Homework 1

Quantum Mechanics
August 29th, 2022 CLAYTON SEITZ

Problem 1. *Problem 1.3 from Sakurai*

Solution.

Let $A = S_x$ and $B = S_y$. The variance $\langle(\Delta S_x)^2\rangle$ in state $|+\rangle_x$ must be zero since $|+\rangle_x$ is an eigenvector of S_x

$$\begin{aligned}\langle(\Delta S_x)^2\rangle &= \langle S_x^2\rangle - \langle S_x\rangle^2 \\ &= \langle +|_x\, S_x^2\, |+\rangle_x - (\langle +|_x\, S_x\, |+\rangle_x)^2 \\ &= \frac{\hbar^2}{4} - \frac{\hbar^2}{4} = 0\end{aligned}$$

Therefore, the LHS of the above inequality is zero. The commutator $[S_x, S_y] = i\hbar S_z$ and

$$\langle S_z\rangle = \langle +|_x\, S_z\, |+\rangle_x = 0$$

Clearly the inequality is satisfied since both sides are zero. Now let $A = S_z$ and $B = S_y$. Since the state is prepared in $|+\rangle_x$, the variance $\langle(\Delta S_z)^2\rangle$ is

$$\begin{aligned}\langle(\Delta S_z)^2\rangle &= \langle S_z^2\rangle - \langle S_z\rangle^2 \\ &= \langle +|_x\, S_z^2\, |+\rangle_x - (\langle +|_x\, S_z\, |+\rangle_x)^2\end{aligned}$$

$$\begin{aligned}
S_z \left|+\right\rangle_x &= \frac{\hbar}{2}\left(\left|+\right\rangle\left\langle+\right| - \left|-\right\rangle\left\langle-\right|\right)\frac{1}{\sqrt{2}}(\left|+\right\rangle + \left|-\right\rangle) \\
&= \frac{\hbar}{2\sqrt{2}}(\left|+\right\rangle - \left|-\right\rangle) = \frac{\hbar}{2}\left|-\right\rangle_x
\end{aligned}$$

and it can be shown by applying it again that $S_z^2 \left|+\right\rangle_x = \left(\frac{\hbar}{2}\right)^2 \left|+\right\rangle_x$. Also, in general, $\left\langle+\right|_x S_z \left|+\right\rangle_x = 0$ which gives us

$$\langle(\Delta S_z)^2\rangle = \left(\frac{\hbar}{2}\right)^2$$

and the variance must be the same for S_y

The commutator $[S_z, S_y] = -i\hbar S_x$ and $\langle S_x\rangle = \frac{\hbar}{2}$. The inequality then reads

$$\begin{aligned}
\left(\frac{\hbar}{2}\right)^2\left(\frac{\hbar}{2}\right)^2 &\geq \frac{1}{4}|\langle[S_z, S_y]\rangle|^2 \\
&= \frac{\hbar^2}{4}|\langle S_x\rangle|^2 \\
&= \left(\frac{\hbar}{2}\right)^2\left(\frac{\hbar}{2}\right)^2
\end{aligned}$$

which is satisfied by the equality.

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Problem 2. *Problem 1.4 from Sakurai*

Solution.

$$\begin{aligned}
\mathrm{Tr}(X) &= \mathrm{Tr}(a_0) + \mathrm{Tr}\left(\sum_k a_k\sigma_k\right) \\
&= 2a_0
\end{aligned}$$

$$
\begin{aligned}
\mathrm{Tr}(\sigma_k X) &= \mathrm{Tr}\left(\sigma_k a_0 + \sigma_k \sum_j a_j \sigma_j\right) \\
&= \mathrm{Tr}\left(\sigma_k a_0 + \sum_j a_j \sigma_k \sigma_j\right) \\
&= \mathrm{Tr}\left(\sum_j a_j \sigma_k \sigma_j\right)
\end{aligned}
$$

We can write out the equation $X = a_0 + \sigma \cdot a$ explicitly

$$
X = \begin{pmatrix} a_0 + a_3 & a_1 - ia_3 \\ a_1 + ia_2 & a_0 - a_3 \end{pmatrix}
$$

Thus we have four equations involving X_{ij}'s and a_k for $k = (1, 2, 3)$. We can manipulate those four equations to show that

$$
\begin{aligned}
a_0 &= \frac{X_{11} + X_{22}}{2} \\
a_1 &= \frac{X_{12} + X_{21}}{2} \\
a_2 &= i\frac{X_{12} - X_{21}}{2} \\
a_3 &= \frac{X_{11} - X_{22}}{2}
\end{aligned}
$$

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Problem 3. *Problem 1.5 from Sakurai*

Solution.

