_{m/b,-n/a} \rangle & nq \equiv (r-k) \bmod p, \\ 0 & \text{otherwise.} \end{cases} \tag{1.5.20}$$

This result has been proved in [Jan95c], see Prop. 1.1.

1.5.4 Wexler-Raz biorthogonality condition in the Zak transform domain

Assume that $g, \gamma \in L^2(\mathbb{R})$. Then the two systems $g_{k/b,l/a}$, $(k,l) \in \mathbb{Z}^2$, and $\gamma_{k/b,l/a}$, $(k,l) \in \mathbb{Z}^2$, are biorthogonal, see (1.4.15), if and only if

$$A^{g\gamma}(t,\nu) = \Phi^g(t,\nu)\,(\Phi^\gamma(t,\nu))^* = I_{p\times p}\ , \text{ a.e. } t,\nu \in \mathbb{R}\ . \tag{1.5.21}$$

This is an immediate consequence of 5.3.

1.5.5 Characterization of minimal dual

Assume that the system $g_{na,mb}$, $(n,m) \in \mathbb{Z}^2$, is a frame. Then there holds

$$\left(\Phi^{^\circ\gamma}(t,\nu)\right)^* = (\Phi^g(t,\nu))^*\ (A^{gg}(t,\nu))^{-1}\ , \text{ a.e. } t,\nu \in \mathbb{R}\ , \tag{1.5.22}$$

and in particular

$$A^{^\circ\gamma^\circ\gamma}(t,\nu) = (A^{gg}(t,\nu))^{-1}\ , \text{ a.e. } t,\nu \in \mathbb{R}\ . \tag{1.5.23}$$

This result readily follows from 5.1 and the fact that $S_g\,{}^\circ\gamma = g$.

1.5.6 Condition A for rational Weyl-Heisenberg systems

We recall that a $g \in L^2(\mathbb{R})$ satisfies condition A when

$$\sum_{k,l} |\langle g, g_{k/b,l/a}\rangle| < \infty\ . \tag{1.5.24}$$

When condition A holds, the system $g_{na,mb}$, $(n,m) \in \mathbb{Z}^2$, has a finite frame upper bound, and the frame operator representation (1.4.24) is unconditional. There holds

$$\begin{aligned} & g \in L^2(\mathbb{R}) \text{ satisfies condition A} \\ \Leftrightarrow\ & \\ & A^{gg}(t,\nu) \text{ has an absolutely convergent Fourier series.} \end{aligned} \tag{1.5.25}$$

Furthermore, when the system $g_{na,mb}$, $(n,m) \in \mathbb{Z}^2$, is a frame, where $g \in L^2(\mathbb{R})$ satisfies condition A, then $^\circ\gamma$ satisfies condition A as well. This has been proved in [Jan95c]. It is not known whether the latter result continues to hold when ab is irrational.

1.5.7 Different choice of the Zak transform

We briefly indicate what the results of this section look like when we choose $\lambda = a$, rather than b^{-1}, in (1.5.1). Denote for $f, h \in L^2(\mathbb{R})$

$$\Psi^f(t,\nu) = p^{-1/2}\Big((\mathcal{Z}_a f)\Big(t - k\,\frac{q}{p}, \nu - \frac{l}{q}\Big)\Big)_{k=0,\ldots,p-1\,;\,l=0,\ldots,q-1}\,, \tag{1.5.26}$$

$$B^{fh}(t,\nu) = \Big(B^{fh}_{kr}(t,\nu)\Big)_{k,r=0,\ldots,p-1} = \Psi^f(t,\nu)\left(\Psi^h(t,\nu)\right)^*\ . \tag{1.5.27}$$

Then there holds

$$\Psi^{S_g f}(t,\nu) = B^{gg}(t,\nu)\,\Psi^f(t,\nu)\ ,\ \text{a.e. } t,\nu \in \mathbb{R}\ . \tag{1.5.28}$$

Also, the matrix elements $B^{fh}_{kr}(t,\nu)$ are 1-periodic in t and q^{-1}-periodic in ν, with Fourier series

$$B^{fh}_{kr}(t,\nu) \sim \sum_{n,m} e_{nmkr}\, e^{-2\pi int - 2\pi imq\nu}\ , \tag{1.5.29}$$

where

$$e_{nmkr} = \frac{q}{p}\, e^{2\pi inr/ab}\,\langle f, h_{\frac{k-r-mp}{b},\,\frac{-n}{a}}\rangle\ . \tag{1.5.30}$$

From these two results the main results of this section can be rederived.

1.6 Time-discrete Weyl-Heisenberg systems

In this section we consider shift-invariant systems and Weyl-Heisenberg systems for the time-discrete case. That is, we consider systems

$$g_{nm} = (g_m(j - nN))_{j\in\mathbb{Z}}\ ,\ n \in \mathbb{Z}\ ,\ m \in \mathcal{I} = \{0,\ldots,M-1\}\ , \tag{1.6.1}$$

with $N \in \mathbb{N}$, $M \in \mathbb{N}$ and $g_m \in l^2(\mathbb{Z})$, $m \in \mathcal{I}$, where for the special case of Weyl-Heisenberg systems we take

$$g_m = \left(e^{2\pi imj/M}\, g(j)\right)_{j\in\mathbb{Z}}\ ,\ m \in \mathcal{I}\ , \tag{1.6.2}$$

with $g \in l^2(\mathbb{Z})$. We shall again be somewhat careless about the phase factors in g_{nm} for Weyl-Heisenberg systems. As in Secs. 1.2–1.5 we consider for these systems the duality condition, frame operator representations, frame bounds, and the characterization and computation of minimal duals in various domains. Furthermore, we present a link with the theory of filter banks.

The developments for the time-discrete case and time-continuous case are to a large extent the same. Nevertheless, we found it necessary to list the time-discrete versions of the results in detail since the correct formulation of them, taking due account of the discretization, is often a non-trivial matter. We omit, however, all proofs since most of them can be found in the existing literature, to which we refer when possible, or consist of a straightforward repeating of the arguments used for the time-continuous case once the correct time-discrete version has been formulated.

1.6.1 Time-discrete shift-invariant systems

We consider $l^2(\mathbb{Z})$ with the inner product norm $\|f\| = \langle f, f\rangle^{1/2}$ where

$$\langle f, h\rangle = \sum_{j=-\infty}^{\infty} f(j)\, h^*(j) \ , \ f, h \in l^2(\mathbb{Z}) \ . \tag{1.6.3}$$

Also, for $l \in \mathbb{Z}$, we let T_l be the time-shift operator defined on $l^2(\mathbb{Z})$ by

$$T_l\, f = (f(j+l))_{j\in\mathbb{Z}} \ , \ f \in l^2(\mathbb{Z}) \ . \tag{1.6.4}$$

Finally, we denote for $h \in l^2(\mathbb{Z})$ by $\hat{h}$ the Fourier transform

$$\hat{h}(\vartheta) = \sum_{j=-\infty}^{\infty} h(j)\, e^{-2\pi i j\vartheta} \ , \ \text{a.e. } \vartheta \in \mathbb{R} \ . \tag{1.6.5}$$

Proposition 1.6.1 *Assume that g_{nm}, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, and γ_{nm}, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, are two shift-invariant systems with finite frame upper bounds B_g, B_γ (no duality assumption). Then*

$$\sum_m |\hat{g}_m(\vartheta)|^2 \le N\, B_g \ , \ \sum_m |\hat{\gamma}_m(\vartheta)|^2 \le N\, B_\gamma \ , \ \text{a.e. } \vartheta \in \mathbb{R} \ . \tag{1.6.6}$$

When furthermore $f, h \in l^2(\mathbb{Z})$, then the function

$$\rho(f,h)(l) = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{M-1} \langle T_l\, f, g_{nm}\rangle \langle \gamma_{nm}, T_l\, h\rangle \tag{1.6.7}$$

is periodic in $l \in \mathbb{Z}$ with period N, and has a Fourier series

$$\rho(f,h)(l) = \sum_{k=0}^{N-1} c_k \, e^{-2\pi i k l/N} \ , \tag{1.6.8}$$

$$c_k = \frac{1}{N} \int_0^1 \hat{f}(\vartheta)\, \hat{h}^*\Big(\vartheta + \frac{k}{N}\Big) \sum_m \hat{g}_m^*(\vartheta)\, \hat{\gamma}_m\Big(\vartheta + \frac{k}{N}\Big)\, d\vartheta \ , \ k \in \mathbb{Z} \ . \tag{1.6.9}$$

Theorem 1.6.2 *Assume that the systems g_{nm}, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, and γ_{nm}, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, have finite frame upper bounds. Then the systems are dual in the sense that*

$$\langle f, h\rangle = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{M-1} \langle f, g_{nm}\rangle \langle \gamma_{nm}, h\rangle \ , \ f, h \in l^2(\mathbb{Z}) \ , \tag{1.6.10}$$

if and only if

$$\sum_{m=0}^{M-1} \hat{g}_m^*(\vartheta - k/N)\, \hat{\gamma}_m(\vartheta) = N\, \delta_{ko} \ , \ \text{a.e. } \vartheta \in \mathbb{R}, \ \ k \in \mathbb{Z} \ . \tag{1.6.11}$$

Proposition 1.6.3 *Assume that the system g_{nm}, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, has a finite frame upper bound B_g. Then the matrix*

$$H_g(\vartheta) := (\hat{g}_m(\vartheta - k/N))_{k=0,\dots,N-1\,;\, m=0,\dots,M-1} \tag{1.6.12}$$

defines for a.e. $\vartheta \in \mathbb{R}$ a bounded linear mapping of $\mathbb{C}^M$, into $\mathbb{C}^N$ with norm $\le (N\, B_g)^{1/2}$. Explicitly, we have for a.e. $\vartheta \in \mathbb{R}$

$$\sum_{k=0}^{N-1} \left| \sum_{m=0}^{M-1} \hat{g}_m(\vartheta - k/N)\, \beta_m \right|^2 \le N\, B_g\, \|\underline{\beta}\|^2 \ , \ \underline{\beta} \in \mathbb{C}^M \ , \tag{1.6.13}$$

where $\|\underline{\beta}\| = (\sum_{m=0}^{M-1} |\beta_m|^2)^{1/2}$ is the norm of $\underline{\beta} \in \mathbb{C}^m$.

Theorem 1.6.4 *Assume that the system g_{nm}, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, has a finite frame upper bound, and let $f \in L^2(\mathbb{R})$. Then we have*

$$\widehat{S_g f}(\vartheta) = \frac{1}{N} \sum_{k=0}^{N-1} d_k(\vartheta)\, \hat{f}(\vartheta - k/N) \ , \ \text{a.e. } \vartheta \in \mathbb{R} \ , \tag{1.6.14}$$

where, see (1.6.12),

$$d_k(\vartheta) = \left(H_g(\vartheta)\, H_g^*(\vartheta)\right)_{ok} = \sum_{m=0}^{M-1} \hat{g}_m(\vartheta)\, \hat{g}_m^*(\vartheta - k/N)\ ,\ \text{a.e. } \vartheta \in \mathbb{R}\ , \tag{1.6.15}$$

for $k = 0, ..., N-1$.

Theorem 1.6.5 *Let* $g_m \in l^2(\mathbb{R})$, $m \in \mathcal{I}$, *and let* $A \geq 0$, $B < \infty$. *Then*

$$A\,\|f\|^2 \leq \sum_{n=-\infty}^{\infty} \sum_{m=0}^{M-1} |\langle f, g_{nm}\rangle|^2 \leq B\,\|f\|^2\ ,\ f \in l^2(\mathbb{Z}) \tag{1.6.16}$$

$$\Leftrightarrow$$

$$N\,A\,I \leq H_g(\vartheta)\, H_g^*(\vartheta) \leq N\,B\,I\ ,\ \text{a.e. } \vartheta \in \mathbb{R}\ ,$$

where I *denotes the identity mapping of* $\mathbb{C}^N$.

As a consequence of this result we see that the system g_{nm}, $(n,m) \in \mathbb{Z}\times\mathcal{I}$, can only be a frame when $N \leq M$ since $H_g(\vartheta)$ has rank $\leq M$.

Theorem 1.6.6 *Assume that the system* g_{nm}, $(n,m) \in \mathbb{Z}\times\mathcal{I}$, *is a frame, and denote by* $\underline{c}(\vartheta) \in \mathbb{C}^m$ *for a.e.* $\vartheta \in \mathbb{R}$ *the least-norm solution* $\underline{c} = (c_m)_{m=0,\ldots,M-1}$ *of the linear system*

$$\sum_{m=0}^{M-1} \hat{g}_m(\vartheta - k/N)\, c_m(\vartheta) = N\,\delta_{ko}\ ,\ k = 0, ..., N-1\ . \tag{1.6.17}$$

Then there holds

$$^\circ\hat{\gamma}_m(\vartheta) = c_m^*(\vartheta)\ ,\ m \in \mathcal{I}\,,\ \text{a.e. } \vartheta \in \mathbb{R}\ . \tag{1.6.18}$$

Theorem 1.6.7 *Assume that the systems* g_{nm}, $(n,m) \in \mathbb{Z}\times\mathcal{I}$, *and* γ_{nm}, $(n,m) \in \mathbb{Z}\times\mathcal{I}$, *have finite frame upper bounds. Then they are dual if and only if*

$$H_g(\vartheta)\, H_\gamma^*(\vartheta) = N\,I\ ,\ \text{a.e. } \vartheta \in \mathbb{R}\ . \tag{1.6.19}$$

Moreover,

$$H_{\circ\gamma}^*(\vartheta) = N\, H_g^*(\vartheta)\ \left(H_g(\vartheta)\, H_g^*(\vartheta)\right)^{-1}\ ,\ \text{a.e. } \vartheta \in \mathbb{R}\ , \tag{1.6.20}$$

so that in particular

$$\frac{1}{N}\, H_{\circ\gamma}(\vartheta)\, H_{\circ\gamma}^*(\vartheta) = \left(\frac{1}{N}\, H_g(\vartheta)\, H_g^*(\vartheta)\right)^{-1}\ ,\ \text{a.e. } \vartheta \in \mathbb{R}\ . \tag{1.6.21}$$

As a consequence of (1.6.21) we note that, see Theorem 6.2, the inverse frame operator $S_g^{-1} = S_{\circ\gamma}$ has the representation

$$\widehat{S_{\circ\gamma} f}(\vartheta) = \sum_{k=0}^{N-1} \left(\left(\frac{1}{N} H_g(\vartheta)\, H_g^*(\vartheta)\right)^{-1}\right)_{ok} \hat{f}(\vartheta - k/N)\ , \text{ a.e. } \vartheta \in \mathbb{R}\ , \tag{1.6.22}$$

where $f \in l^2(\mathbb{Z})$.

The proofs of the results in this section consist of simple adaptations of the proofs in Sec. 1.2 for the corresponding time-continuous versions of the results, except that certain measure theoretic intricacies are absent now due to finite summation ranges.

1.6.2 Time-discrete shift-invariant systems and filter banks

Time-discrete shift-invariant dual systems can be considered as, what are called in engineering literature, perfect reconstructing (PR) filter banks of the FIR- or IIR-type according as the impulse responses g_m, γ_m are finite or infinite. In this view the formula expressing duality,

$$f = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{M-1} \langle f, \gamma_{nm}\rangle\, g_{nm} = T_g^*\, T_\gamma\, f\ ,\ f \in l^2(\mathbb{Z})\ , \tag{1.6.23}$$

see (1.1.3),(1.1.5)–(1.1.7), can be regarded as a representation of time-discrete signals f by means of the expansion coefficients (subband signals) $\langle f, \gamma_{nm}\rangle$, $(n,m) \in \mathbb{Z} \times \mathcal{I}$. Thus the finite-energy input signal f is passed through analysis filters (A) with impulse responses $(\gamma_m^*(-j))_{j\in\mathbb{Z}}$, $m \in \mathcal{I}$, whose outputs

$$\sum_{j=-\infty}^{\infty} f(j)\, \gamma_m^*(j-k)\ ,\ k \in \mathbb{Z}\ , \tag{1.6.24}$$

are downsampled (DS) by a factor N, so that only the samples with $k = nN$, $n \in \mathbb{Z}$, are retained. This operation is usually followed by a quantization step (Q), in which the outputs $\langle f, \gamma_{nm}\rangle$ are quantized to numbers $[\langle f, \gamma_{nm}\rangle]$, and a coding operation (C), after which the resulting data are transmitted or stored. To regenerate a distorted version $[f]$ of the input signal, one has to decode the data, and to apply synthesis filters (S) with impulse responses g_m, $m \in \mathcal{I}$, to upsampled versions of the numbers $[\langle f, \gamma_{nm}\rangle]$. The upsampling operation (US) just amounts to inserting $N-1$ zeros between any two consecutive numbers $[\langle f, \gamma_{nm}\rangle]$ with $n \in \mathbb{Z}$. Hence

$$[f]\,(j) = \sum_{n=-\infty}^{\infty} \sum_{m=0}^{M-1} [\langle f, \gamma_{nm}\rangle]\, g_m(j - nN)\ . \tag{1.6.25}$$

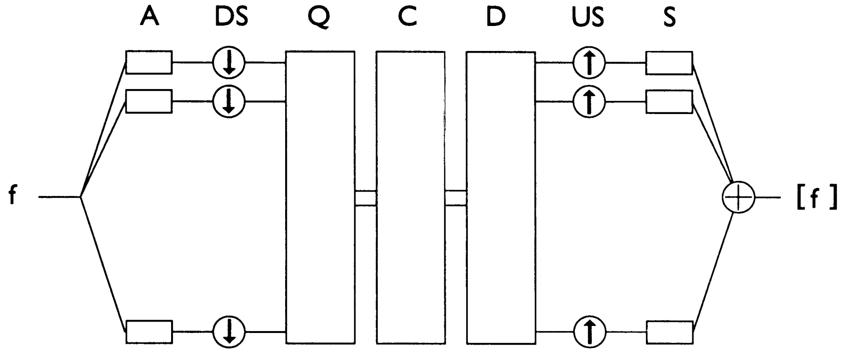


FIGURE 1.6.1. Schematic representation of an analysis-synthesis filter bank, with A analysis filters, DS downsampling by factor N, Q quantization, C coding, D decoding, US upsampling by factor N, S synthesis filters, + summation operation, and f, $[f]$ the signal to be processed, processed signal, respectively.

Schematically one has the situation as depicted in Fig. 1.6.1.

When the QCD-operation in Fig. 1.6.1 does not introduce distortion, so that

$$[\langle f,\gamma_{nm}\rangle] = \langle f,\gamma_{nm}\rangle \ , \ n,m \in \mathbb{Z}\times\mathcal{I} \ , \tag{1.6.26}$$

the formulas (1.6.23), (1.6.25) show that the systems γ_{nm}, $(n,m)\in\mathbb{Z}\times\mathcal{I}$, g_{nm}, $(n,m)\in\mathbb{Z}\times\mathcal{I}$, should be dual so as to achieve perfect reconstruction for all $f\in l^2(\mathbb{Z})$. (Note that for the notion of duality we may interchange the roles of the γ_m's and the g_m's.) When the QCD-operation is imperfect, the question naturally arises how to find for a given $\underline{\beta}\in l^2(\mathbb{Z}\times\mathcal{I})$ an $f\in l^2(\mathbb{Z})$ with minimal value of

$$\sum_{n=-\infty}^{\infty}\sum_{m=0}^{M-1} |\langle f,\gamma_{nm}\rangle - \beta_{nm}|^2 \ . \tag{1.6.27}$$

It turns out that the minimizing f is given by

$$f = \sum_{n=-\infty}^{\infty}\sum_{m=0}^{M-1} \beta_{nm}\,{}^\circ g_{nm} \tag{1.6.28}$$

with ${}^\circ g_m = S_\gamma^{-1}\gamma_m$, the minimal duals corresponding to γ_m, $m\in\mathcal{I}$ (again the roles of the g_m's and the γ_m's have been interchanged). Hence the relevance of minimal dual systems for filter banks.

In data-transmission and/or storage systems one aims at signal representation methods that allow easy implementation while being efficient and robust and that do not introduce perceptually annoying distortion in the processed signals. For the present case one is thus faced with several constraints, in addition to the perfect reconstruction property, on N, M, and the analysis and synthesis filters. Below we list a number of these constraints where it is noted that a particular constraint can be severe for one application and unimportant for another.

1.6.2.1 Implementational constraints

For easy implementation and a fast signal processing it is desirable that M is not too large, that the g_m and γ_m, $m \in \mathcal{I}$, are of short duration, and that N is large. Also, it is convenient that the g_m and γ_m, $m \in \mathcal{I}$, are real and all of the same form, or perhaps also that $g_m = \gamma_m$, $m \in \mathcal{I}$. Furthermore, an integer relation between N, M and the durations of the g_m and γ_m, $m \in \mathcal{I}$, is desirable.

1.6.2.2 Coding-efficiency constraints

Coding efficiency is increased when advantage is taken of the masking effects occurring in the processing of stimuli by human observers. For certain applications this means that the analysis filter should have a reasonable amount of frequency selectivity, so that quantization of the outputs $\langle f, \gamma_{nm} \rangle$ can take place in accordance with their relative amplitudes. It may also be desirable that all but one pair γ_m, g_m satisfy $\hat{g}_m(0) = \hat{\gamma}_m(0) = 0$, so that the DC-component in f is represented by only one term in its expansion. The coding system's complexity is low for low values of M and large values of N. Using larger values of M and/or lower values of N does not necessarily decrease the coding efficiency, but then the coding system must be able to remove the redundancy thus introduced and this increases the coding system's complexity.

1.6.2.3 Perceptual constraints

To avoid blocking effects or pre-echos, the impulse responses g_m of the synthesis filters should be smooth, in particular at the boundaries of their supports. Consequently, because of the perfect reconstruction property, the time-shifted g_m should overlap. On the other hand, the impulse responses g_m should have short length so as to avoid (quantization) errors in the $\langle f, \gamma_{nm} \rangle$ to have an impact on large regions (ringing). To avoid phase errors, the analysis and/or synthesis filters should be linear phase (i.e. the impulse responses, assumed to be real, should be symmetric around the midpoint of their support).

1.6.2.4 Robustness constraints

To avoid (quantization) errors in the $\langle f, \gamma_{nm} \rangle$ to have large detrimental effects, or to avoid that some signals f of unit energy need large coefficients in their expansion, it is desirable that the ratio A/B of the frame bounds is not too small. Even more desirable would be $A/B = 1$ (tight frame), so that $^\circ\gamma_m = \frac{1}{A} g_m$, $m \in \mathcal{I}$, or that g_{nm}, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, is an orthonormal base for $l^2(\mathbb{Z})$.

It is the designer's task to construct perfect reconstructing filter banks such that some or many of the above constraints are satisfied. For instance in [Heu96][1] there are designed, for the purpose of data compression of images, critically sampled ($N = M$), perfect reconstructing, orthonormal ($g_m = \gamma_m$, $m \in \mathcal{I}$), 50% overlapped (support length of the responses equals $2N$), linear phase filters with smooth responses g_m satisfying $\hat{g}_m(0) = 0$ for all $m \in \mathcal{I}$ but one, and with acceptable frequency selectivity. As one can see, the constraints on analysis and synthesis filters are formulated in different domains (time domain, frequency domain, and perhaps also in the time-frequency domain). Some of the constraints are of the hard type (such as the perfect reconstruction property and the linear phase condition) in the sense that they can be formulated as a set of equations involving the filter impulse responses in one of the domains. Other constraints are of the soft type (such as reasonable amount of frequency selectivity, smoothness). Since we have been able to formulate the most significant and most difficult constraint of the hard type, viz. perfect reconstruction, in various domains, a possible strategy would be to introduce a cost functional incorporating (some of) the soft constraints, and to minimize this functional under the condition that perfect reconstruction is satisfied.

We next present a link with, what are called in the theory of filter banks, polyphase matrices. Here one considers matrices

$$P_l = (g_m(j - lN))_{m=0,\ldots,M-1\,;\,j=0,\ldots,N-1} \tag{1.6.29}$$

and puts

$$P_g(e^{2\pi i\vartheta}) = \sum_{l=-\infty}^{\infty} P_l \, e^{2\pi i l\vartheta} \tag{1.6.30}$$

Thus we have

$$P_g(e^{2\pi i\vartheta}) = ((\mathcal{Z}_N \, g_m)(j,\vartheta))_{m=0,\ldots,M-1\,;\,j=0,\ldots,N-1} \ , \tag{1.6.31}$$

[1]The author wishes to thank R. Heusdens for introducing him to the subject of constrained filter design, and for providing him the considerations and insights that led to the above list of constraints.

where $\mathcal{Z}_N$ is a discrete Zak transform, defined for $h \in l^2(\mathbb{Z})$ by

$$(\mathcal{Z}_N\, h)(j,\vartheta) = \sum_{l=-\infty}^{\infty} h(j - lN)\, e^{2\pi i l\vartheta} \ , \ j \in \mathbb{Z}\ , \text{ a.e. } \vartheta \in \mathbb{R}\ . \tag{1.6.32}$$

Also see Subsec. 1.6.5. The matrix $P_g(e^{2\pi i\vartheta})$ is the polyphase matrix corresponding to the M-channel filter bank with impulse responses g_m, $m \in \mathcal{I}$, and downsampling factor N. Here we note that in filter bank theory it is customary to write $z = e^{2\pi i\vartheta}$, so that the Zak transforms in (1.6.31) become Z-transforms.

There holds

$$H_g^T(\vartheta) = \sqrt{N}\, P_g(e^{2\pi i N\vartheta})\, D(\vartheta)\, \mathcal{F}_N^{-1} \ , \ \text{a.e. } \vartheta \in \mathbb{R}\ , \tag{1.6.33}$$

with $H_g(\vartheta)$ given in (1.6.12), and $D(\vartheta)$ and $\mathcal{F}_N^{-1}$ the unitary operators

$$D(\vartheta) : \underline{x} \in \mathbb{C}^N \to (e^{-2\pi i r\vartheta}\, x_r)_{r=0,\dots,N-1} \in \mathbb{C}^N \ , \tag{1.6.34}$$

$$\mathcal{F}_N^{-1} : \underline{x} \in \mathbb{C}^N \to \Big(\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi i kr/N}\, x_r\Big)_{k=0,\dots,N-1} \in \mathbb{C}^N \ . \tag{1.6.35}$$

There is an extensive theory on filter design for analysis-synthesis filter banks by using polyphase matrices, see e.g. [Mal92, Vai93, VK95]. More recently the connection between the theories of DFT filter banks and of (oversampled) Weyl-Heisenberg frames has been established, see e.g. [Cve95a, BHF96c] and Chapter 9 of this volume, while the connections between the theories of more general filter banks and of time-discrete shift-invariant systems are in the process of being established, see [CV, Jan95a, BH95].

We conclude this subsection by a short dictionary[2] of terms used in polyphase filter bank theory and shift-invariant system theory.

1.6.3 Time-discrete Weyl-Heisenberg systems as shift-invariant systems

We consider now time-discrete Weyl-Heisenberg systems g_{nm}, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, with

$$g_m(j) = e^{2\pi i m j/M}\, g(j) \ , \ j \in \mathbb{Z}\ , \tag{1.6.36}$$

and $g \in l^2(\mathbb{Z})$.

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[2]The author wishes to thank H. Bölcskei for advise in preparing this dictionary.

TABLE 1.1.

Filter bank theory	Shift-invariant systems theory
perfect reconstruction bank	the systems $g_{nm}, (n,m) \in \mathbb{Z} \times \mathcal{I}$, and γ_{nm}, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, are dual
paraunitary filter bank	the system g_{nm}, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, is a tight frame ($A_g = B_g$), so that $^\circ\gamma_m = c\, g_m, m \in \mathcal{I}$
biorthogonal filter bank	the system $g_{nm}, (n,m) \in \mathbb{Z} \times \mathcal{I}$, is a frame with $N = M$ and unique dual system γ_{nm}, $(n,m) \in \mathbb{Z} \times \mathcal{I}$
polyphase representation	Zak transform
alias component matrix	the matrix $H_g^T(\vartheta)$, see (1.6.12)
polyphase matrix	the matrix $P_g(e^{2\pi i\vartheta})$, see (1.6.31)
DFT filter bank	the systems g_{nm}, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, and γ_{nm}, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, are Weyl-Heisenberg systems
quadrature mirror filters	the Weyl-Heisenberg system g_{nm}, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, is a frame with $N = M = 2$, so that the duality condition takes a special form

1.6.3.1 Frequency-domain results

Since now

$$\hat{g}_m(\vartheta) = \hat{g}(\vartheta - m/M)\ ,\ \text{a.e. } \vartheta \in \mathbb{R}\ ,\ m \in \mathcal{I}\ , \tag{1.6.37}$$

the results of Subsec. 1.6.1 can be translated straightforwardly to results for time-discrete Weyl-Heisenberg systems. For instance, the two systems g_{nm}, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, and γ_{nm}, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, with finite frame upper bounds are dual if and only if

$$\sum_{m=0}^{M-1} \hat{g}^*\Big(\vartheta - \frac{m}{M} - \frac{k}{N}\Big)\, \hat{\gamma}\Big(\vartheta - \frac{m}{M}\Big) = N\,\delta_{ko}\ ,$$
$$\text{a.e. } \vartheta \in \mathbb{R},\ k = 0, \ldots, N-1\ . \tag{1.6.38}$$

Also, there is the frame operator representation result

$$\widehat{S_g f}(\vartheta) = \frac{1}{N} \sum_{k=-\infty}^{\infty} d_k(\vartheta)\, \hat{f}\Big(\vartheta - \frac{k}{N}\Big)\ ,\ \text{a.e. } \vartheta \in \mathbb{R}\ , \tag{1.6.39}$$

for $f \in l^2(\mathbb{Z})$, where

$$d_k(\vartheta) = \sum_{m=0}^{M-1} \hat{g}\Big(\vartheta - \frac{m}{M}\Big)\, \hat{g}^*\Big(\vartheta - \frac{m}{M} - \frac{k}{N}\Big) \ ,$$

$$\text{a.e. } \vartheta \in \mathbb{R}, \ k = 0, ..., N-1 \ . \qquad (1.6.40)$$

The matrix $H_g(\vartheta)$ in (1.6.12) assumes the form

$$H_g(\vartheta) = \Big(\hat{g}\Big(\vartheta - \frac{m}{M} - \frac{k}{N}\Big)\Big)_{k=0,\ldots,N-1\,;\, m=0,\ldots,M-1} \ , \ \text{a.e. } \vartheta \in \mathbb{R} \ , \qquad (1.6.41)$$

and the system g_{nm}, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, is a frame with frame bounds $A_g > 0$, $B_g < \infty$, if and only if

$$N\, A_g\, I \le H_g(\vartheta)\, H_g^*(\vartheta) \le N\, B_g\, I \ , \ \text{a.e. } \vartheta \in \mathbb{R} \ , \qquad (1.6.42)$$

with I the $N \times N$-identity matrix. Furthermore, the minimal dual ${}^\circ\gamma$ can be computed as

$${}^\circ\hat{\gamma}(\vartheta) = \sum_{k=0}^{N-1} \Big(\Big(\frac{1}{N}\, H_g(\vartheta)\, H_g^*(\vartheta)\Big)^{-1}\Big)_{ok} \hat{g}(\vartheta - k/N) \ , \ \text{a.e. } \vartheta \in \mathbb{R} \ , \qquad (1.6.43)$$

and we have $||{}^\circ\gamma|| \le ||\gamma||$ when γ is another dual (equality if and only if $\gamma = {}^\circ\gamma$ a.e.). Finally, the polyphase matrix $P_g(e^{2\pi i\vartheta})$ of (1.6.30) assumes for a.e. $\vartheta \in \mathbb{R}$ the form

$$P_g(e^{2\pi i\vartheta}) = \Big(e^{2\pi i m j/N}\, (\mathcal{Z}_N\, g)\Big(j, \vartheta - \frac{m}{M}\Big)\Big)_{m=0,\ldots,M-1\,;\, j=0,\ldots,N-1} \ , \qquad (1.6.44)$$

with $\mathcal{Z}_N$ the Zak transform of (1.6.32).

1.6.3.2 Time-domain results

The results of the previous subsection cannot be translated immediately into time-domain results. This is so since, by contrast with the time-continuous case, see Subsection 1.3.2, the (discrete) Fourier transform does not map an infinite-time, discrete Weyl-Heisenberg system onto another such system. Nevertheless, a great deal of the results of the previous subsection have time-domain counterparts.

For instance when we denote for $j \in \mathbb{Z}$ by $M_g(j)$ the matrix

$$M_g(j) = (g(j + nN - lM))_{l\in\mathbb{Z}, n\in\mathbb{Z}} \ , \qquad (1.6.45)$$

then the system g_{nm}, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, is a discrete Weyl-Heisenberg frame with frame bounds $A_g > 0$, $B_g < \infty$ if and only if

$$\frac{1}{N} A_g \, I \leq M_g(j) \, M_g^*(j) \leq \frac{1}{N} B_g \, I \ , \ j = 0, ..., N-1 \ , \tag{1.6.46}$$

where I is the identity operator of $l^2(\mathbb{Z})$. And two Weyl-Heisenberg systems g_{nm}, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, and γ_{nm}, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, are dual if and only if

$$\sum_{n=-\infty}^{\infty} g^*(j + nN - lM) \, \gamma(j + nN) = \frac{1}{M} \, \delta_{lo} \ , \ l \in \mathbb{Z} \ , \tag{1.6.47}$$

for $j = 0, ..., N-1$. Furthermore, there is the frame operator representation result

$$(S_g \, f)(j) = M \sum_{l=-\infty}^{\infty} \left(M_g(j) \, M_g^*(j) \right)_{ol} f(j - lM) \ , \ j \in \mathbb{Z} \ , \tag{1.6.48}$$

valid for $f \in l^2(\mathbb{Z})$. Also, the minimal dual $^\circ\gamma$ can be computed from the least-norm solutions $\underline{c}^*(j) = (\gamma(j+nN))_{n \in \mathbb{Z}}$ of (1.6.47) for $j = 0, ..., N-1$, and there results

$$^\circ\gamma(j + nN) = \sum_{l=-\infty}^{\infty} \left(M \, M_g(j) \, M_g^*(j) \right)_{ol}^{-1} g(j + nN - lM) \tag{1.6.49}$$

for $j = 0, ..., N-1$ and $n \in \mathbb{Z}$. In particular we have, when γ is another dual, that for $j = 0, ..., N-1$

$$\sum_{n=-\infty}^{\infty} |^\circ\gamma(j + nN)|^2 \leq \sum_{n=-\infty}^{\infty} |\gamma(j + nN)|^2 \tag{1.6.50}$$

with equality if and only if $^\circ\gamma(j+nN) = \gamma(j+nN)$, $n \in \mathbb{Z}$. Finally, duality can equivalently be expressed as

$$M_g(j) \, M_\gamma^*(j) = \frac{1}{M} I \ , \ j = 0, ..., N-1 \ , \tag{1.6.51}$$

and the minimal dual $^\circ\gamma$ satisfies

$$M_{^\circ\gamma}^*(j) = M_g^*(j) \, \left(M \, M_g(j) \, M_g^*(j) \right)^{-1} \ , \ j = 0, ..., N-1 \ , \tag{1.6.52}$$

so that in particular

$$M \, M_{^\circ\gamma}(j) \, M_{^\circ\gamma}^*(j) = \left(M \, M_g(j) \, M_g^*(j) \right)^{-1} \ , \ j = 0, ..., N-1 \ . \tag{1.6.53}$$

For the inverse frame operator $S_g^{-1} = S_{\circ\gamma}$ the last formula has as a consequence that

$$(S_{\circ\gamma}\, f)(j) = \sum_{l=-\infty}^{\infty} \left(\left(M\, M_g(j)\, M_g^*(j) \right)^{-1} \right)_{ol} f(j - lM) \ , \ j \in \mathbb{Z} \ , \tag{1.6.54}$$

holds for $f \in l^2(\mathbb{Z})$.

All these results can be proved by carefully carrying out the program of Sec. 1.2 for the present case, where we note that there are certain simplifications since there are no measure theoretic intricacies.

1.6.4 Discrete Weyl-Heisenberg systems in the time-frequency domain

We let for $k \in \mathbb{Z}, \vartheta \in \mathbb{R}$

$$h_{k,\vartheta}(j) = h(j-k)\, e^{2\pi i j \vartheta} \ , \ j \in \mathbb{Z} \ , \tag{1.6.55}$$

when $h \in l^2(\mathbb{Z})$. With M, N as in Subsec. 1.6.3 we use in this section the notation

$$g_{nm}(j) = g_{nN,m/M}(j) = g(j-nN)\, e^{2\pi i m j/M} \ , \ j \in \mathbb{Z} \ , \tag{1.6.56}$$

for $n \in \mathbb{Z}$, $m = 0, ..., M-1$, where, as usual, we do not bother about the factor $\exp(-2\pi i m n N/M)$ omitted at the right-hand side. We also consider the time-frequency shift operators U_{kl} defined for $k, l \in \mathbb{Z}$ by

$$U_{kl}\, h = h_{kM,l/N} \ , \ h \in l^2(\mathbb{Z}) \tag{1.6.57}$$

The proofs of the main results in this subsection can be given by carefully mimicking the arguments presented in [Jan95b] for the time-continuous case; also, they are presented in all detail in [Jan94a], Ch. 3, where we note that also the case of discrete, periodic Weyl-Heisenberg systems is treated in [Jan94a].

1.6.4.1 Fundamental Identity

Let $f^{(1)}, f^{(2)}, f^{(3)}, f^{(4)} \in l^2(\mathbb{Z})$, and assume that at least one of $f^{(1)}$, $f^{(2)}$ and at least one of $f^{(3)}$, $f^{(4)}$ generates a discrete Weyl-Heisenberg system (for the parameters N and M) with a finite frame upper bound. Also assume that

$$\sum_{k=-\infty}^{\infty} \sum_{l=0}^{N-1} |\langle f^{(3)}, f^{(2)}_{kM,l/N} \rangle|\, |\langle f^{(1)}_{kM,l/N}, f^{(4)} \rangle| < \infty \ . \tag{1.6.58}$$

Then there holds

$$\sum_{n=-\infty}^{\infty}\sum_{m=0}^{M-1}\langle f^{(1)},f^{(2)}_{nN,m/M}\rangle\langle f^{(3)}_{nN,m/M},f^{(4)}\rangle) = \tag{1.6.59}$$
$$= \frac{M}{N}\sum_{k=-\infty}^{\infty}\sum_{l=0}^{N-1}\langle f^{(3)},f^{(2)}_{kM,l/N}\rangle\langle f^{(1)}_{kM,l/N},f^{(4)}\rangle\ .$$

1.6.4.2 Wexler-Raz biorthogonality condition for duality

Assume that the two discrete Weyl-Heisenberg $g_{nN,m/M}$, $(n,m)\in\mathbb{Z}\times\mathcal{I}$, and $\gamma_{nN,m/M}$, $(n,m)\in\mathbb{Z}\times\mathcal{I}$, have a finite frame upper bound. Then the systems are dual if and only if

$$\langle\gamma,g_{kM,l/N}\rangle=\frac{N}{M}\,\delta_{ko}\,\delta_{lo}\ ,\ k\in\mathbb{Z}\ ,\ l=0,...,N-1\ . \tag{1.6.60}$$

The condition (1.6.60) can also be written as

$$U_g\,\gamma=\underline{\sigma}\ ;\ \underline{\sigma}=\frac{N}{M}\,(\delta_{ko}\,\delta_{lo})_{k\in\mathbb{Z},l=0,...,N-1}\ , \tag{1.6.61}$$

where U_g is the linear mapping of $l^2(\mathbb{Z})$ defined by

$$U_g\,f=(\langle f,g_{kM,l/N}\rangle)_{k\in\mathbb{Z},l=0,...,N-1}\ ,\ f\in l^2(\mathbb{Z})\ . \tag{1.6.62}$$

1.6.4.3 Frame bounds in the time-frequency domain

For any $A\geq 0$, $B<\infty$ there holds

$$A\,||f||^2\leq\sum_{n=-\infty}^{\infty}\sum_{m=0}^{M-1}|\langle f,g_{nN,m/M}\rangle|^2\leq B\,||f||^2\ ,\ f\in l^2(\mathbb{Z})\ , \tag{1.6.63}$$

$\Leftrightarrow$

$$A\,I\leq\frac{M}{N}\,U_g\,U_g^*\leq B\,I$$

with I the identity operator of $l^2(\mathbb{Z}\times\{0,...,N-1\})$. Also, the discrete Weyl-Heisenberg system $g_{nN,m/M}$, $(n,m)\in\mathbb{Z}\times\mathcal{I}$, has the finite frame upper bound B_g if and only if the system $g_{kM,l/N}$, $k\in\mathbb{Z}$, $l=0,...,N-1$, has the frame upper bound $N\,M^{-1}\,B_g$.

1.6.4.4 Frame operator representation in the time-frequency domain

Assume that the system $g_{nN,m/M}$, $(n,m)\in\mathbb{Z}\times\mathcal{I}$, has a finite frame upper bound. Then the frame operator S_g has the representation

$$S_g=\frac{M}{N}\sum_{k=-\infty}^{\infty}\sum_{l=0}^{N-1}c_{kl}\,U_{kl}\ , \tag{1.6.64}$$

where

$$c_{kl} = (U_g\,U_g^*)_{kl;oo} = \langle g, g_{kM,l/N} \rangle \ , \ k \in \mathbb{Z} \ , \ l = 0, ..., N-1 \ , \tag{1.6.65}$$

in the sense that for any $f, h \in l^2(\mathbb{Z})$ such that

$$\sum_{k=-\infty}^{\infty} \sum_{l=0}^{N-1} |\langle U_{kl}\, f, h \rangle|^2 < \infty \tag{1.6.66}$$

there holds

$$\langle S_g\, f, h \rangle = \frac{M}{N} \sum_{k=-\infty}^{\infty} \sum_{l=0}^{N-1} c_{kl} \langle U_{kl}\, f, h \rangle \ . \tag{1.6.67}$$

In the case that g satisfies Tolimieri and Orr's condition A,

$$E := \sum_{k=-\infty}^{\infty} \sum_{l=0}^{N-1} |\langle g, g_{kM,l/N} \rangle| < \infty \ , \tag{1.6.68}$$

the system $g_{nN,m/M}$, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, has the frame upper bound E, and the representation (1.6.64) is unconditional.

1.6.4.5 Computation and characterization of minimal duals

Assume that the system $g_{nN,m/M}$, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, is a frame. Then the minimal dual ${}^\circ\gamma$ can be computed as

$${}^\circ\gamma = U_g^*(U_g\,U_g^*)^{-1}\,\underline{\sigma} = \frac{N}{M} \sum_{k=-\infty}^{\infty} \sum_{l=0}^{N-1} \left((U_g\,U_g^*)^{-1}\right)_{kl;oo} U_{kl}\, g \ , \tag{1.6.69}$$

see (1.6.61). Also, the two systems $g_{nN,m/M}$, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, and $\gamma_{nN,m/M}$, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, both having a finite frame upper bound, are dual if and only if

$$\frac{M}{N}\, U_g\, U_\gamma^* = I \ , \tag{1.6.70}$$

and the minimal dual ${}^\circ\gamma$ satisfies

$$\frac{M}{N}\, U_{{}^\circ\gamma}^* = U_g^*(U_g\,U_g^*)^{-1} \ , \tag{1.6.71}$$

so that in particular

$$\frac{M}{N}\, U_{{}^\circ\gamma}\, U_{{}^\circ\gamma}^* = \left(\frac{M}{N}\, U_g\, U_g^*\right)^{-1} \ . \tag{1.6.72}$$

As a consequence the inverse frame operator $S_g^{-1} = S_{\circ\gamma}$ has the representation

$$S_{\circ\gamma} = \sum_{k=-\infty}^{\infty} \sum_{l=0}^{N-1} \left(\left(\frac{M}{N} U_g U_g^*\right)^{-1}\right)_{kl;oo} U_{kl} \ , \tag{1.6.73}$$

with the same precautions as in the previous subsection.

1.6.4.6 Connection between $H_g(\vartheta)$ and U_g

Let for $K \in \mathbb{N}$ the Hilbert space $L^2([0, M^{-1}) \times \{0, ..., K-1\}$ be denoted by Φ_K, so that

$$\varphi \in \Phi_K \Leftrightarrow \underline{\varphi}(\vartheta) \in \mathbb{C}^K \ , \ \vartheta \in [0, M^{-1}) \ \& \tag{1.6.74}$$

$$\|\varphi\|^2 := \int_0^{1/M} \sum_{k=0}^{K-1} |\varphi_k(\vartheta)|^2 \, d\vartheta < \infty \ .$$

Assume that the system $g_{nN,m/M}$, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, has a finite frame upper bound, and define a linear mapping $\mathcal{H}_g : \Phi_N \to \Phi_M$ by

$$(\mathcal{H}_g \, \varphi)(\vartheta) = H_g^T(\vartheta) \, \underline{\varphi}(\vartheta) \ , \ \text{a.e. } \vartheta \in \left[0, \frac{1}{M}\right) \ , \tag{1.6.75}$$

for $\varphi \in \Phi_N$. Then there holds

$$U_g^* = \frac{1}{\sqrt{M}} \, \mathcal{F}^{-1} \, R \mathcal{H}_g \, Q \ , \tag{1.6.76}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform, defined by

$$\mathcal{F}^{-1} : H \in L^2([0,1)) \to \mathcal{F}^{-1} H = \left(\int_0^1 e^{2\pi i j \vartheta} \, H(\vartheta) \, d\vartheta\right)_{j \in \mathbb{Z}} \in l^2(\mathbb{Z}) \ , \tag{1.6.77}$$

and R and Q are unitary operators, defined by

$$Q : \underline{\alpha} \in l^2(\mathbb{Z} \times \{0, ..., N-1\}) \to$$
$$\to (Q \, \underline{\alpha})(\vartheta) = \left(\sqrt{M} \sum_{k=-\infty}^{\infty} \alpha_{kl} \, e^{2\pi i k l M/N} \, e^{-2\pi i k M \vartheta}\right)_{l=0,...,N-1} \in \Phi_N \ , \tag{1.6.78}$$

and

$$R : \psi \in \Phi_M \to \psi_{-\lfloor M\vartheta \rfloor}(\vartheta - M^{-1} \lfloor M\vartheta \rfloor) \in L^2([0,1)) \ , \tag{1.6.79}$$

respectively.

1.6.5 *Discrete Weyl-Heisenberg systems in the Zak transform domain*

We now consider discrete Weyl-Heisenberg systems in the Zak transform domain, and we let

$$LCM(N, M) = Nq = Mp \tag{1.6.80}$$

with $p, q \in \mathbb{N}$ and $\mathrm{GCD}(p, q) = 1$. Observe that

$$K := \frac{N}{p} = \frac{M}{q} \in \mathbb{Z} \,. \tag{1.6.81}$$

We define for $L \in \mathbb{N}$ the discrete Zak transform $\mathcal{Z}_L$ by

$$(\mathcal{Z}_L\, f)(j, \vartheta) = \sum_{k=-\infty}^{\infty} f(j - kL)\, e^{2\pi i k \vartheta} \;,\; j \in \mathbb{Z} \,,\; \text{a.e. } \vartheta \in \mathbb{R} \,, \tag{1.6.82}$$

where $f \in l^2(\mathbb{Z})$. This definition agrees with the one given in [Hei89], except that ϑ is replaced by $-\vartheta$. Then we have the following

$$\sum_{j=-\infty}^{\infty} f(j)\, h^*(j) = \sum_{j=0}^{L-1} \int_0^1 (\mathcal{Z}_L\, f)(j, \vartheta)(\mathcal{Z}_L\, h)^*(j, \vartheta)\, d\vartheta \;, \tag{1.6.83}$$

$$f(j) = \int_0^1 (\mathcal{Z}_L\, f)(j, \vartheta)\, d\vartheta \;, j \in \mathbb{Z} \;, \tag{1.6.84}$$

$$(\mathcal{Z}_L\, f)(j + L, \vartheta) = e^{2\pi i \vartheta}(\mathcal{Z}_L\, f)(j, \vartheta) \;, j \in \mathbb{Z}, \text{a.e. } \vartheta \in \mathbb{R}, \tag{1.6.85}$$

$$(\mathcal{Z}_L\, f)(j, \vartheta + 1) = (\mathcal{Z}_L\, f)(j, \vartheta) \;, j \in \mathbb{Z} \,,\; \text{a.e. } \vartheta \in \mathbb{R} \;, \tag{1.6.86}$$

where $f, h \in l^2(\mathbb{Z})$. Moreover, for any $Z \in L^2_{\mathrm{loc}}(\mathbb{Z} \times \mathbb{R})$ such that

$$Z(j + L, \vartheta) = e^{2\pi i \vartheta}\, Z(j, \vartheta) \;,\; Z(j, \vartheta + 1) = Z(j, \vartheta) \;, \quad j \in \mathbb{Z} \,,\; \text{a.e. } \vartheta \in \mathbb{R} \;, \tag{1.6.87}$$

there is a unique $f \in l^2(\mathbb{Z})$ such that $Z = \mathcal{Z}_L\, f$. Observe that there is no discrete version of (1.5.6) since the (discrete) Fourier transform does not map $l^2(\mathbb{Z})$ into itself.

We shall make the choice $L = M$, and suppress the subscript L in $\mathcal{Z}_L$ so that

$$(\mathcal{Z}\, h)(j, \vartheta) = \sum_{k=-\infty}^{\infty} h(j - kM)\, e^{2\pi i k \vartheta} \;,\; j \in \mathbb{Z} \,,\; \text{a.e. } \vartheta \in \mathbb{R} \;, \tag{1.6.88}$$

for $h \in l^2(\mathbb{Z})$. At the end of this subsection we shall indicate how the results given below would look like when the choice $L = N$ instead of $L = M$ were made in $\mathcal{Z}_L$.

We set for $f, h \in l^2(\mathbb{Z})$ and $j \in \mathbb{Z}$, a.e. $\vartheta \in \mathbb{R}$

$$\Phi^f(j,\vartheta) = p^{-1/2}\Big((\mathcal{Z} f)\Big(j - lN, \vartheta + \frac{k}{p}\Big)\Big)_{k=0,\ldots,p-1\,;\,l=0,\ldots,q-1}\ , \tag{1.6.89}$$

$$A^{fh}(j,\vartheta) = \Big(A^{fh}_{kr}(j,\vartheta)\Big)_{k,r=0,\ldots,p-1} = \Phi^f(j,\vartheta)\left(\Phi^h(j,\vartheta)\right)^* . \tag{1.6.90}$$

Observe that (see (1.6.81) for the definition of K)

$$\frac{1}{p}\langle f, h\rangle = \sum_{l=0}^{q-1}\sum_{j=0}^{K-1}\int_0^{p^{-1}} \langle \Phi^f_{.l}(j,\vartheta), \Phi^h_{.l}(j,\vartheta)\rangle\, d\vartheta\ , \tag{1.6.91}$$

where $\Phi^f_{.l}(j,\vartheta)$, $\Phi^h_{.l}(j,\vartheta)$ are the l^{th} column of $\Phi^f(j,\vartheta)$, $\Phi^h(j,\vartheta)$, and the inner product at the right-hand side of (1.6.91) is the inner product of $\mathbb{C}^p$. Formula (1.6.91) follows from (1.6.83), (1.6.85) and (1.6.86), and the fact that the sets

$$\{lN\,|\,l = 0, ..., q-1\} \text{ and } \{lK\,|\,l = 0, ..., q-1\} \tag{1.6.92}$$

agree $\bmod M$ since $\mathrm{GCD}(p, q) = 1$.

1.6.5.1 Frame operator representation

Assume that the system $g_{nN,m/M}$, $(n, m) \in \mathbb{Z} \times \mathcal{I}$, has a finite frame upper bound, and let $f \in l^2(\mathbb{Z})$. Then there holds

$$\Phi^{S_g f}(j,\vartheta) = M\,A^{gg}(j,\vartheta)\,\Phi^f(j,\vartheta)\ ,\ j \in \mathbb{Z}\ ,\ \text{a.e. } \vartheta \in \mathbb{R}\ , \tag{1.6.93}$$

and

$$\langle S_g\, f, f\rangle = \frac{M}{p}\sum_{l=0}^{q-1}\sum_{j=0}^{K-1} \langle A^{gg}\Phi^f_{.l}(j,\vartheta), \Phi^f_{.l}(j,\vartheta)\rangle\, d\vartheta\ . \tag{1.6.94}$$

These formulas follow in the same way as in the time-continuous case.

1.6.5.2 Frame bounds

For any $A \ge 0$, $B < \infty$ there holds

$$A\,||f||^2 \le \sum_{n=-\infty}^{\infty}\sum_{m=0}^{M-1} |\langle f, g_{nN,m/M}\rangle|^2 \le B\,||f||^2\ ,\ f \in l^2(\mathbb{Z})\ ,$$

$$\Leftrightarrow \tag{1.6.95}$$

$$\frac{1}{M}\,A\,I_{p\times p} \le A^{gg}(j,\vartheta) \le \frac{1}{M}\,B\,I_{p\times p}\ ,\ j \in \mathbb{Z}\ ,\ \text{a.e. } \vartheta \in \mathbb{R}\ .$$

This follows readily from (1.6.91) and (1.6.94). As in the time-continuous case there holds

$$\begin{aligned} A^{gg}(j+K,\vartheta) &= F\,A^{gg}(j,\vartheta)\,F^{-1}\ , \\ A^{gg}\Big(j,\vartheta+\frac{1}{p}\Big) &= J\,A^{gg}(j,\vartheta)\,J^{-1} \end{aligned} \tag{1.6.96}$$

with F and J as in Subsec. 1.5.2, so that the right-hand side member of (1.6.95) only needs to be checked for $j = 0, ..., K-1$, $\vartheta \in [0, p^{-1})$.

1.6.5.3 Fourier series expansion of matrix elements

Let $f, h \in l^2(\mathbb{Z})$. Then $A_{kr}^{fh}(j,\vartheta)$ is N-periodic in $j \in \mathbb{Z}$ and 1-periodic in $\vartheta \in \mathbb{R}$ with Fourier series

$$A_{kr}^{fh}(j,\vartheta) \sim \sum_{n=0}^{N-1} \sum_{m=-\infty}^{\infty} d_{nmkr}\, e^{-2\pi i n j/N}\, e^{-2\pi i m\vartheta}\ , \tag{1.6.97}$$

where

$$d_{nmkr} = \begin{cases} \dfrac{q}{pM}\, e^{-2\pi i m r/p} \langle f, h_{mM,-n/N}\rangle & nq \equiv (r-k) \bmod p, \\ 0 & \text{otherwise.} \end{cases} \tag{1.6.98}$$

This result can be derived as [Jan95c], Prop. 1.1.

1.6.5.4 Wexler-Raz biorthogonality condition in the Zak transform domain

Assume that $g, \gamma \in l^2(\mathbb{Z})$. Then the two systems $g_{kM,l/N}$, $(k,l) \in \mathbb{Z} \times \{0, ..., N-1\}$, and $\gamma_{kM,l/N}$, $(k,l) \in \mathbb{Z} \times \{0, ..., N-1\}$ are biorthogonal, see (1.6.60), if and only if

$$A^{g\gamma}(j,\vartheta) = \Phi^g(j,\vartheta)\,(\Phi^\gamma(j,\vartheta))^* = \frac{1}{M} I_{p\times p}\ ,\ j \in \mathbb{Z}\ ,\ \text{a.e. } \vartheta \in \mathbb{R}\ . \tag{1.6.99}$$

This follows immediately from 1.6.5.3.

1.6.5.5 Characterization of minimal dual

Assume that the system $g_{nN,m/M}$, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, is a frame. Then there holds

$$\Big(\Phi^{{}^\circ\gamma}(j,\vartheta)\Big)^* = \frac{1}{M}\,(\Phi^g(j,\vartheta))^*\,\left(A^{gg}(j,\vartheta)\right)^{-1}\ , j \in \mathbb{Z},\ \text{a.e. } \vartheta \in \mathbb{R}\ , \tag{1.6.100}$$

and in particular

$$M\,A^{{}^\circ\gamma{}^\circ\gamma}(j,\vartheta) = \left(M\,A^{gg}(j,\vartheta)\right)^{-1} \tag{1.6.101}$$

This result follows from 6.5.1 and the fact that $S_g\,{}^\circ\gamma = g$.

1.6.5.6 Condition A for discrete Weyl-Heisenberg systems

There holds (see Subsec. ref6.4.4) that

$$
\begin{aligned}
& g \in l^2(\mathbb{Z}) \text{ satisfies condition A} \\
\Leftrightarrow & \\
& A^{gg}(j,\vartheta) \text{ has an absolutely convergent Fourier series.}
\end{aligned}
\tag{1.6.102}
$$

Furthermore, when $g_{nN,m/M}$, $(n,m) \in \mathbb{Z} \times \mathcal{I}$, is a frame and g satisfies condition A, then ${}^\circ\gamma$ satisfies condition A as well. These results can be proved in the same way as [Jan95c], Thm. 1.3 and 1.4.

1.6.5.7 Different choice of the Zak transform

We briefly indicate how the results of this subsection look like when we choose $L = N$, rather than M, in (1.6.82). Denote for $f, h \in l^2(\mathbb{Z})$

$$\Psi^f(j,\vartheta) = p^{-1/2}\Big((\mathcal{Z}_N f)\Big(j - kM, \vartheta - \frac{l}{q}\Big)\Big)_{k=0,\ldots,p-1\,;\,l=0,\ldots,q-1}, \tag{1.6.103}$$

$$B^{fh}(j,\vartheta) = \Big(B^{fh}_{kr}(j,\vartheta)\Big)_{k,r=0,\ldots,p-1} = \Psi^f(j,\vartheta)\,\big(\Psi^h(j,\vartheta)\big)^* \ . \tag{1.6.104}$$

Then there holds

$$\Psi^{S_g f}(j,\vartheta) = N\,B^{gg}(j,\vartheta)\,\Psi^f(j,\vartheta)\ ,\ j \in \mathbb{Z}\ ,\ \text{a.e. } \vartheta \in \mathbb{R}\ . \tag{1.6.105}$$

Also, the matrix elements $B^{fh}_{kr}(j,\vartheta)$ are N-periodic in $j \in \mathbb{Z}$ and q^{-1}-periodic in $\vartheta \in \mathbb{R}$, with Fourier series

$$B^{fh}_{kr}(j,\vartheta) \sim \sum_{n=0}^{N-1} \sum_{m=-\infty}^{\infty} e_{nmkr}\, e^{-2\pi i n j/N}\, e^{-2\pi i m q \vartheta}\ , \tag{1.6.106}$$

where

$$e_{nmkr} = \frac{q}{pN}\, e^{2\pi i n k q/p}\, \langle f, h_{(r-k+mp)M,-n/N}\rangle\ . \tag{1.6.107}$$

From these two results the main results of this subsection can be rederived.

2

Gabor systems and the Balian–Low Theorem

John J. Benedetto, Christopher Heil, and David F. Walnut

"If it is true, it can be proved." – Enrico Fermi

ABSTRACT – The *Balian–Low theorem* (BLT) is a key result in time-frequency analysis, originally stated by Balian and, independently, by Low, as: If a Gabor system $\{e^{2\pi imbt}\,g(t-na)\}_{m,n\in\mathbb{Z}}$ with $ab=1$ forms an orthonormal basis for $L^2(\mathbb{R})$ then

$$\left(\int_{-\infty}^{\infty} |t\,g(t)|^2\,dt\right)\left(\int_{-\infty}^{\infty} |\gamma\,\hat{g}(\gamma)|^2\,d\gamma\right) = +\infty.$$

The BLT was later extended from orthonormal bases to exact frames. This paper presents a tutorial on Gabor systems, the BLT, and related topics, such as the Zak transform and Wilson bases. Because of the fact that $(g')^\wedge(\gamma) = 2\pi i\gamma\,\hat{g}(\gamma)$, the role of differentiation in the proof of the BLT is examined carefully. We include the construction of a complete Gabor system of the form $\{e^{2\pi ib_m t}\,g(t-a_n)\}$ such that $\{(a_n,b_m)\}$ has density strictly less than 1, and an Amalgam BLT that provides distinct restrictions on Gabor systems $\{e^{2\pi imbt}\,g(t-na)\}$ that form exact frames.

2.1 Introduction

2.1.1 History

The *Balian–Low theorem* (BLT) is a key result in time-frequency analysis. It was originally stated by Balian [Bal81] and, independently, by Low [Low85], as: *If a Gabor system* $\{e^{2\pi imbt}\,g(t-na)\}_{m,n\in\mathbb{Z}}$ *with* $ab=1$ *forms an orthonormal basis for* $L^2(\mathbb{R}) = \{f : \int_{-\infty}^{\infty} |f(t)|^2\,dt < \infty\}$ *then*

$$\left(\int_{-\infty}^{\infty} |t\,g(t)|^2\,dt\right)\left(\int_{-\infty}^{\infty} |\gamma\,\hat{g}(\gamma)|^2\,d\gamma\right) \;=\; +\infty,$$

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where $\hat{g}(\gamma) = \int_{-\infty}^{\infty} g(t)\, e^{-2\pi i\gamma t}\, dt$ is the Fourier transform of g, $\mathbb{R}$ is the real line, and $\mathbb{Z}$ is the set of integers. The setting for Balian was Gabor's notion of *information area* in signal analysis as well as the role of the *phase plane* in quantum mechanics [vN55]. The background for Low included completeness properties of classical wave packets in quantum mechanics (for example, [vN55] including von Neumann's work from the late 1920's).

The proofs given by Balian and Low each contained a gap arising from the fact that square-integrability of the partial derivatives of a function does not imply continuity of the function itself in dimensions two and higher. This gap was independently addressed in two ways. Battle provided an elegant and entirely new proof based on the operator theory associated with the Classical Uncertainty Principle Inequality [Bat88]. Daubechies, Coifman, and Semmes retained the original approach of Balian and Low, filling the gap directly [Dau90]. In the process, they extended the result from Gabor systems which form orthonormal bases for $L^2(\mathbb{R})$ to Gabor systems which form *exact frames*—a natural generalization of orthonormal bases that we describe below. Although Battle's uncertainty principle approach to the BLT for orthonormal bases was both elegant and simple, and although the step from orthonormal bases to exact frames seems small, it was a non-trivial matter to adapt Battle's proof to the case of exact frames. This was accomplished by Daubechies and Janssen in [DJ93]. Their proof is beautifully recounted in Daubechies' recent monograph [Dau92]. The Balian–Low theorem for locally compact abelian groups is discussed in Chapter 6 in this book.

In order to describe frames and exact frames, let us first recall and describe orthonormal bases. We begin with the L^2*-norm* of a complex-valued function f, given by

$$||f||_2 = \left(\int |f(t)|^2\, dt\right)^{1/2}.$$

Integrals with unspecified limits are to be taken over the entire real line $\mathbb{R}$. The quantity $||f||_2^2$ is often referred to as the *energy* of f. Thus $L^2(\mathbb{R})$ is the space of all functions f which have *finite energy*. We can define an *inner product* on $L^2(\mathbb{R})$ by

$$\langle f, g\rangle = \int f(t)\, \overline{g(t)}\, dt,$$

the bar denoting complex conjugation.

A sequence $\{f_k\}$ of functions in $L^2(\mathbb{R})$ is *orthonormal* if

$$\langle f_k, f_\ell\rangle = \delta_{k,\ell} = \begin{cases} 1, & \text{if } k = \ell \\ 0, & \text{if } k \neq \ell \end{cases}$$

A sequence $\{f_k\}$ is a *basis* for $L^2(\mathbb{R})$ if

$$\forall f \in L^2(\mathbb{R}), \ \exists \text{ unique scalars } c_k(f) \text{ such that } f = \sum_k c_k(f)\, f_k. \quad (2.1.1)$$

A sequence which is both orthonormal and a basis is an *orthonormal basis*. In this case, the scalars $c_k(f)$ are given simply by inner products of f with the basis elements: $c_k(f) = \langle f, f_k \rangle$. Thus, for an orthonormal basis,

$$\forall f \in L^2(\mathbb{R}), \quad f = \sum_k \langle f, f_k \rangle\, f_k. \tag{2.1.2}$$

Moreover, the *Plancherel formula* states that the energy of f is determined by the "energy" of the coefficients $\langle f, f_k \rangle$:

$$\forall f \in L^2(\mathbb{R}), \quad \|f\|_2^2 = \sum_k |\langle f, f_k \rangle|^2. \tag{2.1.3}$$

The Plancherel formula motivates the definition of frames. A sequence $\{f_k\}$ is a *frame* if there exist constants A, $B > 0$, called the *frame bounds*, so that the following *approximate Plancherel formula* holds:

$$\forall f \in L^2(\mathbb{R}), \quad A\,\|f\|_2^2 \le \sum_k |\langle f, f_k \rangle|^2 \le B\,\|f\|_2^2. \tag{2.1.4}$$

Thus, for a frame, the energy $\|f\|_2^2$ of f is "equivalent" but not necessarily equal to the coefficient energy $\sum |\langle f, f_k \rangle|^2$. If $A = B$ then the frame is said to be *tight*.

Unlike orthonormal bases, this approximate Plancherel formula is the *only* requirement we place on a sequence in order for it to be called a frame. We do not require that a frame form a basis, and in fact there are many useful frames that are not bases. This stems from the surprising fact that every frame provides a series representation of functions f similar to the one given in (2.1.2) for orthonormal bases. In particular, for any frame $\{f_k\}$ there will exist a *dual frame* $\{\tilde{f}_k\}$ so that

$$\forall f \in L^2(\mathbb{R}), \quad f = \sum_k \langle f, \tilde{f}_k \rangle\, f_k = \sum_k \langle f, f_k \rangle\, \tilde{f}_k. \tag{2.1.5}$$

If the frame $\{f_k\}$ happens to form a basis, then the only way to write $f = \sum c_k\, f_k$ is with $c_k = \langle f, \tilde{f}_k \rangle$. If the frame is not a basis then there will exist other choices of c_k so that $f = \sum c_k\, f_k$. Among all these choices, $c_k = \langle f, \tilde{f}_k \rangle$ is still "best" in several ways. For example, it is the choice that minimizes the quantity $\sum |c_k|^2$.

It is the concept of *exactness* that distinguishes between frames that are bases and those that are not. A frame $\{f_k\}$ is *exact* if it ceases to be a frame when any single element f_ℓ is deleted, that is, $\{f_k\}_{k\neq\ell}$ is not a frame for any ℓ. A frame is a basis if and only if it is exact. Therefore, an exact frame satisfies both the approximate Plancherel formula (2.1.4) and has the property that (2.1.5) is the *unique* series representation of functions f in terms of the frame elements f_k or the dual frame elements $\tilde{f}_k$. All orthonormal bases are exact frames, and all exact frames are frames. However, there exist frames that are not exact, and there exist exact frames that are not orthonormal bases. Moreover, tightness and exactness are distinct concepts: there exist tight frames that are exact and there exist tight frames that are inexact.

We can now state the BLT for exact frames. For simplicity of notation, we write

$$g_{p,q}(t) = e^{2\pi ipt}\, g(t-q).$$

We call $\{g_{mb,na}\}$ a *Gabor system*, and we assume that the indices m, n range over $\mathbb{Z}$ unless otherwise specified. Then the BLT is:

Theorem 2.1.1 (BLT) *Let $g \in L^2(\mathbb{R})$, and let a, $b > 0$ satisfy $ab = 1$. If the Gabor system $\{g_{mb,na}\}$ is an exact frame for $L^2(\mathbb{R})$, then*

$$\|tg(t)\|_2\, \|\gamma\hat{g}(\gamma)\|_2 \;=\; +\infty. \tag{2.1.6}$$

We shall see in Section 2.2.2 that the hypothesis $ab = 1$ in the BLT is redundant, following from the hypothesis that the Gabor system $\{g_{mb,na}\}$ forms an exact frame for $L^2(\mathbb{R})$ (Theorem 2.2.2). Moreover, when $ab = 1$, a Gabor system is a frame if and only if it is an exact frame (Theorem 2.3.1d).

It is instructive to compare the BLT to the Classical Uncertainty Principle Inequality. We use the symbol $\hat{\mathbb{R}}$ to denote the real line thought of as the frequency axis, and we let $\mathbb{C}$ denote the set of complex numbers.

Theorem 2.1.2 (Classical Uncertainty Principle Inequality) *Let $(t_0, \gamma_0) \in \mathbb{R} \times \hat{\mathbb{R}}$. Then*

$$\forall\, f \in L^2(\mathbb{R}), \quad \|f\|_2^2 \;\le\; 4\pi\, \|(t-t_0)\, f(t)\|_2\, \|(\gamma-\gamma_0)\, \hat{f}(\gamma)\|_2, \tag{2.1.7}$$

and there is equality in (2.1.7) *if and only if $f(t) = C\, e^{2\pi i t\gamma_0}\, e^{-s(t-t_0)^2}$ for some $C \in \mathbb{C}$ and $s > 0$.*

Thus, if a Gabor system $\{g_{mb,na}\}$ forms an exact frame then the right side of (2.1.7) is infinite when f is replaced by g. In this sense, the BLT *maximizes* the Classical Uncertainty Principle Inequality. For comparison, note that if $f(t) = (\sin 2\pi\Omega t)/(\pi t)$, or if $f \in L^2(\mathbb{R})$ behaves like $|t|^\alpha$ as $t \to \infty$ with $\alpha \in [-3/2, -1/2)$, then the right side of (2.1.7) is infinite.

2.1.2 Outline

In this paper, we present a tutorial on Gabor systems, focusing on the BLT and related phenomena.

We begin in Section 2.2 by presenting background and context. After some brief remarks on frames and Gabor systems, we show in Section 2.2.1 that properties of the Gabor system $\{g_{mb,na}\}$ are related to the "density" of the set $\Lambda = \{(na, mb)\}_{m,n\in\mathbb{Z}} = a\mathbb{Z} \times b\mathbb{Z}$, which is a *rectangular lattice* in $\mathbb{R} \times \hat{\mathbb{R}}$. Small values of a, b correspond to high density for Λ, while large values of a, b correspond to low density. We see that the condition $ab = 1$, corresponding to Λ with density 1, can be interpreted as a Nyquist phenomenon for Gabor systems: a Gabor system can be a frame only when $ab \leq 1$, can be an exact frame only when $ab = 1$, and must be *incomplete* if $ab > 1$. In Section 2.2.2 we turn from considering such "regular" Gabor systems to "irregular" Gabor systems $\{g_{p,q}\}_{(q,p)\in\Lambda}$, where Λ is now a discrete subset of $\mathbb{R} \times \hat{\mathbb{R}}$ which need not form a lattice. We define the density of such irregular sets, and see that properties of irregular Gabor systems are again tied to the density of Λ. We construct in Section 2.2.3 an *irregular* Gabor system that violates the Nyquist phenomenon observed by regular Gabor systems. Specifically, for each integer $K > 0$ we show that there exists an irregular Gabor system $\{g_{p,q}\}_{(q,p)\in\Lambda}$ that is complete in $L^2(\mathbb{R})$ even though Λ has density $2/K$.

In Section 2.3 we develop and apply the *Zak transform*, a major tool in the analysis of Gabor systems. The basic definitions and properties of the Zak transform are laid out in Section 2.3.1. We use the Zak Transform in Section 2.3.2 to prove a variation on the BLT that we call the *Amalgam BLT*. This uses the same hypotheses as the BLT and gives a related but distinct conclusion. The Amalgam BLT has the advantage of possessing an elegantly simple proof that applies both to orthonormal bases and to exact frames. We provide examples illustrating the distinction between the BLT and the Amalgam BLT. In Section 2.3.3 we show that these examples provide insight into the class of Gabor systems that forms *Bessel sequences*, i.e., Gabor systems that satisfy at least the upper frame bound. In Section 2.3.4 we discuss discrete analogues of Gabor systems and the Zak transform and compare BLT phenomena in the continuous and discrete settings.

The BLT places a severe restriction on those functions g such that $\{g_{mb,na}\}$ can form an exact frame. *Wilson bases* were originated as a way to avoid this restriction. They are constructed from linear combinations of Gabor system elements. This seemingly small modification permits Wilson bases to avoid the BLT phenomenon. We devote Section 2.4 to a discussion of Wilson bases. We show how they are related to Gabor systems for which $\Lambda = \{(na, mb)\}$ has density 1/2 and discuss how they can be analyzed using the Zak transform.

We set the stage for the BLT itself in Section 2.5 by presenting a number of elementary but revealing calculations, as well as placing the BLT in the context of some major ideas from classical harmonic analysis. In the category of *calculations* there is a comparison of distributional and ordinary differentiation (Theorem 2.5.1) and an assortment of facts dealing with the BLT as related to types of bounded variation arising in classical harmonic analysis (Example 2.5.1). In the category of *context*, we prove elementary versions of the BLT using Sobolev's Lemma (Remark 2.5) and Wiener amalgam spaces (Theorem 2.5.3 and Theorem 2.5.4).

Finally, in Section 2.6 we consider the BLT from Battle's uncertainty principle point of view. We elaborate on the proof of the BLT for exact frames given by Daubechies–Janssen. This proof is based on the operator theory associated with the Classical Uncertainty Principle Inequality, which is stated in Theorem 2.1.2 and restated in operator-theoretic terms in Theorem 2.6.1. After proving two lemmas related to Battle's original idea, we prove a *Weak BLT* (Theorem 2.6.4). This result is proved by Hilbert space methods alone, but has the feature of highlighting the role of the *position operator* $\mathcal{P}f(t) = tf(t)$ and the *dual function* $\tilde{g}$ in determining if the Gabor system $\{g_{mb,na}\}$ is an exact frame. Battle's BLT for orthonormal bases is a consequence of Theorem 2.6.4, as is the BLT for several other Gabor systems (Example 2.6.1). Further, Theorem 2.6.4 coupled with a differentiation argument gives another proof of the BLT for exact frames (Theorem 2.6.6). The remainder of Section 2.6 deals with examples closely related to the relation between differentiation and the BLT. We isolate the key step at which Hilbert space methods appear to fail to complete the proof of the BLT for exact frames. We close with an open problem related to this key step, and with the observation that a solution of this problem without the use of differentiation would yield a proof of the BLT in which differentiation plays no role.

We use the usual notation of analysis as found in the books by Hörmander [Hör83], Schwartz [Sch66], and Stein and Weiss [SW71]. For the convenience of the reader we have collected in the Appendix a summary of the notation used throughout the chapter.

2.2 Background

2.2.1 *Gabor systems, Gabor frames, and density*

Frames can be defined in any *Hilbert space*, of which $L^2(\mathbb{R})$ is a prime example. We let H denote any Hilbert space with inner product $\langle f, g\rangle$ and norm $\|f\| = \langle f, f\rangle^{1/2}$. A sequence $\{f_k\}$ in H is a *frame* for H if there exist constants A, $B > 0$ such that the approximate Plancherel formula (2.1.4) holds with $L^2(\mathbb{R})$ replaced by H. A, B are the *frame bounds*. The frame is *tight* if $A = B$, and it is *exact* if it is no longer a frame when any one of its elements is removed. A sequence that satisfies the upper frame bound (and which may or may not satisfy the lower frame bound) is called a *Bessel sequence.*

The theory of frames is due to Duffin and Schaeffer [DS52], whose context was the theory of nonharmonic Fourier series. Frame analysis has seen a recent resurgence in both time-frequency analysis and wavelet theory, spurred by the papers [DGM86] and [Dau90]. Expository treatments of frames can be found in [HW89], [Dau92], and [BW94]. Gröchenig [Grö91] determined the nontrivial extension of frames to Banach spaces and developed applications to atomic decompositions. Recent developments in pure frame theory can be found in [CH97, Grö93a, Hol94, Li95]. There are recent applications of frames in signal processing, [BT93], and to the analysis of pseudodifferential operators, [HRT]. More information on frames can be found in Chapter and 5 of this volume.

We define the *translation* $T_x f$ and *modulation* $M_\gamma f$ of a function f by

$$T_x f(t) \;=\; f(t-x) \quad \text{and} \quad M_\gamma f(t) \;=\; e^{2\pi i\gamma t} f(t). \tag{2.2.1}$$

Given $g \in L^2(\mathbb{R})$ and fixed a, $b > 0$, the *Gabor system* corresponding to g, a, and b is the sequence $\{g_{mb,na}\}_{m,n\in\mathbb{Z}}$, where

$$g_{p,q}(t) \;=\; M_p T_q g(t) \;=\; e^{2\pi i p t} g(t-q).$$

Gabor systems are sometimes referred to as *Weyl–Heisenberg* or *windowed Fourier* systems.

A Gabor system that forms a frame for $L^2(\mathbb{R})$ is called a *Gabor frame.* The dual frame of a Gabor frame has a particularly nice form, which follows from the calculation $S(g_{mb,na}) = (Sg)_{mb,na}$, where S is the frame operator

$$Sf = \sum_{m,n} \langle f, g_{mb,na}\rangle\, g_{mb,na}. \tag{2.2.2}$$

Replacing g by $S^{-1}g$ gives $S^{-1}(g_{mb,na}) = (S^{-1}g)_{mb,na}$, so the dual frame of $\{g_{mb,na}\}$ is another Gabor frame $\{\tilde{g}_{mb,na}\}$, generated by the *dual function* $\tilde{g} = S^{-1}g$.

The frame operator S and the dual Gabor frame play a key role in understanding and implementing Gabor analysis. A short list of references taking this approach includes [Jan95b, Jan94b, Li94b, LH94, QCL92, RS95c, TO95, WR90, ZZ93b].

The algebraic structure of the lattice $\Lambda = \{(na, mb)\} = a\mathbb{Z} \times b\mathbb{Z}$ has been exploited to derive necessary conditions for a Gabor system $\{g_{mb,na}\}$ to be complete, a frame, or an exact frame, in terms of the value of the product ab. It is the product rather than the individual values of a and

b that are important because if $h(t) = r^{1/2}g(rt)$ is a dilation of g then $h_{p,q}(t) = r^{1/2}g_{p/r,qr}(rt)$. Hence the Gabor system $\{h_{mb,na}\}$ is complete, a frame, or an exact frame if and only if the same is true of the Gabor system $\{g_{mb/r,nar}\}$.

As a corollary of other results on C^* algebras, Rieffel [Rie81] proved that $\{g_{mb,na}\}$ is incomplete for any g if $ab > 1$. This can be viewed as a Nyquist phenomenon for Gabor systems. Rieffel's proof was nonconstructive, following from the value of the coupling constant of the von Neumann algebra generated by the operators $\{M_{mb}T_{na}\}$. Daubechies [Dau90] proved Rieffel's theorem without recourse to operator algebras for the special case that ab is rational and exceeds 1. Her proof exhibited a function orthogonal to each $g_{mb,na}$. Landau [Lan93] obtained the weaker result that $\{g_{mb,na}\}$ cannot be a frame for $L^2(\mathbb{R})$ if $ab > 1$ and both g and $\hat{g}$ satisfy certain decay conditions. His proof did not rely on the algebraic structure of $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ and in fact applies to more general, "irregular" Λ, a topic we explore further in Section 2.2.2. Janssen [Jan94b] proved that $\{g_{mb,na}\}$ cannot be a frame for any $g \in L^2(\mathbb{R})$ if $ab > 1$, without restrictions on g or ab, but again relying on the algebraic structure of the lattice $a\mathbb{Z} \times b\mathbb{Z}$. Daubechies, Landau, and Landau [DLL95] explore the Rieffel incompleteness result in detail, including the connections to operator algebras and to a remarkable result known as the *Wexler–Raz identity*, which was first stated in [WR90].

2.2.2 Irregular Gabor systems

Landau's work in [Lan93] extended the analysis of Gabor systems from "regular" systems $\{g_{mb,na}\}$, where $\Lambda = \{(na, mb)\} = a\mathbb{Z} \times b\mathbb{Z}$ is a rectangular lattice, to "irregular" systems $\{g_{p,q}\}_{(q,p)\in\Lambda}$, where Λ is a discrete subset of $\mathbb{R} \times \hat{\mathbb{R}}$ but not necessarily a lattice. The crucial quantity is still the density of Λ. We give below a precise definition of density due to Beurling. Density concepts were used by Paley and Wiener [PW34] and by Levinson [Lev40] to study spectral gaps and the uniqueness of nonharmonic Fourier series. Beurling's definition of density was used in the beautiful characterization of complete sets of exponentials due to Beurling and Malliavin [BM62, BM67], and by Landau in his studies of completeness properties of exponentials [Lan64], and the sampling and interpolation of entire functions [Lan67].

We shall define the density of a subset Λ of d-dimensional Euclidean space, which we denote by $\mathbb{R}^d$ (usually $d = 1$ or 2 in this chapter). Assume that $\Lambda \subseteq \mathbb{R}^d$ is *uniformly discrete*, i.e., there is a minimum separation between the elements of Λ. Let B denote the ball of volume 1 in $\mathbb{R}^d$ centered at the origin. Given a positive $r > 0$ and given $x \in \mathbb{R}^d$, let $B(x; r) = \{x + rt : t \in B\}$, so that $B(x; r)$ is the ball of volume r^d centered at x. For each r, let $\nu^+(r)$ and $\nu^-(r)$ denote the largest and smallest number of points of Λ

that lie in any $B(x;r)$:

$$\nu^+(r) = \max_{x\in\mathbb{R}^d} \#\{\lambda : \lambda \in \Lambda \cap B(x;r)\}$$

and

$$\nu^-(r) = \min_{x\in\mathbb{R}^d} \#\{\lambda : \lambda \in \Lambda \cap B(x;r)\}.$$

These quantities are finite because Λ is uniformly discrete. The upper and lower uniform Beurling densities of Λ are then

$$D^+(\Lambda) = \limsup_{r\to\infty} \frac{\nu^+(r)}{r^d} \qquad \text{and} \qquad D^-(\Lambda) = \liminf_{r\to\infty} \frac{\nu^-(r)}{r^d}.$$

Landau has shown that these quantities are unchanged if the ball B is replaced by any other bounded set with measure 1. If $D^+(\Lambda) = D^-(\Lambda)$ then the set Λ is said to have *uniform Beurling density* $D(\Lambda) = D^+(\Lambda) = D^-(\Lambda)$.

Example 2.2.1 Let $d = 1$, and consider the rectangular lattice $\Lambda = a\mathbb{Z} = \{an\}_{n\in\mathbb{Z}}$. Here B is the interval $(-1/2, 1/2)$ with length 1, so $B(x;r) = (x - r/2, x + r/2)$ has length r. When $r > a$, there are at least $[r/a] - 1$ and at most $[r/a]$ points of $a\mathbb{Z}$ in $B(x;r)$, where $[t]$ denotes the largest integer not exceeding t. Therefore $a\mathbb{Z}$ has uniform Beurling density $D(a\mathbb{Z}) = 1/a$.

Now consider the consequences of slightly perturbing $a\mathbb{Z}$. Specifically, suppose that $\Lambda = \{\lambda_n\}$ is such that $|\lambda_n - an| < \varepsilon$ for every n. Then the density is unchanged: Λ has uniform Beurling density $D(\Lambda) = 1/a$. ❑

If Λ_1 is a uniformly discrete subset of $\mathbb{R}^{d_1}$ and Λ_2 is a uniformly discrete subset of $\mathbb{R}^{d_2}$ then $\Lambda_1 \times \Lambda_2$ is a uniformly discrete subset of $\mathbb{R}^{d_1+d_2}$, and $D^+(\Lambda_1 \times \Lambda_2) = D^+(\Lambda_1)\, D^+(\Lambda_2)$. Similar calculations hold for the lower uniform densities. In particular, the density of the lattice $a\mathbb{Z} \times b\mathbb{Z}$ in $\mathbb{R} \times \hat{\mathbb{R}}$ is $D(a\mathbb{Z} \times b\mathbb{Z}) = 1/(ab)$.

Landau proved in [Lan93] that if $g \in L^2(\mathbb{R})$ satisfies $|g(t)| \leq C/(1+|t|^\alpha)$ and $|\hat{g}(\gamma)| \leq C/(1+|\gamma|^\alpha)$ for some $\alpha > 1$, then $\{g_{p,q}\}_{(q,p)\in\Lambda}$ cannot be a frame if $D(\Lambda) < 1$. Landau's proof relied on a phase-space localization theorem of Daubechies [Dau90]. Ramanathan and Steger [RS95a] recently improved Landau's result to apply to all $g \in L^2(\mathbb{R})$. Their proof relied on a simple dimension counting technique. Moreover, if Λ actually forms a rectangular lattice, their technique recovers the full Rieffel incompleteness result. Specifically, they proved the following theorem [RS95a].

Theorem 2.2.1 *Let $g \in L^2(\mathbb{R})$, and let $\Lambda \subseteq \mathbb{R} \times \hat{\mathbb{R}}$ be a uniformly discrete set.*

(a) *If $D^+(\Lambda) < 1$, then $\{g_{p,q}\}_{(q,p)\in\Lambda}$ is not a frame for $L^2(\mathbb{R})$.*

(b) *If* $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ *is a rectangular lattice with uniform Beurling density* $D(\Lambda) = 1/(ab) < 1$, *then* $\{g_{mb,na}\}$ *is incomplete in* $L^2(\mathbb{R})$.

Ramanathan and Steger conjectured in [RS95a] that Theorem 2.2.1 can be further improved to say that $\{g_{p,q}\}_{(q,p)\in\Lambda}$ must be incomplete when Λ is a uniformly discrete set with $D^+(\Lambda) < 1$. We show in Section 2.2.3 that this conjecture is false. Thus Theorem 2.2.1 is the best possible result.

From Theorem 2.2.1, we see that a Gabor system can only be a frame when the upper uniform Beurling density of Λ is at least 1. In the case of a rectangular lattice $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ it is easy to see that if the density $1/(ab)$ equals 1, i.e., $ab = 1$, then $\{g_{mb,na}\}$ will be exact (Theorem 2.3.1d). Ramanathan and Steger proved that density 1 is also a necessary condition for a Gabor system to be a Riesz basis [RS95a], as follows.

Theorem 2.2.2 *Let* $g \in L^2(\mathbb{R})$, *and let* $\Lambda \subseteq \mathbb{R} \times \hat{\mathbb{R}}$ *be a uniformly discrete set. If* Λ *has uniform Beurling density* $D(\Lambda)$ *and if* $\{g_{p,q}\}_{(q,p)\in\Lambda}$ *is an exact frame for* $L^2(\mathbb{R})$, *then* $D(\Lambda) = 1$.

In particular, the hypothesis in the BLT (Theorem 2.1.1) that $ab = 1$ is redundant, being a consequence of the hypothesis that the Gabor system is an exact frame. This fact had not been established prior to [RS95a], even in the case of a rectangular lattice Λ.

Thus, a Gabor frame with uniform Beurling density exceeding 1 will be inexact. Any inexact frame is *overcomplete*, that is, some elements of the frame can be deleted without destroying the completeness of the system. Heil, Ramanathan, and Topiwala have investigated the independence of such overcomplete Gabor systems [HRT96]. In particular, they have shown that, for a broad class of g, every finite subset of the Gabor system $\{g_{p,q}\}_{(q,p)\in\Lambda}$ is linearly independent even though the entire Gabor system is overcomplete.

Feichtinger and Gröchenig have developed a powerful and general theory that applies not only to Gabor frames but to more general *atomic decompositions* defined on function spaces [FG89a, FG89b, Grö91]. For the particular case of Gabor systems it implies that, for a certain class of g, the Gabor system $\{g_{p,q}\}_{(q,p)\in\Lambda}$ will be a frame for $L^2(\mathbb{R})$ if the density of Λ is high enough.

In the opposite direction, the question of how large ab can be in order for $\{g_{mb,na}\}$ to generate a frame has been answered for only a few g. In the case of the Gaussian $g(x) = e^{-x^2}$, the BLT implies that g cannot generate a frame with $ab = 1$. Seip and Wallstén have shown that g does generate a frame for every value of $ab < 1$ [Sei92a, SW92b]. The tractability of the Gaussian case is because the mapping $f \mapsto \langle f, g_{p,q}\rangle\, e^{\pi(p^2+q^2)}$ is an isometry from $L^2(\mathbb{R})$ to a Hilbert space of entire functions on $\mathbb{C}$. This *Bargmann transform* allows techniques from complex analysis to be brought to bear on the problem. A short survey article, including a discussion of related *Bergman transforms* for wavelet systems, can be found in [Sei92b].

2.2.3 Sparse Gabor systems

We can summarize the discussion in Section 2.2.1 and Section 2.2.2 as follows. If an irregular Gabor system $\{g_{p,q}\}_{(q,p)\in\Lambda}$ is a frame for $L^2(\mathbb{R})$, then the upper uniform Beurling density $D^+(\Lambda)$ satisfies $D^+(\Lambda) \geq 1$. If the frame is exact and Λ has uniform Beurling density $D(\Lambda)$, then $D(\Lambda) = 1$. If $\Lambda = \{(na, mb)\} = a\mathbb{Z} \times b\mathbb{Z}$ is a lattice, then the Gabor system is incomplete if $D(\Lambda) < 1$. We show in this section that this last statement does not extend to irregular Gabor systems, that is, there exist irregular Gabor systems that are complete in $L^2(\mathbb{R})$ yet Λ has uniform Beurling density $D(\Lambda) < 1$.

Our construction is an easy consequence of Landau's fundamental work in [Lan64] on the completeness of sets of exponentials in $C(S)$ when S is finite union of intervals in $\mathbb{R}$. Here $C(S)$ is the space of all continuous functions f with domain S. It is a Banach space when equipped with the L^∞ norm $||f||_\infty = \max_{t\in S} |f(t)|$. Landau proved the following theorem [Lan64].

Theorem 2.2.3 *Fix any $\delta > 0$ and let I_n denote the interval*

$$I_n = (n - (\frac{1}{2} - \delta), n + (\frac{1}{2} - \delta)). \tag{2.2.3}$$

Let S be any finite union of the intervals I_n. Then for each $\varepsilon > 0$ there exists a symmetric real sequence $\{\lambda_k\}_{k\in\mathbb{Z}}$ with $|\lambda_k - k| < \varepsilon$ such that $\{e^{2\pi i\lambda_k t}\}_{k\in\mathbb{Z}}$ is complete in $C(S)$.

Now we construct our irregular Gabor system. Fix any $0 < \varepsilon < 1/2$, and also fix any $\delta < 1/4$. Then the intervals I_n defined in (2.2.3) have length $1 - 2\delta > 1/2$. Choose any integer $K > 0$ and define

$$S = \bigcup_{n=0}^{K-1} I_n.$$

Then by Theorem 2.4, there exists a collection $\Lambda = \{\lambda_k\}$ so that $\{e^{2\pi i\lambda_k t}\}$ is complete in $C(S)$. Because $C(S)$ is dense in $L^2(S)$ and because $||f||_2 \leq |S|^{1/2}\,||f||_\infty$ for each $f \in C(S)$, the set $\{e^{2\pi i\lambda_k t}\}$ is complete in $L^2(S)$ as well.

Our Gabor system will be generated by the indicator function of S:

$$g(x) = \mathbf{1}_S(x) = \begin{cases} 1, & x \in S, \\ 0, & x \notin S. \end{cases}$$

We use the following irregular set of time-frequency shifts:

$$\Gamma = \{(Kn, \lambda_m)\}_{m,n\in\mathbb{Z}} \cup \{(Kn+\frac{1}{2}, \lambda_m)\}_{m,n\in\mathbb{Z}} = (K\mathbb{Z}\times\Lambda) \cup ((K\mathbb{Z}+\frac{1}{2})\times\Lambda).$$

Because each element $\lambda_k \in \Lambda$ is within ε of k, the densities of Λ and Γ are easy to compute.

Lemma 2.2.4 (a) *$\Lambda \subseteq \mathbb{R}$ has uniform Beurling density $D(\Lambda) = 1$.*

(b) *$\Gamma \subseteq \mathbb{R} \times \hat{\mathbb{R}}$ has uniform Beurling density $D(\Gamma) = 2/K$.*

Proof: The fact that Λ has uniform Beurling density 1 in $\mathbb{R}$ follows from Example 2.1. Therefore $K\mathbb{Z} \times \Lambda$ and $(K\mathbb{Z} + \frac{1}{2}) \times \Lambda$ each have uniform Beurling density $1/K$ in $\mathbb{R} \times \hat{\mathbb{R}}$. Since these two sets are simply shifts of each other, their union has uniform Beurling density $2/K$. ❑

Finally, we show that the irregular Gabor system generated by $g = \mathbf{1}_S$ and Γ is complete in $L^2(\mathbb{R})$.

Theorem 2.2.5 *The irregular Gabor system*

$$\{g_{p,q}\}_{(q,p)\in\Gamma} \;=\; \{g_{\lambda_m,Kn}\}_{m,n\in\mathbb{Z}} \;\cup\; \{g_{\lambda_m,Kn+\frac{1}{2}}\}_{m,n\in\mathbb{Z}} \tag{2.2.4}$$

is complete in $L^2(\mathbb{R})$.

Proof: Suppose that $f \in L^2(\mathbb{R})$ satisfies $\langle f, g_{\lambda_m,Kn}\rangle = 0$ and $\langle f, g_{\lambda_m,Kn+\frac{1}{2}}\rangle = 0$ for every m, $n \in \mathbb{Z}$. Fix any n. Then for each m, we have

$$\begin{aligned} 0 \;=\; \langle f, g_{\lambda_m,Kn}\rangle &= \int f(t)\, e^{-2\pi i\lambda_m t}\, \mathbf{1}_S(t - Kn)\, dt \\ &= e^{-2\pi i\lambda_m Kn} \int_S f(t + Kn)\, e^{-2\pi i\lambda_m t}\, dt. \end{aligned}$$

However, $\{e^{2\pi i\lambda_k t}\}$ is complete in $L^2(S)$, so, by the definition of completeness, this implies that $f(t+Kn) = 0$ for a.e. $t \in S$. Similarly, $f(t+Kn+\frac{1}{2}) = 0$ for a.e. $t \in S$. Since the shifts of S by Kn and $Kn + \frac{1}{2}$ cover the entire line, we conclude that $f = 0$ a.e. on $\mathbb{R}$. Therefore, the Gabor system in (2.2.4) is complete in $L^2(\mathbb{R})$. ❑

By taking K as large as we like, we can make the density of the Gabor system in (2.2.4) arbitrarily small.

2.3 The Zak Transform and the Amalgam BLT

2.3.1 The Zak Transform

The *Zak transform* of a function $f \in L^2(\mathbb{R})$ is formally defined by

$$\mathcal{Z}f(t,\omega) = F(t,\omega) = \sum_{k\in\mathbb{Z}} f(t+k)\, e^{2\pi ik\omega}, \quad (t,\omega) \in \mathbb{R} \times \hat{\mathbb{R}}.$$

$\mathcal{Z}f$ satisfies the *quasiperiodicity relations*

$$\mathcal{Z}f(t+1,\omega) = e^{-2\pi i\omega}\, \mathcal{Z}f(t,\omega) \quad \text{and} \quad \mathcal{Z}f(t,\omega+1) = \mathcal{Z}f(t,\omega),$$

and therefore $\mathcal{Z}f$ is determined by its values on the cube $Q = [0,1) \times [0,1)$. As with many useful concepts, the Zak transform has been rediscovered many times, and also goes by the name *Weil–Brezin transform* and *k-q transform.* For properties and history of the Zak transform we refer to [AT75, DGM86, Dau90, HW89, Jan82, Jan88, Zak75].

The Zak transform is a unitary mapping of $L^2(\mathbb{R})$ onto $L^2(Q)$, where

$$L^p(Q) = \left\{F : \|F\|_p = \left(\iint_Q |F(t,\omega)|^p\, dt\, d\omega\right)^{1/p} < \infty\right\}.$$

The utility of the Zak transform arises from its action on a Gabor system $\{g_{m,n}\}$ constructed with $a = b = 1$:

$$\mathcal{Z}(g_{m,n})(t,\omega) = e^{2\pi imt}\, e^{2\pi in\omega}\, \mathcal{Z}g(t,\omega) = e_m(t)\, e_n(\omega)\, \mathcal{Z}g(t,\omega), \quad (2.3.1)$$

where $e_\gamma(t) = e^{2\pi i\gamma t}$. The Zak transform is particularly useful in proving the BLT. Although the BLT is stated for Gabor systems satisfying $ab = 1$, the discussion in Section 2.2.1 shows that only the value of the product ab is important; therefore, we may always reduce to the case $a = b = 1$.

Consider the Gabor system $\{g_{m,n}\}$ generated by $g = \mathbf{1}_{[0,1)}$, the indicator function of the interval $[0,1)$. This Gabor system forms an orthonormal basis for $L^2(\mathbb{R})$. Note that g satisfies (2.1.6) since $\|\gamma\hat{g}(\gamma)\|_2 = \infty$, although $\|tg(t)\|_2 < \infty$. The Zak transform of g is $\mathcal{Z}g(t,\omega) = 1$ for $(t,\omega) \in Q$. Therefore $\mathcal{Z}g_{m,n}(t,\omega) = e_m(t)\, e_n(\omega)$. Thus the Zak transform maps the orthonormal basis $\{g_{m,n}\}$ for $L^2(\mathbb{R})$ onto the orthonormal basis $\{e_m(t)\, e_n(\omega)\}$ for $L^2(Q)$. This is one method of verifying that $\mathcal{Z}$ is a unitary mapping of $L^2(\mathbb{R})$ onto $L^2(Q)$.

For general $g \in L^2(\mathbb{R})$, the determination of properties of $\{g_{m,n}\}$ reduces by the Zak transform to the determination of properties of $\{e_m(t)e_n(\omega)\mathcal{Z}g\}$. This set is formed by multiplying each element of the orthonormal basis $\{e_m(t)\, e_n(\omega)\}$ by the single function $\mathcal{Z}g$. We expect then that the behavior of $\mathcal{Z}g$ will be crucial. Parts a, c, d, and e of the following result are well-known, and a proof is given in [HW89]. A proof of Theorem 2.3.1b can be found in [Hei90].

Theorem 2.3.1 *Let $g \in L^2(\mathbb{R})$, and set $a = b = 1$. Then*

(a) $\{g_{m,n}\}$ *is complete in $L^2(\mathbb{R})$ if and only if $\mathcal{Z}g \neq 0$ a.e.*

(b) $\{g_{m,n}\}$ *is minimal and complete in $L^2(\mathbb{R})$ if and only if $1/(\mathcal{Z}g) \in L^2(Q)$.*

(c) $\{g_{m,n}\}$ *is a Bessel sequence for $L^2(\mathbb{R})$ if and only if $\mathcal{Z}g \in L^\infty(Q)$.*

(d) $\{g_{m,n}\}$ *is a frame for $L^2(\mathbb{R})$ with frame bounds A, B if and only if*

$$0 < A \leq |\mathcal{Z}g(t,\omega)|^2 \leq B < \infty \text{ a.e.}$$

In this case, $\{g_{m,n}\}$ is an exact frame for $L^2(\mathbb{R})$.

(e) $\{g_{m,n}\}$ *is an orthonormal basis for $L^2(\mathbb{R})$ if and only if $|\mathcal{Z}g(t,\omega)|^2 = 1$ a.e.*

The equivalences in Theorem 2.3.1 fail for Gabor systems with $ab \neq 1$.

The Zak transform has the remarkable property that if $\mathcal{Z}g$ is continuous on all of $\mathbb{R} \times \hat{\mathbb{R}}$ (not just on the subset Q) then $\mathcal{Z}g$ has a zero in Q. A proof can be found in Theorem 4.3.4 of [HW89]. For original proofs and related results, see [AT75] (p. 18), [BGZ75, BZ81, Jan82, Zak75]. In light of Theorem 2.3.1, the continuity—or lack thereof—of $\mathcal{Z}g$ is a major influence on any approach to proving the BLT. Note for example that if $g = \mathbf{1}_{[0,1)}$ then $\mathcal{Z}g = 1$ on Q. However, by quasiperiodicity, $\mathcal{Z}g$ is not continuous on $\mathbb{R} \times \hat{\mathbb{R}}$.

The Zak transforms of g and $\hat{g}$ are related by the following formulas:

$$\mathcal{Z}\hat{g}(t,\omega) = e^{-2\pi i t\omega}\, \mathcal{Z}g(\omega,-t) \qquad \text{and} \qquad \mathcal{Z}g(t,\omega) = e^{-2\pi i t\omega}\, \mathcal{Z}\hat{g}(-\omega,t). \tag{2.3.2}$$

2.3.2 The Amalgam BLT

In this section we use the Zak transform to prove a variation of the BLT. This *Amalgam BLT* is stated in terms of a *Wiener amalgam space* $W(L,G)$. Amalgam spaces allow descriptions of a function in terms of both its local (L) and global (G) behavior. For example,

$$W(L^p,\ell^q) = \left\{ f : \|f\|_{W(L^p,\ell^q)} = \left(\sum_k \|f \cdot \mathbf{1}_{[k,k+1]}\|_p^q \right)^{1/q} < \infty \right\}.$$

Wiener's original amalgam space is the Segal algebra

$$W(C_0,\ell^1) = \{ f \in W(L^\infty,\ell^1) : f \text{ is continuous } \}.$$

Clearly $W(C_0, \ell^1) \subseteq L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \cap C_0(\mathbb{R})$. We also define

$$W(C_0, \ell^2) \;=\; \{f \in W(L^\infty, \ell^2) : f \text{ is continuous }\}.$$

General Wiener amalgam spaces $W(L, G)$ are defined in [Fei90, FS85]. There have been recent applications of Wiener amalgam spaces to Wiener–Plancherel formulas and Gabor theory [BBE89, Hei90, Wal89b].

If $g \in W(L^p, \ell^1)$ then the series defining $\mathcal{Z}g$ converges absolutely in $L^p(Q)$:

$$\begin{aligned} \|\mathcal{Z}g\|_{L^p(Q)} &\leq \sum_k \|g(t+k)\, e^{2\pi i k\omega}\|_{L^p(Q)} \\ &= \sum_k \|g \cdot \mathbf{1}_{[k,k+1]}\|_{L^p(\mathbb{R})} \;=\; \|g\|_{W(L^p,\ell^1)} \;<\; \infty \end{aligned} \tag{2.3.3}$$

Therefore the Zak transform is a continuous mapping of $W(L^p, \ell^1)$ into $L^p(Q)$. The case $p = \infty$ leads immediately to the following result due to Heil [Hei90].

Theorem 2.3.2 (Amalgam BLT) *Let $g \in L^2(\mathbb{R})$, and let a, $b > 0$ satisfy $ab = 1$. If the Gabor system $\{g_{mb,na}\}$ is an exact frame for $L^2(\mathbb{R})$, then*

$$g \notin W(C_0, \ell^1) \qquad \text{and} \qquad \hat{g} \notin W(C_0, \ell^1).$$

Proof: It suffices to consider $a = b = 1$. If $g \in W(C_0, \ell^1)$ then $g(t+k)\, e^{2\pi i k\omega}$ is continuous for every k. However, by (2.3.3), $\mathcal{Z}g(t,\omega) = \sum g(t+k)\, e^{2\pi i k\omega}$ converges in L^∞ norm, i.e., uniformly on Q. The square $Q = [0,1) \times [0,1)$ can be replaced in this argument by any translate of Q, so $\mathcal{Z}g$ must in fact be continuous on all of $\mathbb{R} \times \hat{\mathbb{R}}$. Therefore $\mathcal{Z}g$ must have a zero, so the Gabor system $\{g_{m,n}\}$ cannot be a frame. Because $(g_{m,n})^\wedge = e^{2\pi i mn}\hat{g}_{-n,m}$, the Gabor system generated by g is a frame if and only if the Gabor system generated by $\hat{g}$ is a frame. Therefore $\hat{g} \notin W(C_0, \ell^1)$ as well. ❑

The BLT and the Amalgam BLT may both be qualitatively stated as "if $\{g_{m,n}\}$ is an exact frame then either g is not smooth or it has poor decay at infinity." However, the two results are distinct. We show in Example 2.3.1 that there exists a function $g \in L^2(\mathbb{R})$ such that $\|tg(t)\|_2\, \|\gamma\hat{g}(\gamma)\|_2 = \infty$ and g, $\hat{g} \in W(C_0, \ell^1)$. This means that the BLT does not imply the Amalgam BLT. We show in Example 2.3.2 that there exists a function $g \in L^2(\mathbb{R})$ such that g, $\hat{g} \notin W(C_0, \ell^1)$ and $\|tg(t)\|_2\, \|\gamma\hat{g}(\gamma)\|_2 < \infty$. This means that the Amalgam BLT does not imply the BLT. Thus Theorem 3.2 is neither a restatement of nor a weak form of the BLT.

Example 2.3.1 We shall construct a function g such that both g, $\hat{g} \in W(C_0, \ell^1)$ while $\|tg(t)\|_2 \, \|\gamma\hat{g}(\gamma)\|_2 = \infty$. Let f be the "tent function" on $[0,1]$, that is,

$$f(t) = \max\{1 - |2t-1|, 0\}.$$

Then f is a continuous, piecewise linear function supported on $[0,1]$. Define

$$g(t) \;=\; \sum_{n=1}^{\infty} n^{-3/2} f(t-n).$$

Then $g \in W(C_0, \ell^1)$. Further,

$$\hat{g}(\gamma) \;=\; \sum_{n=1}^{\infty} n^{-3/2} e^{-2\pi i n\gamma} \hat{f}(\gamma) \;=\; a(\gamma)\,\hat{f}(\gamma) \;=\; a(\gamma)\, 2\, e^{-\pi i\gamma} \left(\frac{\sin(\pi\gamma/2)}{\pi\gamma}\right)^2,$$

where $a(\gamma) = \sum_{n=1}^{\infty} n^{-3/2}\, e^{-2\pi i n t}$ is a continuous, periodic function. Therefore $\hat{g} \in W(C_0, \ell^1)$ as well. Finally,

$$\begin{aligned} \int_n^{n+1} |t\, g(t)|^2\, dt \;&=\; n^{-3} \int_n^{n+1} |t\, f(t-n)|^2\, dt \\ &\geq\; n^{-3} \int_n^{n+1} |n\, f(t-n)|^2\, dt \;=\; n^{-1}\, \|f\|_2^2, \end{aligned}$$

so $\|tg(t)\|_2^2 = \sum_{n=1}^{\infty} \int_n^{n+1} |t\, g(t)|^2\, dt \geq \sum_{n=1}^{\infty} n^{-1}\, \|f\|_2^2 = \infty$. ❑

Example 2.3.2 We shall construct a function $g(t)$ such that $g \in W(C_0, \ell^2) \setminus W(C_0, \ell^1)$, $\hat{g} \notin W(C_0, \ell^1)$, and $\|tg(t)\|_2 \, \|\gamma\hat{g}(\gamma)\|_2 < \infty$.

For k large, say $k \geq N$, let $b_k = k - 1/2$, and let $a_k < b_k$ have the properties ${b_k}^3 - {a_k}^3 \leq k$ and $b_k + \frac{1}{k} \leq a_{k+1} - \frac{1}{k+1}$. Define the continuous functions g_k by

$$g_k(t) \;=\; \begin{cases} \frac{1}{k\,\log k}, & t \in [a_k, b_k], \\ 0, & t \notin [a_k - \frac{1}{k}, b_k + \frac{1}{k}], \\ \text{linear}, & t \in [a_k - \frac{1}{k}, a_k] \cup [b_k, b_k + \frac{1}{k}]. \end{cases}$$

Define $g = \sum_{k=N}^{\infty} g_k$. Direct computations yield

$$\|tg(t)\|_2^2 \;\leq\; \frac{7}{3} \sum_{k=N}^{\infty} \frac{1}{k\,\log^2 k} \;<\; \infty \quad \text{and} \quad \|g'\|_2^2 \;=\; 2 \sum_{k=N}^{\infty} \frac{1}{k\,\log^2 k} \;<\; \infty,$$

where g' is the classical derivative of g, defined except at a countable collection of points. Therefore $\|tg(t)\|_2 \, \|\gamma\hat{g}(\gamma)\|_2 < \infty$. Also,

$$\sum_{k=N}^{\infty} \|g \cdot \mathbf{1}_{[k,k+1]}\|_{\infty} \;=\; \sum_{k=N}^{\infty} \frac{1}{k\,\log k} \;=\; \infty$$

and

$$\sum_{k=N}^{\infty} \|g \cdot \mathbf{1}_{[k,k+1]}\|_\infty^2 = \sum_{k=N}^{\infty} \frac{1}{k^2 \log^2 k} < \infty.$$

Therefore, $g \in W(C_0, \ell^2) \setminus W(C_0, \ell^1)$.

It remains to show that $\hat{g} \notin W(C_0, \ell^1)$. If we did have $\hat{g} \in W(C_0, \ell^1)$ then $\mathcal{Z}\hat{g}$ would be bounded by (2.3.3). By (2.3.2), it would then follow that $\mathcal{Z}g$ is bounded. Therefore, it suffices to show that $\mathcal{Z}g$ is unbounded.

In fact, we can show that for each $M > 0$ there is a set $S = [-\frac{1}{2} - 2\varepsilon, -\frac{1}{2} - \varepsilon] \times [0, \delta]$ such that $|\mathcal{Z}g(t, \omega)| \geq M$ for $(t, \omega) \in S$. Specifically, let K be so large that

$$\sum_{k=N}^{K} (k \log k)^{-1} \geq 2M.$$

Now choose $\varepsilon > 0$ so small that $2\varepsilon \leq |b_k - a_k|$ for $N \leq k \leq K$, and choose $K' \geq K$ so large that $|b_k - a_k + \frac{1}{k}| < \varepsilon$ for $k \geq K'$. Finally, choose $\delta > 0$ so small that $|1 - e^{2\pi i k \omega}| \leq 1/2$ if $\omega \in [0, \delta]$ and $N \leq k \leq K'$. It can then be shown that for $(t, \omega) \in S$ we have $g(t + k) = 0$ if $k < N$ or $k \geq K'$, and $g(t + k) = (k \log k)^{-1}$ if $N \leq k \leq K$. Hence, for $(t, \omega) \in S$,

$$\begin{aligned} |\mathcal{Z}g(t, \omega)| &= \left| \sum_{k=N}^{K'} g(t + k)\, e^{2\pi i k \omega} \right| \\ &\geq \frac{1}{2} \sum_{k=N}^{K'} |g(t + k)| \geq \frac{1}{2} \sum_{k=N}^{K} \frac{1}{k \log k} \geq M. \end{aligned}$$

❏

2.3.3 *Bessel sequences*

In Example 2.3.2 we constructed a function $g \in W(C_0, \ell^2)$ whose Zak transform $\mathcal{Z}g$ was unbounded. Since $g \in W(C_0, \ell^2)$, the periodic function $\lambda(t) = \sum |g(t - n)|^2$ is bounded above:

$$\lambda(t) = \sum_n |g(t - n)|^2 \leq \sum_n \|g \cdot \mathbf{1}_{[n,n+1]}\|_\infty^2 = \|g\|_{W(C_0,\ell^2)}^2 \text{ a.e.}$$

In order for the Gabor sequence $\{g_{m,n}\}$ to form a Bessel sequence for $L^2(\mathbb{R})$, it is necessary that λ be bounded above (see Theorem 2.3.3b below). However, by Theorem 2.3.1c, it is also necessary that $\mathcal{Z}g$ be bounded. Therefore, the boundedness of λ is not sufficient to guarantee that $\{g_{m,n}\}$ will form a Bessel sequence for $L^2(\mathbb{R})$.

The following theorem summarizes the connection between the lower or upper frame bounds and the function λ for arbitrary $g \in L^2(\mathbb{R})$.

Theorem 2.3.3 *Let $g \in L^2(\mathbb{R})$.*

(a) *If there exists a constant $A > 0$ such that*

$$\forall f \in L^2(\mathbb{R}), \quad A\,\|f\|_2^2 \;\le\; \sum_{m,n} |\langle f, g_{m,n}\rangle|^2, \tag{2.3.4}$$

then

$$A \;\le\; \sum_n |g(t-n)|^2 \text{ a.e. and } \; A \;\le\; \textstyle\sum_m |\hat{g}(\gamma - m)|^2 \text{ a.e.} \tag{2.3.5}$$

(b) *If $\{g_{m,n}\}$ is a Bessel sequence, i.e., there exists a constant $B > 0$ such that*

$$\forall f \in L^2(\mathbb{R}), \quad \sum_{m,n} |\langle f, g_{m,n}\rangle|^2 \;\le\; B\,\|f\|_2^2, \tag{2.3.6}$$

then

$$\sum_n |g(t-n)|^2 \le B \text{ a.e. and } \; \textstyle\sum_m |\hat{g}(\gamma - m)|^2 \le B \text{ a.e.} \tag{2.3.7}$$

Theorem 2.3.3 is a generalization of a result stated in [Dau90](Section 2.3.1B) and proved in [HW89] (Proposition 4.1.4). The theorem as stated here was proved in [CS93b] (Theorem 2).

The remainder of this subsection is devoted to examining the converse of Theorem 2.3.3. First, the BLT implies that the converse of Theorem 2.3.3a fails.

Proposition 2.3.4 *There exists a function $g \in L^2(\mathbb{R})$ satisfying* (2.3.5) *but not* (2.3.4).

Proof: Let $g(t) = e^{-\pi t^2}$. Then g certainly satisfies (2.3.5) for some $A > 0$. Moreover, $\{g_{m,n}\}$ is a Bessel sequence, i.e., (2.3.6) holds [Dau90] (Theorem 2.5), [HW89] (Proposition 4.2.1), [Wal92] (Theorem 3.1). However, by the BLT, $\{g_{m,n}\}$ is not a frame. Therefore (2.3.4) cannot hold. ❑

The converse of Theorem 2.3.3b is false as well.

Proposition 2.3.5 *There exists a function $g \in L^2(\mathbb{R})$ satisfying* (2.3.7) *but not* (2.3.6).

Proof: If $g \in L^2(\mathbb{R})$ then, by the Plancherel formula for Fourier series,

$$\sum_n |g(t+n)|^2 \;=\; \int_0^1 \left|\sum_n g(t+n)\, e^{2\pi i n\omega}\right|^2 d\omega \;=\; \int_0^1 |\mathcal{Z}g(t,\omega)|^2\, d\omega \tag{2.3.8}$$

for almost every t. Since $|Z\hat{g}(\omega, t)| = |\mathcal{Z}g(t, -\omega)|$, we also have

$$\sum_m |\hat{g}(\omega + m)|^2 = \int_0^1 |\mathcal{Z}g(t, -\omega)|^2 \, dt \tag{2.3.9}$$

for almost every ω. Let $G(t, \omega) = |t - \omega|^{-1/4}$ for $(t, \omega) \in Q$. Then

$$\iint_Q |G(t, \omega)|^2 \, dt \, d\omega = 6 \, ,$$

so $G \in L^2(Q)$. Therefore $g = \mathcal{Z}^{-1}G \in L^2(\mathbb{R})$, and $G = \mathcal{Z}g$. Since

$$\operatorname*{ess\,sup}_{t \in [0,1]} \int_0^1 |G(t, \omega)|^2 \, d\omega = \operatorname*{ess\,sup}_{\omega \in [0,1]} \int_0^1 |G(t, -\omega)|^2 \, dt = 2\sqrt{2},$$

it follows from (2.3.8) and (2.3.9) that (2.3.7) holds. However, $\mathcal{Z}g = G$ is unbounded on Q. Theorem 2.3.1c therefore implies that (2.3.6) does not hold. ❑

2.3.4 The discrete Zak transform

Our main focus in this chapter is on functions f of a continuously defined independent variable t. However, practical applications often require discrete implementations. We therefore devote this section to discussing discrete analogues of Gabor systems and the Zak transform. Here the variable t will be restricted to discrete values, that is, $t \in \mathbb{Z}$. This amounts to considering sequences $\{f(t)\}_{t \in \mathbb{Z}}$ in place of functions f. Our discussion is based on [Wal89a] and [Hei89]. However, as with the usual Zak transform, the discrete Zak transform has been independently considered by other authors. For example, [AGT92] considers a discrete Zak transform acting on finite–length discrete signals. A survey of discrete Zak transforms appears in [BH96a]

Let $g = \{g(t)\}_{t \in \mathbb{Z}}$ be a sequence in $\ell^2(\mathbb{Z})$, so that $\sum |g(t)|^2 < \infty$. As with the usual Gabor systems, we define new sequences $g_{mb,na}$ by translating and modulating g. We must restrict the values of a and b to those for which a and $N = 1/b$ are integers. Then we define $g_{mb,na}(t) = e^{2\pi imbt} \, g(t - na)$ for $t \in \mathbb{Z}$. The collection of sequences $\{g_{mb,na}\}_{m \in \mathbb{Z}_N, n \in \mathbb{Z}}$, where $\mathbb{Z}_N = \{0, 1, \ldots, N-1\}$, is the *discrete Gabor system* generated by g, a, and b.

Many results for usual Gabor systems have analogues for discrete Gabor systems. For example, there is a *discrete Zak transform* that is especially useful for analyzing the case $ab = 1$. Because the variable t has been restricted to discrete values, it is more convenient to retain the given values of a and b rather than attempting to "dilate" g in order to redefine a and

b so that $a = b = 1$. Therefore, we define the discrete Zak transform of a sequence $f \in \ell^2(\mathbb{Z})$ by

$$\mathcal{Z}_d f(t,\omega) \;=\; \sum_{k\in\mathbb{Z}} f(t+ka)\, e^{2\pi i k\omega}, \qquad (t,\omega) \in \mathbb{Z} \times \hat{\mathbb{R}}.$$

Note that $\mathcal{Z}_d f$ is *quasiperiodic* in the following sense:

$$\mathcal{Z}_d f(t+a,\omega) \;=\; e^{-2\pi i\omega}\, \mathcal{Z}_d f(t,\omega) \qquad \text{and} \qquad \mathcal{Z}_d f(t,\omega+1) \;=\; \mathcal{Z}_d f(t,\omega).$$

Therefore $\mathcal{Z}_d f$ is completely determined by its values on the set $D = \mathbb{Z}_a \times [0,1) = \{0,\dots,a-1\} \times [0,1)$, which is a union of a line segments in the plane. It is easy to show that $\mathcal{Z}_d$ is a unitary mapping of $\ell^2(\mathbb{Z})$ onto $L^2(D)$. Note that if the sequence $\{f(t)\}_{t\in\mathbb{Z}}$ is obtained by sampling a continuous function $F \in L^2(\mathbb{R})$ at times t/a, i.e., $f(t) = F(t/a)$ for $t \in \mathbb{Z}$, then $\mathcal{Z}_d f(t,\omega) = \mathcal{Z}F(t/a,\omega)$ for $(t,\omega) \in \mathbb{Z} \times \hat{\mathbb{R}}$.

Now we apply the discrete Zak transform to a discrete Gabor system satisfying $ab = 1$. In this case $N = 1/b = a$. Therefore, given $g \in \ell^2(\mathbb{Z})$, we compute

$$\mathcal{Z}_d(g_{mb,na})(t,\omega) = e^{2\pi i m t/a}\, e^{2\pi i n\omega}\, \mathcal{Z}_d g(t,\omega).$$

As $\{a^{-\frac{1}{2}}\, e^{2\pi i m t/a}\, e^{2\pi i n\omega}\}_{m\in\mathbb{Z}_a, n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(D)$, there is an exact analogue of Theorem 2.3.1 for discrete Gabor systems. In particular, $\{g_{mb,na}\}_{m\in\mathbb{Z}_a,n\in\mathbb{Z}}$ is an exact frame for $\ell^2(\mathbb{Z})$ if and only if there exist constants A, $B > 0$ such that $A \le |\mathcal{Z}_d g(t,\omega)|^2 \le B$ for a.e. $(t,\omega) \in D$. However, the fact that D is a *disjoint* union of line segments leads to the following conclusion: We can find continuous functions $G \in L^2(\mathbb{R})$ that do not generate exact Gabor frames for $L^2(\mathbb{R})$ but whose sampled versions $\{g(t)\}_{t\in\mathbb{Z}}$ with $g(t) = G(t/a)$ for $t \in \mathbb{Z}$ do generate exact Gabor frames for $\ell^2(\mathbb{Z})$. We can even construct these functions $G \in L^2(\mathbb{R})$ so that $||tG(t)||_2\, ||\gamma\hat{G}(\gamma)||_2 < \infty$ and $G \in W(C_0,\ell^1)$, so that both the BLT and the Amalgam BLT imply that G cannot generate a Gabor frame for $L^2(\mathbb{R})$. The reason is clear: $\mathcal{Z}_d g(t,\omega) = \mathcal{Z}G(t/a,\omega)$ for $(t,\omega) \in D$. Therefore $\mathcal{Z}_d g$ may be bounded above and below on D even though the same is not true for $\mathcal{Z}G$ on $Q = [0,1) \times [0,1)$.

Example 2.3.3 Let G be the Gaussian function $G(t) = e^{-t^2}$. As $G \in \mathcal{S}(\mathbb{R})$, both the BLT and the Amalgam BLT imply that G cannot generate an exact Gabor frame for $L^2(\mathbb{R})$. In fact, this can be verified directly: $\mathcal{Z}G$ is continuous and has a single zero in Q, at the point $(t,\omega) = (1/2,1/2)$. Define $g(t) = G(t/a)$ for $t \in \mathbb{Z}$. Then $\mathcal{Z}_d g(t,\omega) = \mathcal{Z}G(t/a,\omega)$ for $(t,\omega) \in D$. However, if a is odd then $(a/2,1/2) \notin D$. Therefore $\mathcal{Z}_d g$ is continuous *and nonzero* on each of the disjoint line segments making up D. As a consequence, $\mathcal{Z}_d g$ is bounded both above and below on D, and therefore $\{g(t)\}_{t\in\mathbb{Z}}$ generates an exact Gabor frame for $\ell^2(\mathbb{Z})$. ❑

2.4 Wilson bases

In [DJJ91], Daubechies, Jaffard and Journé proved a remarkable result which provides a means to circumvent the restrictions imposed by the BLT. Their construction is a simplification of an earlier construction due to Wilson [Wil87, SRWW87].

Theorem 2.4.1 *Suppose that $g \in L^2(\mathbb{R})$ is such that*

(a) *$\hat{g}$ is real-valued, and*

(b) *$\{g_{m,n/2}\}$ is a tight frame for $L^2(\mathbb{R})$ with frame bound 2.*

Then the collection $\{\psi_{\ell,k}\}_{\ell\geq 0, k\in\mathbb{Z}}$ defined by

$$\begin{aligned}
\psi_{0,k}(t) &= g(t-k), & \ell &= 0,\\
\psi_{\ell,k}(t) &= \sqrt{2}\, g(t-k/2)\cos(2\pi\ell t), & \ell &\neq 0,\ k+\ell \text{ even},\\
\psi_{\ell,k}(t) &= \sqrt{2}\, g(t-k/2)\sin(2\pi\ell t), & \ell &\neq 0,\ k+\ell \text{ odd},
\end{aligned} \tag{2.4.1}$$

is an orthonormal basis for $L^2(\mathbb{R})$.

Note that $\psi_{0,k} = g_{0,k}$ and that $\psi_{\ell,k} = (1/\sqrt{2})\,(g_{\ell,k/2} + (-1)^{k+\ell}\, g_{-\ell,k/2})$ if $k+\ell$ is even and $\psi_{\ell,k} = (i/\sqrt{2})\,(g_{\ell,k/2} + (-1)^{k+\ell}\, g_{-\ell,k/2})$ if $k+\ell$ is odd. Thus, $\{\psi_{\ell,k}\}$ consists of combinations of appropriately chosen elements from the Gabor system $\{g_{m,n/2}\}$. Note that by the discussion in Section 2.2.1, this Gabor system is inexact: it is neither orthogonal nor a basis. However, there does exist a g satisfying the hypotheses of Theorem 2.4.1 that has good smoothness and decay. In particular, we can construct such a g so that $\|tg(t)\|_2\,\|\gamma\hat{g}(\gamma)\|_2 < \infty$. It is therefore possible to construct Wilson orthonormal bases for $L^2(\mathbb{R})$ whose elements have good smoothness and decay. Moreover, these bases are constructed from linear combinations of translates and modulates of a single function. In this sense, Theorem 2.4.1 circumvents the restrictions of the BLT. An example of a function $g \in L^2(\mathbb{R})$ satisfying the hypotheses of Theorem 2.4.1 and such that both g and $\hat{g}$ have exponential decay was constructed in [DJJ91].

Theorem 2.4.1 has been generalized by Auscher [Aus94].

Theorem 2.4.2 *Assume that $|g(t)| \leq C(1+|t|)^{-1-\epsilon}$ for some $\epsilon > 0$. Then $\{\psi_{\ell,k}\}$ forms an orthonormal basis for $L^2(\mathbb{R})$ if and only if the following two conditions are satisfied.*

$$\forall\, k \in \mathbb{Z}, \quad \sum_n \overline{g(t-k-\tfrac{n}{2})}\, g(t-\tfrac{n}{2}) = 2\,\delta_{0,k} \text{ a.e.}, \tag{2.4.2}$$

$$\forall\, k \in \mathbb{Z}, \quad \sum_n (-1)^n\, \overline{g(t-k-\tfrac{n}{2}-\tfrac{1}{2})}\, g(-t-\tfrac{n}{2}) = 0 \text{ a.e.} \tag{2.4.3}$$

If $\hat{g}$ is real-valued then (2.4.3) is automatically satisfied (see [Aus94] (Section 5.2.3)). In this case, (2.4.2) is equivalent to the statement that $\{g_{m,n/2}\}$ forms a tight frame for $L^2(\mathbb{R})$ with frame bound 2 [DJJ91, Aus94]. Therefore, in order to construct a Wilson basis whose elements have good smoothness and decay, it suffices to construct a tight Gabor frame $\{g_{m,n/2}\}$ such that $\hat{g}$ is real-valued and g has both good smoothness and good decay. We shall construct examples of such functions.

Before doing this, we make some general remarks concerning Gabor frames of the form $\{g_{m,n/2}\}$. Recall that the frame operator S is a positive operator, and that $\{S^{-1/2}(g_{m,n/2})\}$ is a tight frame with bound 1. We can show that this latter frame is also a Gabor frame, generated by the function $S^{-1/2}g$. Recall that $\|I - cS\| < 1$, where $c = 2/(A+B)$ and A, B are the frame bounds for $\{g_{m,n/2}\}$. Then $S^{-1/2}$ can be realized as a norm-convergent power series in $I - cS$:

$$S^{-1/2} \;=\; c^{1/2} \sum_{k=0}^{\infty} a_k \, (I - cS)^k, \tag{2.4.4}$$

for some appropriate choice of real coefficients a_k. The frame operator S commutes with each of the operators M_m and $T_{n/2}$. Therefore each partial sum in (2.4.4) commutes with these operators, and therefore the limit $S^{-1/2}$ commutes with each M_m and $T_{n/2}$. Hence $S^{-1/2}(g_{m,n/2}) = (S^{-1/2}g)_{m,n/2}$, so the tight frame is a Gabor frame generated by $S^{-1/2}g$.

The following result regarding general Gabor frames $\{g_{mb,na}\}$ was proved in [Wal92].

Theorem 2.4.3 *Assume $\{g_{mb,na}\}$ is a Gabor frame with frame operator S. If $\hat{g} \in W(L^\infty, \ell^1)$, then*

$$\forall\, f \in L^2(\mathbb{R}), \quad (Sf)^\wedge \;=\; a^{-1} \sum_j \Big(T_{j/a}\hat{f} \cdot \sum_n T_{nb}\big(\hat{g} \cdot T_{j/a}\overline{\hat{g}}\big) \Big). \tag{2.4.5}$$

This says that, assuming good decay of $\hat{g}$, $(Sf)^\wedge$ can be realized solely in terms of translations of $\hat{g}$ and $\hat{f}$. Suppose now that $\hat{g} \in W(C_0, \ell^1)$ and that $\hat{g}$ is real-valued. If $\hat{f}$ is also real-valued then, by (2.4.5), so is $(Sf)^\wedge$. Since the operator $S^{-1/2}$ can be realized as the norm-convergent sum (2.4.4) and since every partial sum is a linear combination of powers of S with real-valued coefficients, $(S^{-1/2}f)^\wedge$ is also real-valued. Applying Theorem 2.4.1, we therefore obtain the following.

Theorem 2.4.4 *Suppose that $h \in L^2(\mathbb{R})$ is such that*

(a) *$\hat{h}$ is real-valued,*

(b) *$\hat{h} \in W(L^\infty, \ell^1)$, and*

(c) *$\{h_{m,n/2}\}$ is a frame for $L^2(\mathbb{R})$.*

Let S be the frame operator for $\{h_{m,n/2}\}$, and define $g = \sqrt{2}\,S^{-1/2}h$. Then the collection $\{\psi_{\ell,k}\}$ defined by (1.4.1) *is an orthonormal basis for $L^2(\mathbb{R})$.*

Example 2.4.1 We shall construct a function $g \in C_c^\infty(\mathbb{R})$ such that $\{\psi_{\ell,k}\}$ is an orthonormal basis for $L^2(\mathbb{R})$. A similar example appears in [DJJ91] and [Aus94]. Let $h \in C_c^\infty(\mathbb{R})$ be given such that

(a) $\mathrm{supp}(h) \subseteq [-1/2, 1/2]$,

(b) h is even and real-valued (so $\hat{h}$ is real-valued as well), and

(c) there exist constants A, $B > 0$ such that $A \le \sum |h(t - n/2)|^2 \le B$.

In this case, $\{h_{m,n/2}\}$ forms a frame for $L^2(\mathbb{R})$ with frame bounds A and B [Dau90, HW89]. The corresponding frame operator is $Sf = f \cdot \lambda$, where $\lambda(t) = \sum |h(t - n/2)|^2$ is bounded, infinitely differentiable, and has no zeros. In addition, $S^{-1/2}f = f \cdot \lambda^{-1/2}$. Defining $g = \sqrt{2}\,h \cdot \lambda^{-1/2}$, we have that $\{g_{m,n/2}\}$ is a tight frame with frame bound 2, and that $g \in C_c^\infty(\mathbb{R})$. Therefore, by Theorem 4.4, $\{\psi_{\ell,k}\}$ is an orthonormal basis for $L^2(\mathbb{R})$, and $\psi_{\ell,k} \in C_c^\infty(\mathbb{R})$ for each ℓ and k.

A similar construction can be performed on the Fourier transform side, resulting in a Wilson basis consisting of smooth, rapidly decaying functions which are bandlimited. Let h be given such that $\hat{h} \in C_c^\infty(\hat{\mathbb{R}})$ and

(a) $\mathrm{supp}(\hat{h}) \subseteq [-1, 1]$,

(b) $\hat{h}$ is real-valued, and

(c) there exist constants A, $B > 0$ such that $A \le \sum |\hat{h}(\gamma - m)|^2 \le B$.

Then $\{\hat{h}_{n/2,m}\}$ forms a frame for $L^2(\mathbb{R})$ with frame bounds A and B. Since $(h_{m,n/2})^\wedge = e^{\pi i m n}\hat{h}_{-n/2,m}$, we conclude that $\{h_{m,n/2}\}$ is a frame for $L^2(\mathbb{R})$. The corresponding frame operator is defined by $(Sf)^\wedge = \hat{f} \cdot \Lambda$, where $\Lambda(\gamma) = \sum |\hat{h}(\gamma - m)|^2$, and $(S^{-1/2}f)^\wedge = \hat{f} \cdot \Lambda^{-1/2}$. If we define g by $\hat{g} = \sqrt{2}\,\hat{h} \cdot \Lambda^{-1/2}$, then $\{g_{m,n/2}\}$ is a tight frame with frame bound 2 for $L^2(\mathbb{R})$. Therefore, $\{\psi_{\ell,k}\}$ is an orthonormal basis for $L^2(\mathbb{R})$. Moreover, $\psi_{\ell,k} \in \mathcal{S}(\mathbb{R})$ and $(\psi_{\ell,k})^\wedge \in C_c^\infty(\mathbb{R})$ for each ℓ and k. ❑

It is possible to describe the frame operator S for a Gabor frame $\{g_{m,n/2}\}$ in terms of the Zak transform [Dau90, Dau92, HW89], cf. Theorem 2.3.1.

Theorem 2.4.5 *If $g \in L^2(\mathbb{R})$ then $\{g_{m,n/2}\}$ is a frame for $L^2(\mathbb{R})$ with frame bounds A, B if and only if*

$$0 < A \le |\mathcal{Z}g(t,\omega)|^2 + |\mathcal{Z}g(t,\omega + 1/2)|^2 \le B < \infty \text{ a.e.} \qquad (2.4.6)$$

In this case, the frame operator S is given by $\mathcal{Z}(Sf) = \mathcal{Z}f \cdot M$, where $M(t,\omega) = |\mathcal{Z}g(t,\omega)|^2 + |\mathcal{Z}g(t,\omega+1/2)|^2$, and the operator $S^{-1/2}$ is given by $\mathcal{Z}(S^{-1/2}f) = \mathcal{Z}f \cdot M^{-1/2}$.

We shall use this result, along with the following theorem of Janssen [Jan82], to construct Wilson bases.

Theorem 2.4.6 *If $f \in \mathcal{S}(\mathbb{R})$, then $\mathcal{Z}f \in C^\infty(\mathbb{R}^2)$ is quasiperiodic. Conversely, if $F \in C^\infty(\mathbb{R}^2)$ is quasiperiodic, then there exists $f \in \mathcal{S}(\mathbb{R})$ such that $F = \mathcal{Z}f$.*

Example 2.4.2 We shall construct a function $g \in \mathcal{S}(\mathbb{R})$ such that $\{\psi_{\ell,k}\}$ is a Wilson basis for $L^2(\mathbb{R})$. Let $h(t) = e^{-\pi t^2}$. Then $h \in \mathcal{S}(\mathbb{R})$, $\hat{h}$ is real-valued, and h satisfies (2.4.6) [Dau90, Dau92]. By Theorem 2.4.6, $\mathcal{Z}h \in C^\infty(\mathbb{R}^2)$ and is quasiperiodic. Moreover, (2.4.6) holds for some A, B, so $M(t,\omega)^{-1/2} = (|\mathcal{Z}g(t,\omega)|^2 + |\mathcal{Z}g(t,\omega+1/2)|^2)^{-1/2}$ is in $C^\infty(\mathbb{R}^2)$, and is periodic with period 1 in both t and ω. Hence the function $G = \mathcal{Z}h \cdot M^{-1/2}$ is in $C^\infty(\mathbb{R}^2)$ and is quasiperiodic. Therefore, by Theorem 2.4.6, $g = \sqrt{2}\,\mathcal{Z}^{-1}G = \sqrt{2}\,S^{-1/2}h \in \mathcal{S}(\mathbb{R})$. Hence $\{\psi_{\ell,k}\}$ is an orthonormal basis for $L^2(\mathbb{R})$ with $\psi_{\ell,k} \in \mathcal{S}(\mathbb{R})$ for each ℓ and k. ❑

A more general investigation of the continuity properties of a Gabor frame operator S and the related operators S^{-1} and $S^{-1/2}$ was conducted in [Wal92] and [Wal93]. The continuity of S, S^{-1}, and $S^{-1/2}$ on weighted spaces was examined in [Wal92]. In particular, for $s \geq 0$ let $w_s(t) = (1+|t|)^s$ or $w_s(t) = e^{s|t|}$, and define the weighted spaces

$$\begin{aligned} L^p_s(\mathbb{R}) &= \{f : \|f\|_{L^p_s} = \|fw_s\|_p < \infty\}, \\ (L^p_s)^\wedge(\mathbb{R}) &= \{f : \|f\|_{(L^p_s)^\wedge} = \|\hat{f}w_s\|_p < \infty\}. \end{aligned}$$

Membership of f in L^p_s quantifies the decay of f, and membership of f in $(L^p_s)^\wedge$ quantifies the smoothness of f. Assume now that $g(t)\,w_s(t) \in W(L^\infty, \ell^1)$, and that $\hat{g}(\gamma)\,w_s(\gamma) \in W(L^\infty, \ell^1)$. Then for all sufficiently small a, $b > 0$ (how small depends on g and s), S and $S^{1/2}$ are continuous bijections of $L^p_s(\mathbb{R})$ and $(L^p_s)^\wedge(\mathbb{R})$ onto themselves for all $1 \leq p < \infty$. In particular, for a and b small enough, $S^{-1/2}g$ has the same smoothness and decay properties as g, as measured by membership in the spaces L^p_s and $(L^p_s)^\wedge$.

For the specific case of $ab = 1/2$, it has been shown in [Wal93] that if $g \in \mathcal{S}(\mathbb{R})$ satisfies (2.4.6) then S and $S^{1/2}$ are continuous bijections of the *modulation spaces* $M^s_{p,q}(\mathbb{R})$ onto themselves for $s \geq 0$ and $1 \leq p$, $q < \infty$. $M^s_{p,q}(\mathbb{R})$ is an example of a Wiener amalgam space, namely $M^s_{p,q} = W((L^q_s)^\wedge, \ell^p)$ [Fei89, Fei90]. Given a function $\theta \in C^\infty_c(\mathbb{R})$ satisfying $\sum \theta(t-n) \equiv 1$, $f \in M^s_{p,q}(\mathbb{R})$ if and only if $\sum \|f \cdot T_n\theta\|^p_{(L^q_s)^\wedge} < \infty$. That is, $f \in$

$M^s_{p,q}(\mathbb{R})$ is characterized by a local smoothness criterion ($f \cdot T_n\theta \in (L^q_s)^\wedge$ for all n), and a global decay criterion ($\{\|f \cdot T_n\theta\|_{(L^q_s)^\wedge}\} \in \ell^p$).

The continuity and invertibility of S and $S^{1/2}$ on $M^s_{p,q}(\mathbb{R})$ for $ab = 1/2$ says that these operators preserve smoothness and decay as measured by membership in the spaces $M^s_{p,q}$. It has been shown that unconditional bases of Wilson-type exist for the modulation spaces [FGW92]. The continuity and invertibility of S and $S^{1/2}$ on the modulation spaces has been used in [GW92] to construct sets of sampling and interpolation in the Bargmann–Fock spaces of entire functions.

A general construction that includes many examples of Wilson bases as well as wavelet bases are the *local trigonometric bases* of Coifman and Meyer [CM91b]. Similar constructions were proposed in [Mal90b] in a signal processing context, and in [Lae90] in the context of bases adapted to arbitrary decompositions of the frequency domain. For more details, and especially the connection with Wilson bases, see [Aus94] and [AWW92].

2.5 Distributional calculations and the continuity of the Zak transform

We devote this section to investigating relationships between differentiation and the continuity of the Zak transform.

Distributional differentiation will be denoted by ∂ on $\mathbb{R}$ or ∂_j on $\mathbb{R}^d$ (the j-th partial). Classical differentiation is denoted by D on $\mathbb{R}$ and D_j on $\mathbb{R}^d$. The Sobolev space $L^2_1(\mathbb{R}^d)$ is $\{f \in L^2(\mathbb{R}^d) : \partial_j f \in L^2(\mathbb{R}^d),\ j = 1, \ldots, d\}$.

If $x = (t, \omega) \in \mathbb{R} \times \hat{\mathbb{R}}$ and $r > 0$ then $Q(x; r) = Q(t, \omega; r)$ is the square centered at x with side r,

$$\begin{aligned} Q(x;r) &= [t - r/2, t + r/2) \times [\omega - r/2, \omega + r/2) \\ &= \{(u, \alpha) \in \mathbb{R} \times \hat{\mathbb{R}} : u \in [t - r/2, t + r/2), \alpha \in [\omega - r/2, \omega + r/2)\}. \end{aligned}$$

Thus the square $Q = [0,1) \times [0,1)$ defined in Section 2.3.1 is $Q = Q(\frac{1}{2}, \frac{1}{2}; 1)$.

The proof of the following result is routine, e.g., [Maz85] (pp. 8–9), [Sch66] (Theorem 2.III (pp. 53–54) and Theorem 2.V (pp. 57–59)).

Theorem 2.5.1 a. *Given $F \in L^2_1(\mathbb{R}^d)$, there is a function $\tilde{F}$ on $\mathbb{R}^d$ such that $\tilde{F} = F$ a.e. and $\tilde{F} \in AC_{\text{loc}}$ on almost all straight lines parallel to the coordinate axes. The classical gradient $\nabla\tilde{F}$ of $\tilde{F}$ exists a.e. on $\mathbb{R}^d$; and the distributional gradient of F is the distribution corresponding to $\nabla\tilde{F}$.*

b. *Given $F \in L^2(\mathbb{R}^d)$, if there is a function $\tilde{F}$ on $\mathbb{R}^d$ such that $\tilde{F} \in AC_{\text{loc}}$ on almost all straight lines parallel to the coordinate axes, $\tilde{F} = F$ a.e., and $D_j\tilde{F} \in L^2(\mathbb{R}^d)$ for $j = 1, \ldots, d$, then $\partial_j F$ is the distribution corresponding to $D_j\tilde{F}$ for $j = 1, \ldots, d$, and, hence, $F \in L^2_1(\mathbb{R}^d)$.*

$\mathbb{R}^d$ can be replaced by any open subset and L_1^2 can be replaced by any L_1^p for $1 \leq p \leq \infty$.

