Homework 2

Quantum Mechanics
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Problem 1. *Problem 1.12 from Sakurai*

Solution.

If we choose the representation such that $|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $|2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ then we can use the definition of the outer product to show that

$$H = a\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

The energy eigenvalues are then found by

$$\begin{aligned}\det(H-\lambda I) &= \det\begin{pmatrix}a-\lambda & a\\ a & -a-\lambda\end{pmatrix}\\ &= (a-\lambda)(-a-\lambda)-a^2\\ &= \lambda^2-2a^2=0\end{aligned}$$

therefore $E_\pm = \pm a\sqrt{2}$. The + eigenvector $|\psi_+\rangle$ is given by the system

$$\begin{pmatrix}a-E_+ & a\\ a & -a-E_+\end{pmatrix}\begin{pmatrix}\psi_1^+\\ \psi_2^+\end{pmatrix} = \begin{pmatrix}a-a\sqrt{2} & a\\ a & -a-a\sqrt{2}\end{pmatrix}\begin{pmatrix}\psi_1^+\\ \psi_2^+\end{pmatrix}=0$$

$$\begin{aligned}(1-\sqrt{2})\psi_1^+ + \psi_2^+ &= 0\\ \psi_1^+ - (1+\sqrt{2})\psi_2^+ &= 0\end{aligned}$$

The second equation is just the first multiplied by $(1-\sqrt{2})$ so we can choose $\psi_1^+ = 1$ giving $\psi_2^+ = \sqrt{2}-1$

The eigenvector $|\psi_-\rangle$ is found similarly

$$\begin{pmatrix} a - E_- & a \\ a & -a - E_- \end{pmatrix} \begin{pmatrix} \psi_1^- \\ \psi_2^- \end{pmatrix} = \begin{pmatrix} a + a\sqrt{2} & a \\ a & -a + a\sqrt{2} \end{pmatrix} \begin{pmatrix} \psi_1^- \\ \psi_2^- \end{pmatrix} = 0$$

$$\begin{aligned} (1+\sqrt{2})\psi_1^+ + \psi_2^+ &= 0 \\ \psi_1^+ + (-1+\sqrt{2})\psi_2^+ &= 0 \end{aligned}$$

Similar to before, the second equation is $(-1+\sqrt{2})$ multiplied by the first, allowing us to set $\psi_1^- = 1$ and $\psi_2^- = -(1+\sqrt{2})$, giving a $|\psi_-\rangle$ that is orthogonal to $|\psi_+\rangle$

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Problem 2. *Problem 1.13 from Sakurai*

Solution.

Writing H out in matrix form gives

$$\begin{aligned} H &= \frac{H_{11}}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{H_{22}}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + H_{12}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \frac{H_{11}+H_{22}}{2} I + \frac{H_{11}-H_{12}}{2}\sigma_x + H_{12}\sigma_z \\ &= aI + b\sigma_x + c\sigma_z \end{aligned}$$

where have made appropriate substitutions of constants for brevity. Now this implies,

$$\begin{aligned} H\,|\psi\rangle &= (aI + b\sigma_x + c\sigma_z)\,|\psi\rangle \\ &= a\,|\psi\rangle + (b\sigma_x + 0\sigma_y + c\sigma_z)\,|\psi\rangle \end{aligned}$$

Any $|\psi\rangle$ is an eigenvector under the identity operation, so what we are really after is an eigenvector of the operator $\boldsymbol{\sigma} \cdot \boldsymbol{a}$ for $\boldsymbol{a} = (b, 0, c)$. In other words, if $|\psi\rangle$ is an eigenvector of $\boldsymbol{\sigma} \cdot \boldsymbol{a}$ then it is also an eigenvector of H. It is useful to work with the unit vector in the direction of $\boldsymbol{a}$ which is $\hat{\boldsymbol{n}} = (b/\sqrt{b^2+c^2}, 0, c/\sqrt{b^2+c^2})$. We already know the eigenvectors of $\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}$

$$
\begin{aligned}
|\psi_+\rangle &= \cos\frac{\beta}{2}\,|+\rangle + \exp(i\alpha)\sin\frac{\beta}{2}\,|-\rangle \\
|\psi_-\rangle &= -\sin\frac{\beta}{2}\,|+\rangle + \exp(i\alpha)\cos\frac{\beta}{2}\,|-\rangle
\end{aligned}
$$

where we take the definition that α is the polar angle and β the azimuthal angle. Therefore

$$
\begin{aligned}
&\alpha = 0 \\
&\beta = \arctan\left(\frac{n_z}{n_x}\right) = \arctan\left(\frac{c}{b}\right) = \arctan\left(\frac{2H_{12}}{H_{11} - H_{12}}\right)
\end{aligned}
$$

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Problem 3. *Problem 1.15 from Sakurai*

