Homework 6

Quantum Mechanics
October 28th, 2022 C Seitz

Problem 1. *Problem 3.12 from Sakurai*

Solution.

In general the ensemble average of an operator $[A]$ is defined as

$$[A] = \sum_i w_i \langle \alpha_i | A | \alpha_i \rangle$$

where $\sum_i w_i = 1$

$$\begin{aligned}
[\sigma_x] &= a \langle + | \sigma_x | + \rangle + (1-a) \langle -; y | \sigma_x | -; y \rangle \\
&= a \langle + | \left(| + \rangle \langle - | + | - \rangle \langle + | \right) | + \rangle + (1-a) \langle -; y | \left(| + \rangle \langle - | + | - \rangle \langle + | \right) | -; y \rangle \\
&= 0
\end{aligned}$$

$$\begin{aligned}
[\sigma_y] &= a \langle + | \sigma_y | + \rangle + (1-a) \langle -; y | \sigma_y | -; y \rangle \\
&= ai \langle + | \left(| + \rangle \langle - | - | - \rangle \langle + | \right) | + \rangle + i(1-a) \langle -; y | \left(| + \rangle \langle - | - | - \rangle \langle + | \right) | -; y \rangle \\
&= i(1-a) \langle -; y | \left(-\frac{i}{\sqrt{2}} | + \rangle - \frac{1}{\sqrt{2}} | - \rangle \right) \\
&= -i(1-a) \langle -; y | +; y \rangle = 0
\end{aligned}$$

$$\begin{aligned}
[\sigma_z] &= a \langle + | \left(| - \rangle \langle - | - | + \rangle \langle + | \right) | + \rangle + i(1-a) \langle -; y | \left(| - \rangle \langle - | - | + \rangle \langle + | \right) | -; y \rangle \\
&= -a + i(1-a) \langle -; y | \left(-\frac{i}{\sqrt{2}} | + \rangle - \frac{1}{\sqrt{2}} | - \rangle \right)
\end{aligned}$$

■

Problem 2. *Problem 3.13 from Sakurai*

Solution.

The state vector in the S_z basis has the form

$$|\alpha\rangle = c_+ |+\rangle + c_- |-\rangle$$

First note that

$$\langle S_z \rangle = |c_+|^2 - |c_-|^2 \quad |c_+|^2 + |c_-|^2 = 1$$

Together, these equations tell us the magnitude of each complex component.

$$|c_+|^2 = \frac{\langle S_z \rangle + 1}{2} \quad |c_-|^2 = \frac{1 - \langle S_z \rangle}{2}$$

$$\begin{aligned}
\langle S_x \rangle &= \langle \alpha | (|+\rangle \langle -| + |-\rangle \langle +|)(c_+ |+\rangle + c_- |-\rangle) \\
&= \langle \alpha | (c_- |+\rangle + c_+ |-\rangle) \\
&= (c_+^* \langle +| + c_-^* \langle -|)(c_- |+\rangle + c_+ |-\rangle) \\
&= c_+^* c_- + c_-^* c_+ \\
&= |c_+||c_-|(e^{i(\theta - \phi)} + e^{i(\phi - \theta)}) \\
&= 2|c_+||c_-| \cos(\theta - \phi)
\end{aligned}$$

$$\begin{aligned}
\langle S_y \rangle &= \langle \alpha | ((i |+\rangle \langle -| - i |-\rangle \langle +|)(c_+ |+\rangle + c_- |-\rangle) \\
&= i \langle \alpha | (c_- |+\rangle - c_+ |-\rangle) \\
&= i(c_+^* \langle +| + c_-^* \langle -|)(c_- |+\rangle - c_+ |-\rangle) \\
&= c_+^* c_- - c_-^* c_+ \\
&= |c_+||c_-|(e^{i(\theta - \phi)} - e^{i(\phi - \theta)}) \\
&= 2i|c_+||c_-| \sin(\theta - \phi)
\end{aligned}$$

So $\langle S_x \rangle$ gives us the phase difference of c_+ and c_-. Then the sign of $\langle S_y \rangle$ tells us whether θ or ϕ is larger, since sine is odd. This is all we can hope to extract from the expectation values, since multiplying by a global phase $e^{i\delta} |\alpha\rangle$ has no effect on the expectation values.

■

Problem 3. *Problem 3.14 from Sakurai*

Solution. ■

Problem 4. *Problem 3.15 from Sakurai*

Solution. ■

Problem 5. *Problem 3.16 from Sakurai*

Solution. ■

Problem 6. *Problem 3.40 from Sakurai*

Solution. ■