Theorem 2.5.2 *If $g \in L^2(\mathbb{R})$, then $\partial g \in L^2(\mathbb{R})$ if and only if $\gamma\hat{g}(\gamma) \in L^2(\hat{\mathbb{R}})$. In this case, $g \in AC_{\mathrm{loc}}$, $Dg = \partial g$ a.e., and*

$$\partial g(t) \;=\; \big(2\pi i\gamma\hat{g}(\gamma)\big)^{\vee}(t). \tag{2.5.1}$$

Proof: a. Suppose $\partial g \in L^2(\mathbb{R})$. Since $L^2(\mathbb{R}) \subseteq \mathcal{S}'(\mathbb{R})$ and $\gamma\hat{g}(\gamma) \in \mathcal{S}'(\mathbb{R})$, we can compute

$$\big\langle (\partial g)^{\wedge}, \varphi \big\rangle \;=\; -\langle g, \partial\check{\varphi}\rangle \;=\; 2\pi i\,\big\langle g(t),\, (\gamma\varphi(\gamma))^{\vee}(t)\big\rangle \;=\; \big\langle 2\pi i\gamma\hat{g}(\gamma),\, \varphi(\gamma)\big\rangle$$

for each $\varphi \in \mathcal{S}(\hat{\mathbb{R}})$. Thus, $\gamma\hat{g}(\gamma) \in L^2(\hat{\mathbb{R}})$ and (2.5.1) holds. The same calculation, this time proceeding right to left, proves that $\gamma\hat{g}(\gamma) \in L^2(\hat{\mathbb{R}})$ implies $\partial g \in L^2(\mathbb{R})$.

b. If $\partial g \in L^2(\mathbb{R})$, then

$$h(t) \;=\; \int_0^t \partial g(u)\,du$$

is absolutely continuous on $[0,1]$. For each $k \in C_c^\infty(\mathbb{R})$ with $\mathrm{supp}(k) \subseteq (0,1)$, we have the following calculations:

$$\int_0^1 g(t)\,Dk(t)\,dt \;=\; \langle g, \overline{Dk}\rangle \;=\; -\langle \partial g, \bar{k}\rangle \;=\; -\int_0^1 \partial g(t)\,k(t)\,dt$$

and

$$\int_0^1 h(t)\,Dk(t)\,dt \;=\; -\int_0^1 Dh(t)\,k(t)\,dt \;=\; -\int_0^1 \partial g(t)\,k(t)\,dt.$$

The first calculation is distributional while the second employs integration by parts and the fundamental theorem of calculus. Thus,

$$\int_0^1 \big(g(t) - h(t)\big)\,Dk(t)\,dt \;=\; 0$$

for all such k, and hence $g - h$ is constant a.e. on $[0,1]$. We conclude that $g \in AC_{\mathrm{loc}}$ and $Dg = \partial g$ a.e. A calculation such as this is involved in proving Theorem 2.5.1a. ❑

Example 2.5.1 a. If g, $tg(t) \in L^2(\mathbb{R})$ and $\gamma\hat{g}(\gamma) \in L^2(\mathbb{R})$ then not only does $G = \mathcal{Z}g \in L^2(Q)$, but also $\partial_1 G$, $\partial_2 G \in L^2_{\mathrm{loc}}(\mathbb{R}^2)$. To see this, first note that

$$\partial_1 G(t,\omega) \;=\; \sum \partial g(t+k)\,e^{2\pi i k\omega} \;\in\; L^2(Q),$$

since $\partial g \in L^2(\mathbb{R})$ by Theorem 2.5.2. Next, we compute

$$\partial_2 G(t,\omega) = 2\pi i \mathcal{Z}\big(tg(t)\big)(t,\omega) - 2\pi i t \mathcal{Z} g(t,\omega) \in L^2_{\text{loc}}(\mathbb{R}^2),$$

since $tg(t) \in L^2(\mathbb{R})$.

b. If $F \in L^1_{\text{loc}}(\mathbb{R}^2)$ and $\partial_1 F, \partial_2 F \in L^2_{\text{loc}}(\mathbb{R}^2)$ then

$$\begin{aligned} \frac{1}{|Q(t,\omega;r)|} &\iint_{Q(t,\omega;r)} |F(u,\alpha) - F_{Q(u,\alpha;r)}|\, du\, d\alpha \\ &\leq 2\left(\iint_{Q(t,\omega;2r)} |\partial_1 F(u,\alpha)|^2\, du\, d\alpha\right)^{1/2} \\ &\quad + 2\left(\iint_{Q(t,\omega;2r)} |\partial_2 F(u,\alpha)|^2\, du\, d\alpha\right)^{1/2}, \end{aligned} \tag{2.5.2}$$

where $Q(t,\omega;r)$ designates the square $\{(u,\alpha) : |u-t|, |\alpha-\omega| \leq r/2\}$ centered at (t,ω) and

$$F_{Q(t,\omega;r)} = \frac{1}{|Q(t,\omega;r)|} \iint_{Q(t,\omega;r)} F(u,\alpha)\, du\, d\alpha.$$

To verify (2.5.2), first write the left side as

$$\begin{aligned} \frac{1}{r^4} &\iint_{Q(t,\omega;r)} \left| \iint_{Q(u,\alpha;r)} \big(F(u,\alpha) - F(v,\beta)\big)\, dv\, d\beta \right| du\, d\alpha \\ &\leq \frac{1}{r^4} \iint_{Q(t,\omega;r)} \iint_{Q(u,\alpha;r)} \big|F(u,\alpha) - F(v,\alpha)\big|\, dv\, d\beta\, du\, d\alpha \\ &\quad + \frac{1}{r^4} \iint_{Q(t,\omega;r)} \iint_{Q(u,\alpha;r)} \big|F(v,\alpha) - F(v,\beta)\big|\, dv\, d\beta\, du\, d\alpha \end{aligned} \tag{2.5.3}$$

Next, by Theorem 2.5.1a, $F \in AC_{\text{loc}}$ on almost all straight lines parallel to the coordinate axes, and the classical and distributional gradients are equivalent. In particular, we can use the fundamental theorem of calculus as follows: for almost all α,

$$\forall\, u, v, \quad |F(u,\alpha) - F(v,\alpha)| = \left| \int_u^v \partial_1 F(s,\alpha)\, ds \right|, \tag{2.5.4}$$

and, for almost all v,

$$\forall\, \alpha, \beta, \quad |F(v,\alpha) - F(v,\beta)| = \left| \int_\alpha^\beta \partial_2 F(v,\theta)\, d\theta \right|.$$

Thus, the right hand side of (2.5.3) is bounded by

$$\begin{aligned} I_1 + I_2 &= \frac{1}{r^4} \iint_{Q(t,\omega;r)} \iint_{Q(u,\alpha;r)} \left| \int_u^v \partial_1 F(s,\alpha)\, ds \right| dv\, d\beta\, du\, d\alpha \\ &\quad + \frac{1}{r^4} \iint_{Q(t,\omega;r)} \iint_{Q(u,\alpha;r)} \left| \int_\alpha^\beta \partial_2 F(v,\theta)\, d\theta \right| dv\, d\beta\, du\, d\alpha. \end{aligned}$$

In I_1, s is between u and v, $|u-v| \le r/2$, and $|u-t| \le r/2$; hence,

$$\begin{aligned} I_1 &\le \frac{r^3}{r^4} \int_{t-r}^{t+r} \int_{\omega-r/2}^{\omega+r/2} |\partial_1 F(s,\alpha)| \, d\alpha \, ds \\ &\le \sqrt{2} \left(\iint_{Q(t,\omega;2r)} |\partial_1 F(s,\alpha)|^2 \, d\alpha \, ds \right)^{1/2}. \end{aligned}$$

In I_2, θ is between α and β, $|\alpha-\beta| \le r/2$, and $|\alpha-\omega| \le r/2$; also, $|v-u| \le r/2$ and $|u-t| \le r/2$. Hence,

$$\begin{aligned} I_2 &\le \frac{r^3}{r^4} \int_{\omega-r}^{\omega+r} \int_{t-r}^{t+r} |\partial_2 F(v,\theta)| \, dv \, d\theta \\ &\le 2 \left(\iint_{Q(t,\omega;2r)} |\partial_2 F(v,\theta)|^2 \, dv \, d\theta \right)^{1/2}. \end{aligned}$$

(2.5.2) now follows.

c. Let $g = \mathbf{1}_{[0,1)}$. Then $G(t,\omega) = \mathcal{Z}g(t,\omega) = e^{-2\pi i n\omega}$ for $t \in [n, n+1)$. Clearly, $g, tg(t) \in L^2(\mathbb{R})$, $Dg = 0$ a.e., $\gamma\hat{g}(\gamma) \notin L^2(\hat{\mathbb{R}})$, and $\partial g = \delta - \delta_1 \notin L^2(\mathbb{R})$. Also, $\{g_{m,n}\}$ is an orthonormal basis for $L^2(\mathbb{R})$.

This data is, of course, consistent with the BLT; and we mention it only because of part b, and especially (2.5.4), which is used in the proof of (2.5.2). In fact, $G(u,\alpha) - G(v,\alpha)$ has the form $e^{-2\pi i n(u)\alpha} - e^{2\pi i n(v)\alpha}$, whereas $D_1 G = 0$ on $\cup\,(n, n+1)$ and

$$\partial_1 G(s,\alpha) = \sum \big(\delta_n(s) - \delta_{n+1}(s)\big) \otimes e_{-n}(\alpha);$$

and, consequently, (2.5.4) fails for $F = G$.

In this example, Dg exists for all but two points, and $Dg \in L^1(\mathbb{R})$; whereas, $\gamma\hat{g}(\gamma) \notin L^2(\hat{\mathbb{R}})$, cf., Theorem 2.5.2. On the other hand, we know that if Df exists for each point of $\mathbb{R}$ and $Df \in L^1_{\mathrm{loc}}(\mathbb{R})$ then $f \in AC_{\mathrm{loc}}$ by the Banach–Zarecki theorem; cf., for example, [Ben76] (Section 4.6).

d. C. Fefferman and E. Stein (Acta Mathematica, 1972) implemented the *sharp maximal function* whose mid-point version is

$$F^{\sharp}(t,\omega) = \sup_{r>0} \frac{1}{|Q(t,\omega;r)|} \iint_{Q(t,\omega;r)} |F(u,\alpha) - F_{Q(t,\omega;r)}| \, du \, d\alpha.$$

This should be compared with the maximal version of the left-hand side of (2.5.2), viz.,

$$F^{\flat}(t,\omega) = \sup_{r>0} \frac{1}{|Q(t,\omega;r)|} \iint_{Q(t,\omega;r)} |F(u,\alpha) - F_{Q(u,\alpha;r)}| \, du \, d\alpha.$$

The sharp maximal function corresponding to F is in $L^\infty(\mathbb{R}^2)$ if and only if $F \in BMO(\mathbb{R}^2)$; and twice the Hardy–Littlewood maximal function bounds $F^{\sharp}$. ❑

Remark: The BLT is reasonable not only from the point of view of modern physics, as demonstrated in [Bal81, Low85], but also in the tradition of classical analysis. We list the following instances.

a. Given g, $tg(t) \in L^2(\mathbb{R})$ and $\gamma\hat{g}(\gamma) \in L^2(\hat{\mathbb{R}})$, and let $G = \mathcal{Z}g$. Suppose

$$\partial_1^2 G, \partial_2^2 G \in L^2_{\text{loc}}(\mathbb{R}^2) \tag{2.5.5}$$

(noting there are no mixed partials). Using Example 2.5.1a and (2.5.5) we have the hypotheses of that form of Sobolev's lemma that allows us to conclude that there is a continuous function $\tilde{G}$ on $\mathbb{R}^2$ for which $\tilde{G} = G$ a.e. By properties of the Zak transform, $\tilde{G}$ has zeros. Therefore, $\{g_{m,n}\}$ is not a frame. We have shown that the phenomenon exhibited by the BLT follows easily with the added hypothesis (2.5.5).

b. In the setting of $\mathbb{R}^2$, Krylov's theorem [Sch66] (pp. 181–185) asserts that if a distribution G has the properties $\partial_1 G, \partial_2 G \in L^2_{\text{loc}}(\mathbb{R}^2)$, then $G^n \in L^1_{\text{loc}}(\mathbb{R}^2)$ for all $n \geq 1$. Consequently, in the case g, $tg(t) \in L^2(\mathbb{R})$ and $\gamma\hat{g}(\gamma) \in L^2(\hat{\mathbb{R}})$, we conclude that $G = \mathcal{Z}g$ almost has the upper bound property for $\{g_{m,n}\}$ to be a frame, that is, G is almost an element of $L^\infty_{\text{loc}}(\mathbb{R}^2)$. This tantalizing suggestion contrary to the Balian–Low phenomenon is balanced by the following fact also based on Sobolev's work: If a distribution G has the properties $\partial_1 G, \partial_2 G \in L^2_{\text{loc}}(\mathbb{R}^2)$ then $G \in L^1_{\text{loc}}(\mathbb{R}^2)$ satisfies an L^p_{loc} Lipschitz condition [Sch66] (pp. 185–188). If G were actually continuous we could use our argument from part a to verify the BLT.

In Remark 2.5a we gave a simple proof of the BLT in the case that additional smoothness, viz., (2.5.5), is assumed. This is a natural procedure to follow in implementing proofs depending on the continuity of the Zak transform. For example, whereas Remark 2.5a invoked Sobolev's lemma, the following results, Theorem 2.5.3 and Theorem 2.5.4, utilize a Wiener amalgam condition.

Theorem 2.5.3 *Given $g \in L^1(\mathbb{R})$, assume*

$$\partial g \in W(L^\infty, \ell^1). \tag{2.5.6}$$

Then $\{g_{m,n}\}$ is not a frame.

Proof: Note that by (2.5.6), $g \in L^1_1(\mathbb{R})$. Therefore, by Theorem 2.5.1, g is equal almost everywhere to a locally absolutely continuous function. Hence, we may assume that g is continuous, and that the Fundamental Theorem of Calculus holds. To prove the theorem, it is sufficient to prove that $g \in W(C_0, \ell^1)$. We do this by verifying that $\sum \sup_{t\in[0,1]} |g(t+k)| < \infty$, i.e,

$$\sum_k |g(t_k + k)| < \infty \tag{2.5.7}$$

for any sequence $\{t_k\} \subseteq [0,1]$.

Since $g \in L^1(\mathbb{R})$ we know that

$$\sum_k |g(t+k)| < \infty \quad \text{a.e. on } [0,1]. \tag{2.5.8}$$

Fixing one such t in (2.5.8) we take the absolute value of the difference of the sums, (2.5.7) and (2.5.8). (To be proper, we consider finite sums.) This expression is bounded by

$$\sum_k |g(t+k) - g(t_k+k)|,$$

which in turn is finite by (2.5.6). The result follows by routine manipulations.

❑

As a consequence of Theorem 2.5.3, we obtain the following result.

Theorem 2.5.4 *Given g, $tg(t) \in L^2(\mathbb{R})$, and $\gamma\hat{g}(\gamma) \in L^2(\hat{\mathbb{R}})$, assume*

$$\sum_n n^2 \, \|\partial g \cdot \mathbf{1}_{[n,n+1]}\|_\infty^2 < \infty \tag{2.5.9}$$

(in particular, $\partial g \in L^\infty(\mathbb{R})$). Then $\{g_{m,n}\}$ is not a frame.

2.6 The Uncertainty Principle approach to the BLT

We shall present a proof of the BLT due to Daubechies and Janssen [DJ93], based on the operator theory associated with the Classical Uncertainty Principle Inequality and inspired by the elegant proof of the BLT for orthonormal bases by Battle [Bat88].

We shall use the basic properties of frames, Gabor systems, and the Zak transform discussed in Section 2.2.1, and Section 2.3.1. In particular, we may consider only Gabor systems with $a = b = 1$. In this case, $\{g_{m,n}\}$ is a frame for $L^2(\mathbb{R})$ if and only if it is an exact frame (Theorem 2.3.1d). Therefore the dual frame $\{\tilde{g}_{m,n}\}$ is biorthogonal to $\{g_{m,n}\}$, that is, $\langle g_{m,n}, \tilde{g}_{m',n'}\rangle = \delta_{m,m'}\delta_{n,n'}$. Also, $\mathcal{Z}g_{m,n}(t,\omega) = e_m(t)\, e_n(\omega)\mathcal{Z}g(t,\omega)$.

2.6.1 Uncertainty Principles

The Classical Uncertainty Principle Inequality was stated in Theorem 2.1.2. We prove it for $f \in \mathcal{S}(\mathbb{R})$:

$$\begin{aligned} \|f\|_2^2 = \int t\,(|f|^2)'(t)\,dt &\leq \int |t|\,(2\,|\overline{f(t)}\,f'(t)|)\,dt \\ &\leq 2\,\|tf(t)\|_2\,\|f'\|_2 \\ &= 4\pi\,\|tf(t)\|_2\,\|\gamma\hat{f}(\gamma)\|_2. \end{aligned}$$

A standard closure argument, e.g., [Ben94] (Remark 7.31), which we also use in Lemma 2.6.2, extends the inequality to all $f \in L^2(\mathbb{R})$. There is a theory of weighted and local uncertainty principles emanating from Theorem 2.1.2 and its Hilbert space analogue, Theorem 2.6.1. This theory is the subject of [Ben94] (Section 7.8), [Ben96b] (Chapter 6).

The Classical Uncertainty Principle Inequality can be formulated in general Hilbert spaces. Given a Hilbert space H with inner product $\langle f, g\rangle$ and norm $\|f\| = \langle f, f\rangle^{1/2}$, and given operators A, B mapping H into H (or mapping domains $D(A)$, $D(B) \subseteq H$ into H), we define the *commutator* of A and B to be

$$[A, B] = AB - BA.$$

If A is *self-adjoint* (i.e., $\langle Af, f\rangle = \langle f, Af\rangle$ for all $f \in H$), then the *expectation* of A at $f \in D(A)$ is $E_f(A) = \langle Af, f\rangle$, and the *variance* of A at $f \in D(A^2)$ is $\sigma_f^2(A) = E_f(A^2) - (E_f(A))^2$.

Theorem 2.6.1 *Given self-adjoint (but not necessarily continuous) operators A, B on a Hilbert space H. If $f \in D(A^2) \cap D(B^2) \cap D(i[A,B])$ and $\|f\| = 1$, then*

$$E_f(i[A,B])^2 \leq 4\,\sigma_f^2(A)\,\sigma_f^2(B).$$

The proof of Theorem 2.6.1 is not difficult (for example, [Ben94] (Theorem 7.32)) and we obtain the Classical Uncertainty Principle Inequality as a corollary, as follows. Define the *position operator* $\mathcal{P}$ and *momentum operator* $\mathcal{M}$ by

$$\mathcal{P}f(t) = t\,f(t) \qquad \text{and} \qquad \mathcal{M}f = (\mathcal{P}\hat{f})^{\vee} = (\gamma\hat{f}(\gamma))^{\vee},$$

when these make sense. Both $\mathcal{P}$ and $\mathcal{M}$ are self-adjoint and if $f \in \mathcal{S}(\mathbb{R})$ then $(f')^{\wedge} = 2\pi i\mathcal{P}\hat{f}$, $f' = 2\pi i\mathcal{M}f$, and $[\mathcal{P}, \mathcal{M}]f = -\frac{1}{2\pi i}f$. By Theorem 2.6.1, we have

$$E_f(-\frac{1}{2\pi}I)^2 \leq \sigma_f^2(\mathcal{P})\,\sigma_f^2(\mathcal{M}),$$

where I is the identity operator. On the other hand, $E_f(I) = \langle If, f\rangle = ||f||_2^2$, and

$$\sigma_f^2(\mathcal{P}) = ||\mathcal{P}f||_2^2 - \langle \mathcal{P}f, f\rangle^2 \leq ||\mathcal{P}f||_2^2 = ||tf(t)||_2^2,$$

and

$$\sigma_f^2(\mathcal{M}) = ||\mathcal{M}f||_2^2 - \langle \mathcal{M}f, f\rangle^2 \leq ||\mathcal{M}f||_2^2 = ||\gamma\hat{f}(\gamma)||_2^2,$$

since both $\langle \mathcal{P}f, f\rangle = \int t\,|f(t)|^2\,dt$ and $\langle \mathcal{M}f, f\rangle = \int \gamma\,|\hat{f}(\gamma)|^2\,d\gamma$ are real, hence have nonnegative squares. The Classical Uncertainty Principle Inequality follows immediately.

2.6.2 The weak BLT

The fact that $[\mathcal{P},\mathcal{M}] = -\frac{1}{2\pi i}I$ forms the core of the uncertainty principle approach to proving the BLT. We state this fact in the following form.

Lemma 2.6.2 *If f, $g \in L^2(\mathbb{R})$ satisfy $\mathcal{P}f$, $\mathcal{P}g \in L^2(\mathbb{R})$ and $\mathcal{P}\hat{f}$, $\mathcal{P}\hat{g} \in L^2(\hat{\mathbb{R}})$, then*

$$\langle \mathcal{P}f, \mathcal{M}g\rangle - \langle \mathcal{M}f, \mathcal{P}g\rangle = \frac{1}{2\pi i}\langle f, g\rangle. \tag{2.6.1}$$

Proof: $\mathcal{M}f$ and $\mathcal{M}g$ are well-defined since $\mathcal{P}\hat{f}$, $\mathcal{P}\hat{g} \in L^2(\hat{\mathbb{R}})$. By standard techniques we can find φ_k, $\psi_k \in \mathcal{S}(\mathbb{R})$ such that $\varphi_k \to f$, $\mathcal{P}\varphi_k \to \mathcal{P}f$, $\mathcal{M}\varphi_k \to \mathcal{M}f$, and $\psi_k \to g$, $\mathcal{P}\psi_k \to \mathcal{P}g$, $\mathcal{M}\psi_k \to \mathcal{M}g$, all in L^2-norm. Since φ_k, $\psi_k \in \mathcal{S}(\mathbb{R})$ and $\mathcal{P}$, $\mathcal{M}$ are self-adjoint, we have

$$\begin{aligned}
\langle \mathcal{P}\varphi_k, \mathcal{M}\psi_k\rangle - \langle \mathcal{M}\varphi_k, \mathcal{P}\psi_k\rangle &= \langle \mathcal{M}\mathcal{P}\varphi_k, \psi_k\rangle - \langle \mathcal{P}\mathcal{M}\varphi_k, \psi_k\rangle \\
&= -\langle [\mathcal{P},\mathcal{M}]\varphi_k, \psi_k\rangle \\
&= \frac{1}{2\pi i}\langle \varphi_k, \psi_k\rangle.
\end{aligned}$$

However, the inner product is continuous, so $\langle \varphi_k, \psi_k\rangle \to \langle f, g\rangle$, $\langle \mathcal{P}\varphi_k, \mathcal{M}\psi_k\rangle \to \langle \mathcal{P}f, \mathcal{M}g\rangle$, and $\langle \mathcal{M}\varphi_k, \mathcal{P}\psi_k\rangle \to \langle \mathcal{M}f, \mathcal{P}g\rangle$. Therefore (2.6.1) holds. ❑

Next, we compute the commutators of $\mathcal{P}$ and $\mathcal{M}$ with the translation and modulation operators T_n and M_m defined in (1.2.3).

Lemma 2.6.3 a. $[M_mT_n, \mathcal{P}] = M_mT_n\mathcal{P} - \mathcal{P}M_mT_n = -n\,M_mT_n$.

b. $[M_mT_n, \mathcal{M}] = M_mT_n\mathcal{M} - \mathcal{M}M_mT_n = -m\,M_mT_n$.

Proof: As the two parts are similar, we prove only part a. We compute

$$\begin{aligned}(M_mT_n\mathcal{P}f)(t) - (\mathcal{P}M_mT_nf)(t) &= e_m(t)\,(T_n\mathcal{P}f)(t) - t\,(M_mT_nf)(t) \\ &= e_m(t)\,(\mathcal{P}f)(t-n) - t\,e_m(t)\,(T_nf)(t) \\ &= e_m(t)(t-n)f(t-n) - te_m(t)f(t-n) \\ &= -n\,e_m(t)\,f(t-n) \\ &= -n\,e_m(t)\,(T_nf)(t) \\ &= -n\,(M_mT_nf)(t).\end{aligned}$$

❑

We can now prove a weak version of the BLT.

Theorem 2.6.4 (Weak BLT) *Assume $g \in L^2(\mathbb{R})$ is such that $\{g_{m,n}\}$ is an exact frame for $L^2(\mathbb{R})$. Then we cannot have all of $\mathcal{P}g$, $\mathcal{P}\tilde{g} \in L^2(\mathbb{R})$ and $\mathcal{P}\hat{g}$, $\mathcal{P}\hat{\tilde{g}} \in L^2(\hat{\mathbb{R}})$, i.e., we must have*

$$\|tg(t)\|_2\, \|\gamma\hat{g}(\gamma)\|_2\, \|t\tilde{g}(t)\|_2\, \|\gamma\hat{\tilde{g}}(\gamma)\|_2 = +\infty.$$

Proof: Assume all four functions were elements of L^2. Note that

$$\forall\, f, h \in L^2(\mathbb{R}), \quad \langle f, h_{m,n}\rangle = \langle f_{-m,-n}, h\rangle.$$

Also, by Lemma 2.6.3a,

$$\forall\, f \in L^2(\mathbb{R}), \quad \mathcal{P}(f_{m,n}) = (\mathcal{P}f)_{m,n} + n\, f_{m,n}.$$

Since $\mathcal{P}$ is self-adjoint and $\{g_{m,n}\}$ is biorthonormal to its dual frame $\{\tilde{g}_{m,n}\}$, we can therefore compute

$$\begin{aligned}\langle \mathcal{P}g, \tilde{g}_{m,n}\rangle = \langle g, \mathcal{P}(\tilde{g}_{m,n})\rangle &= \langle g, (\mathcal{P}\tilde{g})_{m,n})\rangle + n\,\langle g, \tilde{g}_{m,n})\rangle \\ &= \langle g_{-m,-n}, \mathcal{P}\tilde{g}\rangle + n\,\delta_{m,0}\,\delta_{n,0} \\ &= \langle g_{-m,-n}, \mathcal{P}\tilde{g}\rangle.\end{aligned}$$

Now, by the L^2-inversion formula, both $\mathcal{M}g$ and $\mathcal{M}\tilde{g}$ exist and are in $L^2(\mathbb{R})$, so by Lemma 2.6.3b we similarly obtain

$$\langle g_{m,n}, \mathcal{M}\tilde{g}\rangle = \langle \mathcal{M}(g_{m,n}), \tilde{g}\rangle = \langle (\mathcal{M}g)_{m,n}, \tilde{g}\rangle + m\langle g_{m,n}, \tilde{g}\rangle = \langle \mathcal{M}g, \tilde{g}_{-m,-n}\rangle.$$

Since $f = \sum\langle f, g_{m,n}\rangle\, \tilde{g}_{m,n} = \sum\langle f, \tilde{g}_{m,n}\rangle\, g_{m,n}$ for every $f \in L^2(\mathbb{R})$, we therefore have

$$\begin{aligned}\langle \mathcal{P}g, \mathcal{M}\tilde{g}\rangle &= \sum_{m,n}\langle \mathcal{P}g, \tilde{g}_{m,n}\rangle\, \langle g_{m,n}, \mathcal{M}\tilde{g}\rangle \\ &= \sum_{m,n}\langle g_{-m,-n}, \mathcal{P}\tilde{g}\rangle\, \langle \mathcal{M}g, \tilde{g}_{-m,-n}\rangle \\ &= \sum_{m,n}\langle \mathcal{M}g, \tilde{g}_{m,n}\rangle\, \langle g_{m,n}, \mathcal{P}\tilde{g}\rangle \\ &= \langle \mathcal{M}g, \mathcal{P}\tilde{g}\rangle.\end{aligned}$$

Therefore, by biorthonormality and Lemma 2.6.2,

$$1 = \langle g, \tilde{g} \rangle = 2\pi i \Big(\langle \mathcal{P}g, \mathcal{M}\tilde{g} \rangle - \langle \mathcal{M}g, \mathcal{P}\tilde{g} \rangle \Big) = 0,$$

a contradiction. ❑

2.6.3 Equivalence of the weak BLT and the BLT

We can give several special cases illustrating the relationship between the weak BLT and the usual BLT (Theorem 2.1.1).

Example 2.6.1 a. If the Gabor frame set $g_{m,n}$ is actually an orthonormal basis, then $\tilde{g} = g$ and the equivalence is clear. This is precisely Battle's proof of the BLT [Bat88].

b. If g generates a tight exact frame with bounds $A = B$ then $\tilde{g} = S^{-1}g = A^{-1}g$, and the equivalence is again clear. However, any tight exact frame is a multiple of an orthonormal basis.

c. If $\operatorname{supp}(g)$ is contained in an interval of length 1, then the frame operator S is $Sf = f \cdot \lambda$, where $\lambda(t) = \sum |g(t-n)|^2$ (for example, [HW89]). Any Gabor frame must have $A \leq \lambda(t) \leq B$ a.e. (Theorem 3.5), so S is multiplication by an essentially constant function. Therefore, $\tilde{g} = S^{-1}\tilde{g} = g/\lambda$, and thus $\mathcal{P}g \in L^2(\mathbb{R})$ if and only if $\mathcal{P}\tilde{g} \in L^2(\mathbb{R})$. Similarly, if $\operatorname{supp}(\hat{g})$ is contained in an interval of length 1, then $(Sf)^\wedge = \hat{f} \cdot \Lambda$, where Λ is the essentially constant function $\sum |\hat{g}(\gamma - m)|^2$, and so $\mathcal{P}\hat{g} \in L^2(\hat{\mathbb{R}})$ if and only if $\mathcal{P}\hat{\tilde{g}} \in L^2(\hat{\mathbb{R}})$. ❑

The BLT will follow from the weak BLT if we can prove that

$$\mathcal{P}g \in L^2(\mathbb{R}) \Leftrightarrow \mathcal{P}\tilde{g} \in L^2(\mathbb{R}) \qquad \text{and} \qquad \mathcal{P}\hat{g} \in L^2(\hat{\mathbb{R}}) \Leftrightarrow \mathcal{P}\hat{\tilde{g}} \in L^2(\hat{\mathbb{R}}), \tag{2.6.2}$$

whenever $\{g_{m,n}\}$ is an exact frame. We verify (2.6.2) in Theorem 2.6.6. First, however, we compute the Zak transform of the dual function $\tilde{g}$.

Proposition 2.6.5 *If $g \in L^2(\mathbb{R})$ and $\{g_{m,n}\}$ is a frame then*

$$\mathcal{Z}\tilde{g} = 1/\overline{\mathcal{Z}g}.$$

Proof: If $\{g_{m,n}\}$ is a frame then $0 < A \leq |\mathcal{Z}g|^2 \leq B < \infty$ a.e. on Q. Therefore, $h = \mathcal{Z}^{-1}(1/\overline{\mathcal{Z}g}) \in L^2(\mathbb{R})$. Given $m, n \in \mathbb{Z}$ we then compute:

$$\begin{aligned} \langle h, g_{m,n} \rangle = \langle \mathcal{Z}h, \mathcal{Z}g_{m,n} \rangle &= \langle 1/\overline{\mathcal{Z}g}, e_m(t)e_n(\omega)\mathcal{Z}g \rangle \\ &= \langle 1, e_m(t)e_n(\omega) \rangle \\ &= \delta_{m,0}\,\delta_{n,0} \\ &= \langle \tilde{g}, g_{m,n} \rangle. \end{aligned}$$

Since $\{g_{m,n}\}$ is complete in $L^2(\mathbb{R})$ and $h,\ \tilde{g} \in L^2(\mathbb{R})$, it follows that $h = \tilde{g}$. ❑

The following theorem is due to Daubechies and Janssen [DJ93].

Theorem 2.6.6 *If* $\{g_{m,n}\}$ *is an exact frame then* (2.6.2) *holds.*

Proof: First, given any function f, we formally compute:

$$\begin{aligned} \mathcal{Z}\mathcal{P}f(t,\omega) &= \sum_k \mathcal{P}f(t+k)\, e^{2\pi i k\omega} \\ &= t\sum_k f(t+k)\, e^{2\pi i k\omega} \;+\; \sum_k f(t+k)\, k\, e^{2\pi i k\omega} \\ &= t\,\mathcal{Z}f(t,\omega) \;+\; \sum_k f(t+k)\, \frac{1}{2\pi i}\, \partial_2 e^{2\pi i k\omega} \\ &= t\,\mathcal{Z}f(t,\omega) \;+\; \frac{1}{2\pi i}\, \partial_2 \mathcal{Z}f(t,\omega). \end{aligned} \tag{2.6.3}$$

Next, using Proposition 2.6.5, we formally compute:

$$\begin{aligned} \overline{\mathcal{Z}\mathcal{P}\tilde{g}(t,\omega)} &= t\,\overline{\mathcal{Z}\tilde{g}(t,\omega)} \;+\; \overline{\frac{1}{2\pi i}\, \partial_2 \mathcal{Z}\tilde{g}(t,\omega)} \\ &= \frac{t}{\mathcal{Z}g(t,\omega)} \;-\; \frac{1}{2\pi i}\, \partial_2 (1/\mathcal{Z}g)(t,\omega) \\ &= \frac{t}{\mathcal{Z}g(t,\omega)} \;+\; \frac{\frac{1}{2\pi i}\, \partial_2 \mathcal{Z}g(t,\omega)}{\mathcal{Z}g(t,\omega)^2} \\ &= \frac{t\,\mathcal{Z}g(t,\omega) \;+\; \frac{1}{2\pi i}\, \partial_2 \mathcal{Z}g(t,\omega)}{\mathcal{Z}g(t,\omega)^2} \\ &= \frac{\mathcal{Z}\mathcal{P}g(t,\omega)}{\mathcal{Z}g(t,\omega)^2}. \end{aligned} \tag{2.6.4}$$

If these formal calculations are justified, then $\mathcal{Z}\mathcal{P}g \in L^2(Q)$ if and only if $\mathcal{Z}\mathcal{P}\tilde{g} \in L^2(Q)$ since $0 < A \le |\mathcal{Z}g|^2 \le B < \infty$ a.e. As $\mathcal{Z}$ is unitary, this implies $\mathcal{P}g \in L^2(\mathbb{R})$ if and only if $\mathcal{P}\tilde{g} \in L^2(\mathbb{R})$. A similar formal argument involving the partial derivative ∂_1 is used to show that $\mathcal{P}\hat{g} \in L^2(\hat{\mathbb{R}})$ if and only if $\mathcal{P}\hat{\tilde{g}} \in L^2(\hat{\mathbb{R}})$. The critical step is the use of the quotient rule in (2.6.4).

The calculations above are justified as follows. If $f \in L^2(\mathbb{R})$ then (2.6.3) is valid distributionally, i.e., in the $\sigma(D'(\mathbb{R}^2), C_c^\infty(\mathbb{R}^2))$ topology. In particular, (2.6.3) is true distributionally for $f = g$ and $f = \tilde{g}$. Then (2.6.4) follows from Theorem 2.5.1a. In fact, $G = \mathcal{Z}g \in AC_{\text{loc}}$ on almost all straight lines parallel to the coordinate axes since $\partial_1 G,\ \partial_2 G \in L^2_{\text{loc}}(\mathbb{R}^2)$ (for example, Example 2.5.1a). Then, for almost all t, the quotient rule to compute the classical partial derivative $D_2(1/G)$ holds for almost all ω. Distributional and classical differentiation are equivalent in this case by Theorem 2.5.1a. Therefore (2.6.4) is valid distributionally, and so the distributions $\overline{\mathcal{Z}\mathcal{P}\tilde{g}}$ and $\mathcal{Z}\mathcal{P}g/(\mathcal{Z}g)^2$ are equal. Thus, if one of them is square integrable then so is the other. ❑

The combination of the Weak BLT (Theorem 2.6.4) and Theorem 2.6.6 proves the BLT.

2.6.4 The BLT without differentiation

It is interesting to recount that distributional differentiation really initiated the proof of the BLT in [Dau90] before shifting to classical differentiation, e.g., Example 2.5.1a,b; and that, in the proof of the BLT in Theorem 2.6.6, classical differentiation was required to implement the quotient rule in order to proceed with distributional calculations. In both cases, Theorem 2.5.1a played a pivotal role. A proof using neither classical nor distributional differentiation overtly was given in [BHW95].

The final issue we wish to address is the possibility of proving the BLT for exact frames along the operator-theoretic lines of Section 2.6.2, but without invoking differentiation. The proof of Proposition 2.6.5 motivates the argument we have in mind. Let us attempt to use the same technique to prove that $\mathcal{P}g \in L^2(\mathbb{R})$ if and only if $\mathcal{P}\tilde{g} \in L^2(\mathbb{R})$. Assume that $g \in L^2(\mathbb{R})$ generates an exact frame $\{g_{m,n}\}$ and that $\mathcal{P}g \in L^2(\mathbb{R})$. Then $\mathcal{Z}\mathcal{P}g \in L^2(Q)$ and $\mathcal{Z}g$ is essentially constant, so $\mathcal{Z}\mathcal{P}g/(\mathcal{Z}g)^2 \in L^2(Q)$. Therefore, $h = \mathcal{Z}^{-1}\big(\overline{\mathcal{Z}\mathcal{P}g/(\mathcal{Z}g)^2}\big) \in L^2(\mathbb{R})$. Note that

$$\begin{aligned}\langle h, g_{m,n}\rangle &= \langle \mathcal{Z}h, \mathcal{Z}g_{m,n}\rangle = \langle \overline{\mathcal{Z}\mathcal{P}g/(\mathcal{Z}g)^2}, e_m(t)e_n(\omega)\mathcal{Z}g\rangle \\ &= \langle \overline{\mathcal{Z}\mathcal{P}g/\mathcal{Z}g}, e_m(t)e_n(\omega)\rangle.\end{aligned}$$

By Lemma 2.6.3a, $\mathcal{P}g_{m,n} = (\mathcal{P}g)_{m,n} + n\, g_{m,n} \in L^2(\mathbb{R})$, so we can compute

$$\begin{aligned}\langle \tilde{g}, \mathcal{P}g_{m,n}\rangle &= \langle \tilde{g}, (\mathcal{P}g)_{m,n}\rangle + n\,\langle \tilde{g}, g_{m,n}\rangle \\ &= \langle \mathcal{Z}\tilde{g}, \mathcal{Z}(\mathcal{P}g)_{m,n}\rangle + n\,\delta_{m,0}\,\delta_{n,0} \\ &= \langle 1/\overline{\mathcal{Z}g}, e_m(t)e_n(\omega)Z(\mathcal{P}g)\rangle \\ &= \langle h, g_{m,n}\rangle.\end{aligned}$$

Although we have not assumed that $\mathcal{P}\tilde{g} \in L^2(\mathbb{R})$, the integral $\int \mathcal{P}\tilde{g}(t) g_{m,n}(t)\,dt$ is well-defined since it equals $\int \tilde{g}(t)\,\mathcal{P}g_{m,n}(t)\,dt$ and $\tilde{g}\cdot\mathcal{P}g \in L^1(\mathbb{R})$. Thus,

$$\langle \mathcal{P}\tilde{g} - h, g_{m,n}\rangle = \langle \tilde{g}, \mathcal{P}g_{m,n}\rangle - \langle h, g_{m,n}\rangle = 0 \qquad (2.6.5)$$

for all $m,\ n \in \mathbb{Z}$. We would like to conclude from (2.6.5) and the fact that $\{g_{m,n}\}$ is complete in $L^2(\mathbb{R})$ that $\mathcal{P}g - h = 0$. Note that the function $f = \mathcal{P}\tilde{g} - h = f_1 + f_2$ has the property that $f_1 \in t\,L^2(\mathbb{R})$ and $f_2 \in L^2(\mathbb{R})$.

Problem/Examples 2.6.1 Motivated by the preceding discussion, we pose the following uniqueness problem. Let $f \in L^2_{\text{loc}}(\mathbb{R})$ and assume that $\{g_k\}$ is a complete sequence in $L^2(\mathbb{R})$. Suppose that $f\,\overline{g_k} \in L^1(\mathbb{R})$ and that $\int f(t)\,\overline{g_k(t)}\,dt = 0$ for all k. What further conditions must be assumed to conclude that $f = 0$?

The following examples deal with our uniqueness problem.

a. The Haar system $\{h_k\} \subseteq L^2(\mathbb{R})$ is an orthonormal basis of $L^2(\mathbb{R})$; and if $f = 1$, then $f\,\overline{h_k} \in L^1(\mathbb{R})$ and $\langle f, h_k\rangle = 0$ for all k. In fact, this holds for any wavelet orthonormal basis whose elements are integrable.

b. Uniqueness also fails for the Gabor sequence $\{g_{m,n}\}$ generated by $g = \mathbf{1}_{[0,1]} - \mathbf{1}_{[-1,0]}$. In fact, if $f = 1$, then $f\,\overline{g_{m,n}} \in L^1(\mathbb{R})$ and $\langle f, g_{m,n}\rangle = 0$ for all m and n. Further, the Zak transform of g is

$$\mathcal{Z}g(t,\omega) \;=\; 1 \;-\; e^{-2\pi i\omega} \;\neq\; 0 \quad \text{a.e. on } Q,$$

so that $\{g_{m,n}\} \subseteq L^2(\mathbb{R})$ is a complete sequence (for example, Theorem 2.3.1a). In contrast to part a, $\{g_{m,n}\}$ is not even a frame.

c. Let $g = \sum a_k\,\mathbf{1}_{[k-1/2,k+1/2]}$ where $\sum |a_k| < \infty$ and $\sum a_k = 0$. Then $\mathcal{Z}g(t,\omega) = \sum a_k\,e^{2\pi i k\omega}$ for $t \in (-1/2, 1/2)$, so that $\mathcal{Z}g$ is continuous in a neighborhood of $(0,0)$ and $\mathcal{Z}g(0,0) = 0$. Thus $\{g_{m,n}\}$ is not a Gabor frame. On the other hand, if $f = 1$ then $f\,\overline{g_{m,n}} \in L^1(\mathbb{R})$ and $\langle f, g_{m,n}\rangle = 0$ for all m and n.

d. Let $g \in L^2(\mathbb{R}) \setminus (L^1(\mathbb{R}) \cap L^2(\mathbb{R}))$ be the odd function defined as $g = 0$ on $[0, 1/2)$, $g = 0$ on $[2k-1, 2k+1)$ for k even, and $g = -2i/(\pi k)$ on $[2k-1, 2k+1)$ for k odd. We can write g as

$$g(t) \;=\; g(u-k) \;=\; \int_{-1/2}^{1/2} G(u,\omega)\,e^{-2\pi i k\omega}\,d\omega,$$

where $t = u - k$, $u \in [-1/2, 1/2)$, $k \in \mathbb{Z}$, and $G(u,\omega)$ is 1 on $[-1/2,1/2) \times [0,1/2)$ and is -1 on $[-1/2,1/2) \times [-1/2,0)$. Thus, $G = \mathcal{Z}g$ and $\{g_{m,n}\}$ is a Gabor frame. If $f = 1$ then $\langle f, g_{m,n}\rangle = 0$ for all m and n, whereas $f\,\overline{g_{m,n}} \notin L^1(\mathbb{R})$. Clearly, $\mathcal{P}g \notin L^2(\mathbb{R})$. ❑

2.7 Appendix: Notation

The Fourier transform of $f \in L^1(\mathbb{R})$ is $\hat{f}(\gamma) = \int f(t)\,e^{-2\pi i\gamma t}\,dt$ for $\gamma \in \hat{\mathbb{R}}$. The inverse Fourier transform is $\check{f}(\gamma) = \hat{f}(-\gamma)$. The Fourier transform extends to $f \in L^2(\mathbb{R})$. If $f \in L^2(\mathbb{R})$ then $f = (\hat{f})^{\vee}$.

$C(\mathbb{R}^d)$ is the space of continuous functions on $\mathbb{R}^d$. $C_c(\mathbb{R}^d)$ is the space of continuous functions with compact support. $C_c^\infty(\mathbb{R}^d)$ is the space of compactly supported functions infinitely differentiable on $\mathbb{R}^d$. $C_0(\mathbb{R}^d)$ is the space of continuous functions on $\mathbb{R}^d$ vanishing at infinity. $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz class of infinitely differentiable functions which are rapidly decreasing at infinity, and $\mathcal{S}'(\mathbb{R}^d)$ is its topological dual, the space of tempered distributions. $A(\hat{\mathbb{R}}^d)$ is $L^1(\mathbb{R}^d)^\wedge = \{\hat{f} : f \in L^1(\mathbb{R}^d)\}$. AC_{loc} is the space of all functions that are locally absolutely continuous.

If $g \in L^2(\mathbb{R})$ and a, $b > 0$ are given then the Gabor system generated by g, a, and b is $\{g_{mb,na}\}_{m,n\in\mathbb{Z}}$, where $g_{p,q}(t) = e^{2\pi ipt}\, g(t-q)$. If $\{g_{mb,na}\}$ is a frame then the frame operator is $Sf = \sum \langle f, g_{mb,na}\rangle\, g_{mb,na}$. The dual function is $\tilde{g} = S^{-1}g$. The dual frame is $\{\tilde{g}_{mb,na}\}$.

Distributional differentiation is denoted by ∂ on $\mathbb{R}$ or ∂_j on $\mathbb{R}^d$ (the j-th partial). Classical differentiation is denoted by D on $\mathbb{R}$ and D_j on $\mathbb{R}^d$. The Sobolev space $L_1^2(\mathbb{R}^d)$ is $\{f \in L^2(\mathbb{R}^d) : \partial_j f \in L^2(\mathbb{R}^d),\ j = 1, \ldots, d\}$.

If $x = (t, \omega) \in \mathbb{R} \times \hat{\mathbb{R}}$ and $r > 0$ then $Q(x; r) = Q(t, \omega; r)$ is the square centered at x with side r, i.e., $Q(x; r) = [t - r/2, t + r/2) \times [\omega - r/2, \omega + r/2)$. We set $Q = Q(1/2, 1/2; 1) = [0, 1) \times [0, 1)$.

The translation operator is $T_x f(t) = f(t - x)$ and the modulation operator is $M_\gamma f(t) = e^{2\pi i\gamma t} f(t)$. We write $e_\gamma(t) = e^{2\pi i\gamma t}$.

The indicator function of the set E is $\mathbf{1}_E(x) = 1$ if $x \in E$ and 0 if $x \notin E$.

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3

A Banach space of test functions for Gabor analysis

Hans G. Feichtinger and Georg Zimmermann

ABSTRACT – We introduce the Banach space $\boldsymbol{S}_0 \subseteq \boldsymbol{L}^2$ which has a variety of properties making it a useful tool in Gabor analysis. $\boldsymbol{S}_0$ can be characterized as the smallest time-frequency homogeneous Banach space of (continuous) functions. We also present other characterizations of $\boldsymbol{S}_0$ turning it into a very flexible tool for Gabor analysis and allowing for simplifications of various proofs. A careful analysis of both the coefficient and the synthesis mapping in Gabor theory shows that an arbitrary window in $\boldsymbol{S}_0$ not only is a Bessel atom with respect to arbitrary time-frequency lattices, but also yields boundedness between $\boldsymbol{S}_0$ and $\boldsymbol{\ell}^1$. On the other hand, we can study properties of general $\boldsymbol{L}^2$-atoms since they induce mappings from $\boldsymbol{S}_0$ to $\boldsymbol{S}_0'$. This enables us to introduce a new, very natural concept of weak duality of Gabor atoms, applying also to the classical pair of the Gauss-function and its dual function determined by Bastiaans. Using the established results, we show a variety of properties that are desirable in applications, like the continuous dependence of the canonical dual window on the given Gabor window and on the lattice; continuity of thresholding and masking operators from signal processing; and an algorithm for the reconstruction of bandlimited functions from samples of the Gabor transform in a corresponding horizontal strip in the time-frequency plane. We also present an approximate Balian–Low Theorem stating that for close-to-critical lattices, the dual Gabor atoms progressively lose their time-frequency localization.

3.1 Introduction

One of the main difficulties in Gabor theory is the choice of the appropriate window for the short time Fourier transformation, to ensure that the STFT of a function f reflects those properties of f we want to study. In practice, this amounts to the construction of a suitable *Gabor family*, consisting of the time-frequency shifts $(\pi(\lambda)\gamma)_{\lambda\in\Lambda}$ of a *Gabor atom* γ along a *time-frequency lattice* $\lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Then the *Gabor coefficients* $\big(\langle f, \pi(\lambda)\gamma\rangle\big)_{\lambda\in\Lambda}$ are just the values of the STFT of f with window γ sampled on Λ. Thus we need to choose γ and Λ in such a way that the *coefficient mapping* T_γ,

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mapping f to its Gabor coefficients, is well-behaved in the sense that it is bounded between the function space we are interested in and a corresponding sequence space on Λ. Furthermore, we require stable reconstruction of f from its Gabor coefficients. I.e., we want T_γ to have a bounded left inverse, the *synthesis mapping* of the form $T_g^* : (c_\lambda)_\lambda \mapsto \sum_\lambda c_\lambda \pi(\lambda) g$ for some *dual Gabor window g*. Note that the Gabor frame operator is just the composition $T_\gamma^* T_\gamma$.

In this chapter, we introduce the reader to the Banach space $\boldsymbol{S}_0(\mathbb{R}^d)$ and show that it has various useful properties for approaching the problems described above. After establishing some basic results concerning $\boldsymbol{S}_0(\mathbb{R}^d)$ in Section 2, we will give in Section 3 a detailed overview over the continuity properties of the mappings T_γ and T_g^* between several function and distribution spaces, in particular involving $\boldsymbol{L}^2(\mathbb{R}^d)$, $\boldsymbol{S}_0(\mathbb{R}^d)$, and $\boldsymbol{S}_0'(\mathbb{R}^d)$, and corresponding sequence spaces over Λ. Based on these results and on some abstract observations concerning frames and Riesz bases stated in Section 4, we will be able to establish in Section 5 some of the key results in Gabor analysis, such as the Wexler–Raz principle and the existence of a Janssen representation of the frame operator. We also introduce a new concept of weak duality of Gabor atoms. In the last section, we shall show in various examples that it is useful to have dual pairs of Gabor atoms $(g, \gamma) \in \boldsymbol{S}_0 \times \boldsymbol{S}_0(\mathbb{R}^d)$. This will yield continuous dependence of the canonical dual atoms on the lattice constants of Λ; an approximate Balian-Low result for close-to-critical time-frequency lattices; and the possibility of complete reconstruction of bandlimited signals from the knowledge of their Gabor coefficients in some horizontal strip only.

3.1.1 Notation

The following notations and symbols will be used throughout this chapter.

Besides the standard expressions $\boldsymbol{L}^p$ and $\boldsymbol{\ell}^p$ for the respective Banach spaces of functions and sequences, we will use $\boldsymbol{C}_0$ (and $\boldsymbol{c}_0$) for the space of continuous functions (and sequences, respectively) vanishing at infinity. Due to the Riesz representation theorem, its dual space coincides with the space of bounded Radon measures, denoted by $\boldsymbol{M}_b$. For the Schwartz space of smooth functions with rapid decay, we use the symbol $\boldsymbol{\mathcal{S}}$. By $\boldsymbol{A}$ and $\boldsymbol{A}'$, we denote the Fourier image $\mathcal{F}\boldsymbol{L}^1$ of the space of integrable functions and its dual $\mathcal{F}\boldsymbol{L}^\infty$.

It follows from the Riemann-Lebesgue Lemma that $\mathcal{F}\boldsymbol{L}^1$ is a (dense) subalgebra of $\boldsymbol{C}_0$. The subscript c in $\boldsymbol{A}_c$, $\boldsymbol{C}_c$, etc. indicates that we refer to the respective subspace of elements with compact support. The symbol $\boldsymbol{\mathcal{L}}(\boldsymbol{X}, \boldsymbol{Y})$ denotes the space of bounded linear operators $T : \boldsymbol{X} \to \boldsymbol{Y}$ between two Banach spaces, with operator norm $|||T|||_{\mathcal{L}(\boldsymbol{X},\boldsymbol{Y})}$. Furthermore, we abbreviate $\boldsymbol{\mathcal{L}}(\boldsymbol{X}, \boldsymbol{X})$ by $\boldsymbol{\mathcal{L}}(\boldsymbol{X})$.

For a function or distribution f on $\mathbb{R}^d$, we define the *translation* by an element $x \in \mathbb{R}^d$ as $(\mathrm{T}_x f)(y) = f(y-x)$, and the *modulation* by $\xi \in \widehat{\mathbb{R}}^d$ as $(\mathrm{M}_\xi f)(y) = e^{2\pi i y \xi} f(y)$.

The *time-frequency plane* is the set $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, and for $\lambda = (x, \xi) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, we define the *time-frequency shift* to be $\pi(\lambda) = \mathrm{M}_\xi \mathrm{T}_x$. A *time-frequency homogeneous* Banach space is a Banach space $\boldsymbol{X}$ of functions or distributions on $\mathbb{R}^d$ with the property that the action of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ on the elements of $\boldsymbol{X}$ via π is isometric and strongly continuous; i.e., it satisfies $\|\pi(\lambda) f\|_{\boldsymbol{X}} = \|f\|_{\boldsymbol{X}}$ for all $\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, and $\|\pi(\lambda)f - f\|_{\boldsymbol{X}} \to 0$ as $\lambda \to (0,0)$ for each $f \in \boldsymbol{X}$.

A *time-frequency lattice* Λ is a discrete subgroup of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ with compact quotient. Its *redundancy* $\mathrm{red}(\Lambda)$ is the reciprocal value of the measure of a fundamental domain for the quotient $\mathbb{R}^d \times \widehat{\mathbb{R}}^d / \Lambda$. Any such lattice is of the form $\Lambda = M\mathbb{Z}^{2d}$ for some nonsingular $2d \times 2d$-matrix M, and $\mathrm{red}(\Lambda) = (\det M)^{-1}$. For $d = 1$, we have in the separable case $\Lambda = a\mathbb{Z} \times b\mathbb{Z} \lhd \mathbb{R} \times \widehat{\mathbb{R}}$ that $\mathrm{red}(\Lambda) = (ab)^{-1}$.

Dilation by a nonsingular matrix A is defined by $(\mathrm{D}_A f)(y) = f(Ay)$. In case $A = \rho I_d$, we simply write D_ρ. Thus D_A is an isometry on $\boldsymbol{L}^\infty(\mathbb{R}^d)$ and on $\boldsymbol{C}_b(\mathbb{R}^d)$, while $|\det A|^{1/2}\, \mathrm{D}_A$ is unitary on $\boldsymbol{L}^2(\mathbb{R}^d)$, and $|\det A|\, \mathrm{D}_A$ acts isometrically on $\boldsymbol{L}^1(\mathbb{R}^d)$ and on $\boldsymbol{M}_b(\mathbb{R}^d)$. Dilations commute with time-frequency shifts via $\mathrm{D}_A \mathrm{M}_\xi \mathrm{T}_x = \mathrm{M}_{A^T\xi} \mathrm{T}_{A^{-1}x} \mathrm{D}_A$.

In a normed space $\boldsymbol{X}$, we denote the open ball of radius $r > 0$ around $x \in \boldsymbol{X}$ by $B_r(x) = \{y \in \boldsymbol{X} : \|y - x\|_{\boldsymbol{X}} < r\}$.

3.2 Characterizations of the Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$

The function space $\boldsymbol{S}_0(\mathbb{R}^d)$, which we want to define in this section, has a number of different characterizations (e.g., see [Fei81], [FG92a]), some of which extend directly to general locally compact abelian groups. We will discuss several properties of this space, providing simplified proofs based on the approach suggested in [Fei88] for some of them. For other properties of interest, we refer the reader to the literature.

3.2.1 $\boldsymbol{S}_0(\mathbb{R}^d)$ *via the STFT*

To begin with, we define $\boldsymbol{S}_0(\mathbb{R}^d)$ via the short time Fourier transform (STFT) with respect to the Gauss-function. Recall that for $f \in \boldsymbol{L}^2(\mathbb{R}^d)$, the STFT with respect to a window $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ is defined as[1]

$$(\mathcal{V}_g f)(x,\xi) = \int_{\mathbb{R}^d} f(y)\,\overline{g(y-x)}\,e^{-2\pi i y\xi}\,dy = \langle f, \mathrm{M}_\xi \mathrm{T}_x g\rangle . \tag{3.2.1}$$

The book of Folland [Fol89] contains extensive material concerning the Fourier–Wigner transformation which is related to the STFT via

$$V(f,g)(x,\xi) = e^{-i\pi x\xi}\,\mathcal{V}_g f(x,-\xi) . \tag{3.2.2}$$

Thus several results he states obviously hold for the STFT also, like the following ones.

Lemma 3.2.1 *i) For $f,\ g \in \boldsymbol{L}^2(\mathbb{R}^d)$, we have $\mathcal{V}_g f \in \boldsymbol{C}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ with*

$$\|\mathcal{V}_g f\|_{\boldsymbol{C}_0(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} \le \|g\|_{\boldsymbol{L}^2(\mathbb{R}^d)}\,\|f\|_{\boldsymbol{L}^2(\mathbb{R}^d)} . \tag{3.2.3}$$

ii) Moyal's formula:

$$\|\mathcal{V}_g f\|_{\boldsymbol{L}^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} = \|g\|_{\boldsymbol{L}^2(\mathbb{R}^d)}\,\|f\|_{\boldsymbol{L}^2(\mathbb{R}^d)} . \tag{3.2.4}$$

Remark. (i) follows from the definition (3.2.1) using the Cauchy–Schwarz inequality and the strong continuity of time-frequency shifts in $\boldsymbol{L}^2(\mathbb{R}^d)$ (i.e., $\|\mathrm{M}_\xi \mathrm{T}_x f - f\|_{\boldsymbol{L}^2} \to 0$ for $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ as $(x,\xi) \to (0,0)$.)

(ii) can be seen as a sophisticated consequence of Plancherel's Theorem. It implies that f can be reconstructed from $\mathcal{V}_g f$, and the inverse STFT is given – in vector-valued interpretation – by

$$f = \frac{1}{\|g\|^2_{\boldsymbol{L}^2}} \iint_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} \mathcal{V}_g f(x,\xi)\,\mathrm{M}_\xi \mathrm{T}_x g\,dx d\xi . \tag{3.2.5}$$

The space $\boldsymbol{S}_0(\mathbb{R}^d)$ can be characterized as the functions with integrable STFT for "sufficiently nice" windows (compare Theorem 3.2.4 below.) To introduce a standard, we choose the Gauss-function, defined by

$$g_{0,d}(x) = g_0(x) = e^{-\pi\|x\|^2} = e^{-\pi(x_1^2+\cdots+x_d^2)} \tag{3.2.6}$$

for $x = (x_1,\ldots,x_d) \in \mathbb{R}^d$, for several reasons. One could argue – following Gabor in [Gab46] – that g_0 offers the best time-frequency resolution and therefore is best suited as a window for the STFT. Furthermore, the following properties of g_0 will turn out to be convenient for our purposes.

[1] In an attempt to avoid possible confusion, we remark that at some places in the literature, g is used in place of $\overline{g}$ in (3.2.1).

Lemma 3.2.2 ([Fol89])
The Gauss-function $g_0(x) = e^{-\pi\|x\|^2}$ has the following properties.
i) $\widehat{g_0} = g_0$ *with* $\|g_0\|_{L^1} = \|g_0\|_{L^\infty} = 1$ *and* $\|g_0\|_{L^2} = 2^{-d/4}$.
ii) g_0 *is separable, i.e.,* $g_0(x_1, \dots, x_d) = g_0(x_1) \cdots g_0(x_d)$ *for* $x \in \mathbb{R}^d$.
iii) g_0 *is radially symmetric, hence does not depend on the particular choice of orthogonal axes in* $\mathbb{R}^d$.

Using the STFT with respect to the Gauss-function, we can define the space $S_0(\mathbb{R}^d)$ as follows.

Definition. The space $S_0(\mathbb{R}^d)$ is given by

$$S_0(\mathbb{R}^d) = \left\{ f \in L^2(\mathbb{R}^d) : \|f\|_{S_0} = \left\|\mathcal{V}_{g_0} f\right\|_{L^1(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)} < \infty \right\}. \tag{3.2.7}$$

(Note that this means that $S_0(\mathbb{R}^d)$ coincides with the modulation space $M^0_{1,1}$ as discussed in [Fei89], [FGW92], and [FG92a].)

Theorem 3.2.3 (Basic properties of S_0)
$S_0(\mathbb{R}^d)$ *is a time-frequency homogeneous Banach space, continuously embedded as a dense subspace in* $L^2(\mathbb{R}^d)$. *It is also continuously embedded in any time-frequency homogeneous Banach space containing* g_0 *and thus is the smallest (non-trivial) space of this kind.*
Furthermore, it has the following properties.
i) For $f \in S_0(\mathbb{R}^d)$, *also* $\overline{f} \in S_0(\mathbb{R}^d)$ *with* $\|f\|_{S_0} = \|\overline{f}\|_{S_0}$.
ii) For $f \in S_0(\mathbb{R}^d)$, *also* $\hat{f} \in S_0(\mathbb{R}^d)$ *with* $\|f\|_{S_0} = \|\hat{f}\|_{S_0}$,
i.e., the Fourier transformation is an isometry of $S_0(\mathbb{R}^d)$.
iii) $M_b * S_0 \subseteq S_0$ *with* $\|\mu * f\|_{S_0} \le \|\mu\|_{M_b} \|f\|_{S_0}$ *for all* $\mu \in M_b$, $f \in S_0$.
iv) $\widehat{M_b} \cdot S_0 \subseteq S_0$ *with* $\|\hat{\mu} f\|_{S_0} \le \|\mu\|_{M_b} \|f\|_{S_0}$ *for all* $\mu \in M_b$, $f \in S_0$.

Proof. In view of (3.2.3) and (3.2.4), it is clear that

$$\begin{aligned}\|f\|_{L^2}^2 &= \|g_0\|_{L^2}^{-2} \|\mathcal{V}_{g_0} f\|_{L^2}^2 \\ &\le \|g_0\|_{L^2}^{-2} \|\mathcal{V}_{g_0} f\|_{L^\infty} \|\mathcal{V}_{g_0} f\|_{L^1} \le \|g_0\|_{L^2}^{-1} \|f\|_{L^2} \|f\|_{S_0},\end{aligned}$$

and thus

$$\|f\|_{L^2} \le \|g_0\|_{L^2}^{-1} \|f\|_{S_0} \qquad \text{for } f \in S_0(\mathbb{R}^d). \tag{3.2.8}$$

To check completeness of $S_0(\mathbb{R}^d)$, simply verify that for any absolutely convergent series $\sum_{n\in\mathbb{N}} \|f_n\|_{S_0} < \infty$, the L^2-limit $f = \sum_{n\in\mathbb{Z}^d} f_n$ has an STFT satisfying $|\mathcal{V}_{g_0} f| \le \sum_{n\in\mathbb{N}} |\mathcal{V}_{g_0} f_n|$, hence $f \in S_0(\mathbb{R}^d)$ and $\|f\|_{S_0} \le \sum_{n\in\mathbb{N}} \|f_n\|_{S_0}$. The density of S_0 in L^2 will follow from Corollary 3.2.7 below.

The isometric time-frequency invariance of $S_0(\mathbb{R}^d)$ follows from the fact that the STFT transforms time-frequency shifts essentially into translations in the time-frequency plane; namely,

$$\mathcal{V}_g(\mathrm{M}_\xi\,\mathrm{T}_x\,f) = e^{2\pi i x\xi}\,\mathrm{M}_{(0,-x)}\,\mathrm{T}_{(x,\xi)}\,\mathcal{V}_g f\,. \tag{3.2.9}$$

Thus continuity of the action of the time-frequency shifts follows from the strong continuity of the actions of translations and modulations in $L^1(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

If B is any time-frequency homogeneous Banach space containing g_0, we can use the inversion formula of the STFT (3.2.5) to write any $f \in S_0(\mathbb{R}^d)$ as a vector valued integral with values in B. The continuous embedding then follows from the estimate

$$\|f\|_B \le \iint_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} \left|\mathcal{V}_{g_0}f(x,\xi)\right|\,\|\mathrm{M}_\xi\,\mathrm{T}_x\,g_0\|_B\,dx\,d\xi \le \|g_0\|_B\,\|f\|_{S_0}\,.$$

(i) Follows from $(\mathcal{V}_{g_0}\overline{f})(x,\xi) = \overline{(\mathcal{V}_{g_0}f)(x,-\xi)}$.

(ii) Using the Plancherel Theorem, we obtain

$$\langle f, \mathrm{M}_\xi\,\mathrm{T}_x\,g\rangle = \langle \hat{f}, \mathrm{T}_\xi\,\mathrm{M}_{-x}\,\hat{g}\rangle = \langle \hat{f}, e^{2\pi i x\xi}\,\mathrm{M}_{-x}\,\mathrm{T}_\xi\,\hat{g}\rangle\,,$$

or

$$(\mathcal{V}_g f)(x,\xi) = e^{-2\pi i x\xi}(\mathcal{V}_{\hat{g}}\hat{f})(\xi,-x)\,. \tag{3.2.10}$$

In particular, for $g = \hat{g} = g_0$, this implies

$$\left|(\mathcal{V}_{g_0}f)(x,\xi)\right| = \left|(\mathcal{V}_{g_0}\hat{f})(\xi,-x)\right|. \tag{3.2.11}$$

(iii) Follows from the fact that $S_0(\mathbb{R}^d)$ is a homogeneous Banach space.

(iv) Follows from (ii) and (iii). ❑

The following result demonstrates that g_0 does not play a special role in the definition of $S_0(\mathbb{R}^d)$. It also shows that in the above theorem, it suffices to assume that B is a time-frequency homogeneous Banach space having non-zero intersection with $S_0(\mathbb{R}^d)$, to be able to conclude that $S_0(\mathbb{R}^d)$ is continuously embedded in B.

Theorem 3.2.4 ([FG92a], see also [FG88])
For $f \in L^2(\mathbb{R}^d)$, we have that $f \in S_0(\mathbb{R}^d)$ if and only if $\mathcal{V}_{g_1}f \in L^1(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ for some (and then all) $g_1 \in S_0(\mathbb{R}^d) \setminus \{0\}$, and each such g_1 defines an equivalent norm on $S_0(\mathbb{R}^d)$ via

$$\|f\|_{S_0,g_1} = \|\mathcal{V}_{g_1}f\|_{L^1(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)}\,.$$

The above characterizations of $\boldsymbol{S}_0$ are linear in f, but depend on the choice of a particular test function g. The advantage of the following surprising, since quadratic, condition is that it does not depend on the a priori choice of such a test function, but only uses essentially the Wigner distribution of f.

Lemma 3.2.5 ([FG88], Lemma 4.2.ii))
$f \in \boldsymbol{S}_0(\mathbb{R}^d)$ if and only if $\mathcal{V}_f f \in \boldsymbol{L}^1(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

3.2.2 $\boldsymbol{S}_0(\mathbb{R}^d)$ as a Wiener amalgam space

An important class of time-frequency homogeneous Banach spaces are certain *Wiener amalgam spaces* $\boldsymbol{W}(\boldsymbol{X}, \boldsymbol{\ell}^p)$. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ coincides with a space of this type, we shall give a short description of their discrete representation.

The idea behind Wiener amalgam spaces is a separation of local and global properties of a function or distribution. We want a function to belong to $\boldsymbol{W}(\boldsymbol{X}, \boldsymbol{\ell}^p)$, if it belongs locally to $\boldsymbol{X}$, and if its global decay, measured via the local norm in $\boldsymbol{X}$, behaves $\boldsymbol{\ell}^p$-like. To turn this idea into a well-defined mathematical concept, we introduce a standard method to decompose a function into a series of compactly supported pieces.

Definition. A function $\psi \in \boldsymbol{A}_c(\mathbb{R}^d)$ generates a *bounded uniform partition of unity* (BUPU) in $\boldsymbol{A}(\mathbb{R}^d)$, if it satisfies

$$\sum_{n\in\mathbb{Z}^d} \psi(x-n) \equiv 1\,. \tag{3.2.12}$$

Remark. (i) Equation (3.2.12) shows that $\{\mathrm{T}_n\,\psi\}_{n\in\mathbb{Z}^d}$ defines a partition of unity over $\mathbb{R}^d$. Thus we have for any function f on $\mathbb{R}^d$ that

$$f = \sum_{n\in\mathbb{Z}^d} f\;\mathrm{T}_n\,\psi\,,$$

where for each x, the sum has a finite number of non-zero terms. Uniformity refers to the fact that $\operatorname{supp}\mathrm{T}_n\,\psi = n + \operatorname{supp}\psi$, i.e., the supports have uniform size. Finally, $\|\,\mathrm{T}_n\,\psi\|_{\boldsymbol{A}} = \|\psi\|_{\boldsymbol{A}}$ for all $n\in\mathbb{Z}^d$ implies boundedness of the family in $\boldsymbol{A}(\mathbb{R}^d)$.

(ii) Functions $\psi \in \boldsymbol{A}_c(\mathbb{R}^d)$ generating BUPUs, even with respect to more general lattices than $\mathbb{Z}^d$, exist in abundance. Consider, e.g., the triangular function $\psi(x) = (1-|x|)_+$ on $\mathbb{R}$ and take tensor products in $\mathbb{R}^d$. To obtain smooth examples, convolve ψ with $\varphi \in \boldsymbol{C}_c^\infty$ satisfying $\widehat{\varphi}(0) = 1$. We can also obtain partitions of unity using B-splines of arbitrarily high order.

Definition. Let $\boldsymbol{X}(\mathbb{R}^d)$ be a translation invariant Banach space of functions or distributions on $\mathbb{R}^d$ such that $\boldsymbol{A}(\mathbb{R}^d) \cdot \boldsymbol{X}(\mathbb{R}^d) \subseteq \boldsymbol{X}(\mathbb{R}^d)$ with $\|\varphi f\|_{\boldsymbol{X}} \leq \|\varphi\|_{\boldsymbol{A}} \|f\|_{\boldsymbol{X}}$. Given $\psi \in \boldsymbol{A}_c(\mathbb{R}^d)$ generating a BUPU, we define

$$\boldsymbol{W}(\boldsymbol{X},\boldsymbol{\ell}^p) = \left\{ f \in \boldsymbol{X}_{loc} : \|f\|_{\boldsymbol{W}(\boldsymbol{X},\boldsymbol{\ell}^p)} = \left(\sum_{n \in \mathbb{Z}^d} \|f \, \mathrm{T}_n \psi\|_{\boldsymbol{X}}^p \right)^{1/p} < \infty \right\}$$

for $p \in [1,\infty)$, with the usual modification for $p = \infty$.

Remark. Again it is routine to verify completeness of a space of this type. Since the family of time-frequency shifts is uniformly bounded with respect to the norm given above, we can replace this norm by an equivalent one to obtain a time-frequency homogeneous Banach space for $p < \infty$. Moreover, different choices for ψ yield the same space and equivalent norms. For details, e.g., see [Fei80], [FG85], [Fei90], or [Fei91].

In the terminology of Wiener amalgam spaces, we have the following characterization of $\boldsymbol{S}_0(\mathbb{R}^d)$.

Theorem 3.2.6

$$\boldsymbol{S}_0(\mathbb{R}^d) = \boldsymbol{W}(\boldsymbol{A},\boldsymbol{\ell}^1)$$

with equivalence of the corresponding norms. In other words, if $\psi \in \boldsymbol{A}_c(\mathbb{R}^d)$ generates a BUPU, then $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ if and only if $\sum_n \|f \, \mathrm{T}_n \psi\|_{\boldsymbol{A}} < \infty$.

Corollary 3.2.7 *$\boldsymbol{A}_c$ is a dense subspace of $\boldsymbol{S}_0$, or, equivalently, the space of bandlimited integrable functions is dense in $\boldsymbol{S}_0$. Furthermore, $\boldsymbol{\mathcal{D}}(\mathbb{R}^d) = \boldsymbol{C}_c^\infty(\mathbb{R}^d) \hookrightarrow \boldsymbol{S}_0(\mathbb{R}^d)$ and $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \hookrightarrow \boldsymbol{S}_0(\mathbb{R}^d)$ as dense subspaces. On the other hand, we have the dense continuous embeddings $\boldsymbol{S}_0 \hookrightarrow \boldsymbol{W}(\boldsymbol{C},\boldsymbol{\ell}^1)$ and $\boldsymbol{S}_0 \hookrightarrow \boldsymbol{L}^1 \cap \boldsymbol{A}$.*

In [FG85], it has been shown that if $\boldsymbol{A}_c(\mathbb{R}^d)$ is dense in $\boldsymbol{X}(\mathbb{R}^d)$, then $\boldsymbol{W}(\boldsymbol{X},\boldsymbol{\ell}^p)' = \boldsymbol{W}(\boldsymbol{X}',\boldsymbol{\ell}^{p'})$ for $p \in [1,\infty)$. This yields an easy characterization of the dual space of $\boldsymbol{S}_0(\mathbb{R}^d)$.

Corollary 3.2.8 *The space $\boldsymbol{W}(\boldsymbol{A}',\boldsymbol{\ell}^\infty) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty,\boldsymbol{\ell}^\infty)$ is a canonical representation of $\boldsymbol{S}_0'(\mathbb{R}^d)$, the Banach space dual of $\boldsymbol{S}_0(\mathbb{R}^d)$.*

Corollary 3.2.9 *The Poisson Summation Formula*

$$\sum_{n \in \mathbb{Z}^d} f(n) = \sum_{n \in \mathbb{Z}^d} \hat{f}(n) \tag{PSF}$$

holds for all $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, with absolute convergence on both sides.

Proof. Obviously, $\sum_{n\in\mathbb{Z}^d}\delta_n$ is an element of $\boldsymbol{W}(\boldsymbol{A}',\boldsymbol{\ell}^\infty)=\boldsymbol{S}_0'(\mathbb{R}^d)$. Thus, since (PSF) holds on the Schwartz space which is dense in $\boldsymbol{S}_0$, it holds on all of $\boldsymbol{S}_0(\mathbb{R}^d)$. The absolute convergence follows from Lemma 3.2.11 below. ❑

Another standard result concerning Wiener amalgam spaces implies the following corollary, which can be derived directly.

Corollary 3.2.10 *The space $\boldsymbol{W}(\boldsymbol{A},\boldsymbol{\ell}^\infty)$ is a canonical representation of the space of pointwise multipliers on $\boldsymbol{S}_0(\mathbb{R}^d)$, in the sense that $L_h : f\mapsto h\,f$ is in $\boldsymbol{\mathcal{L}}(\boldsymbol{S}_0)$ if and only if $h\in\boldsymbol{W}(\boldsymbol{A},\boldsymbol{\ell}^\infty)$, and then the norms $|||L_h|||_{\boldsymbol{\mathcal{L}}(\boldsymbol{S}_0)}$ and $\|h\|_{\boldsymbol{W}(\boldsymbol{A},\boldsymbol{\ell}^\infty)}$ are equivalent.*

For our purposes, an important consequence of the characterization of $\boldsymbol{S}_0(\mathbb{R}^d)$ as a Wiener amalgam space stems from the following result, since it will turn out to be useful when we discretize the STFT.

Lemma 3.2.11 (Sampling estimates for amalgam spaces)
Let $\Lambda\lhd\mathbb{R}^d\times\widehat{\mathbb{R}}^d$ be a lattice. Then there is a constant C_Λ such that for $p\in[1,\infty)$, we have

$$\|f|_\Lambda\|_{\boldsymbol{\ell}^p(\Lambda)}\le C_\Lambda\,\|f\|_{\boldsymbol{W}(\boldsymbol{C},\boldsymbol{\ell}^p)}\qquad\forall f\in\boldsymbol{W}(\boldsymbol{C},\boldsymbol{\ell}^p).$$

Proof. By definition, $\|f\|^p_{\boldsymbol{W}(\boldsymbol{C},\boldsymbol{\ell}^p)}=\sum_{n\in\mathbb{Z}^d}\|f\ \mathrm{T}_n\psi\|^p_{\boldsymbol{L}^\infty}<\infty$, where we have $\operatorname{supp}\psi\subseteq B_R(0)$ for some $R>0$. Since there is a finite constant $C_\Lambda>0$ such that $\sharp(\Lambda\cap B_R(n))\le C_\Lambda$ for all $n\in\mathbb{Z}^d$, we conclude that

$$\begin{aligned}\sum_{\lambda\in\Lambda}|f(\lambda)|^p&\le\sum_{\lambda\in\Lambda}C_\Lambda^{p-1}\sum_{n\in\mathbb{Z}^d}\left|f(\lambda)\ \mathrm{T}_n\psi(\lambda)\right|^p\\&=C_\Lambda^{p-1}\sum_{n\in\mathbb{Z}^d}\sum_{\lambda\in\Lambda\cap B_R(n)}\left|f(\lambda)\ \mathrm{T}_n\psi(\lambda)\right|^p\\&\le C_\Lambda^p\sum_{n\in\mathbb{Z}^d}\|f\ \mathrm{T}_n\psi\|^p_{\boldsymbol{L}^\infty}=C_\Lambda^p\,\|f\|^p_{\boldsymbol{W}(\boldsymbol{C},\boldsymbol{\ell}^p)}\,.\end{aligned}$$ ❑

3.2.3 Atomic characterizations of $\boldsymbol{S}_0(\mathbb{R}^d)$

From Theorem 3.2.6, we obtain almost immediately an atomic representation for the elements of $\boldsymbol{S}_0(\mathbb{R}^d)$, using time-frequency shifted copies of specific Gabor atoms ψ.

Theorem 3.2.12 *Given $\psi \in \boldsymbol{A}_c(\mathbb{R}^d)$ generating a BUPU, we have for any $b > 0$ sufficiently small that $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ if and only if*

$$f = \sum_{m,n \in \mathbb{Z}^d} a_{n,m} \, \mathrm{M}_{mb} \, \mathrm{T}_n \, \psi \tag{3.2.13}$$

with $\sum_{m,n} |a_{n,m}| < \infty$. We can choose $f \mapsto (a_{n,m})$ to be a bounded linear map $\boldsymbol{S}_0(\mathbb{R}^d) \to \boldsymbol{\ell}^1(\mathbb{Z}^{2d})$, and then $\|f\| = \sum_{m,n} |a_{n,m}|$ is an equivalent norm for $\boldsymbol{S}_0(\mathbb{R}^d)$.

Proof. By choosing $\varphi \in \boldsymbol{A}_c(\mathbb{R}^d)$ with $\varphi \psi = \psi$, we see that

$$f \, \mathrm{T}_n \, \varphi = \Big(\sum_{k=n-n_1}^{n+n_1} f \, \mathrm{T}_k \, \psi \Big) \, \mathrm{T}_n \, \varphi$$

is in $\boldsymbol{A}_c(\mathbb{R}^d)$. Thus we can expand its periodization as an absolutely convergent Fourier series, meaning that for $b > 0$ small enough, there are sequences $(a_{n,m})_{m \in \mathbb{Z}^d} \in \boldsymbol{\ell}^1(\mathbb{Z}^d)$ with $\sum_m |a_{n,m}| \le C_0 \, \|f \; \mathrm{T}_n \, \varphi\|_{\boldsymbol{A}}$ such that

$$f(x) \; \mathrm{T}_n \, \varphi(x) = \sum_{m \in \mathbb{Z}^d} a_{n,m} \, e^{2\pi i m b x} \quad \text{for } x \in \operatorname{supp} \mathrm{T}_n \, \varphi,$$

and the possible choices for b depend on the size of $\operatorname{supp} \varphi$ only (and not on n). Consequently, we have for all $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ that

$$\begin{aligned} f = \sum_{n \in \mathbb{Z}^d} f \; \mathrm{T}_n \, \psi &= \sum_{n \in \mathbb{Z}^d} f \; \mathrm{T}_n \, \varphi \; \mathrm{T}_n \, \psi \\ &= \sum_{n \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d} a_{n,m} \, \mathrm{M}_{mb} \, \mathrm{T}_n \, \psi \end{aligned} \tag{3.2.14}$$

with

$$\sum_{n \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d} |a_{n,m}| \le C_0 \, \|f\|_{S_0} \, . \tag{3.2.15}$$

Vice versa, we obviously have

$$\|f\|_{S_0} \le \sum_{m,n} |a_{n,m}| \, \big\| \mathrm{M}_{mb} \, \mathrm{T}_n \, \psi \big\|_{S_0} \le \|\psi\|_{S_0} \sum_{m,n} |a_{n,m}| \, . \qquad \square$$

Remark. (i) It is worth noting that we have

$$a_{n,m} = \int_{\mathbb{R}^d} f(x) \, \varphi(x-n) \, e^{-2\pi i m b x} \, dx = \langle f, \mathrm{M}_{mb} \, \mathrm{T}_n \, \overline{\varphi} \rangle \, .$$

Thus we have for any such φ that for b sufficiently small, $\overline{\varphi}$ is a dual Gabor window for ψ with respect to $\Lambda = \mathbb{Z}^d \times b\mathbb{Z}^d$.

(ii) This construction essentially coincides with the concept of local Fourier analysis already suggested in 1978 by G. Kaiser (see [Kai94b].)

The above "atomic characterization" of $\boldsymbol{S}_0(\mathbb{R}^d)$ is closely related to the atomic representations of Besov–Triebel–Lizorkin spaces as suggested by Frazier and Jawerth (e.g., see [FJ85], [FJ90]), based on the so-called φ-transform, which turned out to be essentially the continuous wavelet transform (CWT). The difference is that in the present context, dilation is replaced by modulation, i.e., the STFT takes the place of the CWT. This approach to generate $\boldsymbol{S}_0(\mathbb{R}^d)$ from a family of atoms, namely, translates and modulates of a single function g_1, can be generalized considerably. The following characterization of $\boldsymbol{S}_0(\mathbb{R}^d)$ shows very clearly that it is the smallest (non-trivial) Banach space of functions which is isometrically invariant with respect to time-frequency shifts. (Note that this statement is slightly stronger than Theorem 3.2.3, since we omit the homogeneity assumption made there.)

Theorem 3.2.13 ([Fei87], Thm. 1, or [Fei89], Cor. 2)
For any $g_1 \in \boldsymbol{S}_0(\mathbb{R}^d) \setminus \{0\}$, we have that

$$\boldsymbol{S}_0(\mathbb{R}^d) = \left\{ f = \sum_{n=1}^{\infty} a_n \,\mathrm{M}_{\xi_n}\, \mathrm{T}_{x_n}\, g_1 : \xi_n, x_n \in \mathbb{R}^d, \sum_n |a_n| < \infty \right\}, \quad (3.2.16)$$

and the expression $\|f\| := \inf\left\{ \sum_n |a_n| \right\}$, the infimum being taken over all such representations of f, defines an equivalent norm for $\boldsymbol{S}_0(\mathbb{R}^d)$.

This characterization of $\boldsymbol{S}_0(\mathbb{R}^d)$ is useful for a very elementary proof of another invariance property of the space.

Theorem 3.2.14 *Dilation by a non-singular $d \times d$-matrix A is a topological isomorphism of $\boldsymbol{S}_0(\mathbb{R}^d)$, i.e., there exists a constant $C_A > 0$ such that*

$$\|\mathrm{D}_A f\|_{\boldsymbol{S}_0} \leq C_A \|f\|_{\boldsymbol{S}_0}\,. \quad (3.2.17)$$

Proof. Choose an arbitrary $g \in \boldsymbol{A}_c(\mathbb{R}^d) \setminus \{0\}$, then by Theorem 3.2.13, we can represent any $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ as

$$f = \sum_{n=1}^{\infty} a_n \,\mathrm{M}_{\xi_n}\, \mathrm{T}_{x_n}\, g \quad \text{with} \sum_n |a_n| < C_g \|f\|_{\boldsymbol{S}_0}\,.$$

This yields immediately

$$\mathrm{D}_A f = \sum_{n=1}^{\infty} a_n \,\mathrm{M}_{A^T \xi_n}\, \mathrm{T}_{A^{-1} x_n}\, \mathrm{D}_A\, g\,.$$

Since $\operatorname{supp}(\mathrm{D}_A g) = A^{-1}(\operatorname{supp} g)$ is still compact, and we have $\widehat{\mathrm{D}_A g} = |\det A|^{-1} \mathrm{D}_{(A^{-1})^T} \hat{g} \in \boldsymbol{L}^1(\widehat{\mathbb{R}}^d)$ for $\hat{g} \in \boldsymbol{L}^1(\widehat{\mathbb{R}}^d)$, we see that $\mathrm{D}_A g \in \boldsymbol{A}_c(\mathbb{R}^d)$ as well. Thus we obtain

$$\| \mathrm{D}_A f \|_{\boldsymbol{S}_0} \le \Big(\sum_n |a_n| \Big) \, \| \mathrm{D}_A g \|_{\boldsymbol{S}_0} \le \widetilde{C}_{g,A} \| f \|_{\boldsymbol{S}_0} . \qquad \square$$

Remark. For fixed A, the map D_A is a topological isomorphism of $\boldsymbol{S}_0(\mathbb{R}^d)$, but in general, it is not an isometry. Moreover, the family of such dilations is not uniformly bounded.

Next we shall use the atomic decomposition (3.2.16) using the atom g_0 to derive the following result, which will permit another step towards discretization.

Lemma 3.2.15 *For $f, g \in \boldsymbol{S}_0(\mathbb{R}^d)$ we have $\mathcal{V}_g f \in \boldsymbol{W}(\boldsymbol{C}, \boldsymbol{\ell}^1)$, and there exists $C > 0$ such that*

$$\|\mathcal{V}_g f\|_{\boldsymbol{W}(\boldsymbol{C},\boldsymbol{\ell}^1)} \le C \, \|g\|_{\boldsymbol{S}_0} \, \|f\|_{\boldsymbol{S}_0} \quad \forall f, g \in \boldsymbol{S}_0(\mathbb{R}^d) .$$

Proof. By a straightforward calculation we obtain for the Gauss-function

$$\mathcal{V}_{g_0} g_0(x, \xi) = \frac{1}{\sqrt{2}} e^{-\pi i x \xi} \, e^{-\pi (x^2 + \xi^2)/2} , \tag{3.2.18}$$

which implies that $\mathcal{V}_{g_0} g_0 \in \boldsymbol{W}(\boldsymbol{C}, \boldsymbol{\ell}^1)$. It is also easy to show that

$$\mathcal{V}_{\mathrm{M}_\zeta \mathrm{T}_z g}(\mathrm{M}_\eta \mathrm{T}_y f) = e^{2\pi i y(\eta - \zeta)} \, \mathrm{M}_{(\zeta, -y)} \, \mathrm{T}_{(y-z, \eta-\zeta)} \, \mathcal{V}_g f \tag{3.2.19}$$

for any $(y, \eta), (z, \zeta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Consequently,

$$\big\| \mathcal{V}_{\mathrm{M}_\zeta \mathrm{T}_z g_0}(\mathrm{M}_\eta \mathrm{T}_y g_0) \big\|_{\boldsymbol{W}(\boldsymbol{C},\boldsymbol{\ell}^1)} = \|\mathcal{V}_{g_0} g_0\|_{\boldsymbol{W}(\boldsymbol{C},\boldsymbol{\ell}^1)} = C_1 < \infty .$$

By Theorem 3.2.13, there is a constant C_2 such that we can write any $f, g \in \boldsymbol{S}_0(\mathbb{R}^d)$ as

$$f = \sum_{n \in \mathbb{N}} a_n \, \mathrm{M}_{\xi_n} \, \mathrm{T}_{x_n} \, g_0 \quad \text{with} \sum_{n \in \mathbb{N}} |a_n| \le C_2 \, \|f\|_{\boldsymbol{S}_0}$$

$$\text{and} \quad g = \sum_{m \in \mathbb{N}} b_m \, \mathrm{M}_{\eta_m} \, \mathrm{T}_{y_m} \, g_0 \quad \text{with} \sum_{m \in \mathbb{N}} |b_m| \le C_2 \, \|g\|_{\boldsymbol{S}_0} .$$

Thus the sesquilinearity of the STFT yields

$$\|\mathcal{V}_g f\|_{\boldsymbol{W}(\boldsymbol{C},\boldsymbol{\ell}^1)} \le \sum_{m,n \in \mathbb{N}} |a_n| \, |b_m| \, C_1 \le C_1 \, C_2^2 \, \|f\|_{\boldsymbol{S}_0} \, \|g\|_{\boldsymbol{S}_0} . \qquad \square$$

To decide whether a given function f is an element of $\boldsymbol{S}_0(\mathbb{R}^d)$, we do not only have a lot of freedom in the choice of the analyzing window g, as we saw above, but it is also sufficient to know samples of $\mathcal{V}_g f$ on a sufficiently dense lattice.

Theorem 3.2.16 ([FG89a], Thm. 6.1, see also [Fei87])
Let $g \in \boldsymbol{S}_0(\mathbb{R}^d)\backslash\{0\}$. For any sufficiently dense lattice Λ in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, we have for $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ that $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ if and only if $\big(\langle f, \pi(\lambda)g\rangle\big)_{\lambda\in\Lambda} \in \ell^1(\Lambda)$. Furthermore, $\|f\| = \sum_{\lambda\in\Lambda} \big|\langle f, \pi(\lambda)g\rangle\big|$ defines an equivalent norm on $\boldsymbol{S}_0(\mathbb{R}^d)$.

Remark. "Sufficiently dense" in the above theorem does not only mean – as one might assume – sufficiently redundant, but rather dense in the following sense. A discrete subset Λ of the time-frequency plane is ε-*dense*, if it satisfies $\bigcup_{\lambda\in\Lambda} B_\varepsilon(\lambda) = \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Thus we have that for $g \in \boldsymbol{S}_0(\mathbb{R}^d)$, there is $\varepsilon > 0$ such that for any ε-dense lattice, the theorem applies.

3.2.4 Further properties of $\boldsymbol{S}_0(\mathbb{R}^d)$

Sufficient conditions for $f \in \boldsymbol{S}_0(\mathbb{R}^d)$. Several of the characterizations of $\boldsymbol{S}_0(\mathbb{R}^d)$ given above rely on the knowledge whether a given function g_1 belongs to $\boldsymbol{S}_0(\mathbb{R}^d)$. Also, it is desirable to be able to decide quickly whether a given function is in $\boldsymbol{S}_0(\mathbb{R}^d)$. For example, a common assumption in the literature (e.g., see Chapter 1) is that the Gabor family generated by a function with respect to a given time-frequency lattice be a Bessel family. In this situation we can make use of the fact that this holds for elements of $\boldsymbol{S}_0(\mathbb{R}^d)$ with respect to any lattice.

Therefore, we state a few easily verifiable sufficient conditions for a function to be contained in $\boldsymbol{S}_0(\mathbb{R}^d)$. Part (i) is immediate from Theorem 3.2.6, and (ii) follows by Theorem 3.2.3.ii). For parts (iii)–(v), we make use of the family of polynomial weights

$$w_{s,d}(t) = w_s(t) = (1 + \|t\|^2)^{s/2} \tag{3.2.20}$$

on $\mathbb{R}^d$, where $s \in \mathbb{R}$.

Theorem 3.2.17 *The following are sufficient conditions on a function f on $\mathbb{R}^d$ to be an element of $\boldsymbol{S}_0(\mathbb{R}^d)$.*

i) $f \in \boldsymbol{A}_c(\mathbb{R}^d)$.

ii) $f \in \boldsymbol{L}^1(\mathbb{R}^d)$ and bandlimited, i.e., $\hat{f} \in \boldsymbol{A}_c(\widehat{\mathbb{R}}^d)$.

iii) ([Grö96]) *$f\,w_s \in \boldsymbol{L}^p(\mathbb{R}^d)$ and $\hat{f}\,w_r \in \boldsymbol{L}^q(\mathbb{R}^d)$, for some p,q,r,s with $(\frac{s}{d} - \frac{1}{p'})(\frac{r}{d} - \frac{1}{q'}) > \max\{\frac{1}{pq}, \frac{1}{2p}, \frac{1}{2q}, \frac{1}{4}\}$.*
In particular, letting $p = q = 2$ and $r = s$, this becomes

iv) $f\,w_s,\ \hat{f}\,w_s \in \boldsymbol{L}^2(\mathbb{R}^d)$ for some $s > d$;

for $p = q = \infty$ and $r = s$, we have

v) $f\,w_s,\ \hat{f}\,w_s \in \boldsymbol{L}^\infty(\mathbb{R}^d)$ for some $s > \frac{3}{2}d$.

Conditions (iv) and (v) are satisfied e.g. if
vi) $f \in \boldsymbol{S}(\mathbb{R}^d)$.

Remark. In [Grö96], it has been shown that the conditions stated in (iii) are close to necessary for the validity of Poisson's summation formula (see Corollary 3.2.9) and thus also close to necessary for $f \in \boldsymbol{S}_0(\mathbb{R}^d)$.

Approximations of the identity. When we ask for the existence and characterization of so-called Dirac sequences or summability kernels for $\boldsymbol{S}_0(\mathbb{R}^d)$, i.e., sequences of convolution operators converging to the identity, we have results quite similar to the case of $\boldsymbol{L}^1(\mathbb{R}^d)$ or $\boldsymbol{A}(\mathbb{R}^d)$. We will not consider the most general case, though, but only discuss families of dilations of a single function.

Lemma 3.2.18 *i) For* $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ *with* $\hat{g}(0) = 1$, *we have*

$$\lim_{\rho \to 0} \left\| (\rho^{-d} \mathrm{D}_{1/\rho}\, g) * f - f \right\|_{\boldsymbol{S}_0} = 0 \qquad \forall f \in \boldsymbol{S}_0(\mathbb{R}^d). \tag{3.2.21}$$

ii) For $h \in \boldsymbol{A}(\mathbb{R}^d)$ *with* $h(0) = 1$, *we have*

$$\lim_{\rho \to 0} \left\| (\mathrm{D}_\rho\, h)\, f - f \right\|_{\boldsymbol{S}_0} = 0 \qquad \textit{for all } f \in \boldsymbol{S}_0(\mathbb{R}^d). \tag{3.2.22}$$

We leave the proof to the reader, mentioning however that (i) and (ii) are of course equivalent under the Fourier transform. Furthermore, in view of the uniform boundedness of the operators, namely,

$$\begin{aligned} &|||\rho^{-d} \mathrm{D}_{1/\rho}\, g * |||_{\mathcal{L}(\boldsymbol{S}_0)} \le \|\rho^{-d} \mathrm{D}_{1/\rho}\, g\|_{\boldsymbol{L}^1} = \|g\|_{\boldsymbol{L}^1} \\ \text{and} \quad &|||\, \mathrm{D}_\rho\, h \cdot |||_{\mathcal{L}(\boldsymbol{S}_0)} \le \|\, \mathrm{D}_\rho\, h\|_{\boldsymbol{A}} = \|h\|_{\boldsymbol{A}}, \end{aligned}$$

it suffices to show (3.2.21) or (3.2.22) for f in a dense subspace of $\boldsymbol{S}_0$ (such as, e.g., $\boldsymbol{A}_c$). The boundedness of these two types of approximate identities allows combining them in the following manner.

Corollary 3.2.19 *Let g and h as in Lemma 3.2.18. Then we have*

$$\begin{aligned} &\lim_{(\rho,\tau)\to(0,0)} \left\| \mathrm{D}_\rho\, g * (\tau^{-d} \mathrm{D}_{1/\tau}\, h\, f) - f \right\|_{\boldsymbol{S}_0} = 0 \\ \textit{and} \quad &\lim_{(\rho,\tau)\to(0,0)} \left\| \rho^{-d} \mathrm{D}_{1/\rho}\, h\, (\mathrm{D}_\tau\, g * f) - f \right\|_{\boldsymbol{S}_0} = 0 \qquad \forall f \in \boldsymbol{S}_0(\mathbb{R}^d). \end{aligned}$$

Note that since these limits hold in $\boldsymbol{S}_0$ in the norm, they also hold for $f \in \boldsymbol{S}_0'(\mathbb{R}^d)$ in the weak*-sense. Using the theory of Wiener amalgam spaces, it is fairly easy to see that $\boldsymbol{S}_0 \cdot (\boldsymbol{S}_0' * \boldsymbol{S}_0) \subseteq \boldsymbol{S}_0$, which implies the following.

Corollary 3.2.20 *S_0 is weak*-dense in S_0'.*

Further characterizations and extensions of S_0. There are several other possibilities to characterize the space $S_0(\mathbb{R}^d)$. As an example, we mention the approach via the so-called *coorbit spaces.* If we consider the Schrödinger representation of the Heisenberg group $\mathbb{H}^d = \mathbb{R}^d \times \widehat{\mathbb{R}}^d \times \mathbb{T}$, then we obtain $S_0(\mathbb{R}^d)$ as the coorbit $\mathcal{C}o(L^1) \subseteq L^2(\mathbb{R}^d)$. For more details, e.g., see [FG88], [FG89a], or [FG92a].

It is also worth noting that $S_0(\mathbb{R}^d)$ can be characterized as the smallest strongly character invariant (i.e., isometrically modulation invariant) space (see [Fei81], [Los80]) in the class of *Segal algebras* introduced by Reiter (e.g., see [Rei68]), and has been used extensively in [Rei93]. This was also the motivation for the choice of the symbol S_0 in [Fei81], the subscript 0 indicating the minimality.

The Banach space $S_0(\mathbb{R}^d)$ has a number of further useful properties, which we only mention here shortly, since they are not explicitly required for the questions we discuss here.

Besides being a Banach space, another strong advantage of S_0 against $\mathcal{S}(\mathbb{R}^d)$ is the existence of an absolutely convergent basis, the so-called *Wilson basis* (e.g., see [FGW92]), which at the same time turns out to be an orthonormal basis for $L^2(\mathbb{R}^d)$ and a weak* ℓ^∞-Riesz basis for $S_0'(\mathbb{R}^d)$. Also, a kernel theorem holds, stating that any element of $\mathcal{L}\big(S_0(\mathbb{R}^d), S_0'(\mathbb{R}^d)\big)$ has a kernel in $S_0'(\mathbb{R}^{2d})$, and that this correspondence is a bijective isometry (cf., Chapter 7.)

The definition $S_0 = W(A, \ell^1)$ extends directly to a general locally compact abelian group G. If G is compact, then $S_0(G) = A(G)$, while on the other hand, $S_0(G) = \ell^1(G)$ on a discrete group. In view of these properties, it may be seen as the proper generalization of Wiener's Algebra $A(\mathbb{T})$, the space of absolutely convergent Fourier series, to the setting of a general locally compact abelian group (cf., [Fei96].)

Another important feature is the tensor product property satisfied by S_0, meaning that $S_0(G_1 \times G_2) = S_0(G_1)\widehat{\otimes} S_0(G_2)$. It is also stable under restrictions in the sense that for a closed subgroup H, the restriction is a bounded linear map from $S_0(G)$ onto $S_0(H)$. Thus the Poisson Summation Formula holds in its widest sense.

Theorem 3.2.3.ii) also extends to any locally compact abelian group G in the sense that the Fourier transform is an isomorphism between $S_0(G)$ and $S_0(\widehat{G})$. Furthermore, S_0 is stable under several variants of the Fourier transform, like the symplectic Fourier transform, partial Fourier transforms, etc. It is worth noting that Lemma 3.2.15 can be strengthened in the sense that for f, $g \in S_0(\mathbb{R}^d)$, we have $\mathcal{V}_g f \in S_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, which together with the restriction property mentioned above yields a somewhat more elegant proof of Theorem 3.3.1.ii) (a).

This richness of mathematical properties makes the space $S_0(\mathbb{R}^d)$ together with its dual Banach space $S_0'(\mathbb{R}^d)$ not only a very useful tool in pure mathematics, but also a natural space for the discussion of applied problems, including, e.g., stationary stochastic processes (cf., Chapter 7, §5 of [FG92a], or [Fei96].)

3.3 Continuity of Gabor operators

For all practical purposes, we want to be able to sample the STFT, i.e., to reconstruct f from the values of $\mathcal{V}_\gamma f$ on a discrete set Λ. To ensure stability in the reconstruction, we first have to choose the analyzing window γ in such a way that the values of $\mathcal{V}_\gamma f$ on Λ are in an appropriate sequence space, and then the synthesizing window in a manner as to guarantee good convergence properties. In the L^2-case, for example, we wish for ℓ^2-coefficients, which motivates the following definition.

Definition. Given a lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, an element $\gamma \in L^2(\mathbb{R}^d)$ is a *Bessel atom*, if $(\pi(\lambda)\gamma)_{\lambda\in\Lambda}$ is a Bessel family, i.e., if the map $f \mapsto (\langle f, \pi(\lambda)\gamma\rangle)_{\lambda\in\Lambda}$ is in $\mathcal{L}\big(L^2(\mathbb{R}^d), \ell^2(\Lambda)\big)$.

It is important to note that while for the continuous STFT, Moyal's formula (3.2.4) tells us that we may choose any $\gamma \in L^2(\mathbb{R}^d)$ as analyzing window, this does not hold for the discrete case. To make matters even worse, the set of Λ-Bessel atoms varies with Λ, i.e., there exist functions in $L^2(\mathbb{R}^d)$ which are Λ-Bessel atoms for certain choices of Λ, but not for others. The space $S_0(\mathbb{R}^d)$ offers two possible answers to this problem. On the one hand, the elements of S_0 are Bessel atoms for any lattice, and on the other hand, if we want to study properties of the discrete STFT for general L^2-windows γ and g, we can restrict $\mathcal{V}_\gamma$ to $S_0(\mathbb{R}^d)$, and then reconstruction is well-defined in the dual space $S_0'(\mathbb{R}^d)$.

3.3.1 Continuity of the coefficient mapping (analysis)

We begin with a discussion of the continuity properties of the coefficient operator. By choosing γ in the appropriate space, we can obtain the following bounds on the operator norm of T_γ.

Theorem 3.3.1 (Continuity of T_γ)
Let Λ be a lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, and consider the linear operator

$$T_\gamma = T_{\gamma,\Lambda} : f \mapsto \big(\langle f, \pi(\lambda)\gamma\rangle\big)_{\lambda\in\Lambda} = \mathcal{V}_g f\big|_\Lambda . \tag{3.3.1}$$

Then there exists $C_\Lambda > 0$ such that the following hold.

i) For $\gamma \in \boldsymbol{S}_0(\mathbb{R}^d)$, *we have*

(a) $T_\gamma \in \boldsymbol{\mathcal{L}}(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{\ell}^1(\Lambda))$ *with* $|||T_\gamma|||_{\boldsymbol{\mathcal{L}}(\boldsymbol{S}_0,\boldsymbol{\ell}^1)} \leq C_\Lambda \, \|\gamma\|_{\boldsymbol{S}_0}$,

(b) $T_\gamma \in \boldsymbol{\mathcal{L}}(\boldsymbol{L}^2(\mathbb{R}^d), \boldsymbol{\ell}^2(\Lambda))$ *with* $|||T_\gamma|||_{\boldsymbol{\mathcal{L}}(\boldsymbol{L}^2,\boldsymbol{\ell}^2)} \leq C_\Lambda \, \|\gamma\|_{\boldsymbol{S}_0}$
(here $\boldsymbol{S}_0$ *may be replaced by* $\boldsymbol{W}(\boldsymbol{L}^\infty, \boldsymbol{\ell}^1)$*),*

(c) $T_\gamma \in \boldsymbol{\mathcal{L}}(\boldsymbol{S}_0'(\mathbb{R}^d), \boldsymbol{\ell}^\infty(\Lambda))$ *with* $|||T_\gamma|||_{\boldsymbol{\mathcal{L}}(\boldsymbol{S}_0',\boldsymbol{\ell}^\infty)} \leq \|\gamma\|_{\boldsymbol{S}_0}$.

ii) For $\gamma \in \boldsymbol{L}^2(\mathbb{R}^d)$, *we have*

(a) $T_\gamma \in \boldsymbol{\mathcal{L}}(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{\ell}^2(\Lambda))$ *with* $|||T_\gamma|||_{\boldsymbol{\mathcal{L}}(\boldsymbol{S}_0,\boldsymbol{\ell}^2)} \leq C_\Lambda \, \|\gamma\|_{\boldsymbol{L}^2}$,

(b) $T_\gamma \in \boldsymbol{\mathcal{L}}(\boldsymbol{L}^2(\mathbb{R}^d), \boldsymbol{c}_0(\Lambda))$ *with* $|||T_\gamma|||_{\boldsymbol{\mathcal{L}}(\boldsymbol{L}^2,\boldsymbol{c}_0)} \leq \|\gamma\|_{\boldsymbol{L}^2}$.

iii) For $\gamma \in \boldsymbol{S}_0'(\mathbb{R}^d)$, *we have*

$$T_\gamma \in \boldsymbol{\mathcal{L}}(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{\ell}^\infty(\Lambda)) \text{ with } |||T_\gamma|||_{\boldsymbol{\mathcal{L}}(\boldsymbol{S}_0,\boldsymbol{\ell}^\infty)} \leq \|\gamma\|_{\boldsymbol{S}_0'}.$$

Remark. a) It is worth noting that since Λ is discrete, we have $\boldsymbol{\ell}^1(\Lambda) = \boldsymbol{S}_0(\Lambda)$ and thus also $\boldsymbol{\ell}^\infty(\Lambda) = \boldsymbol{S}_0'(\Lambda)$, which makes the relations above even more aesthetically pleasing.

b) Case i) (b) implies that any $g \in \boldsymbol{W}(\boldsymbol{L}^\infty, \boldsymbol{\ell}^1)$ and thus in particular any $g \in \boldsymbol{S}_0$ is a Λ-Bessel atom for any lattice Λ.

c) Obviously, the operator in i) (c) is also weak*-continuous, i.e., weak*-convergent sequences in $\boldsymbol{S}_0'$ are mapped to weak*-convergent sequences in $\boldsymbol{\ell}^\infty$ (which simply means pointwise convergence.)

d) Note that while we can choose γ such that $T_\gamma \in \boldsymbol{\mathcal{L}}(\boldsymbol{L}^2, \boldsymbol{\ell}^2)$, there is no (nontrivial) possibility to have $T_\gamma \in \boldsymbol{\mathcal{L}}(\boldsymbol{L}^1, \boldsymbol{\ell}^1)$. Thus $\boldsymbol{S}_0$ seems to be the appropriate dense subspace of $\boldsymbol{L}^1$ to be considered instead.

Proof. i) (a) is a consequence of Lemma 3.2.15 and the restriction property in Lemma 3.2.11 for $p = 1$.

To show i) (b), we again make use of the Fourier–Wigner transformation (3.2.2) and some of its properties stated in [Fol89]. Folland introduces a twisted convolution $F_1 \natural F_2$ on $\boldsymbol{L}^2(\mathbb{R}^{2d})$ and shows that

$$\left|V(f_1,g_1) \natural V(f_2,g_2)\right| = |\langle f_1, g_2\rangle| \left|V(f_2,g_1)\right|, \tag{3.3.2}$$

$$\text{and} \quad |F_1 \natural F_2| \leq |F_1| * |F_2| \qquad \text{for } F_1,\, F_2 \in \boldsymbol{L}^2(\mathbb{R}^{2d}). \tag{3.3.3}$$

For details, see [Fol89], Prop. 1.47 and Prop. 1.33, respectively. Furthermore, we obviously have $|V(f,g)| = |V(g,f)|$. Using these properties, we obtain for $\gamma \in \boldsymbol{S}_0$ and $f \in \boldsymbol{L}^2$

$$
\begin{aligned}
\left|V(\gamma, f)\right| &= \|g_0\|_{\boldsymbol{L}^2}^{-2} \left|\langle g_0, g_0\rangle\right| \left|V(\gamma, f)\right| \\
&\underset{(3.3.2)}{=} \|g_0\|_{\boldsymbol{L}^2}^{-2} \left|V(g_0, f) \natural V(\gamma, g_0)\right| \\
&\underset{(3.3.3)}{\leq} \|g_0\|_{\boldsymbol{L}^2}^{-2} \left|V(g_0, f)\right| * \left|V(g_0, \gamma)\right| \\
&\in \boldsymbol{L}^2 * \boldsymbol{W}(\boldsymbol{L}^\infty, \boldsymbol{\ell}^1) \quad \text{(by (3.2.4) and Lemma 3.2.15)} \\
&\subseteq \boldsymbol{W}(\boldsymbol{L}^1, \boldsymbol{\ell}^2) * \boldsymbol{W}(\boldsymbol{L}^\infty, \boldsymbol{\ell}^1) \\
&\subseteq \boldsymbol{W}(\boldsymbol{L}^\infty, \boldsymbol{\ell}^2) \quad \text{(see [Fei80])}
\end{aligned}
$$

with all inclusions being bounded. By Lemma 3.2.1.i), we know that $\mathcal{V}_\gamma f$ is continuous and thus even in $\boldsymbol{W}(\boldsymbol{C}, \boldsymbol{\ell}^2)$.

By switching the roles of γ and f, we obtain ii) (a).

To show the remaining cases, consider a Banach space $\boldsymbol{X}(\mathbb{R}^d)$ with dual $\boldsymbol{X}'(\mathbb{R}^d)$, and assume that $\pi(\lambda)$ is an isometry of $\boldsymbol{X}$ (and thus of $\boldsymbol{X}'$) for all λ. Then we have for $f \in \boldsymbol{X}$ and $\gamma \in \boldsymbol{X}'$ that for all $\lambda \in \Lambda$,

$$
\left|(T_\gamma f)(\lambda)\right| = \left|\langle f, \pi(\lambda)\gamma\rangle\right| \leq \|f\|_{\boldsymbol{X}} \, \|\pi(\lambda)\gamma\|_{\boldsymbol{X}'} = \|f\|_{\boldsymbol{X}} \, \|\gamma\|_{\boldsymbol{X}'} \, .
$$

Therefore, $|||T_\gamma|||_{\mathcal{L}(\boldsymbol{X}, \boldsymbol{\ell}^\infty)} \leq \|\gamma\|_{\boldsymbol{X}'}$. Letting $\boldsymbol{X} = \boldsymbol{S}_0$ yields iii). If we choose $\boldsymbol{X} = \boldsymbol{S}_0'$, it suffices to note that $\boldsymbol{S}_0 \subseteq \boldsymbol{S}_0''$ isometrically to obtain i) (c). And since $\boldsymbol{L}^2$ is reflexive, the operator norm bound in ii) (b) follows for $\boldsymbol{X} = \boldsymbol{L}^2$ (recall that $\|\cdot\|_{\boldsymbol{c}_0} = \|\cdot\|_{\boldsymbol{\ell}^\infty}$.) That the image is actually in $\boldsymbol{c}_0(\Lambda)$ in this case is a consequence of Lemma 3.2.1. ❑

Remark. a) C_Λ depends only on the (geometric) density of the lattice Λ in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. In particular, a fixed constant C can be used instead of C_Λ for all subgroups of the form $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$ with lattice constants $a_i \geq a_0 > 0$ and $b_i \geq b_0 > 0$, $i = 1 \ldots d$.

b) It is not difficult to construct examples which show that in general, for $\gamma \in \boldsymbol{L}^2(\mathbb{R}^d)$, one can not expect T_γ to map $\boldsymbol{L}^2(\mathbb{R}^d)$ boundedly into $\boldsymbol{\ell}^2(\Lambda)$. E.g., in the case $d = 1$, the choice $\gamma(x) = (1+|x|)^{-s}$ for some $s > 1/2$ and $f = \gamma$ reveals that even for $\Lambda = \mathbb{Z} \times b\mathbb{Z}$, where $b > 0$ is arbitrary, we obtain coefficients which are not in $\boldsymbol{\ell}^2(\Lambda)$. We even obtain from this family of examples that for any $p < \infty$, there exist f and γ in $\boldsymbol{L}^2(\mathbb{R})$ such that $T_\gamma f \notin \boldsymbol{\ell}^p(\Lambda)$ by choosing s appropriately. Namely, it suffices to consider the values on $\mathbb{Z} \times \{0\} \subseteq \Lambda$.

Thus it becomes clear that we need to impose certain restrictions on γ to ensure that T_γ is a bounded linear map from $\boldsymbol{L}^2(\mathbb{R}^d)$ to $\boldsymbol{\ell}^2(\Lambda)$. Assumptions of this form are frequently made in the literature, mostly without giving practicable sufficient conditions on the analyzing window to ensure them, though (e.g., see [DLL95], [Jan95b].) An explicit condition for the one-dimensional case can be found in [DLL95], Prop. 2.1. There it is assumed that $g(x) \le C\,(1+|x|)^{-1-\varepsilon}$. Since this condition implies that $g \in \boldsymbol{W}(\boldsymbol{L}^\infty, \boldsymbol{\ell}^1)$, this result is a special case of i) (b). For the multivariate case, more refined conditions and also lattice dependent ones are given in [RS97]. That the boundedness of the coefficient mapping may depend extremely on the chosen lattice is discussed in [FJ97] in much detail.

3.3.2 Continuity of the synthesis operator

The results on the analysis operator T_γ can easily be transferred to the analogous question for the synthesis operator, simply by considering the properties of the adjoint operator.

Corollary 3.3.2 (Continuity of T_g^*)
Let Λ be a lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, and consider the linear operator

$$T_g^* = T_{g,\Lambda}^* : (c_\lambda)_{\lambda\in\Lambda} \mapsto \sum_{\lambda\in\Lambda} c_\lambda\, \pi(\lambda) g\,. \tag{3.3.4}$$

Then the following hold.

i) For $g \in \boldsymbol{S}_0(\mathbb{R}^d)$, we have

(a) $T_g^* \in \mathcal{L}(\boldsymbol{\ell}^1(\Lambda), \boldsymbol{S}_0(\mathbb{R}^d))$ with $\|T_g^*\|_{\mathcal{L}(\boldsymbol{\ell}^1,\boldsymbol{S}_0)} \le \|g\|_{\boldsymbol{S}_0}$ and absolute convergence (of the series on the right hand side),

(b) $T_g^* \in \mathcal{L}(\boldsymbol{\ell}^2(\Lambda), \boldsymbol{L}^2(\mathbb{R}^d))$ with $\|T_g^*\|_{\mathcal{L}(\boldsymbol{\ell}^2,\boldsymbol{L}^2)} \le C_\Lambda\, \|g\|_{\boldsymbol{S}_0}$ and unconditional norm-convergence (here $\boldsymbol{S}_0$ may be replaced by $\boldsymbol{W}(\boldsymbol{L}^\infty, \boldsymbol{\ell}^1)$),

(c) $T_g^* \in \mathcal{L}(\boldsymbol{\ell}^\infty(\Lambda), \boldsymbol{S}_0'(\mathbb{R}^d))$ with $\|T_g^*\|_{\mathcal{L}(\boldsymbol{\ell}^\infty,\boldsymbol{S}_0')} \le C_\Lambda\, \|g\|_{\boldsymbol{S}_0}$ and unconditional weak-convergence (on $\boldsymbol{c}_0(\Lambda)$, we even have unconditional norm-convergence.)*

ii) For $g \in \boldsymbol{L}^2(\mathbb{R}^d)$, we have

(a) $T_g^* \in \mathcal{L}(\boldsymbol{\ell}^1(\Lambda), \boldsymbol{L}^2(\mathbb{R}^d))$ with $\|T_g^*\|_{\mathcal{L}(\boldsymbol{\ell}^1,\boldsymbol{L}^2)} \le \|g\|_{\boldsymbol{L}^2}$ and absolute convergence,

(b) $T_g^* \in \mathcal{L}(\boldsymbol{\ell}^2(\Lambda), \boldsymbol{S}_0'(\mathbb{R}^d))$ with $\|T_g^*\|_{\mathcal{L}(\boldsymbol{\ell}^2,\boldsymbol{S}_0')} \le C_\Lambda\, \|g\|_{\boldsymbol{L}^2}$ and unconditional norm-convergence.

iii) For $g \in \boldsymbol{S}_0'(\mathbb{R}^d)$, we have

$T_g^* \in \mathcal{L}(\boldsymbol{\ell}^1(\Lambda), \boldsymbol{S}_0'(\mathbb{R}^d))$ with $\|T_g^*\|_{\mathcal{L}(\boldsymbol{\ell}^1,\boldsymbol{S}_0')} \le \|g\|_{\boldsymbol{S}_0'}$ and absolute convergence.

Proof. The statements i) (a), ii) (a), and iii) simply follow from the convergence of absolutely convergent series in Banach spaces and the isometric time-frequency-invariance of $\boldsymbol{S}_0$, $\boldsymbol{L}^2$, and $\boldsymbol{S}_0'$, respectively.

i) (b), i) (c), and ii) (b) follow from Theorem 3.3.1, parts i) (b), i) (a), and ii) (a), respectively, by duality. The unconditional norm-convergence in these cases is a consequence of the fact that the spaces $\boldsymbol{\ell}^2(\Lambda)$, $\boldsymbol{\ell}^\infty(\Lambda)$, and $\boldsymbol{c}_0(\Lambda)$ are solid (i.e., $|c_\lambda| \leq |d_\lambda|$ for all λ implies $\|c\| \leq \|d\|$), and that T_g^* is continuous. ❑

3.3.3 Continuity of $S_{g,\gamma}$ and Gabor multipliers

We can now easily combine Theorem 3.3.1 and Corollary 3.3.2 to obtain the following properties of the operator $S_{g,\gamma} = T_g^* \circ T_\gamma$.

Corollary 3.3.3 (Continuity of $S_{g,\gamma}$)
Let Λ be a lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, and consider the linear operator

$$S_{g,\gamma} = S_{g,\gamma,\Lambda} : f \mapsto \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma \rangle \, \pi(\lambda) g \, .$$

(In case $g = \gamma$, we will simply write $S_g = S_{g,\Lambda}$ for $S_{g,\gamma}$.) Then the following hold.

i) For $\gamma, g \in \boldsymbol{S}_0(\mathbb{R}^d)$, we have

(a) $S_{g,\gamma} \in \boldsymbol{\mathcal{L}}\big(\boldsymbol{S}_0(\mathbb{R}^d)\big)$ with $|||S_{g,\gamma}|||_{\boldsymbol{\mathcal{L}}(\boldsymbol{S}_0)} \leq C_\Lambda \, \|g\|_{\boldsymbol{S}_0} \, \|\gamma\|_{\boldsymbol{S}_0}$ and absolute convergence (of the series on the right hand side),

(b) $S_{g,\gamma} \in \boldsymbol{\mathcal{L}}\big(\boldsymbol{L}^2(\mathbb{R}^d)\big)$ with $|||S_{g,\gamma}|||_{\boldsymbol{\mathcal{L}}(\boldsymbol{L}^2)} \leq C_\Lambda \, \|g\|_{\boldsymbol{S}_0} \, \|\gamma\|_{\boldsymbol{S}_0}$ and unconditional norm-convergence (here $\boldsymbol{S}_0$ may be replaced by $\boldsymbol{W}(\boldsymbol{L}^\infty, \boldsymbol{\ell}^1)$),

(c) $S_{g,\gamma} \in \boldsymbol{\mathcal{L}}\big(\boldsymbol{S}_0'(\mathbb{R}^d)\big)$ with $|||S_{g,\gamma}|||_{\boldsymbol{\mathcal{L}}(\boldsymbol{S}_0')} \leq C_\Lambda \, \|g\|_{\boldsymbol{S}_0} \, \|\gamma\|_{\boldsymbol{S}_0}$ and unconditional weak-convergence.*

ii) For $\gamma \in \boldsymbol{S}_0(\mathbb{R}^d)$ and $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ (or vice versa), we have

(a) $S_{g,\gamma} \in \boldsymbol{\mathcal{L}}\big(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{L}^2(\mathbb{R}^d)\big)$
with $|||S_{g,\gamma}|||_{\boldsymbol{\mathcal{L}}(\boldsymbol{S}_0, \boldsymbol{L}^2)} \leq C_\Lambda \, \|g\|_{\boldsymbol{L}^2} \, \|\gamma\|_{\boldsymbol{S}_0}$
and absolute convergence
(or $|||S_{g,\gamma}|||_{\boldsymbol{\mathcal{L}}(\boldsymbol{S}_0, \boldsymbol{L}^2)} \leq C_\Lambda \, \|g\|_{\boldsymbol{S}_0} \, \|\gamma\|_{\boldsymbol{L}^2}$
and unconditional norm-convergence, resp.),

(b) $S_{g,\gamma} \in \boldsymbol{\mathcal{L}}\big(\boldsymbol{L}^2(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d)\big)$
with $|||S_{g,\gamma}|||_{\boldsymbol{\mathcal{L}}(\boldsymbol{L}^2, \boldsymbol{S}_0')} \leq C_\Lambda \, \|g\|_{\boldsymbol{L}^2} \, \|\gamma\|_{\boldsymbol{S}_0}$

(or $\|\!|S_{g,\gamma}\|\!|_{\mathcal{L}(\boldsymbol{L}^2,\boldsymbol{S}_0')} \leq C_\Lambda \|g\|_{\boldsymbol{S}_0} \|\gamma\|_{\boldsymbol{L}^2}$*, resp.)*
and unconditional norm-convergence
(in both cases!)

iii) For $\gamma \in \boldsymbol{S}_0(\mathbb{R}^d)$ *and* $g \in \boldsymbol{S}_0'(\mathbb{R}^d)$
(or γ *and* $g \in \boldsymbol{L}^2(\mathbb{R}^d)$*,*
or $\gamma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ *and* $g \in \boldsymbol{S}_0(\mathbb{R}^d)$*), we have*

$$S_{g,\gamma} \in \mathcal{L}\big(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d)\big)$$

with $\|\!|S_{g,\gamma}\|\!|_{\mathcal{L}(\boldsymbol{S}_0,\boldsymbol{S}_0')} \leq C_\Lambda \|g\|_{\boldsymbol{S}_0'} \|\gamma\|_{\boldsymbol{S}_0}$
and absolute convergence
(or $\|\!|S_{g,\gamma}\|\!|_{\mathcal{L}(\boldsymbol{S}_0,\boldsymbol{S}_0')} \leq C_\Lambda \|g\|_{\boldsymbol{L}^2} \|\gamma\|_{\boldsymbol{L}^2}$
and unconditional norm-convergence,
or $\|\!|S_{g,\gamma}\|\!|_{\mathcal{L}(\boldsymbol{S}_0,\boldsymbol{S}_0')} \leq C_\Lambda \|g\|_{\boldsymbol{S}_0} \|\gamma\|_{\boldsymbol{S}_0'}$
and unconditional weak-convergence, resp.)*

Corollary 3.3.4 *For a lattice* Λ *in* $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$*, the mapping* $(g,\gamma) \mapsto S_{g,\gamma}$ *is continuous from* $\boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0(\mathbb{R}^d)$ *to* $\mathcal{L}\big(\boldsymbol{S}_0(\mathbb{R}^d)\big)$*, to* $\mathcal{L}\big(\boldsymbol{L}^2(\mathbb{R}^d)\big)$*, and to* $\mathcal{L}\big(\boldsymbol{S}_0'(\mathbb{R}^d)\big)$*. Thus we also have that the mapping* $g \mapsto S_g$ *is continuous from* $\boldsymbol{S}_0(\mathbb{R}^d)$ *to* $\mathcal{L}\big(\boldsymbol{S}_0(\mathbb{R}^d)\big)$*, to* $\mathcal{L}\big(\boldsymbol{L}^2(\mathbb{R}^d)\big)$*, and to* $\mathcal{L}\big(\boldsymbol{S}_0'(\mathbb{R}^d)\big)$*.*

We stated Corollary 3.3.3 in such a variety of cases to give an overview of the different spaces involved in the description of $S_{g,\gamma}$. In particular, note that case iii) shows that for a pair (g,γ) of general $\boldsymbol{L}^2$-functions, where we can not expect $S_{g,\gamma}$ to be in $\mathcal{L}(\boldsymbol{L}^2)$, we still can study its properties as an element of $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$.

A fairly general sufficient condition for $S_{g,\gamma}$ to be in $\mathcal{L}(\boldsymbol{L}^2)$ is given in i) (b). Indeed, it has been shown in [Wal92] that the assumption $g \in \boldsymbol{W}(\boldsymbol{L}^\infty, \boldsymbol{\ell}^1)$ implies that $S_{g,g} \in \mathcal{L}(\boldsymbol{L}^p)$ for $p \in [1,\infty]$.

Note, however, that for $p \neq 2$, such a result is *not* based on a factorization of $S_{g,\gamma}$ through some sequence space, while a strong advantage of our setup lies in the continuity of the components T_γ and T_g^* of $S_{g,\gamma}$. Consequently, it is possible to modify the coefficients obtained from T_γ before applying T_g^* in our model, while for $p \neq 2$, it turns out that we do not have robustness of the construction under such manipulations.

Typical examples of such operations on the coefficients are (hard or soft) thresholding, resulting in nonlinear operators, or so-called masking, i.e., multiplication of the coefficient sequence by an indicator function. These are frequently used in signal processing. In the following result, we consider linear operators which act on the coefficients by pointwise multiplication with a fixed sequence. From the series of possible statements corresponding to those of Corollary 3.3.3, we only select three special cases of particular interest. The straightforward proof is left to the reader.

Corollary 3.3.5 (Continuity of Gabor multipliers)
Let $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ be a lattice, and $m = (m[\lambda])_{\lambda \in \Lambda}$ a sequence on Λ. Consider the linear operator

$$GM = GM_{g,m,\gamma,\Lambda} : f \mapsto \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma \rangle \, m[\lambda] \, \pi(\lambda) g \, .$$

Then the following hold.

i) For $\gamma, g \in \boldsymbol{S}_0(\mathbb{R}^d)$ and $m \in \boldsymbol{\ell}^\infty(\Lambda)$, we have $GM \in \boldsymbol{\mathcal{L}}(\boldsymbol{L}^2(\mathbb{R}^d))$ with $\|GM_{g,m,\gamma,\Lambda}\|_{\boldsymbol{\mathcal{L}}(\boldsymbol{L}^2)} \leq C_\Lambda \, \|g\|_{\boldsymbol{S}_0} \, \|m\|_{\boldsymbol{\ell}^\infty} \, \|\gamma\|_{\boldsymbol{S}_0}$.

ii) For $\gamma, g \in \boldsymbol{L}^2(\mathbb{R}^d)$ and $m \in \boldsymbol{\ell}^1(\Lambda)$, we have $GM \in \boldsymbol{\mathcal{L}}(\boldsymbol{L}^2(\mathbb{R}^d))$ with $\|GM_{g,m,\gamma,\Lambda}\|_{\boldsymbol{\mathcal{L}}(\boldsymbol{L}^2)} \leq C_\Lambda \, \|g\|_{\boldsymbol{L}^2} \, \|m\|_{\boldsymbol{\ell}^1} \, \|\gamma\|_{\boldsymbol{L}^2}$.

iii) For $\gamma \in \boldsymbol{L}^2(\mathbb{R}^d)$, $g \in \boldsymbol{S}_0(\mathbb{R}^d)$, and $m \in \boldsymbol{\ell}^2(\Lambda)$, we have $GM \in \boldsymbol{\mathcal{L}}(\boldsymbol{S}_0(\mathbb{R}^d))$ with $\|GM_{g,m,\gamma,\Lambda}\|_{\boldsymbol{\mathcal{L}}(\boldsymbol{S}_0)} \leq C_\Lambda \, \|g\|_{\boldsymbol{S}_0} \, \|m\|_{\boldsymbol{\ell}^2} \, \|\gamma\|_{\boldsymbol{L}^2}$.

3.4 Riesz bases and frames for Banach spaces

3.4.1 Bases of Banach spaces

An *unconditional basis* in the topological sense for a Banach space $\boldsymbol{X}$ is a family of vectors $(x_n)_{n \in I}$ with the property that every $x \in \boldsymbol{X}$ can be represented uniquely as

$$x = \sum_{n \in I} a_n(x) \, x_n$$

for complex numbers $a_n(x)$, where the series converges unconditionally, i.e., independent of the ordering of I. The basis is *bounded*, if there are positive constants $C_{1,2}$ such that $C_1 \leq \|x_n\|_{\boldsymbol{X}} \leq C_2$ for all n. We speak of an *absolutely convergent basis*, if the series converges absolutely, i.e., $\sum_n |a_n| \, \|x_n\|_{\boldsymbol{X}} < \infty$, for all $x \in \boldsymbol{X}$.

For each n, the mapping $x \mapsto a_n(x)$ is a linear functional on $\boldsymbol{X}$, and the basis is a *Schauder basis*, if all these functionals are continuous. In this case, we have for each n that $a_n = \langle x, x_n^* \rangle$ for some $x_n^* \in \boldsymbol{X}'$. The family $(x_n^*)_{n \in I}$ has to satisfy $\langle x_m, x_n^* \rangle = \delta_{m,n}$ for all m, $n \in I$ because of the uniqueness of the representation for each x_m. Therefore a Schauder basis is frequently considered to be the biorthogonal system $(x_n, x_n^*)_{n \in I}$ in $\boldsymbol{X} \times \boldsymbol{X}'$ which satisfies

$$x = \sum_{n \in I} \langle x, x_n^* \rangle \, x_n \, .$$

If we define $\boldsymbol{Y}(I)$ to be the (linear) space of all sequences $(a_n)_{n\in I}$ such that $\sum_n a_n\, x_n$ converges unconditionally in $\boldsymbol{X}$, then the *coefficient mapping*

$$T : x \mapsto \big(\langle x, x_n\rangle\big)_{n\in I} \tag{3.4.1}$$

is a linear isomorphism from $\boldsymbol{X}$ onto $\boldsymbol{Y}(I)$ with inverse

$$R : (a_n)_{n\in I} \mapsto \sum_{n\in I} a_n\, x_n\,. \tag{3.4.2}$$

It is well known (e.g., see [Mar69]) that the unconditional convergence of all the series is equivalent to $\boldsymbol{Y}$ being *solid.* This means that for any sequence $a \in \boldsymbol{Y}(I)$, we have that whenever b is a sequence on I with $|b_n| \leq |a_n|$ for all n, then b is an element of $\boldsymbol{Y}$ also. Note that in case $\boldsymbol{Y}$ is a Banach space with the property that pointwise evaluation is continuous, solidity is equivalent to the existence of a constant $C > 0$ such that under the above assumption, we have $\|b\|_{\boldsymbol{Y}} \leq C\,\|a\|_{\boldsymbol{Y}}$.

In case $\boldsymbol{X}$ is a Hilbert space, a basis $(x_n)_{n\in I}$ is usually denoted a *Riesz basis* for $\boldsymbol{X}$, if $\boldsymbol{Y} = \boldsymbol{\ell}^2(I)$ and the mapping T (and thus also R) is a topological isomorphism. We want to extend this concept to other Banach spaces of sequences as follows.

Definition. Consider a Banach space $\boldsymbol{Y}$ of sequences on an index set I. A basis for a Banach space $\boldsymbol{X}$ is a *$\boldsymbol{Y}$-Riesz basis*, if $\boldsymbol{X}$ is topologically isomorphic to $\boldsymbol{Y}$ via T and R.

(For example, an $\boldsymbol{\ell}^2$-Riesz basis is a Riesz basis in the usual sense, and an $\boldsymbol{\ell}^1$-Riesz basis is a bounded absolutely convergent basis.)

Thus a $\boldsymbol{Y}$-Riesz basis for $\boldsymbol{X}$ is equivalent to a bounded linear map $T : \boldsymbol{X} \to \boldsymbol{Y}$ with a bounded inverse $R : \boldsymbol{Y} \to \boldsymbol{X}$, and then both T and R are bounded below also.

3.4.2 Complementability and projections

Before we discuss extensions of the concept of a basis, we first state a few useful lemmata, the proofs of which are left to the reader. Recall that a closed subspace $\boldsymbol{V}$ of a Banach space $\boldsymbol{B}$ is *complementable*, if there is a closed subspace $\boldsymbol{W}$ such that $\boldsymbol{B} = \boldsymbol{V} \oplus \boldsymbol{W}$. Equivalently, there exists a bounded projection in $\boldsymbol{B}$ onto $\boldsymbol{V}$, and the possible complementations and bounded projections are canonically equivalent via $\boldsymbol{W} = \ker P$.

Lemma 3.4.1 *Consider a bounded linear map $T : \boldsymbol{B}_1 \to \boldsymbol{B}_2$ between Banach spaces.*

a) Assume that there exists a bounded linear map $R : \boldsymbol{B}_2 \to \boldsymbol{B}_1$ such that

$$R \circ T f = f \quad \textit{for all } f \textit{ in a dense subspace of } \boldsymbol{B}_1. \tag{3.4.3}$$

Then the following hold.
i) (3.4.3) *holds for all* $f \in \boldsymbol{B}_1$, *i.e.,* R *is a bounded left inverse of* T.
ii) T *is a Banach space isomorphism from* $\boldsymbol{B}_1$ *onto the closed subspace* $T(\boldsymbol{B}_1)$ *in* $\boldsymbol{B}_2$. *In particular, there exist positive constants* $C_1, C_2 > 0$ *such that*

$$C_1^{-1}\|f\|_{\boldsymbol{B}_1} \leq \|Tf\|_{\boldsymbol{B}_2} \leq C_2\|f\|_{\boldsymbol{B}_1}. \tag{3.4.4}$$

iii) R *is a surjective linear map with the property that for each* $f \in \boldsymbol{B}_1$, *there exists* $h \in \boldsymbol{B}_2$ *such that*

$$Rh = f \quad \textit{with} \quad \|h\|_{\boldsymbol{B}_2} \leq C_2\|f\|_{\boldsymbol{B}_1} \tag{3.4.5}$$

(namely, e.g., $h = Tf$.*)*
iv) $P = T \circ R$ *is a bounded projection in* $\boldsymbol{B}_2$ *onto* $T(\boldsymbol{B}_1)$. *In particular,* $T(\boldsymbol{B}_1)$ *is complementable in* $\boldsymbol{B}_2$.

b) Conversely, assume that T *is an isomorphism from* $\boldsymbol{B}_1$ *onto a complementable closed subspace of* $\boldsymbol{B}_2$. *Then it has a bounded left inverse, which can be written as* $R = T^{-1} \circ P$, *where* P *is a bounded projection in* $\boldsymbol{B}_2$ *onto* $T(\boldsymbol{B}_1)$.

In Hilbert spaces, closed subspaces are always complementable. (Actually, a Banach space in which every closed subspace is complementable is topologically isomorphic to a Hilbert space [LT71].) Furthermore, the orthogonal complement along with the orthogonal projection plays a particular role among all possible complementations and their associated bounded projections. Thus we should consider characterizations of orthogonal projections. By definition, a projection is orthogonal, if the induced decomposition is orthogonal, i.e., if $\ker P \perp \operatorname{im} P$.

Lemma 3.4.2 *For a bounded projection* P *in a Hilbert space, the following are equivalent.*

i) P *is orthogonal.*

ii) P *is self-adjoint, i.e.,* $P^* = P$.

iii) $\operatorname{im} P = \operatorname{im} P^*$.

iv) $\ker P = \ker P^*$.

For Hilbert spaces, we can restate Lemma 3.4.1 as follows.

Lemma 3.4.3 *The following properties are equivalent for a bounded linear map* $T : \boldsymbol{H}_1 \to \boldsymbol{H}_2$ *between Hilbert spaces.*

i) T *has a bounded left inverse* R.

ii) T is an isomorphism from $\boldsymbol{H}_1$ onto a closed subspace of $\boldsymbol{H}_2$.

iii) There exist positive constants C_1, $C_2 > 0$ such that

$$C_1^{-1}\|f\|_{\boldsymbol{H}_1} \leq \|Tf\|_{\boldsymbol{H}_2} \leq C_2\|f\|_{\boldsymbol{H}_1}$$

for all f in a dense subset of $\boldsymbol{H}_1$.

Furthermore, the projection $P = T \circ R$ is orthogonal if and only if $\operatorname{im} T = \operatorname{im} R^*$.

Remark. Given T in the situation of Lemma 3.4.3, we can determine the canonical left inverse R making $P = T \circ R$ orthogonal as follows. By (iii), we know that T is injective, hence T^* is surjective and is also bounded, thus $T^* \circ T$ is a topological automorphism of $\boldsymbol{H}_1$. Then the map $R = (T^* \circ T)^{-1} \circ T^*$ has the desired properties, since obviously $R \circ T = \mathrm{Id}$, and $P = T \circ R$ satisfies $P^2 = P$. Furthermore, $\operatorname{im} P = \operatorname{im} T$ and $\operatorname{im} P^* = \operatorname{im} R^* = \operatorname{im} T$, so P is orthogonal by Lemma 3.4.2(iii).

3.4.3 Bases for subspaces versus frames

The two important characteristic properties of a basis are that every element of the space can be represented as a series, and that the representation is unique. These are equivalent to R being surjective and injective, respectively. The inverse is given by T, and the basis is a $\boldsymbol{Y}$-Riesz basis if and only if both R and T are bounded maps between $\boldsymbol{X}$ and $\boldsymbol{Y}$.

Generalizations of this concept can be obtained by giving up one (or several) of the conditions on R. If we give up the surjectivity of R, we obtain the concept of a basis for a subspace $\boldsymbol{X}_0$ of $\boldsymbol{X}$ (*basis in* $\boldsymbol{X}$).

Definition. Let $\boldsymbol{X}$ be a Banach space. A *biorthogonal system in* $(\boldsymbol{X}, \boldsymbol{X}')$ is a family $(x_n, x_n^*)_{n\in I} \subseteq \boldsymbol{X} \times \boldsymbol{X}'$ with the property that $\langle x_{n_1}, x_{n_2}^*\rangle = \delta_{n_1,n_2}$ for all n_1, $n_2 \in I$.

A biorthogonal system is a *projection basis for $\boldsymbol{X}_0$ in $\boldsymbol{X}$*, if it is a basis for a closed subspace $\boldsymbol{X}_0$ of $\boldsymbol{X}$ with the property that

$$P(x) := \sum_{n\in I} \langle x, x_n^*\rangle \, x_n \tag{3.4.6}$$

converges for all $x \in \boldsymbol{X}$.

If a projection basis for $\boldsymbol{X}_0$ in $\boldsymbol{X}$ is a $\boldsymbol{Y}$-Riesz basis for $\boldsymbol{X}_0$, we denote it a *$\boldsymbol{Y}$-Riesz projection basis.*

Lemma 3.4.4 ([Mar69]) *Let $\boldsymbol{X}$ be a Banach space and $(x_n, x_n^*)_{n\in I}$ a projection basis for $\boldsymbol{X}_0$ in $\boldsymbol{X}$. Then the operator P defined in* (3.4.6) *is a bounded projection onto $\boldsymbol{X}_0$.*

In the terminology introduced in 3.4.1, this means that R is injective, but not surjective, and T is a left inverse of R. Thus we have the following commutative diagram.

$$\begin{tikzcd}
\boldsymbol{X} \arrow[d, "P"'] \arrow[dr, "T"] & \\
\boldsymbol{X}_0 \arrow[r, shift left, "T"] & \boldsymbol{Y} \arrow[l, shift left, "R"]
\end{tikzcd}$$

These observations can be expressed as the following result.

Lemma 3.4.5 *A $\boldsymbol{Y}$-Riesz projection basis for $\boldsymbol{X}_0$ in $\boldsymbol{X}$ is equivalent to a bounded linear map $T : \boldsymbol{X} \to \boldsymbol{Y}$ with a bounded right inverse $R : \boldsymbol{Y} \to \boldsymbol{X}_0$, and then R is bounded below also. (T, though, is only bounded below on $\boldsymbol{X}_0$.)*

Note that while for a basis, the dual basis is uniquely determined, this is not the case for a basis for a subspace. In general, given a basis $(x_n)_{n\in I}$ for a closed subspace $\boldsymbol{X}_0 \subseteq \boldsymbol{X}$, there is no unique choice of $(x_n^*)_{n\in I} \subseteq \boldsymbol{X}'$ to obtain a biorthogonal system. Different choices of the $x_n^* \in \boldsymbol{X}'$ will yield different projections. In a Hilbert space, though, there is a canonical dual basis. Namely, by Lemma 3.4.2(iii), the projection is orthogonal if and only if $(x_n^*)_{n\in I} \subseteq \boldsymbol{X}_0$. Under this condition, the x_n^* are unique, since by assumption, the x_n form a basis for $\boldsymbol{X}_0$. It is fairly easy to see that these x_n^* are the ones with minimal norm.

If, on the other hand, we give up the injectivity of R, we obtain the concept of a *Banach frame* for $\boldsymbol{X}$.

Definition. Let $\boldsymbol{X}$ be a Banach space, and $\boldsymbol{Y}$ a Banach space of sequences on some index set I. A family $(x_n^*)_{n\in I} \subseteq \boldsymbol{X}'$ is a *$\boldsymbol{Y}$-frame* for $\boldsymbol{X}$, if the operator $T : x \mapsto \big(\langle x, x_n^*\rangle\big)_{n\in I}$ is bounded and has a bounded left inverse $R : \boldsymbol{Y} \to \boldsymbol{X}$.

Consequently, T is bounded below also. This implies that the norms $\|x\|_{\boldsymbol{X}}$ and $\|T(x)\|_{\boldsymbol{Y}}$ are equivalent, i.e., there are finite, positive constants A and B such that

$$A\,\|x\|_{\boldsymbol{X}} \le \|T(x)\|_{\boldsymbol{Y}} \le B\,\|x\|_{\boldsymbol{X}} \quad \forall x \in \boldsymbol{X}\,. \tag{3.4.7}$$

This property is actually the classical definition for a frame in a Hilbert space (cf., [DS52]). By Lemma 3.4.3, it is equivalent to the existence of a bounded left inverse for T; i.e., for Hilbert spaces, the definition above is equivalent to the classical one. In [Grö91] and in [CH97], frames for

Banach spaces have been defined by postulating (3.4.7) and the existence of a bounded left inverse. But since the latter implies the former, our definition is also equivalent to theirs.

R being a left inverse of T implies that $P = T \circ R$ is a projection in $\boldsymbol{Y}$ onto the range $\boldsymbol{Y}_0$ of T, thus we have the following commutative diagram.

$$\begin{tikzcd}
& \boldsymbol{Y} \arrow[dl, "R"'] \arrow[d, "P"] \\
\boldsymbol{X} \arrow[r, shift right, "T"'] & \boldsymbol{Y}_0 \arrow[l, shift right, "R"']
\end{tikzcd}$$

If we assume that the unit vectors δ_n are elements of $\boldsymbol{Y}$ and that finite sequences are dense in $\boldsymbol{Y}$, we can let $x_n = R(\delta_n)$ to obtain $R(c) = \sum_{n\in I} c_n\, x_n$ for all $c \in \boldsymbol{Y}$. This implies that $x = R \circ T(x) = \sum_{n\in I} \langle x, x_n^* \rangle\, x_n$ for all $x \in \boldsymbol{X}$, and we refer to $(x_n)_{n\in I}$ as a *dual frame* for $(x_n^*)_{n\in I}$. Under these assumptions, we can also consider the frame to be the system $(x_n, x_n^*)_{n\in I} \subseteq \boldsymbol{X} \times \boldsymbol{X}'$. Note, though, that this system is not a biorthogonal one, unless the frame is actually a basis.

In case $\boldsymbol{X}$ is a Hilbert space, we can consider the *frame operator* $T^* \circ T$, which by assumption is a topological automorphism of $\boldsymbol{X}$, to obtain the *canonical dual frame* as $x_n = (T^* \circ T)^{-1} x_n^*$. Then we have $R = (T^* \circ T)^{-1} \circ T^*$, and the projection $P = T \circ R$ becomes orthogonal (compare the remark after Lemma 3.4.3.)

The following simple observations will turn out to be useful (cf., Theorems 3.5.8 and 3.6.4 below.)

Lemma 3.4.6 *Assume that $\boldsymbol{Y}$ is dense in $\overline{\boldsymbol{Y}}$ and that $\boldsymbol{X}$ is dense in $\overline{\boldsymbol{X}}$. Then a $\boldsymbol{Y}$-frame for $\boldsymbol{X}$ with the property that $T \in \boldsymbol{\mathcal{L}}(\overline{\boldsymbol{X}}, \overline{\boldsymbol{Y}})$ and $R \in \boldsymbol{\mathcal{L}}(\overline{\boldsymbol{Y}}, \overline{\boldsymbol{X}})$ is also a $\overline{\boldsymbol{Y}}$-frame for $\overline{\boldsymbol{X}}$. Similarly, a $\boldsymbol{Y}$-Riesz projection basis in $\boldsymbol{X}$ satisfying these assumptions is also a $\overline{\boldsymbol{Y}}$-Riesz projection basis in $\overline{\boldsymbol{X}}$.*

Corollary 3.4.7 *If $\boldsymbol{X}_1 \cap \boldsymbol{X}_2$ is dense in $\boldsymbol{X}_1$ and in $\boldsymbol{X}_2$ and the same holds for $\boldsymbol{Y}_1$ and $\boldsymbol{Y}_2$, then a $\boldsymbol{Y}_1$-frame for $\boldsymbol{X}_1$ is also a $\boldsymbol{Y}_2$-frame for $\boldsymbol{X}_2$, assuming the appropriate boundedness properties for T and R. The same holds for Riesz projection bases.*

Remark. It is worth noting that the concepts of a Banach frame and of a Riesz basis in the general sense introduced above can be seen as *retracts* in the terminology of interpolation theory (e.g., see [BL76a]). The existence of a $\boldsymbol{Y}$-frame for $\boldsymbol{X}$ means that $\boldsymbol{X}$ is a retract of the sequence space $\boldsymbol{Y}$, while a $\boldsymbol{Y}$-Riesz projection basis in $\boldsymbol{X}$ means that the sequence space $\boldsymbol{Y}$ is a retract of $\boldsymbol{X}$.

