The variational autoencoder

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Variational Bayes

The variable $\mathbf{x}$ has a latent representation or code $\mathbf{z}$. We often say that $\mathbf{z}$ is the *causal source* of $\mathbf{x}$. The distribution we are after is the *evidence* in Bayes rule $P(\mathbf{x})$

$$P(\mathbf{x}) = \frac{P_\Phi(\mathbf{x}|\mathbf{z})P_\Omega(\mathbf{z})}{Q_\Psi(\mathbf{z}|\mathbf{x})}$$

It is common to take $P_\Omega(\mathbf{z})$ to be Gaussian. We then try find the model parameters $\Theta = (\Phi, \Psi)$ that maximize the likelihood of the observed data:

$$\Theta^* = \underset{\Theta}{\mathrm{argmin}} - \log P(\mathbf{x}_{\mathrm{obs}})$$

Computing the evidence

We can alternatively write the evidence as

$$
\begin{aligned}
P(\mathbf{x}) &= \int P_\Omega(\mathbf{z}) P_\Phi(\mathbf{x}|\mathbf{z}) d\mathbf{z} \\
&= \int P_\Omega(\mathbf{z}) P_\Phi(\mathbf{x}|\mathbf{z}) \frac{P_\Psi(\mathbf{z}|\mathbf{x})}{P_\Psi(\mathbf{z}|\mathbf{x})} d\mathbf{z} \\
&= \mathbb{E}_{\mathbf{z}\sim P_\Psi(\mathbf{z}|\mathbf{x})} \frac{P_\Omega(\mathbf{z}) P_\Phi(\mathbf{x}|\mathbf{z})}{P_\Psi(\mathbf{z}|\mathbf{x})}
\end{aligned}
$$

We call Ψ and Φ the encoder and decoder, respectively

The evidence lower bound (ELBO)

$$
\begin{aligned}
\log P(\mathbf{x}) &= \log\ \mathbb{E}_{\mathbf{z}\sim P_{\Psi}(\mathbf{z}|\mathbf{x})} \frac{P_{\Omega}(\mathbf{z})P_{\Phi}(\mathbf{x}|\mathbf{z})}{P_{\Psi}(\mathbf{z}|\mathbf{x})} \\
&\geq \mathbb{E}_{\mathbf{z}\sim P_{\Psi}(\mathbf{z}|\mathbf{x})} \log\ \frac{P_{\Omega}(\mathbf{z})P_{\Phi}(\mathbf{x}|\mathbf{z})}{P_{\Psi}(\mathbf{z}|\mathbf{x})}
\end{aligned}
$$

$$
-\log P(\mathbf{x}) \leq \mathbb{E}_{\mathbf{z}\sim P_{\phi}(\mathbf{z}|\mathbf{x})} \log \frac{P_{\Psi}(\mathbf{z}|\mathbf{x})}{P_{\Omega}(\mathbf{z})} - \log P_{\Phi}(\mathbf{x}|\mathbf{z})
$$

The ELBO objective

$$\begin{aligned}\Theta^* &= \underset{\Phi,\Psi}{\operatorname{argmin}} \;\; \mathbb{E}_{\mathbf{x}\sim\mathrm{Pop},\, \mathbf{z}\sim P_\Psi(\mathbf{z}|\mathbf{x})} \log \frac{P_\Psi(\mathbf{z}|\mathbf{x})}{P_\Omega(\mathbf{z})} - \log P_\Phi(\mathbf{x}|\mathbf{z}) \\ &= \underset{\Phi,\Psi}{\operatorname{argmin}} \;\; D_{\mathrm{KL}}\left(P_\Psi || P_\Omega\right) - \mathbb{E}_{\mathbf{x}\sim\mathrm{Pop},\, \mathbf{z}\sim P_\Psi(\mathbf{z}|\mathbf{x})} \log P_\Phi(\mathbf{x}|\mathbf{z})\end{aligned}$$

The first term is a **rate term** to be minimized and the second a **reconstruction term** to be maximized

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