Solution. After the first measurement along $+\hat{z}$, all of our atoms are prepared in the $|+\rangle$ state in the S_z basis. At the next apparatus oriented along $\hat{n}$, more atoms will be filtered out since $|+\rangle$ is not an eigenket of the $\mathbf{S}\cdot\hat{n}$ operator. Recall that $|+\rangle_n$ is

$$
|+\rangle_n = \cos\frac{\beta}{2}\,|+\rangle + \sin\frac{\beta}{2}\,|-\rangle
$$

The probability the state $|+\rangle$ survives is given by the inner product

$$
\begin{aligned}
|\,\langle+|+\rangle_n\,|^2 &= |\,\langle+|\cos\frac{\beta}{2}\,|+\rangle + \langle+|\sin\frac{\beta}{2}\,|-\rangle\,|^2 \\
&= \cos^2\frac{\beta}{2}
\end{aligned}
$$

After this, all atoms are in the $|+\rangle_n$ state. We then filter the atoms one more time with an apparatus along $-\hat{z}$. The fraction that survive this one is given by

$$
\begin{aligned}
|\,\langle-|+\rangle_n\,|^2 &= |\,\langle-|\cos\frac{\beta}{2}\,|+\rangle + \langle-|\sin\frac{\beta}{2}\,|-\rangle\,|^2 \\
&= \sin^2\frac{\beta}{2}
\end{aligned}
$$

Therefore the fraction output is $\cos^2\frac{\beta}{2}\sin^2\frac{\beta}{2}$. We can maximize this function by setting $\beta = \pi/2$

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Problem 4. *Problem 1.16 from Sakurai*

Solution.

We have the observable

$$O = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{aligned}\det(O - \lambda I) &= \det\begin{pmatrix} -\lambda & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda \end{pmatrix} \\ &= -\lambda\left(\lambda^2 - \frac{1}{2}\right) - \frac{1}{\sqrt{2}}\left(-\frac{\lambda}{\sqrt{2}}\right) \\ &= -\lambda^3 + \lambda = 0\end{aligned}$$

Clearly our eigenvalues are $\lambda = 0, \pm 1$. There is no degeneracy.

$$\begin{pmatrix} -\lambda & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = 0$$

For $\lambda = 0$, we have the system

$$\begin{aligned}\frac{1}{\sqrt{2}}\psi_2 &= 0 \\ \frac{1}{\sqrt{2}}\psi_1 + \frac{1}{\sqrt{2}}\psi_3 &= 0 \\ \frac{1}{\sqrt{2}}\psi_2 &= 0\end{aligned}$$

Therefore $\psi_2 = 0$ and we can take $\psi_1 = 1$ and $\psi_3 = -1$ For $\lambda = -1$, we have

$$\frac{1}{\sqrt{2}}\psi_2 - \psi_1 = 0$$
$$\frac{1}{\sqrt{2}}\psi_1 - \psi_2 + \frac{1}{\sqrt{2}}\psi_3 = 0$$
$$\frac{1}{\sqrt{2}}\psi_2 - \psi_3 = 0$$

The second equation can be eliminated since it is just $-1/\sqrt{2}$ times the first plus $-1/\sqrt{2}$ times the second. We are free to set $\psi_2 = 1$ which gives $\psi_1 = \psi_3 = \frac{1}{\sqrt{2}}$. For the second eigenvector we have the system

$$\frac{1}{\sqrt{2}}\psi_2 + \psi_1 = 0$$
$$\frac{1}{\sqrt{2}}\psi_1 + \psi_2 + \frac{1}{\sqrt{2}}\psi_3 = 0$$
$$\frac{1}{\sqrt{2}}\psi_2 + \psi_3 = 0$$

Again, the second equation can be eliminated and $\psi_2 = 1$ and $\psi_1 = \psi_3 = -\frac{1}{\sqrt{2}}$

A physical system where this is all relevant is the spin-1 system, which in general has three possible eigenstates. However, this observable O only has two non-trivial eigenvectors and the observable is limited to a two dimensional subspace of the three dimensional space.

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Problem 5. *Problem 1.23 from Sakurai*

Solution. For the ground state, the position space wavefunction $|\psi\rangle$ is a solution to the eigenvalue equation

$$\begin{aligned} H\,|\psi\rangle &= \left[\frac{\boldsymbol{p}^2}{2m} + \boldsymbol{V}(x)\right]|\psi\rangle \\ &= -\frac{\hbar^2}{2m}\frac{\partial^2\,|\psi\rangle}{\partial x^2} + V(x)\,|\psi\rangle \\ &= E\,|\psi\rangle \end{aligned}$$

We set the boundary conditions $\psi(0) = 0$ and $\psi(a) = 0$ since the wavefunction must vanish at the two walls. Since $V(x) = 0$ inside the well, Schrodinger's equation reduces to

$$H|\psi\rangle = -\frac{\hbar^2}{2m}\frac{\partial^2|\psi\rangle}{\partial x^2} = E|\psi\rangle$$