3.5 Dual pairs and biorthogonal systems

3.5.1 Gabor frames and weak duality

When considering the Gabor expansion of functions in the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$, it seems natural to consider such atoms which induce continuous coefficient mappings into $\boldsymbol{\ell}^2$, i.e., Bessel atoms, only. In contrast to this approach, our results from Section 3.3 allow for a more general definition of a dual pair of Gabor atoms.

Definition. Let Λ be a lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Two elements g and γ of $\boldsymbol{S}_0'(\mathbb{R}^d)$ are a *weakly dual pair* with respect to Λ, if

$$\langle f, h \rangle = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma \rangle \, \langle \pi(\lambda) g, h \rangle \tag{3.5.1}$$

with absolute convergence of the series on the right, for all f, $h \in \boldsymbol{S}_0(\mathbb{R}^d)$.

Remark. a) This definition has been chosen for the following reasons. It is symmetric in g and γ, and the absolute convergence implies unconditional convergence, i.e., the order of summation does not make any difference.

On the other hand, the requirement that the convergence of the series be absolute is not too much of a restriction. It follows from the results in Theorem 3.3.1 that it is guaranteed, e.g., if we have $g, \gamma \in \boldsymbol{L}^2(\mathbb{R}^d)$, or $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ and $\gamma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ (or vice versa.) Furthermore, in these cases, it suffices to check the identity for all f, h in a dense subspace of $\boldsymbol{S}_0(\mathbb{R}^d)$, since the operator mapping the pair (f, h) to the series on the right is continuous on $\boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0(\mathbb{R}^d)$.

b) If we assume that (g, γ) is in $\boldsymbol{L}^2 \times \boldsymbol{L}^2$, in $\boldsymbol{S}_0 \times \boldsymbol{S}_0'$, or in $\boldsymbol{S}_0' \times \boldsymbol{S}_0$, then the operator $S_{g,\gamma}$ is bounded from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0'(\mathbb{R}^d)$, and the pair (g, γ) is weakly Λ-dual if and only if $S_{g,\gamma} = \mathrm{Id}$. In order to verify this it suffices to check the validity of (3.5.1) for f, h in some dense subspace of $\boldsymbol{S}_0(\mathbb{R}^d)$, such as, e.g., $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$.

c) Since $S_{g,\gamma} = \mathrm{Id}$, it is of course extendable as an operator on $\boldsymbol{L}^2(\mathbb{R}^d)$ or $\boldsymbol{S}_0'(\mathbb{R}^d)$, but in general, we can not make any statement about the convergence of the series representation of $S_{g,\gamma}$ on these spaces. The situation is comparable to that of the Fourier transformation on $\boldsymbol{L}^1 \cap \boldsymbol{A}$, where we can represent $\mathcal{F}^{-1} \circ \mathcal{F} = \mathrm{Id}$ by a double integral, and $\mathcal{F}$ extends naturally to a bounded linear operator on $\boldsymbol{L}^2$, but the integral representation does not hold anymore for general $f \in \boldsymbol{L}^2$.

One would expect that the concept of weakly dual pairs is very closely related to that of dual frames, and we want to show that this is in fact so.

Proposition 3.5.1 *The families $(\pi(\lambda)g)_{\lambda \in \Lambda}$ and $(\pi(\lambda)\gamma)_{\lambda \in \Lambda}$ are dual Gabor frames for $\boldsymbol{L}^2(\mathbb{R}^d)$ if and only if (g, γ) is a weakly dual pair of Λ-Bessel* *atoms. Then $S_{g,\gamma,\Lambda} = \mathrm{Id}$ on $\boldsymbol{L}^2$, and* (3.5.1) *holds for all $f,\ h \in \boldsymbol{L}^2(\mathbb{R}^d)$ with absolute convergence of the series.*

Proof. First, assume that $(\pi(\lambda)g)_{\lambda\in\Lambda}$ and $(\pi(\lambda)\gamma)_{\lambda\in\Lambda}$ are dual Gabor frames for $\boldsymbol{L}^2(\mathbb{R}^d)$. Then we have for all $f, h \in \boldsymbol{S}_0(\mathbb{R}^d) \subseteq \boldsymbol{L}^2(\mathbb{R}^d)$ that $(\langle f, \pi(\lambda)\gamma\rangle)_\lambda$ and $(\langle h, \pi(\lambda)g\rangle)_\lambda$ are in $\boldsymbol{\ell}^2(\Lambda)$, so $\sum_\lambda \langle f, \pi(\lambda)\gamma\rangle \, \langle \pi(\lambda)g, h\rangle$ converges absolutely by Cauchy-Schwarz. Furthermore, because of the duality, we know that $f = \sum_\lambda \langle f, \pi(\lambda)\gamma\rangle \, \pi(\lambda)g$ in $\boldsymbol{L}^2$, which implies the desired identity.

On the other hand, if g and γ are Bessel atoms, i.e., T_g, $T_\gamma \in \mathcal{L}(\boldsymbol{L}^2, \boldsymbol{\ell}^2)$, then $T_g^* \in \mathcal{L}(\boldsymbol{\ell}^2, \boldsymbol{L}^2)$ and thus $S_{g,\gamma} = T_g^* \circ T_\gamma \in \mathcal{L}(\boldsymbol{L}^2)$. The weak duality of the pair (g,γ) implies that $S_{g,\gamma} f = f$ for all $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, and consequently, by the continuity of $S_{g,\gamma}$, also for all $f \in \boldsymbol{L}^2(\mathbb{R}^d)$. The claim follows by Lemma 3.5.2 below, which is a simple corollary to Prop. 8.3 in [FG94]. ❑

Lemma 3.5.2 *Let $\boldsymbol{H}$ be a Hilbert space, and assume that $(g_n)_{n\in I}$ and $(\gamma_n)_{n\in I}$ are Bessel families in $\boldsymbol{H}$ such that $\sum_n \langle f, \gamma_n\rangle \, g_n = f$ for all $f \in \boldsymbol{H}$. Then the two families are frames for $\boldsymbol{H}$ which are dual to each other.*

Remark. a) The concept of weak Λ-duality is of particular interest in the case of critical sampling, and there again with Gabor's original approach, where $\Lambda = \mathbb{Z} \times \mathbb{Z}$ and $g = g_0$. This case can be dealt with using the Zak transformation, and Bastiaans [Bas80a] was the first to point out the canonical dual $\widetilde{g_0}$. As was shown by Daubechies and Janssen [DJ93], $\widetilde{g_0}$ is not in $\boldsymbol{L}^2(\mathbb{R})$. It is not even in $\boldsymbol{L}^p(\mathbb{R})$ for any $p < \infty$, but it is an element of $\boldsymbol{L}^\infty(\mathbb{R})$ (according to [Jan82], p. 726), and therefore in $\boldsymbol{S}_0'(\mathbb{R})$. Since the Zak transforms of g_0 and $\widetilde{g_0}$ are a.e. pointwise inverse to each other, the two functions form a weakly dual pair. Thus we can say that Gabor's original expectation turns out to be true in a weak sense.

b) Proposition 3.5.1 above should be compared with Thm. 9.2.3 in Chapter 9. The latter shows that in the discrete setting, one has a very similar situation, namely, that Gabor frames arise as the corresponding uniform filter banks, if the analyzing atom is in $\boldsymbol{\ell}^1(\mathbb{Z})$ and we have perfect reconstruction.

Since on the discrete group $\mathbb{Z}$, we have $\boldsymbol{\ell}^1 = \boldsymbol{S}_0$, and every $\boldsymbol{S}_0$-atom is a Bessel atom, the two results are very similar.

Theorem 3.5.3 (Invariance under transformations)
Assume that (g,γ) is a weakly dual pair with respect to a lattice Λ in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Then the following hold.

i) $(\widehat{g}, \widehat{\gamma})$ is a weakly dual pair with respect to the lattice $J\Lambda$, i.e., the lattice obtained from Λ by multiplication with the matrix $J = \begin{pmatrix} O & I_d \\ -I_d & O \end{pmatrix}$.

ii) For a nonsingular matrix A, the pair $\big(\mathrm{D}_A\, g, |\det A|\, \mathrm{D}_A\,\gamma\big)$ is weakly dual with respect to the lattice $\widetilde{A}\Lambda$, where $\widetilde{A} = \begin{pmatrix} A^{-1} & O \\ O & A^T \end{pmatrix}$.

iii) For $\lambda_0 = (x_0, \xi_0) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, the pair $(\pi(\lambda_0)g, \pi(\lambda_0)\gamma)$ is weakly dual with respect to Λ.

iv) For a lattice $\widetilde{\Lambda}$ in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ containing Λ, the pair $\big(g, \frac{1}{k}\gamma\big)$ is weakly dual with respect to $\widetilde{\Lambda}$, where $k = |\widetilde{\Lambda}/\Lambda|$ (k is necessarily finite since Λ is co-compact.)

Remark. Assertion iv) in the above is essentially just Thm. 9 in [CS93a].

Proof. i) For all $f, h \in \boldsymbol{S}_0(\mathbb{R}^d)$, we have

$$
\begin{aligned}
\langle f, h\rangle = \langle \check{f}, \check{h}\rangle &= \sum_{\lambda\in\Lambda} \langle \check{f}, \pi(\lambda)\gamma\rangle\, \langle \pi(\lambda)g, \check{h}\rangle \\
&= \sum_{\lambda\in\Lambda} \big\langle f, \widehat{\mathrm{M}_\xi\, \mathrm{T}_x\,\gamma}\big\rangle\, \big\langle \widehat{\mathrm{M}_\xi\, \mathrm{T}_x\, g}, h\big\rangle \\
&= \sum_{\lambda\in\Lambda} \big\langle f, e^{2\pi i x\xi}\, \mathrm{M}_{-x}\, \mathrm{T}_\xi\, \widehat{\gamma}\big\rangle\, \big\langle e^{2\pi i x\xi}\, \mathrm{M}_{-x}\, \mathrm{T}_\xi\, \widehat{g}, h\big\rangle \\
&= \sum_{\lambda\in\Lambda} \big\langle f, \pi(J\lambda)\widehat{\gamma}\big\rangle e^{-2\pi i x\xi}\, e^{2\pi i x\xi} \big\langle \pi(J\lambda)\widehat{g}, h\big\rangle \\
&= \sum_{\lambda\in J\Lambda} \langle f, \pi(\lambda)\widehat{\gamma}\rangle\, \langle \pi(\lambda)\widehat{g}, h\rangle\,.
\end{aligned}
$$

(Note that this also implies the required absolute convergence.)

The proofs of the other claims are fairly analogous straightforward calculations and thus left to the reader. ❑

Theorem 3.5.4 *Let $\big((g_n, \gamma_n)\big)_{n\in\mathbb{N}}$ be a sequence of weakly dual pairs with respect to a lattice Λ, and assume that $(g_n, \gamma_n) \to (g, \gamma)$ in $\boldsymbol{L}^2(\mathbb{R}^d) \times \boldsymbol{L}^2(\mathbb{R}^d)$ (or in $\boldsymbol{S}_0 \times \boldsymbol{S}_0'$, or in $\boldsymbol{S}_0' \times \boldsymbol{S}_0$.) Then (g, γ) is a weakly dual pair with respect to Λ also.*

Proof. We have $|\!|\!|T_\gamma - T_{\gamma_n}|\!|\!|_{\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{\ell}^2)} = |\!|\!|T_{\gamma-\gamma_n}|\!|\!|_{\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{\ell}^2)} \le C_\Lambda \|\gamma - \gamma_n\|_{\boldsymbol{L}^2} \to 0$, and thus also $|\!|\!|T_g - T_{g_n}|\!|\!|_{\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{\ell}^2)} \to 0$. Therefore we have for f, $h \in \boldsymbol{S}_0(\mathbb{R}^d)$ that $T_\gamma f$, $T_g h \in \boldsymbol{\ell}^2(\Lambda)$ with $\langle T_\gamma f, T_g h\rangle = \lim_{n\to\infty} \langle T_{\gamma_n} f, T_{g_n} h\rangle = \langle f, h\rangle$.

(The proofs for the other cases are analogous.) ❑

Corollary 3.5.5 *For a function $\gamma \in \boldsymbol{L}^2(\mathbb{R}^d)$ and a (fixed) lattice Λ, the set of all $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ such that (g, γ) is a weakly dual pair with respect to Λ is a closed affine subspace of $\boldsymbol{L}^2(\mathbb{R}^d)$.*

Similarly, for $\gamma \in \boldsymbol{S}_0(\mathbb{R}^d)$ and fixed Λ, the set of weakly dual $g \in \boldsymbol{S}_0'(\mathbb{R}^d)$ is a weak-closed affine subspace.*

Note that in contrast to the concept of weak duality, we can not expect in Theorem 3.5.4 that g and/or γ inherit the Λ-Bessel-atom property from the g_n and/or γ_n, respectively. In other words, there is no analogous statement for sequences of pairs (g_n, γ_n) generating dual Gabor frames. However, we do have invariance of the Bessel property under certain modifications, as the following result shows. The proof follows essentially the same arguments as that of Theorem 3.5.3 and is thus also left to the reader. For the notation, we also refer to Theorem 3.5.3.

Lemma 3.5.6 *Let $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ be a lattice. Then for $g \in \boldsymbol{L}^2(\mathbb{R}^d)$, the following are equivalent.*

i) g is a Λ-Bessel atom.

ii) $\hat{g}$ is a $J\Lambda$-Bessel atom.

iii) $\mathrm{D}_A\, g$ is a $\widetilde{A}\Lambda$-Bessel atom for any nonsingular matrix A.

iv) g is a $\widetilde{\Lambda}$-Bessel atom for any refinement $\widetilde{\Lambda}$ of Λ.

Note that for the case of separable lattices, item (iv) implies that $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ is a Bessel atom for a lattice $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$ for some $0 < a, b \in \mathbb{Q}$ if and only if it is a Bessel atom for any rational lattice. This suggests that the Λ-Bessel atom property might be independent of Λ, but this is in fact not true, as recently has been shown in [FJ97]. Compare also Proposition 3.5.10.

3.5.2 Riesz bases and biorthogonality

Definition. Let Λ° be a lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. A pair (g, γ) in $\boldsymbol{L}^2 \times \boldsymbol{L}^2(\mathbb{R}^d)$ or $\boldsymbol{S}_0 \times \boldsymbol{S}_0'(\mathbb{R}^d)$ satisfies the *Wexler–Raz condition with respect to* Λ°, if

$$T_{\gamma,\Lambda^\circ} g = \mathrm{red}(\Lambda^\circ)\, \delta_0\,, \tag{3.5.2}$$

i.e., $(T_{\gamma,\Lambda^\circ} g)(\lambda^\circ) = \mathrm{red}(\Lambda^\circ)\, \delta_{0,\lambda^\circ}$. Equivalently, the families $(\pi(\lambda^\circ)g)_{\lambda^\circ \in \Lambda^\circ}$ and $(\pi(\lambda^\circ)\gamma)_{\lambda^\circ \in \Lambda^\circ}$ are biorthogonal up to suitable normalization, and thus also $T_{g,\Lambda^\circ}\gamma = \mathrm{red}(\Lambda^\circ)\, \delta_0$.

Lemma 3.5.7 *Let Λ° be a lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, and assume that for some $g \in \boldsymbol{L}^2(\mathbb{R}^d)$, the family $(\pi(\lambda^\circ)g)_{\lambda^\circ \in \Lambda^\circ}$ is a Riesz basis for its closed linear span $\boldsymbol{H}_{g,\Lambda^\circ}$. Then the canonical dual basis is of the form $(\pi(\lambda^\circ)\gamma)_{\lambda^\circ \in \Lambda^\circ}$ for some $\gamma \in \boldsymbol{H}_{g,\Lambda^\circ}$.*

Proof. The canonical dual basis consists of elements γ_{λ° in $\boldsymbol{H}_{g,\Lambda^\circ}$. Letting $\gamma = \gamma_0$, we have $\pi(\lambda^\circ)\gamma \in \boldsymbol{H}_{g,\Lambda^\circ}$ for all $\lambda^\circ \in \Lambda^\circ$, since $\boldsymbol{H}_{g,\Lambda^\circ}$ is invariant under $\pi(\Lambda^\circ)$. By the invariance of the inner product, $(\pi(\lambda^\circ)g, \pi(\lambda^\circ)\gamma)_{\lambda^\circ \in \Lambda^\circ}$

is a biorthogonal system in $\boldsymbol{H}_{g,\Lambda^\circ}$, so by the uniqueness of the canonical dual basis, we necessarily have $\gamma_{\lambda^\circ} = \pi(\lambda^\circ)\gamma$. ❑

Theorem 3.5.8 *Let Λ° be a lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Then for g, $\gamma \in \boldsymbol{L}^2(\mathbb{R}^d)$, the following are equivalent.*

i) $(g, \mathrm{red}\,(\Lambda^\circ)\gamma)$ *is a pair of Λ°-Bessel atoms satisfying the Wexler–Raz condition with respect to Λ°.*

ii) $(\pi(\lambda^\circ)g, \pi(\lambda^\circ)\gamma)_{\lambda^\circ\in\Lambda^\circ}$ *is an $\boldsymbol{\ell}^2$-Riesz projection basis in $\boldsymbol{L}^2(\mathbb{R}^d)$.*

Proof. Assuming (i), g and γ being Λ°-Bessel atoms means that the maps T_{g,Λ° and T_{γ,Λ° are in $\boldsymbol{\mathcal{L}}\big(\boldsymbol{L}^2(\mathbb{R}^d), \boldsymbol{\ell}^2(\Lambda^\circ)\big)$. The Wexler–Raz condition implies that $(\pi(\lambda^\circ)g, \pi(\lambda^\circ)\gamma)_{\lambda^\circ\in\Lambda^\circ}$ is a biorthogonal system, so T^*_{g,Λ° is a bounded right inverse of T_{γ,Λ°. This implies (ii) by Lemma 3.4.5. The converse is trivial. ❑

For an improvement of this result, see Theorem 3.6.4.

3.5.3 Fundamental Identity of Gabor analysis

The continuity properties described in Section 3.3 are particularly useful to obtain an extended range for the validity of the so-called "Fundamental Identity of Gabor Analysis" (cf., Section 1.4.1), which is essentially the Janssen representation of $S_{g,\gamma,\Lambda}$ described as a weak identity, but also referred to as the Wexler–Raz identity in [DLL95], Thm. 3.1. It is furthermore practically equivalent to equation (D2) in [TO95].

Definition. Let Λ be a lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. The *adjoint lattice* is defined to be the set

$$\Lambda^\circ = \{\lambda^\circ \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d : \pi(\lambda)\,\pi(\lambda^\circ) = \pi(\lambda^\circ)\,\pi(\lambda) \quad \forall \lambda\in\Lambda\}\,.$$

Example. For a separable lattice $\Lambda = \Lambda_1 \times \Lambda_2 \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, one finds that $\Lambda^\circ = \Lambda_2^\perp \times \Lambda_1^\perp$. In particular, for $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$, we have $\Lambda^\circ = \frac{1}{b}\mathbb{Z} \times \frac{1}{a}\mathbb{Z}$ (due to our choice of normalization of the modulation operators.) Thus the adjoint lattice Λ° coincides with the dual lattice $\Lambda^\perp$ up to a change of variables for the cases discussed most in the literature (e.g., see [RS97].) In general, we have $\Lambda = A\mathbb{Z}^{2d}$ for some nonsingular matrix A, and then $\Lambda^\circ = \begin{pmatrix} O & I_d \\ -I_d & O \end{pmatrix} \left(A^T\right)^{-1} \mathbb{Z}^{2d}$.

Lemma 3.5.9 (see Lemma 7.7.4) *For a time-frequency lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, the adjoint lattice Λ° is another time-frequency lattice with*

$$\mathrm{red}(\Lambda)\ \mathrm{red}(\Lambda^\circ) = 1\,.$$

The following result shows one aspect of the relation between a lattice and its adjoint lattice (compare the remark following Lemma 3.5.6.)

Proposition 3.5.10 *Let $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ be a lattice. Then $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ is a Λ-Bessel atom if and only if it is a Λ°-Bessel atom.*

Proof. The proof of the proposition in full generality is somewhat technical, hence we omit it for the sake of shortness. The case $d = 1$ with real-valued atoms is stated in Lemma 3.7 in [DLL95]. For rational lattices, which clearly are most interesting for all practical purposes, the statement is a consequence of Lemma 3.5.6.iv). The proof of the general case can be bases on the arguments used in the proof of Thm. 3.1 in [Jan95b]. See also [RS97], where the d-dimensional separable case is covered. ❑

Theorem 3.5.11 *Let Λ be a lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ with adjoint lattice Λ°. Then for (g, γ) in $\boldsymbol{L}^2 \times \boldsymbol{L}^2(\mathbb{R}^d)$ or $\boldsymbol{S}_0 \times \boldsymbol{S}_0'(\mathbb{R}^d)$, the following hold.*

i) (Fundamental Identity of Gabor Analysis)

$$\sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma \rangle \, \langle \pi(\lambda) g, h \rangle = \mathrm{red}(\Lambda) \sum_{\lambda^\circ \in \Lambda^\circ} \langle g, \pi(\lambda^\circ)\gamma \rangle \, \langle \pi(\lambda^\circ) f, h \rangle \qquad (3.5.3)$$

for all $f,\ h \in \boldsymbol{S}_0(\mathbb{R}^d)$, where both sides converge absolutely.

ii) (Wexler–Raz Identity)

$$S_{g,\gamma,\Lambda}\, f = \mathrm{red}(\Lambda)\, S_{f,\gamma,\Lambda^\circ}\, g \quad \textit{in } \boldsymbol{S}_0'(\mathbb{R}^d) \qquad (3.5.4)$$

for all $f \in \boldsymbol{S}_0(\mathbb{R}^d)$.

iii) (Janssen Representation)

$$S_{g,\gamma,\Lambda} = \mathrm{red}(\Lambda) \sum_{\lambda^\circ \in \Lambda^\circ} \mathcal{V}_\gamma g(\lambda^\circ)\, \pi(\lambda^\circ) \quad \textit{in } \boldsymbol{\mathcal{L}}\big(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d)\big), \qquad (3.5.5)$$

where the series converges unconditionally in the strong operator sense.

Remark. a) If we strengthen the assumptions on g and γ, we obtain somewhat stronger results in the sense of better convergence or extended range of validity of the formulas above.

If $(g, \gamma) \in \boldsymbol{S}_0 \times \boldsymbol{L}^2(\mathbb{R}^d)$, then (i) holds for $(f, h) \in \boldsymbol{S}_0 \times \boldsymbol{L}^2$; (ii) holds for $f \in \boldsymbol{S}_0$ in the $\boldsymbol{L}^2$-sense and for $f \in \boldsymbol{L}^2$ in the $\boldsymbol{S}_0'$-sense; and (iii) holds in $\boldsymbol{\mathcal{L}}(\boldsymbol{S}_0, \boldsymbol{L}^2)$ and in $\boldsymbol{\mathcal{L}}(\boldsymbol{L}^2, \boldsymbol{S}_0)$.

If we restrict (g, γ) to $\boldsymbol{S}_0 \times \boldsymbol{S}_0(\mathbb{R}^d)$, then (i) holds for $(f, h) \in \boldsymbol{L}^2 \times \boldsymbol{L}^2$ or $\boldsymbol{S}_0 \times \boldsymbol{S}_0'$; (ii) holds for $f \in \boldsymbol{S}_0$ in the $\boldsymbol{S}_0$-sense, for $f \in \boldsymbol{L}^2$ in the $\boldsymbol{L}^2$-sense, and for $f \in \boldsymbol{S}_0'$ in the $\boldsymbol{S}_0'$-sense; and furthermore (iii) holds in $\boldsymbol{\mathcal{L}}(\boldsymbol{S}_0)$, in $\boldsymbol{\mathcal{L}}(\boldsymbol{L}^2)$, and in $\boldsymbol{\mathcal{L}}(\boldsymbol{S}_0')$.

b) In particular, note that (i) can be read as an identity with four arguments, where any two can be chosen in $\boldsymbol{L}^2$ or any one in $\boldsymbol{S}_0'$, as long as the others are in $\boldsymbol{S}_0$; with absolute convergence on both sides. We have complete symmetry in the arguments since $\Lambda^{\circ\circ} = \Lambda$ and $\mathrm{red}(\Lambda^\circ) = \mathrm{red}(\Lambda)^{-1}$. As shown in Section 1.4.1, formula (3.5.3) is also valid under the assumption that at least one each of f, γ and of g, h is a Bessel atom.

As Janssen has pointed out to the authors in a private communication, it is not sufficient to assume that any two of these four functions are Bessel atoms and the other two general $\boldsymbol{L}^2$-functions. Thus the assumption that any two functions are in $\boldsymbol{S}_0$ yields a somewhat more symmetric result.

Proof. Our strategy will be to start from the assumption that g and γ are in $\boldsymbol{S}_0(\mathbb{R}^d)$, and then to extend the validity by making use of the continuity properties established in Section 3.3 together with the density of $\boldsymbol{S}_0$ in $\boldsymbol{L}^2$, and of the symmetry of the statements, respectively.

In Chapter 7, it is shown that for f, g, $\gamma \in \boldsymbol{S}_0(\mathbb{R}^d)$, we have

$$S_{g,\gamma,\Lambda} f = \mathrm{red}(\Lambda)\, S_{f,\gamma,\Lambda^\circ} g\,,$$

$$\text{i.e.,}\qquad \sum_{\lambda\in\Lambda} T_\gamma f(\lambda)\, \pi(\lambda) g = \mathrm{red}(\Lambda) \sum_{\lambda^\circ\in\Lambda^\circ} T_\gamma g(\lambda^\circ)\, \pi(\lambda^\circ) f \tag{3.5.6}$$

in $\boldsymbol{S}_0(\mathbb{R}^d)$. Taking the inner product in $\boldsymbol{L}^2(\mathbb{R}^d)$ of the above with any $h \in \boldsymbol{S}_0(\mathbb{R}^d)$, we obtain

$$\sum_{\lambda\in\Lambda} T_{\gamma,\Lambda} f(\lambda)\ \overline{T_{g,\Lambda} h(\lambda)} = \mathrm{red}(\Lambda) \sum_{\lambda^\circ\in\Lambda^\circ} T_{\gamma,\Lambda^\circ} g(\lambda)\ \overline{T_{f,\Lambda^\circ} h(\lambda^\circ)}$$

This is just (3.5.3) for g, γ, f, $h \in \boldsymbol{S}_0(\mathbb{R}^d)$. To extend the range of validity of this identity, consider for fixed f, $h \in \boldsymbol{S}_0(\mathbb{R}^d)$ the two mappings

$$\begin{aligned} &(g,\gamma) \mapsto T_{\gamma,\Lambda} f\ \ \overline{T_{g,\Lambda} h} \\ \text{and}\qquad &(g,\gamma) \mapsto \mathrm{red}(\Lambda)\ T_{\gamma,\Lambda^\circ} g\ \ \overline{T_{f,\Lambda^\circ} h}\,. \end{aligned}$$

These two mappings are sesquilinear from $\boldsymbol{S}_0 \times \boldsymbol{S}_0$ into $\boldsymbol{\ell}^1(\Lambda)$ and $\boldsymbol{\ell}^1(\Lambda^\circ)$, respectively; thus it suffices to show that they extend continuously to mappings from $\boldsymbol{L}^2 \times \boldsymbol{L}^2$ as well as from $\boldsymbol{S}_0 \times \boldsymbol{S}_0'$ to $\boldsymbol{\ell}^1$.

For g, $\gamma \in \boldsymbol{L}^2(\mathbb{R}^d)$, we know by Theorem 3.3.1 that we can let $C = C_\Lambda^2\, \|f\|_{\boldsymbol{S}_0}\, \|h\|_{\boldsymbol{S}_0}$ to obtain

$$\begin{aligned} \left\|T_{\gamma,\Lambda} f\ \overline{T_{g,\Lambda} h}\right\|_{\boldsymbol{\ell}^1(\Lambda)} &\le \left\|T_{\gamma,\Lambda} f\right\|_{\boldsymbol{\ell}^2(\Lambda)} \left\|T_{g,\Lambda} h\right\|_{\boldsymbol{\ell}^2(\Lambda)} \\ &\le C\, \|g\|_{\boldsymbol{L}^2(\mathbb{R}^d)}\, \|\gamma\|_{\boldsymbol{L}^2(\mathbb{R}^d)}\,, \\ \text{and}\qquad \left\|T_{\gamma,\Lambda^\circ} g\ \overline{T_{f,\Lambda^\circ} h}\right\|_{\boldsymbol{\ell}^1(\Lambda^\circ)} &\le \left\|T_{\gamma,\Lambda^\circ} g\right\|_{\boldsymbol{\ell}^\infty(\Lambda^\circ)} \left\|T_{f,\Lambda^\circ} h\right\|_{\boldsymbol{\ell}^1(\Lambda^\circ)} \\ &\le C\, \|g\|_{\boldsymbol{L}^2(\mathbb{R}^d)}\, \|\gamma\|_{\boldsymbol{L}^2(\mathbb{R}^d)}\,. \end{aligned}$$

Thus the validity of (3.5.3) follows for g, $\gamma \in \boldsymbol{L}^2(\mathbb{R}^d)$ from the density of $\boldsymbol{S}_0$ in $\boldsymbol{L}^2$.

For the case that g or γ is in $\boldsymbol{S}_0'(\mathbb{R}^d)$, note first that since (3.5.6) holds in $\boldsymbol{S}_0(\mathbb{R}^d)$, we immediately have (3.5.3) for g, γ, $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ and $h \in \boldsymbol{S}_0'(\mathbb{R}^d)$. A straightforward calculation shows that

$$\sum_{\lambda\in\Lambda} T_\gamma f(\lambda)\ \overline{T_g h(\lambda)} = \sum_{\lambda\in\Lambda} \overline{T_f\gamma(\lambda)}\ T_h g(\lambda)\,.$$

But this holds for any lattice, so in particular for Λ°, too, and thus (3.5.3) holds for g, f, $h \in \boldsymbol{S}_0(\mathbb{R}^d)$ and $\gamma \in \boldsymbol{S}_0'(\mathbb{R}^d)$. Taking complex conjugates all over switches the roles of g and γ. ❑

3.5.4 The Wexler–Raz biorthogonality principle

Theorem 3.5.12 (Wexler–Raz principle for time-frequency lattices) *Let Λ be a lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ with adjoint lattice Λ°. Then a pair (g,γ) in $\boldsymbol{L}^2\times\boldsymbol{L}^2(\mathbb{R}^d)$ or $\boldsymbol{S}_0\times\boldsymbol{S}_0'(\mathbb{R}^d)$ is weakly dual with respect to Λ if and only if it satisfies the Wexler–Raz condition with respect to Λ°.*

Proof. Let (g,γ) in $\boldsymbol{L}^2\times\boldsymbol{L}^2(\mathbb{R}^d)$ or $\boldsymbol{S}_0\times\boldsymbol{S}_0'(\mathbb{R}^d)$ be given. Then we know from Corollary 3.3.3 iii) that $S_{g,\gamma,\Lambda} \in \boldsymbol{\mathcal{L}}(\boldsymbol{S}_0, \boldsymbol{S}_0')$. In Chapter 7, it is shown that any such operator has a well-defined spreading function $\eta(S_{g,\gamma,\Lambda}) \in \boldsymbol{S}_0'(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, where η defines an isomorphism between the Banach spaces $\boldsymbol{\mathcal{L}}(\boldsymbol{S}_0, \boldsymbol{S}_0')$ and $\boldsymbol{S}_0'(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, satisfying $\eta(\pi(\lambda)) = \delta_\lambda$. Thus equation (3.5.5) is equivalent to

$$\eta(S_{g,\gamma,\Lambda}) = \mathrm{red}(\Lambda) \sum_{\lambda^\circ\in\Lambda^\circ} \mathcal{V}_\gamma g(\lambda^\circ)\ \delta_{\lambda^\circ}\,, \tag{3.5.7}$$

where the coefficient sequence $(\mathcal{V}_\gamma g(\lambda^\circ))_{\lambda^\circ\in\Lambda^\circ} = T_{\gamma,\Lambda^\circ} g$ is bounded by Theorem 3.3.1. Thus we see that $S_{g,\gamma,\Lambda} = \mathrm{Id}$ if and only if $\eta(S_{g,\gamma,\Lambda}) = \eta(\mathrm{Id}) = \delta_0$, which is equivalent to the Wexler–Raz condition on (g,γ) with respect to Λ°. ❑

Remark. The result above originates in the well-known paper by Wexler and Raz [WR90], who discussed the case of periodic discrete Gabor analysis. The detailed proofs for the continuous setting have been given independently for the case $d = 1$ by Daubechies et al. in [DLL95] and by Janssen in [Jan95b] (see also Section 1.4.2), and furthermore, using different methods, by Ron and Shen [RS97] for separable lattices $\Lambda = \Lambda_1 \times \Lambda_2 \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$.

Note, however, that in all these publications, the equivalence of the duality of Gabor atoms and the Wexler–Raz condition is verified under the additional assumption that the respective atoms are Bessel atoms.

Proposition 3.5.13 *(i) Let Λ be a lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, and $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ a Λ-Bessel atom. Then $(\pi(\lambda)g)_{\lambda\in\Lambda}$ is an $\boldsymbol{\ell}^2$-frame for $\boldsymbol{L}^2(\mathbb{R}^d)$ if and only if there exists some Λ-Bessel atom $\gamma \in \boldsymbol{L}^2(\mathbb{R}^d)$ that is weakly Λ-dual to g.*

Furthermore, the canonical dual frame is the Gabor frame $(\pi(\lambda)\widetilde{g})_{\lambda\in\Lambda}$ generated by $\widetilde{g} = (S_{g,\Lambda})^{-1}g$.

(ii) Let Λ° be a lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, and $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ a Λ°-Bessel atom. Then $(\pi(\lambda^\circ)g)_{\lambda^\circ\in\Lambda^\circ}$ is an $\boldsymbol{\ell}^2$-Riesz basis for its closed linear span in $\boldsymbol{L}^2(\mathbb{R}^d)$ if and only if there exists some other Λ°-Bessel atom $\gamma \in \boldsymbol{L}^2(\mathbb{R}^d)$ such that $(\pi(\lambda^\circ)g, \pi(\lambda^\circ)\gamma)_{\lambda^\circ\in\Lambda^\circ}$ is a biorthogonal system.

Furthermore, the canonical dual basis is a Gabor family of the form $(\pi(\lambda^\circ)\gamma)_{\lambda^\circ\in\Lambda^\circ}$ for some $\gamma \in \boldsymbol{L}^2(\mathbb{R}^d)$.

Proof. i) Assume that $(\pi(\lambda)g)_{\lambda\in\Lambda}$ is an $(\boldsymbol{L}^2, \boldsymbol{\ell}^2)$-frame. The frame operator is given by $S_{g,\Lambda}$, so the canonical dual frame consists of $(S_{g,\Lambda}^{-1}\pi(\lambda)g)_{\lambda\in\Lambda}$. By (3.5.5), $S_{g,\Lambda}$ commutes with $\pi(\lambda)$ for all $\lambda\in\Lambda$, thus the same holds for $S_{g,\Lambda}^{-1}$. So we can let $\widetilde{g} = S_{g,\Lambda}^{-1}g$ to obtain $S_{g,\Lambda}^{-1}\pi(\lambda)g = \pi(\lambda)\widetilde{g}$ as claimed, and thus $\widetilde{g} \in \boldsymbol{L}^2(\mathbb{R}^d)$ is a Bessel atom, weakly Λ-dual to g.

Conversely, if (g, γ) is a pair of weakly Λ-dual Bessel atoms in $\boldsymbol{L}^2(\mathbb{R}^d)$, the map $T_{\gamma,\Lambda}^* : \boldsymbol{\ell}^2 \to \boldsymbol{L}^2$ is a bounded left inverse for $T_{g,\Lambda} : \boldsymbol{L}^2 \to \boldsymbol{\ell}^2$, so $(\pi(\lambda)g)_{\lambda\in\Lambda}$ is an $(\boldsymbol{L}^2, \boldsymbol{\ell}^2)$-frame by definition.

ii) Assume that $(\pi(\lambda^\circ)g)_{\lambda^\circ\in\Lambda^\circ}$ is an $\boldsymbol{\ell}^2$-Riesz basis for a closed subspace $\boldsymbol{X}$ of $\boldsymbol{L}^2$. Let $(\gamma_{\lambda^\circ})_{\lambda^\circ\in\Lambda^\circ}$ be the canonical dual basis in $\boldsymbol{X}$. Denoting $\gamma_{(0,0)} = \gamma$, we must have $\gamma_{\lambda^\circ} = \pi(\lambda^\circ)\gamma$ for all $\lambda^\circ \in \Lambda^\circ$, since the family $(\pi(\lambda^\circ)\gamma)_{\lambda^\circ}$ is in $\boldsymbol{X}$ and is biorthogonal to $(\pi(\lambda^\circ)g)_{\lambda^\circ}$. Thus γ is a Λ°-Bessel atom with the required properties.

Conversely, assume that (g, γ) is a pair of Λ°-Bessel atoms generating a biorthogonal system. Then $T_{\gamma,\Lambda^\circ}^* : \boldsymbol{\ell}^2 \to \boldsymbol{L}^2$ is a bounded right inverse for $T_{g,\Lambda^\circ} : \boldsymbol{L}^2 \to \boldsymbol{\ell}^2$, so $(\pi(\lambda^\circ)g)_{\lambda^\circ\in\Lambda^\circ}$ is an $\boldsymbol{\ell}^2$-Riesz basis in $\boldsymbol{L}^2$ by Lemma 3.4.5. ❑

Corollary 3.5.14 *Let Λ be a lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Then for $g \in \boldsymbol{L}^2(\mathbb{R}^d)$, the following are equivalent.*

i) $(\pi(\lambda)g)_{\lambda\in\Lambda}$ is an $\boldsymbol{\ell}^2$-frame for $\boldsymbol{L}^2(\mathbb{R}^d)$.

ii) $(\pi(\lambda^\circ)g)_{\lambda^\circ\in\Lambda^\circ}$ is an $\boldsymbol{\ell}^2$-Riesz basis for its closed linear span.

Furthermore, we have for $\gamma \in \boldsymbol{L}^2(\mathbb{R}^d)$ that $(\pi(\lambda)\gamma)_{\lambda\in\Lambda}$ is a dual frame for (i) if and only if $\big(\mathrm{red}(\Lambda^\circ)\,\pi(\lambda^\circ)\gamma\big)_{\lambda^\circ\in\Lambda^\circ}$ is a dual basis for (ii). Also, the canonical dual atoms are essentially the same in the sense that if $\widetilde{g} = (S_{g,\Lambda})^{-1}g$, so $(\pi(\lambda)\widetilde{g})_{\lambda\in\Lambda}$ is the canonical dual frame for (i), then $\big(\mathrm{red}(\Lambda^\circ)\,\pi(\lambda^\circ)\widetilde{g}\big)_{\lambda^\circ\in\Lambda^\circ}$ is the canonical dual basis for (ii).

In particular, among all $\gamma \in \boldsymbol{L}^2(\mathbb{R}^d)$ generating a dual Gabor frame for (i), the canonical $\widetilde{g}$ has minimal norm.

Remark. For separable lattices $\Lambda = \Lambda_1 \times \Lambda_2$, the main statements of the above corollary can be found in [RS97], where the authors even consider frames for closed subspaces. (These results have been announced much earlier in [RS93].)

Proof. By Proposition 3.5.13(i), $(\pi(\lambda)g)_{\lambda\in\Lambda}$ is a Gabor frame for $\boldsymbol{L}^2(\mathbb{R}^d)$ if and only if g is a Λ-Bessel atom and is weakly dual to some other Λ-Bessel atom $\gamma \in \boldsymbol{L}^2(\mathbb{R}^d)$. By Proposition 3.5.10 and Theorem 3.5.12, this is equivalent to g and γ being Λ°-Bessel atoms satisfying the Wexler–Raz condition with respect to Λ°. Now we can apply Proposition 3.5.13(ii) to see that this again is equivalent to $(\pi(\lambda^\circ)g)_{\lambda^\circ\in\Lambda^\circ}$ being an $\boldsymbol{\ell}^2$-Riesz basis for its closed linear span.

If $\widetilde{g}$ is the canonical dual Gabor atom for g with respect to Λ, then $(S_{g,\Lambda})^{-1} = S_{\widetilde{g},\Lambda}$. Thus we know from Theorem 3.5.11(ii), that $\widetilde{g} = S_{\widetilde{g},\Lambda}g = \operatorname{red}(\Lambda)\, S_{g,\widetilde{g},\Lambda^\circ}\widetilde{g}$. This implies that $\widetilde{g}$ is in the closed span of $(\pi(\lambda^\circ)g)_{\lambda^\circ\in\Lambda^\circ}$. Since this subspace is invariant under $\pi(\lambda^\circ)$ for all $\lambda^\circ \in \Lambda^\circ$, it contains the family $(\pi(\lambda^\circ)\widetilde{g})_{\lambda^\circ\in\Lambda^\circ}$, so this is the canonical dual basis.

The minimal norm property of the canonical $\widetilde{g}$ among all dual Gabor atoms follows from the norm minimality of the elements of the canonical dual basis. ❑

3.5.5 *Consequences of Wexler–Raz and the Fundamental Identity*

Definition. A pair (g,γ) in $\boldsymbol{L}^2(\mathbb{R}^d)$ satisfies *condition (A'_Λ)*, if one has $\sum_{\lambda^\circ\in\Lambda^\circ} |\mathcal{V}_\gamma g(\lambda^\circ)| < \infty$, i.e., $T_\gamma g \in \boldsymbol{\ell}^1(\Lambda^\circ)$ or equivalently $T_g\gamma \in \boldsymbol{\ell}^1(\Lambda^\circ)$. An element $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ satisfies *condition (A_Λ)*, if (g,g) satisfies condition (A'_Λ).

For the case $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$, the above conditions reduce to conditions (A) and (A') as introduced in [TO95]. Evidently, condition (A'_Λ) is equivalent to the absolute convergence of the Janssen representation (3.5.5) of $S_{g,\gamma,\Lambda}$. Therefore it is of interest to observe that it comes close to the assumption that g (or γ) be in $\boldsymbol{S}_0(\mathbb{R}^d)$ in the following sense.

Lemma 3.5.15 *Let Λ be a lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, and $g \in \boldsymbol{L}^2(\mathbb{R}^d)$. Then $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ if and only if for some nontrivial (and then all) $\gamma \in \boldsymbol{S}_0(\mathbb{R}^d)$, we have that $(g,\pi(\lambda)\gamma)$ satisfies condition (A'_Λ) for all $\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$.*

Proof. Assume that $\gamma \in \boldsymbol{S}_0(\mathbb{R}^d) \setminus \{0\}$ has the desired property, i.e., that $T_{g,\Lambda}(\pi(\lambda)\gamma) \in \boldsymbol{\ell}^1(\Lambda)$ for all $\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Then we know that $T_{g,\widetilde{\Lambda}}\gamma \in \boldsymbol{\ell}^1(\widetilde{\Lambda})$ for all refinements $\widetilde{\Lambda} = \bigcup_{k=1}^n \lambda_k + \Lambda$. The conclusion follows from Theorem 3.2.16. The converse is immediate from Theorem 3.3.1(i)a). ❑

Corollary 3.5.16 *Assume that (g, Λ) generates a Gabor frame, and let $\widetilde{g}$ be the canonical dual Gabor atom. Then the orthogonal projection in $\boldsymbol{L}^2(\mathbb{R}^d)$ onto $\boldsymbol{H}_{g,\Lambda^\circ}$ is given by*

$$P_{g,\Lambda^\circ} f = \operatorname{red}(\Lambda) \sum_{\lambda^\circ \in \Lambda^\circ} \langle f, \pi(\lambda^\circ)\widetilde{g}\rangle \, \pi(\lambda^\circ) g \,.$$

Equivalently, using the Janssen representation,

$$P_{g,\Lambda^\circ} f = \sum_{\lambda \in \Lambda} \mathcal{V}_{\widetilde{g}} g(\lambda) \, \pi(\lambda) f \,,$$

the sum being unconditionally convergent in $\boldsymbol{L}^2$ for all $f \in \boldsymbol{S}_0$, and in $\boldsymbol{S}_0'$ for all $f \in \boldsymbol{L}^2$.

Since given (g, Λ), the closed affine subspace $\boldsymbol{A}_{g,\Lambda}$ of all weakly Λ-dual Gabor atoms $\gamma \in \boldsymbol{L}^2$ obviously contains $\widetilde{g}$ as minimal norm element, the following is a consequence of Theorem 3.5.12.

Corollary 3.5.17 *Assume that (g, Λ) generates a Gabor frame, and let $\widetilde{g}$ be the canonical dual Gabor atom. Then the closed affine subspace of all weakly dual Gabor atoms is given by*

$$\boldsymbol{A}_{g,\Lambda} = \widetilde{g} + \boldsymbol{H}_{g,\Lambda^\circ}^{\perp} \,.$$

Thus the affine orthogonal projection $Q_{g,\Lambda}$ onto $\boldsymbol{A}_{g,\Lambda}$ is given by

$$\begin{aligned} Q_{g,\Lambda} h &= \widetilde{g} + h - \operatorname{red}(\Lambda) \sum_{\lambda^\circ \in \Lambda^\circ} \langle h, \pi(\lambda^\circ)\widetilde{g}\rangle \, \pi(\lambda^\circ) g \\ &= \widetilde{g} + h - \sum_{\lambda \in \Lambda} \mathcal{V}_{\widetilde{g}} g(\lambda) \, \pi(\lambda) h \,. \end{aligned}$$

Remark. The last expression appears in the work of Li and Healy [LH94, LH96], there denoted "parametrized representation" of the set of dual Gabor atoms. However, they do not claim to solve the best approximation problem (by orthogonal projection). Furthermore, they use the concept of pseudo-duals (which also does not require a Bessel condition), which appears to be less concise than the notion of weak Λ-duality introduced above.

Let us mention furthermore that the above statement does not include any claim concerning the preservation of a Λ-Bessel property under the affine projection $Q_{g,\Lambda}$. Indeed, the projection is onto the set of all weakly Λ-dual atoms in $\boldsymbol{L}^2$, not just onto those generating dual Gabor frames.

In case $g, \widetilde{g} \in \boldsymbol{S}_0$, the projection extends to a bounded linear map in $\boldsymbol{S}_0'$.

Proof. Both corollaries are immediate from Proposition 3.5.13 and the results presented in part 3.4.3 on projection bases. The alternative representations of the operators follow from the Janssen Representation (3.5.5). ❑

3.6 Dual pairs in $\boldsymbol{S}_0$

From the results in Section 3.3, it should already be obvious that, when working with a weakly dual pair (g, γ), it is highly desirable to have both g and γ in $\boldsymbol{S}_0(\mathbb{R}^d)$. In this case we will speak of a *dual pair in* $\boldsymbol{S}_0(\mathbb{R}^d)$ with respect to the lattice Λ. In the sequel, we will exhibit further advantages of this situation. We begin with two general statements.

Corollary 3.6.1 *Given a (fixed) lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, the set of dual pairs in $\boldsymbol{S}_0(\mathbb{R}^d)$ with respect to Λ is a closed subset of $\boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0(\mathbb{R}^d)$.*

Proof. This is a direct consequence of Theorem 3.5.4. ❑

If we ask for all elements of $\boldsymbol{S}_0(\mathbb{R}^d)$ generating a Gabor frame with canonical dual also in $\boldsymbol{S}_0$, we obtain the following theorem, the proof of which we have to postpone to the next section.

Theorem 3.6.2 *For a (fixed) lattice Λ, consider the set G_Λ of all $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ such that (g, Λ) is a Gabor frame with $\widetilde{g} = S_{g,\Lambda}^{-1} g \in \boldsymbol{S}_0(\mathbb{R}^d)$. Then G_Λ is open in $\boldsymbol{S}_0(\mathbb{R}^d)$, and the mapping $g \to \widetilde{g}$ is continuous on G_Λ.*

Remark. i) Note that this is not a contradiction to Corollary 3.6.1. It seems feasible that G_Λ is the projection of the set of all Λ-dual pairs in $\boldsymbol{S}_0$. If this is true indeed, we can conclude that the latter set is (closed, but) not compact.

ii) The statement becomes false, if we replace $\boldsymbol{S}_0$ by $\boldsymbol{L}^2$. There the set G_Λ defined accordingly is not open, which can be seen, e.g., by choosing $\Lambda = \mathbb{Z} \times \mathbb{Z}$ and $g = \chi_{[0,1]}$. Then (g, Λ) generates a frame (even an orthonormal basis), but not if we replace g by $\chi_{[0,1-\varepsilon]}$.

3.6.1 Equivalent characterizations of dual pairs in $\boldsymbol{S}_0$

Lemma 3.6.3 *Let Λ be a lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, and $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ such that (g, Λ) generates a Gabor frame. Then the following conditions are equivalent.*

i) $S_{g,\Lambda}$ is (boundedly) invertible on $\boldsymbol{S}_0(\mathbb{R}^d)$.

ii) $\widetilde{g} \in \boldsymbol{S}_0(\mathbb{R}^d)$.

iii) $\widetilde{g}$ *satisfies condition* (A_Λ).

iv) $S_{g,\Lambda}^{-1} = \sum_{\lambda^\circ \in \Lambda^\circ} c_{\lambda^\circ} \pi(\lambda^\circ)$ *for some sequence* $c \in \boldsymbol{\ell}^1(\Lambda^\circ)$.

v) $S_{g,\Lambda}$ *is invertible on all time-frequency-homogeneous Banach spaces.*

Proof. i) $\Rightarrow$ ii). Obvious from the fact that $\widetilde{g} = S_{g,\Lambda}^{-1} g$.

ii) $\Rightarrow$ iii). $\widetilde{g} \in \boldsymbol{S}_0(\mathbb{R}^d)$ implies that $\mathcal{V}_{\widetilde{g}}\widetilde{g} \in \boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, and thus the sequence of sampled values $(\mathcal{V}_{\widetilde{g}}\widetilde{g}(\lambda^\circ))_{\lambda^\circ \in \Lambda^\circ}$ is in $\boldsymbol{\ell}^1(\Lambda^\circ)$.

iii) $\Rightarrow$ iv). Clear since by Janssen's representation (3.5.5),

$$S_{g,\Lambda}^{-1} = S_{\widetilde{g},\Lambda} = \operatorname{red}(\Lambda) \sum_{\lambda^\circ \in \Lambda^\circ} \mathcal{V}_{\widetilde{g}}\widetilde{g}(\lambda^\circ)\, \pi(\lambda^\circ)\,.$$

iv) $\Rightarrow$ v). An absolutely convergent series of time-frequency-shift operators is clearly bounded on any time-frequency-homogeneous Banach space.

v) $\Rightarrow$ i). Trivial. ❑

Remark. Note that if the above conditions are satisfied, we may speak of a canonical dual Gabor pair in $\boldsymbol{S}_0(\mathbb{R}^d)$. In that case the frame operator is not only invertible on $\boldsymbol{S}_0(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\mathbb{R}^d)$, but by Corollary 3.3.3 also extends to a bounded isomorphism of $\boldsymbol{S}_0'(\mathbb{R}^d)$.

Proof. (of Theorem 3.6.2.) From Corollary 3.3.4, we know that for $g \in \boldsymbol{S}_0(\mathbb{R}^d)$, the operator $S_{g,\Lambda}$ is well-defined and the mapping $g \mapsto S_{g,\Lambda}$ is continuous from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{\mathcal{L}}(\boldsymbol{S}_0)$. The proof of Lemma 3.6.3 shows that if $S_{g,\Lambda}$ is invertible on $\boldsymbol{S}_0$ then also on $\boldsymbol{L}^2$, and that implies that (g,Λ) generates a Gabor frame for $\boldsymbol{L}^2(\mathbb{R}^d)$. Therefore, we can describe G_Λ as the set of all $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ with the property that $S_{g,\Lambda}$ is invertible on $\boldsymbol{S}_0(\mathbb{R}^d)$. But this means that G_Λ is the preimage of the open set of invertible elements in $\boldsymbol{\mathcal{L}}(\boldsymbol{S}_0)$ under a continuous mapping and thus is open.

The mapping $g \mapsto \widetilde{g}$ can be factored as $g \mapsto (g, S_{g,\Lambda}) \mapsto (g, S_{g,\Lambda}^{-1}) \mapsto S_{g,\Lambda}^{-1} g = \widetilde{g}$ and therefore has to be continuous. ❑

Theorem 3.6.4 *If* (g,γ) *is a* Λ*-dual pair in* $\boldsymbol{S}_0(\mathbb{R}^d)$*, then* $(\pi(\lambda)g)_{\lambda\in\Lambda}$ *and* $(\pi(\lambda)\gamma)_{\lambda\in\Lambda}$ *are dual* $(\boldsymbol{S}_0, \boldsymbol{\ell}^1)$*-frames,* $(\boldsymbol{L}^2, \boldsymbol{\ell}^2)$*-frames, and* $(\boldsymbol{S}_0', \boldsymbol{\ell}^\infty)$*-frames.*

Furthermore, $\big(\pi(\lambda^\circ)g, \operatorname{red}(\Lambda^\circ)\, \pi(\lambda^\circ)\gamma\big)_{\lambda^\circ\in\Lambda^\circ}$ *is an* $\boldsymbol{\ell}^1$*-,* $\boldsymbol{\ell}^2$*-, and* $\boldsymbol{\ell}^\infty$*-Riesz projection basis in* $\boldsymbol{S}_0$*,* $\boldsymbol{L}^2$*, and* $\boldsymbol{S}_0'$ *(in the weak*-sense), respectively.*

Proof. By Theorem 3.3.1, we know that $T_{g,\Lambda}$ and $T_{\gamma,\Lambda}$ as well as T_{g,Λ° and T_{γ,Λ° are in $\boldsymbol{\mathcal{L}}(\boldsymbol{S}_0, \boldsymbol{\ell}^1)$, in $\boldsymbol{\mathcal{L}}(\boldsymbol{L}^2, \boldsymbol{\ell}^2)$, and in $\boldsymbol{\mathcal{L}}(\boldsymbol{S}_0', \boldsymbol{\ell}^\infty)$. The duality of (g,γ) with respect to Λ implies that $T_{\gamma,\Lambda}^*$ is a left inverse of $T_{g,\Lambda}$ which yields the first claims. The others follow by Corollary 3.5.14. ❑

3.6.2 *Sufficient conditions for* $\widetilde{g} \in \boldsymbol{S}_0$

Note that in Lemma 3.6.3, we do not claim that if $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ generates a frame with respect to some lattice Λ, then $\widetilde{g} \in \boldsymbol{S}_0(\mathbb{R}^d)$ also. In fact, this question is still open at this moment; however, we will describe in the sequel a number of practically relevant conditions where the answer is affirmative.

Theorem 3.6.5 *Let* $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ *and* $g \in \boldsymbol{S}_0(\mathbb{R})$ *such that* (g, Λ) *generates an* $(\boldsymbol{L}^2, \boldsymbol{\ell}^2)$*-Gabor frame. Then each of the following conditions implies that the canonical dual Gabor window* $\widetilde{g}$ *is in* $\boldsymbol{S}_0(\mathbb{R})$.

i) $a/b \in \mathbb{Q}$.

ii) $\operatorname{supp} g \subseteq B_r(x)$, *for some* $x \in \mathbb{R}$ *and* $r \leq 1/2b$.

iii) $\operatorname{supp} \hat{g} \subseteq B_s(\xi)$, *for some* $\xi \in \widehat{\mathbb{R}}$ *and* $s \leq 1/2a$.

Proof. i) is a special case of [FG96], Thm. 3.4. In a way this result is based on Wiener's Inversion Lemma, to which it actually reduces in the case of integer oversampling, i.e., if $\operatorname{red}(\Lambda) = 1/ab \in \mathbb{N}$.

ii) Consider the function $G(t) = \sum_{n\in\mathbb{Z}} |g(t-na)|^2$. Since $g \in \boldsymbol{S}_0(\mathbb{R}^d)$, we have that $|g|^2 = g\,\overline{g} \in \boldsymbol{S}_0(\mathbb{R}^d) = \boldsymbol{W}(\boldsymbol{A}, \boldsymbol{\ell}^1)$, and thus $G \in \boldsymbol{A}(\mathbb{T})$.

By [BW94], Thm. 3.12, the fact that (g, Λ) generates an $(\boldsymbol{L}^2, \boldsymbol{\ell}^2)$-Gabor frame implies that there are positive constants A and B such that $A \leq G \leq B$. Thus we may apply [BW94], Thm. 3.13 and obtain that

$$\widetilde{g} = S^{-1} g = \frac{b}{G}\, g\,.$$

By Wiener's Lemma, $A \leq G$ implies that with G we also have $\frac{1}{G} \in \boldsymbol{A}(\mathbb{T}) \subseteq \boldsymbol{W}(\boldsymbol{A}, \boldsymbol{\ell}^\infty)$, so we know by Corollary 3.2.10 that $\frac{1}{G}$ is a bounded multiplier on $\boldsymbol{S}_0(\mathbb{R}^d)$, and thus $\widetilde{g} \in \boldsymbol{S}_0(\mathbb{R}^d)$ as claimed.

iii) follows from ii) via the Fourier invariance of $\boldsymbol{S}_0$ and Theorem 3.5.3 i). ❑

Theorem 3.6.6 *Let* $g \in \boldsymbol{S}_0(\mathbb{R})$. *Then we have for all* a, b *sufficiently small that the pair* $(g, a\mathbb{Z} \times b\mathbb{Z})$ *generates an* $(\boldsymbol{\ell}^2, \boldsymbol{L}^2)$*-Gabor frame with* $\widetilde{g} \in \boldsymbol{S}_0(\mathbb{R})$.

Proof. $g \in \boldsymbol{S}_0(\mathbb{R})$ implies by Lemma 3.2.15 that $\mathcal{V}_g g \in \boldsymbol{W}(\boldsymbol{C}, \boldsymbol{\ell}^1)$. From (3.5.5), we know that we can write

$$S_{g,a\mathbb{Z}\times b\mathbb{Z}} = (ab)^{-1} \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}} \mathcal{V}_g g\Big(\frac{m}{b}, \frac{n}{a}\Big)\, \mathrm{M}_{n/a}\, \mathrm{T}_{m/b}\,.$$

Thus we have

$$\big\|\operatorname{Id} - ab S_{g,a\mathbb{Z}\times b\mathbb{Z}}\big\|_{\mathcal{L}(\boldsymbol{S}_0)} \leq \sum_{(m,n)\neq(0,0)} \Big|\mathcal{V}_g g\Big(\frac{m}{b}, \frac{n}{a}\Big)\Big| < 1$$

for a, b sufficiently small, and then $S_{g,a\mathbb{Z}\times b\mathbb{Z}}$ is invertible on $\boldsymbol{S}_0(\mathbb{R}^d)$, so $\widetilde{g} = S_g^{-1} g \in \boldsymbol{S}_0(\mathbb{R}^d)$.

The frame property follows by Theorem 3.6.4. ❑

3.6.3 Varying the lattice

We have seen that for a fixed lattice Λ, the dual window $\widetilde{g}$ depends continuously on g in $\boldsymbol{S}_0(\mathbb{R}^d)$. Now we want to fix g and observe how $\widetilde{g}$ changes with small variations of Λ. By now, it should be no surprise anymore that we obtain the most natural results if we use the norm in $\boldsymbol{S}_0$ instead of $\boldsymbol{L}^2$.

To ensure $\widetilde{g} \in \boldsymbol{S}_0$, we will assume conditions related to the family of weight functions $w_{s,d}$ defined in (3.2.20). These functions satisfy the following weak subconvolutivity condition.

Lemma 3.6.7 ([Fei79])
On $\mathbb{R}^d$, we have for $s > d$ that there exists a constant $\widetilde{C}_{s,d} > 0$ such that

$$(w_{-s,d} * w_{-s,d})(x) \leq \widetilde{C}_{s,d}\, w_{-s,d}(x) \qquad \forall x \in \mathbb{R}^d \,. \tag{3.6.1}$$

Using this property, we can establish the following auxiliary result.

Lemma 3.6.8 *i) For $s > 2d$, the set*

$$H_{s,C} = \{g \in \boldsymbol{L}^2(\mathbb{R}^d) : |\mathcal{V}_{g_0} g| \leq C\, w_{-s,2d}\}$$

is a relatively compact subset of $\boldsymbol{S}_0(\mathbb{R}^d)$.

ii) For each $s > 2d$, there is a constant $C_{s,2d} > 0$ such that $g \in H_{s,C}$ implies

$$|\mathcal{V}_g g| \leq C^2\, C_{s,2d}\, w_{-s,2d} \,.$$

Proof. i) For $s > 2d$, we have $w_{-s,2d} \in \boldsymbol{W}(\boldsymbol{C}, \boldsymbol{\ell}^1) \subseteq \boldsymbol{L}^1(\mathbb{R}^{2d})$ and thus $H_{s,C} \subseteq \boldsymbol{S}_0(\mathbb{R}^d)$ by (3.2.7). It is well known that for $\Lambda = \frac{1}{2}\mathbb{Z}^d \times \frac{1}{2}\mathbb{Z}^d$, the canonical dual $\widetilde{g_0}$ of the Gauss-function is a Schwartz function and thus in $\boldsymbol{S}_0(\mathbb{R}^d)$ (compare also Theorem 3.6.9 iii).) To show that $H_{s,C}$ is totally bounded, consider $\varepsilon > 0$ and note that there is a finite set $F = F_\varepsilon \subseteq \Lambda$ such that $\sum_{\lambda \notin F} w_{-s,2d}(\lambda) < \varepsilon / (2\, C\, \|\widetilde{g_0}\|_{\boldsymbol{S}_0})$. Consequently, we have for all $g \in H_{s,C}$ that $\left\|g - \sum_{\lambda \in F} \mathcal{V}_{g_0} g(\lambda)\, \pi(\lambda) \widetilde{g_0}\right\|_{\boldsymbol{S}_0} \leq \varepsilon/2$. Now we can choose for each $\lambda \in F$ finitely many complex numbers in $\{z : |z| \leq C\, w_{-s,2d}(\lambda)\}$ in order to obtain a finite ε-dense set for $H_{s,C}$.

To show (ii), we once more make use of the Fourier–Wigner transformation (3.2.2). Using the properties discussed in Section 3.3, we obtain from $|\mathcal{V}_{g_0} g| \leq C\, w_{-s,2d}$ that

$$\left|V(g, g)\right| \;=\; \|g_0\|_{\boldsymbol{L}^2}^{-2}\, |\langle g_0, g_0 \rangle|\, \left|V(g, g)\right|$$

$$
\begin{aligned}
&\underset{(3.3.2)}{=} \|g_0\|_{L^2}^{-2} \left|V(g_0,g) \natural V(g,g_0)\right| \\
&\underset{(3.3.3)}{\le} \|g_0\|_{L^2}^{-2} \left|V(g_0,g)\right| * \left|V(g_0,g)\right| \\
&\le \|g_0\|_{L^2}^{-2} C^2 w_{-s,2d} * w_{-s,2d} \\
&\underset{(3.6.1)}{\le} \|g_0\|_{L^2}^{-2} C^2 \widetilde{C}_{s,2d} w_{-s,2d}
\end{aligned}
$$

which implies the claim. ❑

Our first result based on these decay conditions is an immediate consequence of one of the main results in [FG89a], but it can also be obtained from Janssen's representation of the frame operator described above. We shall state the result in a slightly restricted form, since it will simplify the technical details considerably. To avoid misunderstandings, we emphasize that our modification of the lattice $a\mathbb{Z} \times b\mathbb{Z}$ by varying the lattice constants a and b is essentially different from the usual perturbations studied, e.g., in the results of Christensen and Heil [CH97] (see also Chapter 5.) In our case, corresponding pairs of label points (ka, lb) and (ka', lb') are arbitrarily far apart for large k or l.

Theorem 3.6.9 *Let $H_{s,C}$ be as in Lemma 3.6.8 for some $s>2d$ and $C>0$. Then the following hold.*

i) For $0 < r, R < \infty$, the family of operators

$$\left\{S_{g,a\mathbb{Z}^d\times b\mathbb{Z}^d} : g \in H_{s,C},\ a,b \in [r,R]\right\} \tag{3.6.2}$$

is uniformly bounded on any isometrically time-frequency-invariant Banach space $\boldsymbol{B}$ (in particular, on $\boldsymbol{S}_0(\mathbb{R}^d)$ or $\boldsymbol{L}^p(\mathbb{R}^d)$ for $p\in[1,\infty]$.) More explicitly, there exists a constant $C_{s,r,R}$ such that

$$\|S_{g,a\mathbb{Z}^d\times b\mathbb{Z}^d} f\|_{\boldsymbol{B}} \le C_{s,r,R}\, C^2\, \|f\|_{\boldsymbol{B}} \qquad \forall f \in \boldsymbol{B}. \tag{3.6.3}$$

ii) For $g \in H_{s,C}$, we have that for fixed $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ (or $\boldsymbol{L}^p(\mathbb{R}^d)$ with $p\in[1,\infty)$), the mapping

$$(a,b) \mapsto S_{g,a\mathbb{Z}^d\times b\mathbb{Z}^d} f$$

is continuous from $\mathbb{R}_+^2$ to $\boldsymbol{S}_0(\mathbb{R}^d)$ (or $\boldsymbol{L}^p(\mathbb{R}^d)$, respectively.)

iii) There exist α, $\beta > 0$ such that $S_{g,a\mathbb{Z}^d\times b\mathbb{Z}^d}$ is invertible on $\boldsymbol{S}_0(\mathbb{R}^d)$ (and on $\boldsymbol{L}^p(\mathbb{R}^d)$ for $p\in[1,\infty]$) for all $(a,b) \in (0,\alpha] \times (0,\beta]$ and all $g \in H_{s,C}$ with $\|g\|_{\boldsymbol{L}^2} = 1$.

Proof. i) For $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$, the adjoint lattice is $\Lambda^\circ = \frac{1}{b}\mathbb{Z}^d \times \frac{1}{a}\mathbb{Z}^d$, and thus the Janssen representation (3.5.5) for the frame operator becomes

$$S_{g,a\mathbb{Z}^d\times b\mathbb{Z}^d} = (ab)^{-d} \sum_{(m,n)\in\mathbb{Z}^d\times\mathbb{Z}^d} \mathcal{V}_g g\left(\frac{m}{b}, \frac{n}{a}\right) \mathrm{M}_{n/a}\, \mathrm{T}_{m/b}\,.$$

Therefore we obtain with Lemma 3.6.8 ii)

$$\begin{aligned}\|S_{g,a\mathbb{Z}^d\times b\mathbb{Z}^d}\|_{\mathcal{L}(\boldsymbol{B})} &\leq (ab)^{-d}\sum_{(m,n)}\left|\mathcal{V}_g g\left(\frac{m}{b},\frac{n}{a}\right)\right| \\ &\leq r^{-2d}\,C^2\,C_s\sum_{(m,n)} w_{-s}\left(\frac{m}{R},\frac{n}{R}\right) = C_{s,r,R}\,C^2 < \infty\,.\end{aligned}$$

ii) Given a_0, $b_0 > 0$, choose $r < a_0, b_0 < R$, then the above calculation shows that for ε>0, there is a finite set $F \subseteq \mathbb{Z}^d \times \mathbb{Z}^d$ such that

$$\left\| S_{g,a\mathbb{Z}^d\times b\mathbb{Z}^d} - \frac{1}{(ab)^d}\sum_{(m,n)\in F}\mathcal{V}_g g\left(\frac{m}{b},\frac{n}{a}\right)\,\mathrm{M}_{n/a}\,\mathrm{T}_{m/b}\right\|_{\mathcal{L}(\boldsymbol{B})} < \varepsilon$$

uniformly for all $a, b \in [r, R]$. Thus it suffices to show the continuity of the individual terms, i.e., of $(a, b) \mapsto \mathrm{M}_{n/a}\,\mathrm{T}_{m/b}\,f$ for fixed $m, n \in \mathbb{Z}^d$. But this follows from the fact that $\boldsymbol{S}_0(\mathbb{R}^d)$ and $\boldsymbol{L}^p(\mathbb{R}^d)$ for $p \in [1, \infty)$ are time-frequency-homogeneous Banach spaces.

iii) Note that $\mathcal{V}_g g(0, 0) = \|g\|_{\boldsymbol{L}^2} = 1$, thus

$$\begin{aligned}\|\,\mathrm{Id} - (ab)^d\,S_{g,a\mathbb{Z}^d\times b\mathbb{Z}^d}\|_{\mathcal{L}(\boldsymbol{B})} &\leq \sum_{(m,n)\neq(0,0)}\left|\mathcal{V}_g g\left(\frac{m}{b},\frac{n}{a}\right)\right| \\ &\leq \sum_{(m,n)\neq(0,0)} w_{-s}\left(\frac{m}{\beta},\frac{n}{\alpha}\right) =: q\,,\end{aligned}$$

and we can ensure that $q < 1$ by choosing α and β sufficiently small. ❑

Theorem 3.6.10 *Let g, α, and β as in Theorem 3.6.9 iii), and consider a sequence $((a_n, b_n))_{n\in\mathbb{N}}$ in $(0, \alpha]\times(0, \beta]$. Denote the canonical dual Gabor atom for g with respect to $\Lambda_n = a_n\mathbb{Z}^d \times b_n\mathbb{Z}^d$ by $\widetilde{g}_{a_n,b_n}$. Assume that $(a_n, b_n) \to (a_0, b_0) \in (0, \alpha]\times(0, \beta]$, then $\widetilde{g}_{a_n,b_n} \to \widetilde{g}_{a_0,b_0}$ in $\boldsymbol{S}_0$.*

For the proof of this theorem, we need the following result attributed to Kantorovich [RM94].

Proposition 3.6.11 *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of invertible operators on a Banach space $\boldsymbol{B}$, and assume that the sequence is uniformly bounded above and below, i.e, that there exist constants $0 < C_{1,2} < \infty$ such that*

$$\frac{1}{C_1}\,\|f\|_{\boldsymbol{B}} \leq \|T_n f\|_{\boldsymbol{B}} \leq C_2\,\|f\|_{\boldsymbol{B}} \qquad \textit{for all } n\in\mathbb{N} \textit{ and } f \in \boldsymbol{B}. \tag{3.6.4}$$

Then $(T_n)_n$ converges in the strong operator topology to an invertible operator if and only if the same is true for $(T_n^{-1})_n$, and then

$$(\lim T_n)^{-1} = \lim\,(T_n^{-1})\,.$$

Proof. Assume that T_n converges to T_0 in the strong operator sense, i.e., that $T_n f \to T_0 f$ for all $f \in \boldsymbol{B}$, where T_0 is invertible. Then T_0 is again bounded above and below by C_1 and C_2. In order to show $T_n^{-1}h \to T_0^{-1}h$ for all $h \in \boldsymbol{B}$, we use $T_0 T_0^{-1}h = h = T_n T_n^{-1}h$ to obtain

$$\begin{aligned}\|T_n^{-1}h - T_0^{-1}h\|_{\boldsymbol{B}} &\leq C_1 \big\|T_n(T_n^{-1}h - T_0^{-1}h)\big\|_{\boldsymbol{B}} \\ &= C_1 \big\|T_0(T_0^{-1}h) - T_n(T_0^{-1}h)\big\|_{\boldsymbol{B}} \to 0,\end{aligned}$$

so T_n^{-1} converges to T_0^{-1} as claimed.

The converse follows by replacing the T_n by their inverses. ❑

Remark. The assumption that $\lim_n T_n$ be invertible can not be omitted. If we only assume that $T_n \to T_0$ in the strong operator topology, then T_0 is of course bounded above and below by C_1 and C_2 respectively, which implies injectivity, but T_0 need not be surjective.

Proof. (of Theorem 3.6.10.) By Theorem 3.6.9, the sequence $(S_{g,\Lambda_n})_{n\in\mathbb{N}}$ satisfies the assumptions of Proposition 3.6.11 with limit S_{g,Λ_0}. Thus we have $\widetilde{g}_{a_n,b_n} = (S_{g,\Lambda_n})^{-1}g \to (S_{g,\Lambda_0})^{-1}g = \widetilde{g}_{a_0,b_0}$ as claimed. ❑

It is plausible due to the increasing redundancy of Gabor frames with smaller and smaller lattice constants, and also confirmed by numerical experiments, that if we let $(a_n, b_n) \to (0,0)$, then the canonical dual atoms $\widetilde{g}_n$, suitably normalized, will tend to g itself. Thus the standard inversion formula for the continuous STFT (3.2.5) can be seen as the limiting case of discrete Gabor expansions using the canonical frame duals.

This fact is stated several times in the literature at different places (e.g., see [Dau90], pp. 980–982), compare also Chapters 1 and 14, but we have not found any formal proof yielding at least uniform convergence of the normalized sequence $\widetilde{g}_n$ to g. Therefore we state the following corollary, which is an easy consequence of the arguments used in the proof of Theorem 3.6.9.

Corollary 3.6.12 *Let g, α, and β as in Theorem 3.6.9 iii), and consider a sequence $((a_n, b_n))_{n\in\mathbb{N}}$ in $(0,\alpha]\times(0,\beta]$ with $(a_n,b_n)\to(0,0)$. Then*

$$\frac{1}{a_n b_n}\widetilde{g}_{a_n,b_n} \to g \quad \text{in } \boldsymbol{S}_0$$

(in particular, in $\boldsymbol{L}^2$ and uniformly on $\mathbb{R}^d$.)

Proof. In the proof of Theorem 3.6.9 iii), we saw that

$$ab\, S_{g,a\mathbb{Z}\times b\mathbb{Z}} \to \mathrm{Id} \qquad ((a,b) \to (0,0),\ a,b>0)$$

in the operator norm. This implies $(ab)^{-1}S_{g,a\mathbb{Z}\times b\mathbb{Z}}^{-1} \to \mathrm{Id}$ in the operator norm, and thus

$$\frac{1}{a_n b_n}\widetilde{g}_{a_n,b_n} = \frac{1}{a_n b_n} S_{g,a_n\mathbb{Z}\times b_n\mathbb{Z}}^{-1} g \to g$$

in $\boldsymbol{S}_0(\mathbb{R})$ as claimed. ❑

3.6.4 An approximate Balian–Low Theorem

The Balian–Low Theorem (e.g., see [DJ93] or Chapter 2) implies that there are no Λ-dual pairs in $\boldsymbol{S}_0(\mathbb{R}^d)$ for the critical lattice $\mathbb{Z} \times \mathbb{Z} \lhd \mathbb{R} \times \widehat{\mathbb{R}}$. On the other hand it is well known (e.g., see [Lyu92] and [SW92a]) that for any pair of lattice constants (a, b) with $ab < 1$, the Gauss-function g_0 generates a frame with respect to $a\mathbb{Z} \times b\mathbb{Z}$ with canonical dual atom $\widetilde{g}$ in $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$.

Consequently, if we let $(a, b) \to (1, 1)$, we expect these dual atoms to exhibit increasingly bad time-frequency localization. We present a statement of this phenomenon in terms of the decay conditions introduced in Lemma 3.6.8.

Theorem 3.6.13 *Consider a sequence of lattices $\Lambda_n = a_n\mathbb{Z}^d \times b_n\mathbb{Z}^d$, together with canonically Λ_n-dual pairs $(g_n, \widetilde{g}_n)$ of Schwartz functions. Thus, given $s > 2d$, there are constants $C_n > 0$ such that $g_n, \widetilde{g}_n \in H_{s,C_n}$ for each n. If $(a_n, b_n) \to (1, 1)$ as $n \to \infty$, then necessarily $C_n \to \infty$.*

Proof. Assume that $(a_n, b_n) \to (1, 1)$, and that there is $s > 2d$ and $C_n \not\to \infty$. After switching to a subsequence and employing Lemma 3.6.8, we may assume that there is $C > 0$ such that $g_n, \widetilde{g}_n \in H_{s,C}$ for all n, and that $(g_n, \widetilde{g}_n)$ converges in $\boldsymbol{S}_0 \times \boldsymbol{S}_0$ to some (g, γ). Letting $\Lambda = \mathbb{Z} \times \mathbb{Z}$, this implies by Corollary 3.3.3 and Theorem 3.6.9 ii) that $S_{g_n,\Lambda_n} \to S_{g,\Lambda}$ and $S_{\widetilde{g}_n,\Lambda_n} \to S_{\gamma,\Lambda}$ in the strong operator sense.

$(g_n, \widetilde{g}_n)$ being canonically Λ_n-dual implies that

$$S_{g_n,\Lambda_n} S_{\widetilde{g}_n,\Lambda_n} = \mathrm{Id}_{\boldsymbol{S}_0}$$

for all $n \in \mathbb{N}$, so by Lemma 3.6.14 below, we may conclude that $S_{g,\Lambda} S_{\gamma,\Lambda} = \mathrm{Id}_{\boldsymbol{S}_0}$ also. But this is a contradiction to the Balian–Low Theorem. ❑

Lemma 3.6.14 *Assume that $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ are uniformly bounded, strongly convergent sequences of linear operators between normed spaces, with limits U_0 and V_0, respectively. Then*

$$\lim_{n\to\infty} U_n V_n = U_0 V_0$$

in the strong operator topology.

Proof. By assumption, we have $|||U_n||| \le C$ for some $C > 0$. For every f in the domain of the V_n, we have

$$\begin{aligned} \left\| U_n V_n f - U_0 V_0 f \right\| &\le \left\| U_n (V_n f - V_0 f) \right\| + \left\| (U_n - U_0) V_0 f \right\| \\ &\le C \left\| (V_n - V_0) f \right\| + \left\| (U_n - U_0) V_0 f \right\|, \end{aligned}$$

where both expressions on the right tend to zero as $n \to \infty$. ❑

3.6.5 Bandlimited signals and Gabor analysis

Most of the literature on Gabor analysis discusses global representation of $\boldsymbol{L}^2$-signals and quadratic mean convergence. In practical situations, however, there is more interest in local representations, preferably both in the time- and the frequency-sense, and – if possible – with uniform and/or absolute convergence of the Gabor expansion for a reasonable class of continuous functions. I.e., we would like to reconstruct a signal reasonably well on a compact interval I from finitely many Gabor coefficients, namely, those related to a somewhat larger interval $J \supseteq I$ in the time domain and only a finite bandwidth in the frequency domain. This should be possible in particular if the signal itself is bandlimited.

Theorem 3.6.15 *Let (g, γ) be a Λ-dual pair in $\boldsymbol{S}_0(\mathbb{R}^d)$. Then for every $\varepsilon > 0$, there exists $R = R(\varepsilon, g, \gamma) > 0$ such that*

$$\Bigg\| f - \sum_{\substack{(x,\xi)\in\Lambda \\ |\xi|\leq R+r}} \langle f, \pi(\lambda)\gamma\rangle\, \pi(\lambda) g \Bigg\|_{\boldsymbol{S}_0} \leq \varepsilon \, \|f\|_{\boldsymbol{S}_0}$$

for all $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ with $\operatorname{supp} \hat{f} \subseteq B_r(0)$.

For bandlimited $\boldsymbol{L}^2$-functions, the same estimate holds in the $\boldsymbol{L}^2$-norm.

Remark. It is worth noting that since $\boldsymbol{S}_{0,c} = \boldsymbol{A}_c$, the bandlimited functions in $\boldsymbol{L}^1$ are the same as in $\boldsymbol{S}_0$, so the first part of the above theorem is also a statement about $\boldsymbol{L}^1(\mathbb{R}^d)$.

Proof. Let $\varepsilon > 0$ be given. Recall that $\|T_\gamma f\|_{\ell^1(\Lambda)} \leq C\, \|\gamma\|_{\boldsymbol{S}_0} \|f\|_{\boldsymbol{S}_0}$ for all γ and f, and note that $\boldsymbol{S}_{0,c}$ is dense in $\boldsymbol{S}_0$. Therefore we can approximate γ by γ_c with $\widehat{\gamma_c} \in \boldsymbol{S}_{0,c}(\widehat{\mathbb{R}}^d)$ such that

$$\|T_{\gamma-\gamma_c} f\|_{\ell^1(\Lambda)} \leq \frac{\varepsilon}{\|g\|_{\boldsymbol{S}_0}} \|f\|_{\boldsymbol{S}_0} \quad \text{for all } f \in \boldsymbol{S}_0(\mathbb{R}^d).$$

So we have $\operatorname{supp} \widehat{\gamma_c} \subseteq B_R(0)$ for some R, and thus if $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ with $\operatorname{supp} \widehat{f} \subseteq B_r(0)$, then $T_{\gamma_c} f(\lambda) = \langle f, \mathrm{M}_\xi \mathrm{T}_x \gamma_c\rangle = \langle \widehat{f}, e^{2\pi i x\xi}\, \mathrm{M}_{-x} \mathrm{T}_\xi \widehat{\gamma_c}\rangle = 0$ for all λ with $|\xi| > R{+}r$.

Since $f = \sum_\lambda \langle f, \pi(\lambda)\gamma\rangle\, \pi(\lambda) g$, we can define $c = (c_\lambda)_{\lambda\in\Lambda}$ with

$$c_\lambda = \begin{cases} \langle f, \pi(\lambda)\gamma\rangle & \text{if } |\xi| > R{+}r, \\ 0 & \text{else,} \end{cases}$$

to obtain $f - \sum_{|\xi|\le R+r}\langle f,\pi(\lambda)\gamma\rangle\,\pi(\lambda)g = \sum_{\lambda\in\Lambda} c_\lambda\,\pi(\lambda)g$. But obviously $|c_\lambda| \le \left|\langle f,\pi(\lambda)\gamma - \pi(\lambda)\gamma_c\rangle\right|$ for all $\lambda\in\Lambda$, and so

$$\begin{aligned}\left\| f - \sum_{|\xi|\le R+r}\langle f,\pi(\lambda)\gamma\rangle\,\pi(\lambda)g\right\|_{S_0(\mathbb{R}^d)} &\le \|c\|_{\ell^1(\Lambda)}\,\|g\|_{S_0(\mathbb{R}^d)}\\ &\le \left\|T_{\gamma-\gamma_c}f\right\|_{\ell^1(\Lambda)}\,\|g\|_{S_0(\mathbb{R}^d)}\\ &\le \varepsilon\,\|f\|_{S_0(\mathbb{R}^d)}\,.\end{aligned}$$

The proof for the L^2-case is analogous. ❑

Theorem 3.6.16 *Consider a closed subspace $\mathcal{H} \subseteq L^2(\mathbb{R}^d)$, and assume that $J \subseteq \Lambda$ is an index set such that for some $\varepsilon < 1$,*

$$\left\| f - \sum_{\lambda\in J}\langle f,\pi(\lambda)\gamma\rangle\,\pi(\lambda)g\right\|_{L^2} \le \varepsilon\,\|f\|_{L^2} \quad \textit{for all } f\in\mathcal{H}.$$

Then $f\in\mathcal{H}$ can be completely reconstructed from $\big(\langle f,\pi(\lambda)\gamma\rangle\big)_{\lambda\in J}$.

Remark. Note that given $\mathcal{H}$, the choice of γ influences the size of J, as we saw in Theorem 3.6.15.

Proof. By assumption, the operator

$$A_J : f \mapsto \sum_{\lambda\in J}\langle f,\pi(\lambda)\gamma\rangle\,\pi(\lambda)g$$

satisfies $\|f - A_J f\|_{L^2} < \varepsilon\,\|f\|_{L^2}$ for all $f\in\mathcal{H}$, and in particular, it maps $\mathcal{H}$ into L^2. Let $P_{\mathcal{H}}$ be the orthogonal projection $L^2 \twoheadrightarrow \mathcal{H}$, then it follows that $|\!|\!|\,\mathrm{Id} - P_{\mathcal{H}}A_J|\!|\!|_{\mathcal{H}} < \varepsilon$. Thus we can use the Neumann series to obtain the operator $T = \sum_{k=0}^{\infty}(\mathrm{Id} - P_{\mathcal{H}}A_J)^k$ which satisfies $TP_{\mathcal{H}}A_J = \mathrm{Id}$ on $\mathcal{H}$. Since $A_J f$ is determined by $\big(\langle f,\pi(\lambda)\gamma\rangle\big)_{\lambda\in J}$ only, this shows that it is possible to recover f from these coefficients. ❑

Corollary 3.6.17 *Given a Λ-dual pair (g,γ) in $S_0(\mathbb{R}^d)$, there exists $R = R(g,\gamma) > 0$ such that any $f\in L^2(\mathbb{R}^d)$ with $\operatorname{supp}\hat{f} \subseteq B_r(0)$ can be reconstructed completely from the Gabor coefficients $\{\langle f,\pi(\lambda)\gamma\rangle : |\xi| \le R{+}r\}$.*

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4

Pseudodifferential operators, Gabor frames, and local trigonometric bases

Richard Rochberg and Kazuya Tachizawa

ABSTRACT – We use Gabor frames to give sufficient conditions on the symbols to ensure that the corresponding pseudodifferential operators belong to the classes $S_{p,q}$. We also give estimates for the size of eigenfunctions of pseudodifferential operators. We use local trigonometric bases to approximately diagonalize elliptic pseudodifferential operators.

4.1 Introduction

This paper is an investigation of some of the particulars of the general theme that localized frames and bases can be used to approximately diagonalize pseudodifferential operators. In this paper we use Gabor frames to investigate sufficient conditions on the symbol to ensure that the corresponding pseudodifferential operator belongs to the Schatten–von Neumann classes $S_{p,q}$. We also give estimates for the size of eigenfunctions of pseudodifferential operators. Finally we indicate how local trigonometric bases can be used to approximately diagonalize a class of elliptic pseudodifferential operators and draw some consequences of that.

For a function $\sigma(x,\xi)$ on $\mathbb{R}^d \times \mathbb{R}^d$ the corresponding pseudodifferential operator $\sigma(x,D)$ is formally defined by

$$\sigma(x,D)f(x) = (2\pi)^{-d} \int\int e^{i(x-y)\xi} \sigma\left(\frac{x+y}{2},\xi\right) f(y)\, dyd\xi \tag{4.1.1}$$

for $f \in \mathcal{S}(\mathbb{R}^d)$. The precise definition of $\sigma(x,D)$ will be given in Section 2.

The correspondence in (4.1.1) is called the Weyl correspondence. It was studied in the quantization problem in quantum mechanics. The symbol $\sigma(x,\xi)$ represents an observable in classical mechanics and the operator $\sigma(x,D)$ represents the corresponding observable in quantum mechanics. One of the virtues of the Weyl correspondence is that if the symbol $\sigma(x,\xi)$ is real, then the operator $\sigma(x,D)$ is formally self-adjoint.

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The theory of pseudodifferential operators using the Weyl correspondence is studied by Folland [Fol89], Hörmander [Hör79], and Howe [How80]. For example, if $\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)$ is bounded for all $|\alpha|+|\beta| \leq 2d+1$ then $\sigma(x,D)$ is bounded on L^2. If additionally $\sigma(x,\xi)$ vanishes at ∞, then $\sigma(x,D)$ is compact on L^2 ([How80]).

In this paper we study the boundedness of $\sigma(x,D)$ on weighted Sobolev spaces $H(w)$ whose precise definition will be given in Section 2. As a corollary we give a result on the L^2 boundedness and compactness of $\sigma(x,D)$.

Next we estimate the singular values of $\sigma(x,D)$ with quantities defined using the symbol $\sigma(x,\xi)$. Using these estimates we give a sufficient condition on the symbol for the corresponding operator belongs to the class $S_{p,q}$ for $0 < p < \infty, 0 < q \leq \infty$.

We recall the definitions of the classes S_p and $S_{p,q}$ (cf. [Sim79]). Let K be a compact operator in $L^2(\mathbb{R}^d)$. Let $\lambda_1 \geq \lambda_2 \geq \cdots$ be the singular values of K, that is, the eigenvalues of the compact, self-adjoint operator $(K^*K)^{1/2}$. For $0 < p < \infty$ we say that K belongs to the Schatten–von Neumann class S_p if $\sum_{i=1}^\infty \lambda_i^p < \infty$.

If

$$\sum_{k=1}^\infty k^{q/p-1}\lambda_k^q < \infty$$

for $0 < p,q < \infty$, then we say that K belongs to the class $S_{p,q}$. If $\lambda_k = O(k^{-1/p})$ as $k \to \infty$, then we say that K belongs to the class $S_{p,\infty}$. By definition we have $S_{p,p} = S_p$.

There are several results about the sufficient conditions on the symbols to ensure that the corresponding pseudodifferential operators belong to the Schatten–von Neumann classes.

If $\sigma(x,\xi) \in L^p(\mathbb{R}^{2d})$ for $1 \leq p \leq 2$, then we have $\sigma(x,D) \in S_{p'}$ where $p^{-1} + p'^{-1} = 1$ ([How80]).

If k is an even integer with $k > 2n(2/p-1)$ for $p \in [1,2]$ and $k > 3n(1-2/p)$ for $p \in (2,\infty)$, and if $\sigma(x,\xi) \in L_k^p(\mathbb{R}^{2d})$, then we have $\sigma(x,D) \in S_p$. Here $L_k^p(\mathbb{R}^{2d})$ denotes the classical Sobolev space ([Ron84]).

If $\sigma(x,\xi) \in L_s^2(\mathbb{R}^{2d})$ and $\mathcal{F}_{x,\xi}\sigma(y,\eta) \in L_s^2(\mathbb{R}^{2d})$, then $\sigma(x,D) \in S_{2n/(n+s),2}$ for each $s \geq 0$([HRT]).

In this paper we investigate a new condition on the symbols.

Next we give an estimate for eigenfunctions of $\sigma(x,D)$. We will show that if f is an eigenfunction of $\sigma(x,D)$ with eigenvalue λ, then f can be written as a finite sum of functions which satisfy uniform estimates with respect to λ. This result is similar to that in [Roc96]. As an application we give an estimate for the L^p norm of f and $\mathcal{F}f$. These theorems are proved using a Gabor frame.

Finally we develop some result about the global elliptic pseudodifferential operators. The second author had studied the global elliptic pseudodifferential operators using the Wilson basis [Tac96]. Here we present similar results using a local trigonometric orthonormal basis.

4.2 Main results

First we give the definition of pseudodifferential operators. We recall the definition of the Wigner transform. For f and g in $\mathcal{S}(\mathbb{R}^n)$ we define the Wigner transform $W(f,g)(x,\xi)$ by

$$W(f,g)(x,\xi) = (2\pi)^{-d} \int e^{-i\xi p} f\left(x + \frac{p}{2}\right) g\left(x - \frac{p}{2}\right)^* dp.$$

We have $W(f,g)(x,\xi) \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$(cf. [Fol89]).

For a symbol $\sigma(x,\xi) \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ and $f \in \mathcal{S}(\mathbb{R}^d)$ we define a pseudodifferential operator $\sigma(x,D)f(x)$ by

$$\langle \sigma(x,D)f, g\rangle = \int\int \sigma(x,\xi) W(f,g)(x,\xi)\, dx d\xi. \tag{4.2.1}$$

We can formally represent $\sigma(x,D)f(x)$ by (4.1.1). It turns out that $\sigma(x,D)$ is a continuous linear operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$.

We will be interested in symbols which satisfy a growth condition. Let $M(x,\xi)$ be a positive continuous function on $\mathbb{R}^d \times \mathbb{R}^d$. We assume that there are constants $C > 0$ and $s \geq 0$ such that

$$M(x+y, \xi+\eta) \leq C(1 + |y| + |\eta|)^s M(x,\xi) \tag{4.2.2}$$

for all $x, y, \xi, \eta \in \mathbb{R}^d$. For example $M(x,\xi) = (1 + |x|^2 + |\xi|^2)^{s/2}$.

Let $\sigma(x,\xi)$ be a function on $\mathbb{R}^d \times \mathbb{R}^d$ and R be a positive integer which is greater than or equal to $[(s+d)/2] + 1$. Here $[x]$ denotes the greatest integer which is less than or equal to x. Let $\mathbb{N}_0$ be the set of all nonnegative integers.

We assume that

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x,\xi)| \leq C_{\alpha,\beta} M(x,\xi) \quad \text{a.e. } (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d \tag{4.2.3}$$

for all $\alpha, \beta \in \mathbb{N}_0^d$ satisfying $|\alpha| \leq 2R$ and $|\beta| \leq 2R$. Here ∂_x and ∂_ξ denote distributional derivatives.

Next we define a Gabor frame of $L^2(\mathbb{R}^d)$. We will use the tight frame defined as follows.

Let $u(x)$ be a function in $C_0^\infty(\mathbb{R})$ satisfying the following conditions.

(a) $\operatorname{supp} u \subset [-1, 1]$.
(b) $0 \leq u(x)$ for all $x \in \mathbb{R}$.

(c) $u(-x) = u(x)$ for all $x \in \mathbb{R}$.
(d) $\sum_{k\in\mathbb{Z}} u(x-k)^2 = 1$ for all $x \in \mathbb{R}$.

The condition (c) will be needed in the definition of a local trigonometric orthonormal basis.

For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ we define

$$g(x) = u(x_1) \times \cdots \times u(x_d). \tag{4.2.4}$$

For $m, n \in \mathbb{Z}^d$ we set

$$g_{mn}(x) = g(x-m)e^{inx}.$$

The family $\{g_{mn}\}_{m,n\in\mathbb{Z}^d}$ satisfies

$$\|g_{mn}\| = 1$$

and

$$\sum_{m,n\in\mathbb{Z}^d} |\langle f, g_{mn}\rangle|^2 = (2\pi)^d \|f\|^2$$

for all $f \in L^2(\mathbb{R}^d)$, where $\|\cdot\|$ is the norm of $L^2(\mathbb{R}^d)$ ([Dau92, p.83]). This means that $\{g_{mn}\}_{m,n\in\mathbb{Z}^d}$ is a tight frame of $L^2(\mathbb{R}^d)$ with the frame constant $(2\pi)^d$.

By frame theory the dual frame $\{\tilde{g}_{mn}\}$ is given by $\tilde{g}_{mn}(x) = (2\pi)^{-d} g_{mn}(x)$ and any $f \in L^2(\mathbb{R}^d)$ has an expansion

$$f = (2\pi)^{-d} \sum_{m,n\in\mathbb{Z}^d} \langle f, g_{mn}\rangle g_{mn}$$

in $L^2(\mathbb{R}^d)$.

We have the following fundamental theorem.

Theorem 4.2.1 *We assume that $\sigma(x,\xi)$ satisfies the condition (4.2.3). Then there is a $C > 0$ such that*

$$|\langle \sigma(x,D)g_{mn}, g_{m'n'}\rangle| \le C \frac{\min\{M(m,n), M(m',n')\}}{(1+|m-m'|)^{2R-s}(1+|n-n'|)^{2R-s}}. \tag{4.2.5}$$

for all $m, m', n, n' \in \mathbb{Z}^d$.

We will study the boundedness of $\sigma(x,D)$ on certain weighted Sobolev spaces. Let $w(x,\xi)$ be a positive, continuous function on $\mathbb{R}^d \times \mathbb{R}^d$. We assume that there are constants $C > 0$ and $t \ge 0$ such that

$$w(x+y, \xi+\eta) \le C(1+|y|+|\eta|)^t w(x,\xi) \tag{4.2.6}$$

for all $x, y, \xi, \eta \in \mathbb{R}^d$.

We define the space $H(w)$ by

$$H(w) = \{f \in \mathcal{S}' : \|f\|_{H(w)} < \infty\},$$

where

$$\|f\|_{H(w)} = \left\{\int\int |\langle f, e^{-(\cdot - x)^2 + i\cdot\xi}\rangle|^2 w(x,\xi)^2 \, dx d\xi\right\}^{1/2}.$$

It turns out that $H(w)$ is a Hilbert space and $H(1)$ coincides with L^2(cf. [FG92a]).

Theorem 4.2.2 *We assume that $\sigma(x,\xi)$ satisfies the condition (4.2.3) for $R = [(s+d)/2] + 1 + t$. Then there is a positive constant C such that*

$$\|\sigma(x,D)f\|_{H(\tilde{w})} \le C\|f\|_{H(w)}$$

for all $f \in \mathcal{S}$, where $\tilde{w}(x,\xi) = w(x,\xi)M(x,\xi)^{-1}$.

Remark: Since M satisfies (4.2.2) and w satisfies (4.2.6), $\tilde{w}$ satisfies (4.2.2) with $s+t$ instead of s.

Using Theorem 4.2.2 we can give a sufficient condition for the boundedness and compactness of $\sigma(x,D)$ in $L^2(\mathbb{R}^d)$.

Theorem 4.2.3 *Let $R = [(s+d)/2] + 1$. Let $\sigma(x,D)$ be the pseudodifferential operator defined as above.*

(1) If M is bounded, then $\sigma(x,D)$ is a bounded operator on L^2.

(2) If $\lim_{|x|+|\xi|\to\infty} M(x,\xi) = 0$, then $\sigma(x,D)$ is a compact operator on L^2.

Theorem 4.2.3 is a corollary of the result by Howe([How80]).

Next we study the case where $\sigma(x,D)$ is a compact operator in $L^2(\mathbb{R}^d)$. Let $\lambda_1 \ge \lambda_2 \ge \cdots$ be the singular values of $\sigma(x,D)$. Let $\mu_1 \ge \mu_2 \ge \cdots$ be the non-increasing rearrangement of $\{M(m,n)\}_{m,n\in\mathbb{Z}^d}$.

Theorem 4.2.4 *We assume $\lim_{|x|+|\xi|\to\infty} M(x,\xi) = 0$. Then there is a positive constant C such that*

$$\lambda_k \le C\mu_k$$

for all $k \in \mathbf{N}$.

Using Theorem 4.2.4 we can give a sufficient condition on $\sigma(x,\xi)$ to ensure that $\sigma(x,D)$ belongs to the class $S_{p,q}$.

We recall the definition of the Lorentz spaces. For a measurable function f on $\mathbb{R}^d$ we define a non-increasing rearrangement f^* by

$$f^*(t) = \inf\{\lambda : |\{x \in \mathbb{R}^d : |f(x)| > \lambda\}| \leq t\}$$

for $0 < t < \infty$.

The Lorentz space $L^{p,q}(\mathbb{R}^d)$ is the set of all f such that

$$\int_0^\infty t^{q/p-1} f^*(t)^q \, dt < \infty, \quad 0 < q < \infty,$$

or

$$\sup_{0<t} t^{1/p} f^*(t) < \infty, \quad q = \infty.$$

Theorem 4.2.5 *Let $\sigma(x, D)$ be the pseudodifferential operator defined as above. For $0 < p < \infty, 0 < q \leq \infty$ we assume $M(x, \xi) \in L^{p,q}(\mathbb{R}^{2d})$. Then $T \in S_{p,q}$.*

Remark: If we assume $M(x, \xi) \in L^{p,q}$, then we conclude that $\lim_{|x|+|\xi|\to\infty} M(x, \xi) = 0$ by (4.2.2). Hence $\sigma(x, D)$ is compact on L^2 by Theorem 4.2.2.

Next we study eigenfunctions of $\sigma(x, D)$. We define $\{(m_i, n_i)\}_{i=1}^\infty \subset \mathbb{Z}^d \times \mathbb{Z}^d$ to be numbers which satisfy $\mu_i = M(m_i, n_i)$. Let $\phi_i = g_{m_i n_i}$.

Theorem 4.2.6 *Let $\sigma(x, D)$ be a pseudodifferential operator as in Theorem 4.2.2. Let $\lim_{|x|+|\xi|\to\infty} M(x, \xi) = 0$. Then there are positive constants C_1, C_2 and C_3 which satisfy the following. Suppose $\sigma(x, D)f = \lambda f$ for $f \in L^2, \|f\| = 1, \lambda \in \mathbf{C}$, and $|\lambda| \neq 0$. Then for N satisfying $\mu_{N+1} < C_1|\lambda|$, there are functions $H_i(x), i = 1, \ldots, N$ such that*

$$\lambda f = \sum_{i=1}^{N} \langle f, \phi_i \rangle M(m_i, n_i) H_i(x),$$

$$|H_i(x)| \leq \frac{C_2}{(1 + |x - m_i|)^{d+1}}, \tag{4.2.7}$$

and

$$|\mathcal{F}H_i(\xi)| \leq \frac{C_3}{(1 + |\xi - n_i|)^{d+1}}.$$

Using Theorem 4.2.6 we obtain the following estimates for eigenfunctions of $\sigma(x, D)$.

Theorem 4.2.7 *Let $\sigma(x,D)$ be a pseudodifferential operator as in Theorem 4.2.2. Let $\lim_{|x|+|\xi|\to\infty} M(x,\xi)=0$. Then there are positive constants C_1, C_2, C_3 and C_4 which satisfy the following. We suppose $\sigma(x,D)f=\lambda f$ for $f\in L^2, \|f\|=1, \lambda\in\mathbf{C}$, and $|\lambda|\neq 0$. Then for $p\in[1,\infty]$ we have*

$$\|f\|_p \leq \frac{C_1}{|\lambda|}\left\{\int\int_{M(x,\xi)\geq C_2|\lambda|} M(x,\xi)^2\, dx d\xi\right\}^{1/2}$$

and

$$\|\mathcal{F}f\|_p \leq \frac{C_3}{|\lambda|}\left\{\int\int_{M(x,\xi)\geq C_4|\lambda|} M(x,\xi)^2\, dx d\xi\right\}^{1/2}.$$

Example: Let $s>d/2$ and $a(x,\xi)=(1+|x|^2+|\xi|^2)^s$. Let A be the pseudodifferential operator with symbol $a(x,\xi)$. Then A has only discrete spectrum $\lambda_1\leq\lambda_2\leq\cdots\to\infty$ (cf. [Hel84], [Shu87]). Let $h_n, n=1,2,\ldots$ be the corresponding eigenfunctions. We have

$$Ah_n=\lambda_n h_n. \tag{4.2.8}$$

It turns out that there are pseudodifferential operators B and R with symbols $b(x,\xi)$ and $r(x,\xi)$, respectively, such that

$$BA=I+R,$$

where I is the identity operator and $b(x,\xi)$ satisfies

$$|\partial_x^\alpha\partial_\xi^\beta b(x,\xi)|\leq C_{\alpha,\beta}(1+|x|+|\xi|)^{-2s-|\alpha|-|\beta|}$$

for all $\alpha,\beta\in\mathbf{N}_0$, and $r(x,\xi)$ is a Schwartz function (cf. [Hel84],[Shu87]).

If we apply B to the equation (4.2.8) from the left, we have

$$\left(B-\lambda_n^{-1}R\right)h_n=\lambda_n^{-1}h_n$$

for $\lambda_n>0$. The symbol of $B-\lambda_n^{-1}R$ is $q(x,\xi)=b(x,\xi)-\lambda_n^{-1}r(x,\xi)$ which satisfies

$$|\partial_x^\alpha\partial_\xi^\beta q(x,\xi)|\leq C_{\alpha,\beta}(1+|x|+|\xi|)^{-2s-|\alpha|-|\beta|}$$

for all $\alpha,\beta\in\mathbf{N}_0$, where $C_{\alpha,\beta}$ is a constant which does not depend on n. Hence we can apply Theorem 4.2.7 and conclude that, for $p\in[1,\infty]$,

$$\|h_n\|_p\leq C\lambda_n$$

for all large n. The asymptotic behavior of the λ_n can be read off, for instance, using Theorem 4.3.6 and Corollary 4.3.7 in Section 3.

4.3 Analysis of elliptic pseudodifferential operators

In this section we give an analysis of global elliptic pseudodifferential operators using a local trigonometric orthonormal basis. The proofs of the following theorems are same as those in [Tac96].

Let $g(x)$ be the function defined in (4.2.4). We set

$$\psi_{mn}(x) = 2^{n/2} g(x-m)\Pi_{i=1}^{d} \cos\{(2n_i+1)\pi(x_i - m_i)/2\}$$

for $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d$, $n = (n_1, \ldots, n_d) \in \mathbf{N}_0^d$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. It turns out that $\{\psi_{mn}\}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$ ([AWW92]).

We consider the symbol $\sigma(x,\xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying the following conditions. We write $z = (x, \xi)$.

(S1) $\sigma(z) \geq 1$ and there are positive constants C, ε and K such that

$$C|z|^\varepsilon \leq \sigma(z) \quad (|z| \geq K, z \in \mathbb{R}^{2d}).$$

(S2) There are positive constants C and γ such that

$$\sigma(z+w) \leq C(1+|z|)^\gamma \sigma(w) \quad (z, w \in \mathbb{R}^{2d}).$$

(S3) There is a constant τ, $0 \leq \tau < 1$, such that for all $\alpha \in \mathbb{N}_0^{2d}, |\alpha| \geq 1$,

$$|\partial_z^\alpha \sigma(z)| \leq C_\alpha \sigma(z)^\tau \quad (z \in \mathbb{R}^{2d}),$$

where C_α is a positive constant.

(S4) We have

$$\sigma(x,\xi) \leq \sigma(x,\xi')$$

for all $x, \xi, \xi' \in \mathbb{R}^d$ such that $|\xi_i| \leq |\xi_i'|$, $i = 1, \ldots, d$, where $\xi = (\xi_1, \ldots, \xi_d)$ and $\xi' = (\xi_1', \ldots, \xi_d')$.

For example, the symbol $\sigma(x,\xi) = (1+|x|^2+|\xi|^2)^s$, $s > 0$, satisfies the conditions (S1), (S2), (S3) and (S4).

For $(m,n) \in \mathbb{Z}^d \times \mathbf{N}_0^d$ we set $\sigma_{mn} = \sigma(m, \pi(2n+\mathbf{1})/2)$ where $\mathbf{1} = (1, \ldots, 1)$.

We have the following theorem.

Theorem 4.3.1 *Suppose that a symbol $\sigma(x,\xi)$ satisfies the conditions (S1), (S2), (S3) and (S4). For every $\alpha, \beta \in \mathbf{N}_0$, there exists a positive constant C such that*

$$|\langle \sigma(x,D)\psi_{mn}, \psi_{m'n'}\rangle - \sigma_{mn}\langle \psi_{mn}, \psi_{m'n'}\rangle| \leq$$
$$\leq C \frac{(\sigma_{mn}\sigma_{m'n'})^{\tau/2}}{(1+|m-m'|^2)^\alpha (1+|n-n'|^2)^\beta}$$

for all $(m,n), (m',n') \in \mathbb{Z}^d \times \mathbf{N}_0^d$.

By Theorem 4.3.1 we have the following *a priori* estimate for $\sigma(x, D)$ on the Sobolev spaces.

Theorem 4.3.2 *Suppose that a symbol $\sigma(x,\xi)$ satisfies the conditions (S1), (S2), (S3) and (S4). Let $s \geq 0$. Then there are positive constants C_1, C_2 and C_3 such that*

$$C_1\|f\|_{H(w)} \leq \|\sigma(x,D)f\|_{\mathcal{H}^s} + C_2\|f\|_{L^2} \leq C_3\|f\|_{H(w)}$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$, where $\mathcal{H}^s$ denotes $H((1+|x|+|\xi|)^s)$ and $w(x,\xi) = \sigma(x,\xi)(1+|x|+|\xi|)^s$.

As a corollary we have the following result.

Corollary 4.3.3 *Suppose that a symbol $\sigma(x,\xi)$ satisfies the conditions (S1), (S2), (S3) and (S4). Suppose that $f \in L^2(\mathbb{R}^n)$ and $\sigma(x,D)f \in \mathcal{H}^s$ for $s \geq 0$. Then $f \in \mathcal{H}^{s+\varepsilon}$ where ε is the constant in (S1).*

By Corollary 4.3.3 we have the following theorem.

Theorem 4.3.4 *Suppose that a symbol $\sigma(x,\xi)$ satisfies the conditions (S1), (S2), (S3) and (S4). Let L_0 be the restriction of $\sigma(x,D)$ to $\mathcal{S}(\mathbb{R}^n)$. Then L_0 is essentially self-adjoint on $L^2(\mathbb{R}^n)$. Let L be a unique self-adjoint extension of L_0 on $L^2(\mathbb{R}^n)$. Then the domain of L is $H(\sigma)$.*

Theorem 4.3.5 *Suppose that a symbol $\sigma(x,\xi)$ satisfies the conditions (S1), (S2), (S3) and (S4). Then the self-adjoint operator L has only discrete spectrum, and the eigenfunctions of L belong to $\mathcal{S}(\mathbb{R}^n)$.*

Let $\lambda_1 \leq \lambda_2 \leq \cdots$ be eigenvalues of L and $N(\lambda)$ the number of λ_k less than or equal to $\lambda > 0$. We can prove upper and lower estimates for large λ_k.

Theorem 4.3.6 *Suppose that a symbol $\sigma(x,\xi)$ satisfies the conditions (S1), (S2), (S3) and (S4). We rearrange the set $\{\sigma_{mn}\}$ in the non-decreasing order and we denote them as $\{\mu_k\}_{k\in\mathbf{N}}$. Then there are positive constants C and K such that*

$$\mu_k - C\mu_k^\tau \leq \lambda_k \leq \mu_k + C\mu_k^\tau$$

for all $k \geq K$.

Corollary 4.3.7 *With the assumptions and notation of the previous theorem, we have*

$$\lim_{k\to\infty} \frac{\lambda_k}{\mu_k} = 1.$$

Finally we have the following estimate on $N(\lambda)$.

Theorem 4.3.8 *Suppose that a symbol $\sigma(x,\xi)$ satisfies the conditions (S1), (S2), (S3) and (S4). For $\lambda > 0$, let $M(\lambda)$ be the number of elements of the set $\{(m,n) \in \mathbb{Z}^d \times \mathbf{N}_0^d : \sigma_{mn} \leq \lambda\}$. Then there are positive constants C and R such that*

$$M(\lambda - C\lambda^{\tau}) \leq N(\lambda) \leq M(\lambda + C\lambda^{\tau})$$

for all $\lambda \geq R$.

4.4 Approximate diagonalization of $\sigma(x, D)$

In this section we prove Theorem 2.1. For simplicity we set $T = \sigma(x, D)$ throughout this section.

By (4.2.1)

$$\begin{aligned}
&\langle T g_{mn}, g_{m'n'} \rangle \\
&= e^{i(n+n')(m-m')/2} \int\int \sigma(x,\xi) e^{i(n-n')x - i(m-m')\xi} \\
&\quad \times W(g,g)\left(x - \frac{m+m'}{2}, \xi - \frac{n+n'}{2}\right) dx d\xi. \qquad (4.4.1)
\end{aligned}$$

Let

$$L_x = 1 - \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$$

and

$$L_\xi = 1 - \sum_{j=1}^{d} \frac{\partial^2}{\partial \xi_j^2}.$$

Then (4.4.1) is equal to

$$\begin{aligned}
&\frac{e^{i(n+n')(m-m')/2}}{(1 + \sum_j |m_j - m_j'|^2)^R (1 + \sum_j |n_j - n_j'|^2)^R} \\
&\quad \times \int\int \{L_x^R L_\xi^R e^{i(n-n')x - i(m-m')\xi}\} \sigma(x,\xi) \\
&\quad \times W(g,g)\left(x - \frac{m+m'}{2}, \xi - \frac{n+n'}{2}\right) dx d\xi. \qquad (4.4.2)
\end{aligned}$$

By integration by parts the integral in (4.4.2) is equal to

$$\int\int e^{i(n-n')x-i(m-m')\xi}(L_x)^R(L_\xi)^R\left\{\sigma(x,\xi)\right.$$
$$\left.\times W(g,g)\left(x-\frac{m+m'}{2},\xi-\frac{n+n'}{2}\right)\right\}dxd\xi.$$

Therefore we have

$$\begin{aligned}&|\langle Tg_{mn},g_{m'n'}\rangle|\\&\leq \frac{C}{(1+|m-m'|)^{2R}(1+|n-n'|)^{2R}}\sum\int\int|\partial_x^\beta\partial_\xi^\gamma\sigma(x,\xi)|\\&\qquad\times\left|\partial_x^{\beta'}\partial_\xi^{\gamma'}W(g,g)\left(x-\frac{m+m'}{2},\xi-\frac{n+n'}{2}\right)\right|dxd\xi,\end{aligned}\tag{4.4.3}$$

where the sum is taken over all $\beta,\beta',\gamma,\gamma'\in\mathbf{N}_0^n$ such that $|\beta|+|\beta'|\leq 2R,|\gamma|+|\gamma'|\leq 2R$.

By the assumptions (4.2.2) and (4.2.3) we have

$$|\partial_x^\beta\partial_\xi^\gamma\sigma(x,\xi)|\leq C\left(1+\left|x-\frac{m+m'}{2}\right|+\left|\xi-\frac{n+n'}{2}\right|\right)^s M\left(\frac{m+m'}{2},\frac{n+n'}{2}\right).$$

Since $W(g,g)(x,\xi)\in\mathcal{S}$, we get

$$\begin{aligned}&\left|\partial_x^{\beta'}\partial_\xi^{\gamma'}W(g,g)\left(x-\frac{m+m'}{2},\xi-\frac{n+n'}{2}\right)\right|\\&\leq C\left(1+\left|x-\frac{m+m'}{2}\right|+\left|\xi-\frac{n+n'}{2}\right|\right)^{-s-2n-1}\end{aligned}$$

Therefore (4.4.3) is bounded by

$$\frac{CM((m+m')/2,(n+n')/2)}{(1+|m-m'|)^{2R}(1+|n-n'|)^{2R}}.$$

By the assumption (4.2.2) on $M(x,\xi)$ the last term is bounded by

$$\frac{CM(m,n)}{(1+|m-m'|)^{2R-s}(1+|n-n'|)^{2R-s}}$$

or

$$\frac{CM(m',n')}{(1+|m-m'|)^{2R-s}(1+|n-n'|)^{2R-s}}.$$

4.5 The boundedness of $\sigma(x, D)$ on the Sobolev spaces

In this section we prove Theorems 2.2 and 2.3. We use the following characterization of $H(w)$ by $\{g_{mn}\}$, where we assume $w(x, \xi)$ satisfies condition (4.2.6).

We have the following proposition.

Proposition 4.5.1 *For $f \in H(w)$ we have an expansion*

$$f = (2\pi)^{-d} \sum_{m,n} \langle f, g_{mn} \rangle g_{mn}$$

in $H(w)$ and

$$\sum_{m,n} |\langle f, g_{mn} \rangle|^2 w(m,n)^2 \leq C \|f\|_{H(w)}^2.$$

Conversely let $\{c_{mn}\}_{m,n \in \mathbb{Z}^d}$ be numbers in $\mathbf{C}$ which satisfies

$$\sum_{m,n} |c_{mn}|^2 w(m,n)^2 < \infty.$$

Then there is a $f \in H(w)$ such that

$$f = (2\pi)^{-d} \sum_{m,n} c_{mn} g_{mn}$$

in $H(w)$ and

$$\|f\|_{H(w)}^2 \leq C \sum_{m,n} |c_{mn}|^2 w(m,n)^2.$$

Proposition 4.5.2 *For $f \in \mathcal{S}$ we have*

$$f = (2\pi)^{-d} \sum_{m,n} \langle f, g_{mn} \rangle g_{mn}$$

in $\mathcal{S}$.

For $f \in \mathcal{S}'$ we have

$$f = (2\pi)^{-d} \sum_{m,n} \langle f, g_{mn} \rangle g_{mn}$$

in $\mathcal{S}'$.

The proof of Propositions 4.5.1 and 4.5.2 are similar to that of Theorem 3.1 in [Tac94].

We prove Theorem 4.2.2. Let $T = \sigma(x, D)$. We shall show that there is a $C > 0$ such that

$$\sum_{m,n} |\langle Tf, g_{mn}\rangle|^2 w(m,n)^2 M(m,n)^{-2} \leq C\|f\|^2_{H(w)}$$

for all $f \in \mathcal{S}$.

By Proposition 4.5.2 we have

$$|\langle Tf, g_{mn}\rangle| = |(2\pi)^{-d} \sum_{m',n'} \langle f, g_{m'n'}\rangle\langle Tg_{m'n'}, g_{mn}\rangle|$$

$$\leq (2\pi)^{-d} \left(\sum_{m',n'} |\langle f, g_{m'n'}\rangle|^2 |\langle Tg_{m'n'}, g_{mn}\rangle| \right)^{1/2} \left(\sum_{m',n'} |\langle Tg_{m'n'}, g_{mn}\rangle| \right)^{1/2},$$

where we used Hölder's inequality in the last inequality.

By Theorem 4.2.1 we have

$$\sum_{m',n'} |\langle Tg_{m'n'}, g_{mn}\rangle| \leq C \sum_{m',n'} \frac{M(m,n)}{(1+|m-m'|)^{d+1}(1+|n-n'|)^{d+1}} \leq CM(m,n)$$

where we used the inequality $2R - s \geq d + 1$.

Therefore,

$$\begin{aligned} &\sum_{m,n} |\langle Tf, g_{mn}\rangle|^2 w(m,n)^2 M(m,n)^{-2} \\ \leq\ & C \sum_{m',n'} |\langle f, g_{m'n'}\rangle|^2 \sum_{m,n} |\langle Tg_{m'n'}, g_{mn}\rangle| w(m,n)^2 M(m,n)^{-1} \quad (4.5.1) \end{aligned}$$

By the assumption (4.2.6) we have

$$w(m,n) \leq C(1 + |m-m'| + |n-n'|)^t w(m', n').$$

Hence Theorem 4.2.1 leads to

$$\begin{aligned} &\sum_{m,n} |\langle Tg_{m'n'}, g_{mn}\rangle| w(m,n)^2 M(m,n)^{-1} \\ \leq\ & C \sum_{m,n} \frac{w(m',n')^2}{(1+|m-m'|)^{d+1}(1+|n-n'|)^{d+1}} \leq Cw(m',n')^2 \end{aligned}$$

where we used the inequality $2R - 2t \geq d + 1$. Therefore (4.5.1) is bounded by

$$C \sum_{m',n'} |\langle f, g_{m'n'}\rangle|^2 w(m',n')^2 \leq C\|f\|^2_{H(w)} < \infty.$$

Hence, by Proposition 4.5.1, there is a $h \in H(\tilde{w})$ such that

$$h = \sum_{m,n} \langle Tf, g_{mn} \rangle g_{mn} \quad \text{in } H(\tilde{w})$$

and

$$\|h\|_{H(\tilde{w})} \leq C\|f\|_{H(w)}.$$

By Proposition 4.5.2 we have

$$Tf = (2\pi)^{-d} \sum_{m,n} \langle Tf, g_{mn} \rangle g_{mn} \quad \text{in } \mathcal{S}'.$$

Hence we conclude $h = (2\pi)^d Tf$ and

$$\|Tf\|_{H(\tilde{w})} \leq C\|f\|_{H(w)}.$$

Proof of Theorem 4.2.3 If M is bounded, then T is a bounded operator on L^2 by Theorem 4.2.2.

We assume $\lim_{|x|+|\xi| \to \infty} M(x, \xi) = 0$. Then, by Theorem 4.2.2, T maps the set $\{f \in L^2 : \|f\| \leq 1\}$ into the set $\{f \in L^2 : \|f\|_{H(M^{-1})} \leq c_1\}$. This set is a subset of $U = \{f \in L^2 : \sum_{m,n} |\langle f, g_{mn} \rangle|^2 M(m,n)^{-2} \leq c_2\}$ by Proposition 4.5.1. It is enough to prove that U is totally bounded in L^2.
Let ε be an arbitrary positive number. Let $f \in U$. Since $\lim_{|x|+|\xi| \to \infty} M(x, \xi) = 0$, there is a positive number N such that

$$c_2 \sup_{|m|+|n|>N} M(m,n)^2 < \varepsilon.$$

Hence for every $f \in U$ we have

$$\sum_{|m|+|n|>N} |\langle f, g_{mn} \rangle|^2 < \varepsilon.$$

On the other hand every $f \in U$ satisfies

$$\sum_{|m|+|n| \leq N} |\langle f, g_{mn} \rangle|^2 \leq c_2 \max_{m,n} M(m,n)^2.$$

Hence there exists a finite set $\{\{a_{mn}^k\}_{|m|+|n| \leq N} : a_{mn}^k \in \mathbf{C}, k = 1, \ldots, K\}$ such that the set

$$\{f \in L^2 : \sum_{|m|+|n| \leq N} |\langle f, g_{mn} \rangle - a_{mn}^k|^2 < \varepsilon \text{ for some } k\}$$

contains U.

We set

$$h_k(x) = (2\pi)^{-d} \sum_{|m|+|n|\leq N} a^k_{mn} g_{mn}(x)$$

for $k = 1, \dots, K$. For every $f \in U$ there is a h_k such that

$$\|f - h_k\|^2 \leq c_3 \sum_{|m|+|n|\leq N} |\langle f, g_{mn}\rangle - a^k_{mn}|^2 + c_3 \sum_{|m|+|n|>N} |\langle f, g_{mn}\rangle|^2 < 2c_3\varepsilon.$$

Hence U is totally bounded in L^2.

4.6 Estimates for singular values

In this section we prove Theorem 2.4 and 2.5. If $\lim_{|x|+|\xi|\to\infty} M(x,\xi) = 0$, then $T = \sigma(x, D)$ is compact by Theorem 4.2.3. Let $\lambda_1 \geq \lambda_2 \geq \cdots$ be singular values of T. We recall that $\{\mu_i\}$ is the non-increasing rearrangement of $\{M(m,n)\}_{m,n\in\mathbb{Z}^d}$.

Proof of Theorem 4.2.4 We have, for $N \in \mathbf{N}_0$,

$$\lambda_{N+1} = \min_{P_N} \|T|_{P_N^\perp}\|,$$

where the minimum is taken over all subspaces P_N in L^2 whose dimension is less than or equal to N.

Let P_N be the linear space which is spanned by $\{\phi_1, \cdots, \phi_N\}$ where $\phi_i = g_{m_i,n_i}$.

For $f \in L^2$ and $f \perp P_N$, we have

$$\|Tf\|^2 \leq C \sum_i |\langle Tf, \phi_i\rangle|^2 = C \sum_{i=1}^\infty \left| \sum_{i'=N+1}^\infty \langle f, \phi_{i'}\rangle \langle T\phi_{i'}, \phi_i\rangle \right|^2$$

$$\leq C \sum_{i=1}^\infty \left(\sum_{i'=N+1}^\infty |\langle f, \phi_{i'}\rangle|^2 |\langle T\phi_{i'}, \phi_i\rangle| \right) \left(\sum_{i'=N+1}^\infty |\langle T\phi_{i'}, \phi_i\rangle| \right). \tag{4.6.1}$$

By Theorem 4.2.1

$$\sum_{i'=N+1}^\infty |\langle T\phi_{i'}, \phi_i\rangle| \leq C \sum_{i'=N+1}^\infty \frac{M(m_{i'}, n_{i'})}{(1 + |m_i - m_{i'}|)^{d+1}(1 + |n_i - n_{i'}|)^{d+1}}$$

$$\leq C\mu_{N+1} \sum_{i'=N+1}^\infty \frac{1}{(1 + |m_i - m_{i'}|)^{d+1}(1 + |n_i - n_{i'}|)^{d+1}} \leq C\mu_{N+1}.$$

Therefore (4.6.1) is bounded by

$$C\mu_{N+1} \sum_{i'=N+1}^{\infty} |\langle f, \phi_{i'} \rangle|^2 \sum_{i=1}^{\infty} |\langle T\phi_{i'}, \phi_i \rangle|. \tag{4.6.2}$$

By Theorem 4.2.1 we have

$$\sum_{i=1}^{\infty} |\langle T\phi_{i'}, \phi_i \rangle| \leq \sum_{i=1}^{\infty} \frac{CM(m_{i'}, n_{i'})}{(1+|m_i - m_{i'}|)^{d+1}(1+|n_i - n_{i'}|)^{d+1}} \leq C\mu_{N+1}$$

since $i' \geq N+1$.

Therefore, by (4.6.2),

$$\|Tf\|^2 \leq C\mu_{N+1}^2 \sum_{i'=N+1}^{\infty} |\langle f, \phi_{i'} \rangle|^2 \leq C\mu_{N+1}^2 \|f\|^2.$$

We conclude that

$$\lambda_{N+1} \leq C\mu_{N+1}.$$

Proof of Theorem 4.2.5 Let $\ell \in \mathbf{N}$ and $(m_\ell, n_\ell) \in \mathbb{Z}^d \times \mathbb{Z}^d$ where $m_\ell = (m_{\ell,1}, \dots, m_{\ell,d})$ and $n_\ell = (n_{\ell,1}, \dots, n_{\ell,d})$. We remark that if $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$ satisfies $|x_i - m_{\ell,i}| \leq 1, |\xi_i - n_{\ell,i}| \leq 1,\ i = 1, \dots, d$, then by (4.2.2) we have

$$M(x,\xi) \geq C(1+|x-m_\ell|+|\xi-n_\ell|)^{-s} M(m_\ell, n_\ell) > C'\mu_\ell.$$

Hence we have

$$|\{(x,\xi) : M(x,\xi) > C'\mu_\ell\}| > \ell. \tag{4.6.3}$$

Let

$$f^*(t) = \inf\{\lambda : |\{(x,\xi) : M(x,\xi) > \lambda\}| \leq t\}$$

for $t > 0$. If $0 < p \leq q < \infty$, then

$$\begin{aligned} \int_0^\infty t^{q/p-1} f^*(t)^q \, dt &= \sum_{k=0}^{\infty} \int_k^{k+1} t^{q/p-1} f^*(t)^q \, dt \\ &\geq \sum_{k=0}^{\infty} k^{q/p-1} f^*(k+1)^q \\ &\geq \sum_{k=0}^{\infty} k^{q/p-1} C'^q \mu_{k+1}^q, \end{aligned}$$

where we used (4.6.3). By Theorem 4.2.4 we conclude $T \in S_{p,q}$.

If $0 < q < p < \infty$, then we have

$$\begin{aligned}
\int_0^\infty t^{q/p-1} f^*(t)^q \, dt &\geq \sum_{k=0}^{\infty} (k+1)^{q/p-1} f^*(k+1)^q \\
&\geq \sum_{k=0}^{\infty} (k+1)^{q/p-1} C'^q \mu_{k+1}^q.
\end{aligned}$$

Hence $T \in S_{p,q}$.

If $q = \infty$, then the assumption $M \in L^{p,\infty}$ implies

$$\sup_{\lambda>0} \lambda^p |\{(x,\xi) : M(x,\xi) > \lambda\}| \leq C < \infty.$$

By (4.6.3) we have

$$\sup_k (C'\mu_k)^p k \leq C$$

which means $T \in S_{p,\infty}$ by Theorem 4.2.4.

4.7 Size estimates for eigenfunctions

The proof of Theorems 4.2.6 and 4.2.7 are similar to arguments in [Roc96].

Proof of Theorem 4.2.6 Since $f = (2\pi)^{-d} \sum_{i=1}^\infty \langle f, \phi_i \rangle \phi_i$ in L^2 and $T = \sigma(x, D)$ is bounded in L^2 by Theorem 4.2.3, we have

$$Tf = (2\pi)^{-d} \sum_{i=1}^{\infty} \langle f, \phi_i \rangle T\phi_i = T_1 f + T_2 f,$$

where

$$T_1 f = (2\pi)^{-d} \sum_{i=1}^{N} \langle f, \phi_i \rangle T\phi_i$$

and

$$T_2 f = (2\pi)^{-d} \sum_{i=N+1}^{\infty} \langle f, \phi_i \rangle T\phi_i.$$

We have the following lemma.

Lemma 4.7.1 *There is a $C > 0$ such that $\|T_2\| \leq C\mu_{N+1}$ for all N.*

The proof of Lemma 4.7.1 is same as that of Theorem 4.2.4.
Since $\lambda f = T_1 f + T_2 f$, we have

$$\lambda\left(I - \frac{T_2}{\lambda}\right) f = T_1 f.$$

Since $\lim_{i\to\infty} \mu_i = 0$, we have

$$\left\|\frac{T_2}{\lambda}\right\| \leq c_1 \frac{\mu_{N+1}}{|\lambda|} < 1 \tag{4.7.1}$$

if N is sufficiently large.

If (4.7.1) is satisfied, then the inverse of $I - \dfrac{T_2}{\lambda}$ exists and it is represented by

$$\left(I - \frac{T_2}{\lambda}\right)^{-1} = \sum_{q=0}^{\infty}\left(\frac{T_2}{\lambda}\right)^q.$$

Hence

$$\lambda f = \left(I - \frac{T_2}{\lambda}\right)^{-1} T_1 f = (2\pi)^{-d}\sum_{i=1}^{N}\langle f, \phi_i\rangle \left(I - \frac{T_2}{\lambda}\right)^{-1} T\phi_i.$$

We set

$$H_i(x) = (2\pi)^{-d} M(m_i, n_i)^{-1}\left(I - \frac{T_2}{\lambda}\right)^{-1} T\phi_i(x).$$

Then we have

$$\lambda f = \sum_{i=1}^{N}\langle f, \phi_i\rangle M(m_i, n_i) H_i(x).$$

We shall show that there are positive constants C_1 and C_2 such that

$$|H_i(x)| \leq \frac{C_1}{(1 + |x - m_i|)^{d+1}},$$

and

$$|\mathcal{F}H_i(\xi)| \leq \frac{C_2}{(1 + |\xi - n_i|)^{d+1}}.$$

We get

$$\left(I - \frac{T_2}{\lambda}\right)^{-1} T\phi_i(x) = \sum_{q=0}^{\infty}\left(\frac{T_2}{\lambda}\right)^q T\phi_i(x). \tag{4.7.2}$$

Now

$$\begin{aligned}
&T_2^q T\phi_i(x) \\
= \quad &(2\pi)^{-dq} \sum_{t_1,\dots,t_q=N+1}^{\infty} \langle T\phi_i, \phi_{t_1}\rangle \langle T\phi_{t_1}, \phi_{t_2}\rangle \cdots \langle T\phi_{t_{q-1}}, \phi_{t_q}\rangle T\phi_{t_q}(x).
\end{aligned}$$

Hence

$$\begin{aligned}
&|T_2^q T\phi_i(x)| \\
\leq \quad &\sum_{t_1,\dots,t_q=N+1}^{\infty} |\langle T\phi_i, \phi_{t_1}\rangle| \cdot |\langle T\phi_{t_1}, \phi_{t_2}\rangle| \cdots |\langle T\phi_{t_{q-1}}, \phi_{t_q}\rangle| \cdot |T\phi_{t_q}(x)|.
\end{aligned}$$

We use the following lemma.

Lemma 4.7.2 *There are positive constants C_1 and C_2 such that*

$$|Tg_{m,n}(x)| \leq C_1 \frac{M(m,n)}{(1+|x-m|)^{d+1}}$$

and

$$|\mathcal{F}(Tg_{m,n})(\xi)| \leq C_2 \frac{M(m,n)}{(1+|\xi-n|)^{d+1}}$$

for all m, n.

The proof of Lemma 4.7.2 will be given later.

By Theorem 4.2.1 and Lemma 4.7.2 we have

$$\begin{aligned}
&|T_2^q T\phi_i(x)| \\
\leq \quad &\sum_{t_1,\dots,t_q=N+1}^{\infty} \frac{c_2 M(m_i,n_i)}{(1+|m_i-m_{t_1}|)^{d+1}(1+|n_i-n_{t_1}|)^{d+1}} \\
&\times \frac{c_2 M(m_{t_1},n_{t_1})}{(1+|m_{t_1}-m_{t_2}|)^{d+1}(1+|n_{t_1}-n_{t_2}|)^{d+1}} \cdots \\
&\times \frac{c_2 M(m_{t_{q-1}},n_{t_{q-1}})}{(1+|m_{t_{q-1}}-m_{t_q}|)^{d+1}(1+|n_{t_{q-1}}-n_{t_q}|)^{d+1}} \frac{c_3 M(m_{t_q},n_{t_q})}{(1+|x-m_{t_q}|)^{d+1}}.
\end{aligned}$$

Hence

$$\begin{aligned}
&|T_2^q T\phi_i(x)| \\
\leq \quad &(c_2\mu_{N+1})^q c_3 M(m_i,n_i) \\
&\times \sum_{t_1,\dots,t_q=N+1}^{\infty} \frac{1}{(1+|m_i-m_{t_1}|)^{d+1}(1+|n_i-n_{t_1}|)^{d+1}} \cdots \\
&\times \frac{1}{(1+|m_{t_{q-1}}-m_{t_q}|)^{d+1}(1+|n_{t_{q-1}}-n_{t_q}|)^{d+1}} \frac{1}{(1+|x-m_{t_q}|)^{d+1}}.
\end{aligned}$$

We can prove the following lemma by an easy calculation.

Lemma 4.7.3 *There is a positive constant C such that*

$$\sum_{m\in\mathbb{Z}^d}\frac{1}{(1+|x-m|)^{d+1}(1+|m-y|)^{d+1}}\le\frac{C}{(1+|x-y|)^{d+1}}$$

for all $x,y\in\mathbb{R}^d$.

By Lemma 4.7.3 we have

$$|T_2^qT\phi_i(x)|\le(c_2\mu_{N+1})^qc_3c_4^{2q-1}c_5\frac{M(m_i,n_i)}{(1+|x-m_i|)^{d+1}},$$

where

$$c_5=\sum_{n\in\mathbb{Z}^d}\frac{1}{(1+|n|)^{d+1}}.$$

Therefore, by (4.7.2),

$$\left|\left(I-\frac{T_2}{\lambda}\right)^{-1}T\phi_i(x)\right|\le\sum_{q=0}^{\infty}(c_2c_4^2\mu_{N+1}|\lambda|^{-1})^qc_3c_4^{-1}c_5\frac{M(m_i,n_i)}{(1+|x-m_i|)^{d+1}}.$$

If N is sufficiently large enough to $c_2c_4^2\mu_{N+1}|\lambda|^{-1}\le 1/2$, then

$$|H_i(x)|\le\frac{C}{(1+|x-m_i|)^{d+1}}.$$

The calculation for $\mathcal{F}H_i(\xi)$ is similar.

Proof of Theorem 4.2.7 Let $R=[(d+s)/2]+1$. We can express Tg_{mn} as an oscillatory integral, that is,

$$Tg_{mn}(x)=\lim_{\varepsilon\to 0+}(2\pi)^{-d}\int\int e^{i(x-y)\xi}\chi(\varepsilon\xi)a\left(\frac{x+y}{2},\xi\right)g(y-m)e^{iny}\,dyd\xi.$$

where χ is a function in $\mathcal{S}$ such that $\chi(0)=1$(cf. [Hel84]). The last integral is equal to

$$\lim_{\varepsilon\to 0+}(2\pi)^{-d}\int\int e^{i\{x(\xi+n)-\xi(y+m)\}}\chi(\varepsilon(\xi+n))a\left(\frac{x+y+m}{2},\xi+n\right)g(y)dyd\xi.$$

By integration by parts the above limit is equal to

$$e^{inx}\int\int e^{i(x-y-m)\xi}\frac{1}{(1+|x-y-m|^2)^R}(1-\Delta_\xi)^R\left\{\frac{1}{(1+|\xi|^2)^R}\right.$$
$$\left.\times(1-\Delta_y)^R\left\{a(\frac{x+y+m}{2},\xi+n)g(y)\right\}\right\}dyd\xi,$$

where we can apply the dominated convergence theorem by the assumptions (4.2.2) and (4.2.3).

Hence, by (4.2.3), we have

$$|Tg_{mn}(x)| \leq C \int\int \frac{M((x+y+m)/2, \xi+n)}{(1+|x-y-m|^2)^R(1+|y|)^{2R}(1+|\xi|^2)^R}\, dyd\xi.$$

By (4.2.2) the last term is bounded by

$$M(m,n) \int\int \frac{(1+|x+y-m|/2+|\xi|)^s}{(1+|x-y-m|^2)^R(1+|y|)^{2R}(1+|\xi|^2)^R}\, dyd\xi$$
$$\leq CM(m,n) \int\int \frac{1}{(1+|x-y-m|)^{2R-s}(1+|y|)^{2R-s}(1+|\xi|)^{2R-s}}\, dyd\xi.$$

The last term is estimated by

$$C\frac{M(m,n)}{(1+|x-m|)^{2R-s}} \leq C\frac{M(m,n)}{(1+|x-m|)^{d+1}}.$$

Next we estimate $\mathcal{F}Tg_{m,n}(\xi)$. By the Parseval formula we have for $\varphi \in \mathcal{S}$

$$\begin{aligned} &\langle \mathcal{F}Tg_{mn}, \mathcal{F}\varphi\rangle = \langle Tg_{mn}, \varphi\rangle \\ = &\int\int \sigma(x,\xi) W(g_{mn},\varphi)(x,\xi)\, dxd\xi \\ = &\int\int \sigma(-\xi,x) W(\mathcal{F}g_{mn},\mathcal{F}\varphi)(x,\xi)\, dxd\xi, \end{aligned}$$

where we used the equality

$$W(\mathcal{F}f,\mathcal{F}g)(x,\xi) = W(f,g)(-\xi,x).$$

Hence we can estimate $\mathcal{F}Tg_{mn}$ by similar arguments as in the case of Tg_{mn}.

Proof of Theorem 4.2.7

Let N be the integer which satisfies $\mu_{N+1} < C_1|\lambda| \leq \mu_N$, where C_1 is the constant in Theorem 4.2.6.

By Theorem 4.2.6 we have

$$f(\xi) = \lambda^{-1}\sum_{i=1}^{N}\langle f,\phi_i\rangle M(m_i,n_i)H_i(\xi).$$

Hence

$$||f||_p \leq C|\lambda|^{-1}\sum_{i=1}^{N}|\langle f,\phi_i\rangle| M(m_i,n_i),$$

where we used the estimate (4.2.7). By Hölder's inequality the last term is bounded by

$$C|\lambda|^{-1}\left(\sum_{i=1}^{N}|\langle f,\phi_i\rangle|^2\right)^{1/2}\left(\sum_{i=1}^{N}M(m_i,n_i)^2\right)^{1/2}.$$

Since $\{\phi_i\}_{i=1}^{\infty}$ is a frame of L^2, we have

$$\sum_{i=1}^{N}|\langle f,\phi_i\rangle|^2 \leq C\|f\|^2.$$

For $i=1,\dots,N$ let

$$Q_i=\{(x,\xi): m_{i,j}\leq x_j<m_{i,j}+1, n_{i,j}\leq \xi_j<n_{i,j}+1, j=1,\dots,n\},$$

where we denote $m_i=(m_{i,1},\dots,m_{i,n}), n_i=(n_{i,1},\dots,n_{i,n})$.

By (4.2.2) we get

$$M(m_i,n_i)\leq C(1+|x-m_i|+|\xi-n_i|)^s M(x,\xi)\leq C'M(x,\xi) \tag{4.7.3}$$

for $(x,\xi)\in Q_i$. Therefore we have

$$\sum_{i=1}^{N}M(m_i,n_i)^2\leq \sum_{i=1}^{N}\int\int_{Q_i}M(m_i,n_i)^2\,dxd\xi$$

$$\leq C\sum_{i=1}^{N}\int\int_{Q_i}M(x,\xi)^2\,dxd\xi\leq C\int\int_{\cup_{i=1}^{N}Q_i}M(x,\xi)^2\,dxd\xi.$$

By (4.7.3) we have

$$M(x,\xi)\geq C'^{-1}M(m_i,n_i)\geq C'^{-1}\mu_N\geq C|\lambda|$$

for $(x,\xi)\in Q_i$ and $i=1,\dots,N$. Therefore

$$\|f\|_p\leq \frac{C}{|\lambda|}\left\{\int\int_{M(x,\xi)\geq C'|\lambda|}M(x,\xi)^2\,dxd\xi\right\}^{1/2}.$$

The estimate for $\|\mathcal{F}f\|_p$ is similar.

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5

Perturbation of frames and applications to Gabor frames

Ole Christensen

ABSTRACT – We prove a stability result for frames, with a perturbation condition that generalizes the Paley-Wiener condition used in the context of bases. We show how the result can be applied to different types of perturbations of a Gabor frame $\{e^{imbx}f(x-na)\}_{m,n\in\mathbb{Z}}$ of $L^2(\mathbb{R})$, e.g., perturbation of f or na.

5.1 Introduction

There is a long tradition for studying the stability of bases under perturbation. Already Paley and Wiener [PW34] proved that if $\{f_i\}_{i=1}^{\infty}$is an orthonormal basis for $L^2(a,b)$, $\{g_i\}_{i=1}^{\infty} \subseteq L^2(a,b)$ and

$$\exists\lambda \in [0;1[: \ ||\sum_{i=1}^{n} c_i(f_i - g_i)|| \leq \lambda||\sum_{i=1}^{n} c_i f_i||$$

for all finite sequences $c_1, c_2, ...c_n (n \in N)$, then $\{g_i\}_{i=1}^{\infty}$ is a Riesz basis for $L^2(a;b)$. Boas [Boa48] showed that the same holds in every Hilbert space, even if $\{f_i\}_{i=1}^{\infty}$ is only assumed to be a Riesz basis.

Paley and Wiener were interested in expansions of functions in $L^2(a,b)$ in what is now called nonharmonic Fourier series. The question is, which conditions on the sequence $\{\lambda_n\}_{n=-\infty}^{\infty}$ ensures that every function $f \in L^2(a,b)$ has an expansion

$$f(x) = \sum_{n=-\infty}^{\infty} c_n(f)e^{i\lambda_n x} \ \text{ in } L^2(a,b),$$

where $\{c_n(f)\}_{n=1}^{\infty} \in \ell^2(\mathbb{Z})$?

The celebrated *Kadec 1/4-theorem* gives a nice answer to this question, cf. [You80]: if $sup_{n\in\mathbb{Z}}|\lambda_n - n| < 1/4$, then

$$||\sum_{n=-\infty}^{\infty} c_n(e^{inx} - e^{i\lambda_n x})||_{L^2(-\pi,\pi)} \leq \lambda[\sum_{n=-\infty}^{\infty} |c_n|^2]^{1/2}$$
$$= \lambda||\sum_{n=-\infty}^{\infty} c_n e^{inx}||_{L^2(-\pi,\pi)}, \ \forall\{c_n\} \in \ell^2(\mathbb{Z})$$

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for some $\lambda < 1$, so since $\{e^{inx}\}_{n=-\infty}^{\infty}$ is an orthonormal basis for $L^2(-\pi,\pi)$, the Paley-Wiener theorem above shows that $\{e^{i\lambda_n x}\}_{n=-\infty}^{\infty}$ is a Riesz basis for $L^2(-\pi,\pi)$. The result is even sharp, in the sense that counter examples exists if $sup_{n\in\mathbb{Z}}|\lambda_n - n| = 1/4$.

