

# **TTIC 31230, Fundamentals of Deep Learning**

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## **The Fundamental Equations of Deep Learning**

# What is a Deep Network?

VGG, Zisserman, 2014

Davi Frossard

138 Million Parameters

## What is a Deep Network?

We assume some set  $\mathcal{X}$  of possible inputs, some set  $\mathcal{Y}$  of possible outputs, and a parameter vector  $\Phi \in \mathbb{R}^d$ .

For  $\Phi \in \mathbb{R}^d$  and  $x \in \mathcal{X}$  and  $y \in \mathcal{Y}$  a deep network computes a probability  $P_\Phi(y|x)$ .

# The Fundamental Equation of Deep Learning

We assume a “population” probability distribution  $\text{Pop}$  on pairs  $(x, y)$ .

$$\Phi^* = \underset{\Phi}{\operatorname{argmin}} E_{(x,y) \sim \text{Pop}} - \ln P_{\Phi}(y|x)$$

This loss function  $\mathcal{L}(x, y, \Phi) = -\ln P_{\Phi}(y|x)$  is called **cross entropy loss**.

## A Second Fundamental Equation

### Softmax: Converting Scores to Probabilities

We start from a “score” function  $s_{\Phi}(y|x) \in \mathbb{R}$ .

$$\begin{aligned} P_{\Phi}(y|x) &= \frac{1}{Z} e^{s_{\Phi}(y|x)}; \quad Z = \sum_y e^{s_{\Phi}(y|x)} \\ &= \operatorname{softmax}_y s_{\Phi}(y|x) \end{aligned}$$

# Note the Final Softmax Layer

Davi Frossard

## How Many Possibilities

We have  $y \in \mathcal{Y}$  where  $\mathcal{Y}$  is some set of “possibilities”.

Binary:  $Y = \{-1, 1\}$

Multiclass:  $Y = \{y_1, \dots, y_k\}$   $k$  manageable.

Structured:  $y$  is a “structured object” like a sentence. Here  $|Y|$  is unmanageable.

## Binary Classification

We have a population distribution over  $(x, y)$  with  $y \in \{-1, 1\}$ .

We compute a single score  $s_\Phi(x)$  where

for  $s_\Phi(x) \geq 0$  predict  $y = 1$

for  $s_\Phi(x) < 0$  predict  $y = -1$



## Softmax for Binary Classification

$$\begin{aligned} P_{\Phi}(y|x) &= \frac{1}{Z} e^{ys(x)} \\ &= \frac{e^{ys(x)}}{e^{ys(x)} + e^{-ys(x)}} \\ &= \frac{1}{1 + e^{-2ys(x)}} \\ &= \frac{1}{1 + e^{-m(y)}} \quad m(y|x) = 2ys(x) \text{ is the margin} \end{aligned}$$

# Logistic Regression for Binary Classification

$$\Phi^* = \operatorname{argmin}_{\Phi} E_{(x,y) \sim P_{\text{op}}} \mathcal{L}(x, y, \Phi)$$

$$= \operatorname{argmin}_{\Phi} E_{(x,y) \sim P_{\text{op}}} -\ln P_{\Phi}(y|x)$$

$$= \operatorname{argmin}_{\Phi} E_{(x,y) \sim P_{\text{op}}} \ln \left( 1 + e^{-m(y|x)} \right)$$

$$\ln \left( 1 + e^{-m(y|x)} \right) \approx 0 \quad \text{for } m(y|x) \gg 1$$

$$\ln \left( 1 + e^{-m(y|x)} \right) \approx -m(y|x) \quad \text{for } -m(y|x) \gg 1$$

## Log Loss vs. Hinge Loss (SVM loss)

## Image Classification (Multiclass Classification)

We have a population distribution over  $(x, y)$  with  $y \in \{y_1, \dots, y_k\}$ .

$$P_{\Phi}(y|x) = \underset{y}{\text{softmax}} \ s_{\Phi}(y|x)$$

$$\begin{aligned} \Phi^* &= \underset{\Phi}{\text{argmin}} \ E_{(x,y) \sim P_{\text{op}}} \mathcal{L}(x, y, \Phi) \\ &= \underset{\Phi}{\text{argmin}} \ E_{(x,y) \sim P_{\text{op}}} \textcolor{red}{- \ln P_{\Phi}(y|x)} \end{aligned}$$