We have the following solution

$$|\psi\rangle = A\sin\left(\frac{n\pi x}{a}\right)$$

This solution comes from applying our boundary conditions $\psi(0) = 0$ and $\psi(a) = 0$. These require that the wavelength must satisfy $ka = n\pi$ which means that $k = \frac{n\pi}{a}$ for integer $n > 0$. Now, it is straightforward to show that

$$\langle\psi|\psi\rangle = \frac{2}{a}\int_0^a \sin^2\left(\frac{n\pi x}{a}\right)dx = 1$$

Giving the eigenkets

$$|\psi\rangle = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$$

We would now like to compute the variance of our position measurement in the ground state ($n = 1$). In general, for an observable O and wavefunction $\psi(x)$, we can compute the variance of O as

$$\begin{aligned}\langle(\Delta O)^2\rangle &= \langle O^2\rangle - \langle O\rangle^2\\ \langle O\rangle = \langle\psi|O|\psi\rangle &= \int_{-\infty}^{+\infty}\psi^*(x)O\psi(x)dx\end{aligned}$$

For $O = x$, we have

$$\begin{aligned}\langle(\Delta x)^2\rangle &= \langle x^2\rangle - \langle x\rangle^2\\ &= \langle\psi|x^2|\psi\rangle - (\langle\psi|x|\psi\rangle)^2\end{aligned}$$

We can immediately write the value of $\langle x \rangle^2 = (\langle \psi | \, x \, | \psi \rangle)^2$ based on the symmetry of the wavefunction

$$\langle x \rangle^2 = \frac{a^2}{4}$$

Let $\alpha = n\pi/a$. The term $\langle x^2 \rangle$ is then given by the integral

$$\begin{aligned}
I = \langle x^2 \rangle &\propto \int_0^a x^2 \sin^2(\alpha x)\, dx \\
&= \int_0^a x^2 \cdot \frac{1 - \cos(2\alpha x)}{2} dx \\
&= \frac{1}{2}\left[\frac{a^3}{3} - \int_0^a x^2 \cos(2\alpha x) dx\right] \\
&= \frac{1}{2}\left[\frac{a^3}{3} - \left[a^2 \cdot \frac{\sin(2\alpha a)}{2\alpha} - \frac{1}{\alpha}\int_0^a x\sin(2\alpha x)\right]\right] \\
&= \frac{1}{2}\left[\frac{a^3}{3} + \frac{1}{\alpha}\left[\int_0^a x\sin(2\alpha x)\right]\right]
\end{aligned}$$

The last integral can be evalulated as follows

$$\begin{aligned}
\int_0^a x\sin(2\alpha x) dx &= -\Big|_0^a \frac{x}{2\alpha}\cos(2\alpha x) + \frac{1}{2\alpha}\int_0^a \cos(2\alpha x) dx \\
&= -\frac{a}{2\alpha}\cos(2\alpha a) + \frac{1}{2\alpha}\sin(2\alpha a) \\
&= -\frac{a}{2\alpha}\cos(2\alpha a) \\
&= -\frac{a^2}{n\pi}\cos(n\pi)
\end{aligned}$$

Combining these results, we get

$$I = \frac{1}{2}\left[\frac{a^3}{3} - \frac{a^3}{n^2\pi^2}\cos(n\pi)\right]$$

Finally, bringing in the normalization factor $2/a$ gives

$$I = \langle x^2 \rangle = a^2 \left[\frac{1}{3} - \frac{1}{2n^2\pi^2} \cos(n\pi) \right]$$

which gives the variance

$$\begin{aligned} \langle (\Delta x)^2 \rangle &= \langle x^2 \rangle - \langle x \rangle^2 \\ &= a^2 \left[\frac{1}{3} - \frac{1}{2n^2\pi^2} \cos(n\pi) \right] - \frac{a^2}{4} \end{aligned}$$

Now for $O = p = -i\hbar\frac{\partial}{\partial x}$, we have

$$\begin{aligned} \langle (\Delta p)^2 \rangle &= \langle p^2 \rangle - \langle p \rangle^2 \\ &= \langle \psi | \, p^2 \, | \psi \rangle - \left(\langle \psi | \, p \, | \psi \rangle \right)^2 \end{aligned}$$

Since the potential in the well is zero, the first term is just

$$\langle p^2 \rangle = 2mE_n = \frac{n^2\pi^2\hbar^2}{a^2}$$

Using the same substitution as for position, the second terms reads

$$\begin{aligned} \langle \psi | \, p \, | \psi \rangle &\propto -i\hbar \int_0^a 2\alpha \sin(\alpha x) \cos(\alpha x) dx \\ &= -i\hbar \int_0^a \sin(2\alpha x) dx \\ &= 0 \end{aligned}$$

Summarizing, we have

$$\langle (\Delta p)^2 \rangle = \frac{n^2\pi^2\hbar^2}{a^2}$$

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Problem 6. *Problem 1.24 from Sakurai*

Solution.

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