Homework 10

Quantum Mechanics
December 6, 2022 C Seitz

Problem 1. *4.7*

Solution.
The wave function in three dimensions for a free particle ($V = 0$), is

$$\psi(\boldsymbol{x}, t) = u(\boldsymbol{x})e^{-iE_n t/\hbar}$$
$$\psi^*(\boldsymbol{x}, -t) = u^*(\boldsymbol{x})e^{-iE_n t/\hbar}$$

where $u(\boldsymbol{x}) = e^{i\vec{p}\cdot\vec{k}}$. Note that the phase remains unchanged under complex conjugation and time reversal. Now if we reverse the direction of momentum i.e. $|p\rangle \to |p'\rangle$ for $\vec{p} \cdot \vec{p'} = -1$,

$$\psi'(\boldsymbol{x}, t) = u'(\boldsymbol{x})e^{-iE_n t/\hbar}$$

Notice that $u'(\boldsymbol{x}) = e^{-i\vec{p'}\cdot\vec{k}} = u^*(\boldsymbol{x})$. Therefore $\psi'(\boldsymbol{x}, t) = \psi^*(\boldsymbol{x}, -t)$

$$\chi_+(\hat{n}) = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\gamma} \end{pmatrix}$$

It is also known that

$$\chi_-(\hat{n}) = \begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\gamma} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

So we just need to show that the given transformation gives this result:

$$-i\sigma_2\chi^*(\hat{n}) = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{-i\gamma} \end{pmatrix} = \begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\gamma} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

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Problem 2. *4.8*

Solution.

First note that

$$H\Theta\left|n\right\rangle = \Theta H\left|n\right\rangle = E_n\Theta\left|n\right\rangle$$

so $\left|n\right\rangle$ and $\Theta\left|n\right\rangle$ have the same energy. If the states are nondegenerate then $\left|n\right\rangle$ and $\Theta\left|n\right\rangle$ represent the same state. Their wavefunctions are then the same:

$$\langle x'|n\rangle = \langle n|x'\rangle^*$$

which occurs if they are real, or have a phase difference independent of x. For this reason the wavefunction $\psi = e^{ip\cdot x/\hbar}$ does not violate time reversal invariance, because it is degenerate with $e^{ip\cdot x/\hbar}$.

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Problem 3. *4.9*

Solution.

$$\begin{aligned}
\Theta\left|\alpha\right\rangle &= \int d^3\boldsymbol{p}\ \Theta\left|\boldsymbol{p}\right\rangle\left\langle\boldsymbol{p}|\alpha\right\rangle^* \\
&= \int d^3\boldsymbol{p}\ \left|-\boldsymbol{p}\right\rangle\left\langle\boldsymbol{p}|\alpha\right\rangle^* \\
&= \int d^3\boldsymbol{p}\ \left|\boldsymbol{p}\right\rangle\left\langle-\boldsymbol{p}|\alpha\right\rangle^* \\
&= \phi^*(-p)
\end{aligned}$$

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Problem 4. *4.10*

Solution.

We first prove that

$$\Theta |j, m\rangle = e^{i\delta}(-1)^m |j, -m\rangle$$

Consider acting on $|j, m\rangle$ with J_z:

$$J_z \Theta |j, m\rangle = -\Theta J_z |j, m\rangle = -m \left(\Theta |j, m\rangle\right)$$

so clearly $\Theta |j, m\rangle$ behaves like $|j, -m\rangle$. The $e^{i\delta}$ part is there because we can always include an arbitrary phase.

$$\begin{aligned} \Theta \mathcal{D}(R) \Theta^{-1} &= \Theta \left(1 - i \frac{J \cdot \hat{n}}{\hbar}\right) \Theta^{-1} \\ &= \left(1 + i \frac{\Theta J \Theta^{-1} \cdot \hat{n}}{\hbar}\right) \\ &= \left(1 - i \frac{J \cdot \hat{n}}{\hbar}\right) = \mathcal{D}(R) \end{aligned}$$

so the time reversed state is just $\mathcal{D}(R)\Theta |j, m\rangle$.

$\blacksquare$

Problem 5. *4.11*

Solution.

Since the Hamiltonian is time reversal invariant, then energy eigenkets transform as $\Theta |\alpha\rangle = e^{i\delta} |\alpha\rangle$,

$$\begin{aligned} \langle \boldsymbol{L} \rangle &= \langle \alpha | \boldsymbol{L} | \alpha \rangle \\ &= e^{i\delta} e^{-i\delta} \langle \alpha | \Theta \boldsymbol{L} \Theta^{-1} | \alpha \rangle \\ &= - \langle \alpha | \boldsymbol{L} | \alpha \rangle \end{aligned}$$

This is only satisfied when $\langle \boldsymbol{L} \rangle = 0$. If the wavefunction is expanded as

$$\sum_l \sum_m F_{lm}(r) Y_l^m(\theta, \phi)$$

We know that when the Hamiltonian is invariant under time-reversal, the eigenkets must be real. Therefore, the phase restriction must satisfy the equality $F_{lm}(r)Y_l^m(\theta,\phi) = F_{lm}^*(r)(Y_l^m(\theta,\phi))^*$.

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Problem 6. *4.12*

Solution.

We were given the Hamiltonian:

$$H = AS_z^2 + B(S_x^2 - S_y^2)$$

This Hamiltonian must be invariant under time reversal because these are all scalar values which are invariant e.g, $\Theta S_x^2 \Theta^{-1} = S_x^2 \Theta\Theta^{-1} = S_x^2$ and A and B are real. The explicit matrix representation is

$$H = \hbar^2 \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix}$$

which according to Mathematica has the following eigenvectors

$$\begin{aligned} |E_1\rangle &= (|+1\rangle + |-1\rangle)/\sqrt{2} \\ |E_{-1}\rangle &= (|+1\rangle - |-1\rangle)/\sqrt{2} \\ |E_0\rangle &= |0\rangle \end{aligned}$$

with eigenvalues $\hbar^2(A+B), \hbar^2(A-B), 0$ respectively. These eigenvectors transform in the following way under time reversal:

$$\begin{aligned} \Theta |E_1\rangle &= (\Theta |+1\rangle + \Theta |-1\rangle)/\sqrt{2} = -(|+1\rangle + |-1\rangle)/\sqrt{2} \\ \Theta |E_{-1}\rangle &= (\Theta |+1\rangle - \Theta |-1\rangle)/\sqrt{2} = -(|+1\rangle - |-1\rangle)/\sqrt{2} \\ \Theta |E_0\rangle &= (-1)^0 |E_0\rangle = |E_0\rangle \end{aligned}$$

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