## Machine Translation (Structured Labeling)

We have a population of translation pairs  $(x, y)$  with  $x \in V_x^*$  and  $y \in V_y^*$  where  $V_x$  and  $V_y$  are source and target vocabularies respectively.

$$P_{\Phi}(w_{t+1}|x, w_1, \dots, w_t) = \underset{w \in V_y \cup \langle \text{EOS} \rangle}{\text{softmax}} s_{\Phi}(w \mid x, w_1, \dots, w_t)$$

$$P_{\Phi}(y|x) = \prod_{t=0}^{|y|} P_{\Phi}(y_{t+1} \mid x, y_1, \dots, y_t)$$

$$\begin{aligned} \Phi^* &= \underset{\Phi}{\text{argmin}} E_{(x,y) \sim \text{Pop}} \mathcal{L}(x, y, \Phi) \\ &= \underset{\Phi}{\text{argmin}} E_{(x,y) \sim \text{Pop}} - \ln P_{\Phi}(y|x) \end{aligned}$$

## Fuundamental Equation: Unconditional Form

$$\Phi^* = \operatorname{argmin}_{\Phi} E_{y \sim P_{\text{op}}} - \ln P_{\Phi}(y)$$

## Entropy of a Distribution

The entropy of a distribution  $P$  is defined by

$$H(P) = E_{y \sim P} [-\ln P(y)] \text{ in units of “nats”}$$

$$H_2(P) = E_{y \sim P} [-\log_2 P(y)] \text{ in units of bits}$$

Example: Let  $Q$  be a uniform distribution on 256 values.

$$E_{y \sim Q} [-\log_2 Q(y)] = -\log_2 \frac{1}{256} = \log_2 256 = 8 \text{ bits} = 1 \text{ byte}$$

$$1 \text{ nat} = \frac{1}{\ln 2} \text{ bits} \approx 1.44 \text{ bits}$$

## The Coding Interpretation of Entropy

We can interpret  $H_2(Q)$  as the number of bits required on average to represent items drawn from distribution  $Q$ .

We want to use fewer bits for common items.

There exists a representation where, for all  $y$ , the number of bits used to represent  $y$  is no larger than  $-\log_2 y + 1$  (Shannon's source coding theorem).

$$H(Q) = \frac{1}{\ln 2} H_2(Q) \approx 1.44 H_2(Q)$$



## Cross Entropy

Let  $P$  and  $Q$  be two distribution on the same set.

$$H(P, Q) = E_{y \sim P} - \ln Q(y)$$

$$\Phi^* = \operatorname{argmin}_{\Phi} H(\text{Pop}, P_{\Phi})$$

$H(P, Q)$  also has a data compression interpretation.

$H(P, Q)$  can be interpreted as 1.44 times the number of bits used to code draws from  $P$  when using the imperfect code defined by  $Q$ .

## Entropy, Cross Entropy and KL Divergence

Let  $P$  and  $Q$  be two distribution on the same set.

$$\text{Entropy :} \quad H(P) = E_{y \sim P} - \ln P(y)$$

$$\text{CrossEntropy :} \quad H(P, Q) = E_{y \sim P} - \ln Q(y)$$

$$\begin{aligned} \text{KL Divergence :} \quad KL(P, Q) &= H(P, Q) - H(P) \\ &= E_{y \sim P} \ln \frac{P(y)}{Q(y)} \end{aligned}$$

We have  $H(P, Q) \geq H(P)$  or equivalently  $KL(P, Q) \geq 0$ .

## The Universality Assumption

$$\Phi^* = \operatorname{argmin}_{\Phi} H(\text{Pop}, P_{\Phi}) = \operatorname{argmin}_{\Phi} H(\text{Pop}) + KL(\text{Pop}, P_{\Phi})$$

Universality assumption:  $P_{\Phi}$  can represent any distribution and  $\Phi$  can be fully optimized.

This is clearly false for deep networks. But it gives important insights like:

$$P_{\Phi^*} = \text{Pop}$$

This is the motivation for the fundamental equation.