For applications, the interest in such results are most easy described via bases for "signal spaces" (i.e., subspaces of $L^2(\mathbb{R})$) consisting of translated versions $\{f(x-\lambda_n)\}$ of a single function $f \in L^2(\mathbb{R})$. Such bases are strongly connected with bases of exponentials, since the Fourier transformation interchanges translation by modulation. Usually, the sampling sequence $\{\lambda_n\}$ are some measured data (e.g. of time), so, due to measurement errors, they are not always exact. The perturbation results described here shows that if $\{f(x-\lambda_n)\}$ form a basis and we know that the "exact data" $\{\mu_n\}$ are close enough to $\{\lambda_n\}$, then $\{f(x-\mu_n)\}$ is also a basis. So even with measurement errors we have a strict mathematical conclusion about the exact data, at least when the errors are known to be sufficiently small.

The Paley-Wiener theorem has been applied in many other interesting situations. For example, we propose that the reader interested in wavelets consults [Sei92b].

Unfortunately, there do not exist bases of the form $\{f(x-\lambda_n)\}$ for $L^2(\mathbb{R})$, cf. [OZ92]. But, combining the operators "translation" and "modulation", there exist bases of the form $\{e^{imbx}f(x-na)\}_{m,n\in\mathbb{Z}}$, at least for some choices of $a, b \in \mathbb{R}$. Families of functions of this form are called *Gabor families* or *Weyl-Heisenberg families.*

One of the striking features of a basis $\{f_i\}_{i=1}^{\infty}$ is that every element f in the underlying space has a unique representation $f = \sum_{i=1}^{\infty} c_i f_i$. But in many situations it is preferable that the family $\{f_i\}_{i=1}^{\infty}$ contains more elements than needed to ensure the existence of such a representation of every element f. Without going in details (for which we refer to [Ben96a]) we mention that this kind of "overcompleteness" of $\{f_i\}_{i=1}^{\infty}$ makes the representation $f = \sum_{i=1}^{\infty} c_i f_i$ more stable against noise on the coefficients $\{c_i\}_{i=1}^{\infty}$.

This is one of the motivations for the introduction of *frames.* Roughly speaking, a frame for a Hilbert space $\mathcal{H}$ is a family $\{f_i\}_{i=1}^{\infty}$ with the property that every element $\mathcal{H}$ can be written as a (infinite) linear combination of the frame elements, with square summable coefficients. The precise definition was given by Duffin and Schaeffer in their paper [DS52], actually in the context of nonharmonic Fourier series. We discuss the basic facts about frames in Section 5.2.

Our aim here is to extend the perturbation results of Paley-Wiener to frames. The first results of this type were due to Heil, cf. [Hei90]. Since then, more general theoretical results have been proved; we discuss those in Section 5.3. We shall not present the most general versions, but concentrate on the ones which we need in order to get conclusions about Gabor frames. This is the topic in Section 5.4, where we present results of Favier and Zalik, which appeared in [FZ95a]. It is shown that the theoretical results can be applied to all kinds of perturbations, e.g. perturbation of the "mother wavelet" f or the "sampling points" $\{(na, mb)\}_{m,n\in\mathbb{Z}}$. We close in Section 5.5 by a short discussion about frames in Banach spaces.

5.2 Frames and Riesz bases

In this paper $\mathcal{H}$ denotes a separable Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ linear in the first entry. I is a countable index set, and $\ell^2(I)$ is the set of square summable sequences on I.
A family $\{f_i\}_{i\in I} \subseteq \mathcal{H}$ is a *Riesz basis* for $\mathcal{H}$ is

$$\exists A, B > 0: \ A\sum_{i\in I} |c_i|^2 \leq ||\sum_{i\in I} c_i f_i||^2 \leq B\sum_{i\in I} |c_i|^2, \ \forall \{c_i\} \in \ell^2(I).$$

A family $\{f_i\}_{i\in I} \subseteq \mathcal{H}$ is called a *Bessel sequence* if

$$\exists B > 0: \ \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B||f||^2, \ \ \forall f \in \mathcal{H}.$$

If $\{f_i\}_{i\in I}$ is a Bessel sequence, we can define a bounded linear operator

$$T: \mathcal{H} \to \ell^2(I), \ \ Tf = \{\langle f, f_i\rangle\}_{i\in I}.$$

Clearly $||T|| \leq \sqrt{B}$. The adjoint operator is

$$T^*: \ell^2(I) \to \mathcal{H}, \ T^*\{c_i\}_{i\in I} = \sum_{i\in I} c_i f_i.$$

It is not difficult to prove that the series converges unconditionally for every $f \in \mathcal{H}$. On the other hand, if the series appearing in the expression for T^* defines a bounded linear operator on $\mathcal{H}$ with norm at most $\sqrt{B}$, then $\{f_i\}_{i\in I}$ is a Bessel sequence with bound B.
Having the discussion from the introduction in mind, we want the operator T^* to be surjective. This is the case if $\{f_i\}_{i\in I}$ is a Riesz basis, or more general, a *frame*, meaning that

$$\exists A, B > 0: \ A||f||^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B||f||^2, \ \ \forall f \in \mathcal{H}.$$

A, B are called *frame bounds*. In the case of a frame, the *frame operator* is defined by

$$S: \mathcal{H} \to \mathcal{H}, \ \ Sf = T^*Tf = \sum_{i\in I} \langle f, f_i\rangle f_i.$$

It is elementary to prove that S is invertible and selfadjoint, which leads to the desired surjectivity of T^*:

$$f = SS^{-1}f = \sum_{i \in I} \langle f, S^{-1}f_i \rangle f_i, \quad \forall f \in \mathcal{H}.$$

This is called the *frame decomposition.* The possibility of representing every $f \in \mathcal{H}$ in this way is the reason for the interest in frames. For more details we refer to [Dau92, HW89]. Here we only present the facts which we need in the present paper:
1) If $\{f_i\}_{i \in I}$ is a frame with bounds A, B, then $\{S^{-1}f_i\}_{i \in I}$ is a frame, with bounds $\frac{1}{B}, \frac{1}{A}$. A proof can be found in [HW89]. $\{S^{-1}f_i\}_{i \in I}$ is usually called the *dual frame.*
2) We already explained the reason for considering families which are not bases. The relation between Riesz bases and frames is as follows:

$$\{f_i\}_{i \in I} \text{ is a Riesz basis}$$
$$\Updownarrow$$
$$\{f_i\}_{i \in I} \text{ is a frame and } \sum_{i \in I} c_i f_i = 0, \{c_i\}_{i \in I} \in \ell^2(I) \Rightarrow c_i = 0, \forall i \in I.$$

In words: a Riesz basis is a frame where the elements are *ω-independent.*
3) A Bessel sequence $\{f_i\}_{i=1}^{\infty}$ is a frame for $\overline{span}\{f_i\}_{i \in I}$ if and only if the operator T^* has closed range. This is proved in [Chr95b].

Frames which can be generated by letting a group representation act on a single element f in the Hilbert space plays a special role. Let π denote a strongly continuous unitary representation of the locally compact group $\mathbf{G}$ on $\mathcal{H}$. A frame of the form $\{\pi(x_i)f\}_{i \in I}$, where $\{x_i\}_{i \in I} \in \mathbf{G}$, $f \in \mathcal{H}$, is called a *coherent frame.* It is interesting that most of the important frames are coherent. For example the Gabor families appear by letting the Schrödinger representation act on a lattice in the Heisenberg group:

Example: Let $\Pi = \{z \in \mathbb{C} \mid |z| = 1\}$. As a set, the Heisenberg group is $G = \mathbb{R} \times \mathbb{R} \times \Pi$ and the Schrödinger representation of G onto $L^2(\mathbb{R})$ is

$$[\pi(y, z, t)g](x) = te^{iz(x-y)}g(x - y), \quad g \in L^2(\mathbb{R}), \quad t \in \Pi, \ x, y, z \in \mathbb{R}.$$

Corresponding to the *lattice parameters* $a, b > 0$ we get the Gabor family $\{e^{imbx}f(x - na)\}$ by letting the representation act on the *lattice* $\{(na, mb, e^{iabmn})\}_{m,n \in \mathbb{Z}}$. A Gabor family in $L^2(\mathbb{R}^d)$ (which we define in section 4) can in the same way be generated using the action of the Schrödinger representation on the d-dimensional Heisenberg group. For more information we refer to [Chr96a].

5.3 Perturbation of frames

In this section we state the abstract perturbation theorems for frames. For notational convenience we use the natural numbers as index set. This does not restrict the generality of the results, since the frame decomposition is known to converge unconditionally. The theorem below first appeared in [Chr95c].

Theorem 5.3.1 *Let $\{f_i\}_{i=1}^{\infty}$ be a frame with bounds A, B. Let $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ and assume that there exist constants $\lambda, \mu \geq 0$ such that $\lambda + \frac{\mu}{\sqrt{A}} < 1$ and*

$$||\sum_{i=1}^{n} c_i(f_i - g_i)|| \leq \lambda||\sum_{i=1}^{n} c_i f_i|| + \mu(\sum_{i=1}^{n} |c_i|^2)^{1/2} \qquad (5.3.1)$$

for all scalars $c_1, ..c_n$ $(n \in N)$. Then $\{g_i\}_{i=1}^{\infty}$ is a frame with bounds

$$A(1 - (\lambda + \frac{\mu}{\sqrt{A}}))^2, \quad B(1 + \lambda + \frac{\mu}{\sqrt{B}})^2.$$

Proof: $\{f_i\}_{i=1}^{\infty}$ is a frame, so we can define a bounded linear operator

$$U : \ell^2(N) \to \mathcal{H}, \quad U\{c_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} c_i f_i,$$

and $||U|| \leq \sqrt{B}$. The condition (5.3.1) implies that

$$||\sum_{i=1}^{n} c_i g_i|| \leq ||\sum_{i=1}^{n} c_i(f_i - g_i)|| + ||\sum_{i=1}^{n} c_i f_i||$$
$$\leq (1 + \lambda)||\sum_{i=1}^{n} c_i f_i|| + \mu(\sum_{i=1}^{n} |c_i|^2)^{1/2}, \ \forall \{c_i\}_{i=1}^{n}.$$

A Cauchy sequence argument now shows that $\sum_{i=1}^{\infty} c_i g_i$ actually converges for all $\{c_i\}_{i=1}^{\infty} \in \ell^2(N)$ and that the similar estimate as above holds for those sequences. So we can define an operator

$$V : \ell^2(N) \to \mathcal{H}, \quad V\{c_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} c_i g_i,$$

and

$$||V\{c_i\}_{i=1}^{\infty}|| \leq (1 + \lambda)||U\{c_i\}_{i=1}^{\infty}|| + \mu||\{c_i\}_{i=1}^{\infty}||$$
$$\leq ((1 + \lambda)\sqrt{B} + \mu)||\{c_i\}_{i=1}^{\infty}||, \ \forall \{c_i\}_{i=1}^{\infty} \in \ell^2(N).$$

This estimate shows that $\{g_i\}_{i=1}^{\infty}$ is a Bessel sequence with the upper bound

$$((1+\lambda)\sqrt{B}+\mu)^2 = B(1+\lambda+\frac{\mu}{\sqrt{B}})^2$$

Now we prove that $\{g_i\}_{i=1}^{\infty}$ has a lower frame bound. Since $\{f_i\}_{i=1}^{\infty}$ is a frame, the frame operator $S = UU^*$ is invertible, and we can define

$$U^{\dagger}: \mathcal{H} \to \ell^2(N), \;\; U^{\dagger}f := U^*(UU^*)^{-1}f = \{\langle f, (UU^*)^{-1}f_i\rangle\}_{i=1}^{\infty}.$$

$\{(UU^*)^{-1}f_i\}_{i=1}^{\infty}$ is the dual frame of $\{f_i\}_{i=1}^{\infty}$, so $||U^{\dagger}f|| \leq \frac{1}{\sqrt{A}}||f||, \forall f \in \mathcal{H}$. Using (5.3.1) on the sequence $\{c_i\}_{i=1}^{\infty} = U^{\dagger}f$ we obtain that

$$||f - VU^{\dagger}f|| \leq \lambda||f|| + \mu||U^{\dagger}f|| \leq (\lambda + \frac{\mu}{\sqrt{A}})||f||, \;\; \forall f \in \mathcal{H}$$

Since $\lambda + \frac{\mu}{\sqrt{A}} < 1$, the operator $VU^{\dagger}$ is invertible, and

$$||VU^{\dagger}|| \leq 1 + \lambda + \frac{\mu}{\sqrt{A}}, \quad ||(VU^{\dagger})^{-1}|| \leq \frac{1}{1-(\lambda+\frac{\mu}{\sqrt{A}})}.$$

Every $f \in \mathcal{H}$ can be written as

$$f = VU^{\dagger}(VU^{\dagger})^{-1}f = \sum_{i=1}^{\infty}\langle (VU^{\dagger})^{-1}f, (UU^*)^{-1}f_i\rangle g_i$$

implying that

$$\begin{aligned}
||f||^4 = \langle f, f\rangle^2 &= |\sum_{i=1}^{\infty}\langle (VU^{\dagger})^{-1}f, (UU^*)^{-1}f_i\rangle\langle g_i, f\rangle|^2 \\
&\leq \sum_{i=1}^{\infty}|\langle (VU^{\dagger})^{-1}f, (UU^*)^{-1}f_i\rangle|^2 \cdot \sum_{i=1}^{\infty}|\langle g_i, f\rangle|^2 \\
&\leq \frac{1}{A}||(VU^{\dagger})^{-1}f||^2 \cdot \sum_{i=1}^{\infty}|\langle g_i, f\rangle|^2 \\
&\leq \frac{1}{A}[\frac{1}{1-(\lambda+\frac{\mu}{\sqrt{A}})}]^2 \cdot ||f||^2 \cdot \sum_{i=1}^{\infty}|\langle g_i, f\rangle|^2, \;\; \forall f \in \mathcal{H}.
\end{aligned}$$

So

$$\sum_{i=1}^{\infty}|\langle g_i, f\rangle|^2 \geq A(1-(\lambda+\frac{\mu}{\sqrt{A}}))^2||f||^2, \;\; \forall f \in \mathcal{H}.$$

$\square$

Remarks:
1) Easy examples shows that the condition $\lambda + \frac{\mu}{\sqrt{A}} < 1$ is needed. For example, let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$ (in particular, a frame with $A = 1$) and define $g_i := e_i + e_{i+1},\ i \in N$. Since

$$||\sum_{i=1}^{n} c_i(e_i - g_i)|| = ||\sum_{i=1}^{n} c_i e_{i+1}|| = ||\sum_{i=1}^{n} c_i e_i|| = [\sum_{i=1}^{n} |c_i|^2]^{1/2},$$

this example corresponds to $\lambda = 1, \mu = 0$ (or $\mu = 1, \lambda = 0$). But with $f_n := \sum_{i=1}^{n}(-1)^{i-1}e_i$, we have $||f_n||^2 = n$, and

$$\langle f_n, g_n\rangle = 1,\ \langle f_n, g_i\rangle = 0,\ i \neq n$$

So

$$\sum_{i=1}^{\infty} |\langle f_n, g_i\rangle|^2 = 1 = \frac{1}{n}||f_n||^2,$$

and we conclude that $\{g_i\}_{i=1}^{\infty}$ is not a frame.
2) Various generalizations of Theorem 5.3.1 are possible. In a joint paper with Peter G. Casazza [CC97] it is shown that the conclusion holds even if a term $\gamma \cdot ||\sum_{i=1}^{n} c_i g_i||, \gamma < 1$, is added on the right hand side of (5.3.1). This is surprising from the point of view that 1) shows the optimality of Theorem 5.3.1 in the sense that the bounds on λ, μ are needed. Our proofs rely on an extension of the classical result that an operator V on $\mathcal{H}$ is invertible if it is close enough to the identity I in the sense that $||I - V|| < 1$.

If $\{f_i\}_{i=1}^{\infty}$ is only a frame for a subspace of $\mathcal{H}$, Theorem 5.3.1 does not hold anymore. However, a more abstract approach relying on the theory for pseudo-inverse operators gives a result which is in some sense only a slight modification. We refer to [Chr96c] for details.
3) Our perturbation condition is extended compared with the Paley-Wiener condition in that we have included the μ term. This term is essential in the context of frames. With $\mu = 0$ the perturbation condition (5.3.1) implies that

$$\sum_{i=1}^{\infty} c_i f_i = 0 \Leftrightarrow \sum_{i=1}^{\infty} c_i g_i = 0. \tag{5.3.2}$$

This is automatically satisfied if $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ are Riesz bases, but for frames it would be a too strong restriction only to consider families $\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty}$ satisfying this condition.

Theorem 5.3.1 leads immediately to an extension of Kadec's 1/4–theorem:

Corollary 5.3.2 *Let $\{\lambda_n\}_{n=1}^{\infty}, \{\mu_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$. Suppose that $\{e^{i\lambda_n x}\}_{n=1}^{\infty}$ is a frame for $L^2(-\pi,\pi)$ with bounds A, B. If there exists a constant $L < 1/4$ such that*

$$|\mu_n - \lambda_n| \leq L \quad and \quad 1 - cos\pi L + sin\pi L < \sqrt{\frac{A}{B}},$$

then $\{e^{i\mu_n x}\}_{n=1}^{\infty}$ *is a frame for* $L^2(-\pi,\pi)$ *with bounds*

$$A(1-\sqrt{\frac{B}{A}}(1-cos\pi L+sin\pi L))^2, B(2-cos\pi L+sin\pi L)^2 .$$

Proof: This is just a trivial adjustment of the standard proof of Kadec's 1/4–theorem. Observing that $||\sum c_n e^{i\lambda_n(\cdot)}||^2 \leq B\sum|c_n|^2$ for all finite sequences $\{c_n\}$, the estimates from [You80], p.42 gives that

$$||\sum c_n(e^{i\lambda_n(\cdot)}-e^{i\mu_n(\cdot)})|| \leq \sqrt{B}(1-cos\pi L+sin\pi L)(\sum|c_n|^2)^{1/2}. \tag{5.3.3}$$

So by Theorem 5.3.1 with $\lambda=0, \mu=\sqrt{B}(1-cos\pi L+sin\pi L)$ we have that $\{e^{i\mu_n x}\}_{n=1}^{\infty}$ is a frame with the bounds above. ❑

This version of Kadec's 1/4–theorem is closely related to [DS, lemma 3], which gives the same conclusion if

$$L < \frac{ln(\frac{A}{4B(e-1)}+1)^{1/2}}{\pi},$$

however without estimates of the frame bounds. For $A=B$ this condition is $L<0.1173....$, where we have the usual Kadec condition $L<1/4$.

The following result is also a consequence of Theorem 5.3.1, but since a direct proof is easier, cf. [CH97], we include it:

Theorem 5.3.3 *Let* $\{f_i\}_{i=1}^{\infty}$ *be a frame with bounds* A, B *and let* $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$*. If*

$$\exists R<A: \quad \sum_{i=1}^{\infty}|\langle f, f_i-g_i\rangle|^2 \leq R\cdot||f||^2, \ \forall f\in\mathcal{H},$$

then $\{g_i\}_{i=1}^{\infty}$ *is a frame with bounds* $A(1-\sqrt{\frac{R}{A}})^2, B(1+\sqrt{\frac{R}{B}})^2$.

Proof: By the triangle inequality in $\ell^2(N)$,

$$[\sum_{i=1}^{\infty}|\langle f,g_i\rangle|^2]^{1/2} \geq [\sum_{i=1}^{\infty}|\langle f,f_i\rangle|^2]^{1/2} - [\sum_{i=1}^{\infty}|\langle f,g_i-f_i\rangle|^2]^{1/2}$$

$$\geq (\sqrt{A}-\sqrt{R})||f||, \ \forall f\in\mathcal{H},$$

from which the lower bound follows. The upper estimate is similar. ❑

Despite its simplicity, Theorem 5.3.3 is very useful in applications, cf. Section 5.4 of the present chapter (and [FZ95a] for further applications). In terms of the operators U, V from the proof of Theorem 5.3.1, the condition in Theorem 5.3.3 is

$$||U^*f-V^*f||^2 \leq R\cdot||f||^2, \ \forall f\in\mathcal{H}$$

with $R < A$, which is equivalent to the norm estimate $||U - V|| < \sqrt{A}$.

Usually it is much more difficult to prove that a family $\{f_i\}_{i=1}^{\infty}$ satisfies the lower frame condition than the upper one. In some sense, Theorem 5.3.3 shows that the "difficult" problem reduces to the "easy" one in the case of perturbation: if $\{f_i\}_{i=1}^{\infty}$ is a frame, then $\{g_i\}_{i=1}^{\infty}$ is also a frame if $\{f_i - g_i\}_{i=1}^{\infty}$ satisfies the upper condition with a sufficiently small bound.

Corollary 5.3.4 *Let $\{f_i\}_{i=1}^{\infty}$ be a frame with bounds A, B and let $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$. If*

$$R := \sum_{i=1}^{\infty} ||f_i - g_i||^2 < A,$$

then $\{g_i\}_{i=1}^{\infty}$ is a frame with bounds $A(1 - \sqrt{\frac{R}{A}})^2, B(1 + \sqrt{\frac{R}{B}})^2$.

Corollary 5.3.4 was actually proved before Theorem 5.3.1 and 5.3.3 cf. [Chr94]. One of the reasons for looking for better results was that Corollary 5.3.4 can not be applied if the element f in a coherent frame $\{\pi(x_i)f\}_{i=1}^{\infty}$ is perturbed, since

$$\sum_{i=1}^{\infty} ||\pi(x_i)f - \pi(x_i)g||^2 = \sum_{i=1}^{\infty} ||f - g||^2 = \infty \text{ if } f \neq g.$$

However, Theorem 5.3.1 and 5.3.3 apply, cf. Section 5.4. On the other hand, Corollary 5.3.4 can be applied if the set of group elements $\{x_i\}$ are perturbed. Since π is assumed to be strongly continuous, the mapping

$$x \to ||\pi(x)f - \pi(x_i)f||$$

is continuous so we can obtain that $\sum_{i=1}^{\infty} ||\pi(x_i)f - \pi(y_i)f||^2 < A$ by chosing $\{y_i\}$ sufficiently close to $\{x_i\}$. We refer again to Section 5.4 for concrete calculations in the case of Gabor frames.

If $\{f_i\}_{i=1}^{\infty}$ is a Riesz basis and the condition in Theorem 5.3.1 is satisfied, then $\{g_i\}_{i=1}^{\infty}$ is a Riesz basis; this can be proved by standard techniques. Here we give a slightly more involved argument, showing that if $\{f_i\}_{i=1}^{\infty}$ consists of a Riesz basis plus finitely many elements, then the same holds for $\{g_i\}_{i=1}^{\infty}$. The results and proofs are joint work with Casazza.

Frames $\{f_i\}_{i=1}^{\infty}$ consisting of a Riesz basis $\{f_i\}_{i\in N-\sigma}$ plus finitely many elements $\{f_i\}_{i\in\sigma}$ are called *near-Riesz bases*, a notion introduced by Holub [Hol94]. The number of elements in σ is called the *excess*. Holub showed that a frame $\{f_i\}_{i=1}^{\infty}$ is a near-Riesz basis if and only if the kernel of the corresponding operator T^* (cf. Section 5.2) is finite-dimensional, in which case the dimension of this space is equal to the excess. If the kernel is infinite-dimensional, two things can happen: the frame consists of a Riesz basis plus infinite many elements, (in which case we talk about a frame with infinite excess) or the frame does not contains a Riesz basis at all.

We need a result which is important in itself. It was originally proved in [CC98]:

Proposition 5.3.5 *Let $\{f_i\}_{i=1}^{\infty}$ be a near-Riesz basis with lower bound A. Given $\epsilon > 0$, there exists a finite set $J \subseteq N$ such that $\{f_i\}_{i\in N-J}$ is a Riesz basis for its closed span, with lower bound $A-\epsilon$.*

Proof: By reindexing we can write $\{f_i\}_{i=1}^{\infty} = \{f_i\}_{i=1}^{n} \cup \{f_i\}_{i=n+1}^{\infty}$, where $\{f_i\}_{i=n+1}^{\infty}$ is a Riesz basis for $\mathcal{H}$. Let $d(\cdot,\cdot)$ denote the distance inside $\mathcal{H}$ (i.e., $d(f,E) = inf_{g\in E}||f-g||$ for $f \in \mathcal{H}, E \subseteq \mathcal{H}$). Given $\epsilon > 0$, choose a number $m > n$ such that

$$d(f_j, span\{f_i\}_{i=n+1}^{m}) < \sqrt{\frac{\epsilon}{n}}, \quad j = 1, ...n.$$

We want to show that $\{f_i\}_{i=m+1}^{\infty}$ is a Riesz basis for its closed span, with lower bound $A-\epsilon$. Let P denote the orthogonal projection onto $\overline{span}\{f_i\}_{i=n+1}^{m}$. Since $||\sum c_i f_i|| \geq ||\sum c_i (I-P) f_i||$ for all sequences $\{c_i\} \in \ell^2(N)$, it suffices to show that $\{(I-P)f_i\}_{i=m+1}^{\infty}$ satisfies the lower Riesz basis condition with bound $A-\epsilon$. Let $f \in (I-P)\mathcal{H}$. Then

$$\sum_{i=m+1}^{\infty} |\langle f,(I-P)f_i\rangle|^2 = \sum_{i=1}^{\infty} |\langle f,(I-P)f_i\rangle|^2 - \sum_{i=1}^{n} |\langle f,(I-P)f_i\rangle|^2$$

$$\geq A||f||^2 - \sum_{i=1}^{n} ||f||^2 \cdot ||(I-P)f_i||^2 \geq (A-\epsilon)||f||^2.$$

Now we only have to show that $\{(I-P)f_i\}_{i=m+1}^{\infty}$ is ω-independent. But if $\sum_{i=m+1}^{\infty} c_i(I-P)f_i = 0$, then $\sum_{i=m+1}^{\infty} c_i f_i = P\sum_{i=m+1}^{\infty} c_i f_i$, implying that both sides are equal to zero, since $P\sum_{i=m+1}^{\infty} c_i f_i \in span\{f_i\}_{i=n+1}^{m}$ and $\{f_i\}_{i=n+1}^{\infty}$ is ω-independent. Therefore $c_i = 0$ for all i. ❑

Theorem 5.3.6 *The assumptions in Theorem 5.3.1 imply that*

$$\{f_i\}_{i=1}^{\infty} \textit{ is a near-Riesz basis} \Leftrightarrow \{g_i\}_{i=1}^{\infty} \textit{ is a near-Riesz basis}$$

in which case $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ have the same excess.

Proof: First assume that $\{f_i\}_{i=1}^{\infty}$ is a near-Riesz basis with excess n. Let m be choosen as in the proof of Proposition 5.3.5, corresponding to an ϵ satisfying the condition $\lambda + \frac{\mu}{\sqrt{A-\epsilon}} < 1$. Let Q denote the orthogonal projection onto $\overline{span}\{f_i\}_{i=m+1}^{\infty}$. Then every element $f \in \mathcal{H}$ can be written $f = (I-Q)f + Qf = (I-Q)f + \sum_{i=m+1}^{\infty} c_i f_i$, for some coefficients c_i. Now define an operator $W : \mathcal{H} \to \mathcal{H}$ by

$$Wf = f, \; f \in \overline{span}\{f_i\}_{i=m+1}^{\infty}{}^{\perp}, \quad Wf_i = g_i, \; i \geq m+1.$$

W is bounded. Given $f \in \mathcal{H}$ we choose a representation as above. Then

$$||(I-W)f|| = ||\sum_{i=m+1}^{\infty} c_i(f_i - g_i)|| \le \lambda \cdot ||\sum_{i=m+1}^{\infty} c_i f_i|| + \mu \cdot \sqrt{\sum_{i=m+1}^{\infty} |c_i|^2}$$

$$\le (\lambda + \frac{\mu}{\sqrt{A-\epsilon}})||\sum_{i=m+1}^{\infty} c_i f_i|| = (\lambda + \frac{\mu}{\sqrt{A-\epsilon}})||Qf|| \le (\lambda + \frac{\mu}{\sqrt{A-\epsilon}})||f||.$$

It follows, that W is an isomorphism of $\mathcal{H}$ onto $\mathcal{H}$. So $\{g_i\}_{i=m+1}^{\infty}$ is a Riesz basis for its closed span, and

$$dim(\overline{span}\{g_i\}_{i=m+1}^{\infty}{}^{\perp}) = dim(\overline{span}\{f_i\}_{i=m+1}^{\infty}{}^{\perp}).$$

So we need to add the same number of elements to $\{f_i\}_{i=m+1}^{\infty}$ and to $\{g_i\}_{i=m+1}^{\infty}$ in order to obtain Riesz bases for $\mathcal{H}$, i.e., $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ have the same excess.

Now assume that $\{g_i\}_{i=1}^{\infty}$ is a near-Riesz basis. By reindexing we may again assume that $\{g_i\}_{i=n+1}^{\infty}$ is a Riesz basis for $\mathcal{H}$. Define a bounded operator $U : \mathcal{H} \to \mathcal{H}$ by $Uf := \sum_{i=1}^{\infty} \langle f, S^{-1}f_i\rangle g_i$. This operator already appeared in the proof of Theorem 5.3.1 (there, the notation was "$VU^{\dagger}$") and it was shown that U is an isomorphism of $\mathcal{H}$ onto $\mathcal{H}$. If we define $U_n : \mathcal{H} \to \mathcal{H}$ by $U_n f = \sum_{i=n+1}^{\infty} \langle f, S^{-1}f_i\rangle g_i$, then this operator has a range with finite codimension in $\mathcal{H}$, which we will write as

$$codim_{\mathcal{H}}(\mathbb{R}_{U_n}) < \infty.$$

Now let $\{e_i\}_{i=1}^{\infty}$ be the natural basis for $\ell^2(N)$, i.e., e_i is the sequence with 1 in the i'th entry, otherwise 0. There exists a bounded invertible operator $V : \mathcal{H} \to \overline{span}\{e_i\}_{i=n+1}^{\infty}$ such that $Vg_i = e_i$ for $i \ge n+1$, and clearly

$$codim_{\overline{span}\{e_i\}_{i=n+1}^{\infty}}(\mathbb{R}_{VU_n}) < \infty.$$

Observe that $VU_n f = \sum_{i=n+1}^{\infty} \langle f, S^{-1}f_i\rangle e_i = \{\langle f, S^{-1}f_i\rangle\}_{i=n+1}^{\infty}$. So

$$(VU_n)^*\{c_i\} = \sum_{i=n+1}^{\infty} c_i S^{-1} f_i = S^{-1} \sum_{i=n+1}^{\infty} c_i f_i.$$

Since $\mathbb{R}_{VU_n}^{\perp} = N_{(VU_n)^*}$ has finite dimension, also $\{c_i\}_{i=n+1}^{\infty} \longmapsto \sum_{i=n+1}^{\infty} c_i f_i$ has a finite dimensional kernel. Therefore

$$T^* : \ell^2(N) \to \mathcal{H}, \quad T^*\{c_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} c_i f_i$$

has a finite dimensional kernel, and now the theorem of Holub implies that $\{f_i\}_{i=1}^{\infty}$ is a near-Riesz basis. By the first part of the Theorem the two frames $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ now have the same excess, and the proof is complete. ❑

Remark: It is important that Theorem 5.3.6 is restricted to the case of finite excess. In fact, as shown in [CC98] there exist examples where the perturbation condition is satisfied and $\{f_i\}_{i=1}^{\infty}$ has infinite excess, but where $\{g_i\}_{i=1}^{\infty}$ does not contain a Riesz basis. A different proof of Theorem 5.3.6 can be obtained using Theorem V.1.6, and V.3.6 from [Gol66].

For families $\{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty}$ we again look at the operators U, V from the proof of Theorem 5.3.1. We already observed that the condition in Theorem 5.3.3 is equivalent to a norm condition on the operator $U - V$, i.e., the operator $\{c_i\}_{i=1}^{\infty} \to \sum_{i=1}^{\infty} c_i(f_i - g_i)$. On could equally well look at other conditions on this operator. In the sequel we assume that $U - V$ is compact, and our result is an immediate consequence of the following lemma, which is a special case of [Gol66], Cor. V.2.2. Remember that an operator U with closed range $\mathbb{R}_U$ has an *index* if the kernel N_U is finite dimensional or if the codimension of the range is finite dimensional; in this case the index is

$$ind(U) = dim(N_U) - codim(\mathbb{R}_U).$$

Lemma 5.3.7 *Let $U : \ell^2(N) \to \mathcal{H}$ be a bounded operator with closed range. Suppose furthermore that U has an index. If $K : \ell^2(N) \to \mathcal{H}$ is compact, then $U + K$ has closed range, and $ind(U + K) = ind(U)$.*

If we combine Lemma 5.3.7 with Remark 3 in Section 5.2 we obtain

Theorem 5.3.8 *Let $\{f_i\}_{i=1}^{\infty}$ be a frame for $\overline{span}\{f_i\}_{i=1}^{\infty}$ and assume that*

$$U : \ell^2(N) \to \mathcal{H}, \ U\{c_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} c_i f_i$$

has an index. If $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ and

$$K\{c_i\}_{i=1}^{\infty} := \sum_{i=1}^{\infty} c_i(g_i - f_i)$$

defines a compact operator from $\ell^2(N)$ into $\mathcal{H}$, then $\{g_i\}_{i=1}^{\infty}$ is a frame for $\overline{span}\{g_i\}_{i=1}^{\infty}$, and $ind(U + K) = ind(U)$.

The index of the operator U has a nice interpretation in terms of the frame $\{f_i\}_{i=1}^{\infty}$: the case $\dim(N_U) < \infty$ means that $\{f_i\}_{i=1}^{\infty}$ is a near-Riesz basis for $\overline{span}\{f_i\}_{i=1}^{\infty}$, and $\text{codim}(\mathbb{R}_U) < \infty$ means that $\{f_i\}_{i=1}^{\infty}$ is a frame for a space of finite codimension. Theorem 5.3.8 says that a compact perturbation might increase the dimension of the spanned space, but the excess will decrease with the same amount. This general result can be illustrated by an easy example in $\mathbb{R}^3$: Let $\{e_i\}_{i=1}^3$ be an orthonormal basis for $\mathbb{R}^3$ and let

$$\{f_i\}_{i=1}^3 = \{e_1, 0, 0\}, \quad \{g_i\}_{i=1}^3 = \{e_1, \frac{1}{2}e_2, 0\}.$$

$\{f_i\}_{i=1}^3$ spans a one-dimensional subspace, and the excess is 2. $\{g_i\}_{i=1}^3$ is a compact perturbation of $\{f_i\}_{i=1}^3$ in the sense of Theorem 5.3.8, $\{g_i\}_{i=1}^3$ spans a 2-dimensional subspace, and the excess is 1.
Even if $\{f_i\}_{i=1}^\infty$ is a frame for $\mathcal{H}$, it can happen that a compact perturbation only gives a frame for a subspace of $\mathcal{H}$. If $\{f_i\}_{i=1}^\infty$ is an orthonormal basis for $\mathcal{H}$, this is the case with the perturbed family $\{g_i\}_{i=1}^\infty = \{0\} \cup \{f_i\}_{i=2}^\infty$. The interested reader can consult [CH97, CC98] for direct proofs of Theorem 5.3.8.

5.4 Applications to Gabor frames

We begin with some definitions. Let $a, b \in \mathbb{R}^d$, and assume that all coordinates $a_j, b_j, j = 1, ..d$, are positive. We define

$$\frac{1}{a} := (\frac{1}{a_1}, \frac{1}{a_2}, ..., \frac{1}{a_d})$$

and

$$P := \prod_{j=1}^d b_j.$$

Given $g \in L^2(\mathbb{R}^d)$, $n, m \in \mathbb{Z}^d$, we define

$$g_{n,m}(x) := e^{2\pi i \langle mb, x\rangle} g(x - na), \quad x \in \mathbb{R}^d,$$

where $mb = (m_1b_1, m_2b_2, .., m_db_d)$, $na = (n_1a_1, n_2a_2, ..., n_da_d)$, i is the complex unit number, and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{R}^d$. The rectangle in $\mathbb{R}^d$ determined by the points $\alpha, \beta \in \mathbb{R}^d$ will be denoted by $[\alpha; \beta]$.

The *Gabor family* corresponding to the triple (g, a, b) is $\{g_{n,m}\}_{m,n\in\mathbb{Z}^d}$. We begin by mentioning conditions implying that $\{g_{n,m}\}_{m,n\in\mathbb{Z}^d}$ is a Gabor frame. The proof can be found in, e.g., [BW94]. For more general results we refer to [Wal89b]

Theorem 5.4.1 *Let $g \in L^2(\mathbb{R}^d)$, $a, b \in \mathbb{R}^d$. If* $\text{supp}(g) \subseteq [0, \frac{1}{b}]$ *and there exist numbers $A, B > 0$ such that $A \leq \sum_{n\in\mathbb{Z}^d} |g(x - na)|^2 \leq B$ a.e., then $\{g_{n,m}\}_{m,n\in\mathbb{Z}^d}$ is a Gabor frame for $L^2(\mathbb{R}^d)$with bounds AP^{-1}, BP^{-1}.*

The first question we want to study is the following:
Suppose that $\{g_{n,m}\}_{m,n\in\mathbb{Z}^d}$ is a frame. Which conditions on h (or $h - g$) implies that $\{h_{n,m}\}_{m,n\in\mathbb{Z}^d}$ is a frame? As mentioned in Section 5.3, this question was one of the main motivations for developing the perturbation

theory. The sufficient condition we present here is due to Favier and Zalik. Our Theorem 5.4.2 is a reformulation of Theorem 15 from [FZ95a] and our proof is an extension of the proof of Theorem 4.3 from [Chr96b].

The result concerns the case where $h - g$ has compact support. Since every compact set in $\mathbb{R}^d$ can be covered by translated versions of $[0, \frac{1}{b}]$, we will only consider compact sets of the form $\cup_{j=1}^k([0, \frac{1}{b}] + c_j)$, where $c_j \in \mathbb{R}^d$.

Theorem 5.4.2 *Let $\{g_{n,m}\}_{m,n\in\mathbb{Z}^d}$ be a frame for $L^2(\mathbb{R}^d)$ with bounds A, B. Let $h \in L^2(\mathbb{R}^d)$. Suppose that*

$$\mathrm{supp}(g-h) \subseteq \cup_{j=1}^k([0, \frac{1}{b}] + c_j) \tag{5.4.1}$$

$$M_j := \operatorname*{ess\,sup}_{x\in[0,\frac{1}{b}]+c_j} \sum_{n\in\mathbb{Z}^d} |(g-h)(x-na)|^2 < \infty, \ j = 1, .., k. \tag{5.4.2}$$

Let $\delta := \sum_{j=1}^k {M_j}^{1/2}$ .If $\delta \le P^{1/2}A^{1/2}$, then $\{h_{n,m}\}_{m,n\in\mathbb{Z}^d}$ is a frame with bounds $A(1-\frac{\delta}{\sqrt{AP}})^2, B(1+\frac{\delta}{\sqrt{BP}})^2$.

Proof: First assume that $k = 1$, i.e., that $supp(g-h) \subseteq [0, \frac{1}{b}] + c_1$. Let $f \in L^2(\mathbb{R}^d)$. The function $x \to f(x) \cdot T_{na}(g-h)(x) := f(x) \cdot (g-h)(x-na)$ is in $L^2(\mathbb{R}^d)$ and has support in $[0, \frac{1}{b}] + c_1 + na$. Since $\{\prod_{j=1}^d b_j^{1/2} e^{2\pi i\langle mb,x\rangle}\}_{m\in\mathbb{Z}^d}$ is an orthonormal basis for $L^2([0, \frac{1}{b}] + c_1 + na)$, we have

$$\begin{aligned} &\sum_{m,n\in\mathbb{Z}^d} |\langle f, e^{2\pi i\langle mb,\cdot\rangle}(g-h)(\cdot - na)\rangle|^2 \\ &= \frac{1}{\prod_{j=1}^d b_j} \sum_{n\in\mathbb{Z}^d} \int_{[0,\frac{1}{b}]+c_1+na} |f(x)\overline{T_{na}(g-h)(x)}|^2 dx \\ &= \frac{1}{P} \sum_{n\in\mathbb{Z}^d} \int_{\mathbb{R}} |f(x)\overline{T_{na}(g-h)(x)}|^2 dx \le \frac{M_1}{P}||f||^2. \end{aligned}$$

Now, by Theorem 5.3.3, $\{h_{n,m}\}_{m,n\in\mathbb{Z}^d}$ is a frame if $\frac{M_1}{P} < A$. In the general case, write $h-g = \sum_{j=1}^k (h-g)\cdot \mathbf{1}_{[0,\frac{1}{b}]+c_j}$, where $\mathbf{1}_{[0,\frac{1}{b}]+c_j}$ denotes the indicator function for the interval $[0, \frac{1}{b}] + c_j$. Define the operators $K_j, \ j = 1, ..., k$ by

$$K_j : \ell^2(\mathbb{Z}^d \times \mathbb{Z}^d) \to \mathcal{H}, \ \ K_j\{c_{n,m}\} = \sum_{m,n\in\mathbb{Z}^d} c_{n,m}[(h-g)\cdot \mathbf{1}_{[0,\frac{1}{b}]+c_j}]_{n,m}.$$

Then

$$||\sum_{m,n\in\mathbb{Z}^d} c_{n,m}(h-g)_{n,m}|| = ||\sum_{j=1}^k K_j\{c_{n,m}\}||$$

$$\le \sum_{j=1}^k ||K_j\{c_{n,m}\}|| = \sum_{j=1}^k || \sum_{m,n\in\mathbb{Z}^d} c_{n,m}[(h-g)\cdot \mathbf{1}_{[0,\frac{1}{b}]+c_j}]_{n,m}||.$$

By the first part of the proof, $\{(h-g)\cdot \mathbf{1}_{[0,\frac{1}{b}]+c_j}\}_{m,n\in\mathbb{Z}^d}$ is a Bessel sequence in $L^2(\mathbb{R}^d)$, with upper bound $\frac{M_j}{P}$. So

$$||\sum_{m,n\in\mathbb{Z}^d} c_{n,m}(h-g)_{n,m}|| \le \sum_{j=1}^{k}\sqrt{\frac{M_j}{P}}\cdot ||f||, \quad \forall f\in\mathcal{H}.$$

That is, $\{(h-g)_{n,m}\}_{m,n\in\mathbb{Z}^d}$ is a Bessel sequence with bound $\frac{\delta^2}{P}$. Now the theorem follows from Theorem 5.3.3. □

Favier and Zalik remarked that if $\{g_{n,m}\}_{m,n\in\mathbb{Z}^d}$ is a Riesz basis and the condition in Theorem 5.4.2 is satisfied, then $\{h_{n,m}\}_{m,n\in\mathbb{Z}^d}$ is also a Riesz basis. This is a consequence of Theorem 5.3.6.

Now we turn to the question about perturbation of the lattice points $\{(na,mb)\}$. We only prove that the frame property is preserved if na is replaced by $\lambda_n a$ (under some conditions on λ_n), but since the Fourier transformation interchanges the role of translation and modulation, it is clear that a similar result holds if mb is perturbed.

We use the notation $g_{n,m}$ as in the beginning of the section. Corresponding to a perturbation $\{\lambda_n\}_{n\in\mathbb{Z}^d}\subseteq\mathbb{R}^d$ of $\{n\}_{n\in\mathbb{Z}^d}$ we define

$$g^p_{n,m}(x) = e^{2\pi i\langle mb,x\rangle}g(x-\lambda_n a).$$

Theorem 5.4.3 is again due to Favier and Zalik:

Theorem 5.4.3 *Let $\{g_{n,m}\}_{m,n\in\mathbb{Z}^d}$ be a Gabor frame for $L^2(\mathbb{R}^d)$, with bounds A,B. Assume that g has support in the box $[\beta,\gamma]$, where $\beta,\gamma\in\mathbb{R}^d$ and $0<\gamma-\beta<\frac{1}{b}$ coordinate-wise. Assume furthermore that there exist $\alpha\in\mathbb{R}^d$ such that*

$$0<\alpha<\frac{1}{2}(\frac{1}{b}+\beta-\gamma), \quad |\lambda_n-n|\le\frac{\alpha}{a} \text{ coordinatewise.}$$

If

$$M_n := esssup_{x\in\mathbb{R}^d}|g(x-na)-g(x-\lambda_n a)|^2<\infty, \ \forall n\in\mathbb{Z}^d,$$

and $M:=\sum_{n\in\mathbb{Z}^d}M_n<AP$, then $\{g^p_{n,m}\}_{m,n\in\mathbb{Z}^d}$ is a Gabor frame with bounds

$$A(1-\sqrt{\frac{M}{AP}})^2, B(1+\sqrt{\frac{M}{BP}})^2.$$

Proof: Let $I := [\beta, \beta + \frac{1}{b}]$, $I_r = I + \frac{r}{b}$ for $r \in \mathbb{Z}^d$. For $f \in L^2(\mathbb{R}^d)$ we have

$$\begin{aligned}
&\sum_{m,n\in\mathbb{Z}^d} |\langle f, g_{n,m} - g^p_{n,m}\rangle|^2 \\
&= \sum_{n\in\mathbb{Z}^d}\sum_{m\in\mathbb{Z}^d} |\int f(x)\overline{e^{2\pi i\langle mb,x\rangle}(g(x-na) - g(x-\lambda_n a))}dx|^2 \\
&= \sum_{n\in\mathbb{Z}^d}\sum_{m\in\mathbb{Z}^d} |\sum_{r\in\mathbb{Z}^d}\int_{I_r} f(x)\overline{e^{2\pi i\langle mb,x\rangle}(g(x-na) - g(x-\lambda_n a))}dx|^2.
\end{aligned}$$

By the change of variable $x \to x - \frac{r}{b}$, we see that

$$\begin{aligned}
&\int_{I_r} f(x)\overline{e^{2\pi i\langle mb,x\rangle}(g(x-na) - g(x-\lambda_n a))}dx \\
&= e^{2\pi i\langle mb,\frac{r}{b}\rangle}\int_I f(x-\frac{r}{b})\overline{e^{2\pi i\langle mb,x\rangle}(g(x-na-\frac{r}{b}) - g(x-\lambda_n a-\frac{r}{b}))}dx \\
&= \int_I f(x-\frac{r}{b})\overline{e^{2\pi i\langle mb,x\rangle}(g(x-na-\frac{r}{b}) - g(x-\lambda_n a-\frac{r}{b}))}dx.
\end{aligned}$$

Define for $n \in \mathbb{Z}^d$ the function

$$F_n(x) := \sum_{r\in\mathbb{Z}^d} f(x-\frac{r}{b})\overline{(g(x-na-\frac{r}{b}) - g(x-\lambda_n a-\frac{r}{b}))}.$$

Observe that the sum is well defined: for every x, we only get a finite sum because g has compact support. Since $F_n \in L^2(I)$ and $\{\prod_{j=1}^d b_j^{\frac{1}{2}} e^{2\pi i\langle mb,x\rangle}\}_{m\in\mathbb{Z}^d}$ is an orthonormal basis for $L^2(I)$, we have

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}^d}\sum_{m\in\mathbb{Z}^d} |\sum_{r\in\mathbb{Z}^d}\int_{I_r} f(x)\overline{e^{2\pi i\langle mb,x\rangle}(g(x-na) - g(x-\lambda_n a))}dx|^2 \\
&= \sum_{n\in\mathbb{Z}^d}\sum_{m\in\mathbb{Z}^d} |\sum_{r\in\mathbb{Z}^d}\int_I f(x-\frac{r}{b})\overline{e^{2\pi i\langle mb,x\rangle}(g(x-na-\frac{r}{b}) - g(x-\lambda_n a-\frac{r}{b}))}dx|^2 \\
&= \sum_{n\in\mathbb{Z}^d}\sum_{m\in\mathbb{Z}^d} |\int_I e^{-2\pi i\langle mb,x\rangle}F_n(x)|^2 dx = \prod_{j=1}^d b_j^{-1}\sum_{n\in\mathbb{Z}^d}\int_I |F_n(x)|^2 dx.
\end{aligned}$$

The reader can check that the assumptions on α implies that the function $x \to g(x-na-\frac{r}{b}) - g(x-\lambda_n a-\frac{r}{b})$ has support in the box $[\beta-\alpha, \gamma+\alpha] + na + \frac{r}{b}$. Since the "size" of this box is $2\alpha < \frac{1}{b} + \beta - \gamma < \frac{1}{b}$, those functions have disjoint supports for different values of r. Therefore

$$\sum_{m,n\in\mathbb{Z}^d} |\langle f, g_{n,m} - g^p_{n,m}\rangle|^2 = \prod_{j=1}^d b_j^{-1} \sum_{n\in\mathbb{Z}^d} \int_I |F_n(x)|^2 dx$$

$$= P^{-1} \sum_{n\in\mathbb{Z}^d} \int_I |\sum_{r\in\mathbb{Z}^d} f(x - \frac{r}{b})\overline{(g(x - na - \frac{r}{b}) - g(x - \lambda_n a - \frac{r}{b}))}|^2 dx$$

$$P^{-1} \sum_{n\in\mathbb{Z}^d} \sum_{r\in\mathbb{Z}^d} \int_I |f(x - \frac{r}{b})(g(x - na - \frac{r}{b}) - g(x - \lambda_n a - \frac{r}{b}))|^2 dx$$

$$\leq P^{-1} \sum_{n\in\mathbb{Z}^d} M_n \sum_{r\in\mathbb{Z}^d} \int_I |f(x - \frac{r}{b})|^2 dx = P^{-1} \sum_{n\in\mathbb{Z}^d} M_n \cdot ||f||^2.$$

Now the result follows from Theorem 5.3.3 ❑

5.5 Banach frames

We close with a few remarks concerning Banach frames. Gröchenig introduced this notion in [Grö91], motivated by his joint work with Feichtinger about atomic decomposition of Banach spaces [FG88, FG89a]. Those papers shows that there exist natural examples of expansions in a large number of Banach spaces, which shares many of the properties of frame decompositions in Hilbert spaces. The paper [Grö91] also contains perturbation results formulated in this abstract setting.

The analogue of Theorem 5.3.1 holds for Banach frames, cf. [CC97]; for other results we refer to [CH97]. Banach frames can even be important in the Hilbert space setting. It was pointed out by Walnut [Wal89b] that for weighted L^2-spaces with nontrivial weights, no Gabor frame $\{g_{n,m}\}$ exist in the Hilbert space sense. But Banach frames of this form exists. For a more quantitative result we refer to [CH97].

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6

Aspects of Gabor analysis on locally compact abelian groups

Karlheinz Gröchenig

ABSTRACT – Motivated by recent formulations of Gabor theory for periodic and for discrete signals, this chapter develops several aspects of Gabor theory on locally compact groups. First an uncertainty principle in terms of the short time Fourier transform is derived (Lieb's inequalities). It captures the intuition that any signal occupies a region in the time-frequency plane of area at least one. Secondly, the Zak transform, introduced on locally compact abelian groups already by A. Weil, is used to analyze Gabor frames in the case of integer-oversampled lattices in the time-frequency plane. In this context it is observed that the Balian–Low theorem depends on the group structure and that the known versions do not hold for discrete and compact groups. In the final section a notion for the density of lattices in defined and necessary conditions for lattices in the time-frequency plane to generate Gabor frames are derived.

6.1 Introduction

Gabor theory tries to express and understand the time-frequency behavior of functions in terms of time-frequency shifts $e^{2\pi i y t} g(t-x)$ of a single (or several) function(s) g. According to the prevailing fashion g is called a window, Gabor atom, Gabor logon, time-frequency template, Weyl-Heisenberg atom, Gabor wavelet among others.

Usually Gabor theory is investigated on $\mathbb{R}$, but recently other settings have been looked at. Working with discrete signals, Gabor theory is done on $\mathbb{Z}$ [Li94a, Jan94a, ML94, ZZ93a], and numerical implementations require to consider finite periodic signals and consequently Gabor theory on finite cyclic groups [FCS95, Orr93a, Orr93b, Pri96b, QC93, RN96, WR90]. In image processing Gabor theory on $\mathbb{R}^2$ or $\mathbb{R}^d$ and in its discrete versions on $\mathbb{Z}^d$ and finite abelian groups is necessary [AZG91, Dau88b, Li94c]. A computer scientist might even argue that the right setting for Gabor theory are the $p--$adic groups, because their group laws imitate the computer arithmetic most closely. Since Gabor theory rests mainly on the structure of translations and modulations, it is possible to extend it to other abelian groups.

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The theory and the formal computations on all these groups are always the same, and their derivation becomes somewhat repetitive, often with unnecessary notational problems. In a sense, present-day Gabor theory resembles the state of abelian harmonic analysis in the 1930's before it was discovered that all but a few theorems on Fourier integrals and Fourier series could be formulated for general locally compact abelian (LCA) groups.

One of the objectives of this chapter is to present some of the main results of Gabor theory for LCA groups. Usually Gabor theory on $\mathbb{R}$ is more difficult than on discrete or compact groups, because technical questions about convergence of series and integrals and boundedness of certain operators do not occur or are much easier to deal with. The generalization of Gabor theory from $\mathbb{R}^d$ to LCA groups is then routine and can be based on standard harmonic analysis on LCA groups [HR63, Rei68]. The results obtained here are "new", but hardly surprising.

Our second objective are the occurring differences due to the group structure. On $\mathbb{R}$ the uncertainty principle and the Balian-Low theorem ("BLT") are fundamental obstructions to ideal time-frequency localization and much of the research in Gabor theory is driven by the need to live with these obstacles. However, on general LCA groups neither the uncertainty principle nor the BLT hold true in their standard formulations. It is still a challenge to find appropriate formulations of these principles that validate our fundamental intuition about time-frequency concentration on other groups than $\mathbb{R}^d$. This leads to new interesting problems, and the transition from the discrete case (necessary for the numerical analysis) to the continuous case (the "real world") with its subtleties seems to need rethinking and many aspects have yet to be fully understood.

Section 2 presents the required facts about harmonic analysis on LCA groups and serves as a warm-up for Gabor theory on LCA groups.

In Section 3 we shall discuss Lieb's inequalities [Lie90] for the short time Fourier transform . They imply that the time-frequency concentration of a signal in the time-frequency plane has an area of size at least one, and thus can be viewed as an alternative formulation of the uncertainty principle . Since the proof uses only properties of the short time Fourier transform , the inequalities of Young and Hausdorff-Young, these inequalities carry over to general LCA groups. Furthermore, on $\mathbb{R}^d$ the minimizing functions are Gaussians as in the classical uncertainty principle . Thus the minimizing functions on general groups could serve as a good substitute for the Gaussian.

Section 4 treats the theory of critically sampled and integer oversampled Gabor frames. As the main tool we use the Zak transform. Although the Zak transform is a widely applied tool in signal analysis, its mathematical origin in A. Weil's work [Wei64] in 1964 seems to be largely forgotten. Currently one can observe the rediscovery of the Zak transform on groups such as $\mathbb{Z}$ or finite cyclic groups, although Weil had already investigated and applied the Zak transform for general LCA groups. In this context it is exciting to look at another version of the uncertainty principle , namely the Balian–Low theorem (BLT in short). Although the usual formulation on $\mathbb{R}$ [Bal81, Low85] requires differentiation, it can be stated in a slightly weaker form that just uses the short time Fourier transform . In this version one might expect the BLT to carry over to LCA groups. However, the BLT depends sensitively on the group structure and any version which I know of is false in discrete and compact groups . In these groups no obstruction to critically sampled Gabor frames with well-localized windows seems to exist – at least if time-frequency localization is understood in the usual qualitative sense (f and $\hat{f}$ "have almost compact support"). This leads to interesting questions of how to formulate quantitative concepts of time-frequency concentration on LCA groups and how to find alternative versions of the BLT that work in any group. We hope that these observations will provoke a new critical investigation of time-frequency analysis for discrete and periodic signals and will lead to a rethinking of the underlying assumptions and goals of Gabor theory .

Section 5 discusses conditions under which a discrete subgroup of the time-frequency plane can give rise to a frame. We give both sufficient conditions depending on the window and necessary conditions in the style of Landau [Lan93] and Ramanathan-Steger [RS95a].

We omit the fundamental topic of how to formulate and prove the Wexler-Raz biorthogonality relations on LCA groups, since some aspects of this question are treated in the Chapter of Feichtinger and Kozek, who have also announced a complete treatment for a future publication.

6.2 Basics on locally compact abelian groups

Here we present some of the fundamental facts about locally compact abelian groups, Fourier transforms, the associated "time-frequency" structure, and a few necessary theorems. All these facts are well-known and can be found for example in the monographs [HR63, Rei68].

By $\mathcal{G}$ we denote a locally compact abelian group, its elements are denoted by $x, y, \ldots$ and the group multiplication is written additively as $x+y$. There always exists a left invariant (Radon) measure, the so-called *Haar measure*, which we denote by λ or by dx in integrals. λ is unique up to multiplicative constants, and finite if and only if $\mathcal{G}$ is compact.

Examples: On $\mathbb{R}^d$ and on the torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ the Haar measure is just Lebesgue measure, on discrete, in particular on finite groups, the Haar measure is the counting measure.

In order to avoid uncountable sums and transfinite arguments, we assume that $L^2(\mathcal{G}) = \{f \text{ measurable } : \|f\|_2 = (\int_{\mathcal{G}} |f(x)|^2 dx)^{1/2} < \infty\}$ is *separable*.

This is equivalent to saying that $\mathcal{G}$ is a countable union of compact sets and metrizable [HR63], Thm. 24.15. As a consequence all Gabor sums and all index sets will be countable.

The *dual group* $\hat{\mathcal{G}}$ consists of the set of all *characters*, i.e. all continuous homomorphisms from $\mathcal{G}$ into the torus $\mathbb{T} \cong \{z \in \mathbb{C} : |z| = 1\}$. Characters will be denoted by Greek "characters" $\omega, \chi, \ldots$. Equipped with a suitable topology $\hat{\mathcal{G}}$ also becomes a LCA group under the the pointwise multiplication $(\omega, \chi) \to \omega\chi$ and thus possesses a Haar measure $d\omega$ or $\tilde{\lambda}$.

Examples: $\widehat{\mathbb{R}^d}$ consists of the characters $\chi_y(x) = e^{2\pi i y \cdot x}$ and can be identified with $\mathbb{R}^d$. Similarly, $\widehat{\mathbb{Z}^d} \cong \mathbb{T}^d$ with characters χ_y as before, but $y \in [0,1)^d$.

The famous *duality theorem* of Pontrjagin says that the character group of $\hat{\mathcal{G}}$, $\hat{\hat{\mathcal{G}}}$ is topologically isomorphic with $\mathcal{G}$ via the group isomorphism $x \to \tilde{x} \in \hat{\hat{\mathcal{G}}}, \tilde{x}(\omega) = \omega(x)$ for $\omega \in \hat{\mathcal{G}}$ [HR63], Vol. 1, Thm. 24.8.

The *Fourier transform* maps functions on $\mathcal{G}$ to functions on $\hat{\mathcal{G}}$ and is defined by

$$\mathcal{F}f(\omega) = \hat{f}(\omega) = \int_{\mathcal{G}} f(x)\overline{\omega(x)}\, dx \quad \text{for } \omega \in \hat{\mathcal{G}} \tag{6.2.1}$$

If $f \in L^1(\mathcal{G})$ and $\hat{f} \in L^1(\hat{\mathcal{G}})$, then f is recovered from $\hat{f}$ by the inversion formula

$$f(x) = \int_{\hat{\mathcal{G}}} \hat{f}(\omega)\omega(x)\, d\omega \tag{6.2.2}$$

Originally $\mathcal{F}$ is well-defined only on $L^1(\mathcal{G})$, but the general version of *Plancherel's theorem* states that the Haar measures on $\mathcal{G}$ and $\hat{\mathcal{G}}$ can be normalized in such a way that

$$\int_{\mathcal{G}} |f(x)|^2\, dx = \int_{\hat{\mathcal{G}}} |\hat{f}(\omega)|^2\, d\omega \tag{6.2.3}$$

whence $\mathcal{F}$ extends to a unitary operator from $L^2(\mathcal{G})$ onto $L^2(\hat{\mathcal{G}})$. We assume throughout this chapter that the Haar measures on $\mathcal{G}$ and $\hat{\mathcal{G}}$ have been chosen so that (6.2.3) holds.

The fundamental operations in Gabor theory are *time-frequency shifts*. Without much ado they can be considered on any LCA group: for *translations* (alias time-shifts) we write

$$T_x f(y) = f(y - x) \quad \text{for } x, y \in \mathcal{G} \tag{6.2.4}$$

and for *modulations* (alias frequency shifts)

$$M_\omega f(y) = \omega(y) f(y) \quad \text{for } y \in \mathcal{G}, \omega \in \hat{\mathcal{G}} \tag{6.2.5}$$

As on $\mathbb{R}$ they satisfy

$$(M_\omega T_x f)\hat{}(\chi) = \omega(x)\overline{\chi(x)}\hat{f}(\chi\overline{\omega}) = T_\omega M_{\bar{x}}^{-1}\hat{f}(\chi)\,. \tag{6.2.6}$$

As expected, time-frequency shifts do not commute, but they satisfy the commutation relations

$$T_x M_\omega = \overline{\omega(x)} M_\omega T_x \tag{6.2.7}$$

In most contexts the phase factor $\omega(x)$ will be irrelevant, and we can choose the most convenient order of the T and M operators.

In keeping the original intuition of $x \in \mathcal{G}$ as "time" and $\omega \in \hat{\mathcal{G}}$ as "frequency", the product group $\mathcal{G} \times \hat{\mathcal{G}}$ is the generalized *time-frequency plane* over which one can now investigate joint time-frequency distributions of a function f. We shall focus on the *short time Fourier transform*, although many other signal transforms, even quadratic ones, can be defined and analyzed, very much in analogy to the theory on $\mathbb{R}$. Given an appropriate window $g \in L^2(\mathcal{G})$ on $\mathcal{G}$, the short time Fourier transform of $f \in L^2(\mathcal{G})$ is defined by

$$\mathcal{V}_g f(x,\omega) = \int_{\mathcal{G}} f(y)\bar{g}(y-x)\overline{\omega(y)}dy = \langle f, M_\omega T_x g\rangle = (f \cdot T_x\bar{g})\hat{}(\omega)\,. \tag{6.2.8}$$

Similar to the properties of the ordinary Fourier transform on $\mathcal{G}$, there is an inversion formula and a Plancherel theorem for the short time Fourier transform.

Theorem 6.2.1 *a) For $f, g \in L^2(\mathcal{G})$*

$$\int_{\mathcal{G}}\int_{\hat{\mathcal{G}}} |\mathcal{V}_g f(x,\omega)|^2 \, dx d\omega = \|g\|_2^2 \, \|f\|_2^2 \tag{6.2.9}$$

b) If $f, g \in L^2(\mathcal{G})$ and $\hat{f}, \hat{g} \in L^1(\hat{\mathcal{G}})$, then

$$f(u) = \frac{1}{\langle h, g\rangle}\int_{\mathcal{G}}\int_{\hat{\mathcal{G}}} \mathcal{V}_g f(x,\omega) M_\omega T_x h(u)\, d\omega dx \quad \textit{for all } u \in \mathcal{G} \tag{6.2.10}$$

As with the inversion formula for the Fourier transform there are many ways to interpret (6.2.10) in terms of limiting procedures or in a weak sense. See for instance [HW89] for a version using approximate identities. That approach works of course on any LCA group.

Proof: (a) We prove (6.2.9) for $g \in L^1 \cap L^\infty$ first, the general case then follows from a standard approximation argument. Writing $\mathcal{V}_g f(x,\omega) = (f \cdot T_x\bar{g})\hat{}(\omega) \in L^2(\mathcal{G})$ for all $x \in \mathcal{G}$ by hypothesis on g, we obtain

$$\int_{\mathcal{G}}\int_{\hat{\mathcal{G}}} |\mathcal{V}_g f(x,\omega)|^2 \, d\omega dx = \int_{\mathcal{G}}\int_{\hat{\mathcal{G}}} |(f \cdot T_x\bar{g})\hat{}(\omega)|^2 \, d\omega dx =$$

$$= \int_{\mathcal{G}} \int_{\mathcal{G}} |f(t) T_x \bar{g}(t)|^2 \, dt dx =$$

$$= \int_{\mathcal{G}} |f(t)|^2 \left(\int_{\mathcal{G}} |g(x-t)|^2 \, dx \right) dt = \|f\|_2^2 \|g\|_2^2 \,,$$

where we have used Plancherel's theorem (6.2.3) in the second equality.

(b) By the ordinary inversion for the Fourier transform on $\mathcal{G}$ the inner integral is

$$\int_{\hat{\mathcal{G}}} \mathcal{V}_g f(x,\omega) \omega(u) \, d\omega = (f \cdot T_x \bar{g})(u) \,,$$

since $f \cdot T_x \bar{g} \in L^1(\mathcal{G})$ by Cauchy-Schwarz and its Fourier transform $\hat{f} * M_{\bar{x}}^{-1} \hat{g} \in L^1(\hat{\mathcal{G}})$ by assumption. Now the integration over x yields

$$\frac{1}{\langle h, g \rangle} \int_{\mathcal{G}} \int_{\hat{\mathcal{G}}} \mathcal{V}_g f(x,\omega) M_\omega T_x h(u) \, d\omega dx =$$

$$= \frac{1}{\langle h, g \rangle} \int_{\mathcal{G}} f(u) \bar{g}(u-x) h(u-x) \, dx = f(u)$$

❑

In Section 4 discrete sets of time-frequency translates will be considered. In this context a discrete subgroup D of $\mathcal{G}$ is called a *lattice*, if the quotient $\mathcal{G}/D$ is a compact group. The *lattice size* is defined as the measure of a fundamental domain of D in $\mathcal{G}$, i.e. we choose a measurable set $U \subset \mathcal{G}$, such that every $x \in \mathcal{G}$ can be uniquely written as $x = d + u$ for some $d \in D$ and $u \in U$. Equivalently, $\mathcal{G}$ is the disjoint union of the translates $d + U, d \in D$. Then the lattice size is defined as $s(D) = \lambda(U)$, the quantity $\lambda(U)^{-1}$ would serve as a measure of the *density* of D. It can be shown that such U always exist and that the density is independent of the particular choice of U.

If we normalize the Haar measure $d\dot{x}$ on $\mathcal{G}/D$ to be a probability measure, then the relation between dx on $\mathcal{G}$ and $d\dot{x}$ is given by a special case of *Weil's formula* [Rei68]: *If $f \in L^1(\mathcal{G})$, then $\sum_{d \in D} f(x+d) \in L^1(\mathcal{G}/D)$ and*

$$\int_{\mathcal{G}} f(x) \, dx = s(D) \int_{\mathcal{G}/D} \left(\sum_{d \in D} f(x+d) \right) d\dot{x} \tag{6.2.11}$$

Example: If $\mathcal{G} = \mathbb{R}$ and $D = \alpha\mathbb{Z}$, then U can be chosen to be $U = [0, \alpha)$ and $s(D) = \alpha$ and the density of D is $1/\alpha$. The Haar measure on $\mathbb{R}/(\alpha\mathbb{Z})$ is $\alpha^{-1} \int_0^\alpha \dots dx$ and Weil's formula amounts to the usual periodization trick $\int_0^\alpha (\sum_{k \in \mathbb{Z}} f(x + \alpha k)) \, dx = \int_{\mathbb{R}} f(x) \, dx$.

In dealing with subgroups, we sometimes have to look at their dual groups. For this recall that the *annihilator* of a set $H \subseteq \mathcal{G}$ is defined as the closed subgroup $H^\perp = \{\omega \in \hat{\mathcal{G}} : \omega(h) = 1 \text{ for } h \in H\}$. Then for subgroups H we have

$$\widehat{H} = \mathcal{G}/H^\perp \quad \text{and} \quad (\mathcal{G}/H)\hat{\ } = H^\perp$$

where the identities have to be taken in the sense of group isomorphisms, see [HR63], Vol. I, Thm. 23.25, and Thm. 24.11.

For discrete subgroups we need the following lemmata:

Lemma 6.2.2 (Poisson Summation Formula) *If $f \in L^1(\mathcal{G})$ and $D \subseteq \mathcal{G}$ is a lattice, then the periodization $\phi(\dot{x}) = \sum_{d\in D} f(x+d)$ is in $L^1(\mathcal{G}/D)$ and for $\chi \in (\mathcal{G}/D)\hat{} = D^\perp$*

$$\hat{\phi}(\chi) = \frac{1}{s(D)}\hat{f}(\chi) \tag{6.2.12}$$

If furthermore $\sum_{\chi\in D^\perp} |\hat{f}(\chi)|^2 < \infty$, then $\phi \in L^2(\mathcal{G}/D)$ and

$$\phi(\dot{x}) = \sum_{d\in D} f(x+d) = \frac{1}{s(D)} \sum_{\chi\in D^\perp} \hat{f}(\chi)\chi(x) \qquad a.\ e. \tag{6.2.13}$$

where the right hand side converges in $L^2(\mathcal{G}/D)$.

Proof: By (6.2.11) we have for $\chi \in D^\perp$, since $\chi(x+d) = \chi(x)$

$$s(D)\hat{\phi}(\chi) = s(D)\int_{\mathcal{G}/D}\Big(\sum_{d\in D} f(x+d)\Big)\overline{\chi(x)}\,d\dot{x} =$$

$$= s(D)\int_{\mathcal{G}/D}\Big(\sum_{d\in D} f(x+d)\overline{\chi(x+d)}\Big)\,d\dot{x} = \int_{\mathcal{G}} f(x)\overline{\chi(x)}\,dx = \hat{f}(\chi).$$

If the Fourier coefficients are in ℓ^2, then by Plancherel's theorem $\phi \in L^2(\mathcal{G}/D)$ and (6.2.13) holds in the L^2-sense. ❑

Lemma 6.2.3 *(a) If D is a lattice in $\mathcal{G}$, then $D^\perp$ is a lattice in $\hat{\mathcal{G}}$ and*

$$s(D)s(D^\perp) = 1\,. \tag{6.2.14}$$

(b) If K is an open compact subgroup of $\mathcal{G}$, then $K^\perp$ is an open compact subgroup of $\hat{\mathcal{G}}$.

Proof: (a) Since $\mathcal{G}/D$ is compact, its dual group $(\mathcal{G}/D)\hat{}\cong D^\perp$ is discrete. On the other hand, $\hat{D} \cong \hat{\mathcal{G}}/D^\perp$ is compact and consequently $D^\perp$ is a lattice in $\hat{\mathcal{G}}$.

By Weil's formula (6.2.11) we have

$$\int_{\mathcal{G}} f(x)\,dx = s(D)\int_{\mathcal{G}/D}\Big(\sum_{d\in D} f(x+d)\Big)\,d\dot{x}\,.$$

This means that we need the Haar measure $s(D)d\dot{x}$ (not normalized) on $\mathcal{G}/D$ and the usual counting measure on D for Weil's formula to work.

Next, by [HR63], (31.1), Plancherel's theorem for the pair $\mathcal{G}/D$ and $(\mathcal{G}/D)\hat{}= D^{\perp}$ requires that the Haar measure on $D^{\perp}$ is $s(D)^{-1}\sum_{\chi\in D^{\perp}}\cdots$, so that

$$s(D)\int_{\mathcal{G}/D} |\phi(\dot{x})|^2\,d\dot{x} = \frac{1}{s(D)}\sum_{\chi\in D^{\perp}} |\hat{\phi}(\chi)|^2$$

On the other hand, again by [HR63], (31.1), Plancherel's theorem for the pair D and $\hat{D} = \hat{\mathcal{G}}/D^{\perp}$ does not require any adjustment of constants, since in (6.2.11) we use the regular counting measure on D and thus we can use the normalized Haar measure on $\hat{D}$.

By [HR63], (31.46c) these normalizations guarantee that Weil's formula holds on $\hat{\mathcal{G}}$ as follows

$$\int_{\hat{\mathcal{G}}} h(\omega)\,d\omega = \int_{\hat{\mathcal{G}}/D^{\perp}} \left(\frac{1}{s(D)}\sum_{\chi\in D^{\perp}} h(\omega\cdot\chi)\right) d\dot{\omega}\,. \tag{6.2.15}$$

Comparing to the normalization which is used in (6.2.11)

$$\int_{\hat{\mathcal{G}}} h(\omega)\,d\omega = s(D^{\perp})\int_{\hat{\mathcal{G}}/D^{\perp}} \left(\sum_{\chi\in D^{\perp}} h(\omega\cdot\chi)\right) d\dot{\omega}$$

shows that $s(D)s(D^{\perp}) = 1$, as desired.

(b) is similar. Since K is open, $\mathcal{G}/K$ is discrete and this its dual $(\mathcal{G}/K)\hat{}= K^{\perp}$ is compact. Since K is compact, its dual $\widehat{K} \cong \hat{\mathcal{G}}/K^{\perp}$ is discrete, or equivalently, $K^{\perp}$ is open. ❑

Example: Every lattice in $\mathbb{R}^d$ is of the form $D = A\mathbb{Z}^d$ for some invertible real-valued $d\times d$-matrix A. Since obviously $A[0,1)^d$ is a fundamental domain for D, we have $s(D) = |\det A|$. On the other hand, $e^{2\pi i Ak\cdot x} = 1$ holds for all $k\in\mathbb{Z}^d$, if and only if $x = (A^T)^{-1}l$ for some $l\in\mathbb{Z}^d$, thus the annihilator (dual lattice) is $D^{\perp} = (A^T)^{-1}\mathbb{Z}^d$ and $s(D^{\perp}) = |\det A|^{-1} = s(D)^{-1}$.

For some arguments we shall need the fundamental structure theorem for LCA groups.

Theorem 6.2.4 ([HR63], Thm. 24.30) *A LCA group $\mathcal{G}$ is topologically isomorphic with $\mathbb{R}^d\times\mathcal{G}_0$, where $\mathcal{G}_0$ is a LCA group containing a compact open subgroup. If $\mathcal{G}$ is compactly generated, then it is isomorphic with $\mathbb{R}^d\times\mathbb{Z}^k\times K$, where K is a compact group. The (real) dimension d is an invariant of $\mathcal{G}$.*

In some sense groups with a compact open subgroup are complementary to $\mathbb{R}^d$. Many statements that are hard for $\mathbb{R}^d$ are much easier to prove or false for $\mathcal{G}_0$. Perhaps the most important theorems in this regard are the inequalities of Hausdorff-Young and Young. For this recall that the conjugate index p' is defined by $1/p + 1/p' = 1$ and the *Babenko-Beckner constants* are

$$A_p = (p^{1/p}/p'^{\,1/p'})^{1/2} \tag{6.2.16}$$

Since by the structure theorem $\mathcal{G}$ is isomorphic with $\mathbb{R}^d \times \mathcal{G}_0$, we can write a function on $\mathcal{G}$ as a function of the two variables $x \in \mathbb{R}^d$ and $y \in \mathcal{G}_0$.

The *Hausdorff-Young inequality* extends the Fourier transform from L^1 to other function spaces.

Theorem 6.2.5 *The Fourier transform extends to a bounded mapping from $L^p(\mathcal{G})$ into $L^{p'}(\hat{\mathcal{G}})$ for $1 \leq p \leq 2$ and*

$$\|\hat{f}\|_{p'} \leq A_p^d \|f\|_p .$$

Equality is attained for functions of the form $f(x,y) = g_0(x) T_d M_\omega \chi_K(y)$, where $g_0(x) = e^{2\pi i v\cdot x} e^{-\pi(x-u)^2}$ is a Gaussian on $\mathbb{R}^d$ and K is a compact open subgroup in $\mathcal{G}_0$.

For two functions $f, g \in L^1(\mathcal{G})$ *convolution* is defined (almost everywhere) by

$$f * g(x) = \int_{\mathcal{G}} f(y) g(x-y)\, dy \tag{6.2.17}$$

and satisfies $\|f * g\|_1 \leq \|f\|_1 \, \|g\|_1$. *Young's Theorem* extends the convolution to other function spaces.

Theorem 6.2.6 *If $f \in L^p(\mathcal{G})$ and $g \in L^q(\mathcal{G})$ and r satisfies $1/p + 1/q = 1 + 1/r$, then $f * g$ is in $L^r(\mathcal{G})$ and*

$$\|f * g\|_r \leq (A_p A_q A_{r'})^d \|f\|_p \, \|g\|_q$$

Equality is attained for the same class of functions as in Theorem 6.2.5

Both theorems are very deep for $\mathbb{R}^d$ and are due to Beckner [Bec75] who also derived the correct form for LCA groups. A different approach was taken by Brascamp-Lieb [BL76b], who characterized *all* functions for which equality is attained for the case of $\mathbb{R}^d$. In these two theorems it is very transparent that groups containing compact open subgroups often behave fundamentally different from $\mathbb{R}^d$. It is very easy to see that for $\mathcal{G}_0$ the best constant is 1, and that characteristic functions of compact open subgroups yield equality. To find *all* maximizing functions for such groups is much harder and this has been done in [HR63], Vol. II, Section 43.

6.3 Uncertainty Principles and Lieb's inequalities

Given a signal f on $\mathbb{R}$, its time duration is usually measured by

$$\Delta_f x = \int_{\mathbb{R}} (x-\bar{x})^2 |f(x)|^2 \, dx)^{1/2}$$

and similarly, its frequency band is

$$\Delta_f \omega = \int_{\mathbb{R}} (\omega-\bar{\omega})^2 |\hat{f}(\omega)|^2 \, d\omega)^{1/2}$$

where $\bar{x}$ and $\bar{\omega}$ are the expected values of time and frequency. Then the *classical uncertainty principle* limits the time-frequency concentration of a signal through the inequality [Fol89]

$$\Delta_f x \, \Delta_f \omega \geq \frac{1}{4\pi} \|f\|_2^2 \tag{6.3.1}$$

In signal analysis this is usually interpreted by saying that time and frequency cannot be localized simultaneously to arbitrary precision. Mathematically it means that a function and its Fourier transform cannot both decay fast. There are various formulations of this statement [CP84, DS89, Fef83, Pri83], some of which generalize to LCA groups [PS88, Smi]. However, on LCA groups the problem of time-frequency localization is more subtle. For instance, on compact or on discrete groups there are functions f with both f and $\hat{f}$ being compactly supported, for example trigonometric polynomials on the torus $\mathbb{T}$. By contrast, on $\mathbb{R}^d$ this is impossible.

Thus as long as time-frequency localization is treated in the usual qualitative sense, for discrete and compact groups the problem of simultaneous time-frequency localization does not arise.

Of course, our basic intuition about time-frequency analysis is correct and the solution is to find alternative and quantitative formulations of the uncertainty principle that work for all LCA groups. Perhaps Lieb's inequalities for the short time Fourier transform are an appropriate substitute. These are rather deep and beautiful inequalities which have not yet received attention from signal engineers although they should be very useful. We state a special case and refer to [Lie90] for extensions and variations.

Theorem 6.3.1 *For all $f, g \in L^2(\mathcal{G})$ we have*

$$\int_{\mathcal{G}} \int_{\hat{\mathcal{G}}} |\mathcal{V}_g f(x,\omega)|^p \, dx d\omega \leq \left(\frac{2}{p}\right)^d (\|f\|_2 \|g\|_2)^p \qquad \text{if } p \geq 2 \tag{6.3.2}$$

and

$$\int_{\mathcal{G}} \int_{\hat{\mathcal{G}}} |\mathcal{V}_g f(x,\omega)|^p \, dx d\omega \geq \left(\frac{2}{p}\right)^d (\|f\|_2 \|g\|_2)^p \qquad \text{if } 1 \leq p \leq 2\,. \tag{6.3.3}$$

Equality is attained for functions as in Theorem 6.2.6.

Before turning to the proof, let us discuss in which sense the Theorem expresses an uncertainty principle . Suppose that the short time Fourier transform is "essentially" supported on a set $U \subset \mathcal{G} \times \hat{\mathcal{G}}$ and assume that roughly $\mathcal{V}_g f = F = \lambda(U)^{-1/2}\chi_U$. Then $\|F\|_2 = \|f\|_2\|g\|_2 = 1$ and $\int |\mathcal{V}_g f|^p = \lambda(U)^{1-p/2}$. If $\lambda(U) < 1$, i.e., $\mathcal{V}_g f$ has a sharp peak on U, then $\int |\mathcal{V}_g f|^p$ would grow arbitrarily large, as $p \to \infty$ and it would become small for $p < 2$. By Lieb's inequality, however, $\int |\mathcal{V}_g f|^p$ is uniformly bounded above for $p \geq 2$ and below for $p \leq 2$, thus $\lambda(U) < 1$ is impossible. Thus these inequalities lead to the same conclusions as the classical uncertainty principle . Intuitively they state that the smallest meaningful cell in the time-frequency plane has area ≥ 1.

Proof: We note that $f \cdot T_x\bar{g} \in L^1(\mathcal{G})$ for all $x \in \mathcal{G}$ by the Cauchy-Schwarz inequality, and in $L^2(\mathcal{G})$ for almost all x by (6.2.9). Therefore $\mathcal{V}_g f(x,\omega) = (f \cdot T_x\bar{g})\hat{} \in L^p(\hat{\mathcal{G}})$ for $p \geq 2$, i.e. $p' \leq 2$, and we apply the inequality of Hausdorff-Young (Thm. 6.2.5) to obtain

$$\left(\int_{\hat{\mathcal{G}}} |\mathcal{V}_g f(x,\omega)|^p \, d\omega\right)^{1/p} \leq A_{p'}^d \left(\int_{\mathcal{G}} |f(y) T_x\bar{g}(y)|^{p'} \, dy\right)^{1/p'}$$

$$= A_{p'}^d \left(\int_{\mathcal{G}} |f(y)|^{p'} |g(y-x)|^{p'} \, dy\right)^{1/p'} = A_{p'}^d \left(|f|^{p'} * |g^*|^{p'}(x)\right)^{1/p'} ,$$

where $g^*(x) = \bar{g}(-x)$ is the usual involution. Doing the x-integration next, we get

$$\|\mathcal{V}_g f\|_p \leq A_{p'}^d \| \, |f|^{p'} * |g^*|^{p'} \|_{p/p'}^{1/p'} .$$

To the latter expression we now apply Young's inequality with the triple (p,q,r) in Thm. 6.2.6 replaced by (r,r,s) with $r = 2/p' \geq 1$ and $s = p/p' > 1$ and arrive at

$$\|\mathcal{V}_g f\|_p \leq A_{p'}^d \left[A_r^{2d} A_{s'}^d \| \, |f|^{p'} \|_r \, \| \, |g|^{p'} \|_r\right]^{1/p'} = A_{p'}^d A_r^{2d/p'} A_{s'}^{d/p'} \, \|f\|_2 \|g\|_2 .$$

The constant $A_{p'}^d A_r^{2d/p'} A_{s'}^{d/p'}$ is readily computed to be $(2/p)^{1/p}$.

The case $1 \leq p < 2$ is done similar, but contains some additional twists. We indicate the argument and necessary tools for $1 < p < 2$. We use again the Hausdorff-Young inequality in the reverse direction $\|f\|_{p'} \leq A_p^d \|\hat{f}\|_p$ and obtain as before for $1 \leq p \leq 2$

$$\left(\int_{\hat{\mathcal{G}}} |\mathcal{V}_g f(x,\omega)|^p \, d\omega\right)^{1/p} = \left(\int_{\hat{\mathcal{G}}} |(f \cdot T_x\bar{g})\hat{}(\omega)|^p d\omega\right)^{1/p} \tag{6.3.4}$$

$$\geq A_p^{-d} \left(\int_{\mathcal{G}} |f(y)|^{p'} |g(y-x)|^{p'} \, dy\right)^{1/p'} = A_p^{-d} \left(|f|^{p'} * |g^*|^{p'}(x)\right)^{1/p'} .$$

At this point we use the converse of Young's inequality [Lei72] with the sharp constants due to Brascamp and Lieb [BL76b]. If f and g are non-negative and the exponents $0 < p, q, r < 1$ (!) satisfy $1/p + 1/q = 1 + 1/r$, then

$$\|f * g\|_r \geq (A_p A_q A_{r'})^d \|f\|_p \, \|g\|_q \tag{6.3.5}$$

where the constants are $A_p = (p^{1/p}/|p'|^{1/p'})^{1/2}$ since now $p' < 0$. If we take the p-norm of (6.3.4) with respect to x and apply the converse of Young's inequality (6.3.5) with (p, q, r) replaced by (r, r, s), where $r = 2/p' < 1, s = p/p' < 1$, we can continue with

$$\|\mathcal{V}_g f\|_p \geq A_p^{-d} \| \, |f|^{p'} * |g^*|^{p'} \|_s^{1/p'} \geq$$

$$A_p^{-d} \left[A_r^{2d} A_{s'}^{d} \| \, |f|^{p'} \|_r \cdot \| \, |g|^{p'} \|_r \right]^{1/p'} = A_{p'}^{d} A_r^{2d/p'} A_s^{d/p'} \|f\|_2 \|g\|_2 \, .$$

Again the constant equals $(2/p)^{1/p}$ as before. The above arguments hold for $1 < p < 2$. The case $p = 1$ requires an additional limiting argument.

It is a long, but routine computation to verify that the optimal functions in Theorems 6.2.5 and 6.2.6 also yield equality in this context. ❑

Remark: In Lieb's paper more general inequalities are discussed that require only $f \in L^a$ and $g \in L^b$ with $1/a + 1/b = 1$. While the details are slightly more complicated, the proof remains the same.

Furthermore, for general LCA groups we have only identified *some* of the minimizing functions. To find all such functions one observes that equality holds if and only if we have equality in all applications of Young's and Hausdorff-Young's inequalities. From this observation one could derive after some manipulations that all minimizing functions are of the indicated form. See [Lie90] for the details on $\mathbb{R}$. The maximizing functions on $\mathcal{G}$ can be thought of as the signals of minimal uncertainty and thus as a sort of Gaussians on LCA groups.

6.4 Zak transform, Gabor frames, and the Balian-Low phenomenon

The Zak transform on $\mathbb{R}$ is a tool to treat critical sampling and some cases of oversampled Gabor frames. It has become important in signal analysis through Janssen [Jan81, Jan88] and others, but has been long known in physics and mathematics. In particular, A. Weil already introduced this transform for general LCA groups and formulated its basic properties [Wei64], pp. 164 – 165, whereas in engineering this knowledge has been rediscovered only recently for simple groups such as $\mathbb{Z}$ or finite cyclic groups

[AGT91, BT94, Hei89, ZG92, BH96a]. In this section we present the aspects of the Zak transform related to Gabor theory on LCA groups. In what follows D is a lattice in $\mathcal{G}$ and by Lemma 6.2.3 its annihilator $D^{\perp}$ is also a lattice in $\hat{\mathcal{G}}$.

Definition 6.4.1 *If f is continuous and compactly supported in $\mathcal{G}$, its* Zak transform *$Z_D f$ with respect to the lattice D is defined as*

$$Z_D f(x, \omega) = \sum_{d \in D} f(x + d)\omega(d) \quad \textit{for } x \in \mathcal{G}, \omega \in \hat{\mathcal{G}} \tag{6.4.1}$$

It is usually clear to which subgroup D is referred and we will write Zf instead of $Z_D f$.

Although this definition is due to A. Weil, to be consistent with the current terminology, we keep using the term "Zak transform" instead of kq-transform (which was used by Zak [Zak67, Zak72]) or Weil-Brezin map, which is often used in mathematics [Fol89].

We first state the main properties of the Zak transform which parallel the known properties on $\mathbb{R}$.

Theorem 6.4.2 *Suppose that f is continuous with compact support and $(k, \chi) \in D \times D^{\perp}$. Then*

*a) (*Quasi-periodicity*)*

$$Zf(x + k, \omega) = \overline{\omega(k)} Zf(x, \omega) \tag{6.4.2}$$

and

$$Zf(x, \omega \cdot \chi) = Zf(x, \omega) \tag{6.4.3}$$

*b) (*Plancherel*): Let $\mathcal{M} = \mathcal{G}/D \times \hat{\mathcal{G}}/D^{\perp}$, then*

$$||Zf||_{L^2(\mathcal{M})} = ||f||_{L^2(\mathcal{G})} \tag{6.4.4}$$

Therefore Z extends to a unitary operator from $L^2(\mathcal{G})$ onto $L^2(\mathcal{M})$, provided that the compact group $\mathcal{M}$ is endowed with the Haar measure $s(D)\, d\dot{x}\, d\dot{\omega}$.

*c) (*Diagonalization*)*

$$Z(T_k M_\chi f)(x, \omega) = \chi(x)\omega(k) Zf(x, \omega) = M_{(k,\chi)} Zf(x, \omega) \tag{6.4.5}$$

Proof: a) We do the simple calculation to verify (6.4.2), (6.4.3) is done similarly:

$$Zf(x+k, \omega) = \sum_{d \in D} f(x+k+d)\omega(d) = \sum_{d \in D} f(x+d)\omega(d-k) = \overline{\omega(k)} Zf(x, \omega)$$

b) Since, for fixed $x \in \mathcal{G}$, $Zf(x, \omega) = \sum_{d \in D} f(x + d)\omega(d)$ is a Fourier series on $\hat{\mathcal{G}}/D^{\perp}$ with coefficients $f(x + d)$, Plancherel's theorem (6.2.3) for the groups $\hat{\mathcal{G}}/D^\perp = \hat{D}$, equipped with the normalized Haar measure, and D, equipped with the counting measure, yields

$$\int_{\hat{\mathcal{G}}/D^\perp} |\sum_{d\in D} f(x+d)\omega(d)|^2 \, d\dot{\omega} = \sum_{d\in D} |f(x+d)|^2$$

for all $x \in \mathcal{G}$. Therefore we obtain, using (6.2.11),

$$s(D)\int_{\mathcal{G}/D}\int_{\hat{\mathcal{G}}/D^\perp} |Zf(x,\omega)|^2 \, d\dot{\omega} d\dot{x} = s(D)\int_{\mathcal{G}/D}\sum_{d\in D} |f(x+d)|^2 \, d\dot{x}$$
$$= \int_{\mathcal{G}} |f(x)|^2 \, dx \, .$$

These arguments make sense for f in any nice dense subspace of $L^2(\mathcal{G})$ and therefore extend to all of $L^2(\mathcal{G})$.

c) follows easily from a change of the summation index $d \to d + k$

$$Z(T_k M_\chi f)(x,\omega) = \sum_{d\in D} \chi(x-k+d) f(x-k+d)\omega(d) =$$

$$= \sum_{d\in D} \chi(x)\chi(-d) f(x+d)\omega(d+k) = \chi(x)\omega(k) Zf(x,\omega)\, ,$$

since $\chi(-d) = 1$ for $\chi \in D^\perp$. □

The reader may want to verify by a simple example that the normalization of the Haar measure on $\mathcal{M}$ is correct. If $\mathcal{G} = \mathbb{R}, D = \alpha\mathbb{Z}$, then $s(D) = \alpha, D^\perp = \alpha^{-1}\mathbb{Z}$ and $\mathcal{M}$ can be identified with $[0,\alpha) \times [0, 1/\alpha)$. The appropriate Haar measure on $\mathcal{M}$ is then $\alpha dx d\omega$.

The fact that the Zak transform diagonalizes simultaneously the unitary operators T_k and M_χ for $(k,\chi) \in D \times D^\perp$, makes it very easy to understand the structure and properties of the Gabor frame operator in the case of critical sampling on $\mathcal{G} \times \hat{\mathcal{G}}$. The *Gabor frame operator* $S = S_{D,g}$ associated to a lattice D and a suitably "nice" window function g is defined as

$$Sf = \sum_{d\in D}\sum_{\omega\in D^\perp} \langle f, M_\omega T_d g\rangle M_\omega T_d g \tag{6.4.6}$$

Then Z_D diagonalizes S as follows

Proposition 6.4.3 *For D, g as above and all $F \in L^2(\mathcal{M})$ we have*

$$Z_D S_{D,g} Z_D^{-1} F = s(D)\, |Z_D g|^2 F\, , \tag{6.4.7}$$

in other words, the Zak transform diagonalizes the Gabor frame operator for critical sampling and the spectrum of S equals the range of $s(D)\,|Zg|^2$.

Proof: Using the fact that Z is unitary and (6.4.5), we write

$$\langle f, M_\chi T_d g\rangle = \langle Zf, Z(M_\chi T_d g)\rangle =$$

$$= s(D) \int_{\mathcal{M}} (Zf \cdot \overline{Zg})(x,\omega)\overline{\chi(x)\omega(d)}\, d\dot{x} d\dot{\omega} = s(D)(Zf \cdot \overline{Zg})\widehat{\ }(\chi, d)$$

as the Fourier coefficient of $Zf\overline{Zg}$ at $(\chi, d) \in \hat{\mathcal{G}} \times \mathcal{G}$. Then ZSf becomes

$$ZSf = \sum_{d,\chi} \langle Zf, Z(M_\chi T_d g)\rangle Z(M_\chi T_d g) =$$

$$= \left(s(D) \sum_{d,\chi} (Zf \cdot \overline{Zg})\widehat{\ }(\chi, d)\chi(x)\omega(d) \right) Zg = Zf \cdot |Zg|^2 s(D)$$

In the last equality we have applied once more Plancherel's theorem, since the Fourier series of $Zf \cdot \overline{Zg}$ on $\mathcal{M}$ is exactly the expression in parenthesis. ❑

From this property one deduces immediately the following criteria which are of course well-known on $\mathbb{R}$.

Corollary 6.4.4 *a) The Gabor frame operator S is bounded if and only if $|Zg| \in L^\infty(\mathcal{M})$.*

b) $\{M_\omega T_d g, (d, \chi) \in D \times D^\perp\}$ is a frame if and only if there are two constants $0 < A \leq B$ such that

$$A \leq |Zg(x,\omega)|^2 \leq B \qquad a.\ a.\ (x,\omega) \in \mathcal{G} \times \hat{\mathcal{G}}$$

In particular,

$$||S||_{op} = s(D) \operatorname*{ess\,sup}_{x,\omega} |Zg(x,\omega)|^2$$

and

$$||S^{-1}||_{op} = (s(D) \operatorname{ess\,inf}_{x,\omega} |Zg(x,\omega)|^2)^{-1}\,.$$

c) $\{M_\omega T_d g, (d, \chi) \in D \times D^\perp\}$ is an orthonormal basis for $L^2(\mathcal{G})$ if and only if $||g||_2 = 1$ and $|Zg(x,\omega)|^2 = s(D)^{-1}$ a. e.

We briefly discuss several extensions. No new ideas are needed and the generalizations are proved as in Proposition 6.4.3.

(A) *Several window functions*: Suppose we consider several windows $g_1, \ldots, g_r$ and the frame operator associated to a lattice $D \subseteq \mathcal{G}$

$$Sf = \sum_{i=1}^{r} S_{D,g_i} f = \sum_{i=1}^{r} \sum_{d \in D} \sum_{\omega \in D^\perp} \langle f, M_\omega T_d g_i\rangle M_\omega T_d g_i$$

Then by Prop. 6.4.3

$$Z_D S Z_D^{-1} F = (s(D) \sum_{i=1}^{r} |Zg_i|^2) F := G_0 \cdot F \tag{6.4.8}$$

and as in the Corollary $\{M_\omega T_d g_i, (d,\omega) \in D \times D^\perp, i = 1,\dots,r\}$ is a frame if and only if G_0 is bounded above and below from zero. See [BB96, ZZ95a] for applications of multiwindow frames.

(B) *Integer Oversampling*: Suppose we consider the frame operator $S = S_{E\times H,g}$

$$Sf = \sum_{d\in E} \sum_{\omega \in H} \langle f, M_\omega T_d g\rangle M_\omega T_d g$$

where $E \subseteq \mathcal{G}$ and $H \subseteq \hat{\mathcal{G}}$ are lattices such that $E \supseteq D$ and $H \supseteq D^\perp$ for some lattice D in $\mathcal{G}$ and such that E/D and $H/D^\perp$ are finite groups.

Choosing $d_i \in E$ so that $E = \cup_{i=1}^r (d_i + D)$ and each coset of E/D contains only one d_i and similarly $\chi_j, j = 1,\dots,s$, as representatives of the cosets of $H/D^\perp$, we can rewrite the frame operator on $E \times H$ with a single window g as a frame operator on $D \times D^\perp$ with the windows $g_{ij} = T_{d_i} M_{\chi_j} g$:

$$Sf = \sum_{i=1}^{r}\sum_{j=1}^{s} S_{D,g_{ij}} f = \sum_{i=1}^{r}\sum_{j=1}^{s}\sum_{d\in D}\sum_{\omega\in D^\perp} \langle f, M_\omega T_d T_{d_i} M_{\chi_j} g\rangle M_\omega T_d T_{d_i} M_{\chi_j} g$$

The diagonalization of the frame operator then becomes

$$ZSZ^{-1}F(x,\omega) = \left(s(D) \sum_{i=1}^{r}\sum_{j=1}^{s} |Zg(x - d_i, \omega \cdot \bar{\chi}_j)|^2 \right) F(x,\omega)$$

where we have used (6.4.8) and $|Z(T_{d_i} M_{\chi_j} g)(x,\omega)| = |Zg(x - d_i, \omega \cdot \bar{\chi_j})|$,

We leave it to the reader to combine integer oversampling with several windows and make the trivial modifications in the above formulas. For more concrete presentations on $\mathbb{R}$ and some applications we refer to [ZZ93b].

(C) *Continuous frames*: The definition of the "Zak transform" given by Weil is not limited to lattices, but works for arbitrary subgroups of $\mathcal{G}$. If $\mathcal{H}$ is a (non-discrete) subgroup of $\mathcal{G}$ with Haar measure dh, then

$$Zf(x,\omega) = \int_{\mathcal{H}} f(x+h)\omega(h)\, dh$$

is well-defined a.e. as a function on the quotient $\mathcal{G}/\mathcal{H} \times \hat{\mathcal{G}}/\mathcal{H}^\perp$ for all $f \in L^1(\mathcal{G})$.

One could then look at continuous frames $\{T_h M_\chi g, h \in \mathcal{H}, \chi \in \mathcal{H}^\perp\}$ and investigate them with the Zak transform Z_H and the meta-theory of [AAG93] as the appropriate tools. This leads to some interesting problems and is potentially useful, but as of now has not been considered. We expect results similar to those discussed above, though technical difficulties might occur.

It is well-known that, on $\mathbb{R}$, Gabor frames at the critical density are not an appropriate tool for time-frequency analysis. The famous *Balian-Low theorem* constitutes an important obstruction to good time-frequency localization of critically sampled Gabor frames and is directly related to the classical uncertainty principle . It is usually formulated as follows: *If* $\{M_k T_n g, k, n \in \mathbb{Z}\}$ *is a Gabor frame for* $L^2(\mathbb{R})$, *then either* $xg(x) \notin L^2$ *or* $g' \notin L^2$. We refer to [Bal81] and the chapter of Benedetto, Heil, and Walnut in this book or their survey [BHW95] for a thorough discussion and the subtleties of the proof.

As with the uncertainty principle this formulation cannot be extended to arbitrary LCA groups because there are no differentiation operators and no canonical weights on $\mathcal{G}$ and $\hat{\mathcal{G}}$. It is therefore important to find a formulation that does not require these concepts that are peculiar to $\mathbb{R}^d$. A solution to this problem is indicated in [FG96], where the short time Fourier transform was used to state a version of the Balian-Low theorem. Specifically, *if* $\{M_k T_n g, k, n \in \mathbb{Z}\}$ *is a Gabor frame for* $L^2(\mathbb{R})$, *then* $V_g g \notin L^1(\mathbb{R}^2)$. Since it can be shown that $|x|^{1+\epsilon} f \in L^2(\mathbb{R})$ and $|\omega|^{1+\epsilon} \hat{f} \in L^2(\mathbb{R})$ for some $\epsilon > 0$ already imply that $V_g g \in L^1(\mathbb{R}^2)$, this is just slightly weaker than the original BLT. No special structure is required in this formulation of the BLT, and one might expect that the BLT holds for general LCA groups in the form as stated above, with $\mathbb{R}^d$ being replaced by $\mathcal{G}$ and the critical lattice $\mathbb{Z}^d \times \mathbb{Z}^d$ by $D \times D^\perp$ for some lattice $D \subseteq \mathcal{G}$. A slightly stronger version using the Wiener algebra on $\mathcal{G}$ occurs in [BHW95].

However, the BLT is false in arbitrary groups. As with some other problems in harmonic analysis, for instance the sharp constants in Young's and Hausdorff-Young's inequalities, only $\mathbb{R}^d$ poses a difficulty. On the other hand, the problem is almost trivial for groups containing a compact open subgroup.

Proposition 6.4.5 *If* $\mathcal{G}$ *is a LCA group containing a compact open subgroup* K, *then there exist a window* $g \in L^2(\mathcal{G})$ *and a discrete set* $X \subseteq \mathcal{G} \times \hat{\mathcal{G}}$, *such that* $\{M_\chi T_d g, (d, \chi) \in X\}$ *is an orthonormal basis for* $L^2(\mathcal{G})$ *and* $V_g g \in L^1(\mathcal{G} \times \hat{\mathcal{G}})$.

Proof: Set $g(x) = \lambda(K)^{-1/2} \chi_K(x)$ and choose a set of coset representatives $D \subseteq \mathcal{G}$ of $\mathcal{G}/K$, i.e., every $x \in \mathcal{G}$ has a unique factorization $x = dk$ for some $d \in D$ and $k \in K$. Since K is compact and open, D is discrete and countable. Then the set $X := D \times K^\perp$ is discrete as well. Now it is easily verified that $\{T_d M_\chi g, d \in D, \chi \in K^\perp\}$ is an orthonormal basis for $L^2(\mathcal{G})$. This construction imitates the ONB $\{M_k T_n \chi_{[0,1]}, k, n \in \mathbb{Z}\}$

of $L^2(\mathbb{R})$. $M_\chi g, \chi \in K^\perp$, is an ONB for $L^2(K)$ by definition of $K^\perp$ and $T_d M_\chi g, \chi \in K^\perp$, is an ONB of $L^2(dK)$. Since $\mathcal{G}$ is the disjoint union of $dK, d \in D$, $L^2(\mathcal{G})$ is the direct orthogonal sum $\oplus_{d\in D} L^2(dK)$, and the claim is proved.

Whereas on $\mathbb{R}$ the short time Fourier transform of $\chi_{[0,1]}$ is obviously not in L^1, and the corresponding ONB is badly localized in frequency, this is not the case for the window g constructed on $\mathcal{G}$.

Claim: $V_g g \in L^1(\mathcal{G} \times \hat{\mathcal{G}})$.

We have

$$V_g g(x,\omega) = \lambda(K)^{-1}\langle \chi_K, T_x M_\omega \chi_K\rangle = \tag{6.4.9}$$

$$= \begin{cases} 0 & \text{if } x \notin K\,, \\ \lambda(K)^{-1}(\chi_K \cdot \chi_{xK})\hat{}(\omega) = \lambda(K)^{-1}(\chi_K)\hat{}(\omega) & \text{if } x \in K\,. \end{cases}$$

Therefore we compute

$$\int_{\mathcal{G}}\int_{\hat{\mathcal{G}}} |V_g g(x,\omega)|\, dx d\omega = \lambda(K)^{-1}\int_K\int_{\hat{\mathcal{G}}} |(\chi_K)\hat{}(\omega)|\, dx d\omega = \int_{\hat{\mathcal{G}}} |(\chi_K)\hat{}(\omega)|\, d\omega.$$

But since $\chi_K * \chi_K(x) = \int_{\mathcal{G}} \chi_K(y)\chi_K(x-y)\, dy = \int_{\mathcal{G}} \chi_{K\cap(x+K)}(y)\, dy = \lambda(K)\chi_K(x)$, we obtain

$$\widehat{\chi_K}(\omega) = \frac{1}{\lambda(K)}\widehat{\chi_K}(\omega)^2$$

by taking Fourier transform . Integrating both sides we obtain $\|\widehat{\chi_K}\|_1 = \lambda(K)^{-1}\|\widehat{\chi_K}\|_2^2 = 1$. (This is the standard argument to show that the Fourier transform of a compact open subgroup is in the Fourier algebra.) We have thus shown that $V_g g \in L^1(\mathcal{G} \times \hat{\mathcal{G}})$ and this proves the proposition. ❏

Remark: This simple statement looks rather startling because it seems to say that in discrete and compact groups – the groups which are used for numerical simulations – there is no problem with the time-frequency localization of Gabor frames at the critical density. This is probably due to a too lax a concept of time-frequency concentration in these groups. From a practical point of view the issues of time-frequency concentration still exist, but in the form of numerical instabilities or bad condition numbers for critically sampled Gabor frames. Clearly this phenomenon is not yet well understood and there are still many open questions. The observed failure of time-frequency localization on finite or discrete groups seems to be more subtle than the BLT on $\mathbb{R}^d$, and the causes for this remain to be seen and analyzed.

6.5 Density conditions

In order to apply Gabor frames, it is an important consideration to find conditions under which a Gabor set $\{M_\chi T_d g, d \in D, \chi \in E\}$ generate a Gabor frame. Here D and E are lattices (or more generally discrete sets) in $\mathcal{G}$ and $\hat{\mathcal{G}}$ respectively. In finite groups this question is not too hard, because it usually amounts to counting the dimension of the generated finite-dimensional vector space. We discuss both necessary and sufficient conditions that work on any LCA group.

On $\mathbb{R}$ various sufficient conditions have been derived in [Dau90, Grö91, Wal93]. Typically these conditions are of a qualitative nature only and say that if the lattice sizes are small enough, then the Gabor set generates a Gabor frame. Some of these conditions are formulated explicitly, but they are hard to evaluate and usually provide only little insight into the required density for a particular window g.

For general LCA groups we are only aware of one result that gives sufficient conditions on the required density [Grö91]. As a typical feature of Gabor theory on LCA groups we observe that the relevant conditions are formulated in terms of the short time Fourier transform and do not make use of any particular property of the group $\mathcal{G}$.

Theorem 6.5.1 *Suppose that $g \in L^2(\mathcal{G})$ satisfies $\|g\|_2 = 1$ and*

$$\int_{\mathcal{G}} \int_{\hat{\mathcal{G}}} |V_g g(x,\omega)| \, dx d\omega < \infty \tag{6.5.1}$$

Given a neighborhood $U \subseteq \mathcal{G} \times \hat{\mathcal{G}}$ of the identity, consider the "oscillation"

$$G_U(x,\omega) = \sup_{(y,\chi)\in U} |V_g g(x+y, \omega\cdot\chi) - V_g g(x,\omega)| \,.$$

Choose U so small that

$$\int_{\mathcal{G}} \int_{\hat{\mathcal{G}}} |G_U(x,\omega)| \, dx d\omega < 1 \,,$$

and any discrete set $X = \{(d,\chi)\} \subseteq \mathcal{G}\times\hat{\mathcal{G}}$ (not necessarily a subgroup), such that $X+U = \cup_{d\in X}(d+U) = \mathcal{G}\times\hat{\mathcal{G}}$. Then the collection $\{M_\chi T_d g, (d,\chi) \in X\}$ is a frame for $L^2(\mathcal{G})$.

This theorem is Theorem T in [Grö91] for the special case of the "Heisenberg group $H(\mathcal{G})$" built on $\mathcal{G}$ and its representation by time-frequency shifts. Hypothesis (6.5.1) becomes very natural in the context of "modulation spaces" on LCA groups, see the chapter of Feichtinger and Zimmermann in this book. If the concept of Gabor frames is extended to a larger class of function spaces on $\mathcal{G}$, then condition (6.5.1) is also necessary for a window to generate a Gabor frame. Furthermore, we remark that it is true, but far from obvious, that with (6.5.1) the oscillation G_U is always in L^1 [FG89b], Lemma 7.2.

On the other hand, there are necessary conditions on the density [Lan93, RS95a] which show that a certain minimal density is necessary for a Gabor frame. The condition resembles the uncertainty principle, and on $\mathbb{R}$ reads as follows: *If $M_{\alpha k}T_{\beta n}g, k, n \in \mathbb{Z}$ is a frame for $L^2(\mathbb{R})$, then $\alpha\beta \le 1$.* By using Beurling's density definition, the statement can be extended to non-uniform samplings of the time-frequency plane [RS95a].

We present here a version of these results for lattices in $\mathcal{G} \times \hat{\mathcal{G}}$. The case of non-uniform sampling in $\mathcal{G} \times \hat{\mathcal{G}}$ will be treated in another place.

Theorem 6.5.2 *Suppose that $D \times E$ is a lattice in $\mathcal{G} \times \hat{\mathcal{G}}$ and that $\{T_d M_\chi, d \in D, \chi \in E\}$ is a frame for $L^2(\mathcal{G})$, then $s(D)s(E) \le 1$ (where $s(D)$ is the measure of a fundamental domain of $\mathcal{G}/D$ as defined in section 2).*

Proof: Let $h = S^{-1}g$ be the dual window of g. Since we started with a frame, h is in $L^2(\mathcal{G})$, and

$$\sum_{d,\chi} |\langle f, M_\chi T_d h\rangle|^2 = \sum_{d,\chi} |(f \cdot T_d\bar{h})\hat{}(\chi)|^2 \le B\|f\|_2^2 < \infty\,, \tag{6.5.2}$$

and

$$f = \sum_{d,\chi \in D\times E} \langle f, M_\chi T_d h\rangle\, M_\chi T_d g\,.$$

The function $f \cdot T_d\bar{h}$ is in $L^1(\mathcal{G})$ and by Lemma 6.2.2 and (6.5.2) its periodization with respect to $E^\perp \subseteq \mathcal{G}$ is in $L^2(\mathcal{G}/E^\perp)$ and

$$\sum_{e\in E^\perp} (f \cdot T_d\bar{h})(x-e) = \frac{1}{s(E^\perp)} \sum_{\chi\in E} (f \cdot T_d\bar{h})\hat{}(\chi)\chi(x)$$

Therefore

$$f(x) = \sum_{d\in D} \Bigl(s(E^\perp) \sum_{e\in E^\perp} f(x-e)\bar{h}(x-e-d) \Bigr) g(x-d)$$

for all $f \in L^2(\mathcal{G})$. Let U be a fundamental domain for $\mathcal{G}/E^\perp$, i.e. $U + E^\perp = \mathcal{G}$, and set $f(x) = \chi_{a+U}(x)$. Then for $x \in a+U$, $\chi_{a+U}(x-e) = 1$ if and only if $e = 0$ and therefore the sum over $e \in E^\perp$ contains only one term and for almost all $x \in a+U$ (more precisely for all points of density in $a+U$)

$$\chi_{a+U}(x) = s(E^\perp)\chi_{a+U}(x) \sum_{d\in D} \bar{h}(x-e-d)g(x-d)$$

This is true for all $a \in E^{\perp}$ and thus

$$1 = s(E^{\perp}) \sum_{d \in D} g(x-d)\bar{h}(x-d) \quad \text{a. a. } x\,.$$

With Weil's formula (6.2.11) we now compute

$$\begin{aligned} s(D) =& s(D) \int_{\mathcal{G}/D} 1\, d\dot{x} \\ =& s(D)s(E^{\perp}) \int_{\mathcal{G}/D} \Big(\sum_{d \in D} g(x-d)\bar{h}(x-d) \Big)\, d\dot{x} = s(E^{\perp})\langle g, h\rangle\,, \end{aligned}$$

or in other words, using (6.2.14)

$$\langle g, h\rangle = \frac{s(D)}{s(E^{\perp})} = s(D)s(E) \tag{6.5.3}$$

From the general frame theory [DS52, Dau92] it is known that if $f \in L^2(\mathcal{G})$ has two representations

$$f = \sum_{d,\chi} \langle f, M_\chi T_d h\rangle M_\chi T_d g = \sum_{d,\chi} c_{d\chi} M_\chi T_d g$$

then

$$\sum_{d,\chi} |\langle f, M_\chi T_d h\rangle|^2 \le \sum_{d,\chi} |c_{d\chi}|^2 \tag{6.5.4}$$

Applying this to $f = g = \sum_{d,\chi} \delta_{d,0}\delta_{\chi,0} M_\chi T_d g$ we obtain in combination with (6.5.3) and (6.5.4)

$$(s(D)s(E))^2 = |\langle g, h\rangle|^2 \le \sum_{d,\chi} |\langle g, M_\chi T_d h\rangle|^2 \le \sum_{d,\chi} \delta_{d,0}^2 \delta_{\chi,0}^2 = 1$$

and the proof of the necessary density is finished. ❑

Remark: 1. Choosing $D \times D^{\perp}$ as the lattice, its density is $s(D)s(D^{\perp}) = 1$ by Lemma 6.2.3. This is the optimal density achievable according to this theorem and therefore justifies the denomination "critical density".

2. The above proof is inspired by the methods of Janssen [Jan94b] and Tolimieri-Orr [TO92]. With a little more effort the same method would furnish a proof of the Wexler–Raz relations, of which we have used the special case (6.5.3).

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7

Quantization of TF lattice–invariant operators on elementary LCA groups

Hans G. Feichtinger and Werner Kozek

ABSTRACT – Elementary locally compact abelian groups $\mathcal{G}$ are a natural setup for an abstract view on time–frequency (TF) analysis. The function space Gelfand triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}'_0)(\mathcal{G})$ is adapted to the sampling and periodization procedures on the abstract TF–plane $\mathcal{G} \times \widehat{\mathcal{G}}$ and it allows the definition of a generalized Kohn–Nirenberg correspondence for a "harmonic analysis and synthesis" of linear operators. We extend the concept of duality and biorthogonality of Gabor atoms to arbitrary discrete subgroups of $\mathcal{G} \times \widehat{\mathcal{G}}$ with compact quotient. The setting of elementary LCA groups is not only an extension of standard Gabor analysis but admits a unified formulation for continuous–time, discrete–time, periodic, and multidimensional signals including the case of nonseparable lattices and/or nonseparable atoms.

7.1 Introduction

Time–frequency (phase space) analysis of signals (functions) is closely connected to the Weyl–Heisenberg (WH) quantization of operators. For the Wigner–Weyl framework one has the well–known fact that the Weyl symbol of the rank–one projection onto an $\mathbf{L}^2$–function f coincides with the Wigner distribution of f. Both the Weyl symbol and the Wigner distribution have gained considerable popularity, particularly in the signal processing community [CM80, Koz92b, Fol89, MHK97]. For the operator symbol of Kohn and Nirenberg [KN65] the connection to time–frequency signal analysis leads to a rather exotic signal representation, the energy distribution of Rihaczek [Rih68]. Moreover, other popular time–frequency concepts such as ambiguity functions, short time Fourier transform are usually not connected to the problem of operator quantization.

The basic idea of (WH) operator quantization is to map functions (called symbols) on the TF plane (the phase space of quantum mechanics) to operators in a way that preserves as much as possible the conceptual power of the Fourier transform from the commutative case of translation–

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invariant operators even for noncommutative operators. There are two classical (unitary) quantization procedures: the Weyl and the Kohn–Nirenberg correspondence. Our main result states that a properly generalized Kohn–Nirenberg symbol (GKNS) is a powerful tool for the study of WH frame operators. Through a symplectic Fourier transform the GKNS $\sigma(K)$ of an operator K is connected to the operator's spreading function $\eta(K)$. The spreading function establishes an infinitesimal decomposition of K into time–frequency shifts. TF–periodization of a decent operator K corresponds to sampling of its continuous spreading function. In particular, we show that the GKNS of a WH frame operator generated by a lattice $\Lambda \triangleleft \mathcal{G} \times \widehat{\mathcal{G}}$ and an atom $g \in \mathbf{S}_0(\mathcal{G})$ belongs to $\mathbf{S}_0\left((\mathcal{G} \times \widehat{\mathcal{G}})/\Lambda\right)$, such operators can be decomposed into an ℓ^1–weighted sum of TF–shifts on an adjoint lattice Λ° (generalized Janssen representation), and we give a compact proof of a generalized Wexler–Raz condition which determines the dual atom(s).

For experts on abstract harmonic analysis the choice of our tools might look unusual if not prejudiced. (A much more "popular" setup would be the classical Schwartz–Bruhat Gelfand triple and the Weyl quantization.) However, our tools admit both a higher degree of generality and a lower sophistication of proofs:

- The Schwartz–Bruhat space of rapidly decreasing smooth functions establishes a too restrictive condition for Gabor atoms, while their topological dual, the tempered distributions, are unnecessarily general to handle the inherent sampling procedures of Gabor analysis in a rigorous manner. Moreover, the Schwartz–Bruhat triple involves the "heavy machinery" of non–Banach–type Frechet spaces with their inductive limit topology while the triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathcal{G})$, as introduced by the first named author, consists of "well–behaved" Banach spaces.

- The Weyl correspondence satisfies a beautiful invariance property concerning symplectic coordinate transforms of the Euclidean TF plane ($\mathbb{R}^{2n}$). This, however, comes at the cost of requiring automorphisms of type $x \to 2x$ when switching to general LCA groups [Hen85]. By this fact, the applicability of the Weyl symbol breaks down on general discrete and compact groups which inevitably appear in Gabor analysis. We shall return to this aspect in the last section.

7.2 Elementary LCA groups and their TF–shift

For the basics of locally compact abelian (LCA) groups see the companion chapter of Gröchenig, our setup and notation is largely consistent (the group is written additively). The single exception is that we restrict our attention to *elementary* LCA (ELCA) groups defined through their topological factors as:

$$\mathcal{G} \cong \mathbb{R}^d \times \mathbb{Z}^p \times \mathbb{T}^q \times \Gamma_m \qquad (d \geq 0, p \geq 0, q \geq 0, m \geq 1), \tag{7.2.1}$$

where $\mathbb{R}^d$ is the Euclidean d–space, $\mathbb{Z}^p$ is the p–dimensional lattice of integers, $\mathbb{T}^q$ is the q–dimensional torus group and Γ_m is a finite Abelian group with cardinality m. The product topology inherited from the factors makes $\mathcal{G}$ to an LCA group. Note that ELCA groups are always separable.

ELCA groups are general enough to cover typical signal processing setups and classical harmonic analysis. As a less obvious signal processing example, take a multichannel video signal. It may be seen as a function on the ELCA group $\mathbb{Z}^p \times \Gamma_m$, where p is the number of channels and m is the pixel number of a single image.

The set of continuous, unimodular complex homomorphisms on $\mathcal{G}$ defines the *dual group* (or group of characters):

$$\widehat{\mathcal{G}} := \{\xi \,|\, \xi(x+y) = \xi(x)\xi(y), |\xi(x)| = 1, x, y \in \mathcal{G}, x \mapsto \xi(x) \text{ continuous}\}$$

The dual of an ELCA group is again ELCA (under pointwise multiplication of characters) and can be naturally identified with

$$\widehat{\mathcal{G}} \cong \mathbb{R}^d \times \mathbb{Z}^q \times \mathbb{T}^p \times \Gamma_m$$

(since one has $\widehat{\mathbb{T}^q} \cong \mathbb{Z}^q$ and $\widehat{\mathbb{Z}^p} \cong \mathbb{T}^p$).

The Fourier transform of a function $f \in \mathbf{L}^1(\mathcal{G})$ is defined as

$$\hat{f}(\xi) = \mathcal{F}_{\mathcal{G}} f(\xi) := \int_{\mathcal{G}} f(x)\overline{\xi(x)}dx,$$

where $\xi \in \widehat{\mathcal{G}}$ and dx denotes the (translation invariant) Haar measure of $\mathcal{G}$.

For $t \in \mathcal{G}, \nu \in \widehat{\mathcal{G}}$ we define a *time–shift* (translation) operator T_t, and a *frequency–shift* (modulation) operator M_ν as unitary mappings on $L^2(\mathcal{G})$:

$$T_t f(x) \;:=\; f(x-t). \tag{7.2.2}$$
$$M_\nu f(x) \;:=\; \nu(x) f(x). \tag{7.2.3}$$

The generalized *time–frequency plane* is a self–dual ELCA group since

$$\mathcal{G} \times \widehat{\mathcal{G}} \cong \mathbb{R}^{2d} \times \mathbb{Z}^{(p+q)} \times \mathbb{T}^{(p+q)} \times \Gamma_m \times \Gamma_m.$$

However, notwithstanding the self–duality it will turn out as important to distinguish between elements of $\mathcal{G} \times \widehat{\mathcal{G}}$ and elements of $\widehat{\mathcal{G} \times \widehat{\mathcal{G}}} = \widehat{\mathcal{G}} \times \mathcal{G}$. This situation is similar to that of a vector space and its dual and therefore the following symbolic description is suggestive and will be used in this chapter. Characters $\mu : \mathcal{G} \times \widehat{\mathcal{G}} \mapsto \mathbb{T}$ are defined by

$$\mu(\lambda) := \begin{pmatrix} \nu \\ t \end{pmatrix} \cdot \begin{pmatrix} x \\ \xi \end{pmatrix} = \nu(x)t(\xi) = \nu(x)\xi(t) \tag{7.2.4}$$

$$\text{with} \quad \lambda := (x, \xi) \in \mathcal{G} \times \widehat{\mathcal{G}}, \quad \mu := (\nu, t) \in \widehat{\mathcal{G}} \times \mathcal{G}.$$

For notational simplicity we introduce a *time-frequency shift operator* as

$$\pi(\lambda) := M_\nu T_t, \qquad \lambda = (t, \nu) \in \mathcal{G} \times \widehat{\mathcal{G}}$$

which acts unitarily on $\mathbf{L}^2(\mathcal{G})$. We shall extensively use the following computational rules:

$$T_t M_\nu = \overline{\nu(t)} M_\nu T_t \tag{7.2.5}$$

$$\pi^*(\lambda) = \overline{\nu(t)} \pi(-\lambda) \tag{7.2.6}$$

$$\pi(\lambda_1)\pi(\lambda_2) = \overline{\nu_2(t_1)} \pi(\lambda_1 + \lambda_2) \tag{7.2.7}$$

$$\pi(\lambda_1)\pi(\lambda_2) = \nu_1(t_2)\overline{\nu_2(t_1)} \pi(\lambda_2)\pi(\lambda_1) = \lambda_2(\mathcal{J}\lambda_1)\pi(\lambda_2)\pi(\lambda_1), \tag{7.2.8}$$

where $\lambda_2(\mathcal{J}\lambda_1)$ denotes a *symplectic character* formulated via the following isomorphism between $\mathcal{G} \times \widehat{\mathcal{G}}$ and $\widehat{\mathcal{G}} \times \mathcal{G}$:

$$\mathcal{J} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathcal{J}^* = \mathcal{J}^{-1} = -\mathcal{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{7.2.9}$$

Moreover, we define a tensor product of time–frequency shift operators by its action on operators:

$$(\pi \otimes \pi^*)(\lambda)H := \pi(\lambda)H\pi^*(\lambda). \tag{7.2.10}$$

Later on we shall see that $(\pi \otimes \pi^*)(\lambda)$ indeed leads to a translation of the operator's symbol defined on $\mathcal{G} \times \widehat{\mathcal{G}}$, hence it makes intuitive sense to characterize the action of $(\pi \otimes \pi^*)(\lambda)$ as *time–frequency shifting of operators.* It is remarkable that (i) $(\pi \otimes \pi^*)(\lambda)$ is a unitary representation of $\mathcal{G} \times \widehat{\mathcal{G}}$ on the subsequently defined Hilbert space of Hilbert–Schmidt operators on $\mathbf{L}^2(\mathcal{G})$ (while $\pi(\lambda)$ is only a projective representation of $\mathcal{G} \times \widehat{\mathcal{G}}$, cf. (7.2.7)), i.e.,

$$(\pi \otimes \pi^*)(\lambda_1)(\pi \otimes \pi^*)(\lambda_2) = (\pi \otimes \pi^*)(\lambda_1 + \lambda_2), \tag{7.2.11}$$

(ii) $\pi(\mu)$ is the "generalized eigenoperator" for any $\mu \in \mathcal{G} \times \widehat{\mathcal{G}}$, and (iii) the symplectic character is the unimodular generalized eigenvalue:

$$(\pi \otimes \pi^*)(\lambda)\pi(\mu) = \mu(\mathcal{J}\lambda)\pi(\mu). \tag{7.2.12}$$

These facts are immediate consequences of the above computational rules for compositions of $\pi(\lambda)$.

7.3 The Gelfand triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathcal{G})$

The short time Fourier transform (STFT) of a function $f \in \mathbf{L}^2(\mathcal{G})$ using an atom (window) $g \in \mathbf{L}^2(\mathcal{G})$ is defined as ($\lambda := (t, \nu)$)

$$\mathcal{V}_g f(\lambda) := \langle f, \pi(\lambda) g \rangle_{\mathbf{L}^2(\mathcal{G})} = \int_{\mathcal{G}} f(y) \overline{g(y-t)\nu(y)} dy.$$

The STFT using an arbitrary, fixed Schwartz class window (e.g. the Gaussian function for $\mathcal{G} = \mathbb{R}^d$) $g \neq 0$ allows to define the Segal algebra $\mathbf{S}_0(\mathcal{G})$ as introduced by the first named author:

$$\mathbf{S}_0(\mathcal{G}) := \left\{ f \in \mathbf{L}^1(\mathcal{G}) | \; \|\mathcal{V}_g f\|_{\mathbf{L}^1(\mathcal{G} \times \widehat{\mathcal{G}})} < \infty \right\}.$$

It can be shown that this is a Banach space and a dense ideal of $\mathbf{L}^1(\mathcal{G})$. The $\mathbf{L}^2(\mathcal{G})$–inner product

$$\langle f, g \rangle_{\mathbf{L}^2(\mathcal{G})} = \int_{\mathcal{G}} f(x) \overline{g(x)} dx$$

is well–defined for $f, g \in \mathbf{S}_0(\mathcal{G})$ and the completion with respect to the associated norm is just $\mathbf{L}^2(\mathcal{G})$ itself. A combination of a Banach space $\mathcal{B}$ of test functions which is densely embedded in a Hilbert space $\mathcal{H}$ which in turn is weak*–continuously and densely embedded in the dual space $\mathcal{B}'$ is a special (Banach) case of a *Gelfand triple* [GW64]. (A general Gelfand triple in the sense of [GW64] involves rather Frechet spaces than Banach spaces.)

The Segal algebra $\mathbf{S}_0(\mathcal{G})$ satisfies many of the nice properties of the Schwartz–Bruhat space of test functions concerning stability under Fourier transform, time–frequency shifting, periodization and periodic sampling:

Lemma 7.3.1 *(Functorial Properties of* $\mathbf{S}_0(\mathcal{G})$*)*
Let Λ denote a closed subgroup of $\mathcal{G}$ and R_Λ the associated restriction operator (sampling operator if Λ is discrete), then one has for $f \in \mathbf{S}_0(\mathcal{G})$,

$$g = \mathcal{F}_{\mathcal{G}} f \implies g \in \mathbf{S}_0(\widehat{\mathcal{G}}) \tag{7.3.1}$$
$$g = \pi(\lambda) f \implies g \in \mathbf{S}_0(\mathcal{G}) \tag{7.3.2}$$
$$g = \sum_{\lambda \in \Lambda} T_\lambda f \implies g \in \mathbf{S}_0(\mathcal{G}/\Lambda) \tag{7.3.3}$$
$$g = R_\Lambda f \implies g \in \mathbf{S}_0(\Lambda) \tag{7.3.4}$$

For a proof we refer to [Fei81], where $\mathbf{S}_0(\mathcal{G})$ is introduced on general LCA groups.

Another property of $\mathbf{S}_0(\mathcal{G})$, relevant for this work (and common with the Schwartz–Bruhat space) is the invariance under automorphism of $\mathcal{G}$:

Lemma 7.3.2 *Let α denote an automorphism of $\mathcal{G}$, and define $\mathcal{A} : f \mapsto \mathcal{A}f$ by $\mathcal{A}f(x) = f(\alpha x)$, $x \in \mathcal{G}$, then*

$$C_1 \|f\|_{\mathbf{S}_0(\mathcal{G})} \leq \|\mathcal{A}f\|_{\mathbf{S}_0(\mathcal{G})} \leq C_2 \|f\|_{\mathbf{S}_0(\mathcal{G})}, \qquad 0 < C_1 \leq C_2 < \infty$$

i.e., any $\mathcal{G}$–automorphism induces an automorphism of $\mathbf{S}_0(\mathcal{G})$.

Proof: See [Fei81]. ❑

In the sense of mathematical sophistication the factors of an ELCA group require quite different levels. This is reflected in the canonical decomposition of $\mathbf{S}_0(\mathcal{G})$ which leads to classical function spaces in all factors except the Euclidean d–space:

$$\mathbf{S}_0(\mathcal{G}) := \mathbf{S}_0(\mathbb{R}^d) \,\hat{\otimes}\, \ell^1(\mathbb{Z}^p) \,\hat{\otimes}\, \mathcal{A}(\mathbb{T}^q) \otimes \mathbb{C}^m .$$

Here, $\ell^1(\mathbb{Z}^p)$ are the absolutely summable sequences of complex p–tupels, $\mathcal{A}(\mathbb{T}^q)$ is the Wiener algebra of absolutely convergent trigonometric series on the q–dimensional torus group, $\mathbb{C}^m$ is the Hilbert space of finite–dimensional complex vectors understood as an ℓ^2–space over a finite group of order m, and $\hat{\otimes}$ denotes the projective tensor product of Banach spaces (see e.g. [Kat68, p.250]). The function space $\mathbf{S}_0(\mathbb{R}^d)$ is the topic of a companion chapter in this book 3.

The Hilbert space $\mathbf{L}^2(\mathcal{G})$ can be decomposed in the standard Hilbert spaces on the factor groups (using the Hilbert space tensor product $\otimes$):

$$\mathbf{L}^2(\mathcal{G}) = \mathbf{L}^2(\mathbb{R}^d) \,\otimes\, \ell^2(\mathbb{Z}^p) \,\otimes\, \mathbf{L}^2(\mathbb{T}^q) \,\otimes\, \mathbb{C}^m .$$

$\mathbf{S}_0'(\mathcal{G})$, the topological dual of $\mathbf{S}_0(\mathcal{G})$ is an isometrically TF–invariant Banach space containing "truly" distributional members such as the Dirac functional defined by point evaluation

$$\langle f, \delta_x \rangle := f(x), \qquad f \in \mathbf{S}_0(\mathcal{G}), x \in \mathcal{G}$$

It is not our intention to study $\mathbf{S}_0'(\mathcal{G})$ as a Banach space per se, rather, as the inevitable part of the Gelfand triple about $\mathbf{L}^2(\mathcal{G})$. In this sense we usually do not consider the norm topology of $\mathbf{S}_0'(\mathcal{G})$. We combine Banach space functional brackets and Hilbert space inner product brackets to a unified "Gelfand bracket" notation. As an instructive example consider the generalized Fourier transform defined on $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathcal{G})$, which yields a generalized Plancherel relation

$$\langle f, g \rangle_{(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathcal{G})} = \left\langle \hat{f}, \hat{g} \right\rangle_{(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\widehat{\mathcal{G}})}, \tag{7.3.5}$$

Through this notation we express three facts about the Fourier transform (i) it is an isomorphism from $\mathbf{S}_0(\mathcal{G})$ to $\mathbf{S}_0(\widehat{\mathcal{G}})$, (ii) it is unitary map between

$\mathbf{L}^2(\mathcal{G})$ and $\mathbf{L}^2(\widehat{\mathcal{G}})$, and (iii) it extends to a weak*–continuous bijection from $\mathbf{S}_0'(\mathcal{G})$ to $\mathbf{S}_0'(\widehat{\mathcal{G}})$. (7.3.5) is valid whenever (f,g) is in $\mathbf{L}^2(\mathcal{G}) \times \mathbf{L}^2(\mathcal{G})$, $\mathbf{S}_0(\mathcal{G}) \times \mathbf{S}_0'(\mathcal{G})$ or $\mathbf{S}_0'(\mathcal{G}) \times \mathbf{S}_0(\mathcal{G})$.

Throughout this chapter various isomorphisms between $\mathbf{L}^2$-spaces are extended to isomorphisms between Gelfand triples of the form $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathcal{G})$. Such extensions are useful, as they allow to apply various transforms not only to elements in certain Hilbert spaces (usually $\mathbf{L}^2$ over some group), where the concept of orthogonality is available and indeed very important. First one may treat such transformations on the smaller space $\mathbf{S}_0$, where they can be described in terms of ordinary integral transforms. Typically equations describing the transformation (and its inverse) are well defined in a pointwise sense, and even Riemannian integrals suffice for a correct interpretation.

On the other hand the linear mappings extend in a natural (and unique) way to the distribution spaces $\mathbf{S}_0'$, which contain a number of "extreme" objects, such as Dirac measures, Shah distributions (also called "comb-functions"), or "pure frequencies", which cannot be handled within a pure Hilbert space framework. The w*–density of $\mathbf{S}_0$ in $\mathbf{S}_0'$ justifies then "formal" arguments at the $\mathbf{S}_0'$-level.

A simple abstract principle can be used to establish — without having to check too many technicalities — such an isomorphism of Gelfand triples. Therefore we describe the general principle here in some detail, and prefer to leave the verification of details for individual transformations to the interested reader. Indeed, for most cases we will just describe the transformations in a pointwise sense at the $\mathbf{S}_0$–level.

Before proceeding let us mention that the ordinary Fourier transform (and its extension to $\mathbf{S}_0'$, which may be called a "generalized Fourier transform") fall into this category. Choosing $U = \mathcal{F}$ below one has an isomorphism at the level of $\mathbf{S}_0$ (due to the Fourier invariance of that space), which extends to a unitary mapping between the corresponding $\mathbf{L}^2$-spaces (Plancherel's theorem), and even to the larger space $\mathbf{S}_0'$. Indeed, one can say that $\mathcal{F}$ is uniquely determined by the fact that it maps pure frequencies into the corresponding Dirac measures, and conversely. Indeed, this way of describing $\mathcal{F}$ makes the analogy between the setup of finite groups (where all this can be handled in a pure ℓ^2-setup) most natural, but clearly requires the use of "generalized functions" as soon as continuous variables are arising, either in the time or in the frequency domain.

Theorem 7.3.3 *(Extension of Unitary Gelfand Triple Isomorphism)*
A unitary mapping U acting from $\mathbf{L}^2(\mathcal{G}_1)$ to $\mathbf{L}^2(\mathcal{G}_2)$ extends to an isomorphism between the Gelfand triples $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathcal{G}_1)$ and $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathcal{G}_2)$ if and only if the restrictions of U and U^* are bounded linear operators between $\mathbf{S}_0(\mathcal{G}_1)$ and $\mathbf{S}_0(\mathcal{G}_2)$ respectively, i.e., if and only if there exists a

constant $C > 0$ *such that*

$$\|Uf\|_{\mathbf{S}_0(\mathcal{G}_2)} \leq C\|f\|_{\mathbf{S}_0(\mathcal{G}_1)}, \qquad \forall \quad f \in \mathbf{S}_0(\mathcal{G}_1), \tag{7.3.6}$$

and

$$\|U^*f\|_{\mathbf{S}_0(\mathcal{G}_1)} \leq C\|f\|_{\mathbf{S}_0(\mathcal{G}_2)}, \qquad \forall \quad f \in \mathbf{S}_0(\mathcal{G}_2). \tag{7.3.7}$$

Proof: If some unitary operator extends to an isomorphism of Gelfand triples, then clearly the restrictions of U and $U^{-1} = U^*$ have to be bounded linear mappings on the corresponding $\mathbf{S}_0$–spaces. Assume conversely that (7.3.7) is satisfied. Then, the definition

$$\langle \overline{U}g, f \rangle = \langle g, U^* f \rangle, \qquad \text{for} \quad g \in \mathbf{S}_0'(\mathcal{G}_1), f \in \mathbf{S}_0(\mathcal{G}_2)$$

yields a bounded linear mapping $g \to \overline{U}g$ which extends the unitary mapping U, defined on $\mathbf{L}^2(\mathcal{G}_1)$. Since $\mathbf{L}^2(\mathcal{G}_1)$ is (boundedly) w^*–dense in $\mathbf{S}_0'(\mathcal{G}_1)$, $\overline{U}$ is uniquely determined as a w^*–w^*–continuous and bounded linear mapping between $\mathbf{S}_0'(\mathcal{G}_1)$ and $\mathbf{S}_0'(\mathcal{G}_2)$, which coincides with U on $\mathbf{L}^2(\mathcal{G}_1)$. The same argument applies for $U^{-1} = U^*$. Thus it is clear that $\overline{U}$ defines also an isomorphism between $\mathbf{S}_0'(\mathcal{G}_1)$ and $\mathbf{S}_0'(\mathcal{G}_2)$ (both equipped with either their norm or their w^*–topologies, respectively). ❑

As illustration, the generalized Fourier transform of the Dirac measure can be determined by (7.3.5) — with the expected result $\hat{\delta}_x(\xi) = \overline{x(\xi)}$ — as follows

$$\langle \hat{f}, \hat{\delta}_x \rangle_{\mathbf{S}_0(\widehat{\mathcal{G}})} = \langle f, \delta_x \rangle_{\mathbf{S}_0(\mathcal{G})} = \left(\mathcal{F}_{\mathcal{G}}^{-1}\hat{f}\right)(x) = \int_{\widehat{\mathcal{G}}} \hat{f}(\xi)\xi(x)dx, \qquad f \in \mathbf{S}_0(\mathcal{G}).$$

This leads to the (formal) continuous expansions of $\mathbf{S}_0(\mathcal{G})$–functions:

$$f = \int_{\mathcal{G}} \langle f, \delta_x \rangle_{\mathbf{S}_0(\mathcal{G})} \delta_x dx = \int_{\widehat{\mathcal{G}}} \langle f, \xi \rangle_{\mathbf{S}_0(\mathcal{G})} \xi d\xi. \tag{7.3.8}$$

Note that $\lim_{n\to\infty} x_n = x_0$ does not imply $\|\delta_{x_n} - \delta_{x_0}\|_{\mathbf{S}_0'(\mathcal{G})} \to 0$ for $x_n \neq x_0$, but $\langle f, \delta_{x_n} \rangle = f(x_n) \to \langle f, \delta_{x_0} \rangle = f(x_0)$ for all $f \in \mathbf{S}_0(\mathcal{G})$, i.e., w^*–convergence is valid and the extended Fourier transform preserves such w^*–convergence.

The following corollary says, in essence, that all one has to care about in establishing Gelfand triple isomorphisms is the behavior of the transform for test functions. The rest is "free" by a density and duality argument.

Corollary 7.3.4 *An isomorphism* $V : \mathbf{S}_0(\mathcal{G}_1) \to \mathbf{S}_0(\mathcal{G}_2)$ *extends to a unitary Gelfand triple isomorphism between* $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathcal{G}_1)$ *and* $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathcal{G}_2)$ *if and only if*

$$\langle f_1, f_2 \rangle_{\mathbf{L}^2(\mathcal{G}_1)} = \langle V f_1, V f_2 \rangle_{\mathbf{L}^2(\mathcal{G}_2)}, \qquad \forall f_1, f_2 \in \mathbf{S}_0(\mathcal{G}_1). \tag{7.3.9}$$

Proof: Clearly, (7.3.9) is a necessary condition for a unitary operator. Conversely (7.3.9) implies

$$\|f\|^2_{\mathbf{L}^2(\mathcal{G}_1)} = \langle f, f\rangle_{\mathbf{L}^2(\mathcal{G}_1)} = \langle Vf, Vf\rangle_{\mathbf{L}^2(\mathcal{G}_2)} = \|Vf\|^2_{\mathbf{L}^2(\mathcal{G}_2)}$$

for all $f \in \mathbf{S}_0(\mathcal{G}_1)$ and thus V extends to an isometric mapping (again called V) from $\mathbf{L}^2(\mathcal{G}_1)$ to $\mathbf{L}^2(\mathcal{G}_2)$ with closed range. Since $V(\mathbf{S}_0(\mathcal{G}_1)) = \mathbf{S}_0(\mathcal{G}_2)$ is dense in $\mathbf{L}^2(\mathcal{G}_2)$, V has a dense range in $\mathbf{L}^2(\mathcal{G}_2)$, hence V is a bijection between $\mathbf{L}^2(\mathcal{G}_1)$ and $\mathbf{L}^2(\mathcal{G}_2)$.