## Asymmetry of Cross Entropy

Consider

$$\Phi^* = \operatorname{argmin}_{\Phi} H(P, Q_{\Phi}) \quad (1)$$

$$\Phi^* = \operatorname{argmin}_{\Phi} H(Q_{\Phi}, P) \quad (2)$$

For (1)  $Q_{\Phi}$  must cover all of the support of  $P$ .

For (2)  $Q_{\Phi}$  concentrates all mass on the point maximizing  $P$ .

## Asymmetry of KL Divergence

Consider

$$\begin{aligned}\Phi^* &= \operatorname{argmin}_{\Phi} KL(P, Q_{\Phi}) \\ &= \operatorname{argmin}_{\Phi} H(P, Q_{\Phi})\end{aligned}\tag{1}$$

$$\begin{aligned}\Phi^* &= \operatorname{argmin}_{\Phi} KL(Q_{\Phi}, P) \\ &= \operatorname{argmin}_{\Phi} H(Q_{\Phi}, P) - H(Q_{\Phi})\end{aligned}\tag{2}$$

If  $Q_{\Phi}$  is not universally expressive we have that (1) still forces  $Q_{\Phi}$  to cover all of  $P$  (or else the KL divergence is infinite) while (2) allows  $Q_{\Phi}$  to be restricted to a single mode of  $P$  (a common outcome).

## Proving $KL(P, Q) \geq 0$ : Jensen's Inequality

For  $f$  convex (upward curving) we have

$$E[f(x)] \geq f(E[x])$$

**Proving**  $KL(P, Q) \geq 0$

$$\begin{aligned} KL(P, Q) &= E_{y \sim P} - \log \frac{Q(y)}{P(y)} \\ &\geq -\log E_{y \sim P} \frac{Q(y)}{P(y)} \\ &= -\log \sum_y P(y) \frac{Q(y)}{P(y)} \\ &= -\log \sum_y Q(y) \\ &= 0 \end{aligned}$$

## Summary

$$\Phi^* = \operatorname{argmin}_{\Phi} H(\text{Pop}, P_{\Phi}) \text{ unconditional}$$

$$\Phi^* = \operatorname{argmin}_{\Phi} E_{x \sim \text{Pop}} H(\text{Pop}(y|x), P_{\Phi}(y|x)) \text{ conditional}$$

$$\text{Entropy :} \quad H(P) = E_{y \sim P} - \ln P(y)$$

$$\text{CrossEntropy :} \quad H(P, Q) = E_{y \sim P} - \ln Q(y)$$

$$\text{KL Divergence :} \quad KL(P, Q) = H(P, Q) - H(P)$$

$$= E_{y \sim P} \ln \frac{P(y)}{Q(y)}$$

$$H(P, Q) \geq H(P), \quad KL(P, Q) \geq 0, \quad \operatorname{argmin}_Q H(P, Q) = P$$



## Appendix: The Rearrangement Trick

$$\begin{aligned} KL(P, Q) &= E_{x \sim P} \ln \frac{P(x)}{Q(x)} \\ &= E_{x \sim P} [(-\ln Q(x)) - (-\ln P(x))] \\ &= (E_{x \sim P} -\ln Q(x)) - (E_{x \sim P} -\ln P(x)) \\ &= H(P, Q) - H(P) \end{aligned}$$

In general  $E_{x \sim P} \ln (\prod_i A_i) = E_{x \sim P} \sum_i \ln A_i$

## Appendix: The Rearrangement Trick

$$\begin{aligned}\text{ELBO} &= E_{z \sim P_{\Psi}(z|y)} \ln \frac{P_{\Phi}(z, y)}{P_{\Psi}(z|y)} \\ &= E_{z \sim P_{\Psi}(z|y)} \ln \frac{P_{\Phi}(z)P_{\Phi}(y|z)}{P_{\Psi}(z|y)} \\ &= E_{z \sim P_{\Psi}(z|y)} \ln \frac{P_{\Phi}(y)P_{\Phi}(z|y)}{P_{\Psi}(z|y)}\end{aligned}$$

Each of the last two expressions can be grouped three different ways leading to six ways of writing the ELBO.

**END**