The rest follows from the extension principle Theorem 7.3.3. ❑

Another tool of fundamental relevance for the rest of this chapter is the existence of a "Gelfand triple basis" :

Lemma 7.3.5 *Let $(e_k)_{k\in I}$ be the tensor product of (i) a Wilson basis on $\mathbb{R}^d$, (ii) the standard basis of $\mathbb{Z}^p$, (iii) the trigonometric polynomials on $\mathbb{T}^q$, and any orthonormal basis of $\mathbb{C}^m$. Then one has:*

(a) $(e_k)_{k\in I}$ is a bounded, absolute basis of $\mathbf{S}_0(\mathcal{G})$, i.e.,

$$A_1\|f\|_{\mathbf{S}_0(\mathcal{G})} \le \sum_{k\in I} |\langle f, e_k\rangle| \le B_1\|f\|_{\mathbf{S}_0(\mathcal{G})}, \qquad 0 < A_1 \le B_1 < \infty,$$

(b) $(e_k)_{k\in I}$ is an orthonormal basis of $\mathbf{L}^2(\mathcal{G})$,

(c) $(e_k)_{k\in I}$ is a weak basis of $\mathbf{S}_0'(\mathcal{G})$.*

Summing up, the mapping $f \mapsto \{\langle f, e_k\rangle\}_{k\in I}$ establishes a unitary Gelfand triple isomorphism as follows:

$$(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathcal{G}) \leftrightarrow (\ell^1, \ell^2, \ell^\infty)(I).$$

Proof: (I) First, we consider the case $\mathcal{G} = \mathbb{R}^d$. Wilson bases have been introduced as orthonormal bases for $\mathbf{L}^2(\mathbb{R}^d)$ in [DJJ91], thus establishing a natural isomorphism with $\ell^2(I)$. This identification can be extended to the family of so-called modulation spaces, mapping them isomorphically onto appropriate sequence spaces. As a special case one has: The restriction of this isomorphism maps $\mathbf{S}_0(\mathbb{R}^d)$ isomorphically onto $\ell^1(I)$ [FGW92], i.e., $f \in \mathbf{L}^2(\mathbb{R}^d)$ belongs to the subspace $\mathbf{S}_0(\mathbb{R}^d)$ if and only if its Wilson coefficients are absolutely summable.
(II) One has by definition $\mathbf{S}_0(\mathbb{Z}) = \ell^1(\mathbb{Z}), \mathbf{S}_0(\mathbb{T}) = A(\mathbb{T})$ and $\mathbf{S}_0(\Gamma^m) = \mathbb{C}^m$. Using these facts and (I) one establishes (b) by the standard Hilbert space argument about tensorization of orthonormal bases [BSU96, Vol.II, p.152].
(III) In order to derive (a) we have to verify that the subspace of $\mathbf{L}^2(\mathcal{G})$ with ℓ^1-coefficients is exactly $\mathbf{S}_0(\mathcal{G})$. Clearly, it suffices to consider the case

of two factors of the ELCA group. Consider $\mathcal{G}_1, \mathcal{G}_2$, let $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$ and denote the bases of $\mathbf{S}_0(\mathcal{G}_i)$ by $\{e_k^{(i)}\}_{k\in I}$, hence

$$A^{(i)}\|f\|_{\mathbf{S}_0(\mathcal{G}_i)} \le \sum_{k\in I} \left|\langle f, e_k^{(i)}\rangle\right| \le B^{(i)}\|f\|_{\mathbf{S}_0(\mathcal{G}_i)}, \quad i = 1, 2.$$

We have to show that $\mathbf{S}_0(\mathcal{G})$ coincides with a new Banach space defined by

$$\mathcal{B}_0(\mathcal{G}) := \left\{ f \mid f = \sum_{k,l\in I} c_{k,l}(e_k^{(1)} \otimes e_l^{(2)}), \sum_{k,l\in I} |c_{k,l}| < \infty \right\},$$
$$\|f\|_{\mathcal{B}_0(\mathcal{G})} := \sum_{k,l\in I} |c_{k,l}| < \infty.$$

By the tensor product property for $\mathbf{S}_0(\mathcal{G})$ [Fei81] we have

$$\mathbf{S}_0(\mathcal{G}) = \mathbf{S}_0(\mathcal{G}_1)\hat{\otimes}\mathbf{S}_0(\mathcal{G}_2) := \left\{ f \mid f = \sum_{m\in I} f_m^{(1)} \otimes f_m^{(2)} \right\}$$
$$\text{with } \|f\|_{\mathbf{S}_0(\mathcal{G})} := \inf\left\{ \sum_{m\in I} \|f_m^{(1)}\|_{\mathbf{S}_0(\mathcal{G}_1)} \|f_m^{(2)}\|_{\mathbf{S}_0(\mathcal{G}_2)} \right\} < \infty,$$

where the infimum is taken over all admissible (= absolutely convergent) representations. First we show $\mathcal{B}_0(\mathcal{G}) \subseteq \mathbf{S}_0(\mathcal{G})$:

$$\begin{aligned}
\|f\|_{\mathbf{S}_0(\mathcal{G})} &= \left\| \sum_{k,l\in I} \langle f, e_k^{(1)} \otimes e_l^{(2)}\rangle\, e_k^{(1)} \otimes e_l^{(2)} \right\|_{\mathbf{S}_0(\mathcal{G})} \\
&\le \sum_{k,l\in I} \left|\langle f, e_k^{(1)} \otimes e_l^{(2)}\rangle\right| \left\|e_k^{(1)}\right\|_{\mathbf{S}_0(\mathcal{G}_1)} \left\|e_l^{(2)}\right\|_{\mathbf{S}_0(\mathcal{G}_2)} \\
&\le B^{(1)}B^{(2)}\|f\|_{\mathcal{B}_0(\mathcal{G})}.
\end{aligned}$$

Conversely, we show that $\mathbf{S}_0(\mathcal{G})$ is embedded into $\mathcal{B}_0(\mathcal{G})$:

$$\begin{aligned}
\|f\|_{\mathcal{B}_0(\mathcal{G})} &= \sum_{m\in I}\sum_{k,l\in I} \left|\langle f, e_k^{(1)} \otimes e_l^{(2)}\rangle\right| \\
&= \sum_{m\in I}\sum_{k,l\in I} \left|\langle f_m^{(1)}, e_k^{(1)}\rangle\right| \left|\langle f_m^{(2)}, e_l^{(2)}\rangle\right| \\
&\le \sum_{m\in I} B^{(1)}B^{(2)}\|f_m^{(1)}\|_{\mathbf{S}_0(\mathcal{G}_1)}\|f_m^{(2)}\|_{\mathbf{S}_0(\mathcal{G}_2)} \\
&= B^{(1)}B^{(2)}\|f\|_{\mathbf{S}_0(\mathcal{G})}.
\end{aligned}$$

This finishes the proof of (a).
(IV) By duality we can identify the elements of $\mathbf{S}_0'(\mathcal{G})$ with ℓ^∞–sequences applied to the Hilbert space basis (in $\mathbf{S}_0(\mathcal{G})$), which shows (c). ❑

As a first application of the canonical basis we study the behavior of the partial Fourier transform (for later use in this work):

Lemma 7.3.6 *The partial Fourier transform of* $f \in (\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathcal{G}_1 \times \mathcal{G}_2)$ *defined by*

$$(\mathcal{F}_2 f)(x, \xi) := \int_{\mathcal{G}_2} f(x, y)\overline{\xi(y)}dy.$$

is a unitary Gelfand triple isomorphism between $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathcal{G}_1 \times \mathcal{G}_2)$ *and* $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathcal{G}_1 \times \widehat{\mathcal{G}}_2)$*:*

$$\langle \mathcal{F}_2 f, \mathcal{F}_2 g\rangle_{(\mathbf{S}_0,\mathbf{L}^2,\mathbf{S}_0')(\mathcal{G}_1\times\widehat{\mathcal{G}}_2)} = \langle f, g\rangle_{(\mathbf{S}_0,\mathbf{L}^2,\mathbf{S}_0')(\mathcal{G}_1\times\mathcal{G}_2)}. \tag{7.3.10}$$

Proof: (I) We show that $\|\mathcal{F}_2 f\|_{\mathbf{S}_0(\mathcal{G}_1\times\widehat{\mathcal{G}}_2)} \leq C\|f\|_{\mathbf{S}_0(\mathcal{G}_1\times\mathcal{G}_2)}$. To this end we introduce the canonical Gelfand–triple bases of $\mathbf{S}_0(\mathcal{G}_1)$ and $\mathbf{S}_0(\mathcal{G}_2)$, denoted by $\{u_k\}_{k\in I}$ and $\{v_k\}_{k\in I}$:

$$f(x, y) = \sum_{k,l\in I} (f)_{k,l} u_k(x) v_l(y).$$

Applying $\mathcal{F}_2$ to the expansion and using the $\mathbf{S}_0$–boundedness of the basis and the $\mathbf{S}_0$–stability of the Fourier transform we immediately have

$$\begin{aligned}\|\mathcal{F}_2 f\|_{\mathbf{S}_0(\mathcal{G}_1\times\widehat{\mathcal{G}}_2)} &\leq \sum_{k,l\in I} |(f)_{k,l}| \|u_k\|_{\mathbf{S}_0(\mathcal{G}_1)} \|\hat{v}_l\|_{\mathbf{S}_0(\widehat{\mathcal{G}}_2)} \\ &\leq C_1 \sum_{k,l\in I} |(f)_{k,l}| \\ &\leq C_2 \|f\|_{\mathbf{S}_0(\mathcal{G}_1\times\mathcal{G}_2)}.\end{aligned}$$

(II) By analog reasoning one can show that for $g \in \mathbf{S}_0(\mathcal{G}_1 \times \widehat{\mathcal{G}}_2)$:

$$\|\mathcal{F}_2^{-1} g\|_{\mathbf{S}_0(\mathcal{G}_1\times\mathcal{G}_2)} \leq C\|g\|_{\mathbf{S}_0(\mathcal{G}_1\times\widehat{\mathcal{G}}_2)}.$$

(III) By the extension principle, Corollary 7.3.4 it remains to show that

$$\langle \mathcal{F}_2 f, \mathcal{F}_2 g\rangle_{\mathbf{L}^2(\mathcal{G}_1\times\widehat{\mathcal{G}}_2)} = \langle f, g\rangle_{\mathbf{L}^2(\mathcal{G}_1\times\mathcal{G}_2)}, \quad \text{for} \quad f, g \in \mathbf{S}_0(\mathcal{G}_1 \times \mathcal{G}_2).$$

We use the orthonormality of the basis (and the fact that all involved functions/sequences are absolutely summable/integrable):

$$\begin{aligned}
\langle \mathcal{F}_2 f, \mathcal{F}_2 g\rangle_{\mathbf{L}^2(\mathcal{G}_1\times\widehat{\mathcal{G}}_2)} &= \int\limits_{\mathcal{G}\times\widehat{\mathcal{G}}} \sum_{k,l\in I} (f)_{k,l} u_k(x)\hat{v}_l(\xi) \overline{\sum_{k',l'\in I} (g)_{k',l'} u_{k'}(x)\hat{v}_{l'}(\xi)}\, dx d\xi \\
&= \sum_{k,l\in I}\sum_{k',l'\in I} (f)_{k,l}\overline{(g)_{k',l'}}\, \underbrace{\langle u_k, u_{k'}\rangle_{\mathbf{L}^2(\mathcal{G}_1)}}_{=\delta_{k,k'}} \underbrace{\langle \hat{v}_l, \hat{v}_{l'}\rangle_{\mathbf{L}^2(\mathcal{G}_2)}}_{=\delta_{l,l'}} \\
&= \sum_{k,l\in I} (f)_{k,l}\overline{(g)_{k,l}} \\
&= \langle f, g\rangle_{\mathbf{L}^2(\mathcal{G}_1\times\mathcal{G}_2)}
\end{aligned}$$

❑

7.4 The operator Gelfand triple $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$

A linear integral operator K acting on $\mathbf{L}^2(\mathcal{G})$ is defined by

$$Kf(x) = \int_{\mathcal{G}} \kappa(K)(x,y) f(y) dy.$$

It is well–known that K is a Hilbert–Schmidt (HS) operator, whenever $\kappa(K) \in \mathbf{L}^2(\mathcal{G}\times\mathcal{G})$ [Gaa73]. HS operators establish a Hilbert algebra $\mathcal{H}$ with the inner product

$$\langle K, L\rangle_{\mathcal{H}} := \langle \kappa(K), \kappa(L)\rangle_{\mathbf{L}^2(\mathcal{G}\times\mathcal{G})}$$

and associated norm $\|K\|_{HS} := \sqrt{\langle K, K\rangle} = \mathrm{tr}(K^*K)$.

The unitary isomorphism between $\mathbf{L}^2(\mathcal{G}\times\mathcal{G})$ and $\mathcal{H}$ suggests the existence of a Gelfand triple of Banach spaces of operators with $\kappa(K) \in (\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathcal{G}\times\mathcal{G})$. In fact, via the isomorphism of $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathcal{G})$ to the Gelfand triple of sequence spaces $(\ell^1, \ell^2, \ell^\infty)(I)$ it suffices to resort to the theory of biinfinite matrices acting on the standard sequence spaces. It is clear that matrices in $(\ell^1, \ell^2, \ell^\infty)(I\times I)$ correspond to (a) bounded operators from $\ell^\infty(I)$ to $\ell^1(I)$, (b) HS operators on $\ell^2(I)$, and (c) bounded operators from $\ell^1(I)$ to $\ell^\infty(I)$. This implies in particular for case (a) that $\ell^1(I\times I)$ establishes trace–class operators on $\ell^2(I)$ and for case (c) validity of a "Schwartz–kernel theorem" for $\mathbf{S}_0(\mathcal{G})$. Kernel theorem means that bounded operators from $\mathbf{S}_0(\mathcal{G})$ to $\mathbf{S}_0'(\mathcal{G})$ correspond uniquely to an element of $\mathbf{S}_0'(\mathcal{G}\times\mathcal{G})$. Note, however, that the "Schwartz–kernel theorem" for $\mathbf{S}_0(\mathbb{R}^d)$ was proved without canonical basis in [FG92a]. First, we study the "small" side of the operator Gelfand triple, i.e., the compact, regular integral operators with $\mathbf{S}_0(\mathcal{G}\times\mathcal{G})$–kernels.

Theorem 7.4.1 *There exists a one-to-one correspondence between the space* $\mathbf{S}_0(\mathcal{G} \times \mathcal{G})$ *and the Banach space* $\mathcal{B}$ *of all bounded linear operators* $K : \mathbf{S}_0'(\mathcal{G}) \mapsto \mathbf{S}_0(\mathcal{G})$ *(with the operator norm) satisfying the extra condition that* w^**-convergent sequences, i.e.,* $\{g_n\}_{n\in I} \subseteq \mathbf{S}_0'(\mathcal{G})$ *with*

$$\langle f, g_n\rangle_{\mathbf{S}_0(\mathcal{G})} \to \langle f, g_0\rangle_{\mathbf{S}_0(\mathcal{G})}, \quad \forall f \in \mathbf{S}_0(\mathcal{G}),$$

are mapped onto $\mathbf{S}_0(\mathcal{G})$*-Cauchy sequences:*

$$\|Kg_n - Kg_0\|_{\mathbf{S}_0(\mathcal{G})} \to 0.$$

Furthermore, $\mathcal{B}$ *is an ideal in the Banach algebra of all bounded linear operators on* $\mathbf{L}^2(\mathcal{G})$*, as well as a proper subalgebra of the trace-class operators on* $\mathbf{L}^2(\mathcal{G})$.

Proof:

(I) We first show that $\kappa(K) \in \mathbf{S}_0(\mathcal{G} \times \mathcal{G})$ implies the claimed properties. K is a HS operator acting on $\mathbf{L}^2(\mathcal{G})$ defined by $\kappa(K)$, the matrix representation of this operator w.r.t. the canonical $\mathbf{S}_0(\mathcal{G})$-basis $\{e_k\}_{k\in I}$ is thus well-defined and by a standard tensor trick seen to be $\ell^1(I \times I)$:

$$\langle Ke_n, e_m\rangle_{\mathbf{L}^2(\mathcal{G})} = \langle \kappa(K), e_m \otimes e_n^*\rangle_{\mathbf{L}^2(\mathcal{G}\times\mathcal{G})}, \tag{7.4.1}$$

because $e_m \otimes e_n^*$ is just another canonical basis for $\mathbf{S}_0(\mathcal{G} \times \mathcal{G})$. Now, a biinfinite ℓ^1-matrix defines a bounded operator from $\ell^\infty(I)$ to $\ell^1(I)$ and represents a trace-class operator on $\ell^2(I)$. Switching back to $\mathbf{L}^2(\mathcal{G} \times \mathcal{G})$ shows that $\kappa(K)$ induces a proper subalgebra[1] of the trace-class operator ideal on $\mathbf{L}^2(\mathcal{G} \times \mathcal{G})$.

(II) Second, we show that to any continuous operator K from $\mathbf{S}_0'(\mathcal{G})$ (equipped with the w*-topology) to $\mathbf{S}_0(\mathcal{G})$ corresponds a kernel $\kappa(K) \in \mathbf{S}_0(\mathcal{G} \times \mathcal{G})$. Since $Ke_m \in \mathbf{S}_0(\mathcal{G}) \subseteq \mathbf{S}_0'(\mathcal{G})$ we can introduce a well-defined a priori ℓ^∞-matrix representation of K just as in (7.4.1)

$$(K)_{m,n} = \langle Ke_n, e_m\rangle_{\mathbf{S}_0'(\mathcal{G})}\,.$$

in the sense that K maps $f = \sum_{n\in I} c_n e_n$ to $Kf = \sum_{m\in I} d_m e_m$ with $d_m = \sum_{n\in I} (K)_{m,n} c_n$ for all $m \in I$. Due to the extra condition this representation is not only valid for f in the closed linear span of $(e_n)_{n\in I}$ in $\mathbf{S}_0'(\mathcal{G})$, but for all of $\mathbf{S}_0'(\mathcal{G})$.

However, we require this $\ell^\infty(I \times I)$ matrix to act continuously from $\ell^\infty(I) = (\ell^1(I))'$ to $\ell^1(I)$, hence it must be $\ell^1(I \times I)$. This in turn implies

[1] To see that this inclusion is proper consider a rank-one projection onto a discontinuous function in $\mathbf{L}^2 \cap \mathbf{L}^1(\mathcal{G})$, e.g., a box function: it establishes a trace-class operator which is not in $\mathcal{B}$, i.e., its matrix representation w.r.t. the $\mathbf{S}_0(\mathcal{G})$-basis is not in $\ell^1(I \times I)$.

that $\kappa(K) \in \mathbf{S}_0(\mathcal{G} \times \mathcal{G})$ since

$$\kappa(K)(x,y) = \sum_{m,n \in I \times I} (K)_{m,n} e_m(x) e_n^*(y).$$

❏

Theorem 7.4.2 *The bounded operators K from $\mathbf{S}_0(\mathcal{G})$ to $\mathbf{S}_0'(\mathcal{G})$ are in one-to-one correspondence with their kernels $\kappa(K)$ in $\mathbf{S}_0'(\mathcal{G} \times \mathcal{G})$. These operators form a Banach space $\mathcal{B}'$ which is the dual of the Banach space $\mathcal{B}$ considered in Theorem 7.4.1. Together with $\mathcal{H}$, the Hilbert space of Hilbert–Schmidt operators on $\mathbf{L}^2(\mathcal{G})$, these Banach spaces form an Gelfand triple isomorphic to the function space Gelfand triple on the product group $\mathcal{G} \times \mathcal{G}$:*

$$K \in (\mathcal{B}, \mathcal{H}, \mathcal{B}') \qquad \longleftrightarrow \qquad \kappa(K) \in (\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathcal{G} \times \mathcal{G})$$

which implies validity of:

$$\langle K, L \rangle_{(\mathcal{B},\mathcal{H},\mathcal{B}')} := tr\{KL^*\} = \langle \kappa(K), \kappa(L) \rangle_{(\mathbf{S}_0,\mathbf{L}^2,\mathbf{S}_0')(\mathcal{G}\times\widehat{\mathcal{G}})} \cdot \tag{7.4.2}$$

Proof: Analogously to the proof of Theorem 7.4.1, the Gelfand triple isomorphism can be shown by using the matrix representations w.r.t. the canonical basis (operators in $\mathcal{B}'$ correspond to $\ell^\infty(I \times I)$–matrices).

The trace in (7.4.2) is well–defined since $K \in \mathcal{B}$ and $L \in \mathcal{B}'$ implies $KL \in \mathcal{B}$. ❏

As a typical application of the isomorphism between Banach spaces of operators and function spaces we mention the following "evaluation rule" for sesquilinear functionals:

$$\begin{aligned}
\langle Kf, g \rangle_{(\mathbf{S}_0,\mathbf{L}^2,\mathbf{S}_0')(\mathcal{G})} &= \langle K, g \otimes f^* \rangle_{(\mathcal{B},\mathcal{H},\mathcal{B}')} \\
&= \langle \kappa(K), \kappa(g \otimes f^*) \rangle_{(\mathbf{S}_0,\mathbf{L}^2,\mathbf{S}_0')(\mathcal{G}\times\mathcal{G})} \\
&= \int_{\mathcal{G}\times\mathcal{G}} \kappa(K)(x,y) \overline{\kappa(g \otimes f^*)(x,y)}\, dx dy,
\end{aligned}$$

where we define a rank–one operator $g \otimes f^*$ by its (usual) kernel, i.e.,

$$\kappa(g \otimes f^*)(x,y) := g(x)\overline{f(y)} \quad \Longleftrightarrow \quad (g \otimes f^*)h = \langle h, f \rangle g \tag{7.4.3}$$

Such a rather formal looking tensor algebra leads to short–cut proofs of prominent results of Gabor theory (and even their generalizations). Replacing K by a rank–one operator shows that (7.4.3) may be seen as the root of all "Moyal type" (sesqui–unitary) properties of TF representations:

$$\langle e \otimes d^*, g \otimes f^* \rangle_{(\mathcal{B},\mathcal{H},\mathcal{B}')} = \langle e, g \rangle_{(\mathbf{S}_0,\mathbf{L}^2,\mathbf{S}_0')(\mathcal{G})} \overline{\langle d, f \rangle}_{(\mathbf{S}_0,\mathbf{L}^2,\mathbf{S}_0')(\mathcal{G})} \cdot \tag{7.4.4}$$

7.5 The generalized KN correspondence

The Fourier transform diagonalizes convolution operators, acting as

$$Kf(x) = \int_{\mathcal{G}} h(y-x)f(y)dy, \tag{7.5.1}$$

in the sense of a Gelfand transform, i.e., the action of a convolution operator on $\mathbf{L}^2(\mathcal{G})$ is mapped onto a pointwise multiplication in the frequency domain:

$$Kf(x) = \int_{\widehat{\mathcal{G}}} \hat{h}(\xi)\hat{f}(\xi)\xi(x)d\xi. \tag{7.5.2}$$

From the linear operator viewpoint, one interprets $\hat{h}$ as a symbol of the operator K rather than just the Fourier transform of the function h. In this spirit, one extends the definition of convolution operators allowing distributional kernels.

For a general linear operator one may suggest a correspondence between the frequency dependent symbol and a time–frequency dependent function:

$$Kf(x) =: \int_{\widehat{\mathcal{G}}} \sigma(K)(x,\xi)\hat{f}(\xi)\xi(x)d\xi \tag{7.5.3}$$

where $\sigma(K)$ is usually called the operator symbol of Kohn and Nirenberg (in the terminology introduced by Folland [Fol89]). However, the Fourier transform is in general mismatched in the sense that one cannot hope to read basic properties (boundedness, invertibility, self–adjointness, positivity etc.) of the operator from its symbol. Yet, there exist important classes of specific non–convolution operators where a properly applied Fourier transform turns out very helpful. Pseudo–differential operators (PDOs) are the most prominent instance of such a situation. For the present context we define a generalized Kohn–Nirenberg correspondence directly as a mapping between functions or distributions.

Theorem 7.5.1 *The generalized Kohn–Nirenberg correspondence is an invertible mapping between the (distributional) kernels $\kappa(K)$ of operators, defined on $\mathcal{G} \times \mathcal{G}$, and their symbols $\sigma(K)$ defined on $\mathcal{G} \times \widehat{\mathcal{G}}$, given by*

$$\sigma(K)(x,\xi) := \int_{\mathcal{G}} \kappa(K)(x,x-y)\overline{\xi(y)}dy, \tag{7.5.4}$$

$$\kappa(K)(x,y) = \int_{\widehat{\mathcal{G}}} \sigma(K)(x,\xi)\xi(x-y)d\xi. \tag{7.5.5}$$

It thus establishes a unitary Gelfand triple isomorphism:

$$K \in (\mathcal{B},\mathcal{H},\mathcal{B}') \longleftrightarrow \sigma(K) \in (\mathbf{S}_0,\mathbf{L}^2,\mathbf{S}_0')(\mathcal{G}\times\widehat{\mathcal{G}}),$$

which implies validity of the associated Gelfand–bracket identity:

$$\langle K, L\rangle_{(\mathcal{B},\mathcal{H},\mathcal{B}')} = \langle \sigma(K), \sigma(L)\rangle_{(\mathbf{S}_0,\mathbf{L}^2,\mathbf{S}_0')(\mathcal{G}\times\widehat{\mathcal{G}})} \,. \tag{7.5.6}$$

Proof:
(I) The mapping $\mathcal{A}_0 : f \mapsto \mathcal{A}_0 f$

$$\mathcal{A}_0 f(x,y) = f(x, x-y)$$

is clearly induced by an automorphism $\alpha : (x,y) \mapsto (x, x-y)$ of $\mathcal{G} \times \mathcal{G}$. By combination of $\mathcal{A}_0$ and partial Fourier transform w.r.t. the second variable (defined as in Lemma 7.3.6) the GKNS can be written as:

$$\sigma(K) = \mathcal{F}_2 \mathcal{A}_0 \kappa(K) \tag{7.5.7}$$

Lemma 7.3.6 and Lemma 7.3.2 imply that the $\kappa(K) \mapsto \sigma(K)$ is a continuous injection from $\mathbf{S}_0(\mathcal{G} \times \mathcal{G})$ into $\mathbf{S}_0(\mathcal{G} \times \widehat{\mathcal{G}})$.
(II) Elementary analysis shows that the inverse Kohn–Nirenberg correspondence has an almost identical structure (since $\alpha^{-1} = \alpha$):

$$\kappa(K) = \mathcal{A}_0 \mathcal{F}_2^{-1} \sigma(K) \tag{7.5.8}$$

such that, again using Lemma 7.3.6 and Lemma 7.3.2, $\sigma(K) \mapsto \kappa(K)$ is a continuous injection from $\mathbf{S}_0(\mathcal{G} \times \widehat{\mathcal{G}})$ into $\mathbf{S}_0(\mathcal{G} \times \mathcal{G})$.

Hence, we have established the isomorphism $\mathcal{B} \leftrightarrow \mathbf{S}_0(\mathcal{G} \times \widehat{\mathcal{G}})$.
(III) Corollary 7.3.4 says that now in order to establish the unitary Gelfand triple isomorphism it suffices to show that for two operators $K, L \in \mathcal{B}$ one has $\langle \sigma(K), \sigma(L)\rangle\rangle_{\mathbf{L}^2(\mathcal{G}\times\widehat{\mathcal{G}})} = \langle \kappa(K), \kappa(L)\rangle_{\mathbf{L}^2(\mathcal{G}\times\mathcal{G})}$. But this follows easily from Lemma 7.3.6 (Plancherel–type property of the partial Fourier transform) and the fact that α is measure–preserving:

$$\begin{aligned}
\langle \sigma(K), \sigma(L)\rangle_{\mathbf{L}^2(\mathcal{G}\times\widehat{\mathcal{G}})} &= \langle \mathcal{F}_2\mathcal{A}_0\kappa(K), \mathcal{F}_2\mathcal{A}_0\kappa(L)\rangle_{\mathbf{L}^2(\mathcal{G}\times\widehat{\mathcal{G}})} \\
&= \langle \mathcal{A}_0\kappa(K), \mathcal{A}_0\kappa(L)\rangle_{\mathbf{L}^2(\mathcal{G}\times\mathcal{G})} \\
&= \langle \kappa(K), \kappa(L)\rangle_{\mathbf{L}^2(\mathcal{G}\times\mathcal{G})} \\
&= \langle K, L\rangle_{\mathcal{B}} \,.
\end{aligned}$$

❑

Remark: For $\mathcal{G} = \mathbb{R}^d$ the GKNS reduces to the operator symbol of Kohn and Nirenberg (which is often referred to just as standard symbol in the pseudo–differential operator literature) [KN65]. In electrical engineering, the GKNS appears as *time–varying transfer function* for $\mathcal{G} = \mathbb{R}$ [Zad50] or $\mathcal{G} = \mathbb{Z}$ [LA84]. The GKNS of a rank–one operator $f \otimes g^*$ is equal to the Rihaczek distribution of f against g [Rih68]:

$$\sigma(f \otimes g^*)(x,\xi) = f(x)\overline{\widehat{g}(\xi) x(\xi)} \tag{7.5.9}$$

Note that $f, g \in \mathbf{S}_0(\mathcal{G})$ implies $f \otimes g^* \in \mathcal{B}$ and in turn, by the above GKNS isomorphism, $\sigma(f \otimes g^*) \in \mathbf{S}_0(\mathcal{G} \times \widehat{\mathcal{G}})$. This shows that the function $(x, \xi) \mapsto x(\xi)$ (a so–called "bicharacter") is a multiplier on $\mathbf{S}_0(\mathcal{G} \times \widehat{\mathcal{G}})$, another functorial property of $\mathbf{S}_0(\mathcal{G})$ [Fei81].

The Dirac measure is an element of $\mathbf{S}_0'(\mathcal{G} \times \widehat{\mathcal{G}})$, hence it makes mathematical sense to ask about that operator whose GKNS is ideally concentrated in a point. Although this does not make physical sense it leads to an interesting mathematical interpretation of the GKNS.

Lemma 7.5.2 *(Operator Decomposition I: Weyl–Heisenberg Expansion)* *Let $K \in \mathcal{B}$, then the GKNS induces a weakly convergent operator decomposition:*

$$K = \int_{\mathcal{G} \times \widehat{\mathcal{G}}} \Big\langle K, (\pi \otimes \pi^*)(\lambda) P_0 \Big\rangle_{\mathcal{B}} (\pi \otimes \pi^*)(\lambda) P_0 \, d\lambda \tag{7.5.10}$$

with the prototype operator $P_0 \in \mathcal{B}'$ acting on $f \in \mathbf{S}_0(\mathcal{G})$ as

$$P_0 f = \left(\int_{\mathcal{G}} f(y) dy \right) \delta_0, \;\; i.e, \qquad \kappa(P_0) = \delta_0 \otimes 1. \tag{7.5.11}$$

Remark: Indeed as one may expect from physical considerations P_0 is a (densely defined) "totally" unbounded operator on $\mathbf{L}^2(\mathcal{G})$.

Proof:

(I) Elementary analysis shows that the action of $(\pi \otimes \pi^*)(\lambda)$ with $\lambda := (t, \nu)$ on operators in $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$ is given by:

$$\kappa \left[(\pi \otimes \pi^*)(\lambda) K \right] = M_{(\nu, -\nu)} T_{(t,t)} \kappa(K) \tag{7.5.12}$$

hence, formally:

$$\kappa \left[(\pi \otimes \pi^*)(\lambda) P_0 \right](x, y) = \delta_t(x) \nu(x - y). \tag{7.5.13}$$

One has $\kappa[(\pi \otimes \pi^*)(\lambda) P_0] \in \mathbf{S}_0'(\mathcal{G} \times \mathcal{G})$ for any $\lambda \in \mathcal{G} \times \widehat{\mathcal{G}}$, such that $K \mapsto \langle K, (\pi \otimes \pi^*)(\lambda) P_0 \rangle_{\mathcal{B}}$ is a continuous functional on $\mathcal{B}$. Its evaluation proves the analysis part of this lemma. For $\lambda = (t, \nu)$ we have

$$\begin{aligned}
\langle K, (\pi \otimes \pi^*)(\lambda) P_0 \rangle_{\mathcal{B}} &= \langle \kappa(K), \kappa \left[(\pi \otimes \pi^*)(\lambda) P_0 \right] \rangle_{\mathbf{S}_0(\mathcal{G} \times \mathcal{G})} \\
&= \int_{\mathcal{G} \times \mathcal{G}} \kappa(K)(x, y) \delta_t(x) \overline{\nu(x - y)} dx dy \\
&= \int_{\mathcal{G}} \kappa(K)(t, t - z) \overline{\nu(z)} dz \\
&= \sigma(K)(t, \nu)
\end{aligned}$$

(II) The operator synthesis part of this lemma follows easily from (7.5.13) and the inversion formula of the GKNS (see (7.5.5)):

$$\kappa(K)(x,y) = \int_{\mathcal{G}}\int_{\widehat{\mathcal{G}}} \sigma(K)(t,\nu)\delta_t(x)\nu(x-y)dt\,d\nu = \int_{\widehat{\mathcal{G}}} \sigma(K)(x,\nu)\nu(x-y)d\nu$$

❑

As a formal application of the preceding lemma we prove the shift–covariance of the GKNS.

Lemma 7.5.3 *(Shift–Covariance)*
The action of $(\pi \otimes \pi^*)(\lambda)$ on $K \in (\mathcal{B}, \mathcal{H}, \mathcal{B}')$ corresponds to a translation of the symbol:

$$\sigma\left[(\pi \otimes \pi^*)(\lambda)K\right] = T_\lambda\left[\sigma(K)\right]. \tag{7.5.14}$$

Proof: By (7.5.12) and the functorial properties of $\mathbf{S}_0(\mathcal{G} \times \mathcal{G})$, (see Lemma 7.3.1) we know that $(\pi \otimes \pi^*)(\lambda)K$ establishes an automorphism of the operator space Gelfand triple $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$. Hence, we can apply $(\pi\otimes\pi^*)(\lambda)K$ to the integrand of Lemma 7.5.2. Recalling that $(\pi \otimes \pi^*)(\lambda)$ is a unitary representation of $\mathcal{G} \times \widehat{\mathcal{G}}$ on $\mathcal{H}$ finishes the proof:

$$\begin{aligned}
(\pi \otimes \pi^*)(\lambda)K &= \int_{\mathcal{G}\times\widehat{\mathcal{G}}} \sigma(K)(\mu)(\pi \otimes \pi^*)(\lambda)(\pi \otimes \pi^*)(\mu)P_0 d\mu \\
&= \int_{\mathcal{G}\times\widehat{\mathcal{G}}} \sigma(K)(\mu)(\pi \otimes \pi^*)(\lambda+\mu)P_0 d\mu \\
&= \int_{\mathcal{G}\times\widehat{\mathcal{G}}} \left[T_\lambda\sigma(K)\right](\rho)(\pi \otimes \pi^*)(\rho)P_0 d\rho
\end{aligned}$$

❑

The Rihaczek distribution (GKNS of a rank–one projection) is known as TF–energy distribution in the sense that the total integral gives the $\mathbf{L}^2(\mathcal{G})$–norm of the signal (although $\sigma(f\otimes f^*)$ has a typically nonvanishing imaginary part). This property can be seen as a consequence of the so–called *trace–formula*:

Lemma 7.5.4 *For $K \in \mathcal{B}$ the operator's trace can be obtained by total integration over the GKNS :*

$$tr(K) = \int_{\mathcal{G}} \kappa(K)(x,x)dx = \int_{\mathcal{G}\times\widehat{\mathcal{G}}} \sigma(K)(\lambda)\,d\lambda$$

Proof: The so–called trace formula follows immediately from (7.5.4). ❑

7.6 Spreading function

Recall that $\lambda \mapsto \pi(\lambda)$ defines only a projective representation of $\mathcal{G} \times \widehat{\mathcal{G}}$ on $\mathbf{L}^2(\mathcal{G})$, hence the nonvanishing commutator of the set $\{\pi(\lambda)\}_{\lambda \in \mathcal{G} \times \widehat{\mathcal{G}}}$ "produces" a *symplectic character* which can be formulated most compactly by the $\mathcal{J}$–isomorphism as defined in (7.2.9). The symplectic character induces a slight modification of the Cartesian Fourier transform on $\mathcal{G} \times \widehat{\mathcal{G}}$:

Definition 7.6.1 *(Symplectic Fourier Transform)*
The symplectic Fourier transform on $\mathcal{G} \times \widehat{\mathcal{G}}$ is formally defined by

$$\mathcal{F}_s f(\lambda) := \int_{\mathcal{G} \times \widehat{\mathcal{G}}} f(\mu) \overline{\lambda(\mathcal{J}^* \mu)} d\mu, \qquad \lambda \in \mathcal{G} \times \widehat{\mathcal{G}},$$

$$\Updownarrow$$

$$\mathcal{F}_s f(x, \xi) = \int_{\mathcal{G} \times \widehat{\mathcal{G}}} f(t, \nu) \overline{\xi(t)} \nu(x) dt\, d\nu, \qquad x, t \in \mathcal{G},\ \xi, \nu \in \widehat{\mathcal{G}}.$$

It can be (uniquely) characterized by the fact that

$$\mathcal{F}_s (f \otimes \hat{g}) = g \otimes \hat{f} \tag{7.6.1}$$

The symplectic Fourier transform is related the conventional (Cartesian) Fourier transform on $\mathcal{G} \times \widehat{\mathcal{G}}$ through

$$[\mathcal{F}_s f](x, \xi) = \hat{f}\,(\mathcal{J}^*(x, \xi)) = \mathcal{F}_{\mathcal{G} \times \widehat{\mathcal{G}}} f(-\xi, x) \qquad (x \in \mathcal{G}, \xi \in \widehat{\mathcal{G}}). \tag{7.6.2}$$

Lemma 7.6.2 *The symplectic Fourier transform is:*
(a) an isomorphism of $\mathbf{S}_0(\mathcal{G} \times \widehat{\mathcal{G}})$,
(b) unitary on $\mathbf{L}^2(\mathcal{G} \times \widehat{\mathcal{G}})$,
(c) an isomorphism of $\mathbf{S}_0'(\mathcal{G} \times \widehat{\mathcal{G}})$,
(d) self–inverse.

In other words, it defines an involutive, unitary automorphism of the Gelfand triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathcal{G} \times \widehat{\mathcal{G}})$.

Proof: The symplectic Fourier transform is coupled to the Cartesian Fourier transform via an elementary isomorphism, hence items (a)–(c) need no further reasoning. Item (d) follows from (7.6.1). ❑

The symplectic Fourier transform of the GKNS leads to an alternative time–frequency representation of linear operators, the *spreading function* as introduced in electrical engineering for $\mathcal{G} = \mathbb{R}$ [Bel63]:

$$\eta(K) := \mathcal{F}_s[\sigma(K)]. \tag{7.6.3}$$

Corollary 7.6.3 *The spreading function $\eta(K)$ of a linear operator $K \in (\mathcal{B}, \mathcal{H}, \mathcal{B}')$ with kernel $\kappa(K)$ is defined as a (generalized) function on $\mathcal{G} \times \widehat{\mathcal{G}}$:*

$$\eta(K)(t, \nu) = \int_{\mathcal{G}} \kappa(K)(x, x - t)\overline{\nu(x)}\, dx \tag{7.6.4}$$

and the inversion formula is given by

$$\kappa(K)(x, y) = \int_{\widehat{\mathcal{G}}} \eta(K)(x - y, \nu) x(\nu)\, d\nu. \tag{7.6.5}$$

The mapping $K \leftrightarrow \eta(K)$ defines a unitary Gelfand triple isomorphism between $(\mathcal{B}, \mathcal{H}, \mathcal{B}')$ and $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathcal{G} \times \widehat{\mathcal{G}})$:

$$\langle K, L\rangle_{(\mathcal{B},\mathcal{H},\mathcal{B}')} = \langle \eta(K), \eta(L)\rangle_{(\mathbf{S}_0,\mathbf{L}^2,\mathbf{S}_0')(\mathcal{G}\times\widehat{\mathcal{G}})} \,. \tag{7.6.6}$$

Proof: By Lemma 7.6.2, the properties of $\eta(K)$ follow immediately from the properties of $\sigma(K)$ (see Theorem 7.5.1). ❑

Theorem 7.6.4 *(Operator Decomposition II: Spreading Representation) The spreading function is the complex weight function of an absolutely convergent, unique decomposition of an operator $K \in \mathcal{B}$ into TF shifts:*

$$K = \int_{\mathcal{G}\times\widehat{\mathcal{G}}} \eta(K)(\lambda)\pi(\lambda) d\lambda, \tag{7.6.7}$$

with

$$\eta(K)(\lambda) = \langle K, \pi(\lambda)\rangle_{\mathcal{B}}\,, \quad \lambda \in \mathcal{G} \times \widehat{\mathcal{G}}.$$

For $K \in \mathcal{B}'$, (7.6.7) holds in the weak sense of bilinear forms on $\mathbf{S}_0(\mathcal{G})$:

$$\langle Kf, g\rangle_{\mathbf{S}_0(\mathcal{G})} = \int_{\mathcal{G}\times\widehat{\mathcal{G}}} \eta(K)(\lambda) \langle \pi(\lambda) f, g\rangle_{\mathbf{S}_0(\mathcal{G})}\, d\lambda, \quad \text{for } f, g \in \mathbf{S}_0(\mathcal{G}). \tag{7.6.8}$$

Proof: (I) The operator analysis part of this theorem follows from the definition of the spreading function in terms of a (distributional) kernel (7.6.5). Note that $\pi(\lambda) \in \mathcal{B}'$ can be characterized by a distributional kernel

$$\kappa[\pi(\lambda)](x, y) = \nu(x)\delta_t(x - y) \tag{7.6.9}$$

hence, the elementary TF–shift $\pi(\lambda)$ is mapped on the Dirac measure at λ:

$$\eta\left[\pi(\lambda)\right] = \delta_\lambda, \tag{7.6.10}$$

and therefore

$$\langle K, \pi(\lambda)\rangle_{\mathcal{B}} = \langle \eta(K), \eta\left[\pi(\lambda)\right]\rangle_{\mathbf{S}_0(\mathcal{G}\times\widehat{\mathcal{G}})} = \langle \eta(K), \delta_\lambda\rangle_{\mathbf{S}_0(\mathcal{G}\times\widehat{\mathcal{G}})} = \eta(K)(\lambda).$$

(II) The operator synthesis part can be obtained by characterizing the action of K via (7.6.5):

$$\begin{aligned} Kf(x) &= \int_{\mathcal{G}} \kappa(K)(x,y)f(y)dy \\ &= \int_{\mathcal{G}} \left(\int_{\widehat{\mathcal{G}}} \eta(K)(x-y,\nu)x(\nu)f(y)d\nu\right) dy \\ &= \int_{\mathcal{G}} \int_{\widehat{\mathcal{G}}} \eta(K)(t,\nu)f(x-t)x(\nu)d\nu dt \\ &= \int_{\mathcal{G}} \int_{\widehat{\mathcal{G}}} \eta(K)(t,\nu)\left(M_\nu T_t f\right)(x)dtd\nu. \end{aligned}$$

(III) Absolute convergence in the strong operator topology on $\mathbf{S}_0(\mathcal{G})$ (or $\mathbf{L}^2(\mathcal{G})$) is easily established by the isometrical invariance of $\mathbf{S}_0(\mathcal{G})$ (or $\mathbf{L}^2(\mathcal{G})$) w.r.t. TF–shifts, estimating a vector–valued integral:

$$\|K\|_{\mathcal{L}(\mathbf{S}_0)} \leq \int_{\mathcal{G}} \int_{\widehat{\mathcal{G}}} |\eta(K)(t,\nu)| \|M_\nu T_t f\|_{\mathcal{L}(\mathbf{S}_0)} dt\, d\nu \leq \|\eta(K)\|_1 \leq C\|\eta(K)\|_{\mathbf{S}_0}$$

and analogously for the action of K on $\mathbf{L}^2(\mathcal{G})$.

(IV) In order to show the validity of the distributional spreading representation (7.6.8) consider first the spreading function of a rank–one operator $g \otimes f^*$ with $g, f \in \mathbf{S}_0(\mathcal{G})$. We have already shown that $g, f \in \mathbf{S}_0(\mathcal{G})$ implies $\sigma(g \otimes f^*) \in \mathbf{S}_0(\mathcal{G} \times \widehat{\mathcal{G}})$ (see (7.5.9) and the following discussion). As the symplectic Fourier transform is an isomorphism on $\mathbf{S}_0(\mathcal{G} \times \widehat{\mathcal{G}})$ we have that $\eta(g \otimes f^*) \in \mathbf{S}_0(\mathcal{G} \times \widehat{\mathcal{G}})$. Using the Gelfand bracket identity (7.6.3) the proof is finished by the $\mathbf{S}_0'(\mathcal{G})$–kernel theorem

$$\langle Kf, g\rangle_{\mathbf{S}_0(\mathcal{G})} = \langle K, g\otimes f^*\rangle_{\mathcal{B}'} = \langle \eta(K), \eta(g\otimes f^*)\rangle_{\mathbf{S}_0'(\mathcal{G}\times\widehat{\mathcal{G}})}.$$

❑

The spreading representation is closely related to but different from the integrated Schrödinger representation (the latter involves an integral over all three coordinates of the reduced Heisenberg group on $\mathcal{G}\times\widehat{\mathcal{G}}\times\mathbb{T}$). The engineering terminology stems from the fact that typically one applies time–frequency localized test signals to a given operator (these are also test functions in the sense of distribution theory) which undergo a broadening both in time ("delay spread") and frequency ("Doppler spread").

For finite groups (where $\mathbf{S}_0(\mathcal{G}) = \mathbf{L}^2(\mathcal{G}) = \mathbf{S}_0'(\mathcal{G})$ and $\mathcal{B} = \mathcal{H} = \mathcal{B}'$) the spreading representation (7.6.7) actually reduces to an orthonormal decomposition of matrices:

$$K = \int_{\mathcal{G}\times\widehat{\mathcal{G}}} \langle K, \pi(\lambda)\rangle_{\mathcal{B}}\, \pi(\lambda) d\lambda = \sum_{\lambda\in\Gamma_n\times\Gamma_n} \langle K, \pi(\lambda)\rangle_{\mathcal{H}}\, \pi(\lambda), \quad \text{for} \quad \mathcal{G} \cong \widehat{\mathcal{G}} = \Gamma_n$$

Important questions about the applicability of the KN calculus can be studied by the spreading function of the composite operator:

Lemma 7.6.5 *(Twisted Convolution)*
Composition of two linear operators $K, L \in \mathcal{B}$ corresponds to the twisted convolution of their spreading functions $\eta(K), \eta(L) \in \mathbf{S}_0(\mathcal{G}\times\widehat{\mathcal{G}})$:

$$\eta(KL)(\lambda) = \int_{\mathcal{G}\times\widehat{\mathcal{G}}} \eta(K)(\lambda')\eta(L)(\lambda-\lambda')\overline{t'(\nu-\nu')}d\lambda' \qquad (7.6.11)$$

Proof:
The spreading representation (7.6.7) of K and L yields:

$$KL = \left(\int_{\mathcal{G}\times\widehat{\mathcal{G}}} \eta(K)(\lambda_1)\pi(\lambda_1)d\lambda_1\right)\left(\int_{\mathcal{G}\times\widehat{\mathcal{G}}} \eta(L)(\lambda_2)\pi(\lambda_2)d\lambda_2\right),$$

which, using the commutation relation

$$\pi(\lambda_1)\pi(\lambda_2) = \overline{\nu_2(t_1)}\pi(\lambda_1+\lambda_2)$$

and a change of variables, leads to the spreading representation of the composite operator

$$KL = \int_{\mathcal{G}\times\widehat{\mathcal{G}}}\left(\int_{\mathcal{G}\times\widehat{\mathcal{G}}} \eta(K)(\lambda')\eta(L)(\lambda-\lambda')\overline{t'(\nu-\nu')}d\lambda'\right)\pi(\lambda)d\lambda. \quad (7.6.12)$$

Now, from Theorem 7.4.1 we know that if that $K \in \mathcal{B}$ and $L \in \mathcal{B}$ then $KL \in \mathcal{B}$. Since the spreading representation of an operator in $\mathcal{B}$ is absolutely convergent in the sense of its kernel we have pointwise equivalence of the spreading representations of both sides in 7.6.12. ❑

An immediate consequence of (7.6.11) is the fact that twisted convolution behaves just like regular convolution with regard to support considerations, one has

$$\mathrm{supp}[\eta(KL)] \subseteq \mathrm{supp}[\eta(K)] + \mathrm{supp}[\eta(L)] \qquad (7.6.13)$$

and even

$$|\eta(KL)| \le |\eta(K)| * |\eta(L)|\,. \qquad (7.6.14)$$

if $\eta(K)$ and $\eta(L)$ are given by locally integrable functions.

The latter relation shows that (additive) subgroups $\Lambda \lhd \mathcal{G}\times\widehat{\mathcal{G}}$ always induce operators algebras of operators K satisfying $\eta(K) \subseteq \Lambda$. The algebra of multiplication or convolution operators correspond to subgroups living on one of the "coordinate axes" of $\mathcal{G}\times\widehat{\mathcal{G}}$, i.e., to $\Lambda = \mathcal{G}\times\{0\}$ and $\Lambda = \{0\}\times\mathcal{G}$, respectively.

7.6.1 Relation to STFT and ambiguity function

For the specific choice of a rank–one projection operator we get an abstract generalization of the *ambiguity function* on elementary LCA groups [Sch84a]:

$$\eta(f \otimes f^*)(\lambda) = \langle f, \pi(\lambda) f \rangle_{\mathbf{L}^2(\mathcal{G})}, \quad \lambda \in \mathcal{G} \times \widehat{\mathcal{G}}.$$

The relevance of the ambiguity function to Gabor expansion theory is well recognized [TO95]. Moreover, one can write the *short time Fourier transform* (STFT) as the spreading function of the rank–one operator $f \otimes g^*$:

$$\mathcal{V}_g f(x, \xi) := \eta(f \otimes g^*)(x, \xi) = \int_{\mathcal{G}} f(y) \overline{g(y-x)\xi(y)} dy \tag{7.6.15}$$

By the validity of the Plancherel formula for the symplectic Fourier transform we obtain a two–line proof of a well–known STFT formula:

$$\begin{aligned} \langle \mathcal{V}_e d, \mathcal{V}_g f \rangle_{(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathcal{G} \times \widehat{\mathcal{G}})} &= \langle \eta(d \otimes e^*), \eta(f \otimes g^*) \rangle_{(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathcal{G} \times \widehat{\mathcal{G}})} \\ &= \langle d \otimes \overline{e}, f \otimes \overline{g} \rangle_{(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathcal{G} \times \mathcal{G})} \\ &= \langle e, g \rangle_{(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathcal{G})} \overline{\langle d, f \rangle_{(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathcal{G})}} \end{aligned} \tag{7.6.16}$$

Corollary 7.6.6 *Let $f, g \in \mathbf{S}_0(\mathcal{G})$, then $\mathcal{V}_g f \in \mathbf{S}_0(\mathcal{G} \times \widehat{\mathcal{G}})$.*

Proof: By (7.6.15) this is a corollary of Theorem 7.6.4 (see in particular step (IV) of the proof). ❑

7.7 TF–Lattice invariant operators

Many important linear systems are invariant with respect to groups of unitary operators. In fact, translation invariant systems are characterized by

$$KT_t = T_t K, \qquad t \in \mathcal{G}$$

while the multiplication operators are characterized by

$$KM_\nu = M_\nu K, \qquad \nu \in \widehat{\mathcal{G}}.$$

Periodically time–varying systems commute with time–shifts on a certain subgroup ("lattice") of $\mathcal{G}$

$$KT_\lambda = T_\lambda K, \qquad \lambda \in \Lambda \triangleleft \mathcal{G}.$$

We now study operators with a more complicated commutation relation.

Definition 7.7.1 *Given a TF–subgroup $\Lambda \lhd \mathcal{G} \times \widehat{\mathcal{G}}$, an operator K is called Λ–invariant if*

$$K\pi(\lambda) = \pi(\lambda)K, \qquad \lambda \in \Lambda.$$

If Λ is discrete and $(\mathcal{G} \times \widehat{\mathcal{G}})/\Lambda$ is compact , i.e., $\mathcal{G} \times \widehat{\mathcal{G}} = Q + \Lambda$ for some compact set $Q \subseteq \mathcal{G} \times \widehat{\mathcal{G}}$, we call Λ a TF–lattice.

The lattice size $s(\Lambda)$ is defined as the measure of the fundamental domain of Λ) (see Chapter 6) and we always assume that the Haar measure on the compact quotient group $(\mathcal{G} \times \widehat{\mathcal{G}})/\Lambda$ is normalized to a probability measure.

With the above convention Weil's formula [Rei68] reads as

$$\int_{\mathcal{G}\times\widehat{\mathcal{G}}} f(\mu)d\mu = s(\Lambda) \int_{(\mathcal{G}\times\widehat{\mathcal{G}})/\Lambda} \left(\sum_{\lambda\in\Lambda} f(x+\lambda) \right) dm_{(\mathcal{G}\times\widehat{\mathcal{G}})/\Lambda}(x), \qquad (7.7.1)$$

where the inner term exists almost everywhere on $(\mathcal{G}\times\widehat{\mathcal{G}})/\Lambda$ if $f \in \mathbf{L}^1(\mathcal{G}\times\widehat{\mathcal{G}})$ (according to [Rei68]). For $f \in \mathbf{S}_0(\mathcal{G} \times \widehat{\mathcal{G}})$ the inner sum is absolutely and uniformly convergent and defines an element in $\mathbf{S}_0\left((\mathcal{G} \times \widehat{\mathcal{G}})/\Lambda\right)$, i.e., a Λ–periodic function on $\mathcal{G} \times \widehat{\mathcal{G}}$ with absolutely convergent Fourier series.

Example. The canonical example of a Λ–invariant operator is the Weyl–Heisenberg frame operator which acts as

$$S_{g,\Lambda} f := \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)g = \sum_{\lambda\in\Lambda} [(\pi \otimes \pi^*)(\lambda)(g \otimes g^*)]\, f,$$

where g is the underlying Gabor atom. More generally, we shall show that operators of the form

$$K = \sum_{\lambda\in\Lambda} (\pi \otimes \pi^*)(\lambda)P, \qquad \text{with} \quad P \in \mathcal{B}$$

are well–defined as elements of $\mathcal{B}'$. This includes the case of multi-window WH frames (where P is a finite–rank projection operator).

The standard lattice on $\mathcal{G} = \widehat{\mathcal{G}} = \mathbb{R}$ is the separable lattice defined by

$$\Lambda = a\mathbb{Z} \times b\mathbb{Z} \qquad (7.7.2)$$

$$s(\Lambda) = ab. \qquad (7.7.3)$$

For $\mathcal{G} = \mathbb{R}^d$ the number $1/s(\Lambda)$ corresponds to the average number of lattice points of Λ in large balls in $\mathbb{R}^{2d}$, and is therefore called the *redundancy* of Λ (or redΛ) in Chapter 3.

Definition 7.7.2 *The* adjoint lattice $\Lambda^\circ \lhd \mathcal{G} \times \widehat{\mathcal{G}}$ *is defined as:*

$$\Lambda^\circ = \left\{ \lambda^\circ \in \mathcal{G} \times \widehat{\mathcal{G}} \mid \pi(\lambda)\pi(\lambda^\circ) = \pi(\lambda^\circ)\pi(\lambda), \forall \lambda \in \Lambda \lhd \mathcal{G} \times \widehat{\mathcal{G}} \right\}. \qquad (7.7.4)$$

Note that the set $\{\pi(\lambda)\}_{\lambda\in\Lambda}$, although generated by the action of the group Λ, is not itself a group of operators. Hence, we cannot a priori assume that Λ° is a subgroup of $\mathcal{G}\times\widehat{\mathcal{G}}$. The following lemma shows that, as anticipated in our terminology, Λ° is indeed a subgroup of $\mathcal{G}\times\widehat{\mathcal{G}}$, closely related to the annihilator subgroup $\Lambda^\perp$.

Lemma 7.7.3 *The adjoint lattice Λ° is the annihilator subgroup of the abstract TF plane $\mathcal{G}\times\widehat{\mathcal{G}}$ w.r.t. symplectic Fourier transform $\mathcal{F}_s$, i.e.,*

$$\Lambda^\circ = \left\{\lambda^\circ \in \mathcal{G}\times\widehat{\mathcal{G}} \mid \lambda^\circ(\mathcal{J}^*\lambda) = 1, \forall\lambda\in\Lambda \lhd \mathcal{G}\times\widehat{\mathcal{G}}\right\}. \tag{7.7.5}$$

It can be described as annihilator subgroups w.r.t. standard Fourier transform and the $\mathcal{J}$-isomorphism as follows:

$$\Lambda^\circ = \mathcal{J}\left(\Lambda^\perp\right) = (\mathcal{J}\Lambda)^\perp. \tag{7.7.6}$$

Proof: (I) The commutator of the set $\{\pi(\lambda)\}_{\lambda\in\mathcal{G}\times\widehat{\mathcal{G}}}$ leads exactly to the character of $\mathcal{F}_s$ (cf. (7.2.8) and note that $\overline{\lambda_2\left(\mathcal{J}^*\lambda_1\right)} = \lambda_2\left(\mathcal{J}\lambda_1\right)$)

$$\|\pi(\lambda_1)\pi(\lambda_2) - \pi(\lambda_2)\pi(\lambda_1)\| = |1 - \overline{\lambda_2\left(\mathcal{J}^*\lambda_1\right)}| = |1 - \nu_1(t_2)\overline{\nu_2(t_1)}|.$$

Hence, (7.7.5) is established.
(II) One has $\mathcal{J}^* = -\mathcal{J}$ hence $\mathcal{J}^*\Lambda = \Lambda$. The annihilator subgroup $\Lambda^\perp \lhd \widehat{\mathcal{G}}\times\mathcal{G}$ corresponding to Λ via standard Fourier transform is defined by

$$\Lambda^\perp = \left\{\mu \in \widehat{\mathcal{G}}\times\mathcal{G} \mid \mu(\lambda) = 1, \forall\lambda\in\Lambda \lhd \mathcal{G}\times\widehat{\mathcal{G}}\right\}.$$

To this definition we apply the $\mathcal{J}$–isomorphism which proves the first part of (7.7.6):

$$\begin{aligned}\mathcal{J}\Lambda^\perp &= \left\{\lambda^\circ \in \mathcal{G}\times\widehat{\mathcal{G}} \mid \left(\mathcal{J}\lambda^\circ\right)(\lambda) = 1, \forall\lambda\in\Lambda \lhd \mathcal{G}\times\widehat{\mathcal{G}}\right\}\\ &= \left\{\lambda^\circ \in \mathcal{G}\times\widehat{\mathcal{G}} \mid \lambda\left(\mathcal{J}\lambda^\circ\right) = 1, \forall\lambda\in\Lambda \lhd \mathcal{G}\times\widehat{\mathcal{G}}\right\}\end{aligned}$$

On the other hand, one has also

$$\begin{aligned}(\mathcal{J}\Lambda)^\perp &= \left\{\lambda^\circ \in \mathcal{G}\times\widehat{\mathcal{G}} \mid \lambda^\circ(\mu) = 1, \forall\mu\in\mathcal{J}\Lambda \lhd \widehat{\mathcal{G}}\times\mathcal{G}\right\}\\ &= \left\{\lambda^\circ \in \mathcal{G}\times\widehat{\mathcal{G}} \mid \lambda^\circ(\mathcal{J}\lambda) = 1, \forall\lambda\in\Lambda \lhd \mathcal{G}\times\widehat{\mathcal{G}}\right\},\end{aligned}$$

thus proving the second part of (7.7.6). ❑

Example. Let $\mathcal{G} = \mathbb{R}^d$ and $\Lambda = M\mathbb{Z}^d \times N\mathbb{Z}^d$, where M, N are invertible $d\times d$ matrices, then $\Lambda^\circ = \Lambda^\perp = N^{-1^*}\mathbb{Z}^d \times M^{-1^*}\mathbb{Z}^d$. (Indeed, by the TF–separability of Λ the symplectic and the standard annihilator group coincide.)

It is well–known [DLL95, AAGM95] that operators invariant w.r.t. the action of subgroups give rise to modular von Neumann algebras, where commutation relations are of fundamental relevance. Within the present framework the coupling constant of such von Neumann algebras appears as a lattice size ratio between Λ° and Λ.

Lemma 7.7.4

$$s(\Lambda)s(\Lambda^\circ) = 1 \tag{7.7.7}$$

Proof: We have $\Lambda \triangleleft \mathcal{G} \times \widehat{\mathcal{G}}$ and, consequently, $\Lambda^\perp \triangleleft \widehat{\mathcal{G}} \times \mathcal{G}$. Since $\Lambda^\circ := J\Lambda^\perp$ we can use Lemma 6.2.3 of Chapter 6 which says that $s(\Lambda)s(\Lambda^\perp) = 1$, whenever Λ is a discrete subgroup with compact quotient of some LCA group $\mathcal{G}_0$ and $\Lambda^\perp \triangleleft \widehat{\mathcal{G}}_0$ its annihilator subgroup. ❑

Theorem 7.7.5 *(Analysis of Λ–invariant operators)*
For any Λ–invariant operator $K \in \mathcal{B}'$, the GKNS $\sigma(K)$ is a Λ–periodic distribution whose symplectic Fourier transform leads to a discrete spreading representation of K supported in the adjoint group $\Lambda^\circ = \mathcal{J}\Lambda^\perp$ and thus

$$K = \sum_{\lambda^\circ \in \Lambda^\circ} (K)_{\lambda^\circ} \pi(\lambda^\circ). \tag{7.7.8}$$

By the usual identification of $\sigma(K)$ with its canonical representative on the quotient group, one has the following unitary Gelfand triple isomorphism

$$\sigma(K) \in (\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')\left((\mathcal{G} \times \widehat{\mathcal{G}})/\Lambda\right) \longleftrightarrow \{(K)_{\lambda^\circ}\}_{\lambda^\circ \in \Lambda^\circ} \in (\ell^1, \ell^2, \ell^\infty)(\Lambda^\circ) \tag{7.7.9}$$

Specifically, when $\sigma(K) \in \mathbf{S}_0\left((\mathcal{G} \times \widehat{\mathcal{G}})/\Lambda\right)$, the series (7.7.8) is absolutely convergent and thus K is a bounded operator on $\mathbf{S}_0(\mathcal{G})$ and $\mathbf{L}^2(\mathcal{G})$.

Proof: (I) By the TF–invariance of $\mathbf{S}_0(\mathcal{G})$ we know that $K \in \mathcal{B}'$ implies $\pi(\lambda)K \in \mathcal{B}'$ and $K\pi(\lambda) \in \mathcal{B}'$. Hence we can reformulate the commutation relation as

$$K = \pi(\lambda)K\pi(\lambda)^* = (\pi \otimes \pi^*)(\lambda)K, \tag{7.7.10}$$

which by the shift–covariance of the GKNS (Lemma 7.5.3) leads to a periodicity condition for $\sigma(K) \in \mathbf{S}_0'(\mathcal{G} \times \widehat{\mathcal{G}})$

$$\sigma(K) = T_\lambda\left[\sigma(K)\right], \qquad \lambda \in \Lambda. \tag{7.7.11}$$

Hence, the GKNS allows to map the fairly abstract operator commutation relation onto a translation invariance property for the symbol $\sigma(K)$.

(II) Under the symplectic Fourier transform on $\mathbf{S}_0'(\mathcal{G} \times \widehat{\mathcal{G}})$ equation (7.7.11) becomes

$$\eta(K)(\mu) = \eta(K)(\mu)\lambda(\mathcal{J}^*\mu).$$

Since $\{\mu \mapsto \lambda(\mathcal{J}^*\mu), \lambda \in \Lambda\}$ is a group of characters on $\mathcal{G} \times \widehat{\mathcal{G}}$ it follows that [Rei68, p.141]

$$\operatorname{supp}\{\eta(K)\} \subseteq (\mathcal{J}\Lambda)^{\perp} = \mathcal{J}\Lambda^{\perp} = \Lambda^{\circ}.$$

Thus we have verified the support properties of $\eta(K)$. One can choose a sufficiently refined bounded uniform partition of unity $(\psi_n)_{n\in I}$ with $supp(\psi_n) \cap \Lambda^{\circ}$ containing at most one point for each $n \in I$ and $\eta(K) = \sum_{n\in I} \eta(K)\psi_n$ (cf. Chapter 3, Cor. 3.2.8). Since $\eta(K)\psi_n \in \mathcal{F}L^{\infty}(\mathcal{G})$ is a pseudo–measure carried by one point it must be a Dirac measure (see [Kat68], Sect. 4.11), i.e., altogether one has[2]:

$$\eta(K) = \sum_{\lambda^{\circ}\in\Lambda^{\circ}} c_{\lambda^{\circ}}\delta_{\lambda^{\circ}}.$$

Finally, one easily verifies that under the given conditions $\|\eta(K)\|_{\mathbf{S}_0'(\mathcal{G}\times\widehat{\mathcal{G}})}$ and $\|\{c_{\lambda^{\circ}}\}\|_{\ell^{\infty}(\Lambda^{\circ})}$ are equivalent norms.

(III) Since one has $\mathbf{S}_0\left((\mathcal{G} \times \widehat{\mathcal{G}})/\Lambda\right) = \mathbf{A}((\mathcal{G} \times \widehat{\mathcal{G}})/\Lambda)$ (due to the compactness of $(\mathcal{G} \times \widehat{\mathcal{G}})/\Lambda$), the symplectic Fourier series with the coefficients defined by:

$$(K)_{\lambda^{\circ}} = \int_{(\mathcal{G}\times\widehat{\mathcal{G}})/\Lambda} \sigma(K)(\mu)\overline{\lambda^{\circ}(\mathcal{J}^*\mu)}d\mu, \tag{7.7.12}$$

is absolutely convergent. Hence, we have established the isomorphism between $\sigma(K) \in \mathbf{S}_0\left((\mathcal{G} \times \widehat{\mathcal{G}})/\Lambda\right)$ and ℓ^1–coefficients. The other isomorphisms follow by analog reasoning.

(IV) Since $\sigma(K) \in \mathbf{S}_0\left((\mathcal{G} \times \widehat{\mathcal{G}})/\Lambda\right)$ implies $\{(K)_{\lambda^{\circ}}\} \in \ell^1(\Lambda^{\circ})$ boundedness of K on $\mathbf{S}_0(\mathcal{G})$ (or $\mathbf{L}^2(\mathcal{G})$) follows from the the fact that $\pi(\lambda)$ acts isometrically on $\mathbf{S}_0(\mathcal{G})$ (or $\mathbf{L}^2(\mathcal{G})$):

$$\|K\| = \|\sum_{\lambda^{\circ}\in\Lambda^{\circ}} (K)_{\lambda^{\circ}}\pi(\lambda^{\circ})\| \leq \sum_{\lambda^{\circ}\in\Lambda^{\circ}} |(K)_{\lambda^{\circ}}|.$$

❑

Theorem 7.7.6 *(Synthesis of Λ–invariant operators)*
Any given Λ–invariant operator K with $\sigma(K) \in \mathbf{A}((\mathcal{G} \times \widehat{\mathcal{G}})/\Lambda)$ is the periodization of some (nonunique) prototype operator $P \in \mathcal{B}$. This periodization

[2] Cf. [Rei68] Chapt. 7, Sect. I.3. for a version applicable to general LCA groups.

corresponds to sampling its (bounded and continuous) spreading function $\eta(P) \in \mathbf{S}_0(\mathcal{G} \times \widehat{\mathcal{G}})$ *on* Λ°, *i.e.,*

$$K := \sum_{\lambda \in \Lambda} (\pi \otimes \pi^*)(\lambda) P = \frac{1}{s(\Lambda)} \sum_{\lambda^\circ \in \Lambda^\circ} \langle P, \pi(\lambda^\circ) \rangle_{\mathcal{B}}\, \pi(\lambda^\circ)$$

Proof: (I) First we check that K is indeed Λ–invariant in the above sense, i.e., we have the condition

$$(\pi \otimes \pi^*)(\lambda') K = K, \qquad \lambda' \in \Lambda.$$

This follows from the group structure of $\{(\pi \otimes \pi^*)(\lambda)\}_{\lambda \in \Lambda}$ since

$$(\pi \otimes \pi^*)(\lambda') \sum_{\lambda \in \Lambda} (\pi \otimes \pi^*)(\lambda) P = \sum_{\lambda \in \Lambda} (\pi \otimes \pi^*)(\lambda + \lambda') P = \sum_{\lambda \in \Lambda} (\pi \otimes \pi^*)(\lambda) P.$$

(II) We have already shown the existence of the discrete spreading representation in the Analysis Theorem 7.7.5. It remains to show that the coefficients are indeed samples of the spreading function of the prototype operator. By $P \in \mathcal{B}$ we have $\sigma(P), \eta(P) \in \mathbf{S}_0(\mathcal{G} \times \widehat{\mathcal{G}})$ which implies absolutely convergent Poisson Summation Formula (in its symplectic version) for periodization of functions on $\mathcal{G} \times \widehat{\mathcal{G}}$. One has

$$\sigma(K) = \sum_{\lambda \in \Lambda} T_\lambda \left[\sigma(P)\right]$$

which corresponds to (R_{Λ° is the sampling operator of the set Λ°) for $\lambda^\circ \in \Lambda^\circ$:

$$\int_{(\mathcal{G} \times \widehat{\mathcal{G}})/\Lambda} \sigma(K)(\mu) \overline{\lambda(\mathcal{J}^* \mu)}\, d\mu = \frac{1}{s(\Lambda)} \left[R_{\Lambda^\circ} \eta(P)\right](\lambda^\circ) = \frac{1}{s(\Lambda)} \langle P, \pi(\lambda^\circ) \rangle_{\mathcal{B}} .$$

❑

Note that the above theorem establishes a *discrete* counterpart to Lemma 7.5.2, where the operators in $\mathcal{B}$ are expanded into *continuously shifted* versions of a specific noncompact prototype operator. Moreover, the analysis and synthesis principle for Λ–invariant operators lead to alternative, compact proofs and generalizations of results of Wexler and Raz [WR90], Janssen [Jan95b], Ron and Shen [RS95b], and Tolimieri and Orr [TO95]. First, we present a necessary and sufficient condition on P such that its Λ–periodization gives the identity.

Corollary 7.7.7 *(WH–Decomposition of the Identity)*
A prototype operator $P \in \mathcal{B}$ *defines a* Λ*–invariant decomposition of the identity (on* $\mathbf{S}_0(\mathcal{G})$ *and* $\mathbf{L}^2(\mathcal{G})$*),*

$$\sum_{\lambda \in \Lambda} (\pi \otimes \pi^*)(\lambda) P = I, \tag{7.7.13}$$

if and only if

$$[R_{\Lambda^\circ}\eta(P)]\,(\lambda^\circ) = s(\Lambda)\delta_{0,\lambda^\circ}, \qquad \forall \lambda^\circ \in \Lambda^\circ, \tag{7.7.14}$$

where R_{Λ° denotes the sampling operator of the set Λ° and δ_{0,λ° is Kronecker's delta.

Proof: Combining the Synthesis Theorem 7.7.6 and our prerequisites we know that periodization of P yields $\sum_{\lambda\in\Lambda}(\pi \otimes \pi^*)(\lambda)P = \pi(0)[= I]$. This proves sufficiency. The necessity of (7.7.14) follows from the Analysis Theorem 7.7.5 which assures uniqueness of the correspondence between I (as a Λ–invariant operator) and δ_{0,λ° (as its coefficient in the discrete spreading representation). ❑

7.7.1 Consequences for Gabor analysis

In the context of (multi-window) WH–frames, the prototype operator P is most generally rank–N, i.e,

$$P = \sum_{k=1}^{N} g_k \otimes \gamma_k^*, \quad g_k, \gamma_k \in \mathbf{S}_0(\mathcal{G}).$$

Then, Corollary (7.7.7) leads to the following "averaged" biorthogonality condition:

$$\sum_{k=1}^{N} \langle g_k, \pi(\lambda^\circ)\gamma_k \rangle = s(\Lambda)\delta_{0,\lambda^\circ}, \quad \forall \lambda^\circ \in \Lambda^\circ \tag{7.7.15}$$

i.e., in order to determine the (identically structured) dual frame to the multi-window frame $(g_k, \Lambda)_{k=1}^N$ one has to solve (7.7.15) for $\{\gamma_k\}_{k=1}^N \subset \mathbf{S}_0(\mathcal{G})$. We do not get involved into the question whether or not such $\gamma_k \in \mathbf{S}_0(\mathcal{G})$ do exist (in analogy to [FG96] one can show that $(g_k, \Lambda)_{k=1}^N$ establishing a frame for $\mathbf{L}^2(\mathcal{G})$ and $s(\Lambda) \in \mathbb{Q}$ is sufficient). Eq. (7.7.15) is a generalization of the biorthogonality condition in [WR90],[Jan95b] to the case of multi-windows, nonseparable subgroups and/or nonseparable multidimensional prototypes.

Example. Let $\mathcal{G} = \mathbb{R}$ and $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$. Then $\Lambda^\circ = b^{-1}\mathbb{Z} \times a^{-1}\mathbb{Z}$ and we get the classical Wexler–Raz biorthogonality condition as a special case of (7.7.15):

$$\langle g, M_{k/a}T_{l/b}\gamma \rangle = ab\delta_{0,k}\delta_{0,l}, \quad \forall k, l \in \mathbb{Z}. \tag{7.7.16}$$

By the above theorems, we also generalize a beautiful result of Tolimieri and Orr [TO95] (also called "Wexler–Raz Identity" in [DLL95] and "Fundamental Identity" in [Jan95a]) :

Corollary 7.7.8 *For $f,g,h \in \mathbf{S}_0(\mathcal{G})$ the action of the Λ–periodization of $f \otimes g^*$ on h is (up to a constant) equal to the action of the Λ°–periodization of $h \otimes g^*$ on f:*

$$\left(\sum_{\lambda\in\Lambda}(\pi\otimes\pi^*)(\lambda)(f\otimes g^*)\right)h = \frac{1}{s(\Lambda)}\left(\sum_{\lambda^\circ\in\Lambda^\circ}(\pi\otimes\pi^*)(\lambda^\circ)(h\otimes g^*)\right)f$$

Proof: By Theorem 7.7.6 and tensor computation, one has:

$$\begin{aligned}
&\left(\sum_{\lambda\in\Lambda}(\pi\otimes\pi^*)(\lambda)(f\otimes g^*)\right)h \\
&= \left(\frac{1}{s(\Lambda)}\sum_{\lambda^\circ\in\Lambda^\circ}\langle f\otimes g^*,\pi(\lambda^\circ)\rangle_{\mathcal{B}}\,\pi(\lambda^\circ)\right)h \\
&= \frac{1}{s(\Lambda)}\sum_{\lambda^\circ\in\Lambda^\circ}\langle f,\pi(\lambda^\circ)g\rangle_{\mathbf{S}_0(\mathcal{G}\times\mathcal{G})}\,\pi(\lambda^\circ)h \\
&= \frac{1}{s(\Lambda)}\sum_{\lambda^\circ\in\Lambda^\circ}\underbrace{(\pi(\lambda^\circ)h\otimes(\pi(\lambda^\circ)g)^*)}_{=\pi(\lambda^\circ)(h\otimes g^*)\pi^*(\lambda^\circ)}f \\
&= \frac{1}{s(\Lambda)}\left(\sum_{\lambda^\circ\in\Lambda^\circ}(\pi\otimes\pi^*)(\lambda^\circ)(h\otimes g^*)\right)f
\end{aligned}$$

❏

Another fact that is well–known [Jan95b] for $\mathcal{G}=\mathbb{R}^d$ and separable Λ, is the existence of a Banach algebra of Λ–invariant operators.

Corollary 7.7.9 *(Discrete twisted convolution)*

The Λ–invariant operators with $\sigma(K)\in\mathbf{S}_0\left((\mathcal{G}\times\widehat{\mathcal{G}})/\Lambda\right)$ form an unital Banach algebra with the product defined by the natural operator product and the norm defined as the ℓ^1–norm of the spreading coefficients $(K)_{\lambda^\circ}$. This Banach algebra corresponds to the discrete twisted convolution in $\ell^1(\Lambda^\circ)$:

$$(KL)_{\lambda^\circ} = \sum_{\lambda^{\circ\prime}\in\Lambda^\circ}(K)_{\lambda^{\circ\prime}}(L)_{(\lambda^\circ-\lambda^{\circ\prime})}\overline{t'(\nu-\nu')} \qquad \lambda^\circ,\lambda^{\circ\prime}\in\Lambda^\circ,\lambda^\circ:=(t,\nu). \tag{7.7.17}$$

Proof: The discrete twisted convolution is simply the adequate reformulation of (7.6.11) for discrete measures supported on Λ°. Its ℓ^1–stability follows directly from the well–known fact that $|(KL)_\lambda|\le|(K)_\lambda| * |(L)_\lambda|$. The identity operator on $\mathbf{S}_0(\mathcal{G})$ is trivially Λ–invariant and its spreading representation is $\delta_{0,\lambda}\in\ell^1(\Lambda^\circ)$. ❏

One can expect an intimate relationship between (the ELCA version of) the Zak transform and the GKNS in the separable case. This is the topic of the following theorem.

Theorem 7.7.10 *Let Λ be a critical and separable TF-lattice, i.e., assume that there exists a lattice $\Lambda_1 \lhd \mathcal{G}$ such that $\Lambda = \Lambda_1 \times \Lambda_1^\perp$. Then for $f, g \in \mathbf{S}_0(\mathcal{G})$ the GKNS of the Λ-periodization of the rank–one operator $f \otimes g^* \in \mathcal{B}$ is given by*

$$\sigma\left(\sum_{\lambda\in\Lambda}(\pi\otimes\pi^*)(\lambda)(f\otimes g^*)\right) = s(\Lambda_1)\mathcal{Z}_{\Lambda_1} f \overline{\mathcal{Z}_{\Lambda_1} g} \tag{7.7.18}$$

where the Zak transform is defined by

$$\mathcal{Z}_{\Lambda_1} f(x,\xi) := \sum_{\mu\in\Lambda_1} f(x+\mu)\overline{\xi(\mu)}, \qquad (x,\xi)\in\mathcal{G}\times\widehat{\mathcal{G}}.$$

Proof: Straightforward application of the Poisson Summation Formula allows to formulate the Zak transform via $\hat{f}$:

$$\mathcal{Z}_{\Lambda_1} f(x,\xi) := \frac{1}{s(\Lambda_1)} \sum_{\mu\in\Lambda_1^\perp} \hat{f}(\xi+\mu)(\xi+\mu)(x),$$

On the other hand, the GKNS of the rank–one projection operator is given by (see (7.5.9))

$$\sigma(f\otimes g^*)(x,\xi) = f(x)\overline{\hat{g}(\xi)x(\xi)},$$

which by Theorem 7.7.6 leads to

$$\begin{aligned}
\sigma\left(\sum_{\lambda\in\Lambda}(\pi\otimes\pi^*)(\lambda)(f\otimes g^*)\right)(x,\xi) &= \sum_{\lambda\in\Lambda}\left[T_\lambda\sigma(f\otimes g^*)\right](x,\xi) \\
&= \sum_{\mu\in\Lambda_1}\sum_{\rho\in\Lambda_1^\perp} f(x-\mu)\overline{\hat{g}(\xi-\rho)(x-\mu)(\xi-\rho)} \\
&= \underbrace{\sum_{\mu\in\Lambda_1} f(x+\mu)\overline{\mu(\xi)}}_{\mathcal{Z}_{\Lambda_1} f(x,\xi)}\ \underbrace{\overline{\sum_{\rho\in\Lambda_1^\perp}\hat{g}(\xi+\rho)(\xi+\rho)(x)}}_{s(\Lambda_1)\overline{\mathcal{Z}_{\Lambda_1} g(x,\xi)}}
\end{aligned}$$

(Note that $\mu\in\Lambda_1, \rho\in\Lambda_1^\perp$ implies $\rho(\mu)=1$.) ❑

7.8 KN versus Weyl quantization

A prominent alternative to the Kohn–Nirenberg quantization is the Weyl quantization. For a given HS operator on $L^2(\mathbb{R}^d)$ with kernel $h(x,y) \in$

$L^2(\mathbb{R}^{2n})$ the Weyl symbol is defined as

$$\sigma_W(K)(x,\xi) := \int_{\mathcal{G}} \kappa(K)(x+t/2, x-t/2)\overline{\xi(t)}dt, \qquad \mathcal{G} = \mathbb{R}^d. \qquad (7.8.1)$$

The Weyl symbol of a rank–one projection operator $f \otimes f^*$ is the Wigner distribution of f [Fol89, CM80]. A comparison with the respective definition of the Kohn–Nirenberg symbol,

$$\sigma_{KN}(K)(x,\xi) := \int_{\mathcal{G}} \kappa(K)(x, x-t)\overline{\xi(t)}dt, \qquad (7.8.2)$$

shows that the Weyl and Kohn–Nirenberg calculus are largely parallel concepts for $\mathcal{G} = \mathbb{R}^d$. There one can define a generalized Weyl symbol by [Koz92a, Koz96a] with $|\alpha| \leq 1/2$

$$\sigma_\alpha(K)(x,\xi) := \int_{\mathcal{G}} \kappa(K)\left(x + \left(\frac{1}{2} - \alpha\right)t, x - \left(\frac{1}{2} + \alpha\right)t\right)\overline{\xi(t)}dt. \quad (7.8.3)$$

This definition corresponds to a generalized time–frequency shift operator corresponding to cross–sections of the Schrödinger representation of the reduced Heisenberg group:

$$\pi_\alpha(t,\nu) := M_{\nu(1/2+\alpha)} T_t M_{\nu(1/2-\alpha)} = T_{t(1/2-\alpha)} M_\nu T_{t(1/2+\alpha)}.$$

By symplectic Fourier transform of $\sigma_\alpha(K)$ one has a generalized spreading function as follows:

$$\eta_\alpha(K)(t,\nu) := \int_{\mathcal{G}} \kappa(K)\left(x + \left(\frac{1}{2} - \alpha\right)t, x - \left(\frac{1}{2} + \alpha\right)t\right)\overline{\nu(x)}dx.$$

Among this family of quantization rules the Weyl correspondence is marked out by two desirable properties which are not true for other choices of α:

- One has
$$\eta_0(K^*)(t,\nu) = \overline{\eta_0(K)(-t,-\nu)}$$
and as a consequence, $\sigma_0(K)$ is real–valued for self–adjoint operators.
- Let $\mathcal{A}$ denote a symplectic coordinate transform of $\mathbb{R}^{2d}$, then
$$\mathcal{A} \circ \eta_0(K) = \eta_0(U_\mathcal{A} K U_\mathcal{A}^*),$$
where $U_\mathcal{A}$ is a unitary representation of $\mathcal{A}$ on $\mathbf{L}^2(\mathbb{R}^d)$ (more details about this aspect can be found in [Fol89].)

However, whenever we consider compact or infinite discrete LCA groups a consistent reformulation of the Weyl correspondence is impossible[3] due

[3]Of course, this is similar to the problem of formulating the Wigner distribution for discrete time signals [CM83].

to the nonexistence of automorphisms of the form $x \to 2x$. Note that this excludes in particular the practically important case of finite groups with even order. In [Hen85] Hennings considers generalizations of the Weyl correspondence on LCA groups with the requirement that $x \to 2x$ is an automorphism of $\mathcal{G}$. For Λ–invariant operators one has a similar situation (for simplicity we consider a separable lattice and $\mathcal{G} = \mathbb{R}$):

Theorem 7.8.1 *Let K, L be bounded operators acting on $\mathbf{L}^2(\mathbb{R})$ such that*

$$M_{bn}T_{am}K = KM_{bn}T_{am}, \quad \textit{with} \quad n, m \in \mathbb{Z}$$

and likewise for L, then one has a "perfect symbol calculus"

$$\sigma_\alpha(K)\sigma_\alpha(L) = \sigma_\alpha(KL)$$

for:

(a) $\alpha = 0$ *and* $ab = \frac{1}{2n}$,

(b) $|\alpha| = 1/2$ *and* $ab = \frac{1}{n}$.

Remark: As one may expect, the restriction of the "perfect symbol calculus" to the compact quotient group $\mathbb{R}^2 / (a\mathbb{Z} \times b\mathbb{Z})$ extends to an $\mathbf{L}^\infty$–functional calculus (in the sense of a Gelfand transform for a C^*–algebra), where one has in particular

$$||K|| = ||\sigma_\alpha(K)||_\infty,$$

which can be proved relatively easy. (The relation of the GKNS to the Zak transform as stated Theorem 7.7.18 proves the $\mathbf{L}^\infty$–functional calculus for separable unit lattices by the properties of the Zak transform as discussed in Chapter 6.) However, now our intention is only to emphasize the better behavior of the KN symbol in comparison to the Weyl symbol.

Proof: The generalized Weyl correspondence can be split into an $\mathbb{R}^{2d}$ automorphism $(x, y) \mapsto \left(x + \left(\frac{1}{2} - \alpha\right) t, x - \left(\frac{1}{2} + \alpha\right) t\right)$ and partial Fourier transform. Hence, on $\mathcal{G} = \mathbb{R}^d$ by using Lemma 7.3.6 and Lemma 7.3.2, one can generalize the Gelfand–triple isomorphism shown for the Kohn–Nirenberg correspondence to any member of the family $\sigma_\alpha(K)$. Furthermore, one has respective generalizations of the Analysis and Synthesis Theorem (Theorem 7.7.5 and 7.7.6) which leads finally to a discrete twisted convolution defined by

$$(KL)^{(\alpha)}_{(k,l)} = \sum_{m,n\in\mathbb{Z}} (K)^{(\alpha)}_{(m,n)}(L)^{(\alpha)}_{(k-m,l-n)}$$

$$. \exp\left\{-2\pi i \frac{1}{ab}\left(ml(\alpha + 1/2) + kn(\alpha - 1/2) - 2mn\alpha\right)\right\},$$

where $(KL)^{(\alpha)}_{(k,l)}$ denotes the (symplectic) Fourier coefficients of $\sigma_\alpha(KL) \in \mathbf{S}_0(\mathbb{R}^2 / (a\mathbb{Z} \times b\mathbb{Z}))$. Whenever the discrete twisted convolution reduces to a conventional convolution one has a perfect symbol calculus. It is easy to see that this is the case for the setups (a) and (b) of the theorem.

❑

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8

Numerical algorithms for discrete Gabor expansions

Thomas Strohmer

"In every mathematical investigation, the question will arise whether we can apply our mathematical results to the real world."
V. I. Arnold, 1983

ABSTRACT – We present a unifying approach to discrete Gabor analysis, based on unitary matrix factorization. The factorization point of view is notably useful for the design of efficient numerical algorithms. This presentation is the first systematic account of its kind. In particular, it is shown that different algorithms for the computation of the dual window correspond to different factorizations of the frame operator. Simple number theoretic conditions on the time-frequency lattice parameters imply additional structural properties of the frame operator, which do not appear in an infinite-dimensional setting. Further the computation of adaptive dual windows is discussed. We point out why the conjugate gradient method is particularly useful in connection with Gabor expansions and discuss approximate solutions and preconditioners.

8.1 Introduction

Taking up the quotation of V. I. Arnold we are confronted with the question how to realize the concepts of Gabor analysis in applications. Crucial for the success of such a realization is the availability of efficient numerical algorithms. Therefore we will investigate in this chapter Gabor analysis from a numerical analysis point of view.

We present a unifying framework for discrete Gabor analysis, our approach heavily relies on unitary matrix factorization, inspired by the book *Computational Frameworks for the Fast Fourier Transform* of van Loan [vL92]. The proposed approach allows to exploit the simplicity and intrinsic beauty of Gabor analysis.

In applications one can only process a finite number of data. Finite and periodic discrete Gabor expansions have been extensively investigated, see e.g. [WR90, AT90, AGT92, QCL92, Orr93a, ZZ93a, Jan94a, RN94, FCS95,

H. G. Feichtinger et al. (eds.), *Gabor Analysis and Algorithms*

YKS95, QF95, LH96, QC96]. Unfortunately no serious attempt has been undertaken to unify the various existing approaches and results. Furthermore it has not been recognized up to now that the Gabor frame operator possesses a rich mathematical structure that is not present in an infinite-dimensional setting. Hence a comprehensive elaboration of the structure of the Gabor frame operator is a major aim of this contribution.

This chapter is organized as follows. An algebraic setting for discrete Gabor analysis is presented in Section 8.2. In Section 8.3 we show that discrete Gabor expansions involve highly structured matrices. A description of these structures requires the language of Kronecker products, and unitary matrices. We demonstrate that most of the existing algorithms for the inversion or diagonalization of the frame operator revolve around the proposed matrix factorization approach. One well known key observation is that the matrix representing the frame operator possesses a periodic as well as regular sparse structure. In Section 8.4 we present a fundamental factorization of the Gabor frame operator. It is shown that simple number theoretic relations of the time-frequency shift parameters and a proper unitary factorization yield additional local periodic and block structures of the frame operator. In the relevant literature these additional properties have only be discussed for the special cases of critical sampling and integer oversampling.

A simple consequence of our number theoretic results is the often celebrated diagonalization of the frame operator by the Zak transform in case of integer oversampling.

The calculation of dual windows which are optimal with respect to other norms than the ℓ^2-norm is discussed in Section 8.5. Using the results of Section 8.3 as starting point, we show in Section 8.6 that several variants of the conjugate gradient method are particularly useful for the computation of dual windows. Finally Section 8.7 is devoted to a short discussion of approximate inverse solutions and efficient preconditioners.

8.2 An Algebraic setting for discrete Gabor theory

The basic operation arising in shift-invariant systems is the convolution of two functions. If f and g are discrete signals, their convolution $f * g$ is defined by

$$(f * g)(n) = \sum_{m} f(m)g(n-m)\,. \tag{8.2.1}$$

If $f, g \in \ell^2(\mathbb{Z})$, then n and m range over $\mathbb{Z}$. However in digital signal and image processing we can only process discrete signals of finite length. Thus for a numerical realization of Gabor analysis, we first have to create a finite-dimensional model. If $f, g \in \mathbb{C}^L$ the indices n, m in equation (8.2.1) range over $\mathbb{C}^L$. In this case, the computation of $f * g$ may ask for $g(-1)$ which is not defined. To extend the signal beyond the boundary we do all arithmetic on sequence indices in the ring $\mathbb{Z}_L = \mathbb{Z} \bmod L$. This circular extension also allows to preserve many important properties of the Heisenberg group.

Clearly any finite-dimensional model will thus introduce some effects at the boundaries of the signal. We do not go into detail here, the reader may consult [SN96, AU96] where various boundary conditions are presented. Concerning the Gabor transform the size of region influenced by boundary effects is limited to the length of the support of the window. Hence without loss of generality we may always assume that the finite-length signal has been extended in a proper way at the boundary before it is periodically extended.

Thus we are working with the finite discrete model below. A discrete signal f of length L is a finite sequence $(f(0), f(1), \ldots, f(L-1)) \in \mathbb{C}^L$. The inner product of $f, g \in \mathbb{C}^L$ is $\langle f, g\rangle = \sum_{n=0}^{L-1} f(n)\overline{g(n)}$ and the ℓ^2-norm of f is given by $\|f\| = \sqrt{\langle f, f\rangle}$. Here and throughout this chapter we extend finite sequences in $\mathbb{C}^L$ to infinite sequences of period L by $f(k+lL) = f(k)$ for $k = 0, \ldots, L-1$ and $l \in \mathbb{Z}$.

We write $T_k f(t) := f(t+k)$ for the translation operator or time shift and $M_l f(t) := f(t)e^{-2\pi i l t/L}$, $l \in \mathbb{Z}$ for the modulation operator or frequency shift. For a given function g the set of Gabor analysis functions $\{g_{m,n}\}$ is given by

$$g_{m,n} = M_{mb} T_{na} g \tag{8.2.2}$$

for $m = 0, \ldots, M-1$, $n = 0, \ldots, N-1$ where $Na = Mb = L$ with $a, b, N, M \in \mathbb{N}$. The parameters a and b represent time and frequency sampling intervals, and are also referred to as time and frequency shift parameters or as time-frequency lattice constants. For convenience we restrict ourselves in this chapter to separable (i.e. rectangular) time-frequency lattices $a\mathbb{Z} \times b\mathbb{Z}$ in $Z_L \times \mathbb{Z}_L$.

Discrete Gabor analysis can be understood in an algebraic setting. Let G be the $L \times MN$ matrix having $g_{m,n}$ as its $(m+nM)$-th column for $m = 0, \ldots, M-1$, $n = 0, \ldots, N-1$. We write $c_{m,n} = \langle f, g_{m,n}\rangle$ and by slightly abusing our notation we treat the $N \times M$ array c also as column vector of length MN by stacking the columns of c. The *Gabor transform* of a function f is given by

$$G^* f = c\,.$$

The case $ab = L$ is usually referred to as *critically sampled* Gabor transform, whereas the choice $ab < L$ yields an *oversampled* Gabor transform, the ratio MN/L giving the oversampling rate or redundancy. If we set $a = b = 1$ the Gabor transform of f coincides with the discrete short time Fourier transform of f.

Under suitable conditions on g, a, b (cf. [Jan94a]) we can reconstruct f from its *Gabor coefficients* $c_{m,n}$ by means of a *dual system* $\{\gamma_{m,n}\}$ via

$$f = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} c_{m,n} \gamma_{m,n}$$

where the set $\{\gamma_{m,n}\}$ can be determined from a single dual function γ analogously to the set $\{g_{m,n}\}$ in equation (8.2.2). Defining Γ as the $L \times MN$ matrix having $\gamma_{m,n}$ as its $(m+nM)$-th column, this duality condition can be expressed as

$$G\Gamma^* = I_L \tag{8.2.3}$$

where I_L denotes the identity matrix on $\mathbb{C}^L$. If $ab < L$ there are many dual functions satisfying condition (8.2.3). Of particular importance is the dual with minimal norm, which we denote by $^\circ\gamma$.

G is a $L \times MN$ matrix and therefore it is a rectangular matrix in case of oversampling, thus G^{-1} does not exist. However we can always compute the pseudoinverse [Str80, Chapter 3] G^+ of G. The rows of G are linear independent, provided that $\{g_{m,n}\}$ constitutes a frame for $\mathbb{C}^L$. In this case G^+ becomes a left inverse which can be computed by $G^+ = G^*(GG^*)^{-1}$.

For perfect reconstruction of f the column rank of G must be L, which is equivalent to assuming that the set $\{g_{m,n}\}$ constitutes a frame for $\mathbb{C}^L$. A necessary condition for this is $ab \leq L$. Let $^\circ\Gamma$ be the matrix containing the elements $^\circ\gamma_{m,n}$ of the dual frame associated with $^\circ\gamma$. Then it follows from the relation of frames and pseudo inverses [Chr95b] that $^\circ\Gamma$ coincides with the pseudo inverse of G, i.e.

$$^\circ\Gamma^* = G^+ .$$

Recall that the Gabor frame operator S is defined as

$$Sf = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \langle f, g_{m,n} \rangle g_{m,n} \cdot$$

S is a positive definite Hermitian $L \times L$ matrix which can be expressed as

$$S = GG^* .$$

It is well-known that the minimal norm dual $^\circ\gamma$ satisfies $S^\circ\gamma = g$, see e.g. [Jan94a]. The computation of $^\circ\gamma$ involves the (pseudo-) inversion of G or, equivalently, the inversion of S. From a numerical analysis point of view we may either choose direct or iterative methods for this purpose. The central theme in the following sections is the idea that different algorithms correspond to different unitary matrix factorizations of the frame operator S and the matrix G, respectively. Efficient algorithms for the computation of dual windows are essentially based on these factorizations.

8.3 Unitary factorizations of the Gabor frame operator

The matrix factorization point of view, so successful in other areas of numerical linear algebra, has rarely entered into the description of discrete Gabor expansions. Nevertheless, such factorizations can be seen as the key for various fast numerical algorithms for discrete Gabor expansions.

The structures that arise in the Gabor frame operator are highly regular. Several approaches have been undertaken to exploit these structures for numerical algorithms [WR90, Orr93a, Yao93, QC93, ML94, MX94, LH96]. These approaches include Zak transform based methods [Jan88, Dau90, AGT91, BO92, AGT92, ZZ93b], iterative methods [BB92, FCS95], and methods utilizing sparse matrix techniques [SPA95, YKS95, QF95, Pri96b].

We mention in passing that the results presented in the following sections can also be derived from a group representation point of view. However since the focus in this chapter is on numerical methods, we prefer a language that immediately gives rise to the design of efficient algorithms for discrete Gabor expansions.

8.3.1 Diagonalization and block-diagonalization

One standard method in numerical analysis to derive fast algorithms for the inversion of a matrix X is based on a unitary factorization of X. A normal matrix X always possesses a unitary factorization $\Sigma = U^*XU$ where Σ is a diagonal matrix. One can look at this diagonalization also as the choice of a new basis, the matrix X becomes Σ in this new representation. For a general $L \times N$ matrix X there exists always a *singular value decomposition* $X = U\Sigma V^*$, where U is a $L \times L$ unitary matrix, V is a $N \times N$ unitary matrix and Σ is a $L \times N$ diagonal matrix (completed with zeros) containing the singular values of X.

For convenience let us assume for a moment that X is normal. If the factorization matrix has been found, the (pseudo-) inverse of X is easily computed by (pseudo-) inverting a diagonal matrix. However in general U will be difficult to find and moreover the evaluation of the matrix product U^*XU is computationally expensive.

Therefore we propose the following approach to reduce the overall computational effort for the (pseudo-) inversion of X. By relaxing the diagonality condition on D somewhat and demanding D to be only "close" to a diagonal matrix, we expect to find unitary matrices U which satisfy following criteria:

- the $L \times L$ unitary factorization matrix U should be constructed in at most $\mathcal{O}(L \log L)$ operations,

- U^*XU should be calculated in at most $\mathcal{O}(L \log L)$ operations
- $D = U^*XU$ is at least block diagonal, such that the blocks of D can be inverted within appropriate computational costs, i.e. the overall costs for the inversion of X (including factorization and inversion) should be substantially below $\mathcal{O}(L^3)$.

Clearly this approach will in general not reduce the computational costs for the inversion of unstructured matrices. It is the goal of the following sections to show that the matrices arising in discrete Gabor expansions allow to address all three items mentioned above.

8.3.2 Unitary matrices, Kronecker products, permutations

Our guiding philosophy is to choose a language that best expresses the underlying structures arising in discrete Gabor expansions. Permutation matrices, Fourier matrices and Kronecker products constitute the vocabulary of this unifying language.

A few diagonal matrix notations are handy. If $d \in \mathbb{C}^L$, then $D = \operatorname{diag}(d) = \operatorname{diag}(d_0, \dots, d_{L-1})$ is the $L \times L$ diagonal matrix with diagonal entries $d_0, \dots, d_{L-1}$. If $D_0, \dots, D_{L-1}$ are $p \times p$ matrices, then $D = \operatorname{diag}(D_0, \dots, D_{L-1})$ is the block diagonal matrix defined by

$$D = \begin{bmatrix} D_0 & 0 & \dots & 0 \\ 0 & D_1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & D_{L-1} \end{bmatrix}.$$

A *permutation matrix* is just the identity with its columns reordered. If P is a permutation and A is a matrix, then PA is a row permuted version of A and AP is a column permuted version of A. Note that $P^* = P^{-1}$. The permutation matrices are thus unitary, forming a subgroup of the unitary group. Among the permutation matrices, the mod p *perfect shuffle permutation* [vL92]

$$\begin{array}{r} p \text{ rows} \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\ P_{p,L} = \\ \\ \\ \\ \\ \end{array}
\begin{bmatrix}
1 & 0 & \dots & 0 & 0 & \dots & 0 \\
 & & & & 1 & & \\
 & 0 & & & & & \\
 & 1 & & & & & \\
\vdots & & & & & & \vdots \\
 & & & 1 & & & \\
 & & & & & & \\
0 & & & & & & 1
\end{bmatrix}$$

$$\underbrace{\qquad\qquad\qquad}_{q \text{ columns}}$$

where $L = pq$, shall play a key role in the factorization of the Gabor frame operator. Its transpose $P^*_{p,L}$ is called the mod p *sort permutation* [vL92]. Observe that $P_{p,L} = P^*_{q,L}$. We illustrate the action of $P_{p,L}$ and $P^*_{p,L}$ by means of simple examples with $L = 15 = 3 \cdot 5 \equiv p \cdot q$:

$$P_{3,15}x = [\, x_0\, x_5\, x_{10} \,|\, x_1\, x_6\, x_{11} \,|\, x_2\, x_7\, x_{12} \,|\, x_3\, x_8\, x_{13} \,|\, x_4\, x_9\, x_{14} \,]^*$$
$$P^*_{3,15}x = [\, x_0\, x_3\, x_6\, x_9\, x_{12} \,|\, x_1\, x_4\, x_7\, x_{10}\, x_{13} \,|\, x_2\, x_5\, x_8\, x_{11}\, x_{14} \,]^* .$$

It is not surprising that the *Fourier matrix* is of fundamental importance in the factorization of the Gabor frame operator. By the Fourier matrix of order L we shall mean the unitary matrix

$$F_L = \sqrt{\frac{1}{L}} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{L-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(L-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{L-1} & \omega^{2(L-1)} & \dots & \omega^{(L-1)(L-1)} \end{bmatrix}$$

where $\omega = e^{-2\pi i/L}$.

Further we will frequently make use of circulant matrices. Recall that a *circulant matrix* is of the form

$$C_L = \begin{bmatrix} c_0 & c_1 & \dots & c_{L-1} \\ c_{L-1} & c_0 & \dots & c_{L-2} \\ \vdots & \vdots & & \vdots \\ c_1 & c_2 & \dots & c_0 \end{bmatrix} .$$

Circulant matrices can be seen as finite analogue of infinite Toeplitz matrices. They form a linear subspace of the set of all matrices of size $L \times L$. It is well known that a circulant matrix C_L is diagonalized by the Fourier matrix F_L. In particular let Π_L, defined by

$$\Pi_L = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix} ,$$

be the *basic circulant permutation matrix* of size $L \times L$ then

$$\Pi_L = F_L \Omega_L F_L^* \tag{8.3.1}$$

where $\Omega_L = \operatorname{diag}(1, \omega, \omega^2, \dots, \omega^{L-1})$ contains the eigenvalues of Π_L, and the columns of F_L are the eigenvectors of Π_L, see [Dav79].

In terms of permutations and Fourier matrices the translation operator T_m can be represented by the basic circulant permutation matrix, i.e.,

$T_m f = \Pi_L^m f$. Since modulating a function means that we shift it in the frequency domain, the modulation operator M_n can be expressed via equation (8.3.1) as $M_n f = \Omega_L^n f$.

If $A \in C^{p\times q}$ and $B \in C^{m\times n}$ then the *Kronecker product* $A \otimes B$ is the p-by-q block matrix

$$A \otimes B = \begin{bmatrix} a_{00}B & \dots & a_{0q-1}B \\ \vdots & & \vdots \\ a_{p-10}B & \dots & a_{p-1q-1}B \end{bmatrix} \in \mathbb{C}^{pm\times qn}.$$

Note that

$$I \otimes B = \begin{bmatrix} B & 0 & \dots & 0 \\ 0 & B & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & B \end{bmatrix}$$

is block diagonal, where I is the identity matrix. If $L = mn$ and $A \in \mathbb{C}^{m\times m}$, then [vL92]

$$P_{m,L}^*(I_n \otimes A) = (A \otimes I_n)P_{m,L}^* . \tag{8.3.2}$$

If A and B are unitary matrices, then $A \otimes B$ is a unitary matrix, since $(A \otimes B)^* = A^* \otimes B^*$ and $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$, see [vL92].

8.3.3 Matrix factorization and rational oversampling

In this section we describe several unitary factorizations of the Gabor frame operator for arbitrary windows and general rational oversampling. In Section 8.4 and Section 8.7 we will show how the presented factorizations simplify, if the lattice parameters or the window satisfy certain properties.

Recall that the entries of the frame operator S are given by

$$S_{m,n} = \sum_{k=0}^{N-1}\sum_{l=0}^{M-1} M_{lb}T_{ka}g(m)\overline{M_{lb}T_{ka}g(n)} .$$

Combining this with the fact that $\sum_{l=0}^{M-1} e^{2\pi i m l b/L} = 0$ if $m \neq M$ leads to the *Walnut representation* [HW89] of the Gabor frame operator for the discrete case [QF95]:

$$S_{mn} = \begin{cases} M\sum_{k=0}^{N-1} T_{ka}g(m)\overline{T_{ka}g(n)} & \text{if } |m-n| \bmod M = 0 \\ 0 & \text{otherwise}. \end{cases} \tag{8.3.3}$$

Expressed in words, only every M-th subdiagonal of S is non-zero. Furthermore the entries along a subdiagonal are a–periodic. These observations suggest two strategies of factorizing S. One strategy utilizes the periodicity, the other one focuses on the regular sparsity of the matrix.

Exploiting the regular sparsity of S we show that if we permute the rows and columns of S according to the perfect shuffle mapping $P_{M,L}$, then a Kronecker block structure emerges.

Theorem 8.3.1 *Let B_k denote the $b \times b$ matrix with entry*

$$(B_k)_{mn} = S_{k+mM,k+nM} \tag{8.3.4}$$

for $m, n = 0, \dots, b-1$ and $k = 0, \dots, M-1$. Then S can be unitarily factorized into the block diagonal matrix $D_S = diag(B_0, \dots, B_{M-1})$ with $P_{M,L}$ as factorization matrix.

Proof: Set $E_k = \text{diag}(e_k)$ where e_k is the k-th unit vector $\in \mathbb{C}^M$. Exploiting the uniform sparsity of the Gabor frame operator we can express S as

$$S = \sum_{k=0}^{M-1} B_k \otimes E_k \,.$$

By some algebraic manipulation of property (8.3.2) of the Kronecker product yields

$$P^*_{M,L}(B_k \otimes E_k)P_{M,L} = E_k \otimes B_k$$

and hence

$$D_S = P^*_{M,L} S \; P_{M,L} = \sum_{k=0}^{M-1} E_k \otimes B_k = \text{diag}(B_0, \dots, B_{M-1}) \,. \tag{8.3.5}$$

❑

A block diagonal matrix can be inverted by inverting each block separately, see e.g. [GL89]. Thus the computation of the $L \times L$ matrix S^{-1} can be reduced to the inversion of b blocks of size $M \times M$.

In many situations the computational effort can be further drastically reduced. Most research papers contain results on simplifications and corresponding efficient algorithms only for the special cases of critical sampling and integer oversampling [ZZ93b, Qiu95, BG96]. Our approach will show clearly – based on standard number theoretic arguments – that there are many other situations where one can show that S belongs to a very simple matrix algebra.

In Theorem 8.3.1 we have utilized the uniform sparsity of S. In the sequel we describe an alternative approach, focusing on the periodicity of the diagonal entries of S. This approach has been proposed by Qiu [Qiu95].

The following observation is fundamental for the alternative factorization of S. It is essentially based on the Walnut representation.

Lemma 8.3.2 *The Gabor frame operator S is a block circulant matrix of the form*

$$S = \begin{bmatrix} A_0 & A_1 & \dots & A_{N-1} \\ A_{N-1} & A_0 & \dots & A_{N-2} \\ \vdots & \vdots & & \vdots \\ A_1 & A_2 & \dots & A_0 \end{bmatrix}$$

where the A_k are non-circulant $a \times a$ matrices with $(A_k)_{m,n} = S_{ka+m,ka+n}$ for $k = 0, \dots, N-1$ and $m = 0, \dots, a-1$, $n = 0, \dots, a-1$.

Proof: By equation (8.3.3), the non-zero entries of S are given by $S_{m,n} = M \sum_{k=0}^{N-1} T_{ka}g(m)\overline{T_{ka}g(n)}$. Hence

$$\begin{aligned} S_{m+la,n+la} &= M \sum_{k=0}^{N-1} T_{ka}g(m+la)\overline{T_{ka}g(n+la)} \\ &= M \sum_{k=0}^{N-1} T_{(k+l)a}g(m)\overline{T_{(k+l)a}g(n)} \\ &= S_{m,n} \end{aligned}$$

by a rearrangement of the summation, $k+l=j$. This a-periodicity together with the periodicity of g yields the desired structure of S. ❑

Observe that S does not have circulant blocks in general, thus it cannot be diagonalized by a straightforward generalization of equation (8.3.1). However we still can derive a diagonalization based on Fourier matrices [Qiu95].

Theorem 8.3.3 *S can be unitarily factorized into a block diagonal matrix via*

$$(F_N \otimes I_a)^* S (F_N \otimes I_a)$$

Proof: Since S is a block circulant matrix, it can be represented as [Dav79]

$$S = \sum_{k=0}^{N-1} \Pi_N^k \otimes A_k \,.$$

Hence

$$\begin{aligned}(F_N \otimes I_a)^* S (F_N \otimes I_a) &= \sum_{k=0}^{N-1} (F_N \otimes I_a)^* (\Pi_N^k \otimes A_k)(F_N \otimes I_a) \\ &= \sum_{k=0}^{N-1} (F_N^* \Pi_N^k F_N) \otimes (I_a A_k I_a) \\ &= \sum_{k=0}^{N-1} \Omega_N^k \otimes A_k\end{aligned}$$

where we have used the properties of the Kronecker product and equation (8.3.1) in the last step. Defining

$$\tilde{A}_l = \sum_{k=0}^{N-1} \omega^{lk} A_k \qquad \text{for } l = 0, \dots N-1 .$$

we obtain that S is unitarily equivalent to $\operatorname{diag}(\tilde{A}_0, \dots, \tilde{A}_{N-1})$. ❑

The block diagonalization described in Theorem 8.3.1 can be also derived from a factorization derived by Prinz [Pri96a]. Extending the approach of Stewart, Potter and Ahalt [SPA95] from critical sampling to rational oversampling, he proposes to directly factorize G instead of S.

Proposition 8.3.4 *The matrix G can be factorized into a block diagonal matrix D_G with M rectangular blocks W_k of size $b \times N$ via*

$$D_G = P_{M,L}^* G (I_N \otimes F_M^*) P_{N,MN}^*$$

where

$$D_G = diag(W_0, \dots, W_{M-1})$$

with

$$(W_k)_{mn} = \sqrt{M} g(k + mM - na)$$

for $m = 0, \dots, b-1$, $n = 0, \dots, N-1$ and $k = 0, \dots, M-1$.

The proof is based on the fact that $(G(I_N \otimes F_M^*))_{m,n} = \sqrt{M} g(n - aM)$, if $|m-n| \bmod M = 0$ and 0 else, analogous to equation (8.3.3). Note that we get the entries of W_k by simple data addressing without extra computation.

The blocks W_k can be established by simple permutations of the entries of g, no Fourier transform has to be performed.

Using basic properties of the Kronecker product and the unitarity of F_M and $P_{N,MN}$ we obtain the relation

$$D_S = D_G D_G^* .$$

In case of oversampling, i.e. $MN > L$, we know that G and consequently D_G is not invertible. Clearly we can compute the pseudoinverse D_G^+ of D_G via $D_G^+ = D_G^*(D_G D_G^*)^{-1}$, which means that we end up with the inversion of the matrix D_S.

8.3.4 Wexler–Raz identity and matrix factorization

Wexler and Raz have obtained an equivalent and elegant formulation of the duality condition of the systems $\{g_{m,n}\}$ and $\{\gamma_{m,n}\}$, see [WR90]. We refer to Chapter 3 and Chapter 1 in this book for a systematic discussion of the Wexler-Raz identity. Whereas the formulation of the duality condition in equation (8.2.3) holds for any frame and its dual, the Wexler-Raz duality condition is a specific characterization of duality for Gabor systems. In our algebraic setting we can describe this condition as follows.

Let H be the $L \times ab$ matrix with entries

$$H_{kl} = \Pi_L^{mM} \Omega_L^{nN} g(k)$$

where $l = m + na$ and $m = 0, \dots, b-1$, $n = 0, \dots, a-1$. Observe that H is of similar structure as G but with a and b replaced by M and N respectively.

Then the Wexler-Raz duality condition for g and γ can be expressed as

$$H^* \gamma = \frac{ab}{L}\, [1, 0, \dots, 0]^T. \tag{8.3.6}$$

The minimal norm solution of (8.3.6) coincides with ${}^\circ\gamma$, see e.g. the Chapter of Janssen in this book.

Applying the arguments used in the proof of Proposition 8.3.1 to H we obtain following result.

Corollary 8.3.5 *Let V_k denote the $N \times b$ matrix with entry*

$$(V_k)_{mn} = \sqrt{a}\, g(k - mM + na)$$

for $m = 0, \dots, N-1$, $n = 0, \dots, b-1$ and $k = 0, \dots, a-1$. Then H can be unitarily factorized into the block diagonal matrix

$$D_H = P_{a,L}^* H (I_b \otimes F_a^*) P_{b,ab}^* \tag{8.3.7}$$

where $D_H = diag(V_0, \dots, V_{a-1})$.

Since D_H is in general not invertible, we compute its left inverse D_H^+ via $D_H^+ = (D_H^* D_H)^{-1} D_H^*$. $D_H^* D_H$ is a $ab \times ab$ block matrix with a blocks of size $b \times b$, whereas the $L \times L$ matrix D_S of equation (8.3.5) consists of M blocks of size b.

It is an easy task to derive all factorizations of the previous sections for the matrix H with the appropriate lattice parameters M, N. The advantage of introducing first the matrix G and S is that G^* represents the Gabor transform and therefore is the natural starting point for an algebraic and numerical treatment of discrete Gabor expansions.

Remark: One might argue, that H is a matrix of smaller size and of smaller rank than G or S, therefore solving for the dual window ${}^\circ\gamma$ via factorizing H should be preferable over factorizing G. However it follows from Propositions 8.4.1, 8.4.2 and the remark at the end of Section 8.4, that the costs for computing G^+ or H^+ are the same, since G and H can be factorized into the same number of submatrices of equal size.

8.3.5 Representation in the Fourier domain

In the equation $S\gamma = g$ the operator S acts on functions given in the time domain. Equivalently we could apply the frame operator in the frequency domain. The following proposition is an easy consequence of Parseval's formula, the Walnut representation and equation (8.3.1).

Proposition 8.3.6 *Denote $\hat{S} = F_L S F_L^*$, then the entries of $\hat{S}$ are given by*

$$\hat{S}_{mn} = \begin{cases} N \sum_{k=0}^{M-1} T_{kb}\hat{g}(m)\overline{T_{kb}\hat{g}(n)} & \text{if } |m-n| \mod N = 0 \\ 0 & \text{otherwise}. \end{cases}$$

In other words if the set $\{g_{mb,na}\}$ constitutes a Gabor frame for $\mathbb{C}^L$, then $\{\hat{g}_{nb,ma}\}$ constitutes also a Gabor frame for $\mathbb{C}^L$, see [QF95].

Clearly all factorizations of S could equally well be applied to $\hat{S}$. We will make use of this fact in Section 8.7 in connection with the calculation of approximate inverses for S.

8.3.6 Multi-dimensional discrete Gabor expansions

The extension of the theorems and algorithms in this chapter to higher dimensions is routine, based on the group theoretical approach presented in Chapter 7 and Chapter 6 in this book. Therefore we restrict this discussion to following remarks:

- Separable case: If the d-dimensional function g can be factorized into a product of one-dimensional functions $g = g_1 \otimes \dots g_d$, and if the sampling set Λ of the time-frequency space can be written as product lattice $\Lambda = \prod_{i=1}^{d} \alpha_i \mathbb{C}^L \otimes \prod_{i=1}^{d} \beta_i \mathbb{C}^L$ then the dual Gabor window γ can also be factorized into $\gamma = \gamma_1 \otimes \dots \gamma_d$, where γ_k is a window dual to g_k.

- Multi-dimensional signals – like 2-D images – are in general not separable. Since one usually tries to match the analysis window and the sampling lattice to the properties of the given (class of) signals, we may have to work with non-separable windows and non-separable sampling lattices [KFPS96]. Then the factorizations of the preceding sections can still be easily derived without further modification by using 2-D Fourier matrices and block permutations. For instance extending Theorem 8.3.1 to two dimensions implies that S can be factorized into a block diagonal matrix, where the blocks themselves are also block diagonal.

8.4 Finite Gabor expansions and number theory

The unitary factorization of S into a block diagonal or block circulant matrix derived in Section 8.3 is based on the periodic and sparse structure of S which follows from the Walnut representation. These factorizations clearly hold in general and do not depend on number theoretic conditions on the lattice parameters a, b.

In this section we show that in a finite setting the Gabor frame operator exhibits an additional structure. Although finite and periodic discrete Gabor expansions have been extensively investigated, see e.g. [WR90, AGT92, Jan94a, QC96, ZZ93a, YKS95, RN94, Orr93a, FCS95], its rich structural properties have not yet been fully recognized. A complete description of the structure of S is the purpose of this section. We will show in Subsection 8.4.2 that the well known diagonalization of the frame operator in the cases of critical and integer oversampling (cf. e.g. [Jan88, Dau90, AGT91, AGT92, ZZ93b]) are simple consequences of Theorem 8.4.3. A first attempt to investigate Gabor expansions from a number theoretic point of view can be found in [Pri96b].

8.4.1 Fundamental factorization of the Gabor frame operator

In the sequel we will frequently use following notation. Let $q \in \mathbb{R}$, then $\lfloor q \rfloor$ denotes the largest integer $\leq q$ and $\lceil q \rceil$ is the smallest integer $\geq q$.

Proposition 8.4.1 *Given a function g of length L and lattice parameters a, b. Denote the greatest common divisor (gcd) of a and M by c. Let W_k be* *the submatrices of the block diagonal factorization of G stated in Proposition 8.3.4, i.e. $(W_k)_{mn} = Mg(k+mM-na)$. Then the W_k satisfy following relation:*

$$\Pi_b^q W_k \Pi_N^l = W_{(k+la) \bmod M} \tag{8.4.1}$$

with $q = \lfloor \frac{lab}{L} \rfloor$. Thus there are (up to permutations) c different submatrices W_k for $k = 0, M-1$.

Proof: Formula (8.4.1) is equivalent to $(W_k)_{m,n} = (W_{(k+la) \bmod M})_{m+q,n+l}$. We have

$$\begin{aligned} (W_k)_{m,n} =& (W_{k+(la \bmod M)})_{m+q,n+l} \\ \Leftrightarrow g(k+mM-na) =& g(k+la+(m+q)M-(n+l)a) \\ \Leftrightarrow g(la) =& g((la \bmod M)+qM-la) \end{aligned}$$

and since the relation

$$n_1 = (n_1 \bmod n_2) + \lfloor n_1/n_2 \rfloor n_2$$

is just the general representation of n_1 in terms of divisor n_2, corresponding quotient and remainder, the assertion follows (with $n_1 = la$ and $n_2 = M$). ❑

Next we show that a block circulant structure emerges in each submatrix W_k.

Proposition 8.4.2 *Given a function g of length L and lattice parameters a, b. Denote $d = \gcd(b, N)$. Then W_k is a block circulant matrix with d generating blocks of size $\frac{b}{d} \times \frac{N}{d}$ for $k = 0, \ldots, M-1$.*

Proof: For fixed k we have to show that

$$(W_k)_{m,n} = (W_k)_{m',n'} \tag{8.4.2}$$
$$\text{where} \quad m' = (m + \frac{b}{d}) \bmod b, \quad n' = (n + \frac{N}{d}) \bmod N\,.$$

A simple calculation shows that condition (8.4.2) is equivalent to

$$M[((m+\frac{b}{d}) \bmod b) - m] = a[((n+\frac{N}{d}) \bmod N) - n] \tag{8.4.3}$$

for $m = 0, \ldots, b-1$, $n-0, \ldots, N-1$, where equality in (8.4.3) is understood modulo L. We consider two cases for the left part of (8.4.3):

$$\text{a) } m < b - \frac{b}{d} : \Rightarrow M[((m + \frac{b}{d}) \bmod b) - m] = \frac{L}{d};$$

$$\begin{aligned}\text{b) } m \geq b - \frac{b}{d} : &\Rightarrow m = b - \frac{b}{d} + c \quad \text{for} \quad c = 0, \ldots, \frac{b}{d} - 1 \\ &\Rightarrow M[((m + \frac{b}{d}) \bmod b) - m] \\ &= M[((b - \frac{b}{d} + c + \frac{b}{d}) \bmod b) - b + \frac{b}{d} - c] = -L + \frac{L}{d}.\end{aligned}$$

Repeating this steps for the right part of (8.4.3) and using the relation $(L/d = L/d - L) \bmod L$, completes the proof. ❑

Based on Proposition 8.4.1 and Proposition 8.4.2, we obtain a complete characterization of the structure of the frame operator and an estimate for the complexity for the computation of S^{-1}.

Theorem 8.4.3 (Fundamental Factorization of S) *The Gabor frame operator S can be unitarily factorized into the block diagonal matrix*

$$I_{\frac{M}{c}} \otimes diag(C_0, \ldots, C_{cd-1}), \tag{8.4.4}$$

where the $\frac{b}{d} \times \frac{b}{d}$ submatrices C_j are defined by

$$diag(C_{dk}, \ldots, C_{d(k+1)-1}) = (F_d \otimes I_{\frac{b}{d}})^* B_k (F_d \otimes I_{\frac{b}{d}}) \tag{8.4.5}$$

for $k = 0, \ldots, c-1$ and B_k is given by (8.3.4).
Thus S can be inverted in $\mathcal{O}(cb(\frac{b^2}{d^2} + b \log d))$ operations.

Proof: Since $B_k = W_k W_k^*$ it follows from (8.4.1) that B_k and B_{k+lc} coincide up to permutation. Further B_k is block circulant, since W_k is block circulant [Dav79]. By Proposition 8.4.2 we can unitarily factorize B_k into a block diagonal matrix with d blocks $C_{dk}, \ldots, C_{d(k+1)-1}$ of size $\frac{b}{d} \times \frac{b}{d}$ and $F_d \otimes I_{\frac{b}{d}}$ as factorization matrix. This proves the first part of the theorem.

The cd different matrices C_j can be computed in $\mathcal{O}(c\frac{b}{d}d \log d)$ operations, since the factorization of one single B_k requires $\mathcal{O}(\frac{b^2}{d} d \log d)$ operations. Inversion of the $\frac{b}{d} \times \frac{b}{d}$ (Hermitian) matrices C_j is of order $\mathcal{O}(\frac{b^3}{d^3})$, thus we end up with an overall computational effort of $\mathcal{O}(cb(\frac{b^2}{d^2} + b \log d))$ operations. ❑

Thus S is unitarily equivalent to a block-diagonal matrix with blocks of size $\frac{b}{\gcd(b,N)} \times \frac{b}{\gcd(b,N)}$ and the blocks have periodicity $\gcd(a, M)$. In the sense of the program described in Subsection 8.3.1 the factorization derived in Theorem 8.4.3 provides an optimal factorization of S.

Using Theorem 8.4.3 we can easily compute the number of different eigenvalues of the frame operator.

Corollary 8.4.4 *S has at most* $\gcd(ab, L)$ *different eigenvalues. Furthermore if g is real-valued, the maximal number of eigenvalues reduces to at most* $\frac{bc}{d}\lceil\frac{d+1}{2}\rceil$.

Proof: Theorem 8.3.1 states that S is unitarily equivalent to the block diagonal matrix D_S of (8.3.4) with blocks B_k, $k = 0, \ldots, M-1$. From Theorem 8.4.3 it follows that the maximum number of up to permutations different blocks B_k is at most $\gcd(a, M)$. B_k is of size $b \times b$ and since interchanging the rows and columns of B_k does not change its eigenvalues, S has at most $\gcd(ab, L)$ different eigenvalues, which proves the first statement.

If g is real-valued, it follows from (8.3.3) that S is real-valued, whence the B_k are real-valued. Let k be fixed. By Theorem 8.4.3 we know that B_k can be block diagonalized into $\operatorname{diag}(C_0, \ldots, C_{d-1}$ via $(F_d \otimes I_{\frac{b}{d}})$. It is well-known that the Fourier transform takes real sequences into conjugate even ones. Since B_k is real, it follows from the properties of block matrices, that the C_j are conjugate even, i.e. $C_j = \overline{C_{d-j}}$.

Thus $(F_d \otimes I_{\frac{b}{d}})^* B_k (F_d \otimes I_{\frac{b}{d}})$ has up to conjugation $\lceil\frac{d+1}{2}\rceil$ different blocks. In combination with the first statement of this theorem, the desired result follows. ❑

There are two special cases of Theorem 8.4.3, which deserve to be mentioned separately. The first case is $\gcd(a, M) = 1$, the second is $\gcd(a, M) = a$. The second choice leads to the case of integer oversampling and will be treated in the next section. The first case provides a simple but nice factorization of S.

Corollary 8.4.5 *Assume that* $\gcd(a, M) = 1$, *then the Gabor frame operator S is unitarily equivalent to $I_M \otimes B_0$.*

The factorization of the Gabor frame operator based on number theoretic conditions on the parameters L, a, b expose a strong (and not surprising) relationship to the mixed radix factorizations in FFT frameworks, based on the Chinese Remainder Theorem appearing in number theory [HS91]. A typical example is following result: Given the Fourier matrix F_L and assume that $pq = L$ and $\gcd(p, q) = 1$, then F_L coincides up to permutation with $F_p \otimes F_q$, see [vL92].

Remark: Applying Proposition 8.4.1 to the block diagonal matrix $D_H = \operatorname{diag}(V_0, \ldots, V_{a-1})$ defined in (8.3.7) yields that the number of (up to permutations) different blocks V_k is given by $\gcd(a, M)$. Hence the costs to compute the dual $^\circ\gamma$ do not depend whether we choose the frame operator factorization of Theorem 8.3.1 or the factorization based on the Wexler-Raz condition. In Subsection 8.6 we will make use of the results presented here in connection with the conjugate gradient method.

8.4.2 Critical sampling and integer oversampling

In this section we show that in the case of critical sampling and integer oversampling G and S respectively can be unitarily diagonalized via properly chosen block Fourier matrices. These diagonalizations of S and G have been discovered by different authors, mostly via the Zak transform. Thus we will also show the analogy of our approach with Zak transform based methods.

The choice $ab = L$ is called *critical sampling*, whereas choosing $abq = L$ for $q \in \mathbb{N}$ is referred to as *integer oversampling.*

As an immediate consequence of Theorem 8.4.1 we obtain following result for critical sampling.

Corollary 8.4.6 *Assume that $ab = L$, then G can be unitarily factorized into a diagonal matrix with $(I_a \otimes F_b)$ as factorization matrix.*

Proof: By Proposition 8.3.4 we know that G is similar to the block diagonal matrix D_G with blocks W_k. Proposition 8.4.2 yields that the W_k are circulant matrices for $ab = L$. Thus $F_b^* W_k F_b$ is a diagonal matrix, which implies that $(I_a \otimes F_b)^* D_G (I_a \otimes F_b)$ is a diagonal matrix. ❑

This result has been derived by different people, mostly via the Zak transform, cf. [AGT91, AGT92, ZZ93b, BG]. See 6 for a formulation in the context of locally compact abelian groups. We show the analogy of the Zak transform approach with our point of view.

Recall that for the parameter a the discrete finite Zak transform of g is a $(a \times N)$ matrix given by

$$(\mathcal{Z}_a g)(k,n) = \frac{1}{\sqrt{L}} \sum_{l=0}^{b-1} g(k - la) e^{2\pi i l a n / L}$$

for $k = 0, \ldots, a-1$, $n = 0, \ldots, N-1$. Observe that the n-th element of the first row of $W_k F_b$ is given by $\sum_{l=0}^{b-1} g(k - la) e^{-2\pi i n l / b}$ which coincides up to conjugation with $(\mathcal{Z}_a g)(k,n)$.

For $abq = L$ it is known that the dual window ${}^\circ\gamma$ satisfies the relation

$$(\mathcal{Z}_a {}^\circ\gamma)(k,n) = \frac{(\mathcal{Z}_a g)(k,m)}{a \sum_{l=0}^{q-1} |(\mathcal{Z}_a g)(k - la, n)|^2} \tag{8.4.6}$$

see [ZZ93b].

For critical sampling equation (8.4.6) reduces to

$$(\mathcal{Z}_a {}^\circ\gamma)(k,n) = \frac{1}{a (\mathcal{Z}_a g)^*(k,n)} \,. \tag{8.4.7}$$

Equation (8.4.7) just describes simultaneous inversion of the circulant matrices W_k by the Fourier matrix F_b. Thus $(\mathcal{Z}_a {}^\circ\gamma)(k,n)$ is equal to the n-th entry of the first row of $W_k^{-1} F_b$.

Similar results hold for integer oversampling. Note that the choice $abq = L$ implies $\gcd(a, M) = a$ and $\gcd(b, N) = b$. For $q > 1$ the matrix G and hence the blocks W_k are not invertible. We compute the left inverses of the blocks W_k by $W_k^+ = W_k^*(W_k W_k^*)^{-1}$. According to Theorem 8.3.1 S can be block-diagonalized with blocks $B_k = W_k W_k^*$. It follows from Theorem 8.4.3 that the $b \times b$ blocks B_k are circulant and therefore can be diagonalized by the Fourier matrix F_b. Summing up we have following corollary.

Corollary 8.4.7 *Assume that $abq = L$ with $q \in \mathbb{N}$. Then S is unitarily diagonalized by $(I_{aq} \otimes F_b)$.*

The analogy to the Zak transform based approach for computing ${}^\circ\gamma$ becomes obvious, when we compute $W_k^*(W_k W_k^*)^{-1}$. Due to the block circulant structure of W_k and the circulant structure of $W_k W_k^*$ the matrix W_k^+ can be calculated efficiently by Fourier transforms of small size.

The following lemma follows immediately from Proposition 8.4.2 and the properties of circulant matrices [Dav79].

Lemma 8.4.8 *In case of integer oversampling (with oversampling rate q) the columns of a block W_k can always be reordered such that*

$$W_k P_{q,bq} = \begin{bmatrix} C_0 & C_1 & \dots & C_{q-1} \end{bmatrix} \tag{8.4.8}$$

where the C_k are circulant $(b \times b)$ matrices.

Hence the entries of the circulant matrix $W_k W_k^*$ can be computed by Fourier transforms of length b. In particular for fixed k let c_j denote the first column of the circulant block C_j for $j = 0, \dots, q-1$ according to (8.4.8) and $\tilde{c_j}(m) = |(F_b c_j)(m)|^2$, then the first column of $W_k W_k^*$ is given by $\sum_{j=0}^{q-1}(F_q^* \tilde{c_j})$. Since $P_{q,bq}$ is unitary, $W_k P_{q,bq} P_{q,bq}^* W_k = W_k W_k^*$. The product $W^*(WW^*)^{-1}$ can be computed in a similar manner. These calculations are exactly the same arising in equation (8.4.6).

There is no doubt that the Zak transform has shed light on several theoretical aspects of Gabor analysis [Jan88, Dau90, AGT91, AGT92]. However from a numerical point of view classical and well–known tools like the FFT, permutations and circulant matrices, are sufficient for the derivation of fast algorithms to compute the dual window γ, especially since efficient numerical implementations of the Zak transform are essentially based on FFT. Moreover the Zak transform is not an appropriate tool to exploit *all* structural properties of S for general rational oversampling and non-separable time-frequency lattices.

8.5 Design of adaptive dual windows

One celebrated advantage of the oversampled Gabor transform is the gain of freedom in the design of the dual window. As already mentioned by Wexler and Raz [WR90] there are cases where alternative dual windows lead to better concentration than the minimal norm dual ${}^\circ\gamma$. Therefore we focus in this section on the computation of dual functions that optimize other norms than the standard ℓ_2-norm. In chapter 1 the reader can find a list of typical constraints for the dual arising in applications such as filter bank design. We refer also to [DLL95] for a discussion of dual Gabor windows satisfying specific properties.

The set of all duals can be easily characterized by the minimal norm dual ${}^\circ\gamma$ and the *null space* of H, which we denote by $\mathcal{N}(H)$, where H is the matrix defined in (8.3.4). It follows from standard linear algebra [Str80], that any dual window γ can now be expressed as $\gamma = {}^\circ\gamma + \nu$ where $\nu \in \mathcal{N}$.

Let us return to the question how to compute a dual window subject to certain regularization constraints. We are interested in finding the dual function γ_R for which $\|\gamma_R\|_R = \|R\gamma_R\|$ is minimal, where the Hermitian positive definite matrix R contains the information about regularization constraints.

Finding the dual window ${}^\circ\gamma_R$ which minimizes $\|\gamma\|_R$ is equivalent to finding the function $\tilde{\gamma}$ with smallest ℓ_2-norm that satisfies

$$HR^{-1}\tilde{\gamma} = \frac{ab}{L}[1, 0, \ldots, 0]^T \tag{8.5.1}$$

and then taking

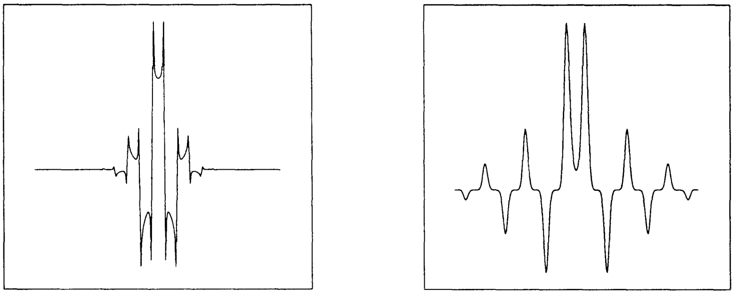
$${}^\circ\gamma_R = R^{-1}\tilde{\gamma} \tag{8.5.2}$$

If R is not invertible, then $\|.\|_R$ becomes a semi-norm. We can for instance want γ to be supported by an interval $[-s, s]$ by setting $R = \mathrm{diag}(r)$ where

$$r(k) = \begin{cases} 1 & \text{for } s \le k \le L - s \text{ where } s = \lfloor \frac{L-ab}{2} \rfloor \\ 0 & \text{else}. \end{cases}$$

Due to the assumption that the $g_{m,n}$ are a frame for $\mathbb{C}^L$, the dimension of $\mathcal{N}(H)$ is $L - ab$. Hence we can force up to $L - ab$ values of γ to be zero to compute the minimal support dual γ_s. In this case it follows from the properties of the pseudo inverse [Str80, Chapter 3] that we do not need to compute $(HR)^+$, but only have to invert the $ab \times ab$ matrix $\tilde{H}$ where $\tilde{H}$ is constructed from H by eliminating those columns of H which correspond to the zero-entries of r.

Note however that if we force γ to have minimal support of length ab then Balian–Low like effects will take place in the behavior of such a minimal support dual γ_s, a typical example is illustrated in Figure 1.



(a) minimal support dual γ_s (b) Fourier transform of γ_s

FIGURE 8.5.1. (a) Minimal support dual γ_s for a Gaussian window of length $L = 240$ and $a = 12, b = 12$, and (b) Fourier transform of γ_s.

Good time-frequency concentration of the dual can be achieved by introducing a weighted ℓ^2-norm. In this case R^{-1} becomes a diagonal matrix $R^{-1} = \operatorname{diag}(r)$, where r typically represents a function with fast decay, e.g. a Gaussian-like function.

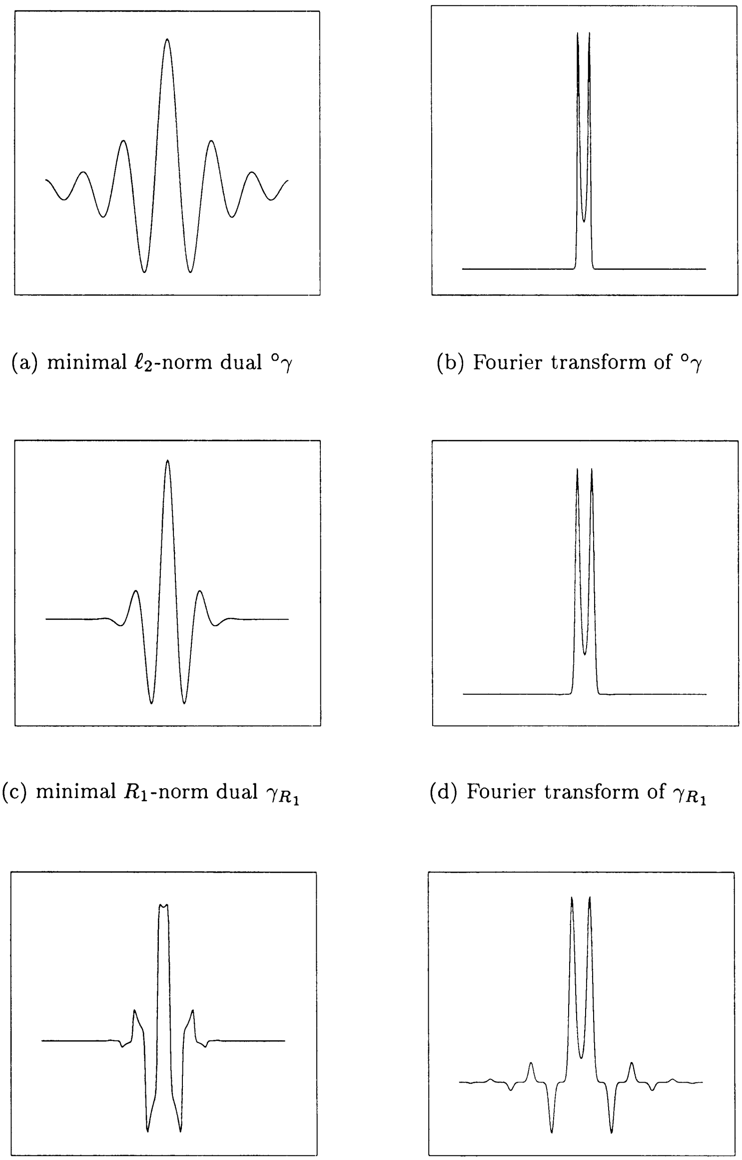
Figure 2 shows several dual windows for different weighted ℓ^2–norms $\|.\|_{R_k}$. In our experiment g is a Gaussian–like window of the form

$$g(n) = e^{-\pi \alpha n^2/L} \tag{8.5.3}$$

with $\alpha = 12, L = 240$ and $n = -(L/2-1), \dots, L/2$. The minimal ℓ^2-norm dual ${}^\circ\gamma$ is displayed in Figure 2(a). The other duals have been computed with respect to weighted ℓ^2-norms. More precisely we $R_k^{-1} = \operatorname{diag}(r_k), k = 1,2$ where r_k is given by equation (8.5.3) with $\alpha_1 = 4$ and $\alpha_2 = 8.6$. The corresponding duals $\gamma_{R_1}, \gamma_{R_2}$ and its Fourier transforms are shown in Figure 2(c)–(e). Although γ_{R_1} is slightly better concentrated than γ_{R_2} in the time domain, $\hat{\gamma}_{R_2}$ is much worse concentrated than $\hat{\gamma}_{R_2}$.

The fundamental law for the inverse of a product, $(BA)^{-1} = A^{-1}B^{-1}$, is not valid for pseudoinverses. In general we must expect that $(BA)^+ \neq A^+B^+$. Thus we cannot compute $(HR)^+$ by computing H^+ and R^+ separately. Unfortunately the product HR does not have the same structural properties as H and therefore the factorization of H as discussed in Corollary 8.3.7 does not hold any more.

The consequence is that the computational effort to compute adaptive duals may increase considerably compared to the computation of the minimal norm dual. This is one of the situations where the conjugate gradient method still allows an efficient computation of dual windows, see also Section 8.6.



(a) minimal ℓ_2-norm dual ${}^{\circ}\gamma$

(b) Fourier transform of ${}^{\circ}\gamma$

(c) minimal R_1-norm dual γ_{R_1}

(d) Fourier transform of γ_{R_1}

(e) minimal R_2-norm dual γ_{R_2}

(f) Fourier transform of γ_{R_2}

FIGURE 8.5.2. Different dual windows for a Gaussian window of length $L = 240$ and $a = 12, b = 12$. The dual shown in (a) has minimal ℓ^2–norm, the duals (c) γ_{R_1} and (e) γ_{R_2} have been computed via equations (8.5.1)– (8.5.2) with respect to weighted ℓ^2–norms. Obviously, γ_{R_1} shows the best time-frequency localization.

8.6 Conjugate Gradient Methods for Gabor Expansions

In this section we discuss an efficient iterative approach for computing ${}^\circ\gamma$, which heavily relies on unitary factorizations of S and H respectively. We have seen in Section 8.4 that S can be factorized into a block diagonal matrix where the block size depends on number theoretic relations of L, a, b. If the blocks are of large size, iterative methods for the computation of ${}^\circ\gamma$ may be preferable. Another reason why we consider iterative methods is that in applications it is often sufficient to compute only an approximate solution to the dual window. Iterative methods for Gabor expansions have also been discussed in [GP92, BB92, BB93, Fei94, FCS95, QF95].

The conjugate gradient method is a particularly efficient method for solving Hermitian positive definite linear systems of equations. S is obviously Hermitian, and it is positive definite whenever the rank of the row space of G equals L. For a detailed introduction to the conjugate gradient method and its variants the reader is referred to [GL89, Bjö96].

Theorem 8.6.1 (Theorem and Algorithm) *Given a function $g \in C^L$, the time- and frequency shift parameters a, b such that $ab \leq L$ and the corresponding frame operator S. Initialize $r_0 = 0, c_0 = 0, n = 0$ and compute γ_n iteratively:*

$$
\begin{array}{l}
\textbf{while } r_n \neq 0 \\
\quad n = n + 1 \\
\quad \textbf{if } n = 1 \\
\qquad q_1 = r_0 \\
\quad \textbf{else} \\
\qquad \beta_n = r_{n-1}^* r_{n-1} / r_{n-2}^* r_{n-2} \\
\qquad q_n = r_{n-1} + \beta_n q_{n-1} \\
\quad \textbf{end} \\
\quad \alpha_n = r_{n-1}^* r_{n-1} / q_n^* S q_n \\
\quad \gamma_n = \gamma_{n-1} + \alpha_n q_n \\
\quad r_n = r_{n-1} - \alpha_n S q_n \\
\textbf{end}
\end{array}
$$

Then, assuming exact arithmetics, γ_n converges in at most $\gcd(ab, L)$ *steps to the minimal norm dual ${}^\circ\gamma$. If in addition the window g is real and symmetric, CG terminates after at most $c\frac{b}{d}\lceil\frac{d+1}{2}\rceil$ iterations.*

Proof: The number of iterations needed to terminate for conjugate gradients is given by the number of different eigenvalues of S, see e.g. [Bjö96].

Corollary 8.4.4 says that the number of different eigenvalues of S is at most $\gcd(ab, L)$ and for real-valued windows at most $c\frac{b}{d}\lceil\frac{d+1}{2}\rceil$. ❑

In each iteration, besides the inner products, a matrix vector multiplication Sq_n is required. It is this product, where the various unitary factorizations of S come into play. For clarity of presentation we restrict ourselves in the sequel to the factorization proposed in Theorem 8.3.1. Since $P_{M,L}$ is a unitary matrix, solving $S{}^\circ\gamma = g$ is equivalent to solving $D_S\tilde{\gamma} = \tilde{g}$ where D_S is the block diagonal matrix of (8.3.5), $\tilde{\gamma} = P^*_{M,L}{}^\circ\gamma$ and $\tilde{g} = P^*_{M,L}g$. Since the M blocks of D_S are of size $b \times b$, the product $D_S q_n$ can be computed in $\mathcal{O}(bL)$ operations. By Theorem 8.6.1 the maximal number of iterations is given by $\gcd(ab, L)$ the overall computational effort for CG is $\mathcal{O}(\gcd(ab, L)bL)$ operations. If the eigenvalues of D_S are additionally clustered (besides the clustering discussed in the proof above), CG will converge much faster.

Remarks:

- Note that we could also apply the conjugate gradient method to the system $H^*\gamma = \frac{ab}{L}[1, 0, \dots, 0]^T$ defined in (8.3.6), see [Bjö96]. However, due to the results in Section 8.4, the computational effort for calculating ${}^\circ\gamma$ is the same as for Algorithm 8.6.1.
- For convergence properties of CG in presence of round off errors we refer to [Bjö96].
- For $ab \ll L$ a good initial guess for the starting value γ_0 is $\gamma_0 = \frac{ab}{L}g$ where $||g|| = 1$, since $\frac{L}{ab}{}^\circ\gamma$ converges to g for $ab \to 1$. For the continuous case it is proven in Chapter 3 that the dual window ${}^\circ\gamma$ converges to g if $(a, b) \to (0, 0)$.
- CG has turned out to be an efficient method in conjunction with irregular Gabor expansions, cf.[FKS95]. For more details on irregular Gabor expansions we refer to [FG92a, FG92b, Grö93b, Lan93, RS95a, BHW95] and Chapter 2 in this book.

8.6.1 Block Conjugate gradient algorithm

Since $D_S = \mathrm{diag}(B_0, \dots, B_{M-1})$, the solution of $D_S\tilde{\gamma} = \tilde{g}$ can be computed by solving each subsystem

$$B_k\tilde{\gamma}_k = \tilde{g}_k \tag{8.6.1}$$

separately. Here $\tilde{g}_k(n) = \tilde{g}(bk + n)$ for $n = 0, \dots, b-1$, $k = 0, \dots, M-1$. Note that the B_k are Hermitian, since S and hence D_S is Hermitian. From Theorem 8.4.3 we know that B_k coincides with B_{k+c} up to permutations, where $c\gcd(a, M)$. That means we can rearrange each subsystem (8.6.1) by permutations, such that the matrix of the k-th system is identical to that of the $k+c$-th system. Thus if $\gcd(a, M) = 1$, we have to invert only one matrix of size $b \times b$, followed by M matrix-vector multiplications to obtain ${}^{\circ}\gamma$. However if we solve the system $D_S\tilde{\gamma} = \tilde{g}$ iteratively by CG, we cannot restrict ourselves to the solution of a single system $B_k\tilde{\gamma}_k = \tilde{g}_k$, since $\tilde{g}_k \neq \tilde{g}_j$ for $k \neq j$ and therefore $\tilde{\gamma}_k \neq \tilde{\gamma}_j$. Thus we have to solve linear systems with multiple right-hand sides but with the same system matrix.

Unfortunately CG does not return the inverse B_k^{-1}, thus we would have to apply CG to each subsystem (8.6.1) separately. Hence the question arises if the CG method can be modified such that we can take profit from the fact that system matrix does not change in (8.6.1). This leads immediately to the so-called *block conjugate gradient method* (BCG) [O'L80].

The block CG method solves the matrix equation

$$AX = Y \tag{8.6.2}$$

in at most $\lceil n/p \rceil$ iterations and may involve less work than applying the conjugate gradient method p times [O'L80]. For our problem, the matrix A in (8.6.2) is given by B_k, X and Y contain permuted versions of the vectors $\tilde{\gamma}_k$ and $\tilde{g}_k$, respectively. More details on the block conjugate gradient method can be found in [O'L80, SPM89, CWar, NY95].

We remark that efficient parallel implementations of direct or iterative algorithms for computing ${}^{\circ}\gamma$ can be easily derived, based on the results presented in this Chapter.

8.7 Preconditioners and Approximate Inverses

The convergence rate of the conjugate gradient method depends on the condition number of the matrix and how clustered the spectrum of the matrix is. If the spectrum of S is not clustered or if the condition number $\kappa(S)$ is large the convergence will be slow. One way to speed up the convergence rate of the method is to precondition the system. Thus instead of solving $S\gamma = g$ we solve the preconditioned system $C^{-1}S\gamma = C^{-1}g$. The preconditioning matrix C should satisfy following criteria

- $\kappa(C^{-1}S) \ll \kappa(S)$.
- The spectrum of $C^{-1}S$ should be clustered.
- C should be constructed within $\mathcal{O}(L \log L)$ operations.
- $Cx = y$ should be solved in $\mathcal{O}(L \log L)$ operations.

Circulant matrices play an important role as preconditioners, see [CN96] and the references cited there for a detailed discussion of preconditioners. Among the circulant preconditioners, T. Chan's optimal Frobenius norm preconditioner, originally proposed for Toeplitz systems [Cha89], is particularly useful. For a Toeplitz matrix S it is defined to be the minimizer of

$$||C - S||_F$$

over all $L \times L$ circulant matrices C. Here $||\,.\,||_F$ denotes the Frobenius norm of a matrix.

Whether or not S is a Toeplitz matrix, the k-th diagonal of $C(S)$ is easily obtained by the arithmetic average of the k-th diagonal of S, where the diagonals of S are wrapped around. More precisely the entries of the k-th diagonal of $C(S)$ is given by

$$c_k = \frac{1}{L} \sum_{m-n=k \bmod L} S_{m,n} \qquad \text{for } k = 0, \dots, L-1.$$

T. Chan's preconditioner is particularly useful in the context of Gabor analysis, due to following observations.

If the window g has small support compared to the size of the time-frequency lattice parameters, then the frame operator S is just a multiplication operator, which is to say a diagonal matrix. This fact was first observed by Walnut, see e.g.[BW94], its discrete analogue can be found in [QFS94].

Proposition 8.7.1 *Given a window g, and lattice parameters a and b. If the length s_g of the support of g satisfies $s_g \le M$, then the frame operator S is a diagonal matrix. If the length $s_{\hat{g}}$ of the support of $\hat{g}$ satisfies $s_{\hat{g}} \le N$ then S is a circulant matrix.*

Proof: We assume without loss of generality that the support of g is located at the first consecutive integers $\{0, 1, \dots, s_g - 1\}$, i.e. $g(k) = 0$ for $s_g \le k \le L-1$. The support condition $s \le M$ implies

$$\sum_{k=0}^{N-1} T_{ka}g(m)\overline{T_{ka}g(n)} = 0, \qquad \text{for } |m-n| \ge M \ge s_g - 1\,,$$

whence $S_{m,n} = 0$ for $|m-n| \ge M$ by (8.3.3), thus S is a diagonal matrix. The second assertion follows from Lemma 8.3.6. ❑

Evidently, if $\hat{g}$ satisfies the "small support" condition, $C(S) = S$. An immediate consequence of Lemma 8.3.1 and Lemma 8.3.6 is

$$C(\hat{S}) = F\,\mathrm{diag}(S) F^*\,.$$

In general g will not satisfy one of the conditions of Proposition 8.7.1 in a strict sense. If we assume that the Fourier transform of g is a rapidly decaying function, such that $\hat{g}(k)$ is small outside the interval $[-\frac{N}{2}, \frac{N}{2}]$, then as a consequence of equation (8.3.3) and Lemma 8.3.6, the off-diagonal entries of $S_{\hat{g}}$ will also be small and $S_{\hat{g}}$ will be strictly diagonal dominant. Since $S = F_L \hat{S} F_L^*$, it follows that the eigenvalues of $C^{-1}(S)S$ will cluster around 1, if the length of the support of $\hat{g}$ approaches N. Thus for a window having most of the energy of its Fourier transform concentrated within the interval $[-\frac{N}{2}, \frac{N}{2}]$, $C^{-1}(S)S$ will serve as an approximate inverse of S. By duality the same arguments hold for a narrow window g and in this case the eigenvalues of $C^{-1}(\mathrm{diag}(S))$ will cluster around 1, if the length of the support of g approaches M.

Note that $C(S)$ can be constructed and inverted in $\mathcal{O}(b \log b)$ operations, thus using the preconditioned conjugate gradients method with $C(S)$ as preconditioner only slightly increases the computational effort per iteration.

Figure 3 illustrates the efficiency of applying the preconditioners $C(S)$ and $C(\mathrm{diag}(S))$. We compute the dual $^\circ\gamma$ by the preconditioned conjugate gradient method (PCG) for two different situations. In the first experiment we have used a Gaussian window which is well localized in the frequency domain ($L = 240, \alpha = 2$ according to equation (8.5.3), lattice parameters are $a = b = 15$). Having in mind the observations discussed above, we can expect that applying the preconditioner $C(S)$ will considerably improve the clustering of the eigenvalues of S. The convergence rates of CG and PCG are displayed in Figure 3(a).

In the second experiment we have used a narrow Gaussian window ($L = 240, \alpha = 8$, lattice parameters are $a = 30, b = 6$). $C(S)$ still improves the clustering of the singular values of S, but evidently $C(\mathrm{diag}(S))$ has the edge in this situation. The corresponding convergence rates are shown in Figure 3(b).

One might argue that for the case where g and $\hat{g}$ are approximately equally well concentrated, a "double preconditioning" strategy might useful. That means we first compute $A = C^{-1}(S)S$ followed by applying $C(\mathrm{diag}(A))$. Note that $C^{-1}(S)S$ is no longer Hermitian and does not induce an inner product, hence CG cannot be applied directly. If we still want to use CG, we can switch to the system of normal equations [GL89]. Since this approach is handicapped by squaring of the condition number, it will considerably reduce the positive effect of the additional preconditioner. However double preconditioning may still be useful to obtain an approximate solution to $^\circ\gamma$ or in connection with the Neumann series.

Acknowledgments: I want to thank Hans G. Feichtinger and Peter Prinz for stimulating discussions on this topic.

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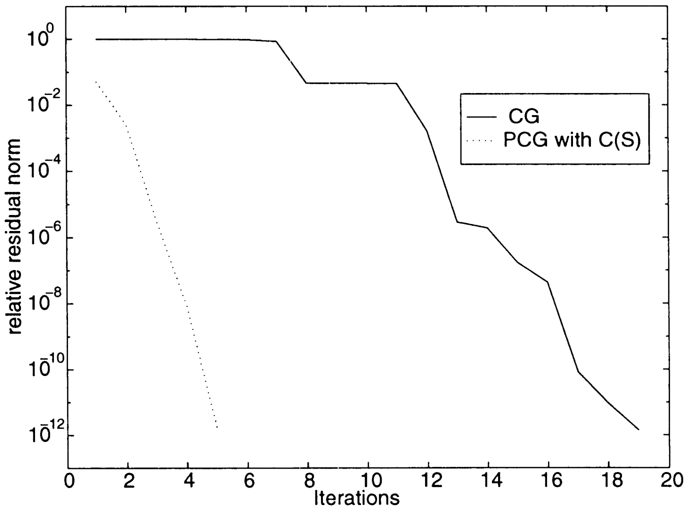
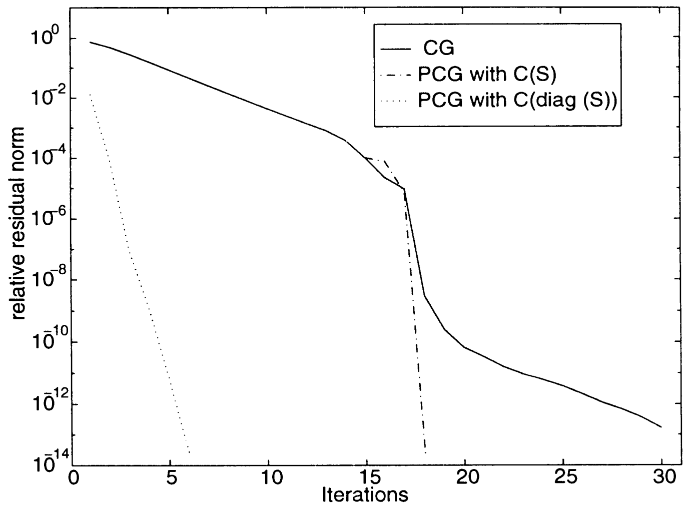


(a)



(b)

FIGURE 8.7.1. Convergence rate of PCG for two different Gabor setups. In the first experiment we have used $\alpha = 2, a = 15, b = 15, L = 240$, in the second experiment we have used $\alpha = 8, a = 30, b = 6, L = 240$.

9

Oversampled modulated filter banks

Helmut Bölcskei and Franz Hlawatsch

ABSTRACT – Oversampled filter banks (FBs) offer increased design freedom and noise immunity as compared to critically sampled FBs. Since these advantages come at the cost of greater computational complexity, oversampled FBs allowing an efficient implementation are of particular interest. In this chapter, we discuss oversampled DFT FBs and oversampled cosine modulated FBs (CMFBs) which allow efficient FFT- or DCT/DST-based implementations. We provide conditions for perfect reconstruction and a frame-theoretic analysis. We show that, concerning both perfect reconstruction properties and frame-theoretic properties, oversampled cosine modulated filter banks are closely related to DFT filter banks with twice the oversampling factor.

9.1 Introduction and outline

Oversampled filter banks (FBs) have recently been found to be attractive due to their increased design freedom and improved noise immunity as compared to critically sampled FBs [BHF96b, CV, BHF96a, BHF96c, Jan95a, Vel93]. The *increased design freedom* corresponds to the nonuniqueness of the synthesis FB satisfying perfect reconstruction (PR) for a given oversampled analysis FB [BHF96b, BH97b]. The *improved noise immunity* corresponds to the fact that oversampled FBs tend to have better frame bounds [BHF96b, BH97b]. Furthermore, oversampled FBs permit the application of *noise shaping techniques* by which considerable noise reduction can be achieved [BH97b]. This makes oversampled FBs interesting for source coding applications with low-resolution quantizers in the subbands. The benefits obtained from using low-resolution quantizers at the cost of increased sample rate are indicated by the popular sigma-delta techniques [Gra87].

These advantages of oversampling come at the cost of increased computational complexity caused by the need to process more subband signal samples per unit of time. Therefore, oversampled FBs allowing efficient implementations are of particular interest. Oversampled *modulated* FBs such as DFT FBs (also known as complex modulated FBs) [CR83, BHF96c,

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Cve95b, BHF96b, CV] and cosine modulated FBs (CMFBs)[BH96c, BH97a] allow efficient FFT- or DCT/DST-based implementations [CR83, BH96b]. Here, CMFBs are advantageous as their subband signals are real-valued if the input signal and the analysis prototype are real-valued.

In this chapter, we discuss oversampled DFT FBs and CMFBs using results from the theory of frames [DS52, HW89, Dau92]. The application of frame theory is based on the close relations (or even equivalences) between modulated FBs on the one hand and Gabor expansions (Weyl–Heisenberg frames) [Bas80b, Jan81, WR90, Jan95b, DLL95] and Wilson expansions [DJJ91, FGW92, Aus94, BFGH96, BFGH97] on the other hand.

This chapter is organized as follows. In Section 2, we discuss oversampled uniform FBs and their relation to frame theory, thereby establishing a basis for our study of modulated FBs in subsequent sections. For a given oversampled analysis FB, we parameterize all synthesis FBs providing PR. We discuss a relation between the FB polyphase matrices and the frame operator, and we find conditions for a FB to provide a frame expansion.

Section 3 considers oversampled DFT FBs and their relation to Weyl-Heisenberg frames and Gabor expansions. We discuss PR conditions and frame-theoretic properties, and we show that the theory of DFT FBs simplifies considerably in the case of integer oversampling.

In Section 4, extending a recent classification of critically sampled CMFBs [Gop96], we consider two classes of oversampled CMFBs. In particular, the class of even-stacked CMFBs allows both PR/paraunitarity and linear phase filters in all channels. We finally show that CMFBs are closely related to PR DFT FBs with twice the oversampling factor.

9.2 Oversampled filter banks and frames

In this section we discuss (uniform) FBs in general, thereby establishing a theoretical basis for our study of modulated FBs in Sections 3 and 4. We extend the polyphase approach proposed in [Vet87, Vai87, Vai93, VK95] for critically sampled (maximally decimated) FBs to the oversampled case [BHF96a, BHF96b, CV, BHF96c, Bön]. We furthermore introduce *uniform filter bank frames* and establish their relation to FBs. Our discussion emphasizes PR and frame-theoretic properties of uniform FBs.

9.2.1 Uniform filter banks

We consider a FB with N channels (or subbands) and subsampling by the integer factor M in each channel, as depicted in Fig. 9.2.1. The FB is as-

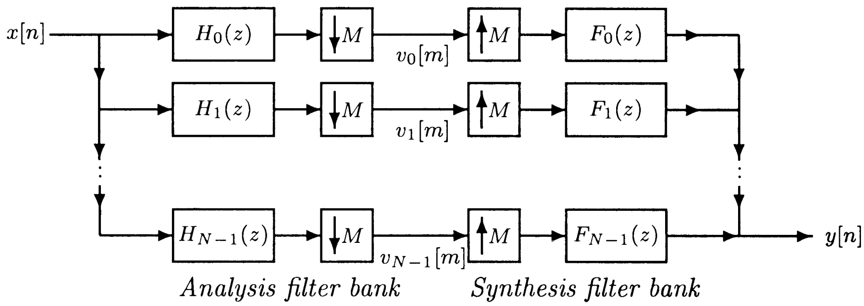

FIGURE 9.2.1. N-channel uniform FB.

sumed to have PR with zero delay[1], so that $y[n] = x[n]$ where $x[n]$ and $y[n]$ denote the input and reconstructed signal[2], respectively. The impulse responses of the analysis and synthesis filters are respectively $h_k[n]$ and $f_k[n]$ ($k = 0, 1, ..., N-1$), with corresponding transfer functions (z-transforms[3]) $H_k(z)$ and $F_k(z)$. The subband signals are given by

$$v_k[m] = \sum_{n=-\infty}^{\infty} x[n]\, h_k[mM - n], \qquad k = 0, 1, ..., N-1, \tag{9.2.1}$$

and the reconstructed signal is

$$y[n] = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} v_k[m]\, f_k[n - mM]. \tag{9.2.2}$$

In a *critically sampled* (or *maximally decimated*) FB we have $N = M$ and thus the subband signals $v_k[m]$ contain exactly as many samples (per unit of time) as the input signal $x[n]$. In the *oversampled* case $N > M$, the subband signals are redundant in that they contain more samples (per unit of time) than the input signal $x[n]$. Finally, the *undersampled* case $N < M$ excludes PR.

The *polyphase decomposition* of the analysis filters $H_k(z)$ reads

$$H_k(z) = \sum_{n=0}^{M-1} z^n\, E_{k,n}(z^M), \qquad k = 0, 1, ..., N-1,$$

---

[1] We note that our theory can easily be extended to nonzero delay.

[2] Signals are usually assumed to be in $l^2(\mathbb{Z})$, the space of square-summable functions $x[n]$, i.e., $\|x\|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$, with inner product $\langle x, y\rangle = \sum_{n=-\infty}^{\infty} x[n]\, y^*[n]$ where * stands for complex conjugation.

[3] For example, $H_k(z) = \sum_{n=-\infty}^{\infty} h_k[n]\, z^{-n}$.

where

$$E_{k,n}(z) = \sum_{m=-\infty}^{\infty} h_k[mM-n]\, z^{-m}, \qquad k=0,1,...,N-1, \;\; n=0,1,...,M-1$$

is the nth polyphase component of the kth analysis filter $H_k(z)$. The $N \times M$ *analysis polyphase matrix* $\mathbf{E}(z)$ is defined as $[\mathbf{E}(z)]_{k,n} = E_{k,n}(z)$. The synthesis filters $F_k(z)$ can be similarly decomposed,

$$F_k(z) = \sum_{n=0}^{M-1} z^{-n} R_{k,n}(z^M)\,, \qquad k=0,1,...,N-1\,,$$

with the synthesis polyphase components

$$R_{k,n}(z) = \sum_{m=-\infty}^{\infty} f_k[mM+n]\, z^{-m}, \qquad k=0,1,...,N-1, \;\; n=0,1,...,M-1\,.$$

The $M \times N$ *synthesis polyphase matrix* $\mathbf{R}(z)$ is defined as $[\mathbf{R}(z)]_{n,k} = R_{k,n}(z)$.

9.2.2 Uniform filter bank frames

FB analysis and synthesis can be interpreted as a signal expansion [VK95, CV94, Vai93, BHF95]. The subband signals in (9.2.1) can be written as the inner products

$$v_k[m] = \langle x, h_{k,m} \rangle \qquad \text{with} \;\; h_{k,m}[n] = h_k^*[mM-n]\,, \qquad k=0,1,...,N-1\,.$$

Furthermore, with (9.2.2) and the PR property $y[n] = x[n]$, we have

$$x[n] = y[n] = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m} \rangle\, f_{k,m}[n] \qquad \text{with} \;\; f_{k,m}[n] = f_k[n-mM]\,.$$

This shows that the FB corresponds to an expansion of the input signal $x[n]$ into the function set $\{f_{k,m}[n]\}$ with $k=0,1,...,N-1$ and $-\infty < m < \infty$. In general the set $\{f_{k,m}[n]\}$ is not orthogonal, so that the expansion coefficients, i.e., the subband signals $v_k[m] = \langle x, h_{k,m} \rangle$, are obtained by projecting the signal $x[n]$ onto a "dual" set of functions $\{h_{k,m}[n]\}$. Critically sampled FBs correspond to orthogonal or biorthogonal signal expansions [VH92], whereas oversampled FBs correspond to redundant (overcomplete) expansions [VK95, BHF96b, CV, BHF96c, BHF96a].

The *theory of frames* [DS52, HW89, Dau92] is a powerful vehicle for the study of redundant signal expansions. We will call the set $\{h_{k,m}[n]\}$ with

$h_{k,m}[n] = h_k^*[mM - n]$ a *uniform filter bank frame* (UFBF) for $l^2(\mathbb{Z})$ if

$$A\|x\|^2 \le \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} |\langle x, h_{k,m}\rangle|^2 \le B\|x\|^2 \qquad \forall\ x[n] \in l^2(\mathbb{Z}) \quad (9.2.3)$$

with the *frame bounds* $A > 0$ and $B < \infty$. Note that the UFBF functions $h_{k,m}[n]$ are generated by uniformly time-shifting the N (conjugated and time-reversed) analysis filter impulse responses $h_k^*[-n]$. The above frame condition (9.2.3) can also be written as $A\|x\|^2 \le \langle \boldsymbol{S}x, x\rangle \le B\|x\|^2$, where $\boldsymbol{S}$ is the *frame operator* defined as

$$(\boldsymbol{S}x)[n] = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m}\rangle\, h_{k,m}[n]\,. \qquad (9.2.4)$$

If (9.2.3) is satisfied, then the frame operator is a positive definite, linear operator that maps $l^2(\mathbb{Z})$ onto $l^2(\mathbb{Z})$. The frame bounds A and B are the infimum and supremum, respectively, of the eigenvalues of $\boldsymbol{S}$ [Dau92]; they determine important numerical properties of the FB (see Section 9.2.6).

For analysis filters $h_k[n]$ such that $\{h_{k,m}[n]\}$ is a UFBF for $l^2(\mathbb{Z})$, a particular PR synthesis set (the PR synthesis set with minimum norm) is given by [HW89, Dau92]

$$f_{k,m}[n] = (\boldsymbol{S}^{-1}h_{k,m})[n]\,,$$

where $\boldsymbol{S}^{-1}$ is the inverse frame operator. If the analysis set $\{h_{k,m}[n]\}$ is a frame, then the synthesis set $f_{k,m}[n] = (\boldsymbol{S}^{-1}h_{k,m})[n]$ is also a frame (the "dual" frame), with frame bounds $A' = 1/B$ and $B' = 1/A$ (note that $B'/A' = B/A$) and frame operator $\boldsymbol{S}^{-1}$. This dual frame can be shown to be *again a UFBF* [BHF96a, BHF96b], i.e., it is obtained by uniformly time-shifting N functions $f_k[n]$,

$$f_{k,m}[n] = f_k[n - mM]\,.$$

The $f_k[n]$ are the synthesis filter impulse responses; they are derived from the analysis filter impulse responses $h_k[n]$ according to[4]

$$f_k[n] = (\boldsymbol{S}^{-1}\tilde{h}_k)[n] \qquad \text{with } \tilde{h}_k[n] = h_k^*[-n]\,. \qquad (9.2.5)$$

A frame is called *snug* if $B'/A' = B/A \approx 1$ and *tight* if $B'/A' = B/A = 1$. For a tight frame we have $\boldsymbol{S} = A\,\boldsymbol{I}$ and $\boldsymbol{S}^{-1} = A'\,\boldsymbol{I}$ (where $\boldsymbol{I}$ is the identity operator on $l^2(\mathbb{Z})$), and hence there is simply $f_k[n] = A'\,h_k^*[-n]$.

[4]There exists a basic difference between FB theory and frame theory: In FB theory one usually specifies the analysis FB $\{h_k[n]\}$ and computes the corresponding synthesis FB $\{f_k[n]\}$ for PR. In frame theory, however, the synthesis set $\{f_{k,m}[n]\}$ is often specified and then the corresponding analysis set $\{h_{k,m}[n]\}$ is calculated. Since this chapter is dealing with FBs, we shall here adopt the FB approach, i.e., assume knowledge of the analysis filters $h_k[n]$ and calculate the synthesis filters $f_k[n]$.

9.2.3 Frame operator and polyphase matrices

The connection between FBs and UFBFs is further expressed by the following representation of the UFBF frame operator in terms of the polyphase matrices of the corresponding FB. This result extends a similar result on continuous-time Weyl-Heisenberg frames [ZZ93b, ZZ95a].

Theorem 9.2.1 *[BHF96a, BHF96b] Consider $u[n] = (\boldsymbol{S}x)[n]$ and $x[n] = (\boldsymbol{S}^{-1}u)[n]$, where $\boldsymbol{S}$ is the frame operator corresponding to a UFBF. Then, the polyphase components $U_n(z) = \sum_{m=-\infty}^{\infty} u[mM+n]\, z^{-m}$ of $U(z)$ and the polyphase components $X_{n'}(z) = \sum_{m=-\infty}^{\infty} x[mM+n']\, z^{-m}$ of $X(z)$ are related as*[5]

$$
\begin{aligned}
U_n(z) &= \sum_{n'=0}^{M-1} S_{n,n'}(z)\, X_{n'}(z) \qquad \textit{with} \quad S_{n,n'}(z) = \sum_{k=0}^{N-1} \tilde{E}_{k,n}(z)\, E_{k,n'}(z) \\
X_{n'}(z) &= \sum_{n=0}^{M-1} S^{-1}_{n',n}(z)\, U_n(z) \qquad \textit{with} \quad S^{-1}_{n',n}(z) = \sum_{k=0}^{N-1} R_{k,n'}(z)\, \tilde{R}_{k,n}(z)\,,
\end{aligned}
$$

or equivalently, the vectors $\mathbf{x}(z) = [X_0(z)\ X_1(z)\ \ldots\ X_{M-1}(z)]^T$ and $\mathbf{u}(z) = [U_0(z)\ U_1(z)\ \ldots\ U_{M-1}(z)]^T$ are related as[6]

$$
\begin{aligned}
\mathbf{u}(z) &= \mathbf{S}(z)\,\mathbf{x}(z) \qquad \textit{with} \quad \mathbf{S}(z) = \tilde{\mathbf{E}}(z)\,\mathbf{E}(z)\,, \\
\mathbf{x}(z) &= \mathbf{S}^{-1}(z)\,\mathbf{u}(z) \qquad \textit{with} \quad \mathbf{S}^{-1}(z) = \mathbf{R}(z)\,\tilde{\mathbf{R}}(z)\,.
\end{aligned}
$$

Thus, the frame operator $\boldsymbol{S}$ is expressed in the polyphase domain by the *$M \times M$ UFBF matrix* $\mathbf{S}(z) = \tilde{\mathbf{E}}(z)\,\mathbf{E}(z)$ defined in terms of the analysis polyphase matrix $\mathbf{E}(z)$. Similarly, the inverse frame operator $\boldsymbol{S}^{-1}$ is expressed by the *$M \times M$ inverse UFBF matrix* $\mathbf{S}^{-1}(z) = \mathbf{R}(z)\,\tilde{\mathbf{R}}(z)$ defined in terms of the synthesis polyphase matrix $\mathbf{R}(z)$.

Specializing to the unit circle ($z = e^{j2\pi\theta}$), it can be shown [BHF96a, BHF96b] that the positive definite $M \times M$ matrices

$$
\mathbf{S}(e^{j2\pi\theta}) = \mathbf{E}^H(e^{j2\pi\theta})\,\mathbf{E}(e^{j2\pi\theta}) \quad \text{and} \quad \mathbf{S}^{-1}(e^{j2\pi\theta}) = \mathbf{R}(e^{j2\pi\theta})\,\mathbf{R}^H(e^{j2\pi\theta})
$$

are *matrix representations* [NS82] of the frame operator $\boldsymbol{S}$ and the inverse frame operator $\boldsymbol{S}^{-1}$, respectively, with respect to the basis $\{b_{n,\theta}[n']\}$ of $l^2(\mathbb{Z})$ given by[7] $b_{n,\theta}[n'] = \sum_{m=-\infty}^{\infty} \delta[n'-n-mM]\, e^{j2\pi\frac{\theta}{M}(n'-n)}$ ($n =$

---

[5] $\tilde{R}_{k,n'}(z) = R^*_{k,n'}(1/z^*)$ denotes the paraconjugate of $R_{k,n'}(z)$ [Vai93].

[6] $\tilde{\mathbf{R}}(z) = \mathbf{R}^H(1/z^*)$ denotes the paraconjugate of the matrix $\mathbf{R}(z)$ [Vai93].

[7] Here, $\delta[n]$ denotes the unit sample ($\delta[0] = 1$ and $\delta[n] = 0$ for $n \neq 0$). The basis $\{b_{n,\theta}[n']\}$ induces the polyphase representation on the unit circle, $\langle x, b_{n,\theta}\rangle = X_n(e^{j2\pi\theta}) = \sum_{m=-\infty}^{\infty} x[mM+n]\, e^{-j2\pi\theta m}$. Equivalently, this is the *Zak transform* of $x[n]$ [Jan88, BHar]. Note that the functions $b_{n,\theta}[n']$ are not in $l^2(\mathbb{Z})$.

$0, 1, ..., M-1,\ 0 \le \theta < 1$). A similar approach based on a matrix representation of the frame operator has been proposed in [RS95b] for the study of shift-invariant function systems. Furthermore, in Chapter 1 of this book Janssen presents an analysis of shift-invariant function systems based on different representations of the FB analysis and synthesis operators.

It is known [NS82] that the eigenvalues of an operator and those of its matrix representation are identical. Let $\lambda_n(\theta) > 0$ with $n = 0, 1, ..., M-1$ denote the eigenvalues of the UFBF matrix $\mathbf{S}(e^{j2\pi\theta}) = \mathbf{E}^H(e^{j2\pi\theta})\,\mathbf{E}(e^{j2\pi\theta})$, defined by the eigenequation $\mathbf{S}(e^{j2\pi\theta})\,\mathbf{e}_n(\theta) = \lambda_n(\theta)\,\mathbf{e}_n(\theta)$ ($n = 0, 1, ..., M-1,\ 0 \le \theta < 1$). Then, any eigenvalue $\lambda_n(\theta)$ is simultaneously an eigenvalue of $\boldsymbol{S}$. Conversely, any eigenvalue of $\boldsymbol{S}$ is simultaneously an eigenvalue of $\mathbf{S}(e^{j2\pi\theta})$. This means that the eigenanalysis of the frame operator $\boldsymbol{S}$ (a matrix of infinite size) is equivalent to that of the UFBF matrix $\mathbf{S}(e^{j2\pi\theta})$ (an $M \times M$ matrix indexed by a real-valued parameter $\theta \in [0, 1)$). Similarly, the eigenvalues of the inverse frame operator $\boldsymbol{S}^{-1}$ are equal to those of the inverse UFBF matrix $\mathbf{S}^{-1}(e^{j2\pi\theta}) = \mathbf{R}(e^{j2\pi\theta})\,\mathbf{R}^H(e^{j2\pi\theta})$, which will be denoted $\lambda'_n(\theta)$ in the following. Since $\mathbf{S}(e^{j2\pi\theta})$ and $\mathbf{S}^{-1}(e^{j2\pi\theta})$ are positive definite matrices, their eigenvalues are positive.

9.2.4 Perfect reconstruction property and design freedom

We shall now derive a PR condition for oversampled FBs and present a parameterization of all synthesis FBs providing PR for a given oversampled analysis FB. Transforming the FB input-output relation $y[n] = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m} \rangle\, f_{k,m}[n]$ into the polyphase domain yields $\mathbf{y}(z) = \mathbf{R}(z)\,\mathbf{E}(z)\,\mathbf{x}(z)$. This gives the following result.

Theorem 9.2.2 *[BHF96a, BHF96b] A FB satisfies the PR property $y[n] = x[n]$ if and only if*

$$\mathbf{R}(z)\,\mathbf{E}(z) = \mathbf{I}_M, \tag{9.2.6}$$

where $\mathbf{I}_M$ is the $M \times M$ identity matrix. In the critically sampled case ($N = M$), $\mathbf{R}(z)$ is uniquely defined by (9.2.6) as [Vet87, Vai87, Vai93, VK95]

$$\mathbf{R}(z) = \mathbf{E}^{-1}(z)\,,$$

where we assumed $\operatorname{rank}\{\mathbf{E}(z)\} = M$ almost everywhere so that $\mathbf{E}^{-1}(z)$ exists. In the oversampled case ($N > M$), $\mathbf{R}(z)$ is not uniquely determined: any solution of (9.2.6) can be written as [Kai80] (still assuming $\operatorname{rank}\{\mathbf{E}(z)\} = M$ almost everywhere)

$$\mathbf{R}(z) \,=\, \mathbf{R}^{(m)}(z) \,+\, \mathbf{P}(z)\left[\mathbf{I}_N - \mathbf{E}(z)\,\mathbf{R}^{(m)}(z)\right], \tag{9.2.7}$$

where $\mathbf{R}^{(m)}(z)$ is the parapseudo-inverse of $\mathbf{E}(z)$, which is a particular solution of (9.2.6) defined as

$$\mathbf{R}^{(m)}(z) = \left[\tilde{\mathbf{E}}(z)\mathbf{E}(z)\right]^{-1} \tilde{\mathbf{E}}(z), \tag{9.2.8}$$

and $\mathbf{P}(z)$ is an $M \times N$ matrix with arbitrary elements $[\mathbf{P}(z)]_{n,k}$ ($n = 0, 1, ..., M-1$, $k = 0, 1, ..., N-1$) satisfying $|[\mathbf{P}(e^{j2\pi\theta})]_{n,k}| < \infty$.

We shall now discuss the above theorem. For critical sampling ($N = M$), $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are square ($M \times M$) matrices and thus (9.2.6) has the unique solution $\mathbf{R}(z) = \mathbf{E}^{-1}(z)$ [Vet87, Vai87, Vai93, VK95]. In the oversampled case ($N > M$), the matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are rectangular ($N \times M$ and $M \times N$, respectively) and thus the solution of (9.2.6) is not uniquely determined; in fact, any *left-inverse* of $\mathbf{E}(z)$ is a valid solution. Expression (9.2.7) is a parameterization of all left-inverses $\mathbf{R}(z)$ in terms of the MN entries $[\mathbf{P}(z)]_{n,k}$ that can be chosen arbitrarily [BHF96b, BH97b]. The nonuniqueness of the synthesis FB for given analysis FB in the oversampled case entails a (desirable) freedom of design that does not exist in the case of critical sampling. In either case, PR requires that $\mathbf{E}(z)$ has full rank.

The particular synthesis polyphase matrix given by the parapseudo-inverse $\mathbf{R}^{(m)}(z) = \left[\tilde{\mathbf{E}}(z)\mathbf{E}(z)\right]^{-1}\tilde{\mathbf{E}}(z)$ corresponds to the synthesis filter impulse responses $f_k^{(m)}[n]$ provided by frame theory via (9.2.5), i.e., $f_k[n] = f_k^{(m)}[n] = (S^{-1}\tilde{h}_k)[n]$ with $\tilde{h}_k[n] = h_k^*[-n]$, or in other words, $\{f_{k,m}[n]\}$ is the UFBF that is dual to $\{h_{k,m}[n]\}$. This frame-theoretic solution minimizes $\sum_{k=0}^{N-1} \|f_k\|^2$ among all left-inverses or, equivalently, all PR synthesis FBs (hence the superscript $^{(m)}$). We note that the relation between pseudo-inverses and frames has been discussed in a more general context in [Chr95a].

The parameterization (9.2.7) can be reformulated in the time domain as

$$f_k[n] \;=\; f_k^{(m)}[n] + p_k[n] - \sum_{l=0}^{N-1} \sum_{m=-\infty}^{\infty} \langle f_k^{(m)}, h_{l,m} \rangle \, p_{l,m}[n], \tag{9.2.9}$$

where $p_k[n]$ is the impulse response of the filter with polyphase components $[\mathbf{P}(z)]_{n,k}$, i.e., $P_k(z) = \sum_{n=0}^{M-1} z^{-n} [\mathbf{P}(z^M)]_{n,k}$, and $p_{k,m}[n] = p_k[n - mM]$. In the z-transform domain, (9.2.7) can be reformulated as

$$F_k(z) \;=\; F_k^{(m)}(z) + P_k(z) - \frac{1}{M} \sum_{i=0}^{M-1} F_k^{(m)}(z\,W_M^i) \left[\sum_{l=0}^{N-1} H_l(z\,W_M^i)\, P_l(z) \right],$$

where $W_M = e^{-j2\pi/M}$. Thus, all PR synthesis filters are parameterized in terms of the N filters $p_k[n] \leftrightarrow P_k(z)$ that can be chosen arbitrarily.

Note that the frame-theoretic, minimum norm solution $f_k^{(m)}[n] \leftrightarrow F_k^{(m)}(z)$ (corresponding to the particular synthesis polyphase matrix $\mathbf{R}^{(m)}(z) = \left[\tilde{\mathbf{E}}(z)\,\mathbf{E}(z)\right]^{-1}\tilde{\mathbf{E}}(z)$) is reobtained for $p_k[n] \equiv 0$ or equivalently $P_k(z) \equiv 0$. In the following we shall mainly use this minimum norm synthesis FB, which will hereafter be denoted simply by $\{f_k[n]\}$ or $\mathbf{R}(z)$.

9.2.5 Frame property

Any synthesis FB of the form (9.2.7) satisfies PR, but it need not correspond to a frame (i.e., UFBF). The frame property is desirable as it guarantees a certain degree of numerical stability (see Subsection 9.2.6). We shall now provide conditions under which a FB corresponds to a frame (UFBF).

Theorem 9.2.3 *[BHF96a, BHF96b] An oversampled or critically sampled FB with BIBO stable*[8] *analysis filters $h_k[n]$ corresponds to a UFBF for $l^2(\mathbb{Z})$, i.e., the analysis set $\{h_{k,m}[n]\}$ is a UFBF for $l^2(\mathbb{Z})$, if and only if the analysis polyphase matrix $\mathbf{E}(z)$ has full rank on the unit circle, i.e.,*

$$\operatorname{rank}\{\mathbf{E}(e^{j2\pi\theta})\} = M \qquad \textit{for} \quad 0 \leq \theta < 1.$$

Since FIR (i.e., finite-length) filters are inherently BIBO stable, an oversampled or critically sampled FB with FIR analysis filters corresponds to a UFBF for $l^2(\mathbb{Z})$ if and only if the analysis polyphase matrix $\mathbf{E}(z)$ has full rank on the unit circle. This condition for the special case of FIR filters has been found previously in [CV].

Alternatively, it can be shown that a FB corresponds to a UFBF for $l^2(\mathbb{Z})$ if $\mathbf{E}(e^{j2\pi\theta})$ has full rank for $0 \leq \theta < 1$ and the $E_{k,n}(e^{j2\pi\theta})$ are continuous and bounded functions of θ [BHF96a, BHF96b]. Yet another condition is phrased in terms of the eigenvalues of the UFBF matrix $\mathbf{S}(e^{j2\pi\theta})$: It can be shown [BHF96a, BHF96b] that an oversampled or critically sampled FB corresponds to a UFBF for $l^2(\mathbb{Z})$ if and only if the eigenvalues $\lambda_n(\theta)$ of $\mathbf{S}(e^{j2\pi\theta}) = \mathbf{E}^H(e^{j2\pi\theta})\,\mathbf{E}(e^{j2\pi\theta})$ satisfy[9]

$$\operatorname*{ess\,inf}_{\theta\,\in\,[0,1),\,n=0,1,\ldots,M-1} \lambda_n(\theta) \;>\; 0 \quad \text{and} \quad \operatorname*{ess\,sup}_{\theta\,\in\,[0,1),\,n=0,1,\ldots,M-1} \lambda_n(\theta) \;<\; \infty.$$

If this condition is satisfied, then the (tightest possible) frame bounds are given by

$$A = \operatorname*{ess\,inf}_{\theta\,\in\,[0,1),\,n=0,1,\ldots,M-1} \lambda_n(\theta)\,, \qquad B = \operatorname*{ess\,sup}_{\theta\,\in\,[0,1),\,n=0,1,\ldots,M-1} \lambda_n(\theta)\,.$$

[8]BIBO (bounded input bounded output) stability means that $h_k[n] \in l^1(\mathbb{Z})$, i.e., $\sum_{n=-\infty}^{\infty} |h_k[n]| < \infty$, for $k = 0, 1, \ldots, N-1$.

[9]ess inf and ess sup denote the essential infimum and essential supremum, respectively.

Similarly, we have

$$A' = \operatorname*{ess\,inf}_{\theta \in [0,1),\, n=0,1,\ldots,M-1} \lambda'_n(\theta)\,, \qquad B' = \operatorname*{ess\,sup}_{\theta \in [0,1),\, n=0,1,\ldots,M-1} \lambda'_n(\theta)\,,$$

where $\lambda'_n(\theta)$ are the eigenvalues of $\mathbf{S}^{-1}(e^{j2\pi\theta}) = \mathbf{R}(e^{j2\pi\theta})\,\mathbf{R}^H(e^{j2\pi\theta})$. Note that in practice the frame bounds have to be estimated by sampling $\mathbf{S}(e^{j2\pi\theta})$ on the unit circle and performing an eigenanalysis of $\mathbf{S}(e^{j2\pi\frac{l}{L}})$ for $l = 0, 1, \ldots, L-1$. Some comments on the quality of this approximate calculation of frame bounds can be found for the FIR case in [Vel93].

The analysis UFBF $\{h_{k,m}[n]\}$ is *tight* if $A = B$ or equivalently $A' = B'$. In this case, $\boldsymbol{S} = A\boldsymbol{I}$ and $\boldsymbol{S}^{-1} = \frac{1}{A}\boldsymbol{I}$ [Dau92]. With (9.2.5), this implies that the frame-theoretic (minimum norm) PR synthesis FB is $f_k[n] = \frac{1}{A}\, h_k^*[-n]$ or $\mathbf{R}(z) = \frac{1}{A}\,\tilde{\mathbf{E}}(z)$. This is precisely the relation between the synthesis and analysis filters in a paraunitary[10] FB [Vai93]. In fact, a FB (oversampled or critically sampled) corresponds to a *tight* UFBF for $l^2(\mathbb{Z})$ if and only if it is *paraunitary*, i.e., $\mathbf{S}(z) = \tilde{\mathbf{E}}(z)\,\mathbf{E}(z) \equiv A\,\mathbf{I}_M$; the frame bound is here $A = [\mathbf{S}(z)]_{n,n} = \sum_{k=0}^{N-1} \tilde{E}_{k,n}(z)\, E_{k,n}(z)$ [BHF96a, BHF96b, CV]. The equivalence of tight Weyl-Heisenberg frames (an important subclass of UFBFs) and paraunitary DFT FBs (cf. Section 9.3) has been noted in [BHF95, Cve95b]. For the special case of FIR oversampled FBs, this equivalence has been stated previously in [CV].

The following theorem describes a method for the derivation of a paraunitary FB from a given nonparaunitary FB. This is an adaptation of a method for the derivation of tight frames from nontight frames [Dau92].

Theorem 9.2.4 *[BHF96a, BHF96b] Let $\mathbf{E}(z)$ and $\mathbf{R}(z)$ be the polyphase matrices of a FB corresponding to a UFBF, and let the $M \times M$ matrix $\mathbf{U}(z)$ be an invertible matrix defined by $\mathbf{U}^2(z) = \tilde{\mathbf{E}}(z)\,\mathbf{E}(z)$ with $\tilde{\mathbf{U}}(z) = \mathbf{U}(z)$. Then the FB with analysis polyphase matrix*

$$\mathbf{E}^{(p)}(z) = \mathbf{E}(z)\,\mathbf{U}^{-1}(z)$$

is paraunitary with frame bound $A = 1$, i.e., $\mathbf{S}^{(p)}(z) = \tilde{\mathbf{E}}^{(p)}(z)\,\mathbf{E}^{(p)}(z) \equiv \mathbf{I}_M$. The corresponding synthesis polyphase matrix is given by $\mathbf{R}^{(p)}(z) = \tilde{\mathbf{E}}^{(p)}(z)$.

9.2.6 Frame bounds and noise sensitivity

The frame bounds A and B or, equivalently, $A' = 1/B$ and $B' = 1/A$ determine important numerical properties of the UFBF $\{h_{k,m}[n]\}$, and thus

---

[10] A FB is *paraunitary* [Vai93] if it satisfies PR and the analysis and synthesis filters satisfy $f_k[n] \propto h_k^*[-n]$; this implies $\mathbf{S}(z) = \tilde{\mathbf{E}}(z)\,\mathbf{E}(z) \propto \mathbf{I}_M$.

also of the associated FB [Dau92]. Due to (9.2.3), the subband signals $v_k[m] = \langle x, h_{k,m} \rangle$ of a FB corresponding to a UFBF satisfy

$$A\|x\|^2 \le \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} |v_k[m]|^2 \le B\|x\|^2 \qquad \forall\, x[n] \in l^2(\mathbb{Z}) \tag{9.2.10}$$

with $0 < A \le B < \infty$. This double inequality generalizes the energy conservation equation $\sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} |v_k[m]|^2 = \|x\|^2$ in orthogonal FBs [CV94], which is reobtained for $A = B = 1$. It also shows that the subband signals $v_k[m]$ are in $l^2(\mathbb{Z})$ if the input signal $x[n]$ is in $l^2(\mathbb{Z})$.

The frame bounds are related to the oversampling factor. It has been shown for the FIR case in [Vel93] and for the general case in [BHF96a, BHF96b] that the analysis filters of a FB corresponding to a UFBF satisfy

$$A \le \frac{1}{M} \sum_{k=0}^{N-1} \|h_k\|^2 \le B .$$

Assuming normalized analysis filters, $\|h_k\|^2 = 1$ for $k = 0, 1, ..., N-1$, this yields the following important relation between the frame bounds and the oversampling factor N/M,

$$A \le \frac{N}{M} \le B .$$

For a tight frame (paraunitary FB) where $A = B$, we obtain

$$A = B = \frac{N}{M} \qquad \text{for } \|h_k\|^2 = 1 . \tag{9.2.11}$$

Hence, the frame bounds of a tight frame (paraunitary FB) with normalized analysis filters equal the oversampling factor N/M.

To show that the frame bounds characterize important numerical properties of a FB, we consider the subband signals $v_k[m]$ corresponding to input signal $x[n]$ and reconstructed signal $y[n]$, and perturbed subband signals, $v'_k[m] = v_k[m] + \Delta v_k[m]$, corresponding to input signal $x'[n] = x[n] + \Delta x[n]$ and reconstructed signal $y'[n] = y[n] + \Delta y[n]$. (In practice, the perturbations $\Delta v_k[m]$ are usually caused by a quantization of the subband signals.) Using the PR property, $y[n] = x[n]$ and $y'[n] = x'[n]$, and the linearity of FB analysis and synthesis, it follows from (9.2.10) that the energy of the reconstruction error $\Delta y[n] = y'[n] - y[n] = x'[n] - x[n]$ is related to the frame bounds A, B and the total energy $\|\Delta v\|^2 = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} |\Delta v_k[m]|^2$ of the subband signal perturbations $\Delta v_k[m]$ as

$$A\,\|\Delta y\|^2 \le \|\Delta v\|^2 \le B\,\|\Delta y\|^2 .$$

With $A' = 1/B$ and $B' = 1/A$, this implies

$$A' \leq \frac{||\Delta y||^2}{||\Delta v||^2} \leq B'.$$

Hence, for given subband perturbation energy $||\Delta v||^2$, the frame bounds A' and B' provide lower and upper bounds on the resulting reconstruction error energy $||\Delta y||^2$. The reconstruction error energy is minimized by making A' as small as possible and B' as close to A' as possible. Thus it is desirable to have $A' \approx B'$ or equivalently $A \approx B$, i.e., a snug frame.

For a tight frame (paraunitary FB), we obtain with (9.2.11)

$$\frac{||\Delta y||^2}{||\Delta v||^2} = \frac{1}{N/M},$$

i.e., *the reconstruction error energy is inversely proportional to the oversampling factor.* We note that a stochastic approach (assuming white uncorrelated noise added to the subband signals) leads to an analogous result [BH97b]. Thus, oversampled FBs feature better noise immunity than critically sampled FBs which, in turn, allows a coarser quantization of the subband signals. A similar result exists for oversampled A/D conversions, where the mean squared error is inversely proportional to the oversampling factor [CMH80, TV94]. The use of noise shaping techniques in oversampled FBs can achieve a further reduction of the reconstruction error [BH97b].

9.3 Oversampled DFT filter banks

DFT FBs (also known as *complex modulated FBs*/) [CR83] are an important class of uniform FBs that can be implemented very efficiently using FFT-based methods [CR83]. In this section, we specialize the results of Section 2 to DFT FBs. We apply the theory of *Weyl-Heisenberg frames* [DGM86, Dau92, BW94] to FIR and IIR, oversampled DFT FBs [Var79, SI87, CR83, Cve95b, BHF96c]. Although the connection between DFT FBs and signal expansions (short time Fourier transforms [Por80, NQ88] or Gabor expansions [Bas80b, Jan81, WR90, Jan95b, DLL95]) is well established [CR83, Vai93, PRV93], a frame-theoretic approach to DFT FBs has been proposed only recently [Cve95b, BHF95, BHF96c].

9.3.1 DFT filter banks and Weyl-Heisenberg sets

In the following we restrict our attention to *even-stacked* DFT FBs [CR83] (*odd-stacked* DFT FBs will be briefly considered in Subsection 9.3.5). The analysis and synthesis filters of an even-stacked DFT FB with N channels and decimation factor M are derived from a single analysis prototype filter $h[n] \leftrightarrow H(z)$ and a single synthesis prototype filter $f[n] \leftrightarrow F(z)$, respectively, as

$$h_k[n] = h[n]\, W_N^{-kn}\,, \qquad f_k[n] = f[n]\, W_N^{-kn}\,, \qquad k = 0, 1, ..., N-1$$

with $W_N = e^{-j2\pi/N}$, or equivalently as

$$H_k(z) = H(zW_N^k)\,, \qquad F_k(z) = F(zW_N^k)\,, \qquad k = 0, 1, ..., N-1\,.$$

The polyphase decomposition of the analysis prototype is given by

$$H(z) = \sum_{n=0}^{M-1} z^n\, E_n(z^M) \qquad \text{with} \qquad E_n(z) = \sum_{m=-\infty}^{\infty} h[mM-n]\, z^{-m}.$$

Note that furthermore $E_{k,n}(z) = W_N^{kn}\, E_n(z\, W_N^{Mk})$, so that the analysis polyphase matrix $\mathbf{E}(z)$ is fully determined by $E_n(z)$ $(n = 0, 1, ..., M-1)$. Similarly, the polyphase decomposition of the synthesis prototype reads

$$F(z) = \sum_{n=0}^{M-1} z^{-n} R_n(z^M) \qquad \text{with} \qquad R_n(z) = \sum_{m=-\infty}^{\infty} f[mM+n]\, z^{-m},$$

and there is $R_{k,n}(z) = W_N^{-kn}\, R_n(z\, W_N^{Mk})$.

The input-output relation of the DFT FB is

$$y[n] = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m} \rangle\, f_{k,m}[n]\,, \tag{9.3.1}$$

where the analysis and synthesis functions are the *Weyl-Heisenberg (WH) sets* [Dau92] generated by $h^*[-n]$ and $f[n]$, respectively,

$$h_{k,m}[n] = h^*[mM-n]\, W_N^{-k(n-mM)}\,, \qquad f_{k,m}[n] = f[n-mM]\, W_N^{-k(n-mM)}$$

with $k = 0, 1, ..., N-1$, $-\infty < m < \infty$.

9.3.2 Perfect reconstruction property and design freedom

If the PR property $y[n] = x[n]$ is satisfied, then (9.3.1) becomes

$$x[n] = \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m} \rangle\, f_{k,m}[n]\,.$$

Hence, a DFT FB with PR provides an expansion of the input signal $x[n]$ into the WH set $f_{k,m}[n] = f[n - mM]\, W_N^{-k(n-mM)}$. This expansion is known as the (discrete-time) *Gabor expansion* [WR90, BHar, Jan94a]. Thus, PR DFT FBs and Gabor expansions are mathematically equivalent [PRV93, BHF95].

A DFT FB (oversampled or critically sampled) with analysis prototype $h[n]$ and synthesis prototype $f[n]$ yields PR if and only if [CR83, WR90, Jan94a]

$$N \sum_{m=-\infty}^{\infty} f[n - mM]\, h[mM - n + lN] \;=\; \delta[l]\,.$$

Equivalently, in the frequency domain the PR condition reads

$$\frac{1}{M} \sum_{m=0}^{N-1} F\Big(e^{j2\pi(\theta - \frac{m}{N})}\Big)\, H\Big(e^{j2\pi(\theta - \frac{m}{N} - \frac{l}{M})}\Big) \;=\; \delta[l]\,.$$

Finally, setting $\frac{N}{M} = \frac{P}{Q}$ where P, Q are relatively prime (i.e., $\gcd(P,Q)=1$ with $\gcd(P,Q)$ denoting the greatest common divisor of P and Q), the PR condition in the polyphase domain is [ZZ93b, Jan94a, Bas95]

$$\frac{M}{Q}\, \mathbf{R}_n(z)\, \mathbf{E}_n(z) = \mathbf{I}_Q\,, \qquad n = 0, 1, ..., M-1\,,$$

where the $Q \times P$ matrices $\mathbf{R}_n(z)$ and the $P \times Q$ matrices $\mathbf{E}_n(z)$ are defined as[11] $[\mathbf{R}_n(z)]_{k,l} = R_{n-kN}(zW_P^l)$ $(k = 0, 1, ..., Q-1,\ l = 0, 1, ..., P-1)$ and $[\mathbf{E}_n(z)]_{k,l} = E_{n-lN}(zW_P^k)$ $(k = 0, 1, ..., P-1,\ l = 0, 1, ..., Q-1)$.

From Subsection 9.2.4 we know that the synthesis FB yielding PR for a given oversampled analysis FB is not uniquely determined. In (9.2.9), all PR synthesis FBs were parameterized in terms of N filters $p_k[n] \leftrightarrow P_k(z)$. In the special case of a DFT analysis FB, the *minimum norm* synthesis FB $\{f_k^{(m)}[n]\}$ is always a DFT FB [BHF96c]; this follows immediately from the fact that the dual frame of a WH frame is again a WH frame [Dau92] (cf. Subsection 9.3.3). Thus we conclude that

$$f_k^{(m)}[n] = f^{(m)}[n]\, W_N^{-kn}\,,$$

where $f^{(m)}[n] = (\boldsymbol{S}^{-1}\tilde{h})[n]$ with $\tilde{h}[n] = h^*[-n]$ is the minimum norm synthesis prototype (cf. Subsection 9.3.3). In general, however, a PR synthesis FB for a given oversampled analysis DFT FB need not have DFT structure, i.e., the $f_k[n]$ need not be modulated versions of a single prototype

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[11] Usually, the polyphase components are defined only for $n = 0, 1, ..., M-1$. However, using $X_{n+lM}(z) = z^l X_n(z)$, where $X_n(z) = \sum_{m=-\infty}^{\infty} x[mM + n]\, z^{-m}$, this definition can be extended to arbitrary $n \in \mathbb{Z}$.

$f[n]$. Li and Healy have shown in the context of WH frames [LH96] that

$$p_k[n] = p[n]\, W_N^{-kn}$$

(with arbitrary $p[n]$) is a sufficient condition for the synthesis FB to have DFT structure. With $p_k[n] = p[n]\, W_N^{-kn}$, (9.2.9) yields

$$\begin{aligned} f[n] &= f^{(m)}[n] + p[n] \\ &- N \sum_{l=-\infty}^{\infty} f^{(m)}[n-lN] \left[\sum_{m=-\infty}^{\infty} h[mM-n+lN]\, p[n-mM] \right], \end{aligned} \tag{9.3.2}$$

which is a parameterization of the synthesis prototype $f[n]$ in terms of the single filter $p[n]$ that may be chosen arbitrarily. Note that the frame-theoretic, minimum norm prototype $f^{(m)}[n]$ is reobtained for $p[n] \equiv 0$.

9.3.3 Frame-theoretic properties

A WH set $\{h_{k,m}[n]\}$ that is a frame (cf. (9.2.3)) is called a *WH frame*. The dual frame can be shown to be again a WH frame [Dau92],

$$f_{k,m}[n] = f[n-mM]\, W_N^{-k(n-mM)},$$

with synthesis prototype $f[n]$ given by

$$f[n] = f^{(m)}[n] = (\boldsymbol{S}^{-1}\tilde{h})[n] \qquad \text{with } \tilde{h}[n] = h^*[-n]\,. \tag{9.3.3}$$

Here, $\boldsymbol{S}^{-1}$ is again the inverse frame operator (cf. (9.2.4)). Among all synthesis prototypes satisfying PR, (9.3.3) defines the synthesis prototype with minimum energy (norm) [Jan94a].

We next provide time, frequency, and polyphase domain expressions for the WH frame operator, and we formulate paraunitarity conditions in the various domains. The *Walnut representation* [Wal92] of the WH frame operator reads

$$(\boldsymbol{S}x)[n] = \sum_{l=-\infty}^{\infty} x[n-lN] \left[N \sum_{m=-\infty}^{\infty} h^*[-n+mM]\, h[-n+mM+lN] \right].$$

(The inverse frame operator can be represented in a similar manner by replacing $h[n]$ with $f^*[-n]$.) Using this representation, it is seen that a DFT FB is paraunitary, i.e., $\{h_{k,m}[n]\}$ is a tight frame for $l^2(\mathbb{Z})$ with frame bound A, or equivalently $\boldsymbol{S} = A\boldsymbol{I}$, if and only if

$$N \sum_{m=-\infty}^{\infty} h^*[-n+mM]\, h[-n+mM+lN] = A\, \delta[l]\,.$$

In the frequency domain, the WH frame operator can be expressed as

$$(\hat{\boldsymbol{S}}X)(e^{j2\pi\theta}) = \sum_{l=0}^{M-1} X\left(e^{j2\pi(\theta-\frac{l}{M})}\right)\left[\frac{1}{M}\sum_{m=0}^{N-1} H^*\left(e^{j2\pi(\theta-\frac{m}{N})}\right) H\left(e^{j2\pi(\theta-\frac{m}{N}-\frac{l}{M})}\right)\right],$$

where $\hat{\boldsymbol{S}} = \boldsymbol{\mathcal{F}}\boldsymbol{S}\boldsymbol{\mathcal{F}}^{-1}$ (with $\boldsymbol{\mathcal{F}}$ denoting the Fourier transform operator) is the frequency domain representation of $\boldsymbol{S}$. (The inverse frame operator can be represented in a similar manner by replacing $H(e^{j2\pi\theta})$ with $F^*(e^{j2\pi\theta})$.) Hence, the FB is paraunitary with frame bound A if and only if

$$\frac{1}{M}\sum_{m=0}^{N-1} H^*\left(e^{j2\pi(\theta-\frac{m}{N})}\right) H\left(e^{j2\pi(\theta-\frac{m}{N}-\frac{l}{M})}\right) = A\,\delta[l]\,.$$

In Subsection 9.2.3, it has been shown that the frame operator $\boldsymbol{S}$ of a general UFBF can be represented in the polyphase domain by the $M \times M$ UFBF matrix $\mathbf{S}(z) = \tilde{\mathbf{E}}(z)\,\mathbf{E}(z)$. For DFT FBs, the frame operator can alternatively be represented in terms of M (often smaller) matrices of size $Q \times Q$, where again $\frac{N}{M} = \frac{P}{Q}$ with $\gcd(P,Q) = 1$. As in Theorem 9.2.1, let $u[n] = (\boldsymbol{S}x)[n]$ and $x[n] = (\boldsymbol{S}^{-1}u)[n]$, and define the polyphase components $U_n(z) = \sum_{m=-\infty}^{\infty} u[mM+n]\,z^{-m}$ and $X_n(z) = \sum_{m=-\infty}^{\infty} x[mM+n]\,z^{-m}$. Then, the polyphase vectors $\mathbf{u}_n(z) = [U_n(z)\; U_{n-N}(z)\; \dots\; U_{n-(Q-1)N}(z)]^T$ and $\mathbf{x}_n(z) = [X_n(z)\; X_{n-N}(z)\; \dots\; X_{n-(Q-1)N}(z)]^T$ can be shown [ZZ93b, Bön] to be related as

$$\begin{aligned}
\mathbf{u}_n(z) &= \mathbf{S}_n(z)\,\mathbf{x}_n(z) \qquad \text{with}\quad \mathbf{S}_n(z) = \frac{M}{Q}\,\tilde{\mathbf{E}}_n(z)\,\mathbf{E}_n(z)\,, \\
\mathbf{x}_n(z) &= \mathbf{S}_n^{-1}(z)\,\mathbf{u}_n(z) \qquad \text{with}\quad \mathbf{S}_n^{-1}(z) = \frac{M}{Q}\,\mathbf{R}_n(z)\,\tilde{\mathbf{R}}_n(z)
\end{aligned}$$

for $n = 0, 1, \dots, M-1$. This representation of $\boldsymbol{S}$ in terms of M matrices $\mathbf{S}_n(z)$ of size $Q \times Q$ is known in WH frame theory as the *Zibulski-Zeevi representation* of the WH frame operator [ZZ93b, BHar]. In particular, the inversion of the frame operator—which, in the general UFBF case, requires the inversion of the $M \times M$ UFBF matrix $\mathbf{S}(z)$—here reduces to the inversion of M matrices of size $Q \times Q$. It can easily be seen that the DFT FB is paraunitary with frame bound A if and only if

$$\mathbf{S}_n(z) = A\,\mathbf{I}_Q \qquad \text{for} \quad n = 0, 1, \dots, M-1.$$

9.3.4 Integer oversampling

For integer oversampled DFT FBs ($N = PM$ with $P \in \mathbb{N}$), it can be shown [BHF96c] that the UFBF matrix $\mathbf{S}(z) = \tilde{\mathbf{E}}(z)\,\mathbf{E}(z)$ is diagonal with diagonal elements $[\mathbf{S}(z)]_{nn} = \Lambda_n(z)$, where

$$\Lambda_n(z) \triangleq M \sum_{r=0}^{P-1} \tilde{E}_n(zW_P^r)\, E_n(zW_P^r)\,, \qquad n = 0, 1, .., M-1.$$

In this case the $Q \times Q$ matrices $\mathbf{S}_n(z)$ reduce to scalars since $Q = 1$ and furthermore $\mathbf{S}(z) = M\,\mathrm{diag}\{\mathbf{S}_n(z)\}_{n=0}^{M-1}$ or equivalently $[\mathbf{S}(z)]_{nn} = M\,\mathbf{S}_n(z)$.

The eigenvalues of $\mathbf{S}(e^{j2\pi\theta})$ follow from the frequency responses $E_n(e^{j2\pi\theta})$ of the analysis prototype's polyphase components according to

$$\lambda_n(\theta) = \Lambda_n(e^{j2\pi\theta}) = M \sum_{r=0}^{P-1} \left| E_n\left(e^{j2\pi(\theta - \frac{r}{P})}\right) \right|^2 .$$

Hence, it follows from Subsection 9.2.5 that an integer oversampled DFT FB corresponds to a WH frame if and only if

$$\operatorname*{ess\,inf}_{\theta \in [0,1),\, n=0,1,\ldots,M-1} \Lambda_n(e^{j2\pi\theta}) > 0, \qquad \operatorname*{ess\,sup}_{\theta \in [0,1),\, n=0,1,\ldots,M-1} \Lambda_n(e^{j2\pi\theta}) < \infty,$$

and that the (tightest possible) frame bounds are given by

$$A = \operatorname*{ess\,inf}_{\theta \in [0,1),\, n=0,1,\ldots,M-1} \Lambda_n(e^{j2\pi\theta}), \quad B = \operatorname*{ess\,sup}_{\theta \in [0,1),\, n=0,1,\ldots,M-1} \Lambda_n(e^{j2\pi\theta}).$$

Note that the frequency responses $E_n(e^{j2\pi\theta})$ of the analysis prototype's polyphase components determine the frame bounds (important numerical properties) of the FB. An integer oversampled DFT FB is paraunitary with frame bound A if and only if

$$\Lambda_n(z) \equiv A \qquad \text{for } n = 0, 1, ..., M-1. \tag{9.3.4}$$

With (9.2.8) it follows that the polyphase components of the minimum norm synthesis prototype are given by

$$R_n(z) = \frac{\tilde{E}_n(z)}{\Lambda_n(z)}\,.$$

Thus, in the case of integer oversampling the synthesis prototype can be calculated in the polyphase domain by simple divisions and the matrix inversion in (9.2.8) is avoided.

According to Theorem 9.2.4, a paraunitary FB can be constructed by factoring the matrix $\mathbf{S}(z)$ of an arbitrary FB corresponding to a frame. For integer oversampled DFT FBs, this reduces to a factorization of polynomials (FIR case) or rational functions (IIR case) in z^{-1}. Let $E_n(z)$ be the analysis polyphase components of an integer oversampled DFT FB corresponding to a WH frame in $l^2(\mathbb{Z})$. Furthermore, let $U_n(z)$ be such that

$$U_n^2(z) = M \sum_{r=0}^{P-1} \tilde{E}_n(zW_P^r)\, E_n(zW_P^r) \qquad \text{and} \qquad \tilde{U}_n(z) = U_n(z)\,. \quad (9.3.5)$$

Then, the DFT FB with analysis polyphase components

$$E_n^{(p)}(z) = \frac{E_n(z)}{U_n(z)}$$

is paraunitary with frame bound $A = 1$, i.e., $\tilde{\mathbf{E}}^{(p)}(z)\,\mathbf{E}^{(p)}(z) = \mathbf{I}_M$.

In the case of critical sampling ($P = 1$), we have

$$\Lambda_n(z) = M\, \tilde{E}_n(z)\, E_n(z)\,, \qquad \lambda_n(\theta) = \Lambda_n(e^{j2\pi\theta}) = M\, |E_n(e^{j2\pi\theta})|^2\,,$$

and the above relations simplify accordingly. In particular, (9.3.4) becomes

$$\tilde{E}_n(z)\, E_n(z) = \frac{A}{M} \qquad \text{for } n = 0, 1, ..., M-1\,,$$

or $|E_n(e^{j2\pi\theta})|^2 \equiv A/M$, which means that the polyphase filters $E_n(z)$ are allpass filters. Thus, the design of a critically sampled paraunitary DFT FB reduces to finding an arbitrary set of M allpass filters. Furthermore, (9.3.5) simplifies to $U_n^2(z) = M\, \tilde{E}_n(z)\, E_n(z)$.

For $P = 2$, a paraunitary DFT FB can be constructed by choosing polyphase filters satisfying the power symmetry conditions [Vai93]

$$\tilde{E}_n(z)\, E_n(z) + \tilde{E}_n(-z)\, E_n(-z) = \frac{A}{M} \qquad \text{for } n = 0, 1, ..., M-1.$$

In [SV86, Vet87] it has been shown that for critical sampling, a DFT FB with FIR filters in both the analysis and the synthesis section is possible only if all the polyphase filters are pure delays. In the oversampled case this restriction is relaxed. A necessary and sufficient condition for a FB to have FIR analysis and FIR minimum norm synthesis filters is $\det[\tilde{\mathbf{E}}(z)\mathbf{E}(z)] = C$ with $C \neq 0$ [CV]. For an integer oversampled DFT FB, $\tilde{\mathbf{E}}(z)\mathbf{E}(z)$ is a diagonal matrix and thus the condition reads

$$\det[\tilde{\mathbf{E}}(z)\mathbf{E}(z)] = \prod_{n=0}^{M-1} [\tilde{\mathbf{E}}(z)\mathbf{E}(z)]_{n,n} = \prod_{n=0}^{M-1} \left[M \sum_{r=0}^{P-1} \tilde{E}_n(zW_P^r) E_n(zW_P^r) \right] = C.$$

For example, for $P = 2$ the above condition can be satisfied by choosing the polyphase filters $E_0(z)$ and $E_1(z)$ such that the power symmetry conditions

$$\tilde{E}_n(z)\, E_n(z) + \tilde{E}_n(-z)\, E_n(-z) = C_n, \qquad n = 0, 1$$

hold with some C_n. These polyphase filters are not necessarily pure delays.

In the general case (i.e., general oversampling) the UFBF matrix $\mathbf{S}(z)$ is not diagonal. However, by imposing restrictions on the length or bandwidth of $h[n]$, it is nevertheless possible to obtain simple expressions for the frame bounds and for the synthesis prototype (see [BHF96c]).

9.3.5 *Odd-stacked DFT filter banks*

The distinction between *even-stacked* and *odd-stacked* DFT FBs has been introduced in [NP78]. For even-stacked DFT FBs (considered so far) the subbands are centered about frequencies $\theta_k = \frac{k}{N}$ ($k = 0, 1, ..., N-1$); in particular, the subband corresponding to frequency index $k = 0$ is centered about $\theta_0 = 0$. For odd-stacked DFT FBs the subbands are centered about frequencies $\theta_k = \frac{k+1/2}{N}$ ($k = 0, 1, ..., N-1$); in particular, the subband corresponding to frequency index $k = 0$ is centered about $\theta_0 = \frac{1}{2N}$.

The impulse responses and transfer functions of the analysis and synthesis filters in an odd-stacked DFT FB with N channels and decimation factor M are given by

$$h_k[n] = h[n]\, W_N^{-(k+1/2)n}, \qquad f_k[n] = f[n]\, W_N^{-(k+1/2)n},$$

$$H_k(z) = H(zW_N^{k+1/2}), \qquad F_k(z) = F(zW_N^{k+1/2})$$

($k = 0, 1, ..., N-1$), and the corresponding polyphase components are

$$\begin{aligned} E_{k,n}(z) &= W_N^{(k+1/2)n} E_n\left(zW_N^{M(k+1/2)}\right) \\ R_{k,n}(z) &= W_N^{-(k+1/2)n} R_n\left(zW_N^{M(k+1/2)}\right). \end{aligned}$$

The FB's input-output relation is (9.3.1) with analysis functions $h_{k,m}[n]$ and synthesis functions $f_{k,m}[n]$ given by

$$\begin{aligned} h_{k,m}[n] &= h^*[mM - n]\, W_N^{-(k+1/2)(n-mM)} \\ f_{k,m}[n] &= f[n - mM]\, W_N^{-(k+1/2)(n-mM)}. \end{aligned}$$

It can be shown that an odd-stacked DFT FB with prototypes $h[n]$ and $f[n]$ is PR or paraunitary if and only if the associated even-stacked DFT FB with the same prototypes is PR or paraunitary, respectively. Thus, the PR and paraunitarity conditions provided for even-stacked DFT FBs in Subsections 9.3.2–9.3.4 are also applicable to odd-stacked DFT FBs.

9.3.6 *Simulation results*

Fig. 9.3.1(a) shows an analysis prototype in a DFT FB with $N = 64$ and $M = 8$ (oversampling factor $P = 8$). The corresponding minimum norm

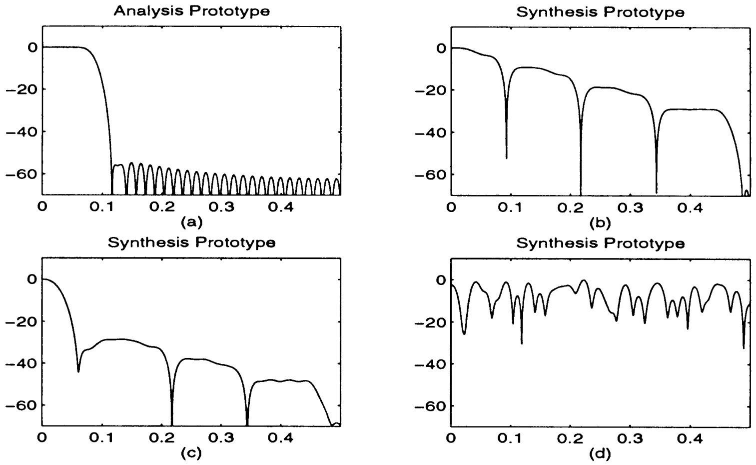

FIGURE 9.3.1. 64-channel DFT FB with oversampling factor $P = 8$: Transfer function magnitude of (a) analysis prototype, (b) minimum norm PR synthesis prototype, (c) PR synthesis prototype with improved frequency selectivity, and (d) "random" PR synthesis prototype.

PR synthesis prototype, depicted in Fig. 9.3.1(b), is seen to have poor frequency selectivity[12]. In Fig. 9.3.1(c) a PR synthesis prototype with improved frequency selectivity is shown. Finally, Fig. 9.3.1(d) shows a PR synthesis prototype which was obtained by a random choice of the parameter function $p[n]$ in (9.3.2). This variety of quite different synthesis prototypes—all satisfying PR—demonstrates the extent of design freedom existing for oversampling factors as high as 8.

9.4 Oversampled cosine modulated filter banks

Cosine modulated FBs (CMFBs) are often preferred over DFT FBs since their subband signals are real-valued if the input signal and the analysis prototype are real-valued. It seems that so far only critically sampled CMFBs have been considered in the literature [Mal92, Rot83, Chu85, KV92, RT91, GB95, NK96, Vai93, Gop96, VK95, LV95]. This section introduces and studies *oversampled* CMFBs. We note that CMFBs can be efficiently implemented using the DCT and DST [Mal92, Vai93, VK95,

[12]Frequency selectivity of the synthesis filters is important in image coding applications where high frequency components are often coarsely quantized. It is here important that the resulting quantization error does not affect low frequency components, which would cause perceptually annoying artifacts in the reconstructed image.

Gop96, BH96b].

9.4.1 Odd-stacked cosine modulated filter banks

We first extend the conventional type of CMFBs [Mal92, Rot83, Chu85, KV92, RT91, GB95, NK96, Vai93, VK95] to the oversampled case [BH97a, BH96c]. For critical sampling, these CMFBs have been termed "class B CMFBs" in [Gop96]; however, we shall here call them "odd-stacked" due to their close relation to odd-stacked DFT FBs (see Subsection 9.4.3).

In the general case with N channels and decimation factor M (note that the CMFB is oversampled for $N > M$), the analysis and synthesis filters of an odd-stacked CMFB are derived from an analysis prototype $h[n]$ and a synthesis prototype $f[n]$, respectively, as[13]

$$h_k^{\text{C-o}}[n] = \sqrt{2}\, h[n] \cos\!\left(\frac{(k+1/2)\pi}{N} n + \phi_k^{\text{o}}\right), \tag{9.4.1}$$

$$f_k^{\text{C-o}}[n] = \sqrt{2}\, f[n] \cos\!\left(\frac{(k+1/2)\pi}{N} n - \phi_k^{\text{o}}\right) \tag{9.4.2}$$

for $k = 0, 1, ..., N-1$. Extending the phase definition given for critical sampling ($N = M$) by Gopinath and Burrus [GB95] to the oversampled case, we define the phases ϕ_k^{o} as

$$\phi_k^{\text{o}} = -\alpha \frac{\pi}{2N}\left(k + \frac{1}{2}\right) + r\frac{\pi}{2} \qquad \text{with } \alpha \in \mathbb{Z}\,,\ r \in \{0, 1\}\,.$$

The choice $r = 1$ corresponds to replacing the cos in (9.4.1) and (9.4.2) by $-$sin and sin, respectively. The above phase expression contains the phases proposed in [Chu85, Rot83, Mal90a, RT91, KV92] as special cases.

The transfer functions of the analysis filters are

$$H_k^{\text{C-o}}(z) = \frac{1}{\sqrt{2}}\left[H\!\left(zW_{2N}^{k+1/2}\right) e^{j\phi_k^{\text{o}}} + H\!\left(zW_{2N}^{-(k+1/2)}\right) e^{-j\phi_k^{\text{o}}}\right]$$

for $k = 0, 1, ..., N-1$. A similar expression exists for the transfer functions of the synthesis filters. Note that the channel frequencies in an odd-stacked CMFB are $\theta_k = \frac{k+1/2}{2N}$, as depicted in Fig. 9.4.1(a). In particular, the channel with index $k = 0$ is centered at frequency $\theta_0 = \frac{1}{4N}$.

An important disadvantage of odd-stacked CMFBs is that the channel filters do not have linear phase even if the prototypes have linear phase [Gop96]. (Linear phase filters are especially important in image coding applications [MS74].)

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[13]The superscripts C-o and C-e indicate that the respective quantity belongs to an odd- and even-stacked CMFB, respectively.

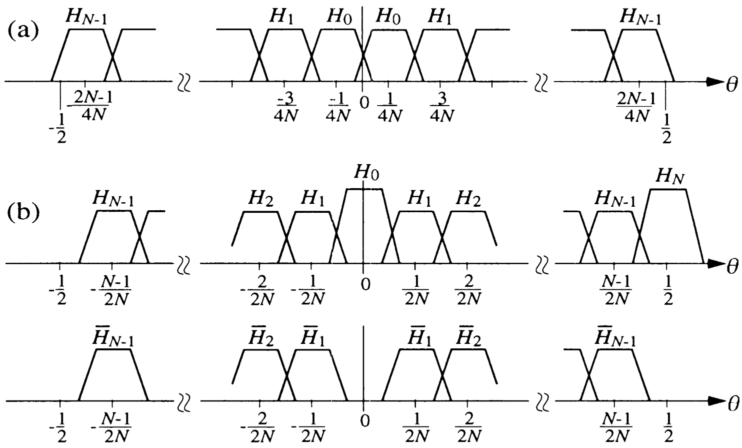

FIGURE 9.4.1. Transfer functions of the channel filters in (a) an N-channel odd-stacked CMFB and (b) a $2N$-channel even-stacked CMFB.

9.4.2 Even-stacked cosine modulated filter banks

We next generalize the "class A" CMFBs recently proposed for critical sampling by Gopinath [Gop96] to the oversampled case [BH96c, BH96b, BH97a]. We call this CMFB type "even-stacked" due to its close relation to even-stacked DFT FBs (see Subsection 9.4.3). The CMFBs recently introduced (for critical sampling) by Lin and Vaidyanathan [LV95] and the recently proposed Wilson FBs [BH96c, BH97a] (corresponding to the discrete-time Wilson expansion [BFGH96]) are special even-stacked CMFBs.

The analysis FB in an even-stacked CMFB with $2N$ channels and decimation factor $2M$ (the CMFB is oversampled for $N > M$) consists of two partial FBs $\{h_k^{\text{C-e}}[n]\}_{k=0,..,N}$ and $\{h_k^{\text{C-e}'}[n]\}_{k=1,..,N-1}$ derived from an analysis prototype $h[n]$ as [BH96c, BH96b, BH97a]

$$h_k^{\text{C-e}}[n] = \begin{cases} h[n-rM], & k=0 \\ \sqrt{2}\,h[n]\cos\left(\frac{k\pi}{N}n+\phi_k^{\text{e}}\right), & k=1,..,N-1 \\ h[n-sM]\,(-1)^{n-sM}, & k=N \end{cases}$$

$$h_k^{\text{C-e}'}[n] = \sqrt{2}\,h[n-M]\,\sin\left(\frac{k\pi}{N}(n-M)+\phi_k^{\text{e}}\right), \quad k=1,..,N-1\,.$$

Similarly, the synthesis FB consists of two partial FBs $\{f_k^{\text{C-e}}[n]\}_{k=0,..,N}$

and $\{f_k^{\text{C-e}'}[n]\}_{k=1,..,N-1}$ defined in terms of a synthesis prototype $f[n]$ as

$$f_k^{\text{C-e}}[n] = \begin{cases} f[n+rM], & k=0 \\ \sqrt{2}\, f[n] \cos\left(\frac{k\pi}{N}n - \phi_k^{\text{e}}\right), & k=1,..,N-1 \\ f[n+sM]\,(-1)^{n+sM}, & k=N \end{cases}$$

$$f_k^{\text{C-e}'}[n] = -\sqrt{2}\, f[n+M] \sin\left(\frac{k\pi}{N}(n+M) - \phi_k^{\text{e}}\right), \quad k=1,..,N-1.$$

Here, extending the phase definition given for critical sampling in [Gop96], we define the phases as

$$\phi_k^{\text{e}} = -\alpha\frac{\pi}{2N}k + r\frac{\pi}{2} \qquad \text{with } \alpha \in \mathbb{Z}\,,\ r \in \{0,1\}\,;$$

furthermore, $s \in \{0,1\}$ with $s=r$ for α even and $s=1-r$ for α odd.

The transfer functions of the analysis filters are

$$H_k^{\text{C-e}}(z) = \begin{cases} z^{-rM}\,H(z), & k=0 \\ \frac{1}{\sqrt{2}}\left[H(zW_{2N}^{k})\,e^{j\phi_k^{\text{e}}} + H(zW_{2N}^{-k})\,e^{-j\phi_k^{\text{e}}}\right], & k=1,..,N-1 \\ z^{-sM}H(-z), & k=N \end{cases}$$

$$H_k^{\text{C-e}'}(z) = \frac{z^{-M}}{j\sqrt{2}}\left[H(zW_{2N}^{k})\,e^{j\phi_k^{\text{e}}} - H(zW_{2N}^{-k})\,e^{-j\phi_k^{\text{e}}}\right], \quad k=1,..,N-1.$$

Similar expressions exist for the transfer functions of the synthesis filters. Note that an even-stacked CMFB has $2N$ channels but there are only $N+1$ different channel frequencies $\theta_k = \frac{k}{2N}$ ($k=0,...,N$), as depicted in Fig. 9.4.1(b). In particular, the $k=0$ channel is centered at frequency $\theta_0 = 0$.

For *any* choice of the parameters $\alpha \in \mathbb{Z}$ and $r \in \{0,1\}$, all analysis filters have linear phase if the analysis prototype $h[n]$ satisfies the linear phase (symmetry) property $h[\alpha+(2l-1)N-n] = h[n]$ for some $l \in \mathbb{Z}$. Similarly, all synthesis filters have linear phase if the synthesis prototype satisfies $f[-\alpha-(2l-1)N-n] = f[n]$ for some $l \in \mathbb{Z}$. This linear phase property of even-stacked CMFBs is an important advantage over odd-stacked CMFBs. For the special case of critical sampling the linear phase property of even-stacked (class A) CMFBs has first been recognized by Gopinath [Gop96].

9.4.3 Perfect reconstruction property

Our discussion of the PR property for CMFBs will be based on an important relation between any CMFB and a corresponding DFT FB of the same stacking type but with twice the CMFB's oversampling factor. This relation is epitomized by the following fundamental decomposition of the reconstructed signal $y[n]$ in a CMFB.

Theorem 9.4.1 *[BH96b, BH96c, BH97a] The reconstructed signal in a CMFB (odd- or even-stacked, oversampled or critically sampled) can be decomposed as*[14]

$$y[n] = \frac{1}{2}\left[\left(\boldsymbol{S}_{\mathrm{D}}^{(h,f)}x\right)[n] + \left(\boldsymbol{T}_{\mathrm{D}}^{(h,f)}x\right)[n]\right]. \tag{9.4.3}$$

Here, $\boldsymbol{S}_{\mathrm{D}}^{(h,f)}$ *is the input-output operator of a DFT FB with* $2N$ *channels and decimation factor* M*,*

$$\left(\boldsymbol{S}_{\mathrm{D}}^{(h,f)}x\right)[n] = \sum_{k=0}^{2N-1}\sum_{m=-\infty}^{\infty}\left\langle x, h_{k,m}^{\mathrm{D}}\right\rangle f_{k,m}^{\mathrm{D}}[n],$$

and $\boldsymbol{T}_{\mathrm{D}}^{(h,f)}$ *is given by*

$$\left(\boldsymbol{T}_{\mathrm{D}}^{(h,f)}x\right)[n] = \sum_{k=0}^{2N-1}\sum_{m=-\infty}^{\infty} e^{j2\phi_k} c_m \left\langle x, h_{k,m}^{\mathrm{D}}\right\rangle f_{k,m}^{\mathrm{D}\dagger}[n],$$

where for an odd-stacked CMFB $h_{k,m}^{\mathrm{D}}[n] = h^*[mM-n]\,W_{2N}^{-(k+1/2)(n-mM)}$, $f_{k,m}^{\mathrm{D}}[n] = f[n-mM]\,W_{2N}^{-(k+1/2)(n-mM)}$, $f_{k,m}^{\mathrm{D}\dagger}[n] = f_{2N-k-1,m}^{\mathrm{D}}[n]$, $\phi_k = \phi_k^{\mathrm{o}}$, *and* $c_m = 1$, *and for an even-stacked CMFB* $h_{k,m}^{\mathrm{D}}[n] = h^*[mM-n]\,W_{2N}^{-k(n-mM)}$, $f_{k,m}^{\mathrm{D}}[n] = f[n-mM]\,W_{2N}^{-k(n-mM)}$, $f_{k,m}^{\mathrm{D}\dagger}[n] = f_{2N-k,m}^{\mathrm{D}}[n]$, $\phi_k = \phi_k^{\mathrm{e}}$, *and* $c_m = (-1)^m$.

We emphasize that the first component, $\left(\boldsymbol{S}_{\mathrm{D}}^{(h,f)}x\right)[n]$, is the output signal of a DFT FB of the same stacking type as the CMFB but with $2N$ channels and decimation factor M, i.e., with twice the oversampling factor (cf. (9.3.1) with N replaced by $2N$). Time, frequency, and polyphase domain expressions of the operators $\boldsymbol{S}_{\mathrm{D}}^{(h,f)}$ and $\boldsymbol{T}_{\mathrm{D}}^{(h,f)}$ are provided in [BH96b].

The above decomposition is the basis for the following fundamental PR condition.

Theorem 9.4.2 *[BH96b, BH96c, BH97a] A CMFB (odd- or even-stacked, oversampled or critically sampled) satisfies the PR property* $y[n] = x[n]$ *if and only if*

$$\boldsymbol{S}_{\mathrm{D}}^{(h,f)} = 2\boldsymbol{I} \qquad \textit{and} \qquad \boldsymbol{T}_{\mathrm{D}}^{(h,f)} = \boldsymbol{O},$$

where $\boldsymbol{I}$ *and* $\boldsymbol{O}$ *denote the identity and zero operator, respectively, on* $l^2(\mathbb{Z})$.

If the second PR condition, $\boldsymbol{T}_{\mathrm{D}}^{(h,f)} = \boldsymbol{O}$, is satisfied, the CMFB's input-output relation (9.4.3) reduces to $y[n] = \frac{1}{2}\left(\boldsymbol{S}_{\mathrm{D}}^{(h,f)}x\right)[n]$, which is (up to

[14]The subscript or superscript D indicates that the respective quantity belongs to a DFT FB.

the constant factor 1/2) the input-output relation of a DFT FB with $2N$ channels and decimation factor M. This DFT FB is odd-stacked (even-stacked) for an odd-stacked (even-stacked) CMFB. Thus, we conclude that *any CMFB with PR corresponds to a PR DFT FB of the* same *stacking type and with twice the oversampling factor.* In view of this correspondence, it is not surprising that the first PR condition, $\boldsymbol{S}_{\mathrm{D}}^{(h,f)} = 2\boldsymbol{I}$ is (up to the constant factor 2) the PR condition for a DFT FB with $2N$ channels and decimation factor M. This PR condition is *the same* for odd-stacked and even-stacked CMFBs (cf. the explicit time, frequency, and polyphase domain formulations in Subsection 9.3.2). Furthermore, explicit time, frequency, and polyphase domain formulations of the second PR condition, $\boldsymbol{T}_{\mathrm{D}}^{(h,f)} = \boldsymbol{O}$, are provided in [BH96b]. We note that the idea of constructing CMFBs from DFT FBs has been previously used in the case of critical sampling and near-PR (see for example [Vai93]).

9.4.4 Frame-theoretic properties

The frame operator $\boldsymbol{S}_{\mathrm{C}}$ of a CMFB is given by

$$(\boldsymbol{S}_{\mathrm{C}}x)[n] = \sum_{k=0}^{N'} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m}^{\mathrm{C}} \rangle \, h_{k,m}^{\mathrm{C}}[n] \,,$$

where in the odd-stacked case $N' = N-1$ and

$$h_{k,m}^{\mathrm{C}}[n] = h_k^{\mathrm{C\text{-}o}^*}[mM - n] \,,$$

and in the even-stacked case $N' = N$ and[15]

$$h_{k,m}^{\mathrm{C}}[n] \;=\; \begin{cases} h_k^{\mathrm{C\text{-}e}^*}[2\mu M - n] \,, & m = 2\mu, \;\; k = 0, 1, ..., N \\ h_k^{\mathrm{C\text{-}e}'^*}[2\mu M - n] \,, & m = 2\mu - 1, \;\; k = 1, 2, ..., N-1. \end{cases}$$

Our frame-theoretic analysis of CMFBs will be based on the following fundamental decomposition of the CMFB frame operator.

Theorem 9.4.3 *[BH96b, BH97a] The frame operator of a CMFB (odd- or even-stacked, oversampled or critically sampled) can be decomposed as*

$$\boldsymbol{S}_{\mathrm{C}} = \frac{1}{2} \left(\boldsymbol{S}_{\mathrm{D}} + \boldsymbol{T}_{\mathrm{D}} \right) . \tag{9.4.4}$$

[15]Note that in the even-stacked case no analysis functions $h_{k,m}^{\mathrm{C}}[n]$ exist for $k = 0$, $m = 2\mu - 1$ and $k = N$, $m = 2\mu - 1$. Furthermore note that for the sake of simplicity we here choose an indexing of the analysis functions $h_{k,m}^{\mathrm{C}}[n]$ of even-stacked CMFBs that is not strictly consistent with the UFBF format in Subsection 9.2.2.

Here, $\boldsymbol{S}_{\mathrm{D}}$ is the frame operator of a DFT FB with $2N$ channels and decimation factor M,

$$(\boldsymbol{S}_{\mathrm{D}}x)[n] = \sum_{k=0}^{2N-1} \sum_{m=-\infty}^{\infty} \langle x, h_{k,m}^{\mathrm{D}} \rangle \, h_{k,m}^{\mathrm{D}}[n] \,,$$

and $\boldsymbol{T}_{\mathrm{D}}$ is given by

$$(\boldsymbol{T}_{\mathrm{D}}x)[n] = \sum_{k=0}^{2N-1} \sum_{m=-\infty}^{\infty} e^{j2\phi_k} c_m \langle x, h_{k,m}^{\mathrm{D}} \rangle \, h_{k,m}^{\mathrm{D}\dagger}[n] \,,$$

with $h_{k,m}^{\mathrm{D}}[n]$, $h_{k,m}^{\mathrm{D}\dagger}[n]$, ϕ_k, and c_m as defined in Theorem 9.4.1.

We emphasize that $\boldsymbol{S}_{\mathrm{D}}$ is the frame operator of a DFT FB of the same stacking type as the CMFB but with $2N$ channels and decimation factor M, i.e., with twice the oversampling factor of the CMFB. Furthermore, $\boldsymbol{S}_{\mathrm{D}} = \boldsymbol{S}_{\mathrm{D}}^{(h,\tilde{h})}$ and $\boldsymbol{T}_{\mathrm{D}} = \boldsymbol{T}_{\mathrm{D}}^{(h,\tilde{h})}$ with $\tilde{h}[n] = h^*[-n]$ (cf. Theorem 9.4.1).

Based on the above decomposition it can be shown [BH96b, BH97a] that, under the condition $\boldsymbol{T}_{\mathrm{D}} = \boldsymbol{O}$, the CMFB inherits the frame-theoretic properties of the corresponding DFT FB.

Theorem 9.4.4 *[BH96b, BH97a] Let $h[n]$ and $f[n]$ denote the analysis and synthesis prototype, respectively, in an odd-stacked CMFB with N channels and decimation factor M, or in an even-stacked CMFB with $2N$ channels and decimation factor $2M$. Let $h[n]$ be such that*[16] *$\{h_{k,m}^{\mathrm{D}}[n]\}$ is a frame for $l^2(\mathbb{Z})$, i.e.,*

$$A_{\mathrm{D}}\|x\|^2 \le \langle \boldsymbol{S}_{\mathrm{D}}x, x \rangle \le B_{\mathrm{D}}\|x\|^2 \qquad \forall\, x[n] \in l^2(\mathbb{Z}) \,.$$

Furthermore, let $h[n]$ be such that $\boldsymbol{T}_{\mathrm{D}} = \boldsymbol{O}$. Then, the following holds:

(i) The CMFB analysis functions $\{h_{k,m}^{\mathrm{C}}[n]\}$ are a frame for $l^2(\mathbb{Z})$ with frame bounds $A_{\mathrm{C}} = A_{\mathrm{D}}/2$ and $B_{\mathrm{C}} = B_{\mathrm{D}}/2$, i.e.,

$$\frac{A_{\mathrm{D}}}{2}\|x\|^2 \le \langle \mathbf{S}_{\mathrm{C}}x, x \rangle \le \frac{B_{\mathrm{D}}}{2}\|x\|^2 \qquad \forall\, x[n] \in l^2(\mathbb{Z}) \,.$$

(ii) For $f[n] = 2\left(\mathbf{S}_{\mathrm{D}}^{-1}\tilde{h}\right)[n]$ with $\tilde{h}[n] = h^*[-n]$, the synthesis CMFB $\{f_k^{\mathrm{C}}[n]\}$ constructed from $f[n]$ is the PR synthesis CMFB with minimum norm filters.

The following interpretations and conclusions apply for $\boldsymbol{T}_{\mathrm{D}} = \boldsymbol{O}$.

[16] Note that the $h_{k,m}^{\mathrm{D}}[n]$ are differently defined for odd-stacked and even-stacked CMFBs (see Theorem 9.4.1).

- Eq. (9.4.4) implies $\boldsymbol{S}_{\mathrm{C}} = \frac{1}{2}\boldsymbol{S}_{\mathrm{D}}$, which means that the CMFB frame operator reduces to the frame operator of the corresponding DFT FB. Since $\left(\boldsymbol{S}_{\mathrm{D}}^{-1}\tilde{h}\right)[n]$ is the minimum-norm synthesis prototype of the corresponding DFT FB [BHF96c], the minimum norm PR synthesis prototype in the CMFB, $f[n] = 2\left(\boldsymbol{S}_{\mathrm{D}}^{-1}\tilde{h}\right)[n]$, is equal (up to a constant factor) to the minimum norm PR synthesis prototype in the corresponding DFT FB. Thus, for $\boldsymbol{T}_{\mathrm{D}} = \boldsymbol{O}$ the design of a CMFB reduces to that of a DFT FB of the same stacking type and with twice the oversampling factor.

- The CMFB frame bounds $A_{\mathrm{C}} = A_{\mathrm{D}}/2$ and $B_{\mathrm{C}} = B_{\mathrm{D}}/2$ are trivially related to the frame bounds A_{D} and B_{D} of the corresponding DFT FB. Since $B_{\mathrm{C}}/A_{\mathrm{C}} = B_{\mathrm{D}}/A_{\mathrm{D}}$, the CMFB inherits important numerical properties (noise sensitivity [BH97b]) of the corresponding DFT FB even though it has just half the oversampling factor of the DFT FB. This is remarkable, since usually a decrease of redundancy leads to a deterioration of the numerical properties of a frame.

- In particular, if the DFT FB is paraunitary ($A_{\mathrm{D}} = B_{\mathrm{D}}$), then the corresponding CMFB is paraunitary as well ($A_{\mathrm{C}} = B_{\mathrm{C}}$).

All of these results hinge on the condition $\boldsymbol{T}_{\mathrm{D}} = \boldsymbol{O}$. Time, frequency, and polyphase domain versions of this condition are provided in [BH96b]. Furthermore, for odd-stacked CMFBs with arbitrary integer oversampling factor P, and for even-stacked CMFBs with odd P, the symmetry property

$$h^*[\alpha + (2l+1)PM - n] \;=\; h[n] \qquad (\text{with some } l \in \mathbb{Z}) \tag{9.4.5}$$

can be shown to be a sufficient condition for $\boldsymbol{T}_{\mathrm{D}} = \boldsymbol{O}$ [BH96b, BH96c, BH97a]. This condition implies that $h[n]$ has linear phase. Thus, PR (with linear phase filters in the case of an even-stacked CMFB) is achieved by choosing $h[n]$ according to (9.4.5) and $f[n] = 2\left(\boldsymbol{S}_{\mathrm{D}}^{-1}\tilde{h}\right)[n]$. In particular, the CMFB will be *paraunitary* with frame bound $A = 1$ if $\boldsymbol{S}_{\mathrm{D}} = 2\boldsymbol{I}$.

9.5 Conclusion

Oversampled filter banks (FBs) have several attractive properties such as increased design freedom and numerical stability. In this chapter, we studied oversampled uniform FBs using a frame-theoretic approach. Special attention has been given to DFT FBs and cosine modulated FBs (CMFBs), which are practically important due to the existence of efficient implementations. Our analysis has emphasized perfect reconstruction and frame-theoretic properties. Among other results, we showed that oversampled even-stacked CMFBs allow perfect reconstruction and paraunitarity as well as linear phase filters in all the channels, and that CMFBs can be derived from DFT FBs with twice the oversampling factor.

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10

Adaptation of Weyl–Heisenberg frames to underspread environments

Werner Kozek

ABSTRACT – Underspread environments provide an operator theoretic framework for slowly time–varying linear systems with finite memory and for the second–order modeling of quasistationary random processes. We consider the adaptation of continuous and discrete Weyl–Heisenberg (WH) frames to trace–class underspread operators in the sense of approximate diagonalization. The atom optimization criteria are formulated in terms of the ambiguity function of the atom and the spreading function of the operator. The theoretical results are demonstrated by a numerical experiment.

10.1 Introduction

One of the fundamental motivations for the use of a local transform like Gabor or wavelet expansion is the concept of approximate diagonalization of slowly time–varying (almost convolution) operators. Based on the knowledge that sinusoids are generalized eigenfunctions of linear translation–invariant (LTI) operators it is intuitively appealing to consider windowed sinusoids as approximate eigenfunctions of almost convolution operators. The mathematical realization of this thought is, however, nontrivial and highly context–dependent in its details.

The early roots of approximate diagonalization go back to Schrödinger who created the concept of a "Wellenpaket" (wave packet) as approximate solution of his famous equation [Sch26]. Calderon et al, studied continuity properties of certain singular integral operators with the help of structured, approximate eigenexpansions [Mey93]. Related to the present context, remarkable extensions and generalizations of Calderon's basic technique are due to Cordoba, Howe and Fefferman [CF78, How80, Fef83]. Moreover, the development of wavelet theory was to some extent motivated by the idea of approximate operator diagonalization and, on the other hand, gave an important impetus to this technique [Dau88a, DP88].

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Among the more recent, mathematical work the present chapter is particularly influenced by Rochberg [Roc90, Roc94] (also see Chapter 4 of this book) who developed an operator decomposition point of view related to approximate diagonalization. Moreover, we mention the work of Heil, Ramanathan and Topiwala [HRT] who used Weyl–Heisenberg frame decompositions to study the decay of singular values of certain pseudodifferential operators.

In the statistical context we mention the work of Mallat, Papanicolaou and Zhang where a statistically adapted local cosine bases is used for covariance estimation of a locally stationary random process [MPZ96]. While approximate operator diagonalization is a central idea of [MPZ96], the techniques and results differ considerably from the present approach: Mallat et al. consider general, adaptive tree–structured tilings of the time–frequency (TF) plane whereas we shall exclusively consider (uniform) lattice tilings. In the applications we have in mind (e.g. signal set design for next generation mobile communication systems) non–lattice type tilings are not admissible. Also in the context of covariance estimation one pays for increased adaptivity by loss of robustness. For the classical problem of deterministic TF signal analysis our approach is immediately applicable while [MPZ96] seems to work only for multiple realizations.

The chapter is structured as follows. In Section 10.2 we review the spreading function and the Kohn-Nirenberg (KN) symbol of a linear operator. These tools are extensively used for the formulation and proofs of our main results. The next section is devoted to the analysis and synthesis of operators via the short time Fourier transform (STFT). We define an STFT of a trace–class operator which induces the mapping of the STFT of an input signal to the STFT of the output signal. The lower/upper WH–symbols are 2D operator symbols related to the STFT–based analysis/synthesis of linear operators. In Section 10.4 we derive two different atom adaptation criteria which aim at optimum diagonalization of a trace–class operator via *continuous* WH frames. Section 10.5 reviews *underspread* operators, their relevance in engineering and statistics, the validity of an approximate symbol calculus and the TF sampling principle. Furthermore, we specialize the variational criteria of Section 10.4 to a single parameter adaptation valid for underspread operators. In Section 10.6 we present theoretical applications of adapted atoms by using the associated continuous WH frame to derive mathematical properties of the KN symbol for underspread operators. The key result is the existence of a continuous set of almost *eigenpairs*, i.e., any (x, ξ)–TF–shifted version of a proper TF–localized prototype function is an approximate eigenfunction of an underspread operator and the KN symbol evaluated at (x, ξ) gives the approximate eigenvalue. Section 10.7 reformulates the atom optimization theory of Section 10.4 for *discrete* WH frames. We assume an a priori choice of the lattice constants in a heuristically adapted way motivated either by a frame–theoretical result of

Tolimieri and Orr [TO95] or the theory of underspread operators. Finally, we demonstrate the conceptual power of the adaptation theory by a simple signal analysis experiment.

10.2 Time–frequency operator representation

The time–shift operator

$$T_t f(x) := f(x-t) \tag{10.2.1}$$

and the frequency–shift operator

$$M_\nu f(x) := f(x) e^{2\pi i \nu x} \tag{10.2.2}$$

can be taken as the unitary building blocks for a quite general linear operator K acting on $\mathbf{L}^2(\mathbb{R})$ according to the following infinitesimal decomposition[1]:

$$K = \iint \eta(K)(t,\nu) M_\nu T_t \, d\nu \, dt. \tag{10.2.3}$$

We adopt the engineering terminology and call $\eta(K)$ *spreading function* of the operator K. Whenever K can be formulated as a (regular or singular) integral operator acting as

$$Kf(x) = \int \kappa(K)(x,y) f(y) dy,$$

then $\eta(K)$ is given by a simple, Fourier–based transform of the kernel:

$$\eta(K)(t,\nu) := \int \kappa(K)(x, x-t) e^{-2\pi i \nu x} dx. \tag{10.2.4}$$

In the statistical context operators are induced by the (auto)correlation kernel of a zero–mean, nonstationary random process p:

$$\kappa(R_p)(x,y) := \mathrm{E}\left\{ p(x)\overline{p(y)} \right\}. \tag{10.2.5}$$

Here, the spreading function can be interpreted as a *time–frequency correlation function* (expected ambiguity function) of p [Koz96b, MHK97].

The convergence of the spreading representation (10.2.3) varies depending on the restrictions imposed on $\kappa(K)$. The function space $\mathbf{S}_0(\mathbb{R})$ consists

[1] All integrals are over $\mathbb{R}$ unless otherwise stated, an overline denotes complex conjugation, $\hat{f}$ is the Fourier transform of f, and E denotes the expectation operator.

of (necessarily continuous) functions with absolutely–integrable STFT and it establishes a "safe" yet fairly general setup both for operators and atoms (the Schwartz class $\mathcal{S}(\mathbb{R})$ of rapidly decreasing smooth functions is a proper subset of $\mathbf{S}_0(\mathbb{R})$, see Chapter 3). Operators with $\kappa(K) \in \mathbf{S}_0(\mathbb{R}^2)$ are trace–class and thus Hilbert–Schmidt (HS) (they form a dense subalgebra of the Hilbert algebra of HS operators, see Chapter 7). The trace–class membership implies a helpful reformulation of (10.2.4):

$$\eta(K)(t,\nu) := \operatorname{tr}\{KT_{-t}M_{-\nu}\}, \quad \text{with} \quad \operatorname{tr}K := \int \kappa(K)(x,x)\,dx \tag{10.2.6}$$

which, together with (10.2.3), stresses the role of the TF–shift operators as building blocks for the harmonic analysis and synthesis of linear operators. Formulas (10.2.3) and (10.2.4) extend accordingly to arbitrary Hilbert–Schmidt operators and more generally to bounded operators from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}'(\mathbb{R})$ (with convergence in the weak sense of bilinear forms).

Via symplectic Fourier transform (denoted by $\mathcal{F}_s$) of $\eta(K)$ one can define the *Kohn–Nirenberg (KN) symbol* of K [KN65]:

$$\sigma(K)(x,\xi) = (\mathcal{F}_s\eta(K))\,(x,\xi) := \iint \eta(K)(t,\nu)e^{-2\pi i(\xi t - x\nu)}\,dtd\nu \tag{10.2.7}$$

$$= \int \kappa(K)(x,x-t)e^{-2\pi i\xi t}\,dt. \tag{10.2.8}$$

In the linear system context, the KN symbol is usually called *time–varying transfer function* [Zad50] and in statistics (according to (10.2.5)) $\sigma(R_p)$ can be interpreted as a *time–varying power spectrum* of the random process p [Fla89]. It is helpful to consider $\sigma(K)$ as a "TF–parametrized eigenvalue distribution" of K, although this interpretation is in conflict with Heisenberg uncertainty (and the spectral theory of operators). The *trace formula* supports the eigenvalue interpretation in a global sense:

$$\iint \sigma(K)(x,\xi)\,dx\,d\xi = \eta(K)(0,0) = \operatorname{tr}K, \tag{10.2.9}$$

i.e., $\sigma(K)$ distributes the eigenvalues of a normal trace–class operator over the TF–plane (similar to "energetic" TF signal representations like the spectrogram which distribute the energy of a signal over the TF–plane). In the present context it is appropriate to mention two other properties of the KN symbol: (i) Unitarity in a HS–sense

$$\langle K,L\rangle_{HS} = \operatorname{tr}\{KL^*\} = \langle\sigma(K),\sigma(L)\rangle_{\mathbf{L}^2(\mathbb{R}^2)} = \langle\eta(K),\eta(L)\rangle_{\mathbf{L}^2(\mathbb{R}^2)}, \tag{10.2.10}$$

and (ii) shift–covariance in the sense that a translation of the symbol leads to a unitarily equivalent operator:

$$T_{(t,\nu)}\left[\sigma(K)\right] = \sigma\left(M_\nu T_t K T_{-t} M_{-\nu}\right) \tag{10.2.11}$$

Important questions about the KN symbol lead to the so–called "twisted convolution", i.e., the relation between $\eta(KL)$ and $\eta(K), \eta(L)$ [Fol89]:

$$\eta(KL)(t,\nu) = \iint \eta(K)(t',\nu')\eta(L)(t-t',\nu-\nu')e^{-2\pi i t'(\nu-\nu')}dt'\,d\nu'. \tag{10.2.12}$$

Whenever K, L are such that this twisted convolution reduces to a regular convolution, one has a "perfect symbol calculus" $\sigma(KL) = \sigma(K)\sigma(L)$ (by (10.2.7), $\sigma(K)\sigma(L)$ corresponds to $\eta(K) * \eta(L)$). Both exact and approximate validity of the symbol calculus are relevant for the theory and application of WH frames:

- *Perfect symbol calculus* seems to be possible only for non–compact operators whose (distributional) $\eta(K)$ is restricted to discrete or simply connected subgroups of the (t,ν)–plane, such as translation invariant operators or the WH–frame operator for integer oversampling (see Chapter 7).
- *Approximate symbol calculus* for operators with smooth symbols corresponding to rapidly decaying $\eta(K)$ [Vor78, Koz97]. The *underspread operators* discussed in this chapter are one canonical way to impose a (Paley–Wiener type) smoothness condition on $\sigma(K)$. We shall show that WH frames give an almost diagonalization of such underspread operators.

10.3 Operator analysis and synthesis via STFT

The short time Fourier transform (STFT), defined as

$$\mathcal{V}_g f(x,\xi) := \langle f, M_\xi T_x g\rangle_{\mathbf{L}^2(\mathbb{R})} = \int f(y)\overline{g(y-x)}e^{-2\pi i \xi y}dy, \tag{10.3.1}$$

establishes the classical "non–parametric" tool for the analysis and synthesis of linear time–varying systems [Por80]. For trace–class operators K with $\kappa(K) \in \mathbf{S}_0(\mathbb{R}^2)$ one can define an "STFT of the operator" as a 4D kernel representation:

$$\mathcal{W}_g K(x,x',\xi,\xi') := \langle K M_{\xi'} T_{x'} g, M_\xi T_x g\rangle\,. \tag{10.3.2}$$

This is the kernel of the integral operator acting on $\mathbf{L}^2(\mathbb{R}^2)$, which maps the STFT of the input signal f to the STFT of the output signal Kf:

$$\mathcal{V}_g K f(x,\xi) = \iint \mathcal{W}_g K(x,x',\xi,\xi')\mathcal{V}_g f(x',\xi')\,dx'\,d\xi'. \tag{10.3.3}$$

The "STFT of an operator" is related to the STFT of the operator's kernel $\kappa(K)$ by a trivial isomorphism:

$$\begin{aligned}\mathcal{W}_g K(x,x',\xi,\xi') &= \left(\mathcal{V}_{(g\otimes\overline{g})}\kappa(K)\right)(x,x',\xi,-\xi')\\ &= \iint \kappa(K)(y,y')\overline{g(y-x)}g(y'-x')e^{-2\pi i(y\xi-y'\xi')}dy\,dy'\end{aligned}$$

where $(g\otimes\overline{g})(x,y) := g(x)\overline{g(y)}$ plays the role of a (separable) atom function. Hence, one has the following Moyal type formula:

$$\langle \mathcal{W}_g K, \mathcal{W}_g L\rangle_{\mathbf{L}^2(\mathbb{R}^4)} = \langle \kappa(K),\kappa(L)\rangle_{\mathbf{L}^2(\mathbb{R}^2)} = \langle K,L\rangle_{\mathrm{HS}}\,, \quad \|g\|=1. \tag{10.3.4}$$

10.3.1 Reformulation via ambiguity function

The (asymmetrical) ambiguity function of g is of fundamental relevance for the formulation and derivation of our main results. It can be formulated as either the spreading function of the rank–one operator with kernel $\kappa(g\otimes g^*)(x,y) := (g\otimes\overline{g})(x,y) = g(x)\overline{g(y)}$ or the STFT of g using g:

$$\eta(g\otimes g^*)(t,\nu) = \mathcal{V}_g g(t,\nu) = \langle g, M_\nu T_t g\rangle_{\mathbf{L}^2(\mathbb{R})}\,. \tag{10.3.5}$$

The following lemma formulates an operator's STFT in terms of its spreading function and the ambiguity function of the atom.

Lemma 10.3.1 *Let K be a trace–class operator with $\kappa(K)\in \mathbf{S}_0(\mathbb{R}^2)$ and consider an atom $g\in\mathbf{S}_0(\mathbb{R})$. Then the STFT of K (see (10.3.2)) can be formulated by a twisted convolution of $\eta(K)$ and $\mathcal{V}_g g$ as follows:*

$$\begin{aligned}\mathcal{W}_g K(x,x-t,\xi,\xi-\nu) = e^{-2\pi i(x-t)\nu}&\iint \eta(K)(t',\nu')\\ &\cdot \mathcal{V}_g g(t-t',\nu-\nu')e^{2\pi i[(x+t'-t)\nu'-\xi t')]}dt'd\nu'.\end{aligned} \tag{10.3.6}$$

Proof: By a standard tensorization trick, we reformulate the sesquilinear form as a HS operator inner product

$$\langle KM_{\xi'}T_{x'}g, M_\xi T_x g\rangle_{\mathbf{L}^2(\mathbb{R})} = \left\langle K, M_\xi T_x g\otimes (M_{\xi'}T_{x'}g)^*\right\rangle_{\mathrm{HS}}\,.$$

This inner product can be transferred to the η–domain via (10.2.10):

$$\langle K,L\rangle_{\mathrm{HS}} = \langle \eta(K),\eta(L)\rangle_{\mathbf{L}^2(\mathbb{R}^2)}\,.$$

A tedious, but straightforward calculation gives

$$\begin{aligned}\eta\left\{M_\xi T_x g\otimes (M_{\xi'}T_{x'}g)^*\right\}(t,\nu) &= \mathcal{V}_{M_{\xi'}T_{x'}g}M_\xi T_x g(t,\nu)\\ &= e^{2\pi i[(\xi-\xi'-\nu)x+\xi' t]}\mathcal{V}_g g(t+x'-x,\nu+\xi'-\xi).\end{aligned}$$

The spreading function of the adjoint operator K^* can be computed by (10.2.3) and the canonical commutation relation ($M_\nu T_t = e^{2\pi i t\nu} T_t M_\nu$):

$$\eta(K^*)(t,\nu) = \overline{\eta(K)(-t,-\nu)} e^{-2\pi i t\nu}, \tag{10.3.7}$$

applied to $(g \otimes g^*)^* = g \otimes g^*$ finishes the proof. ❑

Note that (10.3.6) implies a shift–invariant upper bound for the off–diagonal contributions of $\mathcal{W}_g K$ ($*$ denotes convolution):

$$|\mathcal{W}_g K(x, x-t, \xi, \xi-\nu)| \leq \left(|\eta(K)| * |\mathcal{V}_g g|\right)(t,\nu). \tag{10.3.8}$$

This formula suggests that the magnitude of $\eta(K)$ is of main relevance for the off–diagonal decay of $\mathcal{W}_g K$.

10.3.2 Lower WH–symbol

A four–dimensional operator representation is too detailed in most practical applications. Particularly, from the operator diagonalization point of view we hope that $M_\xi T_x g$ is an approximate eigenfunction of K,

$$K M_\xi T_x g \approx \lambda_{\xi,x} M_\xi T_x g,$$

and, as a consequence, the diagonal of $\mathcal{W}_g K$ should uniquely[2] determine K. We call this diagonal *lower WH–symbol*, it is defined as:

$$\rho_L(K)(x,\xi) := \langle K M_\xi T_x g, M_\xi T_x g\rangle = \mathcal{W}_g K(x,x,\xi,\xi), \tag{10.3.9}$$

In the statistical context (cf. (10.2.5)) $\rho_L(R_p)$ is known as the ***physical spectrum*** of the (nonstationary) random process p [Mar70, Fla89]:

$$\rho_L(R_p) = \mathrm{E}\left\{|\mathcal{V}_g p|^2\right\}.$$

In quantum mechanics, the lower WH–symbol is known as *Berezin symbol* [Per86], while in linear system theory it has been introduced as *short time transfer function* of an LTV system [Koz96a, KFS96]. The terminology "lower WH–symbol" is motivated by the fact that (i) $\rho_L(K)$ satisfies a trace–formula (analogous to (10.2.9))

$$\iint \rho_L(K)(x,\xi)\, dx\, d\xi = \mathrm{tr} K, \tag{10.3.10}$$

[2]By function theoretical methods one can indeed show that the diagonal of $\mathcal{W}_g K$ uniquely determines any bounded operator on $\mathbf{L}^2(\mathbb{R})$ if g has Gaussian shape [Fol89, p.42]. Note, however, that unique determination in the sense of function theory does not mean the existence of a practically stable reconstruction of K for given $\rho_L(K)$.

(ii) the shift–covariance (analogous to the KN symbol)

$$T_{(t,\nu)}\left[\rho_L(K)\right] = \rho_L\left(M_\nu T_t K T_{-t} M_{-\nu}\right), \tag{10.3.11}$$

and (iii) the symbol is "lower" in the sense that:

$$\|\rho_L(K)\|_\infty \leq \|K\|. \tag{10.3.12}$$

Note that such a stability condition is an advantage compared to the KN symbol (which may be unbounded for a bounded operator). Another useful property not shared by the KN symbol is the fact that $K \mapsto \rho_L(K)$ preserves the natural involutions, i.e.,

$$\rho_L(K^*) = \overline{\rho_L(K)}, \tag{10.3.13}$$

i.e., in particular, self–adjoint operators have real–valued symbols. Note, however, that $K \mapsto \rho_L(K)$ can never be an algebraic $*$–homomorphism by the noncommutativity of general trace–class operators.

As a consequence of the shift–covariance (10.3.11) we have a convolution relation with the unitary KN symbol [Fol89] ($\tilde{\sigma}(K)(x,\xi) := \sigma(K)(-x,-\xi)$):

$$\rho_L(K) = \sigma(K) * \overline{\tilde{\sigma}(g \otimes g^*)}. \tag{10.3.14}$$

This convolution carries over to an η–domain multiplication:

$$\mathcal{F}_s \rho_L(K) = \eta(K) \cdot \overline{\mathcal{V}_g g}, \tag{10.3.15}$$

which shows that invertibility of the map $K \mapsto \rho_L(K)$ depends on the support of $\eta(K)$ and $\mathcal{V}_g g$ ($\mathcal{F}_s$ is defined in (10.2.7)).

10.3.3 Upper WH–symbol

From an operator *synthesis* point of view, one can define another nonunitary, STFT–based WH–symbol, denoted by $\rho_U(K)(x,\xi)$, via multiplicative modification of the STFT:

$$K = \iint \rho_U(K)(x,\xi) M_\xi T_x (g \otimes g^*) T_{-x} M_{-\xi}\, dx\, d\xi, \tag{10.3.16}$$

i.e., K is obtained by (i) STFT analysis using g, (ii) multiplicative modification by $\rho_U(K) \in \mathbf{L}^1(\mathbb{R}^2)$, and (iii) STFT synthesis using g. This is a classical method for the realization of linear time–varying systems [Por80]. Operators of the form (10.3.16) are a special case of so–called *Toeplitz operators* [Roc90]. The function $\rho_U(K)$ is a "WH–symbol" in our sense by satisfying (i) a trace–formula (see (10.3.10)), (ii) the shift–covariance (see (10.3.11)) and is "upper" in the sense that [Fol89, Roc90]:

$$\|K\| \leq \|\rho_U(K)\|_\infty. \tag{10.3.17}$$

Analogous to the lower symbol (where self–adjoint operators lead to real–valued symbols), real–valued symbols produce self–adjoint operators.

The invertibility of the map $\rho_U(K) \mapsto K$ corresponds to the engineering problem of synthesizing a given LTV system via multiplicative modification of the STFT. For trace–class operators with $\kappa(K) \in \mathbf{S}_0(\mathbb{R}^2)$ this issue can be studied via the unitary representations $\sigma(K)$ and $\eta(K)$. Applying the KN correspondence to (10.3.16) leads to a σ–domain/η–domain convolution/multiplication relation

$$\sigma(K) = \rho_U(K) * \sigma(g \otimes g^*) \tag{10.3.18}$$

$$\eta(K) = \{\mathcal{F}_s \rho_U(K)\} \cdot \mathcal{V}_g g. \tag{10.3.19}$$

The situation is similar to the lower symbol: Given an $\eta(K)$ with compact support and $\mathcal{V}_g g$ does not vanish on this support, then K can be realized by multiplicative STFT modification with well-defined multiplicator $\rho_U(K)$.

For the special case of Gaussian atoms, the lower/upper WH–symbols correspond to the so–called *Wick/anti–Wick symbols* introduced in quantum field theory (via identifying the TF plane with the complex plane) [Fol89]. Note, moreover, that the above considerations can be extended to quite general Lie groups leading to generalized coherent states, where one calls $\rho_L(K)$ *covariant* and $\rho_U(K)$ *contravariant* symbol of K [AAGM95].

10.4 Adaptation of continuous WH frames

The short time Fourier transform $\mathcal{V}_g f$ is a highly redundant representation. This redundancy leads to a certain "inner symmetry" similar to that of analytic functions, i.e., an arbitrary $\mathbf{L}^2(\mathbb{R}^2)$–function is not necessarily the STFT for any g, f and those that are cannot be characterized by simple smoothness or symmetry conditions. Notwithstanding this infinite redundancy of the STFT it makes sense to consider the STFT–expansion set $\{M_\xi T_x g\}_{(x,\xi)\in\mathbb{R}^2}$ as a *continuous frame* [AAG93].

In what follows we derive various atom optimization criteria which adapt a ($\mathbf{L}^2(\mathbb{R})$–normalized) prototype function g to a trace–class operator K in the sense of formal approximate diagonalization. The diversity of the approaches can be attributed to our incomplete understanding of the "inner symmetry" of the STFT.

10.4.1 Adaptation Via off–diagonal seminorm

One natural way of adaptation is to define an off–diagonal seminorm as

$$M_w(K,g) := \iiiint \left|\mathcal{W}_g K(x, x-t, \xi, \xi-\nu)\right|^2 w(t,\nu)\, dt\, d\nu\, dx\, d\xi,$$

where w is a weight function with $0 \leq w \leq 1$ and $w(t,\nu) = w(-t,-\nu)$, penalizing off–diagonal contributions, say $w(t,\nu) = 1 - exp[-(|t| + |\nu|)]$.

Fortunately, it turns out that minimization of $M_w(K,g)$ is equivalent to a compact optimization problem in terms of the magnitude–squared spreading/ambiguity function of K and g.

Theorem 10.4.1 *Assume that K is an integral operator acting on $\mathbf{L}^2(\mathbb{R})$ with $\kappa(K) \in \mathbf{S}_0(\mathbb{R}^2)$. Then*

$$g_1 = \arg\min_g M_w(K,g) = \arg\max_g \left\langle |\eta(K)|^2 * (1-w), |\mathcal{V}_g g|^2 \right\rangle_{\mathbf{L}^2(\mathbb{R}^2)}, \tag{10.4.1}$$

$$\textit{subject to} \qquad \|g\|_2 = 1 \quad \textit{and} \quad g \in \mathbf{S}_0(\mathbb{R}).$$

Proof. By (10.3.4) we have for $\|g\|_2 = 1$:

$$\iiiint |\mathcal{W}_g K(x, x-t, \xi, \xi-\nu)|^2 \, dt\, d\nu\, dx\, d\xi = \|K\|_{\mathrm{HS}}^2 \,.$$

We split off this atom–independent part and use (10.3.6) to introduce η–domain representations of both g and K. Since, one–variable restrictions of $\mathbf{S}_0(\mathbb{R}^2)$–functions are in $\mathbf{S}_0(\mathbb{R})$ [Fei81], we can make implicit use of the unique Fourier inversion on $\mathbf{S}_0(\mathbb{R})$ by the Dirac measure as an element of $\mathbf{S}_0'(\mathbb{R})$, see Chapter 3 (and we define $w' := 1-w$ for notational convenience):

$$M_w(K,g) - \|K\|_{\mathrm{HS}}^2 = \iiiint w'(t,\nu)$$
$$\cdot \left\{ \iint \eta(K)(t_1,\nu_1)\mathcal{V}_g g(t-t_1, \nu-\nu_1) e^{2\pi i[(x+t_1-t)\nu_1 - \xi t_1]} dt_1\, d\nu_1 \right\}$$
$$\cdot \left\{ \iint \overline{\eta(K)(t_2,\nu_2)\mathcal{V}_g g(t-t_2, \nu-\nu_2)} e^{-2\pi i[(x+t_2-t)\nu_2 - \xi t_2]} dt_2\, d\nu_2 \right\}$$
$$dt\, d\nu\, dx\, d\xi$$

$$= \iint \iiiint w'(t,\nu)\eta(K)(t_1,\nu_1)\mathcal{V}_g g(t-t_1,\nu-\nu_1)$$
$$\cdot \overline{\eta(K)(t_2,\nu_2)\mathcal{V}_g g(t-t_2,\nu-\nu_2)} e^{2\pi i[(t_1-t)\nu_1 - (t_2-t)\nu_2]}$$
$$\cdot \delta(t_1-t_2)\delta(\nu_1-\nu_2) dt_1\, d\nu_1\, dt_2\, d\nu_2\, dt\, d\nu$$

$$= \iiiint w'(t,\nu) \left|\mathcal{V}_g g(t-t_1,\nu-\nu_1)\right|^2 \left|\eta(K)(t_1,\nu_1)\right|^2 dt_1\, d\nu_1\, dt\, d\nu$$

$$= \iint |\mathcal{V}_g g(t',\nu')|^2 \iint w'(-t,-\nu) \left|\eta(K)(t'-t,\nu'-\nu)\right|^2 dt\, d\nu\, dt'\, d\nu'.$$

By the assumed symmetry of w we have $w'(-t,-\nu) = w'(t,\nu)$ which completes the proof. ❑

10.4.2 Adaptation Via orthogonality principle

Given one individual element of the set $\{M_\xi T_x g\}_{(x,\xi)\in\mathbb{R}^2}$, its deviation from an eigenfunction of K can be measured by studying the action of K:

$$KM_\nu T_t g(x) = c(t,\nu)M_\nu T_t g(x) + \varepsilon(t,\nu,x), \tag{10.4.2}$$

where $c(t,\nu)$ plays the role of an "almost eigenvalue" of K. The split–up gets unique by minimizing the $\mathbf{L}^2$–norm of $\varepsilon(t,\nu,.)$ by the orthogonality principle:

$$\langle M_\nu T_t g\,,\,\varepsilon(t,\nu,.)\rangle_{\mathbf{L}^2(\mathbb{R})} = 0. \tag{10.4.3}$$

Hence, by Pythagoras the minimum $\mathbf{L}^2(\mathbb{R})$–norm is given by:

$$\|\varepsilon(t,\nu,.)\|_2^2 = \|KM_\nu T_t g\|_2^2 - \left|\langle KM_\nu T_t g, M_\nu T_t g\rangle_{\mathbf{L}^2(\mathbb{R})}\right|^2. \tag{10.4.4}$$

To measure the adaptation of the whole set $\{M_\xi T_x g\}_{(x,\xi)\in\mathbb{R}^2}$ we integrate over the $\mathbf{L}^2(\mathbb{R})$–norm of the ε–contributions of all members:

$$M_\delta(K,g) := \iint \|\varepsilon(t,\nu,.)\|_2^2\, dt\, d\nu \tag{10.4.5}$$

Similar to the off–diagonal adaptation one can reformulate $M_\delta(K,g)$ in a useful way by switching to the η–domain.

Theorem 10.4.2 *Assume that K is an integral operator acting on $\mathbf{L}^2(\mathbb{R})$ with $\kappa(K)\in\mathbf{S}_0(\mathbb{R}^2)$. Then the minimization of $M_\delta(K,g)$ is given by*

$$g_2 = \arg\min_g M_\delta(K,g) = \arg\max_g \left\langle |\eta(K)|^2, |\mathcal{V}_g g|^2\right\rangle_{\mathbf{L}^2(\mathbb{R}^2)}, \tag{10.4.6}$$
$$\textit{subject to} \qquad \|g\|_2 = 1 \quad \textit{and} \quad g\in\mathbf{S}_0(\mathbb{R}).$$

Proof: (I) As a bounded operator on a Hilbert–space, K has a well-defined adjoint operator such that $\|Kf\|_2^2 = \langle K^*Kf, f\rangle_{\mathbf{L}^2(\mathbb{R})}$. Applying this and (10.3.6) shows that the first term of $M_\delta(K,g)$ does not depend on the ($\mathbf{L}^2(\mathbb{R})$–normalized) atom g:

$$\iint \|KM_\nu T_t g\|_2^2\, dt\, d\nu = \iint \langle K^*KM_\nu T_t g, M_\nu T_t g\rangle_{\mathbf{L}^2(\mathbb{R})}\, dt\, d\nu$$
$$= \iint \mathcal{W}_g(K^*K)(t,t,\nu,\nu)dt\, d\nu = \eta(K^*K)(0,0) = \mathrm{tr}(K^*K) = \|K\|_{\mathrm{HS}}^2$$

(II) For the integration over the second term in (10.4.4) we use again (10.3.6) and proceed similar to the proof of Theorem 10.4.1:

$$\begin{aligned}
&\iint \left|\langle K M_\nu T_t g, M_\nu T_t g\rangle_{\mathbf{L}^2(\mathbb{R})}\right|^2 dt\, d\nu = \iint |\mathcal{W}_g K(t,t,\nu,\nu)|^2\, dt\, d\nu \\
&= \iint \left| \iint \eta(K)(t',\nu')\mathcal{V}_g g(-t',-\nu') e^{2\pi i[t\nu' - \nu t' + t'\nu']} dt'\, d\nu' \right|^2 dt\, d\nu \\
&= \iint \iiiint \eta(K)(t_1,\nu_1)\mathcal{V}_g g(-t_1,-\nu_1)\overline{\eta(K)(t_2,\nu_2)\mathcal{V}_g g(-t_2,-\nu_2)} \\
&\qquad \cdot e^{-2\pi i[t(\nu_1-\nu_2)-\nu(t_1-t_2)+t_1\nu_1 - t_2\nu_2]} dt_1\, d\nu_1\, dt_2\, d\nu_2\, dt\, d\nu \\
&= \iiiint \eta(K)(t_1,\nu_1)\mathcal{V}_g g(-t_1,-\nu_1)\overline{\eta(K)(t_2,\nu_2)\mathcal{V}_g g(-t_2,-\nu_2)} \\
&\qquad \cdot \delta(t_1-t_2)\delta(\nu_1-\nu_2) dt_1\, d\nu_1\, dt_2\, d\nu_2 \\
&= \iint |\eta(K)(t,\nu)|^2\, |\mathcal{V}_g g(t,\nu)|^2\, dt\, d\nu.
\end{aligned}$$

In the last step we have used a well-known symmetry of the magnitude of ambiguity functions [Fol89]: $|\mathcal{V}_g g(t,\nu)| = |\mathcal{V}_g g(-t,-\nu)|$. ❑

10.4.3 Discussion

The atom optimization criteria of the previous sections are structurally equivalent, leading to a nonlinear optimization problem of the form

$$g_{opt} = \arg\max_g \left\langle c, |\mathcal{V}_g g|^2 \right\rangle_{\mathbf{L}^2(\mathbb{R}^2)}, \qquad ||g||_2 = 1,$$

where $c \in \mathbf{L}^2(\mathbb{R}^2)$ is a target function and $\mathcal{V}_g g$ is the ambiguity function of g. Both criteria aim at matching the atom g of a continuous WH–frame $\{M_\xi T_x g\}_{(x,\xi)\in\mathbb{R}^2}$ to a given trace–class operator K.

Closed form solutions of these nonlinear optimization criteria are possible only for very specific symmetries (we shall return to this point along with the discussion of underspread operators in Section 10.5) [Koz96a]. Despite these difficulties one can get an intuitive understanding of the behavior of ambiguity functions (for typical Gaussian–like g), by considering (i) the classical "Radar uncertainty principle":

$$\mathcal{V}_g g(0,0) = 1, \quad \text{for} \quad ||g||_2 = 1, \tag{10.4.7}$$

$$\iint |\mathcal{V}_g g(t,\nu)|^2 dt\, d\nu = 1, \quad \text{for} \quad ||g||_2 = 1; \tag{10.4.8}$$

and, (ii) the fact that the curvature of $\mathcal{V}_g g$ at the origin gives essentially the centralized second–order moments of g and $\hat{g}$:

$$-\frac{1}{\pi}\frac{\partial^2 \mathcal{V}_g g}{\partial \nu^2}(0,0) = 4\pi \int x^2 \left|g(x)\right|^2 dx =: C_g^2, \qquad (10.4.9)$$

$$-\frac{1}{\pi}\frac{\partial^2 \mathcal{V}_g g}{\partial t^2}(0,0) = 4\pi \int \xi^2 \left|\hat{g}(\xi)\right|^2 d\xi =: D_g^2. \qquad (10.4.10)$$

(Near the origin $\mathcal{V}_g g$ is dominated by its real part, hence these relations essentially characterize also $|\mathcal{V}_g g|$.) Moreover, (10.4.9), (10.4.10) relate the shape of $\mathcal{V}_g g$ to Heisenberg's uncertainty relation, which in term of C_g, D_g reads as follows:

$$C_g D_g \geq 1. \qquad (10.4.11)$$

Hence, for nonpathological g (finite C_g, D_g) $\mathcal{V}_g g$ is never highly peaked nor arbitrarily flat about the origin. Having this in mind, consider the residual orthogonal distortion measure achieved by the optimized atom:

$$M_\delta(K, g_2) = \iint \left|\eta(K)(t,\nu)\right|^2 \left(1 - \left|\mathcal{V}_{g_2} g_2(t,\nu)\right|^2\right) dt\, d\nu.$$

It is obvious that $M_\delta(K, g_2)$ can never vanish, but well–concentrated $\eta(K)$ ("underspread") leads to much smaller $M_\delta(K, g_2)$ compared to operators which introduce large time and frequency shifts. This is a first indication that underspread operators are matched to WH function systems in the sense of approximate diagonalization.

Moreover, by (10.4.7) it is clear that g_1 (where $c = |\eta(K)|^2 * w'$) can be seen as a regularized version of g_2 (where $c = |\eta(K)|^2$). By regularized, we mean that whenever $|\eta(K)|^2$ is highly localized (K tending to an identity operator) g_2 tends to get ambiguous due to (10.4.7), while g_1 is still determined by the choice of w' (corresponding to the off–diagonal penalization). When $|\eta(K)|$ is a nice, Gaussian–like function we suggest to use the unweighted atom optimization criterion (10.4.6).

The optimization criteria feature a remarkable double ambiguity in the sense that neither the operator nor the atom are uniquely determined by the cost function. First, TF shifting of the operator in the sense of its symbol leaves the cost function invariant, because

$$|\eta(K)| = |\eta(M_\xi T_x K T_{-x} M_{-\xi})|.$$

Such an ambiguity is not undesired, because it allows the determination of a proper atom in case of incomplete a priori knowledge about an operator. This is particularly relevant in mobile communication problems where incomplete a priori knowledge of the channel (a linear operator) in the form of the magnitude or even only the support of $\eta(K)$ is available. Note that such incomplete a priori knowledge prevents the a priori design of exact eigenbases, because one cannot determine eigenfunctions of a normal trace–class operator without exact knowledge of its kernel.

On the other hand, the inherent ambiguity of the optimum atom is clearly undesired (using a TF shifted atom leads to a falsification of the "absolute" TF localization of the STFT). However, in an iterative numerical solution of the atom optimization it suffices to start with a proper localized function in order to circumvent "TF–biased" atoms.

10.5 Underspread operators

Operators which deviate from strict translation invariance appear in many important applications. Examples are slowly time–varying linear systems in the engineering context or quasistationary random processes for the statistical context. Clearly, there are quite different methods to measure the deviation from strict convolution. In essence all of the "nonparametric" methods[3] impose smoothness conditions on the WH–symbol. One way to measure smoothness of the symbol is by restricting the support of its Fourier transform. Mathematical rigor makes it necessary to distinguish between finite–rank approximable (compact) operators (with "smooth" $\eta(K)$) and noncompact operators (with distributional $\eta(K)$). To operators with compactly supported spreading function defined by

$$\operatorname{supp}\{\eta(K)\} \subseteq Q(t_0, \nu_0), \qquad \text{with } Q(a,b) := [-a,a] \times [-b,b] \quad (10.5.1)$$

one can associate a natural Gelfand triple [GW64]:

(a) *Trace–class operators* [4] *:* $\eta(K) \in \mathbf{A}_{(t_0,\nu_0)}(\mathbb{R}^2) \subset \mathbf{S}_0(\mathbb{R}^2)$,
(b) *Hilbert-Schmidt operators:* $\eta(K) \in \mathbf{L}^2(\mathbb{R}^2)$,
(c) *Noncompact operators:* $\eta(K) \in \mathbf{S}_0'(\mathbb{R}^2)$.

In the present work we put the focus on (a), but we conjecture that all of our results extend with little technical effort to (b). Noncompact operators include the important case of convolution and multiplication operators, hence are relevant for engineering applications, but their treatment would require techniques beyond the scope of this chapter.

Compactly supported $\eta(K)$ leads to a number of interesting mathematical consequences. For $\eta(K) \in \mathbf{L}^2(\mathbb{R}^2)$ the KN symbol $\sigma(K)$ is an element

[3]By nonparametric we exclude methods where, based on a physical model, the linear operator can be characterized by a finite set of coefficients corresponding to e.g. a combination of differential operators and multiplication by polynomials.

[4]$\mathbf{A}_{(t_0,\nu_0)}(\mathbb{R}^2)$ denotes the Banach space of functions f with (i) $\operatorname{supp}\{f\} \subseteq Q(t_0,\nu_0)$ and (ii) $\hat{f} \in \mathbf{L}^1(\mathbb{R}^2)$. Note that one has $\mathbf{A}_{(t_0,\nu_0)}(\mathbb{R}^2) \subset \mathbf{S}_0(\mathbb{R}^2)$ (see Chapter 3).

of a standard Paley–Wiener space, i.e., a reproducing kernel Hilbert space (RKHS) $\subset \mathbf{L}^2(\mathbb{R}^2)$ determined by (t_0, ν_0). The σ–domain RKHS corresponds to an operator domain RKHS, which (i) is a closed subspace of the Hilbert space of HS operators and (ii) consists of pseudodifferential operators of order ∞. The operators which additionally satisfy $\eta(K) \in \mathbf{S}_0(\mathbb{R}^2)$ form a dense subspace of this RKHS of HS operators.

However, while all of this features hold for arbitrarily large $t_0\nu_0$ the intuitive picture of slowly time–varying operators and the naive applicability of frequency–domain methods in general correspond to underspread operators defined by a restricted product of $t_0\nu_0$ as follows:

Definition 10.5.1 *Let $t_0, \nu_0 \in \mathbb{R}_+$, then we define the following subset of the trace–class operators on $\mathbf{L}^2(\mathbb{R})$:*

$$\mathcal{U}(t_0, \nu_0) = \left\{K \in \mathcal{B}(\mathbf{L}^2(\mathbb{R})) \,|\, \eta(K) \in \mathbf{A}_{(t_0,\nu_0)}(\mathbb{R}^2)\right\},$$

where $\mathcal{B}(\mathbf{L}^2(\mathbb{R}))$ denotes the algebra of bounded linear operators on $\mathbf{L}^2(\mathbb{R})$.
In particular, we call operators in $\mathcal{U}(t_0, \nu_0)$ underspread *if*

$$s := 4t_0\nu_0 < 1;$$

and overspread *if $s \geq 1$.*

The set $\mathcal{U}(t_0, \nu_0)$ is a linear vector space but definitely no algebra. Its behavior w.r.t. operator products is reminiscent of pseudodifferential operators (where the product of two operators of order m is of order $2m$):

Lemma 10.5.2 *If $K, L \in \mathcal{U}(t_0, \nu_0)$, then $KL \in \mathcal{U}(2t_0, 2\nu_0)$.*

Proof: Operators with $\eta(K) \in \mathbf{S}_0(\mathbb{R}^2)$ form a Banach algebra, see Chapter 7, Theorem 7.4.1 and twisted convolution (see (10.2.12)) enlarges the support just as regular convolution. ❑

The underspread/overspread terminology seems to originate from the context of time–varying multipath wave propagation channels ("fading channels") where t_0 corresponds to the maximum time–delay and ν_0 is determined by the maximum narrowband Doppler shift [Ken69]. This topic has found renewed interest in the context of digital communication over wireless channels [Pro95, KM97, Koz97].

In the statistical setting underspread processes have been more recently defined via the product of maximum temporal and spectral correlation width. In essence, the so–defined underspread processes are equivalent to the quasistationary processes of Papoulis in [Pap84] or to the locally stationary processes of Priestley [Pri81]. The natural connection to underspread systems is via the innovations representation: Stationary white noise excitation of a $\mathcal{U}(t_0, \nu_0)$ system with $s < 1/4$ leads to an underspread process whose correlation operator is in $\mathcal{U}(2t_0, 2\nu_0)$.

Note that the requirement $\eta(K) \in \mathbf{S}_0(\mathbb{R}^2) \subset \mathbf{L}^1(\mathbb{R}^2)$ is sufficient but not necessary for the trace-class membership of K. Moreover, $K \in \mathcal{U}(2t_0, 2\nu_0)$ implies the continuity of $\eta(K)$, hence it must necessarily approach zero at the boundaries of its support. This setup leads to absolute convergence of various continuous and discrete representations of underspread operators.

10.5.1 Two properties of underspread operators

The optimization theory for continuous WH frames as discussed in Section 10.4 is in principle valid for general trace–class operators. We have, however, pointed out that continuous WH frames obviously perform better for well–concentrated spreading functions. The following property is another —more precise— connection between underspread operators and discrete WH frames:

Theorem 10.5.3 *Let $K \in \mathcal{U}(t_0, \nu_0)$, then sampling of $\sigma(K)$ on a rectangular lattice adapted according to*

$$ab < \frac{1}{4t_0\nu_0} \qquad \text{and} \qquad \frac{a}{b} = \frac{t_0}{\nu_0}, \tag{10.5.2}$$

leads to an absolutely convergent "discrete WH expansion" of K analogous to the continuous WH expansion (10.3.16):

$$K = \sum_{k,l\in\mathbb{Z}} \sigma(K)(ka, lb) M_{lb} T_{ka} P T_{-ka} M_{-lb} \tag{10.5.3}$$

with the prototype operator defined by $\eta(P) \in \mathbf{A}_{(a/2,b/2)}(\mathbb{R}^2)$ and

$$\eta(K) = \frac{1}{ab}\eta(K)\eta(P). \tag{10.5.4}$$

Proof: The standard sampling theorem on the symbol level leads to

$$\sigma(K)(x,\xi) = \sum_{k,l\in\mathbb{Z}} \sigma(K)(ka, lb)\sigma(P)(x - ka, \xi - lb),$$

where $\sigma(P)$ is any admissible interpolation function in $\mathbf{S}_0(\mathbb{R}^2)$ determined by (10.5.4) (recall that $\eta(K)$ is the symplectic Fourier transform of $\sigma(K)$). By the shift–covariance of the KN symbol interpolation on the level of symbols carries over to (10.5.3). It follows from standard sampling theory [Hig96] that $\{\sigma(K)(ka, lb)\}_{k,l\in\mathbb{Z}}$ is in $l^1(\mathbb{Z})$ which implies the claimed mode of convergence. ❑

If the prototype operator P has finite–rank, then the discrete WH expansion (10.5.3) indeed corresponds to a multiatom Gabor multiplication system. However, a finite–rank P is mathematically impossible, yet P is always compact and can be approximated by finite rank operators with arbitrary precision (in HS norm and in turn also operator norm on $\mathbf{L}^2(\mathbb{R})$).

Via (10.5.2) the *critical spread* of underspread operator theory $s = 1$ corresponds to the critical Gabor density $ab = 1$. The critical spread is the threshold value for the (nonparametric) frequency domain identification of LTV systems [Kai62, KF97]. The connection between operators with $s = 1$ and WH systems with $ab = 1$ is certainly an interesting future research topic. However, in the present work we concentrate on properties of $\mathcal{U}(t_0, \nu_0)$–operators that are practically relevant for $s \ll 1$ and break down at $s \approx 1$. The following theorem is a first example for this range of applicability and it is a key building block for later developments.

Theorem 10.5.4 *Let $K, L \in \mathcal{U}(t_0, \nu_0)$, then for $s := 4t_0\nu_0 < 1$ one has*

$$\|\sigma(KL) - \sigma(K)\sigma(L)\|_\infty < 2\sin\left(\frac{\pi}{4}s\right) \|\eta(K)\|_1\|\eta(L)\|_1 \tag{10.5.5}$$

$$\|\sigma(KL) - \sigma(K)\sigma(L)\|_2 < 4\sqrt{s}\sin\left(\frac{\pi}{2}s\right) \|K\|_{HS}\|L\|_{HS} \tag{10.5.6}$$

Proof: (I) The symplectic Fourier transform $\mathcal{F}_s$ maps σ–domain multiplication onto η–domain convolution and one has $\|f\|_\infty \leq \|\mathcal{F}_s f\|_1$. Moreover, we introduce the twisted convolution expression (10.2.12) (and we can use Fubini since $\eta(K), \eta(L) \in \mathbf{L}^1(\mathbb{R}^2)$) :

$$\begin{aligned}
&\|\sigma(KL) - \sigma(K)\sigma(L)\|_\infty \leq \|\eta(KL) - \eta(K) * \eta(L)\|_1 \\
&= \iint \left| \iint \eta(K)(t',\nu')\eta(L)(t-t',\nu-\nu') \left(e^{-2\pi i t'(\nu-\nu')} - 1\right) dt'\, d\nu' \right| dt\, d\nu \\
&\leq 2\int_{-t_0}^{t_0}\int_{-\nu_0}^{\nu_0}\int_{-t_0}^{t_0}\int_{-\nu_0}^{\nu_0} |\eta(K)(t_1,\nu_1)\eta(L)(t_2,\nu_2)| \left|e^{-2\pi i \pi t_1\nu_2} - 1\right| dt_1\, d\nu_1\, dt_2\, d\nu_2 \\
&= 2\int_{-t_0}^{t_0}\int_{-\nu_0}^{\nu_0}\int_{-t_0}^{t_0}\int_{-\nu_0}^{\nu_0} |\eta(K)(t_1,\nu_1)\eta(L)(t_2,\nu_2)|\, |\sin \pi t_1\nu_2|\, dt_1\, d\nu_1\, dt_2\, d\nu_2 \\
&< \max_{(t_k,\nu_k)\in Q(t_0,\nu_0)} \{|\sin \pi t_1 \nu_2|\}\, \|\eta(K)\|_1\|\eta(L)\|_1 \\
&= 2\sin\left(\frac{\pi}{4}s\right) \|\eta(K)\|_1\|\eta(L)\|_1 .
\end{aligned}$$

(II) The proof of (10.5.6) goes along the same lines [Koz97]. ❑

10.5.2 Asymptotic atom adaptation

We now return to the atom adaptation theory and discuss a simple low cost adaptation rule valid for $\mathcal{U}(t_0, \nu_0)$–operators with $s \ll 1$. The principle is to adapt the duration (scale) of an atom with fixed shape (e.g. Gaussian or Hamming) given the spreading ratio $\frac{t_0}{\nu_0}$.

It is well–known that the contour lines of ambiguity functions of TF–localized functions can be approximated by ellipses (in the sense of a Taylor approximation about (0,0)) whose diameters are given by the temporal and spectral moments (see (10.4.9),(10.4.10)) [Pap84]. By elementary analysis one can show that [Koz96a] for $s \to 0$:

$$\left(\frac{C_g}{D_g}\right)_{\text{opt}} = \frac{t_0}{\nu_0} \tag{10.5.7}$$

determines the optimum duration of an atom with given shape (optimum both in the sense of (10.4.1) and (10.4.6)).

The unique function which leads to equality in Heisenberg's uncertainty relation (10.4.11) is the Gaussian function [Fol89],

$$g_\alpha(x) = \left(\frac{2}{\alpha}\right)^{1/4} \exp\left(-\pi\frac{x^2}{\alpha}\right) \tag{10.5.8}$$

it is completely determined by its centralized second order moments:

$$\frac{C_g}{D_g} = \alpha \qquad \text{and} \qquad C_g D_g = 1, \tag{10.5.9}$$

where C_g and D_g are defined in (10.4.9) and (10.4.10). For later use, we also mention that the (asymmetrical) ambiguity function of the Gaussian has Gaussian magnitude[5]:

$$\eta(g_\alpha \otimes g_\alpha^*)(t,\nu) = \exp -\frac{\pi}{2}\left(\frac{1}{\alpha}t^2 + \alpha\nu^2 + i2t\nu\right) \tag{10.5.10}$$

When $|\eta(K)(t,\nu)|$ is of elliptical symmetry, i.e.,

$$|\eta(K)(t,\nu)| = f\left(\left(\frac{t}{t_0}\right)^2 + \left(\frac{\nu}{\nu_0}\right)^2\right)$$

one can prove that the optimum atom is Gaussian and under this circumstances (10.5.7) is exactly valid for arbitrary $t_0, \nu_0 \in \mathbb{R}_+$ [Koz96a] .

10.6 Applying adapted continuous frames

The class of underspread operators is TF shift–invariant in the sense that the mapping $K \mapsto M_\xi T_x K T_{-x} M_{-\xi}$ establishes an isomorphism of $\mathcal{U}(t_0, \nu_0)$

[5]The asymmetrical ambiguity function is related to the KN calculus just as the symmetrical ambiguity function is related to the Weyl calculus. The mutual relation of these ambiguity functions is given by multiplication with $\exp(i\pi t\nu)$.

as it preserves the η–domain support since

$$|\eta(M_\xi T_x K T_{-x} M_{-\xi})| = |M_{(\xi,-x)}\,[\eta(K)]\,|\,.$$

Hence, one can expect that WH frames have the proper structure for the practical and theoretical treatment of underspread operators. The following theorems are applications of adapted continuous WH frames. It turns out that the combination of (nonunitary) symbols using adapted atoms leads to interesting results about the (unitary) KN symbol of underspread operators.

Lemma 10.6.1 *Let $K \in \mathcal{U}(t_0,\nu_0)$, and let $g \in \mathbf{S}_0(\mathbb{R})$ with (i) $\|g\|_2 = 1$ and (ii) "proper TF–localization" defined by*

$$|\mathcal{V}_g g - 1|\,\chi_{Q(2t_0,2\nu_0)} \leq \varepsilon_g < 1, \tag{10.6.1}$$

and assume that $\rho_U(K) \in \mathbf{S}_0(\mathbb{R}^2)$ and $supp\{\widehat{\rho}_U(K)\} \subseteq Q(t_0,\nu_0)$. Then the linear mappings $\sigma(K) \mapsto \rho_L(K)$ and $\rho_U(K) \mapsto \sigma(K)$ have continuous inverse (on $\mathbf{A}_{(t_0,\nu_0)}(\mathbb{R}) \subset \mathbf{S}_0(\mathbb{R}^2)$), and the upper, lower and unitary WH–symbols of K get almost equivalent:

$$\|\rho_L(K) - \sigma(K)\|_\infty < \varepsilon_g \|\eta(K)\|_1 \tag{10.6.2}$$

$$\|\rho_U(K) - \sigma(K)\|_\infty < \frac{\varepsilon_g}{1-\varepsilon_g}\|\eta(K)\|_1 \tag{10.6.3}$$

$$\|\rho_U(K) - \rho_L(K)\|_\infty < \frac{2\varepsilon_g - \varepsilon_g^2}{1-\varepsilon_g}\|\eta(K)\|_1 \tag{10.6.4}$$

Proof: (I) As mentioned in Sect. 10.3 the mappings $\sigma(K) \mapsto \rho_L(K)$ and $\rho_U(K) \mapsto \sigma(K)$ are convolutions with $\sigma(g \otimes g^*) = \mathcal{F}_s \mathcal{V}_g g$. By (10.6.1) $\mathcal{V}_g g$ does not vanish on $Q(2t_0, 2\nu_0)$ and $g \in \mathbf{S}_0(\mathbb{R})$ implies that $\sigma(g \otimes g^*) \in \mathbf{S}_0(\mathbb{R}^2) \subset \mathbf{A}(\mathbb{R}^2)$ (see Chapter 7). Hence the claimed invertibility follows by Wiener's lemma [Kat68].
(II) The symplectic Fourier transform $\mathcal{F}_s$ maps the convolution relation of $\rho_L(K)$ and $\sigma(K)$ to a multiplication and one has $\|f\|_\infty \leq \|\mathcal{F}_s f\|_1$:

$$\|\rho_L(K) - \sigma(K)\|_\infty \leq \left\|\eta(K)\overline{\mathcal{V}_g g} - \eta(K)\right\|_1$$

$$= \int_{-t_0}^{t_0}\int_{-\nu_0}^{\nu_0} \left|\eta(K)(t,\nu)\left(\overline{\mathcal{V}_g g(t,\nu)} - 1\right)\right| dt\, d\nu < \varepsilon_g \|\eta(K)\|_1$$

(III) To proof (10.6.3) we proceed similar:

$$\|\rho_U(K) - \sigma(K)\|_\infty \leq \int_{-t_0}^{t_0}\int_{-\nu_0}^{\nu_0} \left|\frac{\eta(K)(t,\nu)}{\mathcal{V}_g g(t,\nu)} - \eta(K)(t,\nu)\right| dt\, d\nu$$

$$\leq \max_{(t,\nu)\in Q(t_0,\nu_0]} \left\{\left|\frac{\mathcal{V}_g g(t,\nu) - 1}{\mathcal{V}_g g(t,\nu)}\right|\right\} \|\eta(K)\|_1 < \frac{\varepsilon_g}{1-\varepsilon_g}\|\eta(K)\|_1$$

(IV) The inequality (10.6.4) follows by combination of (10.6.2) and (10.6.3). ❑

Note that proper localization in the sense of (10.6.1) establishes a coarse atom adaptation aspect which is, of course, asymptotically equivalent to the atom matching criteria in Section 10.4. In this sense, proper localization of g can be obtained by our standard matching rule (see (10.5.7)):

$$\left(\frac{C_g}{D_g}\right)_{opt} = \frac{t_0}{\nu_0},$$

where C_g and D_g are the duration and bandwidth of g (measured by temporal and spectral moments, see (10.4.9) and (10.4.10)). For the matched Gaussian function one has an explicit estimate for ε_g in terms of the total spread s:

$$\varepsilon_g < 1 - \exp\left(-\pi s\right) + \sin\left(\frac{\pi}{4}s\right); \tag{10.6.5}$$

hence, asymptotically: $\varepsilon_g = \mathcal{O}(s)$ for $s \to 0$.

Applying this to derive the asymptotic $\mathbf{L}_\infty$–difference between the upper and lower symbol (cf. (10.6.4)) we obtain:

$$\frac{\|\rho_U(K) - \rho_L(K)\|_\infty}{\|\eta(K)\|_\infty} = \mathcal{O}(s^3), \qquad s \to 0 \tag{10.6.6}$$

by using $\|\eta(K)\|_1 \leq s\|\eta(K)\|_\infty$, which holds for $K \in \mathcal{U}(t_0, \nu_0)$.

Theorem 10.6.2 *Let $K \in \mathcal{U}(t_0, \nu_0)$, then for $s := 4t_0\nu_0 \to 0$*

$$\frac{\left\|\sigma(K^*) - \overline{\sigma(K)}\right\|_\infty}{\|\eta(K)\|_\infty} = \mathcal{O}(s^2) \tag{10.6.7}$$

$$\frac{|\,\|K\| - \|\sigma(K)\|_\infty\,|}{\|\eta(K)\|_\infty} = \mathcal{O}(s^2) \tag{10.6.8}$$

Proof: (I) We have already mentioned that the lower (and upper) WH–symbol preserve the natural involutions, i.e., $\rho_L(K^*) = \overline{\rho_L(K)}$. This fact combined with (10.6.2) leads to the following estimates:

$$\|\rho_L(K) - \sigma(K)\|_\infty < \varepsilon_g\|\eta(K)\|_1 \tag{10.6.9}$$

$$\left\|\overline{\rho_L(K)} - \sigma(K^*)\right\|_\infty < \varepsilon_g\|\eta(K)\|_1. \tag{10.6.10}$$

The combination of these inequalities leads to

$$\left\|\overline{\sigma(K)} - \sigma(K^*)\right\|_\infty < 2\varepsilon_g\|\eta(K)\|_1 \tag{10.6.11}$$

Switching to asymptotics analogously to (10.6.6) proves (10.6.7).
(II) By the fundamental property of the lower and upper symbol we have

$$\|\rho_L(K)\|_\infty \leq \|K\| \leq \|\rho_U(K)\|_\infty,$$

which in combination with (10.6.4) gives

$$\left| \|\rho_L(K)\|_\infty - \|K\| \right| < \frac{2\varepsilon_g - \varepsilon_g^2}{1-\varepsilon_g} \|\eta(K)\|_1.$$

Similar to step (I), we couple these inequalities to the KN symbol via (10.6.2) and obtain

$$\left| \, \|K\| - \|\sigma(K)\|_\infty \, \right| < \frac{3\varepsilon_g - 2\varepsilon_g^2}{1-\varepsilon_g} \|\eta(K)\|_1.$$

Using the matched Gaussian atom and switching to asymptotics finishes the proof (see (10.6.5) and (10.6.6)). ❑

Theorem 10.6.3 *(Existence of a continuous frame of "almost eigenfunctions"). Let $K \in \mathcal{U}(t_0, \nu_0)$ and $g \in \mathbf{S}_0(\mathbb{R})$ with (i) $\|g\|_2 = 1$ and (ii) proper localization:*

$$|\mathcal{V}_g g - 1| \, \chi_{Q(2t_0, 2\nu_0)} \leq \varepsilon_g < 1; \tag{10.6.12}$$

then $M_\xi T_x g$ and $\sigma(K)(x,\xi)$ form an "almost eigenpair" of K ($s := 4t_0\nu_0$):

$$\|K M_\xi T_x g - \sigma(K)(x,\xi) M_\xi T_x g\|_2^2 < \left[2\sin\left(\frac{\pi}{4}s\right) + 5\varepsilon_g \right] \|\eta(K)\|_1^2 \tag{10.6.13}$$

Proof. (I) We define two TF parametrized auxiliary functions:

$$p(x,\xi) := \langle K M_\xi T_x g, M_\xi T_x g \rangle_{\mathbf{L}^2(\mathbb{R})} - \sigma(K)(x,\xi),$$

$$q(x,\xi) := \|K M_\xi T_x g\|_2^2 - |\sigma(K)(x,\xi)|^2,$$

and split up the $\mathbf{L}^2(\mathbb{R})$–norm in (10.6.13) via the polar identity:

$$\begin{aligned}
&\|K M_\xi T_x g - \sigma(K)(x,\xi) M_\xi T_x g\|_2^2 \\
&= \|K M_\xi T_x g\|_2^2 - 2\mathrm{Re}\left\{ \overline{\sigma(K)(x,\xi)} \, \langle K M_\xi T_x g, M_\xi T_x g\rangle_{\mathbf{L}^2(\mathbb{R})} \right\} \\
&\quad + |\sigma(K)(x,\xi)|^2 \underbrace{\|M_\xi T_x g\|_2^2}_{=1} \\
&= 2\,|\sigma(K)(x,\xi)|^2 + q(x,\xi) - 2\mathrm{Re}\left\{ \overline{\sigma(K)(x,\xi)} \, [\sigma(K)(x,\xi) + p(x,\xi)] \right\} \\
&= q(x,\xi) - 2\mathrm{Re}\left\{ \overline{\sigma(K)(x,\xi)} p(x,\xi) \right\} .
\end{aligned}$$

(II) Recall that $\langle KM_\xi T_x g, M_\xi T_x g\rangle_{\mathbf{L}^2(\mathbb{R})} =: \rho_L(x,\xi)$. Hence, by (10.6.9) we have $\|p\|_\infty \le \varepsilon_g \|\eta(K)\|_1$.
(III) More effort is necessary to estimate $\|q\|_\infty$: Using (10.6.11) we obtain

$$\begin{aligned}\left\|\sigma(K^*)\sigma(K) - |\sigma(K)|^2\right\|_\infty &\le \|\sigma(K)\|_\infty \left\|\sigma(K^*) - \overline{\sigma(K)}\right\|_\infty \\ &< 2\varepsilon_g \|\sigma(K)\|_\infty \|\eta(K)\|_1 \\ &< 2\varepsilon_g \|\eta(K)\|_1^2,\end{aligned}$$

this in combination with (10.5.5) leads to

$$\left\|\sigma(K^*K) - |\sigma(K)|^2\right\|_\infty \le \left[2\sin\left(\frac{\pi}{4}s\right) + 2\varepsilon_g\right] \|\eta(K)\|_1^2.$$

Now, we are ready to bound $\|q\|_\infty$:

$$\begin{aligned}|q(x,\xi)| &= \left|\|KM_\xi T_x g\|_2^2 - |\sigma(K)(x,\xi)|^2\right| \\ &= \left|\langle \eta(K^*K), \eta[M_\xi T_x g \otimes (M_\xi T_x g)^*]\rangle_{\mathbf{L}^2(\mathbb{R}^2)} - |\sigma(K)(x,\xi)|^2\right| \\ &= \left|\int_{-2t_0}^{2t_0}\int_{-2\nu_0}^{2\nu_0} \eta(K^*K)(t,\nu)\overline{V_g g(t,\nu)} e^{2\pi i(\nu x - t\xi)} dt\,d\nu - |\sigma(K)(x,\xi)|^2\right| \\ &\le \left|\sigma(K^*K)(x,\xi) - |\sigma(K)(x,\xi)|^2\right| \\ &\quad + \int_{-2t_0}^{2t_0}\int_{-2\nu_0}^{2\nu_0} |\eta(K^*K)(t,\nu)| \left|1 - \overline{V_g g(t,\nu)}\right| dt\,d\nu \\ &< \left[2\sin\left(\frac{\pi}{4}s\right) + 2\varepsilon_g\right] \|\eta(K)\|_1^2 + \varepsilon_g \int_{-2t_0}^{2t_0}\int_{-2\nu_0}^{2\nu_0} |\eta(K^*K)(t,\nu)|\, dt\,d\nu \\ &\le \left[2\sin\left(\frac{\pi}{4}s\right) + 3\varepsilon_g\right] \|\eta(K)\|_1^2.\end{aligned}$$

Here, we have used the fact that $|\eta(K^*K)| \le |\eta(K^*)| * |\eta(K)|$ which, by (10.3.7) implies $\|\eta(K^*K)\|_1 \le \|\eta(K)\|_1^2$.

(IV) Inserting the results of (II) and (III) in the last line of (I) we can finish the proof as follows:

$$\begin{aligned}\left|q(x,\xi) - 2\mathrm{Re}\left\{\overline{\sigma(K)(x,\xi)}p(x,\xi)\right\}\right| &\le \|q\|_\infty + 2\|\sigma(K)\|_\infty \|p\|_\infty \\ &< \left[2\sin\left(\frac{\pi}{4}s\right) + 5\varepsilon_g\right] \|\eta(K)\|_1^2. \qquad \square\end{aligned}$$

The asymptotic message of the theorem can be compactly formulated by using (i) $\|\eta(K)\|_1 \leq 4t_0\nu_0\|\eta(K)\|_\infty$ and (ii) Taylor expansion of $\mathcal{V}_g g$ about $(0,0)$. One has:

$$\frac{\|KM_\xi T_x g - \sigma(K)(x,\xi)M_\xi T_x g\|_2^2}{\|\eta(K)\|_\infty^2} = \mathcal{O}\left(s^3\right) . \qquad (10.6.14)$$

10.7 Adaptation of discrete WH frames/bases

In the previous section we have shown that TF–shifting of a properly TF–localized atom leads to a continuous frame of "almost eigenfunctions" of an underspread operator. Now, we exclusively consider underspread operators. We have two basic motivations to consider TF–discretization:

- In many applications the redundancy of the STFT is undesired and the natural way to reduce this redundancy is to sample the underlying continuous frame on a rectangular lattice. Provided that the atom and the lattice are properly chosen, one obtains a discrete WH frame as originally suggested by Gabor. However, in applications such as digital communication or standard surveillance radar there is no necessity for completeness of a WH set of functions, rather one is interested to have linear independence[6]. Hence, we a priori consider general sets of the form $\{M_{lb}T_{ka}g\}_{k,l\in\mathbb{Z}}$ leaving open whether it establishes a WH frame of $\mathbf{L}^2(\mathbb{R})$ $(ab < 1)$ or a Riesz basis for its span $(ab > 1)$.
- The KN symbol of an underspread operator is a 2D lowpass function, as such it is uniquely determined by samples on a grid with sufficient density. It is remarkable that this is standard sampling theory on the level of operator symbols with no resort to frame theory or to Gabor's original idea of information transmission.

The adaptation of a discrete WH frame is more complicated because the design freedom includes the choice of the sampling grid in addition to the choice of the atom. Although we expect a good STFT atom to be a good Gabor atom too, it will typically not be optimum in any specific sense of diagonalization.

[6]Digital communication via the so–called orthogonal frequency division multiplex (OFDM) scheme is one example where pulses shifted along a separable time–frequency lattice are used. As the name suggests, the traditional OFDM concept implies the use of an orthonormal WH basis generated by a critical Gabor grid, $ab = 1$. However, recently published works in essence suggest the use of a coarser grid $ab > 1$, together with good TF–localized atoms [San96].

We start with defining the 4D matrix representation of an $\mathcal{U}(t_0, \nu_0)$–operator using the function set $\{M_{lb}T_{ka}g\}_{k,l\in\mathbb{Z}}$:

$$G_gK(k,k',l,l') := \mathcal{W}_gK(ka,k'a,lb,l'b) = \langle KM_{l'b}T_{k'a}g\,,\, M_{lb}T_{ka}g\rangle\,. \tag{10.7.1}$$

By (10.3.8) we can bound the off–diagonal contributions of G_gK in terms of $\eta(K)$ and $\mathcal{V}_g g$:

$$|G_gK(k,k-m,l,l-n)| \le (|\eta(K)| * |\mathcal{V}_g g|)\,(na,mb). \tag{10.7.2}$$

This shows that for $K \in \mathcal{U}(t_0,\nu_0)$ (which implies compact support of $\eta(K)$) the off–diagonal decay is faster than any polynomial whenever g is in $\mathcal{S}(\mathbb{R})$ (as e.g. the Gaussian). Hence, in this sense WH frames achieve an almost diagonalization of underspread operators for any choice of grid and atom. However, decay properties are a coarse way to determine diagonalization performance. In what follows, we first suggest a grid–adaptation rule and then formulate rigid measures for the off–diagonal decay of G_gK leading to atom adaptation criteria similar to that in Section 10.4.

10.7.1 Grid adaptation

A joint optimization of atom and grid in whatever sense of diagonalization seems to be too complicated to get practically useful results. Hence, we proceed in a "semiheuristic" way suggesting an adapted grid by:

$$\left(\frac{a}{b}\right)_{\text{opt}} = \frac{t_0}{\nu_0} \tag{10.7.3}$$

which we take as the basis for the following discrete atom optimization.

There are two lines of reasoning that both lead to (10.7.3):

- Given a trace–class operator where $|\eta(K)|$ has elliptical symmetry the matched Gaussian g, defined by (C_g, D_g are the spectral/temporal moments of g):

$$\left(\frac{C_g}{D_g}\right)_{\text{opt}} = \frac{t_0}{\nu_0} \tag{10.7.4}$$

  establishes the optimum *continuous* WH frame (in the sense of the orthogonality principle in Section 10.4). Now, by a deep result of Tolimieri and Orr [TO95] it is known that, in the sense of frame condition number, the best grid constants for a given Gaussian function g and even–integer oversampling are given by

$$\left(\frac{a}{b}\right)_{\text{opt}} = \frac{C_g}{D_g}. \tag{10.7.5}$$

Hence, by combination of (10.7.5) and (10.7.4), we have a "proof" for (10.7.3).

- Secondly, the sampling principle for underspread operators leads immediately to (10.7.3) when one considers critical sampling. This argument, however, loses most of its relevance for underspread operators with $s \ll 1$, where the critical grid for sampling $\sigma(K)$ is far too coarse to establish (single atom) WH frames. For $s \ll 1$ the choice of the grid constants seem to be less critical in general and can be determined mainly by frame theoretical considerations (provided that closed form analysis or numerical computation of the frame bounds is possible).

10.7.2 Atom adaptation via off–diagonal seminorm

In the discrete setting an off–diagonal seminorm can be defined as

$$M_w(K,g) := \sum_{k,l,m,n\in\mathbb{Z}} |G_gK(k,k-m,l,l-n)|^2\, w(m,n), \qquad (10.7.6)$$

where $w(m,n)$ is a positive weight function designed in a way such that it penalizes off–diagonal entries of G_gK. The simplest choice is $w(m,n) = 1-\delta_{m0}\delta_{n0}$. The minimization of $M_w(K,g)$ leads to a optimization criterion that is structurally equivalent to that in Section 10.4.

Theorem 10.7.1 *Let $K \in \mathcal{U}(t_0,\nu_0)$ and assume grid constants adapted to K according to*

$$b < \frac{1}{2t_0}, \qquad a < \frac{1}{2\nu_0} \qquad (10.7.7)$$

then

$$\arg\min_g M_w(K,g,a,b) = \arg\max_g \left\langle c_{a,b,K,w}\,,\, |\mathcal{V}_g g|^2\right\rangle_{\mathbf{L}^2(\mathbb{R}^2)}$$
$$\textit{subject to} \qquad \|g\|_2 = 1 \quad \textit{and} \quad g\in \mathbf{S}_0(\mathbb{R}),$$

where the target function is defined as:

$$c_{a,b,K,w}(t,\nu) := 1 - \sum_{m,n\in\mathbb{Z}} |\eta(K)(t-ma,\nu-nb)|^2 w(m,n)$$

Proof. We reformulate $M_w(K,g)$ via η–domain representations by (10.3.6) and use twice Poisson's Summation Formula in its $\mathbf{S}_0'(\mathbb{R})$–version (recall that according to our assumptions both $\eta(K)$ and $\mathcal{V}_g g$ are necessarily continuous $\mathbf{S}_0(\mathbb{R}^2)$–functions with absolute convergence of the Poisson Summation Formula, see Chapter 3, Corollary 3.2.9).

$$
\begin{aligned}
&\sum_{k,l,m,n\in\mathbb{Z}} \left|G_g K(k,k-m,l,l-n)\right|^2 w(m,n) \\
&= \sum_{k,l,m,n\in\mathbb{Z}} w(m,n) \left| \iint \eta(K)(t,\nu)\mathcal{V}_g g(ma-t,nb-\nu) \right. \\
&\qquad \left. \cdot\, e^{2\pi i\{[(k-m)a+t]\nu - lbt\}} dt d\nu \right|^2 \\
&= \sum_{k,l,m,n\in\mathbb{Z}} w(m,n) \iiiint \eta(K)(t_1,\nu_1)\mathcal{V}_g g(ma-t_1,nb-\nu_1) \\
&\qquad \cdot \overline{\eta(K)(t_2,\nu_2)\mathcal{V}_g g(ma-t_2,nb-\nu_2)} \\
&\qquad \cdot e^{2\pi i\{(k-m)a(\nu_1-\nu_2)+lb(t_1-t_2)+t_1\nu_1-t_2\nu_2\}} dt_1\, d\nu_1\, dt_2\, d\nu_2 \\
&= \frac{1}{ab} \sum_{k,l,m,n\in\mathbb{Z}} w(m,n) \iiiint \eta(K)(t_1,\nu_1)\mathcal{V}_g g(ma-t_1,nb-\nu_1) \\
&\qquad \cdot \overline{\eta(K)(t_2,\nu_2)\mathcal{V}_g g(ma-t_2,nb-\nu_2)} e^{-2\pi i\{ma(\nu_1-\nu_2)-t_1\nu_1+t_2\nu_2\}} \\
&\qquad \cdot \delta\left(\nu_1-\nu_2-\frac{k}{a}\right)\delta\left(t_1-t_2-\frac{l}{b}\right) dt_1\, d\nu_1\, dt_2\, d\nu_2 \\
&= \frac{1}{ab} \sum_{k,l,m,n\in\mathbb{Z}} w(m,n) \iint \underbrace{\eta(K)(t_1,\nu_1)\overline{\eta(K)\left(t_1-\frac{l}{b},\nu_1-\frac{k}{a}\right)}}_{|\eta(K)(t_1,\nu_1)|^2\delta_{0,l}\delta_{0,k}} \\
&\qquad \cdot \mathcal{V}_g g(ma-t_1,nb-\nu_1)\overline{\mathcal{V}_g g\left(ma-t_1+\frac{l}{b},nb-\nu_1+\frac{k}{a}\right)} \\
&\qquad \cdot e^{-2\pi i(t_1\frac{k}{a}+\nu_1\frac{l}{b}-\frac{kl}{ab})} dt_1\, d\nu_1 \\
&= \frac{1}{ab} \sum_{m,n\in\mathbb{Z}} w(m,n) \iint \left|\eta(K)(t,\nu)\right|^2 \left|\mathcal{V}_g g(ma-t,nb-\nu)\right|^2 dt\, d\nu \\
&= \frac{1}{ab} \iint \underbrace{\sum_{m,n\in\mathbb{Z}} w(m,n) \left|\eta(K)(t-ma,\nu-nb)\right|^2}_{=1-c_{a,b,K,w}(t,\nu)} \left|\mathcal{V}_g g(t,\nu)\right|^2 dt\, d\nu.
\end{aligned}
$$

This finishes the proof because by the side–constraint $\|g\|_2 = 1$ we have $\| \left|\mathcal{V}_g g\right|^2 \|_1 = 1$ such that

$$
\arg\min_g \left\langle q, \left|\mathcal{V}_g g\right|^2 \right\rangle_{\mathbf{L}^2(\mathbb{R}^2)} = \arg\max_g \left\langle 1-q, \left|\mathcal{V}_g g\right|^2 \right\rangle_{\mathbf{L}^2(\mathbb{R}^2)} . \qquad \square
$$

Remark: Note that (10.7.7) can always be satisfied by choosing (i) the grid ratio according to our adaptation rule (10.7.3) and (ii) sufficiently dense sampling $ab < \frac{1}{4t_0\nu_0}$.

10.7.3 Atom adaptation via orthogonality principle

Analogously to the continuous case we measure the orthogonal distortion of the individual set members by

$$\|\varepsilon(t,\nu,.)\|_2^2 = \|KM_\nu T_t g\|_2^2 - \left|\langle KM_\nu T_t g, M_\nu T_t g\rangle_{\mathbf{L}^2(\mathbb{R})}\right|^2 .$$

and get the total orthogonal distortion of the set $\{M_{lb}T_{ka}g\}_{k,l\in\mathbb{Z}}$ by summing over all individual contributions:

$$M_\delta(K,g,a,b) := \sum_{k,l\in\mathbb{Z}} \|\varepsilon(ak,bl,.)\|_2^2 . \tag{10.7.8}$$

It turns out that $\arg\min_g M_\delta(K,g,a,b) = \arg\min_g M_\delta(K,g)$ hence the optimum atom is independent from a,b and equal to the optimum STFT atom (assuming that a,b are within a prescribed, reasonable interval).

Theorem 10.7.2 *Let $K \in \mathcal{U}(t_0,\nu_0)$ and assume a sufficiently dense and matched grid, i.e., a,b satisfy (10.7.7), then*

$$\arg\min_g M_\delta(K,g,a,b) = \arg\max_g \left\langle |\eta(K)|^2, |\mathcal{V}_g g|^2 \right\rangle_{\mathbf{L}^2(\mathbb{R}^2)} \tag{10.7.9}$$
$$\textit{subject to} \qquad \|g\|_2 = 1 \quad \textit{and} \quad g \in \mathbf{S}_0(\mathbb{R}).$$

Proof. By Lemma 10.5.2 we have

$$\operatorname{supp}\{\eta(K^*K)\} \subseteq Q(2t_0, 2\nu_0).$$

Hence by (10.7.7) we have

$$\eta(K^*K)\left(\frac{l}{b},\frac{k}{a}\right) = \eta(K^*K)(0,0)\delta_{k0}\delta_{l0}.$$

Based on this relation we proceed similar to the proof of Theorem (10.7.1).

$$\begin{aligned}
&\sum_{k,l\in\mathbb{Z}} \|KM_{lb}T_{ka}g\|_2^2 = \sum_{k,l\in\mathbb{Z}} \langle K^*KM_{lb}T_{ka}g, M_{lb}T_{ka}g\rangle_{\mathbf{L}^2(\mathbb{R})} \\
&= \sum_{k,l\in\mathbb{Z}} \mathcal{W}_g(K^*K)(ka,ka,lb,lb) \\
&= \sum_{k,l\in\mathbb{Z}} \iint \eta(K^*K)(t',\nu')\mathcal{V}_g g(-t',-\nu') e^{2\pi i[(ka+t')\nu' - lbt']}\, dt'\, d\nu' \\
&= \frac{1}{ab}\sum_{k,l\in\mathbb{Z}} \iint \eta(K^*K)(t',\nu')\mathcal{V}_g g(-t',-\nu') \cdot \delta\left(\nu' - \frac{k}{a}\right)\delta\left(t' - \frac{l}{b}\right) dt'\, d\nu' \\
&= \frac{1}{ab}\sum_{k,l\in\mathbb{Z}} \eta(K^*K)\left(\frac{l}{b},\frac{k}{a}\right)\mathcal{V}_g g\left(-\frac{l}{b},-\frac{k}{a}\right)
\end{aligned}$$

$$= \frac{1}{ab}\mathrm{tr}K^*K = \frac{1}{ab}\|K\|_{HS}$$

The summation over the second term of $\|\varepsilon(ka, lb, .)\|_2^2$ goes similar:

$$\sum_{k,l\in\mathbb{Z}} \left|\langle KM_{lb}T_{ka}g, M_{lb}T_{ka}g\rangle_{\mathrm{L}^2(\mathbb{R})}\right|^2$$
$$= \sum_{k,l\in\mathbb{Z}} |\mathcal{W}_g K(ka, ka, lb, lb)|^2$$
$$= \sum_{k,l\in\mathbb{Z}} \left|\iint \eta(K)(t',\nu')\mathcal{V}_g g(-t',-\nu')e^{2\pi i[(ka+t')\nu'-lbt']}\,dt'\,d\nu'\right|^2$$
$$= \sum_{k,l\in\mathbb{Z}} \iiiint \eta(K)(t_1,\nu_1)\mathcal{V}_g g(-t_1,-\nu_1)\overline{\eta(K)(t_2,\nu_2)}$$
$$\cdot \overline{\mathcal{V}_g g(-t_2,-\nu_2)}e^{2\pi i[(\nu_1-\nu_2)ka-(t_1-t_2)lb+t_1\nu_1-t_2\nu_2]}dt_1\,d\nu_1\,dt_2\,d\nu_2$$
$$= \frac{1}{ab}\sum_{k,l\in\mathbb{Z}} \iiiint \eta(K)(t_1,\nu_1)\mathcal{V}_g g(-t_1,-\nu_1)\overline{\eta(K)(t_2,\nu_2)\mathcal{V}_g g(-t_2,-\nu_2)}$$
$$\cdot\, e^{2\pi i(t_1\nu_1-t_2\nu_2)}\delta\left(t_1-t_2-\frac{l}{b}\right)\delta\left(\nu_1-\nu_2-\frac{k}{a}\right)dt_1\,d\nu_1\,dt_2\,d\nu_2$$
$$= \frac{1}{ab}\sum_{k,l\in\mathbb{Z}} \iint \eta(K)(t,\nu)\overline{\eta(K)\left(t-\frac{l}{b},\nu-\frac{k}{a}\right)}\mathcal{V}_g g(-t,-\nu)$$
$$\cdot\,\overline{\mathcal{V}_g g\left(-t+\frac{l}{b},-\nu+\frac{k}{a}\right)}e^{2\pi i\left(t\frac{l}{b}+\nu\frac{k}{a}-\frac{kl}{ab}\right)}\,dt\,d\nu$$
$$= \frac{1}{ab}\iint |\eta(K)(t,\nu)\mathcal{V}_g g(t,\nu)|^2\,dt\,d\nu\,. \qquad \square$$

10.8 Numerical simulation

Any deterministic signal can be interpreted as a realization of an *underspread random process*, i.e., a process whose covariance operator belongs to $\mathcal{U}(t_0,\nu_0)$ with $4t_0\nu_0 < 1$. However, for the estimation of t_0, ν_0 based upon a single realization only more or less heuristic procedures exist [KF97, Koz93].

For the following experiment we estimated the temporal and spectral correlation width t_0, ν_0 based on thresholding standard autocorrelation estimates of both the signal and its Fourier transform. Based on these estimates of t_0 and ν_0 we determined the matched grid according to (10.7.3) and a matched Gaussian by (10.5.7). This setup was used to compute a (highly) oversampled Gabor representation of a three component test signal. For comparison, we also computed Gabor coefficients using a deliberately long/short atom.

The length of the signal is 2048 samples, while the computational support of the atoms is 512 samples, the oversampling factor was 35. Fig. 10.8.1(a) shows the underlying signal and Fig. 10.8.1(b) its components (a windowed sinusoid, quadratically frequency modulated signal and a modulated, comparatively sharp pulse). We show these components only for illustrational purpose. For the atom adaptation or for the computation of the Gabor coefficients the individual components were not used. Contour (level line) plots of the (magnitude–squared) Gabor coefficients are shown in Fig. 10.8.1(c-d) and Fig. 10.8.1(f) shows the underlying atom functions. Using the "long" atom hides the sharp peak while using a "short" atom the windowed sinusoid cannot be recognized on the contour plot. The adapted atom features a good compromise between temporal and spectral resolution. The numerical standard criterion which measures the concentration of a distribution is the entropy

$$e_c := \sum_{k\in\mathbb{Z}} -c_k \log c_k, \qquad 0 < c_k \leq 1$$

The entropy of the Gabor coefficients was 1.7 for the optimum atom, 2.1 for the "long" atom and 2.5 for the "short" atom. This is the expected result: The adapted atom leads to optimum concentration.

Clearly, for the above setup an experienced signal analyst can find a proper window by trial and error. We emphasize, however, that with the adaptation theory such a window is determined automatically. Note, furthermore, that we have included a signal analysis experiment mainly for its illustrational value and the fact that signal analysis is the classical application of the Gabor expansion.

The relevance of the approximate diagonalization approach goes far beyond the signal *analysis* application: We expect that the presented adaptation theory may help to improve performance in any of the modern applications of Weyl–Heisenberg function systems such as OFDM–type digital communication, speech processing and system identification.

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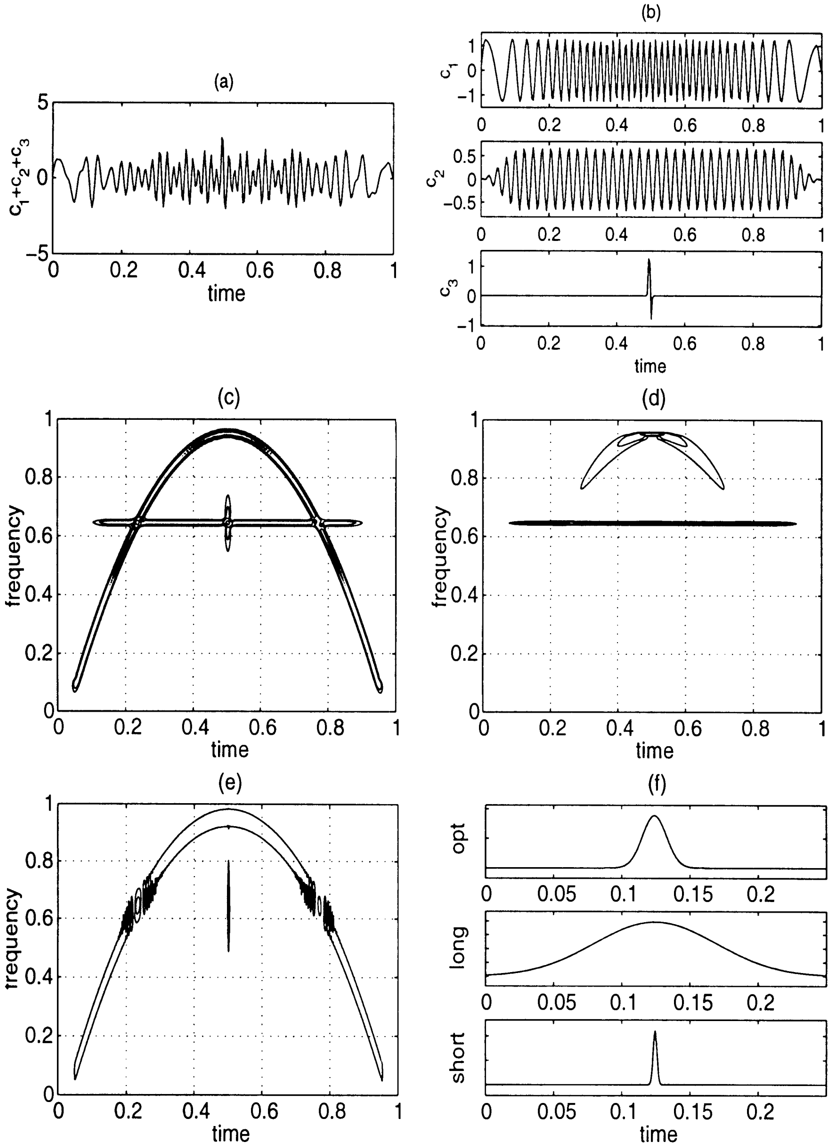

FIGURE 10.8.1. Gabor analysis of three-component signal: (a) The signal, (b) the components; (c–e) Gabor expansion using (c) the adapted atom, (d) a "long" atom, (e) a "short" atom; (f) the underlying atoms.

11

Gabor representation and signal detection

Ariela Zeira and Benjamin Friedlander

ABSTRACT – A common assumption in studies on Gabor based signal detection is that the Gabor representation of the signal is sparse, in the sense that the vector of Gabor coefficients contains only a few nonzero entries at known locations. Under this assumption the problem of detecting a Gabor transient becomes the problem of detecting a subspace signal in background noise. Extending the theory of matched subspace detection to complex signals, we derive matched subspace detectors for this problem and discuss their optimality. We investigate the sensitivity of matched subspace detectors to mismatch in the model parameters. Motivated by the sensitivity analysis, we develop robust matched subspace detectors and analyze their performance.

11.1 Introduction

In recent years, the problem of signal detection based on linear time-frequency representations has received considerable attention [FP89, PF92, LS95, LS97]. A transient signal can often be described as a sum of several components, each localized both in time and frequency. Time-frequency representations map the signal into the time-frequency plane. Since a typical transient signal occupies a relatively small number of time-frequency cells, time-linear representation may provide an efficient tool for data reduction, i.e representing the data vector by another vector of smaller dimensions. Such data reduction is possible, of course, only when there is some prior information restricting the signal to a certain part of the time-frequency plane. If such prior knowledge does not exist, time-frequency representations increase the dimensionality of the problem. Linear time-frequency representations such as the short time Fourier transform (STFT), the wavelet transform and the Gabor representation are preferable over non-linear ones, because they do not produce cross-terms which tend to complicate the representation. Among the linear transforms, the Gabor representation is considered the most promising, as the synthesis window function can be chosen according to prior information available on the

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shape of the signal components. For example, the one-sided exponential function was extensively used for transient detection, as it may represent the jump discontinuity and the gradual decay characterizing many physical phenomena.

All previous studies in the area of Gabor based signal detection considered detection in the transform domain, assuming that the detector performs the following two-step procedure.

1. Data reduction: A Gabor-based linear transformation is applied to the data vector resulting in a vector of smaller dimension.

2. Transform domain detection: A test statistic is applied to the transformed data vector and compared to a threshold, determined so as to achieve a fixed probability of false alarm. Typically, the test statistic is a quadratic function of the transformed data.

The Gabor representation of signals provides, however, a linear model of the signal that can be used for detection in the data domain. A data domain procedure is usually preferable, as it enables derivation of optimal detectors. The transformation step cannot improve the optimal performance, and in some cases can cause performance degradation. Under the framework for transient detection used in [FP89], [PF92] and [LS95] the problem of detecting a Gabor transient (a transient modeled via the Gabor representation) becomes the problem of detecting a subspace signal (a signal that lies in a lower dimensional subspace of the measurement space) in background noise.

The general problem of detecting subspace signal in background noise was thoroughly investigated in [Sch91] and [SF94]. In this chapter we revisit the problem of Gabor based detection, focusing on data domain detection. We extend the theory developed in [Sch91] and [SF94] to the complex data case, as required by the Gabor representation, and apply it to derive and study data domain Gabor based subspace detectors.

The chapter is structured as follows: In Section 11.2 we provide the background for this chapter. We present the Discrete time Gabor representation and formulate the detection problem. In Section 11.3 we review previous results in the area of transform domain detection of Gabor transients, extending them to the complex signal case. In Section 11.4 we review the theory of subspace detection in the context of Gabor based detection, and extend it to the complex signal case. Section 11.5 investigates the sensitivity of matched subspace detectors to mismatch in the model parameters. In Section 11.6 we develop robust data-domain detectors, and analyze their performance. We summarize our results in Section 11.7.

11.2 Background

11.2.1 Gabor representation for discrete time signals

The Gabor expansion for a finite (or periodic) discrete-time sequence with length L is defined as [WR90]

$$x(k) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} C_{m,n} g_{m,n}(k) \quad k = 0, \cdots, L-1 \tag{11.2.1}$$

The set of synthesis functions $g_{m,n}$ is given by

$$\begin{aligned} g_{m,n}(k) &= \tilde{g}(k - ma) W_L^{nbk} \\ W_L^{nbk} &= e^{j(2\pi nb/L)k} \end{aligned} \tag{11.2.2}$$

where $\tilde{g}(k)$ is the periodic extension of the window function $g(k)$, $0 \leq k \leq L-1$ and $Ma = Nb = L$. a and b represent time and frequency sampling intervals, respectively. For stable reconstruction it is required that $ab \leq L$. $ab = L$ is the critically sampled case, while $ab < L$ is the oversampled case.

Eq. (11.2.1) can be rewritten in the following matrix form,

$$\mathbf{x} = G\mathbf{c} \tag{11.2.3}$$

where $\mathbf{x} = [x(0), \cdots, x(L-1)]^T$,

$$\mathbf{c} = [C_{0,0}, \cdots, C_{M-1,0}, \cdots, C_{0,N-1}, \cdots, C_{M-1,N-1}]^T \tag{11.2.4}$$

and G is an L-by-MN matrix. The $(nM + m + 1)$-th column of G is $[g_{m,n}(0), \cdots, g_{m,n}(L-1)]^T$.

In the presence of noise the measured signal is

$$\mathbf{y} = \mathbf{x} + \mathbf{w} = G\mathbf{c} + \mathbf{w} \tag{11.2.5}$$

where $\mathbf{w}$ is an L-dimensional vector representing the noise samples.

11.2.2 Framework for transient signal detection

We use the framework for transient signal detection introduced in [FP89]. The fundamental assumption behind this framework is that the signal is sparse, in the sense that the vector of Gabor coefficients $\mathbf{c}$ contains only a few nonzero entries.

Under the sparseness assumption we can consider several cases, each corresponding to a different detection scheme. The simplest case is when the signal $\mathbf{x}$ is known. In this case the optimal detection test statistic is simply the matched filter $\mathbf{x}^*\mathbf{y}$. A more difficult case is when the location of the nonzero coefficients are known, but not their values. The next difficulty level occurs in the case where we assume that the number of nonzero coefficients are known, but neither their locations nor their values. Finally, the most difficult case is when not even the number of nonzero coefficients is known.

The detection problem we consider is the detection of a signal whose vector of Gabor coefficients contains a known number of nonzero entries, at known locations, but the values of the nonzero coefficients are unknown. In other words we want to decide between the following two hypothesis:

H_0) No signal present, $\mathbf{y} = \mathbf{w}$

H_1) The Gabor representation of the signal contains exactly $K < L$ nonzero components, whose locations are known, but their values are unknown.

Since the K nonzero components are assumed to have known locations, there is no loss of generality in taking them to be the first K components of $\mathbf{c}$, i.e,

$$\mathbf{c} = [\mathbf{c}_1^T, 0]^T \tag{11.2.6}$$

where $\mathbf{c}_1$ is a K-dimensional vector. Similarly we can reorder the columns of G such that $G = [G_1, G_2]$ and $G_1 \in \mathbb{R}^{L \times K}$ contains the columns of G corresponding to the nonzero components of $\mathbf{c}$. It follows that under the assumptions of the detection problem we can write the signal as

$$\mathbf{x} = G_1\mathbf{c}_1 \tag{11.2.7}$$

In this case the transient signal $x \in \mathbb{R}^N$ lies in a K-dimensional subspace of $\mathbb{R}^L$, which we denote by $\mathbf{G}_1$. It is spanned by the columns of the matrix G_1. Thus, the detection problem under consideration can be formulated as the following subspace-signal detection problem:

$$\mathbf{y} = \mu\mathbf{x} + \mathbf{w} \tag{11.2.8}$$

where $\mu = 0$ under H_0 and $\mu > 0$ under H_1, and the signal $\mathbf{x}$ obeys the linear subspace model (11.2.7). Assuming that the noise is a multivariate zero mean white complex circular normal with variance σ^2 we can write the detection problem as a test of distributions: $\mathbf{y}$ is distributed as $N[\mu G_1 \mathbf{c}_1, \sigma^2 I]$ where $\mu = 0$ under H_0 and $\mu > 0$ under H_1, and $\mathbf{c}_1$ is unknown.

The above detection problem is a problem of composite binary hypothesis testing. The performance of a detector (test) $\phi(\mathbf{y})$ is characterized by the probability of detection P_D for a given probability of false alarm P_{FA}. The probability of detection is often referred to as the power of the test denoted by β, while the false alarm probability is referred to as the size of test, denoted α. Denote by $\boldsymbol{\theta}$ the vector of unknown parameters, belonging to a parameter space Θ. Let $\Theta = \Theta_1 \cup \Theta_2$ be a disjoint covering of the parameter space. Under H_0 $\boldsymbol{\theta} \in \Theta_0$, while under H_1 $\boldsymbol{\theta} \in \Theta_1$. Then

$$\begin{aligned} P_{FA} &= \alpha = \sup_{\boldsymbol{\theta} \in \Theta_0} E_{\boldsymbol{\theta}} \phi(\mathbf{y}) = \sup P[t(\mathbf{y}) > \eta | H_0) \\ P_D &= \beta(\boldsymbol{\theta}) = E_{\boldsymbol{\theta}} \phi(\mathbf{y}) = P[t(\mathbf{y}) > \eta | H_1) \ \boldsymbol{\theta} \in \Theta_1 \end{aligned} \tag{11.2.9}$$

where $t(\mathbf{y})$ is the test statistic.

Ideally, it is desired to have a detector (test) that for a given size α, its power is uniformly greater than the power of any other test whose size is less or equal to α. Such a test is called a uniformly most powerful (UMP) test, and is formally defined by

Definition 11.2.1 *Let $\phi(\mathbf{y})$ be a test of size α, $\sup_{\boldsymbol{\theta} \in \Theta_0} E_{\boldsymbol{\theta}} \phi(\mathbf{y}) = \alpha$. Denote by $\phi'(\mathbf{y})$ a test whose size is less or equal to α, $\sup_{\boldsymbol{\theta} \in \Theta_0} E_{\boldsymbol{\theta}} \phi'(\mathbf{y}) \leq \alpha$. $\phi(\mathbf{y})$ is UMP if $E_{\boldsymbol{\theta}} \phi(\mathbf{y}) \geq E_{\boldsymbol{\theta}} \phi'(\mathbf{y})$, for all $\boldsymbol{\theta} \in \Theta_1$.*

It is straight forward to show, however, that there exists no UMP test for the detection problem above. Using the concept of invariant testing, it is possible to show that the Generalized Likelihood Ratio Test GLRT is UMP within a constrained class of tests [Sch91], [SF94]. The constrained class of tests is formed by requiring that all tests in this class will be invariant to a group of transformations $\mathcal{T}$. A hypothesis test $\phi(\mathbf{y})$ is said to be invariant to $\mathcal{T}$ if $\phi(T(\mathbf{y})) = \phi(\mathbf{y})$ for all transformations $T \in \mathcal{T}$. An invariant test may be constructed via a maximal invariant statistic, which is defined below.

Definition 11.2.2 *A statistic $M(\mathbf{y})$ is said to be a maximal $\mathcal{T}$-invariant statistic if $M[T(\mathbf{y})] = M[\mathbf{y}]$ for all $T \in \mathcal{T}$, and $M[\mathbf{y}_1] = M[\mathbf{y}_2]$ if and only if there exists $T \in \mathcal{T}$ such that $\mathbf{y}_2 = T[\mathbf{y}_1]$.*

Every invariant test may be written as a function of a maximal invariant statistic, $\phi(\mathbf{y}) = \phi[M(\mathbf{y})]$. If the distribution of a maximal $\mathcal{T}$-invariant statistic is monotone in a single scalar parameter then, according to the Karlin-Rubin Theorem [Sch91], the test is UMP with respect to all the tests that are invariant to $\mathcal{T}$. In other words, the test is UMP $\mathcal{T}$-invariant.

Using the above ideas and the results in [Sch91] and [SF94] it can be shown that the GLRT for the above detection problem is UMP invariant within a certain class of tests. Following [Sch91] and [SF94] we will argue that all acceptable tests should belong to that constrained class and therefore the GLRT is optimal. In this chapter we focus therefore on the GLRT for the detection problem above.

An alternative approach for this detection problem was considered in [FP89] and [PF92] where a data reduction transformation is applied to the measurement vector, and the detection test is applied to the transformed data. In the following section we review the main results of [FP89] and [PF92].

11.3 Detection in the transform domain

Many detection techniques have the following two step structure. In the first step the signal is sampled and collected in relatively long batches. Then a transformation, typically a linear one, is applied to the data vector resulting in a vector of smaller dimension. In the second step a GLRT or other test statistic is applied to the transformed data.

The discrete Gabor representation of signals is obtained by a linear transformation of the data vector. However, for the critical sampling and oversampling cases, this transformation does not reduce the dimension of the data. The dimension of the data vector remains unchanged in the critical sampled representation, and increases by the oversampling ratio in the oversampled representation. Data reduction is possible therefore only in the case where it is known that the signal can be represented by subset of the Gabor basis functions whose dimension is smaller than the dimension of the original data vector L. In this case we can model the measured signal by,

$$\mathbf{y} = \mathbf{x} + \mathbf{w} = \tilde{G}\tilde{\mathbf{c}} + \mathbf{w} \tag{11.3.1}$$

where $\tilde{G} \in \mathbb{R}^{L \times J}$ is constructed by selecting J columns of G.

In this case an obvious choice of the transformation R is a left inverse of $\tilde{G}$, i.e., a matrix that satisfies $R\tilde{G} = I$. Since $L > J$, the matrix $\tilde{G}$ will have an infinite number of left inverses. A special left inverse is the pseudo inverse given by

$$\tilde{G}^{+} = (\tilde{G}^{*}\tilde{G})^{-1}\tilde{G}^{*} \tag{11.3.2}$$

Under the assumptions of this chapter the measurement model under H_1 is

$$\mathbf{y} = G_1\mathbf{c}_1 + \mathbf{w} \tag{11.3.3}$$

Let $\tilde{G}$ and R be partitioned compatibly with $\tilde{\mathbf{c}}$, i.e.,

$$\begin{aligned} \tilde{G} &= [G_1, G_2] \\ R &= \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \end{aligned} \tag{11.3.4}$$

The transformed data can be represented by

$$\mathbf{z} = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} = \begin{bmatrix} R_1 G_1 \tilde{\mathbf{c}}_1 \\ R_2 G_1 \tilde{\mathbf{c}}_1 \end{bmatrix} + \mathbf{w} \tag{11.3.5}$$

When R is a left inverse of $\tilde{G}$ Eq. (11.3.5) becomes,

$$\mathbf{z} = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 \\ 0 \end{bmatrix} + \mathbf{w} \tag{11.3.6}$$

The transform-domain GLRT derived in [PF92] assumes that R is indeed a left inverse of $\tilde{G}$. Extending the results in [PF92] to the complex case it can be shown that the transform-domain GLRT statistic is given by,

$$t_{td} = \frac{2}{\sigma^2}[\mathbf{z}^*(RR^*)^{-1}\mathbf{z} - \mathbf{z}_2^*(R_2R_2^*)^{-1}\mathbf{z}_2] \tag{11.3.7}$$

and that it has chi-square distribution with $2K$ degrees of freedom. Under H_0 the distribution is central, while under H_1 it is noncentral with noncentrality parameter ν.

$$\nu = \frac{2}{\sigma^2}(G_1\mathbf{c}_1)^* Q (G_1\mathbf{c}_1) \tag{11.3.8}$$

where

$$Q = P_{R_2}^{\perp} R_1^* (R_1 P_{R_2}^{\perp} R_1^*)^{-1} R_1 P_{R_2}^{\perp} \tag{11.3.9}$$

and

$$P_{R_2}^{\perp} = I - R_2^*(R_2R_2^*)^{-1}R_2 \tag{11.3.10}$$

As indicated in [PF92] $P_{R_2}^{\perp}$ is the projection operator on the orthogonal complement of the column space of R_2^*, and Q is the projection operator on the column space of $P_{R_2}^{\perp} R_1^*$. Thus the noncentrality ν is proportional to the norm of the projection of the signal on the column space of $P_{R_2}^{\perp} R_1^*$.

In the special case where R is a left inverse of $\tilde{G}$ we have,

$$\nu = \frac{2}{\sigma^2}(\mathbf{c}_1)^*(R_1 P_{R_2}^{\perp} R_1^*)^{-1}\mathbf{c}_1 \tag{11.3.11}$$

Finally, if R is the pseudo inverse of $\tilde{G}$, then

$$\nu = \nu_0 = \frac{2}{\sigma^2}(\mathbf{c}_1)^*(G_1^* G_1)\mathbf{c}_1 \tag{11.3.12}$$

It is shown in [PF92] that $\nu_0 \geq \nu$. This inequality means that the noncentrality of the transform domain GLRT based on the pseudo inverse is not smaller than that of a transform domain GLRT based on any other left inverse. For a given probability of false alarm (given size), the detection probability (power) is a monotonically increasing function of the noncentrality parameter. The transform-domain GLRT based on the pseudo inverse is therefore uniformly more powerful than a transform domain GLRT based on any other left inverse.

11.4 Detection in the data domain

The detection problem of (11.2.7)–(11.2.8) may be described as the problem of detecting a subspace signal in noise. The term *subspace signal* means that the signal obeys a linear subspace model in the form of (11.2.7). This problem was studied in [Sch91] and [SF94] for the real signal case, where the authors developed a theory of matched subspace detectors based on the construction of invariant statistics. In this section we extend the results in [Sch91] and [SF94] for the case of complex signals. We assume that the noise $\mathbf{w}$ is a multivariate zero mean white complex circular normal with variance σ^2. We consider the following two situations: known noise variance and unknown noise variance. The theory of transform domain detection was developed only for the case of known noise level. For this case we show that the transform domain GLRT based on the pseudo inverse is equivalent to the data domain matched subspace detector.

As discussed earlier, there exists no UMP test for the detection problem above. However, using the concept of invariant testing, it is possible to show that the Generalized Likelihood Ratio Test (GLRT) is UMP within a constrained class of tests. We will argue that all acceptable tests should belong to that constrained class and therefore the GLRT is optimal. We start by deriving the GLRT.

The probability density of a complex vector $\mathbf{z}$ is given by [Pic94]

$$p(\mathbf{z}) = p(\mathbf{z}_r, \mathbf{z}_i) = \pi^{-L}[\det(R_z)]^{-1} \exp\{-(\mathbf{z} - \mathbf{m}_z)^*(R_z)^{-1}(\mathbf{z} - \mathbf{m}_z)\} \tag{11.4.1}$$

where $\mathbf{z} = \mathbf{z}_r + j\mathbf{z}_i$, $\mathbf{m}_z = E\{\mathbf{z}\}$, $R_z = E\{\mathbf{z}\mathbf{z}^*\}$, and $E\{\mathbf{z}\mathbf{z}^T\} = 0$. Under the assumptions of this chapter, the probability density of the data vector $\mathbf{y}$ is given by,

$$p(\mathbf{y}|\sigma^2, \mu\mathbf{c}_1) = (\pi\sigma^2)^{-L} \exp\{-\frac{\mathbf{n}^*\mathbf{n}}{\sigma^2}\} \tag{11.4.2}$$

where

$$\mathbf{n} = \mathbf{y} - \mu G_1 \mathbf{c}_1 \tag{11.4.3}$$

In this case the likelihood function is

$$l(\sigma^2, \mathbf{c}_1; \mathbf{y}) = (\pi\sigma^2)^{-L} \exp\{-\frac{\mathbf{n}^*\mathbf{n}}{\sigma^2}\} \tag{11.4.4}$$

where the parameter μ was absorbed in $\mathbf{c}_1$.

The generalized likelihood ratio (GLR) is defined as

$$\hat{l}(\mathbf{y}) = \frac{l(\hat{\boldsymbol{\theta}}_1; \mathbf{y})}{l(\hat{\boldsymbol{\theta}}_0; \mathbf{y})} \tag{11.4.5}$$

where $\boldsymbol{\theta}_i$ is the vector of unknown parameters and $\hat{\boldsymbol{\theta}}_i$ is its Maximum Likelihood (ML) estimate. The GLR for our problem is therefore

$$\hat{l}(\mathbf{y}) = \frac{\hat{\sigma}_1^2}{\hat{\sigma}_0^2} \exp\{-\frac{1}{\hat{\sigma}_1^2}\hat{\mathbf{n}}_1^*\hat{\mathbf{n}}_1 + \frac{1}{\hat{\sigma}_0^2}\hat{\mathbf{n}}_0^*\hat{\mathbf{n}}_0\} \tag{11.4.6}$$

where $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_0$ are the ML estimates

$$\begin{aligned} \hat{\mathbf{n}}_1 &= \mathbf{y} - G_1\hat{\mathbf{c}}_1 \\ \hat{\mathbf{n}}_0 &= \mathbf{y} \end{aligned} \tag{11.4.7}$$

and $\hat{\mathbf{c}}_1$ is the ML estimate of $\mathbf{c}_1$.

When σ^2 is known, the GLR can be simplified as

$$\hat{l}(\mathbf{y}) = \exp\{\frac{1}{\sigma^2}[\hat{\mathbf{n}}_0^*\hat{\mathbf{n}}_0 - \hat{\mathbf{n}}_1^*\hat{\mathbf{n}}_1]\} \tag{11.4.8}$$

In this case it is convenient to use as the test statistic the logarithmic GLR

$$t_1(\mathbf{y}) = 2\ln\hat{l}(\mathbf{y}) = \frac{2}{\sigma^2}[\hat{\mathbf{n}}_0^*\hat{\mathbf{n}}_0 - \hat{\mathbf{n}}_1^*\hat{\mathbf{n}}_1] \tag{11.4.9}$$

The rationale behind this choice of test statistic will become apparent later.

When σ^2 is unknown we use the ML estimates

$$\hat{\sigma}_i^2 = \frac{1}{L}\hat{\mathbf{n}}_i^*\hat{\mathbf{n}}_i \; i = 0, 1 \tag{11.4.10}$$

and Eq. (11.4.6) becomes,

$$\hat{l}(\mathbf{y}) = \left(\frac{\hat{\mathbf{n}}_0^*\hat{\mathbf{n}}_0}{\hat{\mathbf{n}}_1^*\hat{\mathbf{n}}_1}\right)^L \tag{11.4.11}$$

Then it is convenient to use as the test statistic the L-root GLR,

$$t_2(\mathbf{y}) = \frac{\hat{\mathbf{n}}_0^*\hat{\mathbf{n}}_0}{\hat{\mathbf{n}}_1^*\hat{\mathbf{n}}_1} \tag{11.4.12}$$

The generalized likelihood test (GLRT) is given by,

$$\phi(\mathbf{y}) = \begin{cases} 1 & t(\mathbf{y}) > \eta \\ 0 & t(\mathbf{y}) < \eta \end{cases} \tag{11.4.13}$$

where 1 stands for H_1, 0 stands for H_0, and $t(\mathbf{y})$ is $t_1(\mathbf{y})$ or $t_2(\mathbf{y})$, depending on whether or not σ^2 is known.

In the following part of this section we study in detail the GLRT above. First we consider the case of known noise level, and then the case where the noise level is unknown.

11.4.1 Known noise level

In this case we use the test statistic t_1 given by Eq. (11.4.9). It is straightforward to show that the ML estimates of n_0 and n_1 are given by

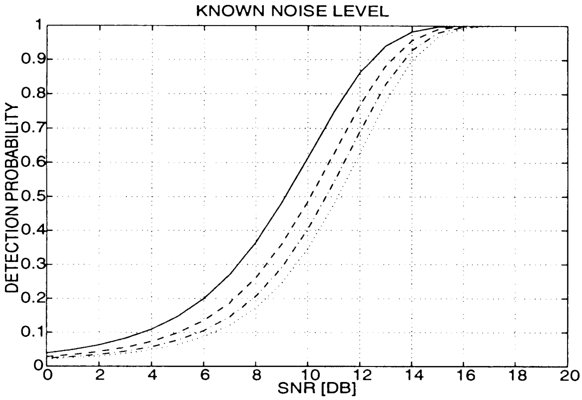

FIGURE 11.4.1. ROC curves for subspace signal in noise of known level. $k = 1$ (solid line), $k = 2$ (dashed line), $k = 3$ (dash-dot line), $k = 4$ (dotted line).

$$\begin{aligned} \hat{\mathbf{n}}_1 &= P_{G_1}^{\perp}\mathbf{y} \\ \hat{\mathbf{n}}_0 &= \mathbf{y} \end{aligned} \tag{11.4.14}$$

where

$$P_{G_1} = G_1(G_1^* G_1)^{-1} G_1^* \tag{11.4.15}$$

is the projection operator on the column space of G_1, and $P_{G_1}^{\perp} = I - P_{G_1}$ is projection operator on its orthogonal complement. It follows that

$$\begin{aligned} t_1(\mathbf{y}) &= \frac{2}{\sigma^2}[\hat{\mathbf{n}}_0^* \hat{\mathbf{n}}_0 - \hat{\mathbf{n}}_1^* \hat{\mathbf{n}}_1] \\ &= \frac{2}{\sigma^2}\mathbf{y}^* P_{G_1}\mathbf{y} \end{aligned} \tag{11.4.16}$$

To proceed we need the following lemma which extends a well known lemma [SS89] to the case of complex signals with non-zero mean.

Lemma 11.4.1 *Let* $\mathbf{y}$ *be an* L *dimensional white complex circular normal vector with mean* $\mathbf{m}_y$ *and variance* σ^2*, and let* P *be an* L*-by-*L *projection matrix of rank* K*. Then* $(2/\sigma^2)\mathbf{y}^* P\mathbf{y}$ *has a chi-square distribution with* $2K$ *degrees of freedom and noncentrality parameter* $\lambda^2 = (2/\sigma^2)[\mathbf{m}_y^* P\mathbf{m}_y]$.

Proof: Since P is Hermitian we can write it as $P = U^*\Lambda U$ where U is unitary with $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_L)$, a diagonal matrix formed from the eigenvalues of P [HJ85]. Since P is idempotent, all its eigenvalues are either zero or one. Therefore Λ has the form

$$\Lambda = \begin{pmatrix} I_K & 0 \\ 0 & 0 \end{pmatrix} \tag{11.4.17}$$

Let

$$\mathbf{v} = \frac{\sqrt{2}}{\sigma} U\mathbf{y} = \boldsymbol{\xi} + j\boldsymbol{\rho} \tag{11.4.18}$$

where $\boldsymbol{\xi} = \mathrm{Re}\{\mathbf{v}\}$ and $\rho = \mathrm{Im}\{\mathbf{v}\}$. Note that $\boldsymbol{\xi}$ and $\boldsymbol{\rho}$ are normal random variables with

$$\begin{aligned} E[(\boldsymbol{\xi} - E(\boldsymbol{\xi}))(\boldsymbol{\xi} - E(\boldsymbol{\xi}))^T] = I \\ E[(\boldsymbol{\rho} - E(\boldsymbol{\rho}))(\boldsymbol{\rho} - E(\boldsymbol{\rho}))^T] = I \end{aligned} \tag{11.4.19}$$

Furthermore,

$$\mathbf{v}^*\Lambda\mathbf{v} = \boldsymbol{\xi}^T\Lambda\boldsymbol{\xi} + \boldsymbol{\rho}^T\Lambda\boldsymbol{\rho} \tag{11.4.20}$$

It follows that

$$\begin{aligned} \frac{2}{\sigma^2}\mathbf{y}^*P\mathbf{y} &= \frac{2}{\sigma^2}\mathbf{y}^*U^*\Lambda U\mathbf{y} \\ &= \mathbf{v}^*\Lambda\mathbf{v} \\ &= \boldsymbol{\xi}^T\Lambda\boldsymbol{\xi} + \boldsymbol{\rho}^T\Lambda\boldsymbol{\rho} \\ &= \sum_{k=1}^{K}[\xi_k^2 + \rho_k^2] \end{aligned} \tag{11.4.21}$$

$\frac{2}{\sigma^2}\mathbf{y}^*P\mathbf{y}$ is a sum of $2K$ unity variance uncorrelated real normal random variables. Hence it has a noncentral chi-square distribution with $2K$ degrees of freedom and noncentrality parameter ζ^2, where

$$\begin{aligned} \zeta^2 &= \sum_{k=1}^{K}[E^2(\xi_k) + E^2(\rho_k)] \\ &= E[\boldsymbol{\xi}]^T\Lambda E[\boldsymbol{\xi}] + E[\boldsymbol{\rho}]^T\Lambda E[\boldsymbol{\rho}] \\ &= E[\mathbf{v}]^*\Lambda E[\mathbf{v}] \\ &= \frac{2}{\sigma^2}E[\mathbf{y}]^*(U^*\Lambda U)E[\mathbf{y}] \\ &= \frac{2}{\sigma^2}E[\mathbf{y}]^*PE[\mathbf{y}] \end{aligned} \tag{11.4.22}$$

❑

It follows from the lemma that $t_1(\mathbf{y})$ has chi-square distribution with $2K$ degrees of freedom and noncentrality parameter λ^2, $t_1(\mathbf{y}) \sim \chi^2_{2K}(\lambda^2)$ where

$$\begin{aligned} \lambda^2 &= \frac{2}{\sigma^2}\mathbf{c}_1^* G_1^* P_{G_1} G_1 \mathbf{c}_1 \\ &= \frac{2}{\sigma^2}\mathbf{c}_1^* G_1^* G_1 \mathbf{c}_1 \end{aligned} \tag{11.4.23}$$

The statistic $t_1(\mathbf{y})$ is invariant to transformations $T \in \mathcal{T}_1$ that rotate $\mathbf{y}$ within $\mathbf{G}_1$ and add a bias component in $\mathbf{G}_1^{\perp}$. It can be shown that if $t_1(\mathbf{y}_1) = t_1(\mathbf{y}_2)$, then there exists a transformation $T \in \mathcal{T}_1$ such that $\mathbf{y}_2 = T(\mathbf{y}_1)$. The statistic t_1 is therefore a maximal $\mathcal{T}_1$-invariant statistic, and every $\mathcal{T}_1$-invariant test of H_0 versus H_1 must be a function of it. Since the distribution of the statistic $t_1(\mathbf{y})$ is monotone in the noncentrality parameter λ^2, the test

$$\phi(\mathbf{y}) = \begin{cases} 1 & t_1(\mathbf{y}) > \eta \\ 0 & t_1(\mathbf{y}) \le \eta \end{cases} \tag{11.4.24}$$

is UMP $\mathcal{T}_1$-invariant for testing H_0 versus H_1, meaning that it is uniformly more powerful than all other tests that are invariant to $\mathcal{T}_1$. We argue that all acceptable detectors must be invariant to $\mathcal{T}_1$. As $\mathbf{c}_1$ is unknown and unconstrained, the signal to be detected can lie anywhere in $\mathbf{G}_1$. No signal of given energy in $\mathbf{G}_1$ should be any more detectable than any other, so the test statistic should be invariant to rotations in $\mathbf{G}_1$. Any detector should be invariant to measurement components orthogonal to $\mathbf{G}_1$. We conclude therefore that the GLRT (11.4.24) is optimal, in the sense that it cannot be improved upon by any detector which shares these natural invariances. Following [Sch91] and [SF94] we call this detector it matched subspace detector. This name reflects the fact that the test statistics is proportional to the energy of $P_{G_1}\mathbf{y}$ which is matched to the subspace $\mathbf{G}_1$.

Comparing the distribution of the statistic $t_1(\mathbf{y})$ to that of the statistic of the transform-domain GLRT based on the pseudo inverse, we observe that the two statistics have exactly the same distribution. This is not surprising, since this two tests are equivalent, as shown in the following. To show this equivalence we first prove the following lemma.

Lemma 11.4.2 *if R is the pseudo inverse of $\tilde{G}$, i.e if $R = \tilde{G}^+$, then*

$$R^*(RR^*)^{-1}R - R_2^*(R_2R_2^*)^{-1}R_2 = P_{G_1} \tag{11.4.25}$$

Proof: We start by expressing R in terms of G_1 and G_2. Recall that

$$\begin{aligned} R &= [\tilde{G}^*\tilde{G}]^{-1}\tilde{G} \\ &= \begin{bmatrix} G_1^*G_1 & G_1^*G_2 \\ G_2^*G_1 & G_2^*G_2 \end{bmatrix}^{-1} \begin{bmatrix} G_1^* & G_2^* \end{bmatrix} \end{aligned} \tag{11.4.26}$$

Using the well known expression for the inverse of partitioned matrix we have,

$$\begin{aligned} &[\tilde{G}^*\tilde{G}]^{-1} \\ &= \begin{bmatrix} (G_1^*G_1)^{-1} + G_1^+G_2AG_2^*(G_1^+)^* & -G_1^+G_2A \\ -AG_2^*(G_1^+)^* & A \end{bmatrix} \end{aligned} \tag{11.4.27}$$

where $G_1^+ = [G_1^*G_1]^{-1}G_1^*$ is the pseudo inverse of G_1, and

$$\begin{aligned} P_{G_1}^{\perp} &= I - P_{G_1} = I - G_1(G_1^*G_1)^{-1}G_1^* \\ A &= (G_2^*P_{G_1}^{\perp}G_2)^{-1} \end{aligned} \tag{11.4.28}$$

$P_{G_1}^{\perp}$ is the projection operator on the orthogonal complement of the column space of G_1.

It follows that

$$\begin{aligned} R &= \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \\ &= \begin{bmatrix} G_1^+ - G_1^+G_2(G_2^*P_{G_1}^{\perp}G_2)^{-1}G_2^*P_{G_1}^{\perp} \\ (G_2^*P_{G_1}^{\perp}G_2)^{-1}G_2^*P_{G_1}^{\perp} \end{bmatrix} \end{aligned} \tag{11.4.29}$$

Next we derive an expression for $R^*(RR^*)^{-1}R$.

$$\begin{aligned} R^*(RR^*)^{-1}R &= \tilde{G}[\tilde{G}^*\tilde{G}]^{-1}\tilde{G}^* \\ &= [G_1G_2]\begin{bmatrix} G_1^*G_1 & G_1^*G_2 \\ G_2^*G_1 & G_2^*G_2 \end{bmatrix}^{-1}\begin{bmatrix} G_1^* \\ G_2^* \end{bmatrix} \end{aligned} \tag{11.4.30}$$

Substituting (11.4.27) into (11.4.30) we get

$$R^*(RR^*)^{-1}R = P_{G_1} + P_{G_1}^{\perp}G_2(G_2^*P_{G_1}^{\perp}G_2)^{-1}G_2^*P_{G_1}^{\perp} \tag{11.4.31}$$

Finally we derive an expression for $R_2^*(R_2R_2^*)^{-1}R_2$. Applying (11.4.29) it is straight-forward to show that

$$(R_2 R_2^*)^{-1} = G_2^* P_{G_1}^{\perp} G_2 \tag{11.4.32}$$

and

$$R_2^*(R_2 R_2^*)^{-1} R_2 = P_{G_1}^{\perp} G_2 (G_2^* P_{G_1}^{\perp} G_2)^{-1} G_2^* P_{G_1}^{\perp} \tag{11.4.33}$$

Combining (11.4.31) and (11.4.33) we get

$$R^*(RR^*)^{-1} R - R_2^*(R_2 R_2^*)^{-1} R_2 = P_{G_1} \tag{11.4.34}$$

❑

Next consider the transform domain GLRT statistic given by Eq. (11.3.7). Substituting $\mathbf{z} = R\mathbf{y}$ we get,

$$t_{td} = \frac{2}{\sigma^2}\mathbf{y}^*[R^*(RR^*)^{-1} R - R_2^*(R_2 R_2^*)^{-1} R_2]\mathbf{y} \tag{11.4.35}$$

According to the lemma

$$R^*(RR^*)^{-1} R - R_2^*(R_2 R_2^*)^{-1} R_2 = P_{G_1}$$

Therefore,

$$t_{td} = \frac{2}{\sigma^2}\mathbf{y}^* P_{G_1}\mathbf{y} \tag{11.4.36}$$

which is equivalent to the matched subspace test statistic t_1 in Eq. (11.4.16).

The performance of the GLRT detector of (11.4.24) is determined by the false alarm and detection probabilities,

$$\begin{aligned} P_{FA} &= P[\chi_{2K}^2(0) > \eta] \\ P_D &= P[\chi_{2K}^2(\lambda^2) > \eta] \end{aligned} \tag{11.4.37}$$

The receiver operating characteristics (ROC) curves for this detector are given in Figure 11.4.1. The false alarm probability is fixed at $P_{FA} = 0.01$ and the dimension of the subspace $\mathbf{G}_1$ is varied from $k = 1$ to $k = 4$.

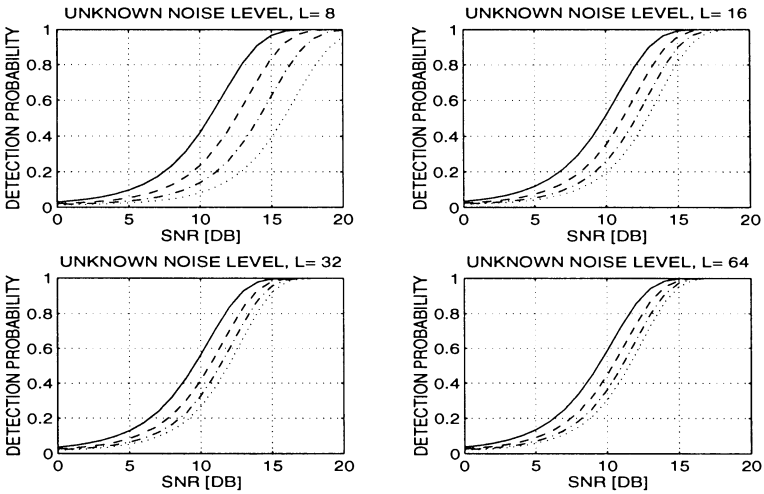

FIGURE 11.4.2. ROC curves for subspace signal in noise of unknown level. $k = 1$ (solid line), $k = 2$ (dashed line), $k = 3$ (dash-dot line), $k = 4$ (dotted line).

11.4.2 Unknown noise level

In the case of unknown noise level we use the test statistic t_2 given by Eq. (11.4.12). The ML estimates of n_0 and n_1 are the same as for the case of known noise level and are given by Eq. (11.4.14). Then $t_2(\mathbf{y})$ becomes,

$$t_2(\mathbf{y}) = \frac{\mathbf{y}^*\mathbf{y}}{\mathbf{y}^* P_{G_1}^{\perp} \mathbf{y}} \tag{11.4.38}$$

It is convenient to replace $t_2(\mathbf{y})$ by $t_2'(\mathbf{y}) = \frac{2(L-K)}{2K}(t_2(\mathbf{y}) - 1)$ which is a monotone function of the original $t_2(\mathbf{y})$. The choice of this new statistic will become apparent shortly. It is given by,

$$t_2'(\mathbf{y}) = \frac{2K - L}{2K} \frac{(2/\sigma^2)\mathbf{y}^* P_{G_1} \mathbf{y}}{(2/\sigma^2)\mathbf{y}^* P_{G_1}^{\perp} \mathbf{y}} \tag{11.4.39}$$

Consider the quadratic forms $(2/\sigma^2)\mathbf{y}^* P_{G_1}\mathbf{y}$ and $(2/\sigma^2)\mathbf{y}^* P_{G_1}^{\perp}\mathbf{y}$. According to Lemma 1

$$\begin{aligned} (2/\sigma^2)\mathbf{y}^* P_{G_1}\mathbf{y} &\sim \chi^2_{2K}(\lambda^2)\ H_1 \\ (2/\sigma^2)\mathbf{y}^* P_{G_1}^{\perp}\mathbf{y} &\sim \chi^2_{2K}(0)\ H_0 \end{aligned} \tag{11.4.40}$$

where

$$\lambda^2 = \frac{2}{\sigma^2}\mathbf{c}_1^* G_1 G_1 \mathbf{c}_1 \tag{11.4.41}$$

Furthermore, these quadratic forms may be regarded as the squared norms of the random vectors $(\sqrt{2}/\sigma)P_{G_1}\mathbf{y}$ and $(\sqrt{2}/\sigma)P_{G_1}^{\perp}\mathbf{y}$. Since these two normal random vectors are uncorrelated, they are independent. It follows that the quadratic forms above are independent chi-square random variables. Thus $t_2'(\mathbf{y})$ has an F distribution with $2K$ and $2(L-K)$ degrees of freedom and noncentrality parameter λ^2.

The statistic $t_2'(\mathbf{y})$ is invariant to transformations $T \in \mathcal{T}_2$ that rotate $\mathbf{y}$ within $\mathbf{G}_1$ and non-negatively scale $\mathbf{y}$. It can be shown that if $t_2'(\mathbf{y}_1) = t_2'(\mathbf{y}_2)$, then there exists a transformation $T \in \mathcal{T}_2$ such that $\mathbf{y}_2 = T(\mathbf{y}_1)$. The statistic t_1 is therefore a maximal $\mathcal{T}_2$-invariant statistic, and every $\mathcal{T}_2$-invariant test of H_0 versus H_1 must be a function of it.

Since the F distribution of the statistic $t_2'(\mathbf{y})$ is monotone in the noncentrality parameter λ^2, the test

$$\phi(\mathbf{y}) = \begin{cases} 1 & t_2'(\mathbf{y}) > \eta \\ 0 & t_2'(\mathbf{y}) \leq \eta \end{cases} \tag{11.4.42}$$

is UMP $\mathcal{T}_2$-invariant for testing H_0 versus H_1, meaning that it is uniformly more powerful than all tests that are invariant to $\mathcal{T}_2$. Again, we argue that all acceptable detectors for this problem must be invariant to $\mathcal{T}_2$. The test statistic should be invariant to rotations in $\mathbf{G}_1$ from the same reasons given for the case of known noise level. Any detector should be invariant to scalings that introduce unknown variances. We conclude therefore that the GLRT (11.4.42) is optimal, in the sense that it cannot be improved upon by any detector which shares these natural invariances. Following [Sch91] we will call this detector a *CFAR matched subspace detector*. This name is motivated by the fact that the detector has a constant false alarm rate independent of σ^2.

The ROC curves for this detector are given in Figure 11.4.2. The false alarm probability is fixed at $P_{FA} = 0.01$. Each part of the figure corresponds to a different length of the measurement vector varying from $L = 8$ to $L = 64$ in powers of 2. In each part, the dimension of the subspace $\mathbf{G}_1$ is varied from $k = 1$ to $k = 4$. As the length of the measurement vector increases, the ROC curves for the case of unknown noise level approach those for the case of known noise level, as dictated by the relation between the F and chi-square distributions.

11.5 Sensitivity to mismatch

In this section we analyze the sensitivity of the of the data domain detectors to mismatch in the model parameters. The sensitivity to mismatch of transform domain detectors was investigated in [FP89], [PF92] and [LS95]. We will consider two types of mismatch: mismatch in the shape of the synthesis function g and mismatch due to errors in the locations of the nonzero Gabor coefficients.

A mismatch occurs when there is a discrepancy between the assumed model and the received signal. Suppose that the assumed model is

$$\mathbf{y} = \mathbf{x}_0 + \mathbf{w} = G_1\mathbf{c}_1 + \mathbf{w} \tag{11.5.1}$$

while the received signal is given by

$$\mathbf{y} = \mathbf{x} + \mathbf{w} = F_1\mathbf{d}_1 + \mathbf{w} \tag{11.5.2}$$

where $F_1 \in \mathbb{R}^{L\times K}$ has the same structure as G_1, but the columns of F_1 are given by $\{f_{m,n}\}$ rather then $\{g_{m,n}\}$.

The detector for the case of known noise level is given by (11.4.24), while the detector for the case of unknown noise level is given by (11.4.42). The test statistic t_1 and t_2' are given by (11.4.16) and (11.4.39) respectively. Using the arguments of Section 11.4 it can be shown that for the above kind of mismatch $t_1(\mathbf{y})$ has chi-square distribution with $2K$ degrees of freedom and noncentrality parameter λ_m^2 where

$$\begin{aligned}\lambda_m^2 &= \frac{2}{\sigma^2}\mathbf{d}_1^* F_1^* P_{G_1} F_1 \mathbf{d}_1 \\ &= \frac{2}{\sigma^2}\mathbf{x}^* P_{G_1}\mathbf{x}\end{aligned} \tag{11.5.3}$$

Similarly, $t_2'(\mathbf{y})$ has an F distribution with $2K$ and $2(L-K)$ degrees of freedom and noncentrality parameter λ_m^2 given by Eq. (11.5.3). In the ideal (no mismatch) case, the noncentrality parameter is the signal to noise ratio (SNR). Thus, this type of mismatch causes an effective loss in SNR. ΔSNR below expresses the SNR loss in dB.

$$\begin{aligned}\Delta\mathrm{SNR} &= 10\log\frac{\mathbf{c}_1^* F_1^* F_1 \mathbf{c}_1}{\mathbf{c}_1^* F_1^* P_{G_1} F_1 \mathbf{c}_1} \\ &= 10\log\frac{\mathbf{x}^*\mathbf{x}}{\mathbf{x}^* P_{G_1}\mathbf{x}}\end{aligned} \tag{11.5.4}$$

Consider for example the case where the rank of G_1 (and F_1) is one. Then there is only one nonzero Gabor coefficient c_{m_0,n_0}. The detection problem in this case is detecting a known signal (with unknown amplitude and phase) in noise. In the case of known noise level the test statistic is simply the matched filter statistic. It has a chi-square distribution with two degrees of freedom for both the ideal and mismatched cases. The noncentrality parameter for the ideal case is

$$\begin{aligned} \lambda^2 &= \frac{2|c_{m_0,n_0}|^2}{\sigma^2} \sum_{l=0}^{L-1} |f_{m_0,n_0}(l)|^2 \\ &= \frac{2|c_{m_0,n_0}|^2}{\sigma^2} E_f \end{aligned} \tag{11.5.5}$$

where $E_f = \sum_{l=0}^{L-1} |f_{m_0,n_0}(l)|^2$. The noncentrality parameter for the mismatched case is given by,

$$\lambda_m^2 = \frac{2|c_{m_0,n_0}|^2}{\sigma^2} E_f |R_{f,g}|^2 \tag{11.5.6}$$

where

$$R_{f,g} = \frac{\sum_{l=0}^{L-1} f_{m_0,n_0} \bar{g}_{m_0,n_0}}{(E_f E_g)^{1/2}} \leq 1 \tag{11.5.7}$$

In the case where the noise level is unknown, the test statistic is the CFAR version of the matched filter statistic. It has an F-distribution with 2 and $2(L-1)$ degrees of freedom. The noncentrality parameters for the ideal and the mismatched case are given by (11.5.5) and (11.5.6) respectively. For both cases of known and unknown noise level the SNR loss in dB is given by,

$$\Delta\text{SNR} = 10 \log \left[\frac{1}{|R_{f,g}|^2} \right] = -20 \log |R_{f,g}| \tag{11.5.8}$$

which is a well known result for the (mismatched) matched filter.

11.5.1 *Mismatch in the shape of the synthesis function*

In this section we apply our results to the case of mismatch in the shape of the synthesis window $g(k)$. We use a Gabor transient of length $L = 64$, and an oversampled Gabor representation with $M = 16$ time intervals and

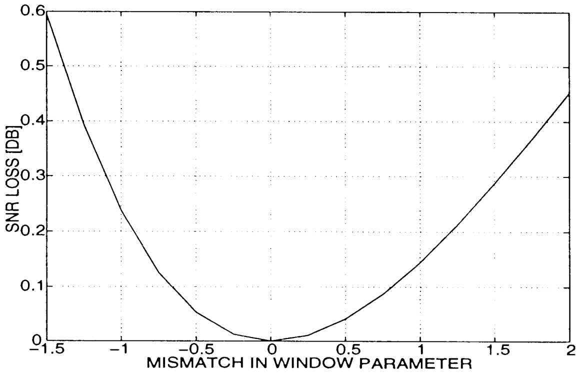

FIGURE 11.5.1. SNR loss as a function of the mismatch in the window shape parameter.

$N = 16$ frequency intervals. In this case the oversampling factor is 4, the time step is $a = 4$ and the frequency step is $b = 4$. We use the one-sided exponential $g_\alpha(k)$ as the synthesis window g,

$$g_\alpha(k) = e^{-\alpha k/L} \quad k = 0, \cdots, L-1 \tag{11.5.9}$$

and assume that the Gabor transient has two nonzero coefficients: $c_{8,0} = c_{2,3} = 1$. We model the mismatch in the window shape by a mismatch in the window parameter α. Thus, the Gabor transient is given by

$$x(k) = g(k-32) + g(k-8)e^{j2\pi 12k/64} \tag{11.5.10}$$

where in the assumed model $g = g_{\alpha_0}$ and in reality $g = g_\alpha$.

In the example of Figure 11.5.1 $\alpha_0 = 2$ and we vary α in the range $[0.5, 4]$. We plot the SNR loss in dB as a function of the mismatch in α. For these values of mismatch, the SNR loss is less or equal to 0.6 dB, indicating that the matched sub-space detectors are quite tolerant to mismatch in the synthesis window parameter. Similar observations were made for transform-domain GLRT detector [FP89].

11.5.2 Mismatch due to errors in the locations of the Gabor coefficients

Suppose that there are errors in the assumed locations of some of the nonzero Gabor coefficients. Partition and reorder G_1 and F_1 in the following way

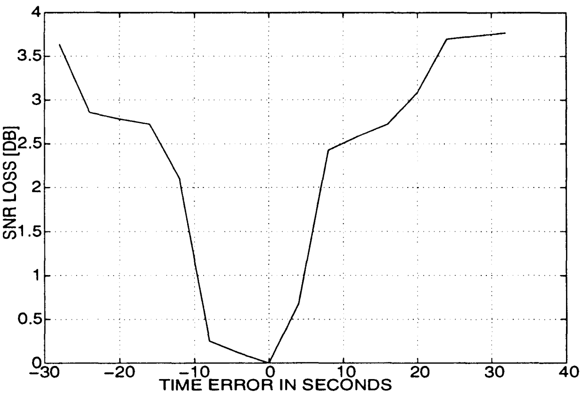

FIGURE 11.5.2. SNR loss as a function of the error in the transient time.

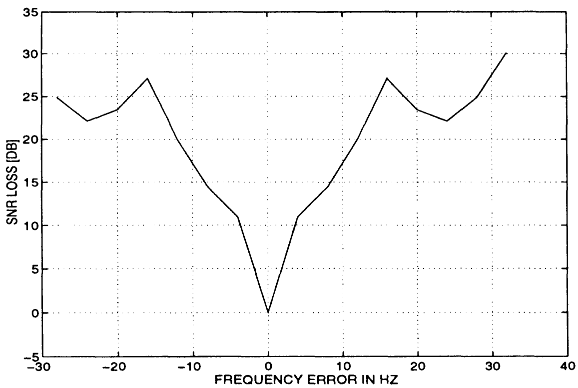

FIGURE 11.5.3. SNR loss as a function of the error in the transient frequency.

$$\begin{aligned} G_1 &= [H, S] \\ F_1 &= [H, T] \end{aligned} \tag{11.5.11}$$

where H corresponds the Gabor coefficients whose assumed locations are equal to their true locations, while S and T correspond to the Gabor coefficients whose true locations differ from their assumed locations. Partition $\mathbf{c}_1$ and $\mathbf{d}_1$ in a similar fashion, i.e.,

$$\begin{aligned} \mathbf{c}_1 &= [\mathbf{c}_H^T, \mathbf{c}_S^T]^T \\ \mathbf{d}_1 &= [\mathbf{d}_H^T, \mathbf{d}_T^T]^T \end{aligned} \tag{11.5.12}$$

Let $\mathbf{x} = \mathbf{x}_H + \mathbf{x}_T$ where $\mathbf{x}_H = H\mathbf{c}_H$ and $\mathbf{x}_T = T\mathbf{c}_T$.

As discussed earlier the mismatch causes an effective reduction in the SNR expressed by the decrease in the value of the noncentrality parameter. Let us consider the noncentrality parameter for the above mismatch. Since $\mathbf{H} \subset \mathbf{G}_1$, $P_{G_1}H = H$. I follows that

$$P_{G_1}F_1 = P_{G_1}[H, T] = [H, P_{G_1}T] \tag{11.5.13}$$

and

$$\begin{aligned} F_1^* P_{G_1} F_1 &= \left[\begin{array}{c} H^* \\ T^* \end{array} \right] [H P_{G_1} T] \\ &= \left[\begin{array}{cc} H^* & H^* P_{G_1} T \\ T^* & T^* P_{G_1} T \end{array} \right] \\ &= \left[\begin{array}{cc} H^* & H^* T \\ T^* & T^* P_{G_1} T \end{array} \right] \end{aligned} \tag{11.5.14}$$

The noncentrality parameter for this type of mismatch is given therefore by,

$$\begin{aligned} \lambda^2 &= \frac{2}{\sigma^2} [\mathbf{c}_H^*, \mathbf{c}_T^*] \left[\begin{array}{cc} H^* & H^* T \\ T^* & T^* P_{G_1} T \end{array} \right] \left[\begin{array}{c} \mathbf{c}_H \\ \mathbf{c}_T \end{array} \right] \\ &= \frac{2}{\sigma^2} [\mathbf{c}_H^* H^* H \mathbf{c}_H + \mathbf{c}_H^* H^* T \mathbf{c}_T + \mathbf{c}_T^* T^* H \mathbf{c}_H + \mathbf{c}_T^* T^* P_{G_1} T \mathbf{c}_T] \\ &= \frac{2}{\sigma^2} [\mathbf{x}^* \mathbf{x} - \mathbf{x}_T^* P_{G_1}^{\perp} \mathbf{x}] \end{aligned} \tag{11.5.15}$$

Thus, the decrease in the effective SNR is determined by the energy of the projection of the mismatched components onto $\mathbf{G}_1^{\perp}$. The SNR loss in dB is given by,

$$\begin{aligned}\Delta\text{SNR} &= 10\log\frac{\mathbf{x}^*\mathbf{x}}{\mathbf{x}^*\mathbf{x} - \mathbf{x}_T^* P_{G_1}^{\perp}\mathbf{x}_T} \\ &= \frac{\mathbf{x}_H^*\mathbf{x}_H + \mathbf{x}_T^*\mathbf{x}_T}{\mathbf{x}_H^*\mathbf{x}_H + \mathbf{x}_T^* P_{G_1}\mathbf{x}_T}\end{aligned} \tag{11.5.16}$$

To demonstrate our results we use a signal of length $L = 64$ and an oversampled Gabor representation with $M = 16$ time intervals and $N = 16$ frequency intervals. In this case the oversampling factor is 4, the time step is $a = 4$ and the frequency step is $b = 4$. We use the one-sided exponential $g_2(k)$ given by Eq. (11.5.9) as the synthesis window g.

First, we consider errors in the times of the Gabor transients. According to the assumed model the Gabor transient has four components corresponding to $c_{7,6}$, $c_{7,7}$, $c_{7,8}$ and $c_{7,9}$, which are all equal to one. In the mismatched cases we assume a Gabor transient with four components at $c_{m,6}$, $c_{m,7}$, $c_{m,8}$ and $c_{m,9}$, where m varies from 0 to $M - 1 = 15$. In Figure 11.5.2 we plot the SNR loss in dB as a function of the error in the arrival time. We observe that in this example the SNR loss is always smaller than 4 dB. It appears that the matched subspace detectors are reasonably tolerant to errors in the arrival time.

Next, we consider errors in the frequencies of the Gabor transients. The assumed signal consists of four components corresponding to $c_{0,7}$, $c_{3,7}$, $c_{6,7}$ and $c_{9,7}$, which are all equal to one. In the mismatched cases we assume a Gabor transient with four components corresponding to $c_{0,n}$, $c_{3,n}$, $c_{6,n}$ and $c_{9,n}$, where n varies from 0 to $N - 1 = 15$. Figure 11.5.3 shows the SNR loss in dB as a function of the error in frequency. We observe large values of the SNR loss, indicating that the matched subspace detectors are not tolerant to errors in the transient frequencies. Similar results are reported in [FP89] for the transform domain detector based on the biorthogonal left inverse.

11.6 Robust matched subspace detectors

In the previous section we have studied the effects of errors in the frequencies and arrival times of the Gabor transients on the performance of matched subspace detectors. Such errors often result from uncertainty in the exact location of the nonzero Gabor coefficients in the time-frequency plane. In this section we derive matched subspace detectors that are robust with respect to uncertainties in the frequencies and arrival times of the Gabor transients.

To derive these detectors we assume that some information is available on the number and locations of the non-zero Gabor coefficients. Namely,

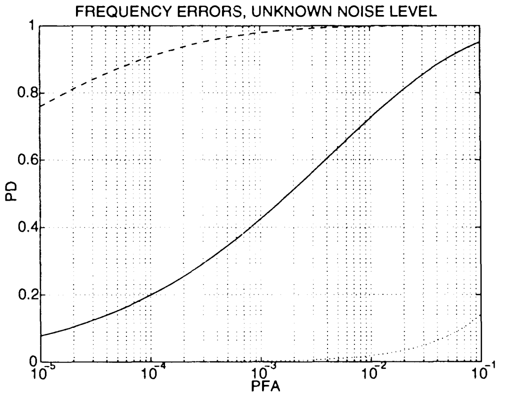

FIGURE 11.6.1. Probability of detection as a function of false alarm probability for robust matched subspace detector (solid line), ideally matched subspace detector (dashed line), and mismatched subspace detector (dash-dot line). Unknown noise level. Mismatch due to frequency errors.

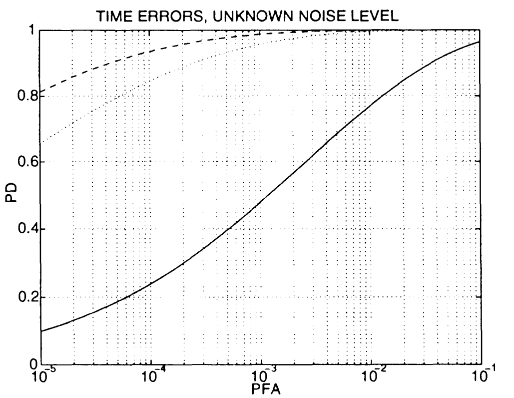

FIGURE 11.6.2. Probability of detection as a function of false alarm probability for robust matched subspace detector (solid line), ideally matched subspace detector (dashed line), and mismatched subspace detector (dash-dot line). Unknown noise level. Mismatch due to time errors.

we assume that the Gabor transient lies in a J-dimensional subspace of $\mathbb{R}^L$, denotes by $\tilde{G}$. The actual number of nonzero components K is smaller (usually much smaller) than J, but the exact value of K and the locations of the nonzero components are unknown. Thus, according to the assumed model the measured signal can be described by

$$\mathbf{y} = \tilde{G}\tilde{\mathbf{c}} + \mathbf{w} \tag{11.6.1}$$

In reality only a part of the entries of $\tilde{\mathbf{c}}$ are not equal to zero and the received signal satisfies,

$$\mathbf{y} = G_1\mathbf{c}_1 + \mathbf{w} \tag{11.6.2}$$

where $\mathbf{G}_1 \subset \tilde{\mathbf{G}}$.

Let

$$\phi(\mathbf{y}) = \begin{cases} 1 & t(\mathbf{y}) > \eta \\ 0 & t(\mathbf{y}) < \eta \end{cases} \tag{11.6.3}$$

be the matched subspace detector corresponding to the model (11.6.1). For the case of known noise level

$$t(\mathbf{y}) = t_1(\mathbf{y}) = \frac{2}{\sigma^2}\mathbf{y}^* P_{\tilde{G}_1}\mathbf{y} \tag{11.6.4}$$

while for the case of unknown level

$$t(\mathbf{y}) = t_2'(\mathbf{y}) = \frac{2K - L}{2K}\frac{\mathbf{y}^* P_{\tilde{G}_1}\mathbf{y}}{\mathbf{y}^* P^{\perp}_{\tilde{G}_1}\mathbf{y}} \tag{11.6.5}$$

In the following we study the performance of the above detectors, and compare it to the performance of lower rank detectors corresponding to the model (11.6.2). We will consider both the ideal and mismatched case.

Assume that we apply a robust detector corresponding to the model (11.6.1) to a signal satisfying (11.6.2). In this case $t_1(\mathbf{y})$ has a chi-square distribution with $2J$ degrees of freedom and noncentrality parameter λ^2 where

$$\lambda^2 = \frac{2}{\sigma^2}\mathbf{c}_1^* G_1^* G_1 \mathbf{c}_1 \tag{11.6.6}$$

Recall that the test statistic associated with the detector based on the model (11.6.2) has a chi-square distribution with $2K$ degrees of freedom and noncentrality parameter given by Eq. (11.6.6). Thus, the performance loss caused by the use of the robust higher rank matched subspace detector is expressed by the increase in the number of degrees of freedom of the test statistic.

Similarly, when we apply a robust detector corresponding to the model (11.6.1) to a signal satisfying (11.6.2), $t_2'(\mathbf{y})$ has an F distribution $2J$ and $2(L-J)$ degrees of freedom and noncentrality parameter λ^2. The test statistic associated with the detector based on (11.6.2) has an F distribution with $2K$ and $2(L-K)$ degrees of freedom and noncentrality parameter λ^2. This time, the performance loss caused by the use of the robust higher rank matched subspace detector is expressed by the increase in the number of the nominator degrees of freedom, accompanied of course by a decrease in number of the denominator degrees of freedom.

A robust transform-domain detector was proposed in [LS97]. The test statistic for this detector is given by,

$$\begin{aligned} t_3(\mathbf{y}) &= \mathbf{z}^*(RR^*)^{-1}\mathbf{z} \\ &= \mathbf{y}^* R^*(RR^*)^{-1} R\mathbf{y} \end{aligned} \tag{11.6.7}$$

It is straight-forward to show that if R is the pseudo inverse of $\tilde{G}$, $t_3(\mathbf{y})$ is equivalent to the data domain robust test statistic $t_1(\mathbf{y})$ matched to $\tilde{G}$. Thus, the robust transform domain detector based on the pseudo inverse is equivalent to the data domain robust matched subspace detector for the case of known noise level.

In the following examples we demonstrate the performance of the robust matched subspace detectors, focusing on cases where there is a mismatch between the actual and assumed signal. We will compare the performance of the robust detector to that of a mismatched subspace detector. The ideally matched subspace detector is, of course, superior to both detectors. The performance degradation for the robust subspace detector is caused by the increased number of degrees of freedom of the test statistic, while the performance degradation associated with the mismatched subspace detector is caused by the lower value of the noncentrality parameter.

We use a signal of length $L = 64$ and an oversampled Gabor representation with $M = 16$ time intervals and $N = 16$ frequency intervals. In this case the oversampling factor is 4, the time step is $a = 4$ and the frequency step is $b = 4$. As in the previous examples we use the one-sided exponential $g_2(k)$ given by Eq. (11.5.9) as the synthesis window g.

In Figure 11.6.1 we consider errors in the frequencies of the Gabor transients, while in Figure 11.6.2 we consider errors in the times of the Gabor transients. In both figures we assume that the noise level is unknown.

In Figure 11.6.1 we assume that the Gabor transient has two components corresponding to $c_{5,4}$ and $c_{7,6}$. We plot the detection probability as a function of the false alarm probability for the following three detectors: A robust matched subspace detector matched to the subspace corresponding to $c_{4:7,4:7}$, a matched subspace detector matched to the actual signal, and a mismatched subspace detector matched to a Gabor transient with components corresponding to $c_{5,5}$ and $c_{7,7}$. We observe that the mismatched subspace detector completely fails, while the robust subspace detector performs reasonably well, but not as good as the ideally matched subspace detector.

In Figures 11.6.2 we assume that the Gabor transient has two components corresponding to $c_{4,5}$ and $c_{6,7}$. We plot the probability of detection as a function of the false alarm probability for the following three detectors: A robust matched subspace detector matched to the subspace corresponding to $c_{4:7,4:7}$, a matched subspace detector matched to the actual signal, and a mismatched subspace detector matched to a Gabor transient with components corresponding to $c_{5,5}$ and $c_{7,7}$. In contrast with the previous example, the mismatched subspace detector outperforms the robust matched subspace detector. These observations agree with our previous results indicating that matched subspace detectors are quite tolerant to errors in the arrival times, but not tolerant to errors in the transients frequencies. It appears that if there is uncertainty in the transient frequencies, the robust matched subspace detector performs significantly better than the mismatched subspace detector. However, if the uncertainty is limited to the arrival times, the lower rank subspace detector is preferable, even in the case of mismatch.

11.7 Summary and conclusions

We have revisited the problem of Gabor based signal detection, focusing on data domain detectors. Following previous studies in the area of matched subspace detection, we presented data domain generalized likelihood ratio tests (GLRT) for two related problems: Detection in background noise of known and unknown noise level. These tests are optimal in the sense of uniformly most powerful (UMP) invariant testing.

Transform domain Gabor based detectors were derived only for the case of known noise level. In this case we have shown that transform domain GLRT detector based on the pseudo inverse data reduction transformation is equivalent to the data domain subspace detector. Thus, this transform-domain GLRT is UMP invariant as well, where the invariance class is the same as for the subspace detector.

Next, we have developed tools to analyze the sensitivity of data domain subspace detectors to mismatch in the model assumptions. By applying these tools we demonstrated that data domain subspace detectors tend to be quite tolerant to mismatch in the synthesis window shape and the arrival times of the Gabor components. However, the performance loss due to mismatch in the frequencies of the Gabor components is severe. These results agree with previous results for transform domain detectors.

To overcome the performance degradation due to mismatch we have derived a data domain robust subspace detector and studied its performance. Our results seem to indicate that the robust detector should be used whenever there is some uncertainty in the frequencies of the Gabor components.

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12

Multi-window Gabor schemes in signal and image representations

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ABSTRACT – Motivated by biological vision, schemes of signal and image representation by localized Gabor-type functions are introduced and analyzed. These schemes, suitable for information representation in a combined frequency-position space are investigated through signal decomposition into a set of elementary functions. Utilizing the Piecewise Zak transform (PZT), the theory of the multi-window approach is given in detail based on the mathematical concept of frames. The advantages of using more than a single window are analyzed and discussed. Applications to image processing and computer vision are presented with regard to texture images, and considered in the context of two typical tasks: image representation by partial information and pattern recognition. In both cases the results indicate that the multi-window approach is efficient and superior in major aspects to previously available methods. It is concluded that the new multi-window Gabor approach could be integrated efficiently into practical techniques of signal and image representation.

12.1 Motivation for using Gabor-type schemes

One of the basic problems encountered in vision and image representation is the high dimensionality of the image space. If for example the fidelity of the displayed information requires a resolution of 1 arc minute over a visual field subbanding 100 degrees in diameter, even a simple image of 1 bit per pixel requires for its representation 10^8 bits. It is obviously difficult to manipulate and compute in real time such a large data set by either a biological or machine vision system. To cope with this problem, biological visual systems of higher species employ position dependent sampling and preprocessing to reduce the bit rate by about 100 folds [KZ85]. Considering the performance of the visual system which projects in this way only partial information, it is tempting to implement this principle in machine vision. This requires the development of a processing scheme based on nonuniform

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sampling and localized operators. In fact, the nonstationary structure of natural images (and other signals like speech) calls for the implementation of localized processing schemes. The most efficient way of performing this task appears in both cases (of biological and computer vision systems) to be by repetition of similar localized operations as are implemented in VLSI [MC80, EH86], and evident from studies of cortical organization [Hub82].

The Gabor approach [Gab46] is readily available for such processing since the Gabor elementary functions (GEF) are by their very nature localized. These localized operators are also well suited for multiresolution edge detection, similar in that sense to processing with the Laplacian operators [BA83, Ros84], or with a variety of compactly supported wavelets [Dau88a]. The localized operators can also serve as oriented-edge operators [Dau85a]. Most importantly, the completeness of the Gabor scheme in the mathematical sense [Hig77] lends itself to various interesting possibilities of image representation such as quantization of localized phase, oversampling, representation by (localized) phase only, and representation by a generalized scheme that incorporate several window functions.

Since the rational of our approach is to learn as much as possible from biological visual systems and implement the principles that emerge in computational vision and image representation, it is interesting to note that both physiological findings [KZ85], [Mar80]-[Dau85b] and psychophysical aspects of adaptation and masking [CR68, GN71, BCA78] indicate that the biological processing is based on matching localized frequency signatures having a form reminiscent of Gabor operators [Mac81, ZD82]. Indeed, preliminary studies reveal interesting possibilities of pattern analysis based on these operations for the purpose of computational vision [Wat83],[DK],[PZ].

The Gabor scheme was introduced as a discrete set of GEFs by means of which signals can be represented [Gab46]. While Gabor's original paper dates back to 1946 [Gab46], no analytic solution suitable for determination of the expansion coefficients was available until a method limited to the one-dimensional case was presented first by Bastiaans [Bas81] in 1980. The purpose of this study is to develop a generalized scheme for image representation in the combined frequency-position space in order to better understand the organizational principles of the visual system [ZP84, PZ85]. We therefore generalize the scheme to multi-window representations which, as a special case, incorporate scaling. We then investigate the advantages inherent in such schemes in so far as image processing and computational vision are concerned.

12.1.1 Frequency-position representations

The fundamental principle of uncertainty of signal representation imposes basic constraints on the structure of functions that can be realized in signal representation by a universal set of elementary functions (a basis). Considering the combined frequency-position (or frequency-time) space, the most widely used sets of functions are comprised of the singular (Dirac) functions distributed along the spatial (or temporal) axis, or the spectral lines (representing harmonic functions) distributed along the frequency axis. Each of these functions requires a support of an infinite extent along the complementary axis (Fig. 1), a condition which can not be realizable in the strict sense.

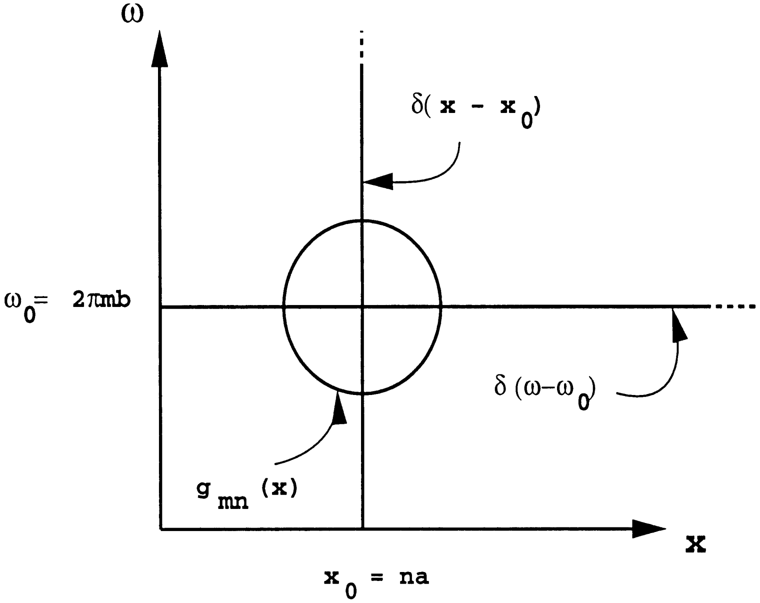

FIGURE 12.1.1. Signal representation in the combined frequency-position space. Shown are the two limiting cases of singular functions along the spatial axis ($\delta(x - x_0)$) and along the frequency axis (the spectral line $\delta(\omega - \omega_0)$). The transition from one limiting case to the other is depicted by the ellipse, which represents the effective supports of a function centered at the coordinates ($x = na, \omega = 2\pi mb$). This figure illustrates the trade-off that exists between the effective bandwidth and effective spatial extent constrained by the basic principle of uncertainty of signal representation.

In our approach we employ localized functions of the type employed by Gabor [Gab46]. These functions are confined in the combined frequency-position space in the sense of being limited in their effective (2nd moment) spatial and spectral extents. The spatial and spectral singular functions are the limiting cases of the trade-off that exists between the effective spatial width and the effective spectral width of all possible elementary functions represented in the combined space.

Considering compact representation of a signal (image), periodic signals are best represented by means of harmonic functions in the sense of requiring the smallest number of functions for reconstruction according to a given quality criterion. Spatially-limited signals (or signals that are zero almost everywhere) call for a presentation by means of spatially localized elementary functions which in the extreme case take the form of Dirac functions. It is reasonable to assume (and obviously impossible to prove) that most natural signals and images are neither periodic nor comprised of a small set of singular (as a limiting case of localized) functions and can therefore be expected to be best represented by elementary functions which are confined along both of the spatial and spectral axes.

For the sake of clarity we present first the formalism in the context of one-dimensional functions. Let $g(x)$ be a window function centered at the origin. The Gabor-type elementary function of order (m,n) is defined by:

$$g_{m,n}(x) = g(x-na)e^{j2\pi mbx}, \tag{12.1.1}$$

where m,n are integers. The harmonic function $g_{m,n}$, represented in the combined frequency-position space, is centered at $\omega = 2\pi mb, x = na$. The Gaussian for $g(x)$ is optimal in that it minimizes the combined effective spread in the positional-spectral plane, compared to the so-called joint entropy achieved by any other window function (Fig. 1).

For a certain class of window functions (which contains the Gaussian), if the condition of optimal information cell size (so-called critical sampling), $ab = 1$, is satisfied, the set of functions $\{g_{m,n}\}$ is complete [Hig77]. A signal $f(x)$ can be expressed by these elementary functions, using a set of signal-specific weighting coefficients $\{c_{m,n}\}$ describing the relative weight of each GEF:[1]

$$f(x) = \sum_{m,n} c_{m,n} g_{m,n}(x). \tag{12.1.2}$$

Since the GEFs are not orthogonal, the analytic formalism for calculating the coefficients employs an auxiliary function $\gamma(x)$ [Bas81]:

$$c_{m,n} = \int f(x) \cdot \overline{\gamma(x-na)} e^{-j2\pi mbx} dx\,, \tag{12.1.3}$$

which is bi-orthogonal [Hig77] to $g(x)$, and can be found by solving the equation

$$\int \gamma(x)\overline{g(x-na)} e^{-j2\pi mbx} dx = \delta_m \delta_n. \tag{12.1.4}$$

[1] Unless otherwise stated, all integrations and summations in this chapter extend from $-\infty$ to $+\infty$.

In view of the duality that exists between $\gamma(x)$ and $g(x)$, their roles can be interchanged. Note however, that representation (12.1.2) is not always stable. For example, in the case of a Gaussian and critical sampling [Dau90], the set $\{g_{m,n}\}$ is complete, but does not constitute a Riesz basis or a frame (frames and Riesz bases are equivalent in the case of critical sampling). In such a case $\gamma(x) \notin L^2(\mathbb{R})$.

A finite set of expansion coefficients $\{c_{m,n}\}$ provides a compact representation of a signal. Since the expansion coefficients fully describe a signal, they can be considered as the means for discrete image representation (by either the visual cortex or machine). According to physiological studies, cells in the visual cortex exhibit localized sensitivity and are specific to spatial frequency bands [KMB82],[PR83] as well as to the phase [PR82]. In terms of GEFs, this means maximum sensitivity in a specific position (na) to spectral order $(m2\pi b)$, and to a specific phase [PR82]. According to this approach, each pair of adjacent cells in the visual cortex represents the real and imaginary parts of one coefficient $c_{m,n}$, corresponding to the GEF, $g_{m,n}$.

The basic trade-off between the effective spatial width and the effective spectral width permits the selection of one out of many (theoretically infinite) possible schemes of tessellation of the space confined by the global effective spatial extent and frequency band. This fact is further elaborated in the subsequent sections. Thus, the finite scheme requires a fixed number of Gabor components, but permits preselection of any desired number of spectral (Gabor) components for spanning the global frequency bandwidth. This explains how a relatively small number of about 4-6 such filters can suffice in vision [SB82].

12.2 Generalized Gabor-type schemes

The growing need for sequences of functions which can be used in time-frequency (or position-frequency) representations (see for example [MZ93]), suggests that a generalization of the Gabor scheme, by using several windows instead of a single one, may be both interesting and useful. For such a generalization, the representation of a given signal $f(x) \in L^2(\mathbb{R})$ is given by:

$$f(x) = \sum_{r=0}^{R-1} \sum_{m,n} c_{r,m,n} g_{r,m,n}(x), \tag{12.2.1}$$

where

$$g_{r,m,n}(x) = g_r(x - na) e^{j2\pi mbx}, \tag{12.2.2}$$

and $\{g_r(x)\}$ is a set of R distinct window functions. Such a set can incorporate, for example, Gaussian windows of various widths. In this case with proper oversampling one can overcome the limitations of either having a high temporal (spatial) resolution and low frequency resolution or vice versa. This can be instrumental in applications such as detection of transients of signals [FP89] or detection of various prototypic textures and other features in images [PZ89].

In the more general case, the windows do not have to be of either the same shape or effective width. This type of a generalization permits the design of Gabor-type schemes which are best matched to the structure of the signal or image and of the task under consideration. It remains to be shown that in the general case of the Gabor-type scheme presented by eqs. (12.2.1) and (12.2.2), one can characterize the set of the representation functions, and efficiently determine the coefficients of the representation.

12.2.1 Preliminaries and notation

We first review the mathematical tools of frames and Zak transform (ZT), which we use in order to characterize the properties of the sequence $\{g_{r,m,n}\}$.

The concept of frames was originally introduced by Duffin and Schaeffer [DS52], in the context of nonharmonic Fourier series. We review, without proofs, some important properties of frames which are essential to the understanding of the present work. For further details see [DS52, HW89].

Definition 12.2.1 *A sequence $\{\psi_n\}$ in a Hilbert Space H constitutes a* **frame** *if there exist positive numbers A,B, called* **frame bounds**, *$0 < A \le B < \infty$ such that for all $f \in H$ we have:*

$$A\|f\|^2 \le \sum_n |\langle f, \psi_n\rangle|^2 \le B\|f\|^2,$$

where $\langle \cdot, \cdot \rangle$ denotes an inner-product in Hilbert space.

Definition 12.2.2 *Given a frame $\{\psi_n\}$ in a Hilbert space H, the* **frame operator** *$S : H \to H$ is defined by :*

$$Sf \triangleq \sum_n \langle f, \psi_n\rangle \psi_n.$$

Corollary 12.2.3 *The frame operator S possesses the following properties:*

(i) S is self-adjoint bounded linear operator with $A\mathcal{I} \le S \le B\mathcal{I}$.
(ii) S is invertible and $B^{-1}\mathcal{I} \le S^{-1} \le A^{-1}\mathcal{I}$.

(iii) $\{S^{-1}\psi_n\}$ *is a frame with bounds* B^{-1}, A^{-1}, *called the* **dual frame** *of* $\{\psi_n\}$.
(iv) Every $f \in H$ *can be presented by means of the frame or the dual frame as follows:*

$$f = \sum_n \langle f, S^{-1}\psi_n \rangle \psi_n = \sum_n \langle f, \psi_n \rangle S^{-1}\psi_n.$$

The inequality $A\mathcal{I} \leq S$ means $\langle A\mathcal{I}f, f\rangle \leq \langle Sf, f\rangle$ for all $f \in H$, where $\mathcal{I}$ is the identity operator.

In order to analyze the properties of the sequence $\{g_{r,m,n}\}$, we shall later on express and examine the frame operator associated with the sequence.

The Zak transform

The ZT is a signal transform that maps $L^2(\mathbb{R})$ unitarily onto $L^2([0,1)^2)$. Since a detailed presentation of the ZT is available in [Jan88], we only review its important relevant properties and introduce the *Piecewise Zak Transform* (PZT), cf. also [ZZ92a, ZZ92b, Dau90, Mun92] and Chapter9 in this book.

The ZT of a signal $f(x)$ is defined as follows:

$$(\mathcal{Z}^\alpha f)(x,u) \triangleq \alpha^{1/2} \sum_{k\in\mathbb{Z}} f[\alpha(x+k)] e^{-j2\pi uk}, \quad -\infty < x, u < \infty, \tag{12.2.3}$$

with a fixed parameter $\alpha > 0$. We often omit the superscript α, i.e., write $(\mathcal{Z}f)(x,u)$, whenever it is clear from the text what the value of α is, or whenever the value of α is irrelevant.

The ZT satisfies the following periodic relation:

$$(\mathcal{Z}f)(x, u+1) = (\mathcal{Z}f)(x,u), \tag{12.2.4}$$

and the following quasiperiodic relation:

$$(\mathcal{Z}f)(x+1, u) = e^{j2\pi u}(\mathcal{Z}f)(x,u). \tag{12.2.5}$$

As a consequence of these two relations, the ZT is completely determined by its values over the unit square $(x,u) \in ([0,1)^2)$. This is the essence of this unitary mapping.

Based on the ZT defined by (12.2.3), we define the PZT as a vector-valued function $F(x,u)$ of order p:

$$F(x,u) = [(F)_0(x,u), (F)_1(x,u) \ldots, (F)_{p-1}(x,u)]^T, \tag{12.2.6}$$

where

$$(F)_i(x,u) = (\mathcal{Z}f)(x, u + \frac{i}{p}), \quad 0 \leq i \leq p-1,\ i \in \mathbb{Z}. \tag{12.2.7}$$

The vector-valued function $F(x,u)$ belongs to $L^2([0,1)\times[0,1/p);\mathbb{C}^p)$, which is a Hilbert space with the inner-product:

$$\langle F,G\rangle = \int_0^1 dx \int_0^{1/p} du \sum_{i=0}^{p-1} (F)_i(x,u)\overline{(G)_i(x,u)}.$$

Since the ZT is a unitary mapping from $L^2(\mathbb{R})$ to $L^2([0,1)^2)$, the PZT is a unitary mapping from $L^2(\mathbb{R})$ to $L^2([0,1)\times[0,1/p);\mathbb{C}^p)$. As a consequence we obtain the following inner-product preserving property:

$$\langle f,g\rangle_{L^2(R)} = \langle \mathcal{Z}f,\mathcal{Z}g\rangle_{L^2([0,1)^2)} = \langle F,G\rangle_{L^2([0,1)\times[0,1/p);C^p)},$$

or in an explicit form:

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx &= \int_0^1\int_0^1 (\mathcal{Z}f)(x,u)\overline{(\mathcal{Z}g)(x,u)}dxdu \\ &= \int_0^1 dx \int_0^{1/p} du \sum_{i=0}^{p-1} (F)_i(x,u)\overline{(G)_i(x,u)}.\end{aligned}$$

This unitary property of the PZT allows us the translation from $L^2(\mathbb{R})$ to $L^2([0,1)\times[0,1/p);\mathbb{C}^p)$, where issues regarding Gabor representation are often easier to deal with.

12.2.2 Continuous-variable one-dimensional scheme

The characterization of the sequence $\{g_{r,m,n}\}$ can be divided into three categories according to the sampling density of the combined space (the so-called phase space density) defined by $d \triangleq R(ab)^{-1}$:

- undersampling: $d<1$.
- critical sampling: $d=1$,
- oversampling: $d>1$.

For the single window Gabor scheme, it is known that in the case of undersampling, the sequence of representation functions is incomplete and therefore does not constitute a frame [Dau90]. In the case of critical sampling, the representation is not always stable; for example when $g(x)$ is a Gaussian function. This instability is expressed formally by the so-called Balian Low Theorem (see [Dau90] for a detailed discussion of problems involved in critical sampling). Such problems are overcome by oversampling, in which case the representation (if it exists) is not unique.

The same is true for the multi-window scheme. That is, in the case of undersampling the sequence $\{g_{r,m,n}\}$ is incomplete, and, in the case of critical sampling there still exists a problem of stability [ZZ97a, ZZ97b]. We will later consider the way this stability problem can be overcome with multi-windows.

Proofs of the theorems presented in this section can be found in [Zib95, ZZ97a].

The frame operator

In order to examine the properties of the sequence $\{g_{r,m,n}\}$, we consider the operator:

$$Sf = \sum_{r=0}^{R-1} \sum_{m,n} \langle f, g_{r,m,n} \rangle g_{r,m,n}. \tag{12.2.8}$$

This is clearly the frame operator if $\{g_{r,m,n}\}$ constitutes a frame, but we call it a frame operator anyway. For the single-window Gabor scheme, this operator was examined by straightforward methods [Dau90, HW89], where application of the ZT was mostly restricted to the case $ab = 1$. In the present study we show that if the product ab is a rational number it might be advantageous to examine this operator in $L^2([0,1) \times [0,1/p); \mathbb{C}^p)$ by using the PZT. Unless explicitly stated otherwise, we use the ZT with the parameter

$$\alpha = 1/b.$$

Note that the same examination can be done with $\alpha = a$, but with a differently defined PZT wherein the x variable of the ZT is considered in a piecewise manner.

The application of the PZT to the multi-window Gabor-type frame operator simplifies the analysis significantly in that the frame operator (12.2.8) can be represented in terms of matrix-algebra, and be characterized by the properties of a matrix-valued function. This is formulated in the following theorem :

Theorem 12.2.4 *Let $ab = p/q$, $p, q \in \mathbb{N}$, and let S_z be the operator that maps the PZT of f to the PZT of Sf. The action of S_z in $L^2([0,1) \times [0,1/p); \mathbb{C}^p)$ is then given by the following matrix algebra:*

$$(S_z F)(x,u) = \mathbf{S}(x,u) F(x,u), \tag{12.2.9}$$

where $\mathbf{S}(x,u)$ is a $p \times p$ matrix-valued function whose entries are given by:

$$(\mathbf{S})_{i,k}(x,u) = \frac{1}{p} \sum_{r=0}^{R-1} \sum_{l=0}^{q-1} \mathcal{Z}g_r\left(x - l\frac{p}{q}, u + \frac{i}{p}\right) \overline{\mathcal{Z}g_r\left(x - l\frac{p}{q}, u + \frac{k}{p}\right)}; \quad i,k = 0, \cdots, p-1, \tag{12.2.10}$$

and the vector-valued function $F(x,u)$ is given by (12.2.6) and (12.2.7).

It is important to note that (12.2.9) is an isometrically isomorphic representation of the frame operator S (12.2.8), since the PZT is a unitary transform from $L^2(\mathbb{R})$ into $L^2([0,1) \times [0,1/p); \mathbb{C}^p)$. Moreover, representation (12.2.9) of the frame operator in terms of matrix algebra enables us to interpret the properties of the sequence $\{g_{r,m,n}\}$ in terms of the properties of the matrix-valued function $\mathbf{S}(x,u)$. It also implies that:

$$\sum_{r=0}^{R-1} \sum_{m,n} |\langle f, g_{r,m,n} \rangle|^2 = \langle Sf, f \rangle = \int_0^1 dx \int_0^{1/p} du F^*(x,u)\mathbf{S}(x,u)F(x,u). \tag{12.2.11}$$

The cases of critical sampling and oversampling

Using the PZT and the matrix representation of the operator S, we examine the properties of the sequence $\{g_{r,m,n}\}$ for a rational ab.

The following theorem examines the completeness of $\{g_{r,m,n}\}$ in the context of the structure of the matrix-valued function $\mathbf{S}(x,u)$.

Theorem 12.2.5 *Given $g_r \in L^2(\mathbb{R})$, $0 \le r \le R-1$, and a matrix-valued function $\mathbf{S}(x,u)$, $(x,u) \in ([0,1) \times [0,1/p))$ as in (12.2.10), the sequence $\{g_{r,m,n}\}$ associated with $\{g_r\}$, $ab = p/q$, $p, q \in \mathbb{N}$ is complete if and only if $\det(\mathbf{S})(x,u) \neq 0$ a.e. on $[0,1) \times [0,1/p)$.*

Next we discuss the frame property of the sequence $\{g_{r,m,n}\}$. Representation (12.2.9), of the frame operator in $L^2([0,1) \times [0,1/p); \mathbb{C}^p)$, as a matrix-valued function, relates the frame bounds A, B to the eigenvalues of the matrix $\mathbf{S}(x,u)$. We define

$$\lambda_{max}(\mathbf{S}) \triangleq \operatorname*{ess\,sup}_{(x,u)\in([0,1)\times[0,1/p))} \max_{1\le i\le p} \lambda_i(\mathbf{S})(x,u) \tag{12.2.12}$$

$$\lambda_{min}(\mathbf{S}) \triangleq \operatorname{ess\,inf}_{(x,u)\in([0,1)\times[0,1/p))} \min_{1\le i\le p} \lambda_i(\mathbf{S})(x,u), \tag{12.2.13}$$

where $\lambda_i(\mathbf{S})(x,u)$ are the eigenvalues of the matrix $\mathbf{S}(x,u)$. Then, we obtain for the upper frame-bound B:

$$B = \lambda_{max}(\mathbf{S}), \tag{12.2.14}$$

and in the same manner for the lower frame-bound A:

$$A = \lambda_{min}(\mathbf{S}). \tag{12.2.15}$$

which is formalized in the following theorem:

Theorem 12.2.6 *The sequence* $\{g_{r,m,n}\}$ *associated with* $\{g_r\}$, $g_r \in L^2(\mathbb{R})$, $0 \le r \le R-1$, *and* $ab = p/q$, $p, q \in \mathbb{N}$ *constitutes a frame if and only if* $0 < \lambda_{min}(\mathbf{S}) \le \lambda_{max}(\mathbf{S}) < \infty$.

Finding the eigenvalues of a matrix-valued function may be a difficult task. We therefore propose an alternative approach to determining whether $\{g_{r,m,n}\}$ constitutes a frame. First, we introduce a lemma, which formulates a necessary and sufficient condition for the existence of an upper frame bound $B < \infty$.

Lemma 12.2.7 *The sequence* $\{g_{r,m,n}\}$ *associated with* $\{g_r\}$, $g_r \in L^2(\mathbb{R})$, $0 \le r \le R-1$ *and* $ab = p/q$, $p, q \in \mathbb{N}$, *has an upper frame bound* $B < \infty$ *if and only if* $(\mathcal{Z}g_r)(x, u)$ *are all bounded a.e. on* $(0,1]^2$ *(*$\mathcal{Z}g_r \in L^\infty((0,1]^2)$*).*

Second, we present a Theorem which determines whether the sequence $\{g_{r,m,n}\}$ constitutes a frame, when an upper frame bound exists, and which does not necessitate calculation of the eigenvalues of $\mathbf{S}(x, u)$.

Theorem 12.2.8 *Given* $g_r \in L^2(\mathbb{R}), 0 \le r \le R-1$, *such that there exists an upper frame bound* $B < \infty$ *for the sequence* $\{g_{r,m,n}\}$ *associated with* $\{g_r\}$, *and* $ab = p/q$, $p, q \in \mathbb{N}$. *The sequence* $\{g_{r,m,n}\}$ *constitutes a frame if and only if* $0 < K \le \det(\mathbf{S})(x, u)$ *a.e. on* $[0, 1) \times [0, 1/p)$, *where the matrix-valued function* $\mathbf{S}(x, u)$ *is as in (12.2.10).*

By combining Lemma 12.2.7 and Theorem 12.2.8 we obtain a necessary and sufficient condition for the sequence $\{g_{r,m,n}\}$ to constitute a frame.

A sequence $\{\psi_n\}$ in a Hilbert space H constitutes a *tight frame* if $\sum_n |\langle f, \psi_n\rangle|^2 = A||f||^2$ for all $f \in H$ (i.e. $A = B$). For example, an orthonormal basis is a particular case of a tight frame with $A = 1$. The following Theorem considers tight frames and the matrix representation of the frame operator.

Theorem 12.2.9 *Given* $g_r \in L^2(\mathbb{R})$, $0 \le r \le R-1$, *and a matrix-valued function* $\mathbf{S}(x, u)$ *as in (12.2.10), the sequence* $\{g_{r,m,n}\}$ *associated with* $\{g_r\}$, $ab = p/q$, $p, q \in \mathbb{N}$ *constitutes a tight frame if and only if* $\mathbf{S}(x, u) = A\mathbf{I}$ *a.e., where* $\mathbf{I}$ *is the identity matrix, and* $A = \frac{q}{p}\sum_{r=0}^{R-1} ||g_r||^2$.

The dual frame

In order to use a representation such as suggested by property (iv) of Corollary 12.2.3, we need a way for obtaining the dual frame. In general one can use operator techniques in order to find the dual frame [DS52, Dau90]. For the single window scheme finding the dual frame $\{S^{-1}g_{m,n}\}$ is simplified since it is generated by a single dual frame window function and it is of the form of $\{g_{m,n}\}$ [HW89, Dau90]. In fact, this is also the case for the multi-window scheme, i.e., let $\{\gamma_{r,m,n}\}$ denote the dual frame of $\{g_{r,m,n}\}$, then $\{\gamma_{r,m,n}\}$ is generated by a finite set of R dual frame window functions $\{\gamma_r\}$:

$$\gamma_{r,m,n}(x) = \gamma_r(x - na)e^{j2\pi mbx}, \; 0 \leq r \leq R-1,$$

where

$$\gamma_r = S^{-1}g_r. \tag{12.2.16}$$

Based on (12.2.16) and using the matrix representation (12.2.9) of the frame operator, the PZT of γ_r, is:

$$\Gamma_r(x,u) = \mathbf{S}^{-1}(x,u)G_r(x,u), \tag{12.2.17}$$

i.e., $\Gamma_r(x,u), G_r(x,u)$ are vector-valued functions in $L^2([0,1) \times [0,1/p); \mathbb{C}^p)$ and $\mathbf{S}^{-1}(x,u)$ is the inverse matrix of $\mathbf{S}(x,u)$ (for example $\mathbf{S}^{-1}(x,u) = [\det(\mathbf{S})(x,u)]^{-1}\operatorname{adj}(\mathbf{S})(x,u))$.

The expansion coefficients can be found by calculating, in either the signal or ZT domain, the inner-product of the signal with the dual frame: $c_{r,m,n} = \langle f, \gamma_{r,m,n}\rangle$. An example of $p=2$ for the single window scheme can be found in [ZZ93b].

Extension of the Balian-Low Theorem to the case of multi-windows

In case of critical sampling, the following theorem of Balian and Low indicates that a wide range of "well behaved", rapidly decaying and smooth, functions $g(x)$ are excluded from being proper candidates for generators of frames, see e.g. [Dau90, BHW95, DJ93] and Chapters2 and 6 in this volume.

Theorem 12.2.10 *Given $g \in L^2(\mathbb{R})$, $a > 0$ and $ab = 1$, if the sequence $\{g_{m,n}\}$ constitutes a frame, then either $xg(x) \notin L^2(\mathbb{R})$ or $g'(x) \notin L^2(\mathbb{R})$.*

Note that $g'(x) \in L^2(\mathbb{R}) \Leftrightarrow \omega\hat{g}(\omega) \in L^2(\mathbb{R})$, where $\hat{g}$ is the Fourier transform of g.

One of the solutions for these problems is oversampling. In fact, it was recently proven that in a case of a Gaussian the $\{g_{m,n}\}$ constitutes a frame for all $ab < 1$ [SW92a, Jan94b].

In the case of critical sampling of the multi-window scheme, an interesting question is whether we can "beat" the Balian Low theorem by utilizing several windows. If all the windows in the set $\{g_r\}$ are well behaved rapidly decaying and smooth functions, it isn't possible to overcome the restrictive Balian Low condition, as indicated by the following theorem.

Theorem 12.2.11 *Given $g_r \in L^2(\mathbb{R})$, $0 \le r \le R-1$, $a > 0$ and $R(ab)^{-1} = 1$, if the sequence $\{g_{r,m,n}\}$, as in (12.2.2), constitutes a frame, then either $xg_r(x) \notin L^2(\mathbb{R})$ or $g_r'(x) \notin L^2(\mathbb{R})$ for at least one of the g_r's.*

In fact, Theorem 12.2.11 applies to a wider class of representation functions with a critical sampling of the combined space [ZZ97a, ZZ97b].

One of the advantages of using more than one window is the possibility to overcome in a way the constraint imposed by the Balian-Low theorem on the choice of window functions, by adding an extra window function of proper nature such that the resultant scheme of critical sampling constitutes a frame. Whether one can find a non-well-behaved window function, complementary to a set of well-behaved window functions, such that the inclusive set will generate a frame for critical sampling, depends on the nature of the set of the well-behaved window functions as indicated by the following proposition.

Proposition 12.2.12 *Let a set, $\{g_r\}$, $0 \le r \le R-2$, of $R-1$ window functions be given. Denote by $\mathbf{G}^0(x,u)$ the $R-1 \times R$ matrix-valued function with entries $(\mathbf{G}^0)_{r,k}(x,u) = \overline{\mathcal{Z} g_r(x, u + \frac{k}{R})}$, and $\mathbf{P}(x,u) = \mathbf{G}^0(x,u)\mathbf{G}^{0^*}(x,u)$. There exists a window function $g_{R-1}(x)$ such that the inclusive set $\{g_r\}, 0 \le r \le R-1$ generates a frame for the critical sampling case, if and only if $0 < K \le \det(\mathbf{P})(x,u)$ a.e. on $[0,1) \times [0,1/R)$.*

An example of $R-1$ well-behaved window functions, which satisfy $0 < K \le \det(\mathbf{P})(x,u)$ a.e. on $[0,1) \times [0,1/R)$, can be constructed in the following manner. Take a window function $g(x)$ such that the sequence $\{g(x - n/b)e^{j2\pi mx/a}\}$ constitutes a frame for $ab = R/(R-1)$. Note, that this is an oversampling scheme ($1/(ab) = (R-1)/R < 1$) and that there exist, therefore, well-behaved window functions $g(x)$ such that $\{g(x - n/b)e^{j2\pi mx/a}\}$ constitutes a frame (for example the Gaussian function). Construct the following $R-1$ window functions:

$$g_r(x) = g\left(x - \frac{rR}{b(R-1)}\right).$$

Clearly these are well-behaved window functions. Moreover, we obtain $(\mathcal{Z} g_r)(x,u) = (\mathcal{Z} g)(x - r\frac{R}{R-1}, u)$, and the matrix-valued function $\mathbf{G}^0(x,u)$ equals the matrix-valued function $\mathbf{G}(x,u)$ which corresponds to the sequence $\{g(x-na)e^{j2\pi mbx}\}$ (which we denote by $\{g_{m,n}\}$). In this case $\{g_{m,n}\}$ corresponds to an undersampling scheme. By the duality principle, as presented in [RS97, Theorem 2.2(e)], since $\{g(x - n/b)e^{j2\pi mx/a}\}$ constitutes a frame, $\{g_{m,n}\}$ constitutes a Riesz basis for a sub-space of $L^2(\mathbb{R})$. It can therefore be shown that $0 < K \le \det(\mathbf{P})(x,u)$ a.e. on $[0,1) \times [0,1/R)$. This point will be further discussed elsewhere. We assume that there, indeed, exist other sets of well-behaved window functions, which are not generated from a single well-behaved function, which satisfy $0 < K \leq \det(\mathbf{P})(x,u)$ a.e. on $[0,1) \times [0,1/R)$.

A wavelet-type example

Consider a generalization of the sequence $\{g_{r,m,n}\}$ where for each window function $g_r(x)$ there is a different set of parameters a_r, b_r. Explicitly, we have

$$g_{r,m,n}(x) = g_r(x - na_r)e^{j2\pi m b_r x}, \tag{12.2.18}$$

and the sampling density of the combined space is:

$$d \triangleq \sum_{r=0}^{R-1} (a_r b_r)^{-1}.$$

Similarly to the previously presented analysis, the characterization of the newly-defined sequence $\{g_{r,m,n}\}$ can be divided into the categories of undersampling, critical sampling and oversampling according to $d < 1$, $d = 1$, $d > 1$, respectively. It is shown in [ZZ97a, ZZ97b] that the properties of the newly-defined sequence $\{g_{r,m,n}\}$ can be analyzed by the matrix-algebra approach, presented earlier, in accordance with a procedure which increases the number of utilized windows.

Utilizing the degrees of freedom of choosing a different set of parameters a_r, b_r for each window function g_r, we construct a wavelet-type scheme. Let α, β be positive, real numbers. Given a window function $g(x)$, let

$$g_r(x) = \alpha^{-r/2} g(\alpha^{-r} x).$$

Also, let $a_r = \beta\alpha^r$, and $a_r b_r = R/d$ for all r, where d is the sampling density of the combined space. We then have

$$g_{r,m,n}(x) = \alpha^{-r/2} g(\alpha^{-r} x - n\beta) e^{j2\pi \frac{mxR}{\beta\alpha^r d}}.$$

In this scheme the width of the window is proportional to the translation step a_r, whereas, the product $a_r b_r$ is constant. As such, this scheme incorporates scaling which is characteristic of wavelets and of the Gabor scheme with the logarithmically-distorted frequency axis [PZ88]. However, in contrast with wavelets, this scheme has a finite number of window functions i.e., resolution levels, and each of the windows is modulated by the infinite set of functions defined by the kernel (a complex exponential or any other proper kernel). Each subset for fixed m can be considered as a finite (incomplete) wavelet-type set, in that it obeys the properties of scaling and translation of a complex "mother wavelet". For example, if R/d is an integer, the mother wavelet is defined by $g(x)e^{j2\pi \frac{mxR}{\beta d}}$, if not, this definition holds within a "complex phase". Thus, all the functions corresponding to each of these mother wavelets are self-similar. We illustrate in Fig. 12.2.1 an example with a Gaussian window function and $\alpha = 2, \beta = 1, d = 1, R = 4$. The real part of $g_{r,m,n}(x)$ is depicted in Fig. 12.2.1(a)-(c), for $m = 0, 1$ and 2, respectively. Each of the three parts of Fig. 12.2.1 shows the four representation functions, $r = 0, 1, 2, 3,\ n = 1$, superimposed on each other.

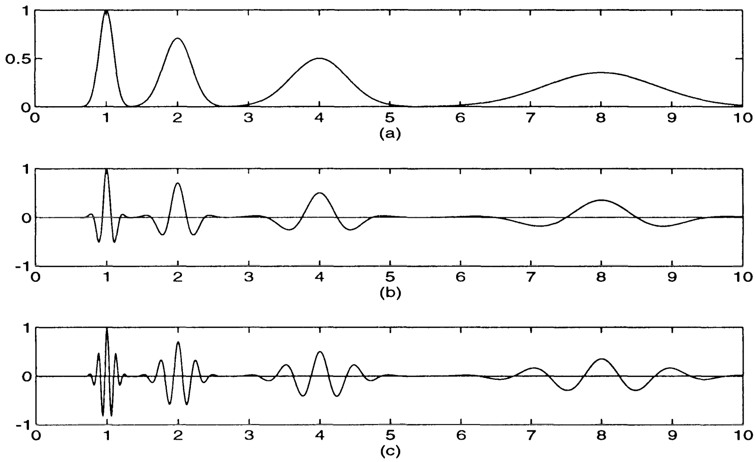

FIGURE 12.2.1. Example of wavelet-type Gaborian representation functions with a Gaussian window. Each of the three displays of this figure, (a)-(c), illustrates the real part of the four functions, $r = 0, 1, 2, 3,\ n = 1$, superimposed on each other, with a different value of m: (a) $m = 0$ (b) $m = 1$ (c) $m = 2$. The self-similarity of the functions within each subset of representation functions is evident.

12.2.3 Continuous-variable multidimensional scheme

A straightforward generalization to multidimensions is possible by considering a separable scheme [PZ88], i.e., a scheme where the multidimensional window functions are comprised of a product of 1D window functions. As a simple example consider the 2D case of a single-window continuous-variable scheme. For $f \in L^2(\mathbb{R}^2)$ a separable scheme of representation is

$$f(x_1, x_2) = \sum_{m_1, m_2, n_1, n_2} c_{m_1,m_2,n_1,n_2} g_{m_1,m_2,n_1,n_2}(x_1, x_2), \tag{12.2.19}$$

where

$$g_{m_1,m_2,n_1,n_2}(x_1, x_2) = g(x_1 - n_1 a_1, x_2 - n_2 a_2) e^{j2\pi(m_1 b_1 x_1 + m_2 b_2 x_2)},$$

and

$$g(x_1, x_2) = g_1(x_1) g_2(x_2).$$

The analysis of the properties of the sequence $\{g_{m_1,m_2,n_1,n_2}(x_1,x_2)\}$ in $L^2(\mathbb{R}^2)$ is straightforward and is based on the properties of the two sequences $\{g_1(x-na_1)e^{j2\pi mb_1x}\}$ and $\{g_2(x-na_2)e^{j2\pi mb_2x}\}$ in $L^2(\mathbb{R})$ as indicated by the following theorem, which is proved in [Zib95].

Theorem 12.2.13 *Let $\{\phi_m(x)\}$ and $\{\psi_m(x)\}$ constitute frames in $L^2(\mathbb{R})$, with dual frames $\{\tilde{\phi}_m(x)\}$ and $\{\tilde{\psi}_m(x)\}$ respectively. Then, $\{\phi_m(x)\psi_n(y)\}$ constitutes a frame in $L^2(\mathbb{R}^2)$ and its dual frame is $\{\tilde{\phi}_m(x)\tilde{\psi}_n(y)\}$. Moreover, let A_ϕ and B_ϕ be the lower and upper frame bounds of $\{\phi_m(x)\}$ respectively, and A_ψ, B_ψ be the lower, upper frame bounds of $\{\psi_m(x)\}$ respectively, then $A_\phi A_\psi$, $B_\phi B_\psi$, are the lower, upper, frame bounds of $\{\phi_m(x)\psi_n(y)\}$ respectively.*

Theorem 12.2.13 implies that the dual frame of a separable sequence $\{g_{m_1,m_2,n_1,n_2}(x_1,x_2)\}$ is generated by a single dual frame window γ given by $\gamma(x_1,x_2) = \gamma_1(x_1)\gamma_2(x_2)$, where $\gamma_1(x)$ and $\gamma_2(x)$, generate the dual frames of $\{g_1(x-na_1)e^{j2\pi mb_1x}\}$ and $\{g_2(x-na_2)e^{j2\pi mb_2x}\}$ respectively. This theorem can be generalized to multidimensions.

Next we introduce multidimensional non-separable schemes. In particular an arbitrary lattice, and not just a rectangular grid, is utilized in order to sample each coordinate of the combined space.

Non-separable schemes

The sampling of each coordinate of the combined space on a lattice is performed in the same way as regular multidimensional sampling of bandlimited signals [DM84]. Consider the K-dimensional case. Let

$$\begin{aligned}
\mathbf{a}_1 &= (a_{11}, a_{21}, \ldots, a_{K1})^T, \\
\mathbf{a}_2 &= (a_{12}, a_{22}, \ldots, a_{K2})^T, \\
&\vdots \\
\mathbf{a}_K &= (a_{1K}, a_{2K}, \ldots, a_{KK})^T,
\end{aligned}$$

be the K linearly independent vectors which define the sampling points in the position (time) domain:

$$\begin{aligned}
x_1 &= a_{11}n_1 + a_{12}n_2 + \ldots + a_{1K}n_K \\
x_2 &= a_{21}n_1 + a_{22}n_2 + \ldots + a_{2K}n_K \\
&\vdots \\
x_K &= a_{K1}n_1 + a_{K2}n_2 + \ldots + a_{KK}n_K
\end{aligned}$$

or in matrix notation:

$$\mathbf{x} = \mathbf{A}\mathbf{n}$$

where

$$\begin{aligned} \mathbf{x} &= (x_1, x_2, \ldots, x_K)^T \in \mathbb{R}^K, \\ \mathbf{n} &= (n_1, n_2, \ldots, n_K)^T \in \mathbb{Z}^K, \\ \mathbf{A} &= [\mathbf{a}_1 | \mathbf{a}_2 | \cdots | \mathbf{a}_K]. \end{aligned}$$

In the frequency domain the sampling points are defined by the matrix $\mathbf{B}$ and the index vector $\mathbf{m}$. For $f \in L^2(\mathbb{R}^K)$, the multidimensional multi-window Gabor-type representation is:

$$f(\mathbf{x}) = \sum_{r=0}^{R-1} \sum_{\mathbf{m},\mathbf{n}} c_{r,\mathbf{m},\mathbf{n}} g_{r,\mathbf{m},\mathbf{n}}(\mathbf{x}) \tag{12.2.20}$$

where

$$g_{r,\mathbf{m},\mathbf{n}}(\mathbf{x}) = g_r(\mathbf{x} - \mathbf{A}\mathbf{n}) \exp(j2\pi \mathbf{x}^T \mathbf{B}\mathbf{m}), \quad \mathbf{m}, \mathbf{n} \in \mathbb{Z}^K,$$

and $\{g_r\}$ is a set of R window functions.

As in the 1D case, the sampling density of the combined space is defined by

$$d \triangleq \frac{R}{|\det \mathbf{A} \det \mathbf{B}|},$$

and the three cases of sampling are: undersampling: $d < 1$, critical sampling $d = 1$, and oversampling $d > 1$. However, such a terminology might be misleading in the multidimensional case, since for separable schemes, one can easily construct examples where the sequence $\{g_{r,\mathbf{m},\mathbf{n}}\}$ is not complete for critical sampling or oversampling for any choice of windows $\{g_r\}$. Such a simple particular example for the 2D separable single-window case, where the scheme of representation is as in (12.2.19), can be constructed by taking $a_1 b_1 = 1/2, a_2 b_2 = 2$. Then $d = 1$, but clearly $\{g_{\mathbf{m},\mathbf{n}}\}$ is not complete in $L^2(\mathbb{R}^2)$, since the sequence $\{g_2(x - na_2)e^{j2\pi m b_2 x}\}$ is not complete in $L^2(\mathbb{R})$. Let $\Omega_{\mathbf{A}}$ denote any fundamental domain of the lattice generated by $\mathbf{A}$. A parallelepiped with the vectors $\{\mathbf{a}_i\}$ as its edges is one such standard choice (called the fundamental parallelepiped [Vai93, p. 561]), in which case $\Omega_{\mathbf{A}} = \mathbf{A}\mathbf{x},\ \mathbf{x} \in [0, 1)^K$. For the single-window case, in order to avoid a situation where $d \geq 1$ and $\{g_{\mathbf{m},\mathbf{n}}\}$ is not complete for any choice of the window function g, one should consider the cases where

$$\bigcap_{\mathbf{n}} (\mathbf{B}^{-T}\mathbf{n} + \Omega_{\mathbf{A}}) = \emptyset. \tag{12.2.21}$$

Eq. (12.2.21) is the condition imposed to avoid aliasing when a multidimensional function, band-limited to a region $\Omega_{\mathbf{A}}$, is sampled on a lattice defined by a matrix $\mathbf{B}$. This, in turn, implies that a sequence $\{g_{\mathbf{m},\mathbf{n}}\}$, with g an indicator function of the region $\Omega_{\mathbf{A}}$ (i.e., $g(\mathbf{x}) = 1$ inside the region $\Omega_{\mathbf{A}}$ and vanishes elsewhere) is complete if (12.2.21) is satisfied. We conjecture that if (12.2.21) is not satisfied, the sequence $\{g_{\mathbf{m},\mathbf{n}}\}$ is not complete for any choice of the window g (see [RS95a] for further discussion).

Conceptually, there is a great deal of similarity between the analysis of the nonseparable multidimensional case and the 1D case although it is not trivial as the separable case. Moreover, a detailed analysis, including analysis in the ZT domain, of the continuous-variable multidimensional multi-window Gabor-type scheme is available in [RS97]. Therefore, we limit our analysis to the single-window case where $\mathbf{B}$ generates the so-called dual lattice of the lattice generated by $\mathbf{A}$. This case emphasizes the usefulness of the multidimensional ZT. For such lattices we have

$$(\mathbf{A}\mathbf{n})^T\mathbf{B}\mathbf{m} \in \mathbb{Z},\ \forall \mathbf{m}, \mathbf{n} \in \mathbb{Z}^K.$$

However, without loss of generality we may assume $\mathbf{A}^T\mathbf{B} = \mathbf{I}$. We conjecture that this is the only "true" critical sampling case, i.e., the only case where $d = 1$ and (12.2.21) holds.

The multidimensional ZT

The multidimensional ZT is defined as follows:

$$(\mathcal{Z}^\Lambda f)(\mathbf{x}, \mathbf{u}) = (\det \Lambda)^{1/2} \sum_{\mathbf{k}} f\left(\Lambda(\mathbf{x} + \mathbf{k})\right) \exp(-j2\pi \mathbf{u}^T\mathbf{k}),$$

where Λ is a fixed matrix such that $\det \Lambda > 0$. [We often omit the superscript Λ and write $(\mathcal{Z}f)(\mathbf{x}, \mathbf{u})$.] The ZT satisfies the following periodic and quasiperiodic properties for $\mathbf{n} \in \mathbb{Z}^K$:

$$(\mathcal{Z}f)(\mathbf{x}, \mathbf{u} + \mathbf{n}) = (\mathcal{Z}f)(\mathbf{x}, \mathbf{u}), (\mathcal{Z}f)(\mathbf{x} + \mathbf{n}, \mathbf{u}) = \exp(-j2\pi \mathbf{u}^T\mathbf{n})(\mathcal{Z}f)(\mathbf{x}, \mathbf{u}).$$

Therefore the ZT is determined by its values on the 2K-dimensional unit cube $(\mathbf{x}, \mathbf{u}) \in ([0,1)^{2K})$. In fact, the multidimensional ZT is a unitary mapping of $L^2(\mathbb{R}^K)$ onto $L^2([0,1)^{2K})$.

Define the following translation operators for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^K$,

$$T_{\mathbf{a}}f(\mathbf{x}) = f(\mathbf{x} - \mathbf{a}),\ E_{\mathbf{b}}f(\mathbf{x}) = \exp(j2\pi \mathbf{x}^T\mathbf{b})f(\mathbf{x}). \tag{12.2.22}$$

The multidimensional ZT possesses accordingly the following two fundamental properties:

$$(\mathcal{Z}^\Lambda T_{\mathbf{a}}f)(\mathbf{x}, \mathbf{u}) = (\mathcal{Z}^\Lambda f)(\mathbf{x} - \Lambda^{-1}\mathbf{a}, \mathbf{u}),$$

$$(\mathcal{Z}^\Lambda E_{\mathbf{b}}f)(\mathbf{x}, \mathbf{u}) = \exp(-j2\pi \mathbf{x}^T\Lambda^T\mathbf{b})(\mathcal{Z}^\Lambda f)(\mathbf{x}, \mathbf{u} - \Lambda^T\mathbf{b}). \tag{12.2.23}$$

The single-window case with critical sampling

Utilizing the translation operators defined in (12.2.22), we can write for the single window case: $g_{\mathbf{m},\mathbf{n}} = E_{\mathbf{Bm}} T_{\mathbf{An}} g$. Therefore, based on properties (12.2.23) of the ZT, we have

$$(\mathcal{Z}^\Lambda g_{\mathbf{m},\mathbf{n}})(\mathbf{x},\mathbf{u}) = \exp(-j2\pi \mathbf{x}^T \Lambda^T \mathbf{Bm})(\mathcal{Z}^\Lambda g)(\mathbf{x} - \Lambda^{-1}\mathbf{An}, \mathbf{u} - \Lambda^T \mathbf{Bm}).$$

As noted, we limit our analysis to the critical sampling case such that $\mathbf{A}^T\mathbf{B} = \mathbf{I}$. Choosing $\Lambda = \mathbf{A} = \mathbf{B}^{-T}$ we obtain

$$(\mathcal{Z} g_{\mathbf{m},\mathbf{n}})(\mathbf{x},\mathbf{u}) = \exp(-j2\pi \mathbf{x}^T \mathbf{m}) \exp(j2\pi \mathbf{u}^T \mathbf{n})(\mathcal{Z} g)(\mathbf{x},\mathbf{u}).$$

Therefore, in the ZT domain, representation (12.2.20) becomes

$$(\mathcal{Z} f)(\mathbf{x},\mathbf{u}) = (\mathcal{Z} g)(\mathbf{x},\mathbf{u}) \sum_{\mathbf{m},\mathbf{n}} c_{\mathbf{m},\mathbf{n}} \exp(-j2\pi \mathbf{x}^T \mathbf{m}) \exp(j2\pi \mathbf{u}^T \mathbf{n}).$$

Since $\{\exp(-j2\pi \mathbf{x}^T \mathbf{m}) \exp(j2\pi \mathbf{u}^T \mathbf{n})\}$ constitutes an orthonormal basis for $L^2([0,1)^{2K})$, assuming $\mathcal{Z} g$ does not vanish, i.e. does not have a zero over $(\mathbf{x},\mathbf{u}) \in [0,1)^{2K}$, the expansion coefficients are given by

$$c_{\mathbf{m},\mathbf{n}} = \int_{\mathbf{x}\in[0,1)^K} \int_{\mathbf{u}\in[0,1)^K} \frac{(\mathcal{Z} f)(\mathbf{x},\mathbf{u})}{(\mathcal{Z} g)(\mathbf{x},\mathbf{u})} \exp(j2\pi \mathbf{x}^T \mathbf{m}) \exp(-j2\pi \mathbf{u}^T \mathbf{n}) d\mathbf{x} d\mathbf{u}.$$

Moreover, one can easily show that

$$\sum_{\mathbf{m},\mathbf{n}} |\langle f, g_{\mathbf{m},\mathbf{n}}\rangle|^2 = \int_{\mathbf{x}\in[0,1)^K} \int_{\mathbf{u}\in[0,1)^K} |\mathcal{Z} f(\mathbf{x},\mathbf{u})|^2 |\mathcal{Z} g(\mathbf{x},\mathbf{u})|^2 d\mathbf{x} d\mathbf{u},$$

which implies that $\{g_{\mathbf{m},\mathbf{n}}\}$ constitutes a frame if and only if $0 < A \leq |\mathcal{Z} g(\mathbf{x},\mathbf{u})|^2 \leq B < \infty$.

Such an analysis of the multidimensional single-window case with critical sampling, is a straightforward generalization of the well-known 1D case. Generalization to multidimensions of the multi-window arbitrary sampling case is presented in [RS97]. A major portion of [RS97] is dedicated to the analysis of scheme properties utilizing a matrix approach in the signal domain. The matrices involved are infinite-dimensional matrix-valued functions. Analysis in the ZT domain is also presented. The rational sampling rate of the 1D case, which enables the analysis with finite-order matrix-valued functions in the ZT domain, corresponds to the so-called compressible systems in the multidimensional case. [A compressible system possesses a certain relation between the two lattices generated by $\mathbf{A}$ and $\mathbf{B}$.]

12.3 Applications in image processing and computer vision

Assuming that the primitives of natural textures are indeed localized frequency components in the form of GEF's, texture analysis takes the form of inner product or correlation of such primitives with textured images. In addition to the positional information defined along the Gabor lattice in the combined position-frequency space, features characterizing texture are defined in relation to the GEF parameters. In the following sub-sections we illustrate the application of the Gabor approach to texture analysis and synthesis using a set of GEF's.

12.3.1 Image reconstruction from partial information

The example presented in Fig. 3 illustrates the application of the multi-window scheme in image reconstruction from partial information. The image (Fig. 12.3.1 (a)) of size 240×256, is a mosaic made of 8 natural textures. Reconstructions (approximations) by columns, where each column is reconstructed as a 1D signal using the 100 coefficients of the highest magnitude, are compared for three different schemes. The image represented and reconstructed by means of a single-window scheme, with a narrow Gaussian window (10 pixels effective width) and density $d = 1.5$, is shown in Fig. 12.3.1(b). A similar representation and reconstruction by means of a single-window scheme with a wide Gaussian window (effective width 40 pixels) is shown in Fig. 12.3.1(c). Finally, the application of a double-window scheme with both narrow and wide windows, is shown in Fig. 12.3.1(d). In all of these reconstructions, the same number of representation functions have been used. A careful scrutiny indicates that the reconstructed image is better in the case of the representation by a double-window scheme.

12.3.2 Feature extraction

As a simple case of texture discrimination, six localized features are extracted from a given texture or textured image. The discrimination is based on some metrics defined along the three dimensions of Gaborian features: spatial frequency along the preferred orientation, orientation of the spatial frequency and intensity information. Along each of these three dimensions the first and second moments are calculated to obtain six-dimensional features defined by the: dominant localized frequency (denoted by F), variance of the dominant localized frequency (VF), dominant orientation (T), variance of the dominant orientation (VT), mean of the localized intensity level (L) and variance of the localized intensity level (VL).

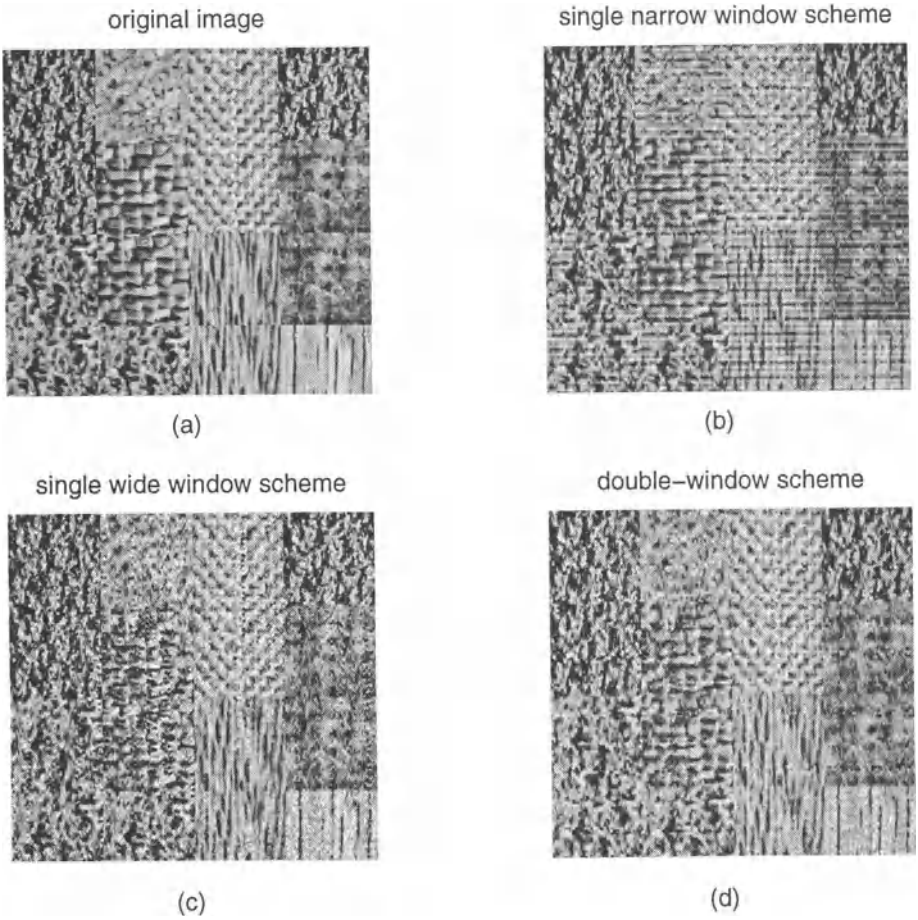

FIGURE 12.3.1. Reconstructions of the image utilizing the 100 coefficients of the highest magnitude. (a) Original image. (b) Single narrow window scheme. (c) Single wide window scheme. (c) Double-window scheme.

A textured image is first represented by its 2D Gabor coefficients denoted as $\{c_{m_1 m_2 n_1 n_2}\}$. Coefficients having common indices of n_1 and n_2 represent the subset of all Gaborian spectral components characterizing an effective local area centered at the point $x = n_1 a, y = n_2 a$. The related spatial frequencies are determined by the harmonic numbers m_1 and m_2, i.e., $\omega_x = m_1 2\pi b$ and $\omega_y = m_2 2\pi b$. Thus, according to this representation, the absolute value of a coefficient specified by the indices m_1, m_2, n_1, n_2 is proportional to the power of the related spatial frequency components (ω_x, ω_y) over the effective area determined by n_1, n_2 and a. The two-dimensional frequency components (ω_x, ω_y) are conveniently expressed by separating the periodicity along the preferred orientation and its orientation, using a polar coordinate representation.

It is assumed that the orientational parameter is fundamental to the definition of texture structure. This assumption is based on the observation that orientation is the most salient parameter characterizing the receptive field organization of cortical cells [Hub82]. It is further assumed that spatial frequency is an additional parameter characterizing the physiology of cortical cells [PR83, MF73, DDY79]. These two parameters are expressed in the polar coordinates by the spatial frequency $\omega = \sqrt{\omega_x^2 + \omega_y^2}$ and orientation θ, where $tan(\theta) = \omega_y / \omega_x$. Note that the preferred orientation defined in this paper is perpendicular to the one often used in neurophysiology of cortical cells [Hub82].

The dominant localized frequency (F) in the area designated by n_1, n_2 is defined by the weighted average of all the 2D Gaborian spectral components characterizing this local area:

$$F_{n_1 n_2} \triangleq \frac{\sum_{m_1=1}^{N-1} \sum_{m_2=1}^{N-1} |c_{m_1 m_2 n_1 n_2}| \sqrt{m_1^2 + m_2^2}}{\sum_{m_1=1}^{N-1} \sum_{m_2=1}^{N-1} |c_{m_1 m_2 n_1 n_2}|}, \tag{12.3.1}$$

where the number of spectral components (along the frequency coordinates) is determined by the sampling rate of the digitized image. Note that the term $\sqrt{m_1^2 + m_2^2}$ represents the harmonic number of the spatial frequency ω related to m_1 and m_2, and that expression (30) is reminiscent of a center of gravity. Since the spatial frequencies are accounted for in this expression regardless of their orientations, this feature is inherently rotation-invariant [KK86] and generally insensitive to translation. This is approximately so as long as the spatial sampling interval (a) along the Gabor lattice is large compared to the texture periodicity. This is not true however for scaling, in which case the resultant feature is directly proportional to the scaling factor. Thus the application of the dominant localized frequency is limited to cases where scaled versions of the same texture are considered different. The sensitivity of F to scaling is exploited in the definition of a scale invariant feature (Eq. 3.3).

The second moment (variance) of the localized frequency VF is defined by:

$$VF_{n_1 n_2} \triangleq \frac{\sum_{m_1=1}^{N-1} \sum_{m_2=1}^{N-1} \left| \sqrt{m_1^2 + m_2^2} - F_{n_1 n_2} \right|}{N^2} . \tag{12.3.2}$$

It provides a measure of regularity of textures and represents the bandwidth of the localized spatial frequency (in units of $2\pi b$). Since the bandwidth is affected when the whole texture is scaled, it is useful to normalize VF by the respective scaling factor which appears implicitly in the dominant localized frequency F. The normalized second moment of the frequency, accordingly defined by the coefficient of variation

$$\widetilde{VF}_{n_1 n_2} = \frac{VF_{n_1 n_2}}{F_{n_1 n_2}} , \tag{12.3.3}$$

generates a scale invariant feature. Like F, VF (or $\widetilde{VF}$) too is insensitive to rotation and/or translation by virtue of its definition. As will be demonstrated later, $\widetilde{VF}$ is instrumental in the classification of textures such as shown in Fig. 3a, where five of the eight textures are discriminated using this feature only. Two additional features are determined by the orientation characterizing the 2D spatial frequencies. Since $\omega_x = m_1 2\pi b$ and $\omega_y = m_2 2\pi b$, one can express the orientation by either ω_y / ω_x or m_2/m_1. Defining the orientation $\theta(m_1, m_2)$, related to a Gabor function indicated by m_1 and m_2 (for certain mx, my), as $\tan\theta(m_1, m_2) = (m_2/m_1)$ for $m_1 \neq 0$ and $\tan\theta(m_1, m_2) = \pi/2$ for $m_1 = 0$, the dominant local orientation is defined similarly to the central frequency by the center of gravity, using the Gabor coefficients representing the effective area specified by n_1, n_2:

$$T_{n_1 n_2} \triangleq \frac{\sum_{m_1=1}^{N-1} \sum_{m_2=1}^{N-1} |c_{m_1 m_2 n_1 n_2}| \theta(m_1, m_2)}{\sum_{m_1=1}^{N-1} \sum_{m_2=1}^{N-1} |c_{m_1 m_2 n_1 n_2}|} , \tag{12.3.4}$$

and the variance of the local orientation is defined by:

$$VT_{n_1 n_2} \triangleq \frac{\sum_{m_1=1}^{N-1} \sum_{m_2=1}^{N-1} |\theta(m_1, m_2) - T_{n_1 n_2}|}{N^2} . \tag{12.3.5}$$

The dominant local orientation is by its very nature sensitive to rotation, and as such is used in detection of edges (or contours) generated by rotations of the same texture. It should be pointed out, however, that the variance is a rotational-invariant feature, and that the dominant local orientation as well as its variance, are insensitive to scaling and/or translation.

The local mean intensity (L) is extracted in order to characterize differences with respect to smoothness of localized areas. Clearly, smooth segments of an image, such as sky or a uniform wall for example, cannot be discriminated from each other using spatial-frequency information since they are represented by almost constant functions. In this case the grey level information (or the color in the general case) is the only way to separate these regions. It appears as though these parameters are extracted in biological systems along channels other than those that are involved in Gabor-like processing (DC information is not mediated by X-type retinal ganglion cells, but rather by the non-linear Y-cells [HS76]).

The locality is accomplished in this case by multiplying the image with the two-dimensional window function. Then, by averaging the pixels in the resultant area we define:

$$L_{n_1 n_2} \triangleq \frac{1}{K} \sum_{x,y \in A(n_1 n_2)} I(x, y) , \tag{12.3.6}$$

defined by the window function centered according to n_1, n_2, and $I(x, y)$ is the intensity function. As may be concluded, this feature is insensitive to rotation, translation and to the scaling of the textures.

The variance of the intensity level VL is calculated to further facilitate the discrimination of textures. This is in particular instrumental in cases of irregular textures (the extreme case of which is the white noise). This is defined by:

$$VL_{n_1 n_2} \triangleq \frac{1}{K} \sum_{x,y \in A(n_1 n_2)} |I(x, y) - L_{n_1 n_2}| , \tag{12.3.7}$$

where $A(n_1 n_2)$ is defined as before. It is not related to the spatial-frequency (as may be easily demonstrated by the fact that the pixels designated by the area $A(n_1 n_2)$ can be rearranged without affecting the variance) and as such provides extra information for the classification process.

The first stage of discrimination is to obtain the position-dependent feature vector $(F, VF, T, VT, L, VL)_{n_1 n_2}$. Thus, for each localized area defined by n_1, n_2, a six-dimensional feature space is specified for which a classification scheme can be applied.

12.3.3 Classification results

The proposed method was tested using the eight textures shown in Figure 3a. Noisy versions of these textures were employed as input data for testing the stability of this classification process in the presence of noise. Square window functions of different sizes: 8x8, 16x16 and 32x32 pixels were used in the experiment.

The discrimination was tested also on rotated and translated versions of the textures shown in Figure 3a. VF and VL were calculated for six of the textures rotated in 49 different orientations (49x6 different samples in total). Even a simple minimum-distance classifier produces good classification results. The usual approach to evaluate a classifier performance is to create the so-called "confusion matrix" and consider the ratio of diagonal (correct classification) over off-diagonal (wrong classification) elements. The results presented in Table 12.3.3 indicate that the classification results are indeed good according to this criterion.

The accuracy obtained is indicated in the table for each texture. The main difficulty reflected in these matrices, is the poor discrimination between grass and bark in the presence of high noise levels. Indeed, the human observer also experiences the same difficulties.

12.4 Summary and discussion

Our original work on the Gabor scheme [PZ88] was motivated by findings related to biological vision and some understanding of the structure of natural images. In particular, there is plenty of evidence that biological processing of visual information is localized up to a certain stage of the visual pathway. This makes sense from the viewpoint of representation and processing of natural images since in the general case they are nonstationary, i.e., the structure of images varies across the visual field. Further, the structure of cortical receptive fields exhibits some ringing and decay as a function of the distance from the center of the field, and resembles the shape of a Gaborian elementary function. Indeed, analysis and synthesis of textural information indicates that Gaborian elementary functions are proper components for dealing with textures. In our multi-window Gabor scheme [ZZ97a, ZZ97b] we have generalized the original work to increase the degrees of freedom and thus permit the development of schemes which

(a)	Noise free							
	bark	44	0	0	0	0	0	(100%)
	wool	0	43	0	0	0	1	(098%)
	wood	0	0	38	5	1	0	(086%)
	textile	0	0	2	42	0	0	(095%)
	cork	0	0	0	0	43	1	(098%)
	grass	4	1	0	0	0	39	(088%)
	classification accuracy = 94.2%							
(b)	Noise variance = 7							
	bark	44	0	0	0	0	0	(100%)
	wool	0	43	0	0	0	1	(098%)
	wood	0	0	38	5	1	0	(086%)
	textile	0	0	2	42	0	0	(095%)
	cork	0	0	0	0	41	3	(093%)
	grass	8	0	0	0	0	36	(082%)
	classification accuracy = 92.3%							
(c)	Noise variance = 15							
	bark	44	0	0	0	0	0	(100%)
	wool	0	39	0	0	0	5	(087%)
	wood	0	0	37	7	0	0	(084%)
	textile	0	0	0	44	0	0	(100%)
	cork	2	0	0	0	41	1	(093%)
	grass	23	0	0	0	0	21	(048%)
	classification accuracy = 85.8%							

TABLE 12.1. Classification with reference to VF and VL using minimum distance classifier. The results presented in (a), (b) and (c) were achieved by classifying the testing samples of Figure 3a (noise free) and two additional noisy versions with noise variance of 7 and 15 respectively.

are most suitable for the analysis of images and the representation of visual information. In particular, as was stressed by the work on wavelets, scaling is important in dealing with images because of some resemblance to fractal structure which lends itself to pyramidal representation.

The multi-window Gabor scheme can incorporate any number and structure of windows. If one selects a geometrical sequence of identically shaped windows (e.g. all the windows having a Gaussian shape) and an exponential kernel, the resultant Gaborian scheme resembles some properties of wavelets and in particular scaling (see Fig. 2). This can be considered as either Gabor-type or wavelet-type scheme. In fact, it is a hybrid type of a scheme. In the case of special applications, such as separation of multi-component signal [ZZ97a, ZZ97b] the windows should be selected to best match the image structure. In this case, multi-window scheme can overcome in a way the constraint imposed by the basic uncertainty principle. In other words, by using both narrow and wide windows, one can combine high resolution in both space (time) and spatial (temporal) frequency. In this chapter we provided some of the formalism of this scheme, whereas specific examples can be found elsewhere [ZZ97a, ZZ97b].

The multi-window Gabor-type scheme can be further generalized to incorporate kernels other than the complex exponential [ZZ97a, ZZ97b]. This does not appear to be relevant to either vision or image representation, which is our main interest and motivation in the development of the scheme. However, in other applications of signal processing one may find better match between a scheme generated by kernels other than the complex exponential and the specific structure of the signals.

Most of the research and publications related to the Gabor scheme has been limited to one- or two-dimensional spaces. It is therefore important to note that generalization of a method or a scheme such as the Gabor-scheme, or wavelets for this matter, incorporates degrees of freedom which are not available at the lower-dimensional space. This may sound as truism and simple observation, but it is a deep observation of far reaching implications insofar as representation and processing of signals is concerned. For example two-dimensional compactly supported orthogonal wavelets can be designed to achieve linear phase such as two-fold symmetry or four-fold symmetry, whereas with one-dimensional wavelets it is impossible to accomplish all these requirements. We have therefore addressed in this chapter the issue of multi-dimensional Gaborian spaces.

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13

Gabor kernels for affine–invariant object recognition

Jezekiel Ben-Arie and Zhiqian Wang

ABSTRACT – We present an approach for affine-invariant object/target recognition by iconic recognition of image patches that correspond to object surfaces that are roughly planar. Each surface is recognized separately invariant to its 3D pose, employing novel Affine-Invariant Spectral Signatures (AISSs). The 3D-pose invariant recognition is achieved by correlating the image with a novel configuration of Gabor kernels[1] and extracting local spectral signatures. The local spectral signature of each image patch is then matched against a set of iconic models using multi-dimensional indexing in the frequency domain. Affine-invariance of the signatures is achieved by a new configuration of Gabor kernels with modulation in two orthogonal axes. The proposed configuration of kernels is *Cartesian* with varying aspect ratios in two orthogonal directions. The kernels are organized in subsets where each subset has a distinct orientation. Each subset spans the entire frequency domain and provides invariance to slant (foreshortening), scale and translation within the region of support of the kernels. The union of differently oriented subsets is utilized to achieve invariance in two additional degrees of freedom, i.e. swing and tilt. Hence, complete affine-invariance is achieved by the proposed set of kernels. The indexing method provides robustness in partial distortion, background clutter, noise, illumination effects and lower image resolution. The localized nature of the Gabor kernels allows independent recognition of adjacent shapes that correspond to object parts which could have different poses. The method yields 100% correct recognition rates in experiments over a wide range of slant, scale, swing, and tilt[2] with a dataset of 26 gray-level and infra-red models, in the presence of noise, clutter and other degradations.

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[1]We use the terminology "Gabor Kernels" to refer to modulated Gaussian kernels. Other people may use it with other meaning. Also, this configuration can be interpreted as multi-window Gabor scheme. A theoretical analysis of multi-window Gabor schemes can be found in Chapter 12.

H. G. Feichtinger et al. (eds.), *Gabor Analysis and Algorithms*

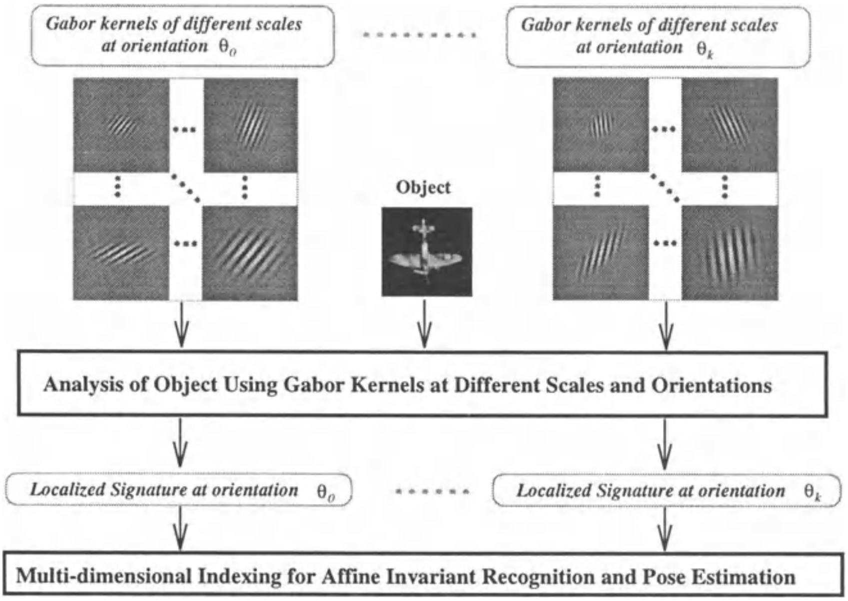


FIGURE 13.1.1. Block diagram of iconic recognition scheme.

13.1 Introduction

One could classify image understanding into two general approaches. The first approach involves preprocessing the image and further analysis deals with abstract information in the form of features such as lines, junctions, interest points, etc. Such simplified pictorial information is then used for interpretation and understanding of the image. However, in practical imagery, feature extraction necessitates prior segmentation that commits subsequent interpretation stages into analyzing information that might be incomplete and might also include a large amount of irrelevant clutter. Also, the large number of small features most often leads to a combinatorial explosion in the complexity of algorithms required for matching image data to stored models.

The method presented in this paper belongs to the second approach called iconic (or pictorial) representation. Here, the given image is directly projected onto a set of kernel functions. This approach is also equivalent to linear shift-invariant filtering (correlation) of the image with a set of filters represented as kernels. The coefficients obtained from this linear operation form a signature (feature vector) that yields a rich and localized representation of the image shape. In addition to describing the local shape, the signatures can also contain local frequency characteristics, which represent texture and other image attributes. In the frequency domain, our kernels divide the whole frequency range into exponentially increased subbands.

The width of the subbands has a geometric progression with a typical step of $1/4 - 1/8$ of an octave. As elaborated in Section 13.2, in order to achieve affine invariance, we also sample the frequency domain in different directions. Thus, our method could be regarded as a multi-wavelet approach. Iconic representation eliminates the need for a simplified feature image, and thus early and sometimes premature decisions (such as thresholding of edge detector outputs etc.) are completely avoided. Methods based on iconic representation can directly 'grasp' the entire iconic representation of local features and shapes. The localized nature of the Gabor kernels employed here allows independent recognition of adjacent shapes that correspond to object surfaces which could have different poses. Hence, a configuration of local signatures can characterize and identify a wide variety of 3D objects.

We have performed a literature survey which reveals that several previous iconic approaches have used localized kernels such as Gabor functions [BLv+92, Dau88b, WB94, Wil91, HN94] or Gaussian derivatives [RB95], and have achieved success with representation and recognition of image shapes. However, the main drawback of previous methods is that they can handle only similarity-transformed shapes (2D swing, translation and scale) and do not provide invariance to complete affine transformations that is required when a 2D shape is depicted in a 3D scene. As explained in detail in Section 13.2, we use the term 'affine transformation' to refer to orthographic projection and scaling, which quite accurately represents the perspective transformation of a planar shape in a 3D pose when viewed from a distance. Here, we seek an iconic representation that is invariant to scaling, slant, tilt, and 3D rotation and 3D translation of the shape.

In this paper, we present a method that yields a representation that is invariant to affine transformations. Such a shape representation is achieved by correlating the image with a new set of kernels and extracting spectral signatures which are affine-invariant. The new kernels are based on Gaussians with two-dimensional modulation, i.e. 2D Gabor kernels. The configuration of kernels suggested in this paper is *Cartesian* with varying aspect ratios in two orthogonal directions. The kernels are organized in subsets where each subset has a distinct orientation. Each subset spans the entire frequency domain and provides invariance to slant[2], scale and limited translation[3]. The union of differently oriented subsets, which now has a redundancy in the frequency domain is utilized to achieve invariance in two additional degrees of freedom, i.e. swing (image rotation around the optical axis) and tilt. Hence, complete affine-invariance is achieved by the entire set of kernels. In our literature survey, we find that previous

[2]Formal definitions of slant, tilt, swing and scale are included in Section 13.2

[3]Signatures obtained don't vary significantly with respect to the relative position of image patch within a limited range (approximately $\pm\sigma$)

approaches tessellate the entire frequency domain in a *polar* fashion and are based on polar configuration of kernels with constant aspect ratios [BLv+92, Dau88b, WB94, Wil91, HN94]. In contrast to the method introduced here (which provides invariance to all geometrical distortions in scaled orthographic projections), previous iconic approaches provide only scale and swing invariance, and are unable to compensate for slant (which causes shape foreshortening) due to the configuration of their kernels. Furthermore, the Cartesian organization described in this paper - in which each subset of kernels has a uniform orientation - provides a more plausible explanation to the orientational columns found in the visual cortex. In Section 13.2, we describe the new set of kernels, and the affine-invariant properties of the spectral signatures.

We have used the spectral signatures obtained from the kernels for affine-invariant recognition of image patches that correspond to object surfaces that are roughly planar [BAWR96c, BAWR96b, BAWR96a]. The process of recognition also includes estimation of the 3D pose. The recognition is based on a Multi-Dimensional Indexing (MDI) [CM94] scheme in the frequency domain. The indexing method provides robustness in partial distortion, background clutter, noise, and illumination effects, and image degradations due to lower resolution. Multi-dimensional indexing has several advantages over conventional low-dimensional hashing [YLW88]. Specifically, by adding more dimensions to the indexing scheme, one can use very coarse quantization (and thus gain robustness), and still eliminate overcrowding of bins in the hash table, without reducing discrimination. One can also construct datasets with significantly larger retrieval size using MDI.

Experimental results of invariant recognition using MDI of the spectral signatures are presented in Section 13.4. Experiments with a model dataset of 26 gray-level models yield close to 100% correct recognition rates over more than 3 octaves of scale, 360 degrees of swing and tilt, and more than 80 degrees of slant. Additional experiments also reveal that the system is very robust to additive noise (colored and white) at levels of up to -5 dB SNR, and is also robust to lowered resolution of the objects and background clutter. Initial experiments with recognition of objects extracted from actual Forward-Looking Infra-Red (FLIR) imagery also yield successful view-invariant recognition.

13.2 Affine-invariant spectral signatures (AISSs)

Our overall approach is based on iconic recognition of image patches that correspond to object surfaces that are approximately planar. As elaborated later, the object surfaces can be recognized in a general 3D pose. The class of objects that can be recognized is not limited to convex objects and also

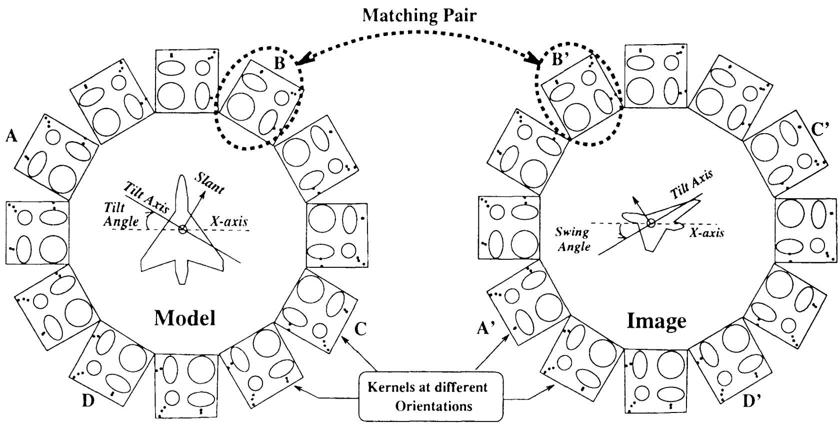


FIGURE 13.2.1. Invariance of signatures to swing and tilt of the shape. For the displayed swing and tilt of the airplane shape, the oriented kernel pairs **A-A'**, **B-B'**, **C-C'**, and **D-D'** yield invariant signatures. Note that since the kernels have symmetry properties, it is required to implement only one quadrant of kernels spanning 90 degrees of orientation to achieve swing and tilt invariance.

can include concave objects. Convexity is not a necessary condition, as long as an object has at least one approximately planar surface with distinctive features, it can be recognized by this approach. Furthermore, as experimental results demonstrate in Section 13.4, many non-planar objects which have approximately flat shapes such as hands, airplanes, etc. are robustly recognized with our approach as well. We also assume that the dimensions of object surfaces are relatively small with respect to the viewing distance. Thus, perspective transformations can be well approximated by affine transformations of surface's points [RW95].

We use the affine transformation to simulate transformation of a planar shape that undergoes 3D rotation and 3D translation, and is then orthographically projected onto the image plane and scaled (reduced or increased in size). A point $\underline{R} = (x, y)^T$ in the coordinate system of the shape is affine-transformed to a point in imaging plane's coordinate system $\underline{R}_a = (x_a, y_a)^T$ according to the following formula:

$$\begin{pmatrix} x_a \\ y_a \end{pmatrix} = P \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} \tag{13.2.1}$$

where matrix P represents tilt and slant operation, $(c, d)^T$ denotes translation. This general formulation represents any orthographic projection plus scaling of planar shapes. Such a projection approximates perspective projections quite accurately if the viewing distance of the shape is relatively large with respect to the shape's dimensions. Based on Singular Value Decomposition (SVD), the matrix P can be decomposed as follows:

$$P = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{pmatrix} \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix} \tag{13.2.2}$$

where λ_1 and λ_2 are eigenvalues of PP^T, α and β are angles related to eigenvectors of PP^T and P^TP. In practical situations, P is usually a nonsingular matrix, so λ_1 and λ_2 have positive values. If we arrange the eigenvalues so that $\lambda_1 \geq \lambda_2$, the eigenmatrix $\Lambda^{1/2}$ can be posed as

$$\Lambda^{1/2} = \sqrt{\lambda_1} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\lambda_2/\lambda_1} \end{pmatrix} \tag{13.2.3}$$

According to Eq. (13.2.1) and Eq. (13.2.2), any orthographic projection of points on a plane can be represented by a sequence of transformations which include translation, tilt, slant, scale and swing (image rotation). To represent 3D rotation of a plane it is necessary to use slant and tilt transformations in which the shape is posed on a plane which is slanted and tilted with respect to the image plane. Slant angle is measured between the normals of the image and shape planes. Tilt is defined as the angle between the the X-axis in the image plane and line L created by the intersection of the image and shape planes (the tilt axis L). Here, this angle is defined as β in Eq. (13.2.2). α in Eq. (13.2.2) represents shape swing within the image plane. The slant angle τ corresponds to shape foreshortening in the image plane along the axis normal to the line L. The foreshortening ratio is equal to $\cos(\tau) = \sqrt{\lambda_2/\lambda_1}$. In contrast to slanting, scaling causes uniform foreshortening (or enlargement) in the image plane in all directions. The scale factor is equal to $\sqrt{\lambda_1}$ in Eq. (13.2.3). The above parameters of translation, swing, scale, slant and tilt completely represent scaled orthographic projection.

When a planar object undergoes affine transformation, the frequency spectrum of its image is also transformed by a similar set of transformations. Given a function $f(\underline{R})$ with Fourier Transform as $\hat{F}(u,v)$, its affine transformed version $f_a(\underline{R}) = f(P^{-1}\underline{R} - P^{-1}(c,d)^T)$ has the frequency spectrum as follows:

$$|\hat{F}_a(u_a, v_a)| = |P||\hat{F}(u,v)|$$

$$\begin{pmatrix} u_a \\ v_a \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} \lambda_1^{\frac{-1}{2}} & 0 \\ 0 & \lambda_2^{\frac{-1}{2}} \end{pmatrix} \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{13.2.4}$$

where $|P|$ denotes the determinant of matrix P. Thus, in the frequency domain, the effect of the affine transform on the spectrum is almost the same as that of the affine transform on the object in spatial domain except for two major differences: First, the spectrum is inversely scaled and slanted. Secondly, shape translations parallel to the image plane do not affect the spectrum.

We define a tilted coordinate system (u_t, v_t) as

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{13.2.5}$$

When the shape is merely tilted, sampling the shape's spectrum $|\hat{F}_t(u_t, v_t)|$ $= |\hat{F}(u, v)|$ along (u_t, v_t) axes in logarithmic configuration results in a spectral representation we call spectral signature $S_t(\delta_1, \delta_2)$, where $\delta_1 = \log_\gamma(u_t)$ and $\delta_2 = \log_\gamma(v_t)$. A tilted, slanted and scaled shape which has a spectrum of $\hat{F}_a(u_t, v_t) = |P|\hat{F}_t(u_t/\sqrt{\lambda_1}, v_t/\sqrt{\lambda_2})$ is transformed by logarithmic sampling of $\hat{F}_a(u_t, v_t)$ into a shifted spectral signature $S_a(\delta_1, \delta_1) =$ $|P|S_t(\delta_1 - \alpha_1, \delta_2 - \alpha_2)$, where $\alpha_1 = \log_\gamma(\sqrt{\lambda_1})$ and $\alpha_2 = \log_\gamma(\sqrt{\lambda_2})$. We note that the signature is not changed due to slanting and scaling but only is translated in (δ_1, δ_2) space (see in Fig. 13.2.4 and Fig. 13.2.4).

It is noted here that a 2D Cartesian - frequency domain version of the Mellin transform which is defined as

$$M_t(\zeta_1, \zeta_2) = \iint |\hat{F}_t(u_t, v_t)| u_t^{-j\zeta_1 - 1} v_t^{-j\zeta_2 - 1} du_t\, dv_t \tag{13.2.6}$$

also achieves invariance to slanting and scaling that results in linear phase shifts proportional to $\ln(\sqrt{\lambda_1})$ and $\ln(\sqrt{\lambda_2})$.

$$M_a(\zeta_1, \zeta_2) = |P|(\frac{1}{\sqrt{\lambda_1}})^{-j\zeta_1}(\frac{1}{\sqrt{\lambda_2}})^{-j\zeta_2} M_t(\zeta_1, \zeta_2) \tag{13.2.7}$$

$M_a(\zeta_1, \zeta_2)$ is the Mellin transform of $\hat{F}_a(u_t, v_t) = |P|\hat{F}_t(u_t/\sqrt{\lambda_1}, v_t/\sqrt{\lambda_2})$ and $M_t(\zeta_1, \zeta_2)$ is the Mellin transform of the unslanted and unscaled spectrum $\hat{F}_t(u_t, v_t)$.

Hence, the signature from the affine transformed object is a shifted version of the signature from the object itself except for a scalar $|P|$. The shift in the 2D signature space (δ_1, δ_2) directly depends on the slant and the scale included in the affine transformation. In our scheme, we generate for the affine transformed object a set of signatures which have equally spaced orientations and which span the range of 360 degrees. Among the set of signatures generated, there exists one which matches the signature of the original object except for a translation in the (δ_1, δ_2) space (that represents scale and slant differences). The estimation of translation from $S(\delta_1, \delta_2)$ is much easier than direct estimation of slant and scale from phase shifts in $M(\zeta_1, \zeta_2)$.

Figure 13.1.1 displays a block diagram of the overall system. The image is correlated with a set of Gabor kernels. The frequencies of the kernels are derived from a logarithmic sampling according Eq. (13.2.8) and

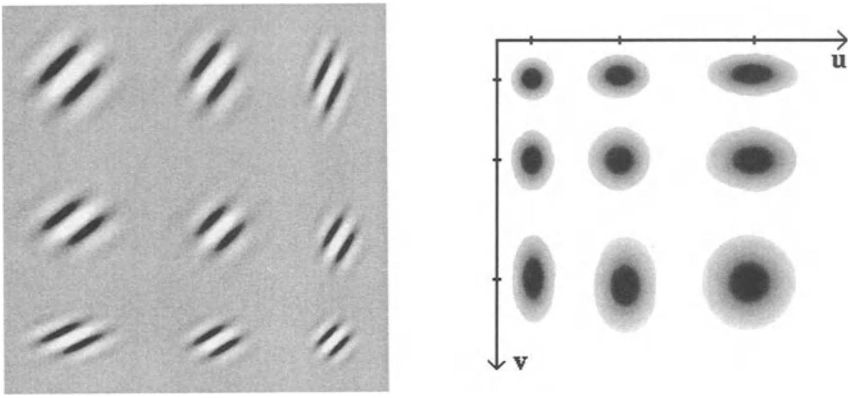


FIGURE 13.2.2. Partial Subset of Kernels $K[\theta_l]$ ($f_x = 1$ and $f_y = 1$) for orientation $\theta_l = 0$ degrees in spatial domain (left) and the configuration of corresponding kernels in frequency domain (right). Each subset $K[\theta_l]$ completely spans the frequency domain.

Eq. (13.2.9). This set is centered at various 'interest locations' which correspond to centers of prominent image patches. A set of spectral signatures is then generated. Each signature represents a local image patch. These signatures are then independently recognized using Multidimensional Indexing (MDI). The 3D pose (slant, tilt, swing and scale) of each recognized patch is also obtained as a by-product. This scheme can be extended to recognize configurations of patches as objects/parts using a hierarchical MDI scheme.

The affine-invariant representation presented in this paper is based on a set of elliptical 2D Gabor kernels defined as

$$g_{m,n}[f_x, f_y, \theta_l](x,y) = \exp(-j2\pi(\frac{x_l f_x}{\sigma_{X_m}} + \frac{y_l f_y}{\sigma_{Y_n}})) \cdot \exp(-\frac{x_l^2}{2\sigma_{X_m}^2} - \frac{y_l^2}{2\sigma_{Y_n}^2})$$

$$\begin{pmatrix} x_l \\ y_l \end{pmatrix} = \begin{pmatrix} \cos\theta_l & \sin\theta_l \\ -\sin\theta_l & \cos\theta_l \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{13.2.8}$$

where f_x, f_y are frequency coefficients, $f_x, f_y = 1...N_f$. The standard deviations σ_{X_m} and σ_{Y_n} of these elliptical kernels vary in a geometrical progression with the indices m and n as

$$\sigma_{X_m} = \gamma^{m-1}\sigma_0;\ \sigma_{Y_n} = \gamma^{n-1}\sigma_0;\ m,n = 1...N_\sigma \tag{13.2.9}$$

where the geometric ratio $\gamma > 1$ and the smallest standard deviation σ_0 are constants. The indices m and n define the sampling points in the (δ_1, δ_2) space. In addition, the Gabor in Eq. (13.2.8) is modulated in two orthogonal axes which have orientation θ_l (denoted by X_l and Y_l) by complex sinusoid with periods proportional to the deviations of corresponding Gaussian profile. The parameter θ_l denotes the orientation of the kernel and uniformly spans the range $[0, 360)$ degrees in discrete steps $l = 1...N_\theta$.

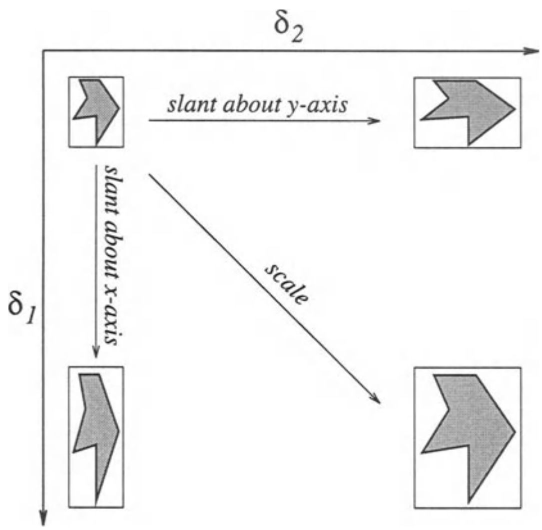


FIGURE 13.2.3. Shifting property of the spectral signature in the (δ_1, δ_2) plane with respect to scaling and slanting of an arbitrary shape.

There are several reasons to use the Gabor kernels suggested in this paper. Firstly, Gabor functions yield the smallest conjoint space-bandwidth product permitted by the uncertainty principle of Fourier analysis [Dau88b, Wec90]. Arbitrary signals can be represented as convergent series of building blocks from such Gabor kernels. Also, Gabor kernels are very similar to the profiles of the receptive fields found in the mammalian visual cortex [HW74, Dau88b]. Moreover, the suggested organization of the kernels - where each subset of kernels spans the entire frequency domain but has uniform orientation selectivity (expressed in the direction of modulation of the Gaussians) - conforms well with the organization of uniform 'columns' found in the visual cortex [HW74].

The above scheme generates a subset of modulated Gaussian kernels $K[f_x, f_y, \theta_l] = \{g_{m,n}[f_x, f_y, \theta_l]\ ;\ m, n = 1...N_\sigma\}$ with identical orientation θ_l and identical frequency coefficients (denoted by f_x and f_y), but *with varying aspect ratio and size* (indexed by σ_{X_m} and σ_{Y_n}). For each orientation θ_l, we have a cumulative subset $K[\theta_l]$ of kernels which includes all the frequency coefficients, i.e., $f_x, f_y = 1...N_f$. The complete set of kernels K consists of the union of all the subsets $K[\theta_l]$ swung to different orientations $\theta_l\ ;\ l = 1..N_\theta$ that uniformly span 360 degrees. In practice it is required only to generate kernels that span one quadrant (90 degrees) of orientation. All the other kernels can be constructed from this reduced set using symmetry properties. An example of one subset of kernels $K[\theta_l]$ with $\theta_l = 0$ degrees is illustrated in Fig. 13.2.2. The frequency spectrum of this

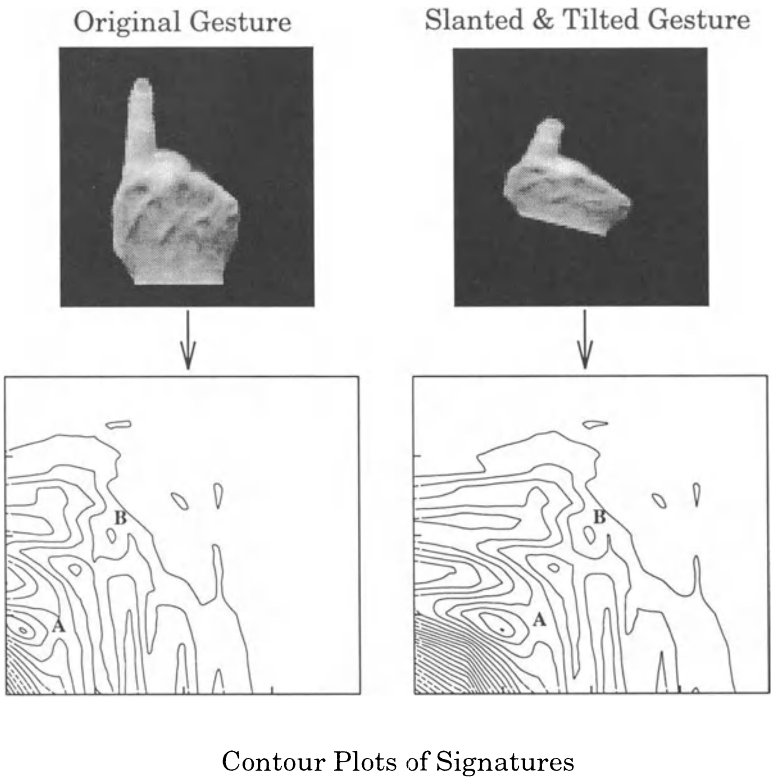


FIGURE 13.2.4. Contour plots of the signature for the original gesture image (left) and shifted spectral signature obtained for the slanted and tilted gesture (right). The labels **A** and **B** illustrate the shift in the signature.

subset of kernels is also illustrated in Fig. 13.2.2 (right), and shows that each subset of kernels $K[\theta_l]$ completely spans the band-limited frequency domain of interest and is logarithmically spaced as requested. However, the reason for introducing the over complete set K is that the redundancy in the complete set K is utilized to overcome tilt and swing. Also, the spectral signatures obtained are invariant to translation within the receptive field.

When a local image patch $I(x, y)$ is correlated with this configuration of kernels, it generates a set of multi-dimensional spectral signatures

$$\{S[f_x, f_y, \theta_l]; f_x, f_y = 1...N_f \ , \ l = 1...N_\theta\}$$

composed of the correlation (projection) coefficients of all the kernels. Mathematically,

$$S[f_x, f_y, \theta_l](\sigma_{X_m}, \sigma_{Y_n}) = |\langle g_{m,n}[f_x, f_y, \theta_l](x, y), I(x, y)\rangle| \qquad (13.2.10)$$

for $m, n = 1...N_\sigma$, where $|\cdot|$ denotes modulus of a complex number. The indices m and n define the sampling points in the (δ_1, δ_2) space. The dimensions of the spectral signature correspond to (δ_1, δ_2). Hence the signatures differ in their orientation and frequencies, while we have a set of aspect ratios along the X_l, Y_l axes within each signature.

The *principle of slant-invariance* is derived from the fact that when the image patch corresponds to a slanted shape, say in axis X_l, the correctly oriented signature $S[f_x, f_y, \theta_l]$ *shifts* in the direction of the slant, i.e. δ_2, with respect to the signature of the unslanted shape. Also, when the shape is scaled, all the signatures $\{S[f_x, f_y, \theta_l]\ ;\ l = 1...N_\theta\}$ shift equally, i.e. diagonally, in the (δ_1, δ_2) plane. Hence, any combined slant and scale results in a corresponding shift in the (δ_1, δ_2) plane. Fig. 13.2.3 illustrates these properties of the signature. Fig. 13.2.4 displays contour plots of the signature of the hand model and the corresponding signature when the hand is slanted by 60 degrees with a tilt of 15 degrees. It is easily observed (see the labels **A** and **B** on the plots displayed for easy registration) that the signature does not change except for a translation in the (δ_1, δ_2) plane. The translation between a model signature and the image signature can be used to compute the relative pose between the two. The difference in δ_1 and δ_2 can be directly translated into relative slant and scale. The other angular pose parameters of tilt and swing can also be retrieved as described below. The positional parameters of the pose are also retrievable. The X and Y coordinates are derived from the image, and the depth parameter can be derived from the scale.

Since shapes can be slanted and tilted in any orientation in space, one has to generate a subset of kernels for each tilt direction and for each orientation, which forms two rotational degrees of freedom. These two degrees of freedom are dealt with by using the complete set of kernels K both for the model signature and for the image signature. This is demonstrated in Fig. 13.2.1, where it is shown that even if the model is tilted and swung, there is exact correspondence between four of the model signatures (marked by labels **A** through **D**) and four of the image signatures (marked by labels **A'** through **D'**). This invariance to swing and tilt is possible only because both the model and the image are processed by subsets of kernels at different orientations. Using the matching model and image orientations one can estimate the relative swing and tilt between the model and the image. We note that in practice it is required only to generate kernels that span one quadrant (90 degrees) of orientation. All the other kernels can be constructed from this reduced set using symmetry properties. In Section 13.4, it is experimentally found that sampling of 7.5 degrees in θ_l is a sufficient interpolation to accommodate any intermediate values of tilt and swing.

From Eq. (13.2.10), we see that the signatures are related only to the magnitudes of the complex correlation coefficients, the phase information being completely eliminated. Thus, the signatures obtained are - to a large extent - invariant to the translation of the object within the localized Gabor windows (receptive field).

Hence, the combined set of kernels K, composed of all the subsets $K[\theta_l]$ sufficiently covers scale, slant, tilt, swing, and translation, i.e. all affine transformation parameters that simulate the scaled orthographic projection.

13.3 Affine-invariant recognition by multi-dimensional indexing (MDI)

Our recognition scheme is based on the affine-invariant nature of the spectral signatures described in Section 13.2. As explained above, when the shape is slanted with a tilt axis of orientation θ_l, the signature $S[f_x, f_y, \theta_l]$ - which corresponds to the tilt orientation θ_l - undergoes simple shifts in the (δ_1, δ_2) plane that correspond to scale and slant transformations. The purpose of the indexing scheme is to robustly identify the image patch from its set of signatures. Each signature $S[f_x, f_y, \theta_l]$ corresponds to a combination of orientation θ_l and the frequency coefficients f_x, f_y. A robust recognition scheme is required since the signatures could be partially distorted due to illumination variations, due to the discrete nature of the orientation or due to the limited range of scales. Furthermore, irrelevant clutter in the receptive field and partial occlusion can result in additional distortions.

In order to overcome these signature distortions, we implement a voting scheme using the spectral signatures, based on MDI [CM94]. MDI basically relies on the same principles as the geometric hashing method [YLW88]. The main difference is that the indices for the hash table have many dimensions which correspond to many invariant shape characteristics. The low dimensionality of geometric hashing causes overcrowding of bins, and the hash table sometimes saturates even with a small number of objects. On the other hand, MDI improves the robustness of the recognition (which is expressed as the ratio of the highest vote to the next highest vote). This result was also observed by [RB95]. The innovation of our indexing scheme is that it is implemented in the frequency domain using spectral signatures. Additional merits of MDI are that the retrieval size of the dataset is considerably increased, the overcrowding of bins in the hash table is almost eliminated, and coarser quantization can be used without reducing discrimination.

In our indexing scheme, the hash table is updated by each model using all its signatures $S[f_x, f_y, \theta_l]$. 20-dimensional indices are generated for the models using combinations of frequencies and orientations, and ratio of amplitudes of the corresponding signatures. The indices for the MDI consist of all the combinatorial pairs of points in the (σ_1, σ_2) plane. Each such pair of points provides five dimensions to the index, consisting of their relative offset, their relative amplitudes via their ratio, and two parameters based on local neighborhood values. A hash table is used to store all the indices of the models. The relative pose, which corresponds to difference in δ_1 and δ_2, and the orientation θ_l, is also stored. These numbers can subsequently be translated to the relative slant, tilt, scale and swing between the image patch and the model.

The ratio of amplitudes is used as part of the multi-dimensional index. In Eq. (13.2.4), we see that the affine transform introduces a pose dependent shift of the signatures as well as a scalar $|P|$. Using the ratio eliminates the effect of this scalar and also yields invariance to variations of illumination.

In our experiments (see Section 13.4), we use 20-dimensional indices, and have obtained 100% correct affine-invariant recognition in a range of more than 3 octaves of scaling and slant angles of more than 80 degrees, with image swing and shape tilt of 360 degrees in a library of 26 gray-level models. Experiments also reveal that the method works with severe additive white and colored noise (SNR of -17 dB to 5 dB), partial occlusion, and background clutter.

13.4 Experimental results

This section describes experimental results using the above mentioned approach for affine-invariant recognition. In these experiments, according to the notation of Eq. (13.2.10) in Section 13.2, the kernels $g_{m,n}[f_x, f_y, \theta_l](x, y)$ employ a set of Standard Deviations $\{\sigma_{X_m}, \sigma_{Y_n} = 8.49, 10.09, 12, 14.27, 16.97, 20.18, 24\}$, a set of frequency coefficients $\{(f_x, f_y) = (1,1)(2,2)(4,4)(7,7)\}$, and 24 orientations θ_l in steps of 7.5 degrees. For a given image patch $I(x, y)$, a set of spectral signatures $S[f_x, f_y, \theta_l]$ is generated by correlating it with the above kernels.

As elaborated in Section 13.2, these signatures are used along with a MDI scheme for affine-invariant recognition. Each index is created for 3 relative frequency combinations, and has 20 dimensions for a given θ_l. For each model to be included in the hash table, signatures are generated using the kernels $g_{m,n}[f_x, f_y, \theta_l](x, y)$, and the set of 20-dimensional indices are computed. Each index is included as an entry in the hash table along with the pose parameters of the model, represented by δ_1, δ_2 and θ_l. Given an image patch to be recognized invariant to affine transformation, its signatures and 20-dimensional indices are generated in an identical fashion. These indices are then compared with the hash table and each matching index adds one vote for the corresponding model in that entry. In addition, the values of δ_1, δ_2 and θ_l stored in the entry is used to vote for the pose of each model in the entry. The total number of votes accumulated by each model (with pose) over all the indices of the test image is the matching

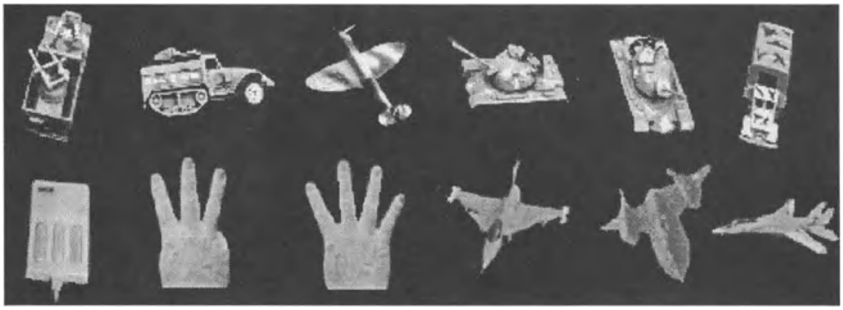


FIGURE 13.4.1. Twelve of the 26 model objects in the library. Close to 100% recognition is achieved over a wide range of slant, tilt, scale, and swing. Note that many of the models (such as the hands) are quite similar in appearance and still are correctly recognized.

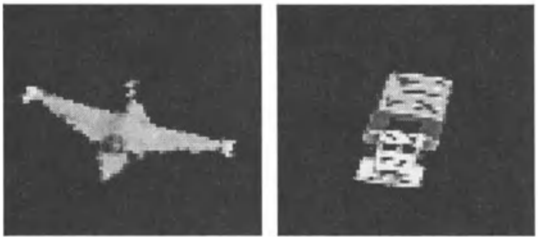


FIGURE 13.4.2. Two test images that correspond to affine-transformed models (airplane and truck).

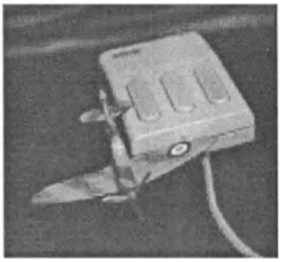


FIGURE 13.4.3. Recognition of neighboring objects (airplane and mouse) in background clutter.

score for that model.

We use a library of 26 objects in our initial experiments, 12 of which are displayed in Fig. 13.4.1. Since the experiments are performed to test the recognition scheme in this paper, we consider every object in the library as a single patch. As one can see, these models consist of real gray-level images (128×128) of objects with some amount of texture as well. As described above, a hash table is created using a single set of signatures from each model. Test images are obtained by affine transforming the model images and adding white or colored noise. Experiments were performed under varied conditions of slant, tilt, scale and swing and yielded 100% correct recognition rates. Also, high Discriminative Voting Ratio (DVR) (defined as the ratio of the highest vote to the next highest vote) was obtained in the experiments. An average DVR of 6.3 was obtained in the recognition experiments. In addition, the pose of each model was also estimated correctly in all experiments. Complete recognition was achieved over a range of more

TABLE 13.1. Correct Recognition Rate Versus Scale Factor (in Octave)

Scale (Octave)	1.0	0.875	0.75	0.625	0.5	0.375
Recognition (%)	100	100	100	100	100	100
Scale (Octave)	0.25	0.125	0	-0.125	-0.25	-0.375
Recognition (%)	100	100	100	100	100	100
Scale (Octave)	-0.5	-0.625	-0.75	-0.875	-1	-1.125
Recognition (%)	100	100	100	100	100	100
Scale (Octave)	-1.25	-1.375	-1.5	-1.625	-1.75	-1.875
Recognition (%)	100	100	100	100	100	100
Scale (Octave)	-2	-2.125	-2.25	-2.375	-2.5	-2.625
Recognition (%)	100	100	100	100	100	92.3
Scale (Octave)	-2.75	-2.875	-3			
Recognition (%)	92.3	80.8	61.5			

TABLE 13.2. Correct Recognition Rate Versus Slant Angle (in Degrees)

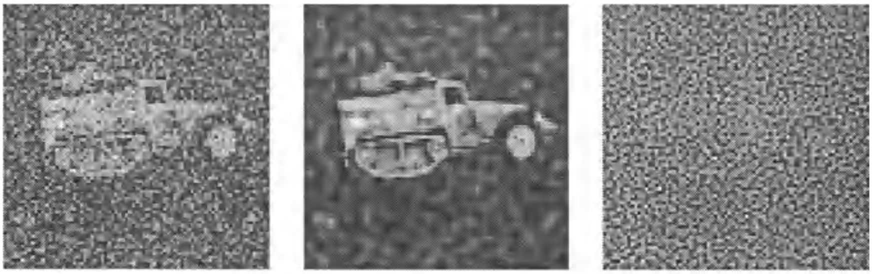
Slane (Degrees)	[0.0 ... 82.8]	83.4	84.0	84.5	84.9	85.4
Recognition (%)	100	92.3	92.3	88.5	80.8	57.7

than 3 octaves of scaling, slant angles of more than 80 degrees (foreshortening ratio of 1:7), and image swing and shape tilt of 360 degrees. Two of the successfully recognized test images are displayed in Fig. 13.4.2. We have also obtained 100% recognition with higher DVR with similar experiments on binary shapes over a wide range of affine transformations.

In scale experiments, we can see from Table 13.1 that we achieve 100% recognition rate even when images are down scaled by 2.5 octaves. It should be noted that the images scaled halfway in between the decimation interval for our Gabor kernels are still correctly recognized. The maximum error in pose estimation is 1.09 of scale factor. In slant experiments, the images are scaled only in one dimension. Minimal scale factor is around 0.0743, which corresponds to a slant angle of 85.4 degrees. Table 13.2 shows the correct recognition rates for different slant angles.

The test image of Fig. 13.4.3 displays two slanted and tilted models in close neighborhood with some background clutter. Both the trackball and the airplane models are successfully recognized when two independent signatures are centered on each object, and their 3D poses are accurately estimated. This experiment demonstrates that the scheme is basically robust to neighboring shapes and background clutter in the image.

Figure 13.4.4 illustrates three test images that are noisy versions of the corresponding model in Fig. 13.4.1. Experiments are carried out with additive white noise, low-frequency colored noise (normalized low pass cutoff frequency $= \pi/2$), and high-frequency colored noise (normalized high pass



a b c

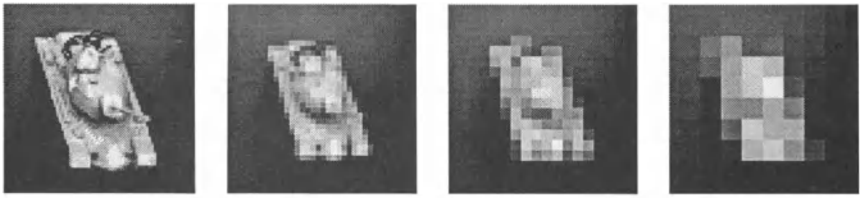
FIGURE 13.4.4. (a) Successfully recognized test image with additive white noise (SNR=-1.8 dB). (b) Successfully recognized test image with additive low-frequency colored noise (SNR=5.0 dB). (c) The test image is recognized even though it is hardly seen (additive high-frequency colored noise of SNR=-17.0 dB).

TABLE 13.3. Correct Recognition Rate Versus Signal to Noise Ratio (dB)

SNR (db)	25	20	15	10	5
Recognition (%)	100	100	100	100	100
SNR (db)	0	-1.8	-5	-10	-15
Recognition (%)	100	100	92.3	30.8	7.7

cutoff frequency = $\pi/2$). For each kind of noise, we experimentally find the largest noise level at which successful recognition with correct pose estimation is obtained. As seen in Fig. 13.4.4, the test image is successfully recognized in all three cases at very high noise levels, demonstrating that the scheme is quite robust to additive noise. The Gabor kernels capture the image information mostly in the low and middle frequencies, and thus the scheme is almost insensitive to high-frequency noise (Fig. 13.4.4.c, SNR=-17 dB) since this noise is outside the frequency range of the kernels. The scheme is also quite resistant to white noise (up to SNR=-1.8 dB), and slightly less resistant (up to SNR=5dB) to low-frequency noise for the same reason. Thus, we can conclude that the overall recognition scheme is quite robust to the effects of additive noise and clutter. Table 13.3 gives the correct recognition rates for white-noise corrupted images with different levels of SNR.

Figures 13.4.5.b-d illustrate test images that have reduced resolution with respect to the model image in Fig. 13.4.5.a (which is 128×128 in size). Experiments were performed over all the 26 models for each of these resolutions. Apart from the reduced resolution, all the test images correspond to model images and are scaled by a factor of 0.8 and swung by 30 degrees. Over all the 26 models in the library, the medium resolution (64×64) set of test images (see Fig. 13.4.5.b) yields 100% successful recognition with the correct pose estimated in all tests. In the low resolution (32×32) experi-



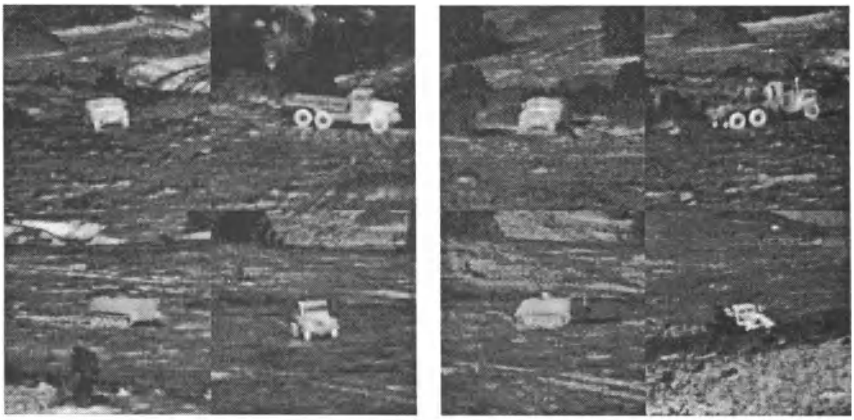
a b c d

FIGURE 13.4.5. (a) Original high resolution model (128 × 128). (b) Medium resolution test image (64 × 64). (c) Low resolution test image (32 × 32). (d) Very low resolution test image (16 × 16).

TABLE 13.4. Correct Recognition Rate Versus Resolution Reduction Factor

Resolution	1	1/2	1/3	1/4	1/5	1/6	1/7	1/8
Recognition (%)	100	100	100	96.2	100	88.5	69.2	42.3

ments of Fig. 13.4.5.c, 96% of the test images were successfully recognized along with accurate pose estimation. When the image resolution is drastically reduced to very low resolution (16×16) as in Fig. 13.4.5.d, the scheme is still able to successfully recognize 42% of the test set. These results show that the representation and recognition scheme is quite robust to significant degradation that correspond to lower resolution. Such degradation could occur from large viewing distances. In Table 13.4, the correct recognition rates under different levels of resolution reduction are given.



a b

FIGURE 13.4.6. (a) The four Forward-Looking Infra Red (FLIR) models. (b) The four successfully recognized FLIR objects.

Initial experiments also reveal that the system also successfully recognizes objects in Forward-Looking Infra-Red (FLIR) imagery. Figure 13.4.6.a displays the four FLIR models and Fig. 13.4.6.b displays the four test FLIR objects. The affine-invariant recognition scheme successfully recognizes all the four objects despite the substantial change in view and background clutter. Currently, the computation mainly required is in the generation of model signatures and corresponding hash tables. A typical recognition time for one object is about 15 seconds with 150MHz Pentium processor with a dataset of 26 models. We are currently working on improving the efficiency of our approach.

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14

Gabor's signal expansion in optics

Martin J. Bastiaans

ABSTRACT – In this chapter some applications of Gabor's signal expansion in the field of optics are considered. After a preparatory treatment of some necessary optics fundamentals and the translation of relevant concepts of time-dependent signals to signals that depend on spatial variables, Gabor's signal expansion and its companion – the Gabor transform – are introduced in the field of optics. Special attention is paid to Gaussian windows, which are related to the well-known concept of Gaussian light beams in optics. The case of critical sampling is considered, in particular in its relation to the degrees of freedom of an optical signal and to the space-bandwidth product of an optical system; to do this, the propagation of Gabor's expansion coefficients through optical systems is considered. The case of integer oversampling is considered and it is shown how in that case the Gabor transform can be transformed into a product of Zak transforms. It is demonstrated how this product of Zak transforms can form the basis of a coherent-optical setup for generation of the Gabor transform and it is shown how this setup can be used for an approximate generation of the windowed Fourier transform.

14.1 Introduction

In his original paper, Gabor suggested the representation of a time signal in a combined time-frequency domain. Actually he proposed to represent the signal as a superposition of shifted and modulated versions of a so-called elementary signal. Moreover, as an elementary signal he chose a Gaussian signal, because such a signal has a good localization, both in the time domain and in the frequency domain.

In this chapter we will consider some applications of Gabor's ideas in the field of optics. In optics, signals not only depend on the time variable t, but also on the space vector $\boldsymbol{r} = (x, y, z)$. In fact, the space dependence is often much more important than the time dependence. To see how Gabor's ideas can be translated to the spatial domain, we shall confine ourselves in this chapter to strictly time-harmonic optical signals with temporal frequency ω. Such an optical signal can be described, for instance,

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by $\Re\{\varphi(x,y,z)\exp(-j\omega t)\}$, where $\Re$ denotes the real part and where the complex amplitude $\varphi(x,y,z)$ contains the relevant spatial information of the signal.

Very often the optical signal propagates through some optical medium from a certain input plane $z = z_i$, say, to an output plane $z = z_o$. In that case it suffices to know – in an arbitrary plane z = constant – the complex amplitude $\varphi(x,y)$ that depends on the transverse coordinates x and y only; we will see in Section 14.2 that the z-dependence of the complete signal $\varphi(x,y,z)$ follows from the properties of the medium in which the optical signal is propagating. In Section 14.2 we will also show how we can interpret the two-dimensional Fourier transform $\hat{\varphi}(u,v)$ of a function $\varphi(x,y)$ in physical terms. Throughout this entire chapter we will denote the Fourier transform of a function by the same symbol as the function itself, but marked by a hat on top of the symbol.

In Section 14.3 we will consider Gabor's signal expansion and its inverse – the Gabor transform, with the help of which Gabor's expansion coefficients can be determined – for optical signals. For convenience we will consider optical signals that depend on one transverse coordinate x only, and that do not depend on y. In that case we can restrict ourselves to the one-dimensional function $\varphi(x)$ and its Fourier transform $\hat{\varphi}(u)$. The extension to the more general, two-dimensional case is rather straightforward. We will pay special attention to the case of a Gaussian elementary signal, which is intimately related to the optically important Gaussian beam. Moreover, we will use Section 14.3 to do some preparatory, theoretical work on critical sampling, on integer oversampling and the product form of the Gabor transform in terms of the Zak transform, and on the windowed Fourier transform, expressed as an interpolation of the Gabor transform; the results of this work will then be used in Sections 14.4 and 14.5.

The propagation of an optical signal, described in terms of its Gabor coefficients, will be treated in Section 14.4. In this section we will restrict ourselves to the case of critical sampling of the space-frequency domain, and we will study in more detail the concept of degrees of freedom of a signal.

Finally, in Section 14.5 a coherent-optical setup will be considered, with which Gabor's expansion coefficients can be generated.

14.2 Some optics fundamentals

In this section we will show how the exponential $\exp[j(k_x x + k_y y)]$ that plays a central role in the Fourier transform and in Gabor's expansion of an optical signal, can be given a physical interpretation. We will derive such an interpretation considering only one of the simplest systems in which an optical signal can propagate, viz. vacuum.

Since an optical signal is in essence an electromagnetic phenomenon, described by an electric field vector $\boldsymbol{E}$ and a magnetic field vector $\boldsymbol{H}$, its propagation is governed by *Maxwell's equations.* In vacuum – and in our special case of a harmonic time dependence – these equations take the form (see, for instance, [Goo96])

$$\begin{aligned} \nabla \times \boldsymbol{E} &= j\omega\mu\boldsymbol{H} \\ \nabla \times \boldsymbol{H} &= -j\omega\epsilon\boldsymbol{E} \\ \nabla \cdot \epsilon\boldsymbol{E} &= 0 \\ \nabla \cdot \mu\boldsymbol{H} &= 0. \end{aligned} \tag{14.2.1}$$

Here μ and ϵ are the permeability and permittivity, respectively, of the medium in which the optical signal is propagating, $\times$ and $\cdot$ represent a vector cross product and a vector dot product, respectively, while ∇ represents the gradient operator, which in Cartesian coordinates takes the form $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$.

From Maxwell's equations (14.2.1) and using the vector identity

$$\nabla \times (\nabla \times \boldsymbol{E}) = \nabla(\nabla \cdot \boldsymbol{E}) - (\nabla \cdot \nabla)\boldsymbol{E},$$

we can easily derive the equation

$$(\nabla^2 + k^2)\boldsymbol{E} = 0 \tag{14.2.2}$$

for the electric field vector $\boldsymbol{E}$, and a similar one for the magnetic field vector $\boldsymbol{H}$; in Eq. (14.2.2) we have introduced the Laplacian operator $\nabla^2 = \nabla \cdot \nabla$, and the wave number $k = \omega/c$, with $c = 1/\sqrt{\mu\epsilon}$ the velocity of propagation in vacuum. An equation of the form (14.2.2) is known as a *Helmholtz equation* (see, for instance, [Goo96]).

We note that an equation of the form (14.2.2) is obeyed by both $\boldsymbol{E}$ and $\boldsymbol{H}$; hence an identical scalar equation is obeyed by all components of those vectors. Therefore it is possible to summarize the behaviour of all components of $\boldsymbol{E}$ and $\boldsymbol{H}$ through a single *scalar* Helmholtz equation

$$(\nabla^2 + k^2)\varphi(x, y, z) = 0, \tag{14.2.3}$$

where $\varphi(x, y, z)$ represents any of the components of the electric or the magnetic field vector.

A basic solution of the Helmholtz equation (14.2.3) is formed by the signal

$$\varphi(x, y, z) = A\, e^{j\boldsymbol{k}\cdot\boldsymbol{r}} = A\, e^{j(k_x x + k_y y + k_z z)}, \tag{14.2.4}$$

where a *wave vector* $\boldsymbol{k} = (k_x, k_y, k_z)$ has been introduced, which should satisfy the condition

$$\boldsymbol{k}\cdot\boldsymbol{k} = k_x^2 + k_y^2 + k_z^2 = k^2. \tag{14.2.5}$$

In the important case that the wave vector $\boldsymbol{k}$ has real components, the signal (14.2.4) represents a (uniform) *plane wave*: on a plane $\boldsymbol{k}{\cdot}\boldsymbol{r}$ = constant – which plane is perpendicular to the wave vector $\boldsymbol{k}$ – the signal has a constant phase. And, of course, *the direction of the plane wave is determined by the wave vector.* It would be outside the scope of this chapter to consider non-uniform plane waves, for which the wave vector is complex.

We remark that it is sufficient to specify a plane wave in a plane $z = 0$, say, in which case the optical signal (14.2.4) reduces to

$$\varphi(x, y, 0) = A\ e^{j(k_x x + k_y y)}. \tag{14.2.6}$$

The complete signal (14.2.4) follows from including again the additional term $k_z z$ in the exponential, while knowing that k_z is related to k_x and k_y by the relation $k_z^2 = k^2 - (k_x^2 + k_y^2)$, see Eq. (14.2.5). Without going into more detail, we note that such a behaviour is not restricted to plane waves and not restricted to propagation in vacuum, but holds for any optical signal: if the signal $\varphi(x, y, 0)$ is specified in a plane $z = 0$, the complete signal $\varphi(x, y, z)$ follows from the propagation properties of the medium. It is crucial, however, that we are allowed to represent the optical signal by means of a scalar function instead of a vectorial one. And although this might not always be the case, such a scalar treatment is appropriate in a large number of cases (see, for instance, [Goo96]).

The exponential $\exp[j(k_x x + k_y y)]$ that arises in Eq. (14.2.6) resembles the exponential $\exp[j2\pi f t]$ that is normally used in the case of time signals. Analogously to the way in which a time signal can be represented in terms of its frequency spectrum via an inverse Fourier transformation, a (two-dimensional) space signal can be represented by an integral of the form

$$\varphi(x, y) = \iint \hat{\varphi}\left(\frac{k_x}{2\pi}, \frac{k_y}{2\pi}\right) e^{j(k_x x + k_y y)} \frac{dk_x}{2\pi}\frac{dk_y}{2\pi}, \tag{14.2.7}$$

where $\hat{\varphi}(k_x/2\pi, k_y/2\pi)$ is the (two-dimensional) spatial Fourier transform of the space signal $\varphi(x, y)$. Unless otherwise stated, all integrations and summations in this chapter extend from $-\infty$ to $+\infty$; moreover, we will throughout assume that the operands are such that these integrations and summations exist.

The exponential that arises in Eqs. (14.2.6) and (14.2.7) represents in fact a (cross-section of a) plane wave [see Eq. (14.2.4)] with wave vector $\boldsymbol{k} = (k_x, k_y, [k^2 - k_x^2 - k_y^2]^{1/2})$ [see Eq. (14.2.5)], and the inverse Fourier transformation (14.2.7) thus represents the optical signal as *a superposition of plane waves.*

14.3 Gabor's signal expansion in optics

The previous section has lead us to some similarities between time signals, which formed the subject in Gabor's original paper, and space signals, which form the relevant functions in optics. It is now clear that the analogy of *time* is *space*, and the analogy of *temporal frequency* is *spatial frequency.* And where temporal frequency corresponds to the *pitch of a tone*, spatial frequency corresponds to the *direction of a plane wave.*

In the remainder of this chapter we will, for convenience, confine ourselves to space signals that – while propagating in the z-direction – depend on the transverse coordinate x only, and do not depend on y, implying that the y-component k_y of the wave vector $\boldsymbol{k}$ is identically zero. In that case we can restrict ourselves to the one-dimensional function $\varphi(x)$ and its spatial Fourier transform

$$\hat{\varphi}(u) = (\mathcal{F}\varphi)(u) = \int \varphi(x) e^{-j2\pi ux} dx. \tag{14.3.1}$$

The extension to the more general, two-dimensional case is rather straightforward.

14.3.1 Gabor's signal expansion and the Gabor transform

Instead of describing a signal in a space domain [by means of $\varphi(x)$] or in a spatial-frequency domain [by means of $\hat{\varphi}(u)$], Gabor's signal expansion represents the signal in a combined space-frequency domain. With $g(x)$ the *elementary signal* or *synthesis window*, and with a space shift X and a spatial-frequency shift U that satisfy the conditions $UX \leq 1$, the expansion reads

$$\varphi(x) = \sum_m \sum_k a_{mk} g(x - mX) e^{j2\pi kUx}. \tag{14.3.2}$$

Throughout this chapter we will consistently use the variable m (and later on M and n) in connection with a space shift (mX, for instance), and the variable k (and later on K and l) in connection with a spatial-frequency shift (kU, for instance).

Suppose that the elementary signal $g(x)$ is a function that is concentrated in the space domain around the position $x = 0$ and that its Fourier transform $\hat{g}(u)$ is concentrated in the spatial-frequency domain around the spatial frequency, or direction, $u = 0$. If we let this function propagate in vacuum, it will behave more or less like an *optical ray*, remaining concentrated around the z-axis. Likewise, the shifted and modulated versions $g(x - mX)\exp(j2\pi kUx)$ of the elementary signal behave like optical rays passing through the positions $x = mX$ and having directions $u = kU$.

Gabor's signal expansion can thus be considered as representing an optical signal as a *superposition of rays*; moreover, the Gabor coefficient a_{mk} represents the complex amplitude of the ray passing through the position mX with direction kU.

With $\gamma(x)$ an *analysis window* that corresponds to the synthesis window $g(x)$, the minimal norm Gabor coefficients a_{mk} can be determined by means of the *Gabor transform*

$$a_{mk} = \int \varphi(x)\gamma^*(x - mX)e^{-j2\pi kUx}dx. \tag{14.3.3}$$

The Gabor transform can, of course, be considered as an inner product of the signal $\varphi(x)$ and a shifted and modulated version of the analysis window. Note, however, that the Gabor transform can also be considered as a sampled version of the *windowed Fourier transform* $\mathcal{W}_\varphi(x,u)$ of the signal $\varphi(x)$,

$$\mathcal{W}_\varphi(x,u) = \int \varphi(x')\gamma^*(x' - x)e^{-j2\pi ux'}dx', \tag{14.3.4}$$

on the rectangular lattice $(x = mX, u = kU)$: $a_{mk} = \mathcal{W}_\varphi(mX, kU)$.

14.3.2 Gaussian elementary signal and Gaussian beams

Gabor's original choice of the elementary signal was a Gaussian

$$g(x) = 2^{\frac{1}{4}}e^{-\pi(x/X)^2}. \tag{14.3.5}$$

The big advantage of a Gaussian elementary signal is that it has a good localization, both in the space and in the spatial-frequency domain. In optics, a Gaussian choice is very attractive, because the interpretation of such a Gaussian elementary signal as an optical ray is very appealing. And, indeed, much knowledge has been gained in optics (see, for instance, [Sie86]) about the propagation of so-called *Gaussian beams*, i.e., signals of the form

$$\varphi(x,z) = A\sqrt{q(z)}e^{j\pi q(z)x^2}, \tag{14.3.6}$$

where the imaginary part of the *beam parameter* $q(z)$ is related to the width of the beam in the transversal x-direction and the real part of $q(z)$ is related to the convergence or divergence of the beam in the longitudinal z-direction. For a large class of optical systems, a Gaussian beam retains its Gaussian character (14.3.6), and the propagation is completely determined by the z-dependent beam parameter $q(z)$.

It is obvious that a Gaussian beam reduces – in the plane $z = 0$, say – to the form of the Gaussian elementary signal (14.3.5), if the beam parameter

$q(0)$ is strictly imaginary with an imaginary part that is positive. Hence, Gabor's signal expansion can thus be considered as representing an optical signal as a superposition of (cross-sections of) Gaussian beams. Since for a large class of optical systems it is not too difficult to determine the propagation of Gaussian beams, it is now straightforward to determine the propagation of an arbitrary optical signal: we let all the Gaussian beams that build up the optical signal in the input plane, propagate through the optical medium, and superpose the beams in the output plane with their proper Gabor coefficients. And although this superposition in the output plane is no longer in the form of a Gabor expansion, it remains possible to determine the optical signal in the output plane.

With these ideas in mind, several authors have expressed the optical signal inside an aperture in terms of a Gabor expansion [ERS86, EHF87, Wol89] and have studied the propagation of light in homogeneous and weakly inhomogeneous media [MMTZ86, ER88, MF89, FKLG91, SHF91c, SHF91b, KFLG92], in layered media [MF90], and in the focal region of a parabolic reflector [DA94]. Whereas most of these papers deal with time-harmonic signals, the case of pulsed signals (where the time-pulse has again a Gaussian shape) has been considered, as well [ER87, SHF91a, SH91]. Note that if both the time and the space behaviour of a (temporal-spatial) elementary signal are Gaussian, Gabor's signal expansion leads to an expansion in a set of Gaussian *wave packets*, concentrated at certain time moments and certain positions, modulated by certain temporal frequencies and traveling into certain spatial directions. A description of an aperture field in terms of exponential elementary beams instead of Gaussian beams has been reported, as well [Ein88].

14.3.3 Critical sampling and oversampling

In the case of *critical sampling*, i.e., $UX = 1$, there exists a unique relationship between the synthesis window $g(x)$ that arises in Gabor's signal expansion (14.3.2) and the analysis window $\gamma(x)$ that arises in the Gabor transform (14.3.3). Hence, given a certain analysis window $\gamma(x)$, the corresponding synthesis window $g(x)$ can uniquely be determined (see, for instance, [Bas80a, Bas81, Bas93]).

In the case of *oversampling*, i.e., $UX < 1$, such a unique relationship does not exist, and – given an analysis window – the synthesis window is not unique anymore; very often we choose $g(x)$ such that it has minimum L_2 norm. In that case, the synthesis window $g(x)$ resembles best (in a minimum L_2 norm sense, again) the analysis window $\gamma(x)$. In the case of *infinite* oversampling, i.e., $(U, X) \downarrow (0, 0)$, Gabor's signal expansion (14.3.2) resembles the relationship

$$\varphi(x')\int|\gamma(x)|^2dx=\iint \mathcal{W}_\varphi(x,u)\gamma(x'-x)e^{j2\pi ux'}dxdu, \tag{14.3.7}$$

which is, in fact, one possible inversion formula for the windowed Fourier transform (14.3.4); note that in this case the synthesis window is indeed proportional to the analysis window see Chapter 3 in this book. Several ways are described in the literature to determine the synthesis window in the case of oversampling (see, for instance, [Dau90, BG96, ZZ93b]).

In the case of critical sampling, $UX=1$, Gabor's signal expansion is related to the *degrees of freedom* of a signal: each expansion coefficient a_{mk} represents one complex degree of freedom. If a spatial signal $\varphi(x)$ is, roughly, limited to the space interval $|x|<\frac{1}{2}a$, and its spatial Fourier transform $\hat{\varphi}(u)$ to the frequency interval $|u|<\frac{1}{2}b$, the number of degrees of freedom equals the number of Gabor coefficients in the space-frequency rectangle with area ab, which number is about equal to the *space-bandwidth product* ab. In Section 14.4 we will restrict ourselves to this case of critical sampling and study the degrees of freedom in more detail.

14.3.4 Integer oversampling - Gabor transform as a product of Zak transforms

It is well known that a correlation and a convolution can be brought into product form by means of the Fourier transform. We now try to bring the Gabor transform (14.3.3) in a product form, as well. We therefore introduce the Fourier transform $\hat{a}(\xi,\eta)$ of the two-dimensional array a_{mk} through the definition

$$\hat{a}(\xi,\eta)=(\mathcal{F}a)(\xi,\eta)=\sum_m\sum_k a_{mk}e^{-j2\pi(m\eta-k\xi)}. \tag{14.3.8}$$

We substitute from the Gabor transform (14.3.3) and rearrange factors

$$\hat{a}(\xi,\eta)=\sum_m\left[\int\varphi(x)\gamma^*(x-mX)\left\{\sum_k e^{-j2\pi k(Ux-\xi)}\right\}dx\right]e^{-j2\pi m\eta};$$

we will assume that here – and at other places in this chapter – such a rearranging of factors is allowed. We replace the sum of exponentials by a sum of Dirac functions and rearrange factors again

$$\hat{a}(\xi,\eta)=\frac{1}{U}\sum_m\left[\sum_k\int\varphi(x)\gamma^*(x-mX)\delta\left(x-\frac{\xi+k}{U}\right)dx\right]e^{-j2\pi m\eta}.$$

We evaluate the integral and rearrange factors again

$$\begin{aligned}\hat{a}(\xi,\eta) &= \frac{1}{U}\sum_k \varphi\left(\frac{\xi}{U}+\frac{k}{U}\right) e^{-j2\pi k(1/U)(\eta/X)} \\ &\times \left[\sum_m \gamma^*\left(\frac{\xi}{U}+\frac{k}{U}-mX\right) e^{j2\pi(k/U-mX)(\eta/X)}\right].\end{aligned}$$

In the case of *integer* oversampling, i.e., $1/U = pX$ with p a positive integer, the latter expression can be written as

$$\begin{aligned}\hat{a}(\xi,\eta) &= pX\sum_k \varphi(\xi pX + kpX)e^{-j2\pi k(pX)(\eta pU)} \\ &\times \left[\sum_m \gamma(\xi pX + [kp-m]X)e^{-j2\pi(kp-m)X(\eta pU)}\right]^*.\end{aligned}$$

In the right-hand side of the latter relationship, we recognize the expression

$$\mathcal{Z}_\varphi(x,u;\Delta) = \sum_m \varphi(x+m\Delta)e^{-j2\pi m\Delta u} \tag{14.3.9}$$

for $\varphi(x)$ [with $x = \xi pX$, $u = \eta pU$, and $\Delta = pX$], and a similar expression $\mathcal{Z}_\gamma(x,u;\Delta)$ for $\gamma(x)$ [with $x = \xi pX$, $u = \eta pU$, and $\Delta = X$]. The expression (14.3.9) is known as the *Zak transform* [Zak67, Zak72, Jan82, Jan88]; note that we have explicitly stated the step size Δ, which may take different values for $\mathcal{Z}_\varphi$ and $\mathcal{Z}_\gamma$. In terms of Zak transforms, the Fourier transform $\hat{a}(\xi,\eta)$ can thus be expressed as

$$\hat{a}(\xi,\eta) = pX\,\mathcal{Z}_\varphi(\xi pX,\eta pU;pX)\mathcal{Z}_\gamma^*(\xi pX,\eta pU;X), \tag{14.3.10}$$

which is the product form of the Gabor transform that we are looking for.

We remark that a Fourier transform like $\hat{a}(\xi,\eta)$ [see Eq. (14.3.8)] is periodic in the time variable ξ and the frequency variable η with period 1: $\hat{a}(\xi+m,\eta+k) = \hat{a}(\xi,\eta)$; hence, in considering such a Fourier transform we can restrict ourselves to the *fundamental Fourier interval* $(-\frac{1}{2} < \xi \le \frac{1}{2}, -\frac{1}{2} < \eta \le \frac{1}{2})$. We remark further that a Zak transform like $\mathcal{Z}_\varphi(x,u;\Delta)$ [see Eq. (14.3.9)] is periodic in the frequency variable u with period $1/\Delta$ and *quasi-periodic* in the space variable x with quasi-period Δ:

$$\mathcal{Z}_\varphi\left(x+m\Delta,u+\frac{k}{\Delta};\Delta\right) = \mathcal{Z}_\varphi(x,u;\Delta)e^{j2\pi m\Delta u}; \tag{14.3.11}$$

hence, in considering a Zak transform we can restrict ourselves to the *fundamental Zak interval* $(-\frac{1}{2} < x/\Delta \le \frac{1}{2}, -\frac{1}{2} < u\Delta \le \frac{1}{2})$.

The Zak transform $\mathcal{Z}_\varphi(x,u;\Delta)$ can also be expressed in terms of the Fourier transform $\hat{\varphi}(u)$ of $\varphi(x)$ and then takes the form

$$\Delta\ \mathcal{Z}_\varphi(x,u;\Delta) = e^{j2\pi ux} \sum_k \hat{\varphi}\left(u + \frac{k}{\Delta}\right) e^{j2\pi(k/\Delta)x}. \tag{14.3.12}$$

From the latter relation we conclude that for a signal that is band-limited to the frequency interval $-\frac{1}{2} < u\Delta \leq \frac{1}{2}$, the Zak transform takes the form (in this fundamental frequency interval)

$$\Delta\ \mathcal{Z}_\varphi(x,u;\Delta) = e^{j2\pi ux}\hat{\varphi}(u). \tag{14.3.13}$$

In Fig. 14.3.1 we have depicted the Zak transform $\mathcal{Z}_\gamma(x,u;\Delta)$ of a Gaussian window [cf. Eq. (14.3.5)] for several values of Δ. Note that for small values of Δ/X the sampling frequency $1/\Delta$ is sufficiently high and the above-mentioned property (14.3.13) holds.

If we consider in Eq. (14.3.10) the domains of the functions $\mathcal{Z}_\varphi(\xi pX, \eta pU; pX)$ and $\mathcal{Z}_\gamma(\xi pX, \eta pU; X)$ in the fundamental Fourier interval ($-\frac{1}{2} < \xi \leq \frac{1}{2}, -\frac{1}{2} < \eta \leq \frac{1}{2}$) of the function $\hat{a}(\xi,\eta)$, we note that, whereas the Fourier transform $\hat{a}(\xi,\eta)$ appears only *once* in the fundamental Fourier interval, the Zak transforms $\mathcal{Z}_\varphi(\xi pX, \eta pU; pX)$ and $\mathcal{Z}_\gamma(\xi pX, \eta pU; X)$ appear *p-fold*: $\mathcal{Z}_\varphi(\xi pX, \eta pU; pX)$ as p identical horizontal stripes with height $1/p$ and width 1, and $\mathcal{Z}_\gamma(\xi pX, \eta pU; X)$ as p vertical stripes with width $1/p$ and height 1, which stripes are identical to each other apart from the factor $\exp(j2\pi m\eta)$ [cf. the quasi-periodicity property (14.3.11) of the Zak transform].

Note that the product form (14.3.10) of the Gabor transform (14.3.3) enables us to determine this transform in a different way:

- we first determine the Zak transform $\mathcal{Z}_\varphi(\xi pX, \eta pU; pX)$ and the Zak transform $\mathcal{Z}_\gamma(\xi pX, \eta pU; X)$ of the signal $\varphi(x)$ and the window function $\gamma(x)$, respectively, by means of definition (14.3.9);
- we then find the Fourier transform $\hat{a}(\xi,\eta)$ by means of the product rule (14.3.10);
- we finally determine the Gabor transform a_{mk} via an inverse Fourier transformation.

This way of determining the Gabor transform resembles the way of calculating correlations and convolutions via the Fourier domain. Since the Zak transform is essentially a Fourier transformation, fast algorithms can be used when we are dealing with discrete-time signals, and the algorithm described above then resembles the fast convolution, well-known in digital signal processing [BG96].

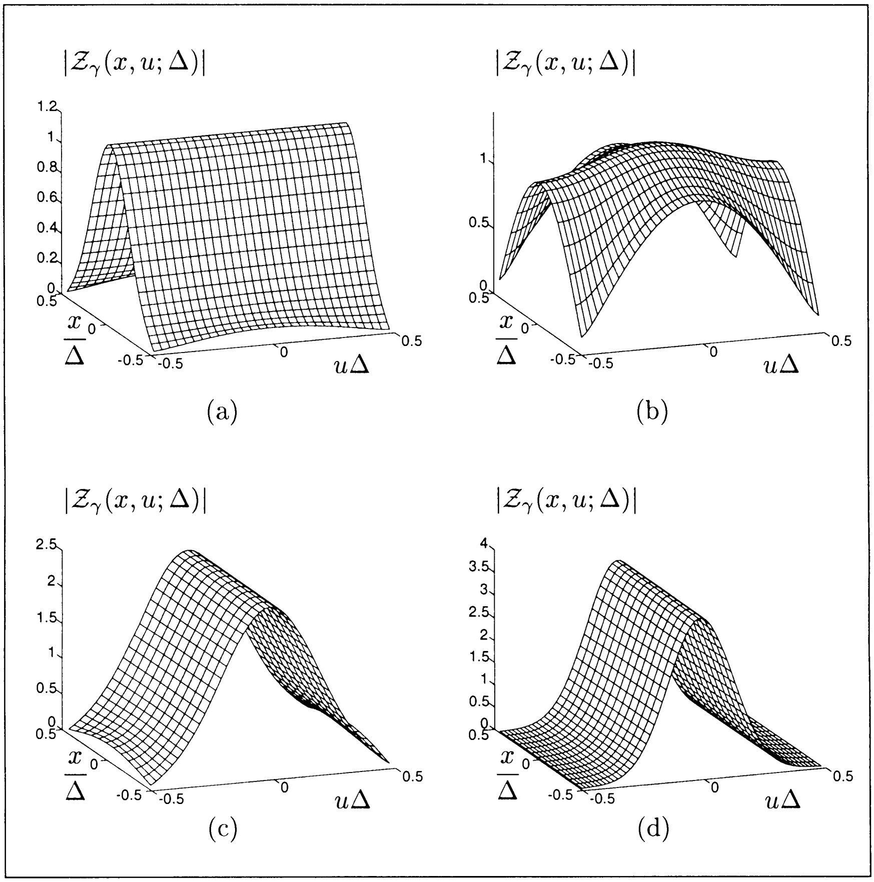

FIGURE 14.3.1. The Zak transform $\mathcal{Z}_\gamma(x,u;\Delta)$ that corresponds to a Gaussian window function $\gamma(x) = 2^{1/4}\exp[-\pi(x/X)^2]$, for different values of Δ: (a) $\Delta = 2X$, (b) $\Delta = X$, (c) $\Delta = X/2$, and (d) $\Delta = X/3$.

The product form also suggests a coherent-optical generation of the Gabor transform, since the basic mathematical operations – Fourier transformation and multiplication – are perfectly suited to be performed by optical means. We will show in Section 14.5 how the Gabor transform can be generated on a rectangular lattice in the output plane of the optical system. Since the Gabor transform is a sampled version of the windowed Fourier transform, it will also be interesting to know whether the optical signal in this output plane represents the windowed Fourier transform at positions that are not on the rectangular lattice. To treat this problem, we will spend some words on the windowed Fourier transform in the next subsection.

14.3.5 Windowed Fourier transform - interpolation of the Gabor transform

indexGabor transform!interpolation of

Since the signal can be represented by its Gabor expansion, we can easily derive an interpolation procedure for the windowed Fourier transform. We start with the windowed Fourier transform (14.3.4) and substitute from Gabor's signal expansion (14.3.2)

$$\mathcal{W}_\varphi(x,u) = \int \left[\sum_m \sum_k a_{mk} g(x' - mX) e^{j2\pi kUx'} \right] \gamma^*(x' - x) e^{-j2\pi ux'} dx'.$$

We rearrange factors and substitute $\xi = x' - mX$

$$\mathcal{W}_\varphi(x,u) = \sum_m \sum_k a_{mk} \int g(\xi) \gamma^*(\xi - [x - mX]) e^{-j2\pi(u - kU)(\xi + mX)} d\xi.$$

We rearrange factors again and recognize the windowed Fourier transform of the synthesis window $g(x)$

$$\mathcal{W}_\varphi(x,u) = \sum_m \sum_k a_{mk} \mathcal{W}_g(x - mX, u - kU) e^{-j2\pi(u - kU)mX}. \tag{14.3.14}$$

The latter relationship can be considered as an interpolation formula for the windowed Fourier transform in terms of the Gabor coefficients a_{mk}, where the *interpolation kernel* is in fact the windowed Fourier transform $\mathcal{W}_g(x,u)$ of the synthesis window $g(x)$. Since the synthesis window is not unique in the case of oversampling, the same remark applies to the interpolation kernel.

We might look for the interpolation kernel $\mathcal{W}_g(x,u)$ for which the L_2 norm takes its minimum value. From Moyal's formula [Moy49]

$$\iint |\mathcal{W}_g(x,u)|^2 dx du = \left(\int |g(x)|^2 dx\right)\left(\int |\gamma(x)|^2 dx\right) \qquad (14.3.15)$$

we conclude that the minimum L_2 norm of $\mathcal{W}_g(x,u)$ is reached when the synthesis window $g(x)$ has minimum L_2 norm itself.

In the special case $UX = 1$ (Gabor's case of critical sampling) it has been shown [Jan82] that the interpolation kernel that arises in the case of the Gaussian window [cf. Eq. (14.3.5)] can be expressed in the form

$$\mathcal{W}_g(x,u)\ e^{j\pi ux} = \frac{\sigma(2K_0\zeta)}{2K_0\zeta} e^{-\frac{1}{2}\pi|\zeta|^2}, \qquad (14.3.16)$$

where, for convenience, we have introduced the short-hand notation

$$\zeta = uX + j\frac{x}{X}, \qquad (14.3.17)$$

where the constant $K_0 = \frac{1}{4}\pi^{-\frac{1}{2}}[\Gamma(\frac{1}{4})]^2 = 1.85407468$ is the complete elliptic integral for the modulus $\frac{1}{2}\sqrt{2}$ (see, for instance, [WW27], Section 22.8, The lemniscate functions), and where $\sigma(2K_0\zeta)$ represents Weierstrass' sigma function [WW27], expressible as

$$\sigma(2K_0\zeta) = \left(\frac{\pi}{K_0}\right)^{\frac{1}{2}} e^{\frac{1}{2}\pi\zeta^2}\ 2\sum_{n=0}^{\infty}(-1)^n e^{-\pi(n+\frac{1}{2})^2} \sin[2\pi(n+\tfrac{1}{2})\zeta].$$

The interpolation function has been depicted in Fig. 14.3.2a. We remark that in this case of critical sampling, the interpolation kernel has the property $\mathcal{W}_g(mX,kU) = \delta_m\delta_k$, where δ_m represents the Kronecker delta; this property is in accordance with the fact that the two window functions should be *biorthogonal* [Bas80a, Bas81, Bas93].

In the limiting case $(U,X) \downarrow (0,0)$ (infinite oversampling) the optimum synthesis window $g_{opt}(x)$ – i.e., the one with minimum L_2 norm – becomes proportional to the analysis window $\gamma(x)$, and it is not difficult to show that the interpolation kernel that arises in the case of the Gaussian window [cf. Eq. (14.3.5)] then takes the form

$$\mathcal{W}_g(x,u)\ e^{j\pi ux} = UXe^{-\frac{1}{2}\pi|\zeta|^2}. \qquad (14.3.18)$$

The interpolation function has been depicted in Fig. 14.3.2b.

In Section 14.5 we will study the interpolation of the Gabor transform in more detail.

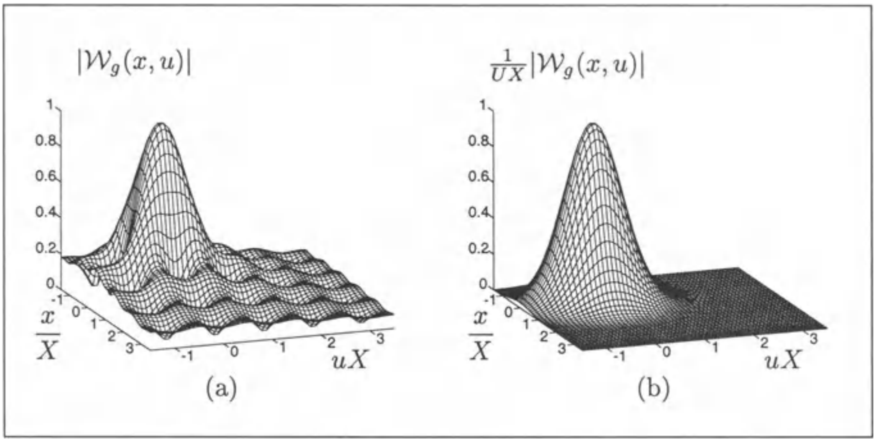

FIGURE 14.3.2. The interpolation function $\mathcal{W}_g(x,u)$ in the case of (a) critical sampling $UX = 1$, and (b) infinite oversampling $UX \downarrow 0$.

14.4 Degrees of freedom of an optical signal

In his original paper [Gab46] Gabor not only chose "Gaussian elementary signals which occupy the smallest possible area in the information diagram," but he also constructed this information (or space-frequency) diagram such that "each elementary signal can be considered as conveying exactly one datum, or one quantum of information." In this case of critical sampling, Gabor's signal expansion is thus intimately related to the degrees of freedom of a signal.

In this section we consider an optical system that truncates both the space and the spatial-frequency content of the input signal. Whereas the input signal of such an optical system may have an infinite number of degrees of freedom, the number of complex degrees of freedom of the output signal is limited to the space-bandwidth product of the system. This can be proved elegantly by expanding the signal in prolate spheroidal wave functions, which are eigenfunctions of this system [Sle76], but we will present a different, more physically oriented proof, by showing that the number of non-vanishing Gabor coefficients of the output signal is equal to the space-bandwidth product of the system [Bas82a].

14.4.1 Representation of a linear optical system

A linear optical system that transforms an input signal φ_i into an output signal φ_o, can be described in several ways, depending on whether we describe the input and the output signal in the space or in the frequency domain. We thus have four equivalent input-output relationships,

$$\varphi_o(x_o) = \int h_{xx}(x_o, x_i)\varphi_i(x_i)dx_i, \tag{14.4.1}$$

$$\hat{\varphi}_o(u_o) = \int h_{ux}(u_o, x_i)\varphi_i(x_i)dx_i, \tag{14.4.2}$$

$$\varphi_o(x_o) = \int h_{xu}(x_o, u_i)\hat{\varphi}_i(u_i)du_i, \tag{14.4.3}$$

$$\hat{\varphi}_o(u_o) = \int h_{uu}(u_o, u_i)\hat{\varphi}_i(u_i)du_i, \tag{14.4.4}$$

in which the four *system functions* h_{xx}, h_{ux}, h_{xu}, and h_{uu} are completely determined by the system. Relation (14.4.1) is the usual system representation in the space domain (see, for instance [Goo96]) by means of the *impulse response* $h_{xx}(x_o, x_i)$, which is also known as the (coherent) *point spread function* in Fourier optics: the function $h_{xx}(x, x_i)$ is the space domain response of the system at point x due to the input impulse signal $\varphi_i(x) = \delta(x - x_i)$. Relation (14.4.4) is a similar system representation in the frequency domain: the function $h_{uu}(u, u_i)$ is the frequency domain response of the system at frequency u due to the input $\hat{\varphi}(u) = \delta(u - u_i)$, which is the Fourier transform of the harmonic input signal $\varphi(x) = \exp(j2\pi u_i x)$. In Fourier optics such a harmonic signal is a representation of the space dependence of a uniform, obliquely incident, time-harmonic plane wave; in this context we might call $h_{uu}(u_o, u_i)$ the *wave spread function* of the system. Relations (14.4.2) and (14.4.3) are hybrid system representations, since the input and the output signal are described in different domains.

We remark that there is a similarity between the four system functions h_{xx}, h_{ux}, h_{xu}, and h_{uu} and the four *Hamilton characteristics* [BW75] that can be used to describe geometric-optical systems. Indeed, for a geometric-optical system the *point characteristic* is nothing but the phase of the point spread function; similar relations hold between the *angle characteristic* and the wave spread function, and between the *mixed characteristics* and the hybrid system representations.

Unlike the *four* system representations (14.4.1-14.4.4), there is only *one* system representation when we describe the input and the output signal by their Gabor expansions. Let us therefore choose a synthesis window $g_i(x)$ [with a corresponding analysis window $\gamma_i(x)$] to represent the input signal, and a (possibly different) synthesis window $g_o(x)$ [with a corresponding analysis window $\gamma_o(x)$] to represent the output signal. The space shift X and the frequency shift $U = 1/X$ are chosen identical in the input and the output plane. Since, as an example, we will consider the hybrid system representation (14.4.2) later on, we describe the input signal $\varphi_i(x)$ and the Fourier transform of the output signal $\hat{\varphi}_o(u)$ of a linear system by their Gabor expansions (14.3.2) with expansion coefficients a^i_{mk} and a^o_{mk}, respectively,

$$\varphi_i(x) = \sum_m \sum_k a^i_{mk} g_i(x - mX) e^{j2\pi kUx} \quad (14.4.5)$$

$$\hat{\varphi}_o(u) = \sum_m \sum_k a^o_{mk} \hat{g}_o(u - kU) e^{-j2\pi muX}, \quad (14.4.6)$$

where the expansion coefficients follow from the Gabor transforms (14.3.3)

$$a^i_{mk} = \int \varphi_i(x) \gamma^*_i(x - mX) e^{-j2\pi kUx} dx \quad (14.4.7)$$

$$a^o_{mk} = \int \hat{\varphi}_o(u) \hat{\gamma}^*_o(u - kU) e^{j2\pi muX} du. \quad (14.4.8)$$

Note that it is only under the condition of critical sampling $UX = 1$, that the expressions for the Gabor expansion and the Gabor transform take the forms (14.4.6) and (14.4.8), respectively.

14.4.2 Propagation of Gabor's expansion coefficients

It is not difficult to derive how Gabor's expansion coefficients propagate through a linear system; as an example we will choose the hybrid system representation (14.4.2). When we combine the input-output relation (14.4.2) with the Gabor expansion (14.4.5) and the Gabor transform (14.4.8) we can easily derive a relationship between the output and the input expansion coefficients a^o_{mk} and a^i_{nl}, reading

$$a^o_{mk} = \sum_n \sum_l c_{mk,nl} a^i_{nl}, \quad (14.4.9)$$

where the coefficients $c_{mk,nl}$ are completely determined by the system and by the input and output window functions through the relationship

$$c_{mk,nl} = \iint h_{ux}(u, x) \hat{\gamma}^*_o(u - kU) g_i(x - nX) e^{j2\pi(muX + lUx)} dx du. \quad (14.4.10)$$

Of course, similar relations can be found for the other system functions.

As an example we consider the basic coherent-optical system depicted in Fig. 14.4.1, consisting of a so-called $4f$-arrangement with rectangular apertures of width a and b in the input plane and the Fourier plane, respectively (see, for instance, [Goo96]).

The input plane is located in the front focal plane of a positive lens with focal distance f, while the output plane is located in the back focal plane of a second positive lens. The back focal plane of the first lens and the

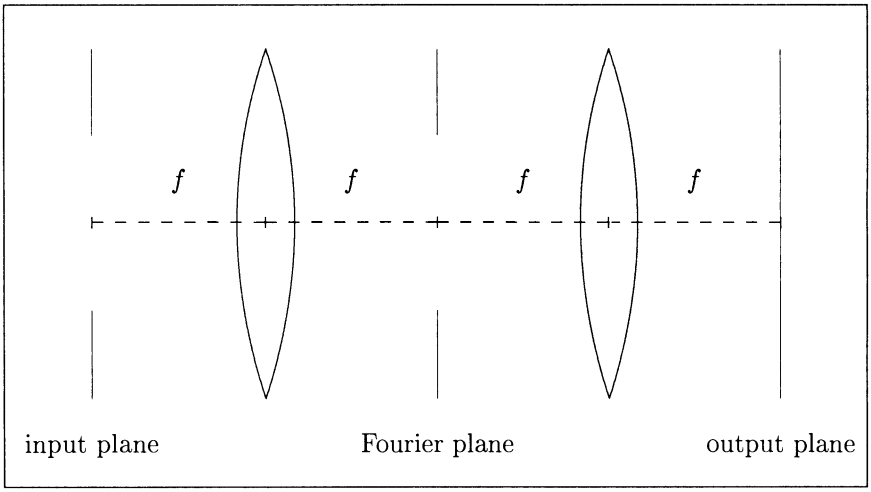

FIGURE 14.4.1. A 4-f arrangement with rectangular apertures.

front focal plane of the second lens coincide, and form the Fourier plane, in which the Fourier transform of the input signal occurs and in which a transparency (in our case: a simple rectangular aperture) may be located. The output signal can thus be considered as a filtered version (in our case: a low-pass filtered version) of the input signal. Such a system can most easily be described by a system function $h_{ux}(u,x)$, which in this case takes the form

$$h_{ux}(u,x) = \text{rect}\left(\frac{x}{a}\right) \text{rect}\left(\frac{u}{b}\right) e^{j2\pi ux}. \tag{14.4.11}$$

For convenience, we choose the widths of the apertures in the input and the Fourier plane equal to an odd multiple of the space and the frequency shift X and U, respectively; thus

$$a = (2M+1)X \quad \text{and} \quad b = (2K+1)U, \tag{14.4.12}$$

with M and K integers. When we substitute from relations (14.4.11) and (14.4.12) into relation (14.4.10), we conclude that the array of coefficients $c_{mk,nl}$ can be expressed as a 4-dimensional convolution of two arrays $d_{mk,nl}$ and $e_{mk,nl}$, where the coefficients $d_{mk,nl}$ are defined by

$$d_{mk,nl} = \begin{cases} \delta_{m-n}\delta_{k-l} & \text{for } |m| \leq M \text{ and } |k| \leq K \\ 0 & \text{elsewhere,} \end{cases} \tag{14.4.13}$$

and the coefficients $e_{mk,nl}$ are defined by

$$\begin{aligned} e_{mk,nl} &= \iint \text{rect}\left(\frac{x}{X}\right) \text{rect}\left(\frac{u}{U}\right) e^{j2\pi ux} \\ &\times \hat{\gamma}_o^*(u-kU) g_i(x-nX) e^{j2\pi(muX+lUx)} dx du \end{aligned} \tag{14.4.14}$$

14.4.3 Space-bandwidth product - degrees of freedom

A system whose Gabor coefficients $c_{mk,nl}$ would have the form (14.4.13) is *ideal* in the sense that the Gabor coefficients of the output signal vanish outside the space-frequency rectangle with area ab. Hence, whereas the input signal of such an ideal system may have an infinite number of degrees of freedom, the number of degrees of freedom of the output signal, i.e., the number of non-vanishing Gabor coefficients, is equal to the *space-bandwidth product* ab. However, our system under consideration is not ideal: to find its Gabor coefficients $c_{mk,nl}$, the ideal array $d_{mk,nl}$ must be smeared out by convolving it with the array $e_{mk,nl}$. The latter array is, in fact, the array of Gabor coefficients of the elementary system described by the system function (14.4.2), with the special choice $a = X$ and $b = U$, i.e., $M = K = 0$.

Depending on the choice of the window functions in the input and the output plane, the array of coefficients $e_{mk,nl}$ can be strongly concentrated. To show this we choose a rectangular window function in the input plane and a sinc-shaped window function in the output plane, thus

$$g_i(x) = \text{rect}\left(\frac{x}{X}\right) \quad \text{and} \quad \hat{\gamma}_o(u) = \text{rect}\left(\frac{u}{U}\right). \tag{14.4.15}$$

We then find $e_{00,00} = 0.873$, and the strong concentration becomes apparent by noting that

$$\sum_m \sum_k \sum_n \sum_l |e_{mk,nl}|^2 = 1.$$

In general the value of $e_{00,00}$ for this elementary system is given by

$$e_{00,00} = \iint \text{rect}\left(\frac{x}{X}\right) \text{rect}\left(\frac{u}{U}\right) e^{j2\pi ux} \hat{\gamma}_o^*(u) g_i(x) dx du.$$

Furthermore, the identities

$$\sum_m \sum_k \sum_n \sum_l e_{mk,nl} = \left(\sum_k \hat{\gamma}_o(kU)\right)^* \left(\sum_n g_i(nX)\right)$$

and

$$\sum_m \sum_k \sum_n \sum_l |e_{mk,nl}|^2 = \left(\int |\hat{\gamma}_o(u)|^2 du\right) \left(\int |g_i(x)|^2 dx\right)$$

can be derived in a straightforward way, using the basic relation

$$\sum_n e^{j2\pi nUx} = X \sum_n \delta(x - nX).$$

The ratio

$$\frac{|e_{00,00}|^2}{\sum_m \sum_k \sum_n \sum_l |e_{mk,nl}|^2} \tag{14.4.16}$$

can be considered as a *degree of concentration* of the array $e_{mk,nl}$ around the coefficient $e_{00,00}$. By applying a variational principle to the expression (14.4.16), it is not difficult to show that the degree of concentration has a stationary value when $g_i(x)$ and $\hat{\gamma}_o(u)$ are chosen according to

$$g_i(x) = \psi_{2m}\left(\frac{x}{X}\right) \text{rect}\left(\frac{x}{X}\right) \quad \text{and} \quad \hat{\gamma}_o(u) = \psi_{2m}\left(\frac{u}{U}\right) \text{rect}\left(\frac{u}{U}\right), \tag{14.4.17}$$

where the functions $\psi_n(\xi)$ are the *prolate spheroidal wave functions* (see, for instance [Sle76]) defined by the eigenfunction equation

$$\int \psi_n(\xi)\text{rect}(\xi)e^{-2\pi j\xi\eta}d\xi = j^n\sqrt{\lambda_n}\psi_n(\eta) \quad (n = 0, 1, ...) \tag{14.4.18}$$

and normalized according to

$$\int |\psi_n(\xi)|^2\text{rect}(\xi)d\xi = 1. \tag{14.4.19}$$

If we choose the window functions as in relations (14.4.17), the corresponding stationary value of the degree of concentration is equal to λ_{2m}. An optimum value is attained for $m = 0$, for which the degree of concentration takes the value $\lambda_0 = 0.783$. This is a slightly better result than choosing the window functions as in relations (14.4.15), in which case the degree of concentration takes the value 0.762.

We conclude that for a proper choice of the window functions the array $e_{mk,nl}$ can be strongly concentrated. Since the Gabor coefficients $c_{mk,nl}$ of the basic optical system under consideration can be found by convolving the ideal array $d_{mk,nl}$ with the strongly concentrated array $e_{mk,nl}$, the array of system coefficients $c_{mk,nl}$ is very similar to the array $d_{mk,nl}$. Hence, the number of degrees of freedom of the output signal of this system is equal to the space-bandwidth product ab. We remark that the way in which we have proved this has a clear physical interpretation. Roughly speaking, with the Gabor expansion of the input signal in mind, only those shifted and modulated versions of the synthesis window that can pass both the input plane aperture and the Fourier plane aperture, will reach the output plane and will contribute to the output signal.

A slightly more general system than the one described by relation (14.4.11) is the one whose system function $h_{ux}(u, x)$ takes the form

$$h_{ux}(u, x) = \sum_m \sum_k h_{mk}\text{rect}\left(\frac{x - mX}{X}\right) \text{rect}\left(\frac{u - kU}{U}\right) e^{j2\pi ux}. \tag{14.4.20}$$

The array of system coefficients $c_{mk,nl}$ can now be expressed as a 4-dimensional convolution of the arrays $h_{mk}\delta_{m-n}\delta_{k-l}$ and $e_{mk,nl}$. In the case that the array $e_{mk,nl}$ is again strongly concentrated around the element $e_{00,00}$, the Gabor coefficients of the input and the output signal are related by the simple relation

$$a^o_{mk} \simeq h_{mk} a^i_{mk}. \tag{14.4.21}$$

For the special system described by relation (14.4.11), we easily find that the array h_{mk} equals unity in the interval $(|m| \leq M, |k| \leq K)$ and vanishes outside that interval.

14.5 Coherent-optical generation of the Gabor transform via the Zak transform

The product form (14.3.10) of the Gabor transform suggests a generation of this transform by coherent-optical means [Bas82b, LZ92]. Apart from being able to realize ideal imaging, coherent optics is well suited for two specific signal operations, viz. multiplication by a constant function (like in a slide projector, for instance) and Fourier transformation (between the back and the front focal plane of a lens, for instance). These two operations are exactly the ones that are needed for generation of the Gabor transform.

14.5.1 Coherent-optical setup

Let a plane wave of monochromatic laser light be normally incident upon a transparency situated in the input plane of a coherent-optical system, see Fig. 14.5.1. The transparency contains the time signal $\varphi(x)$ in a rastered format. With $X_o = pX$ being the width of this raster and $p\mu X_o$ (with $\mu > 0$) being the spacing between the raster lines, the light amplitude $\varphi_i(x_i, y_i)$ just behind the transparency reads

$$\varphi_i(x_i, y_i) = \operatorname{rect}\left(\frac{x_i}{X_o}\right) \sum_n \varphi\left(\frac{x_i + nX_o}{X_o} pX\right) \delta(y_i - np\mu X_o). \tag{14.5.1}$$

An anamorphic optical system between the input plane and an intermediate plane performs a Fourier transformation in the y-direction and an ideal imaging (possibly with inversion) in the x-direction. Such an anamorphic system can be realized, for instance, using a combination of a spherical and

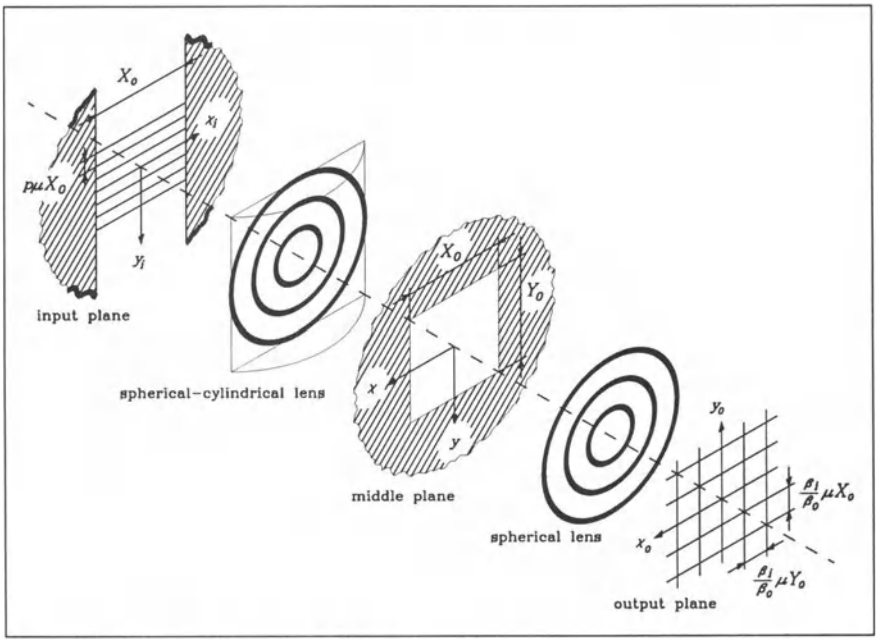

FIGURE 14.5.1. Coherent-optical setup for generation of the Gabor transform.

a cylindrical lens. The anamorphic operation results in the light amplitude

$$\begin{aligned} \varphi_1(x,y) &= \iint \varphi_i(x_i,y_i)e^{-j2\pi\beta_i y y_i}\delta(x-x_i)dx_i dy_i \\ &= \operatorname{rect}\left(\frac{x}{X_o}\right)\mathcal{Z}_\varphi\left(\frac{x}{X_o}pX,\frac{y}{Y_o}pU;pX\right) \end{aligned} \tag{14.5.2}$$

just in front of the intermediate plane. The parameter β_i contains the effect of the wavelength λ of the laser light and the focal length f_i of the spherical and the cylindrical lens: $\beta_i = 1/\lambda f_i$; moreover, the vertical distance Y_o is defined through $\beta_i\mu X_o Y_o = 1$ and, as usual, the frequency step U is related to the space step X through $pUX = 1$.

A transparency with amplitude transmittance

$$m(x,y) = \operatorname{rect}\left(\frac{x}{X_o}\right)\operatorname{rect}\left(\frac{y}{Y_o}\right)pX\mathcal{Z}_\gamma^*\left(\frac{x}{X_o}pX,\frac{y}{Y_o}pU;X\right) \tag{14.5.3}$$

is situated in the intermediate plane. Just behind this transparency, the light amplitude takes the form

$$\varphi_2(x,y) = m(x,y)\varphi_1(x,y) = \operatorname{rect}\left(\frac{x}{X_o}\right)\operatorname{rect}\left(\frac{y}{Y_o}\right)\hat{a}\left(\frac{x}{X_o},\frac{y}{Y_o}\right), \tag{14.5.4}$$

where use has been made of the product form (14.3.10). Note that the aperture $\mathrm{rect}(x/X_o)\mathrm{rect}(y/Y_o)$ contains *one* period of the periodic Fourier transform $\hat{a}(x/X_o, y/Y_o)$, p horizontal periods of the (periodic) Zak transform

$$\mathcal{Z}_\varphi\left(\frac{x}{X_o}pX, \frac{y}{Y_o}pU; pX\right),$$

and p vertical quasi-periods of the (quasi-periodic) Zak transform

$$\mathcal{Z}_\gamma^*\left(\frac{x}{X_o}pX, \frac{y}{Y_o}pU; X\right).$$

Finally, a two-dimensional Fourier transformation is performed between the intermediate plane and the output plane. Such a Fourier transformation can be realized, for instance, using a spherical lens. The light amplitude in the output plane then takes the form

$$\begin{aligned}\varphi_o(x_o, y_o) &= \frac{1}{X_oY_o}\iint \varphi_2(x,y)e^{-j2\pi\beta_o(x_ox - y_oy)}dxdy \\ &= \sum_m\sum_k a_{mk}\ \mathrm{sinc}\left(\frac{\beta_o}{\beta_i}\frac{y_o}{\mu X_o} - m\right)\mathrm{sinc}\left(\frac{\beta_o}{\beta_i}\frac{x_o}{\mu Y_o} - k\right), \quad (14.5.5)\end{aligned}$$

where the sinc-function $\mathrm{sinc}(z) = \sin(\pi z)/(\pi z)$ has been introduced; the parameter β_o, again, contains the effects of the wave length λ of the laser light and the focal length f_o of the spherical lens: $\beta_o = 1/\lambda f_o$. We conclude that the Gabor transform appears on a rectangular lattice of points

$$a_{mk} = \varphi_o\left(k\frac{\beta_i}{\beta_o}\mu Y_o, m\frac{\beta_i}{\beta_o}\mu X_o\right) \quad (14.5.6)$$

in the output plane. Note that relationship (14.5.5) represents the output signal as an interpolated version of the Gabor transform, where the interpolation kernel consists of two sinc-functions, in accord with the rectangular aperture in the intermediate (Fourier) plane.

To compare the function $\varphi_o(x_o,y_o)$ with the interpolation formula (14.3.14) we write Eq. (14.5.5) in the form

$$\varphi_o\left(\frac{u}{U}\frac{\beta_i}{\beta_o}\mu Y_o, \frac{x}{X}\frac{\beta_i}{\beta_o}\mu X_o\right) = \sum_m\sum_k a_{mk}\ \mathrm{sinc}\left(\frac{x - mX}{X}\right)\mathrm{sinc}\left(\frac{u - kU}{U}\right). \quad (14.5.7)$$

We remark that this relationship – which represents in fact a *band-limited* interpolation – can never have the form of the *exact* interpolation formula (14.3.14). However, if the degree of oversampling p is sufficiently high,

$\varphi(x_o, y_o)$ forms a good approximation of the windowed Fourier transform, as we will show in the next section.

The technique described in this section to generate the Gabor transform (and the windowed Fourier transform), fully utilizes the two-dimensional nature of the optical system, its parallel processing features, and the large space-bandwidth product possible in optical processing. The technique exhibits a resemblance to *folded spectrum* techniques, where space-bandwidth products in the order of 300 000 are reported [Cas78].

14.5.2 Interpolation of the Gabor transform revisited

It is elucidating to consider the windowed Fourier transform $\mathcal{W}_\varphi(x,u)$ as a two-dimensional function, and determine its two-dimensional Fourier transform $\hat{\mathcal{W}}_\varphi(x,u)$, defined by

$$\hat{\mathcal{W}}_\varphi(x,u) = (\mathcal{F}\mathcal{W}_\varphi)(x,u) = \iint \mathcal{W}_\varphi(x',u')e^{-j2\pi(ux' - u'x)}dx'du'. \tag{14.5.8}$$

We substitute from Eq. (14.3.4) and rearrange factors

$$\hat{\mathcal{W}}_\varphi(x,u) = \iint \varphi(x'')\gamma^*(x'' - x')e^{-j2\pi ux'}dx''dx' \int e^{-j2\pi u'(x'' - x)}du'.$$

We evaluate the integral over u' and subsequently integrate over the variable x''

$$\hat{\mathcal{W}}_\varphi(x,u) = \int \varphi(x)\gamma^*(x - x')e^{-j2\pi ux'}dx'.$$

After evaluating the integral over x' we get the result

$$\hat{\mathcal{W}}_\varphi(x,u) = \varphi(x)\hat{\gamma}^*(u)e^{-j2\pi ux}. \tag{14.5.9}$$

Since the Gabor transform a_{mk} is a sampled version of the windowed Fourier transform $\mathcal{W}_\varphi(x,u)$, it is worthwhile to compare $\hat{a}(\xi,\eta)$ [the Fourier transform of the two-dimensional array a_{mk}, see Eq. (14.3.8)] with $\hat{\mathcal{W}}_\varphi(x,u)$ [the Fourier transform of the windowed Fourier transform $\mathcal{W}_\varphi(x,u)$, see Eq. (14.5.8)]. Note that a factor $1/p = UX$ will arise from the mere fact that $a_{mk} = \mathcal{W}_\varphi(mX, kU)$ is a *sampled* version of $\mathcal{W}_\varphi(x,u)$, and that this factor automatically arises as the proportionality factor between the Fourier transforms $\hat{a}(\xi,\eta)$ and $\hat{\mathcal{W}}_\varphi(x,u)$.

We start with the Gabor transform in product form [cf. Eq. (14.3.10)],

$$\frac{1}{p}\hat{a}\left(\frac{x}{pX}, \frac{u}{pU}\right) = X\mathcal{Z}_\varphi(x,u;pX)\mathcal{Z}^*_\gamma(x,u;X), \tag{14.5.10}$$

and try to bring this expression into a form that is comparable with Eq. (14.5.9); in particular we want to know if the expression (14.5.10) may be confined to the fundamental Fourier interval. If $\gamma(x)$ is band-limited , and if X has been chosen sufficiently small in order to have the property [cf. Eq. (14.3.13)]

$$X\mathcal{Z}_\gamma(x,u;X) = \hat{\gamma}(u)e^{j2\pi ux},$$

the right-hand side of Eq. (14.5.10) takes the form

$$\mathcal{Z}_\varphi(x,u;pX)\hat{\gamma}^*(u)e^{-j2\pi ux}, \tag{14.5.11}$$

which expression shows already some resemblance to the right-hand side of Eq. (14.5.9). Note that indeed no aliasing will occur if $\hat{\gamma}(u)$ vanishes outside the interval $-\frac{1}{2} < uX \leq \frac{1}{2}$, in which case we can restrict ourselves to one u-period of the periodic function $\hat{a}(x/pX, u/pU)/p$.

We now substitute into the expression (14.5.11) from the alternate definition (14.3.12) for the Zak transform $\mathcal{Z}_\varphi(x,u;pX)$ and get

$$\frac{1}{pX}\hat{\gamma}^*(u)\sum_k \hat{\varphi}(u+kU)e^{j2\pi kUx}. \tag{14.5.12}$$

To see whether the summation over k in the latter expression has the same effect as the factor $\varphi(x)\exp(-j2\pi ux)$ that appears in the right-hand side of Eq. (14.5.9), we apply a Fourier transformation to the expression (14.5.12) with respect to x over the *finite* time interval $-\frac{1}{2} < x/pX \leq \frac{1}{2}$, thus taking into account only one x-period of the periodic function $\hat{a}(x/pX, u/pU)/p$:

$$\frac{1}{pX}\hat{\gamma}^*(u)\int_{-\frac{1}{2}pX}^{\frac{1}{2}pX}\left[\sum_k \hat{\varphi}(u+kU)e^{j2\pi kUx}\right]e^{-j2\pi u'x}dx.$$

We rearrange factors

$$\frac{1}{pX}\hat{\gamma}^*(u)\sum_k \hat{\varphi}(u+kU)\int_{-\frac{1}{2}pX}^{\frac{1}{2}pX} e^{j2\pi(kU-u')x}dx$$

and evaluate the integral over x

$$\hat{\gamma}^*(u)\sum_k \hat{\varphi}(u+kU)\mathrm{sinc}\left(\frac{u'-kU}{U}\right). \tag{14.5.13}$$

The latter expression should now be compared with the Fourier transform of the right-hand side of Eq. (14.5.9) with respect to x, which reads

$$\hat{\gamma}^*(u)\hat{\varphi}(u+u'); \tag{14.5.14}$$

in particular, we should compare the summation over k that appears in expression (14.5.13) with $\hat{\varphi}(u+u')$. Now, if U is sufficiently small, the summation can be considered as an approximation of the integral

$$\int \hat{\varphi}(u+u'')\mathrm{sinc}\left(\frac{u'-u''}{U}\right)\frac{du''}{U}$$

and when the compressed sinc-function can be considered as an approximation of a Dirac function, we get the final result

$$\int \hat{\varphi}(u+u'')\delta(u'-u'')du'' = \hat{\varphi}(u+u').$$

We already concluded that for a sufficiently small value of X we were allowed to restrict ourselves to the frequency interval $-\frac{1}{2} < uX \leq \frac{1}{2}$. From the previous paragraph we also conclude that for a sufficiently small value of U we can restrict ourselves to the time interval $-\frac{1}{2} < x/pX \leq \frac{1}{2}$. These intervals form exactly the fundamental Fourier interval that we consider in the intermediate (Fourier) plane in the coherent-optical arrangement described in the previous section. We therefore conclude that the signal in the output plane of the optical system is a good approximation of the windowed Fourier transform, as even if the interpolation is not in accord with the exact interpolation formula (14.3.14).

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