To simplify the notation, let $\theta = \phi/2$. The matrix exponential can be expanded as a power series

$$
\begin{aligned}
\exp(i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})\theta) &= I + i\theta(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}) + \frac{(i\theta(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}))^2}{2!} + \frac{(i\theta(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}))^3}{3!} + ... \\
&= I + i\theta(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}) - \frac{\theta^2}{2!} + \frac{\theta^3(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})}{3!} + ... \\
&= \left(I - \frac{\theta^2}{2!} + ...\right) + i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})\left(\theta - \frac{\theta^3}{3!} + ...\right) \\
&= \cos\theta I + i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})\sin\theta
\end{aligned}
$$

and similarly $\exp(-i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})\theta) = \cos\theta I - i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})\sin\theta$. We can use this result to write $\exp(\pm i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})\theta)$ out more explicitly:

$$
\exp(i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})\theta) = \begin{pmatrix} \cos\theta + in_z\sin\theta & (-in_x + n_y)\sin\theta \\ (in_x - n_y)\sin\theta & \cos\theta - in_z\sin\theta \end{pmatrix}
$$

$$
\exp(-i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})\theta) = \begin{pmatrix} \cos\theta - in_z\sin\theta & (-in_x - n_y)\sin\theta \\ (-in_x + n_y)\sin\theta & \cos\theta + in_z\sin\theta \end{pmatrix}
$$

Now, we were given the transformation

$$
\boldsymbol{\sigma}\cdot\boldsymbol{a}' = \exp\left(\frac{i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\phi}{2}\right)\boldsymbol{\sigma}\cdot\boldsymbol{a}\exp\left(-\frac{i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}}\phi}{2}\right)
$$

and would like to show that

$$
\det(\boldsymbol{\sigma}\cdot\boldsymbol{a}') = \det(\boldsymbol{\sigma}\cdot\boldsymbol{a})
$$

To see this, notice that the determinant of $\det(\boldsymbol{\sigma}\cdot\boldsymbol{a}')$ can be written as a product of three determinants. The two determinants coming from terms $\exp(\pm i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})\theta)$ will multiply to unity

$$
\det(\exp(i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})\theta)) \cdot \det(\exp(-i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})\theta)) = 1
$$

Leaving only $\det(\boldsymbol{\sigma}\cdot\boldsymbol{a})$. In the case that $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, the matrices $\exp(\pm i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}})\theta)$ reduce to

$$
\exp(i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{z}})\theta) = \begin{pmatrix} \exp(i\theta) & 0 \\ 0 & \exp(-i\theta) \end{pmatrix}
$$

$$\exp(-i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{z}})\theta) = \begin{pmatrix} \exp(-i\theta) & 0 \\ 0 & \exp(i\theta) \end{pmatrix}$$

Now using these matrices above in some simple matrix operations and substituting back $\phi = 2\theta$, we can show that

$$\exp(i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{z}})\theta)\ \sigma_z\ \exp(-i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{z}})\theta) = \sigma_z$$

$$\exp(i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{z}})\theta)\ \sigma_x\ \exp(-i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{z}})\theta) = \sigma_x\cos\phi - \sigma_y\sin\phi$$

$$\exp(i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{z}})\theta)\ \sigma_y\ \exp(-i(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{z}})\theta) = \sigma_x\sin\phi + \sigma_y\cos\phi$$

which means that $a'_z = a_z$, $a'_y = a_x\sin\phi + a_y\cos\phi$, and $a'_x = a_x\cos\phi - a_y\sin\phi$. This is a rotation about $\hat{\mathbf{z}}$ by an angle ϕ.

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Problem 4. *Problem 1.8 from Sakurai*

Solution.

$$A(|i\rangle + |j\rangle) = i\,|i\rangle + j\,|j\rangle$$

If we have degenerate eigenvalues i.e., $i = j$ then

$$A(|i\rangle + |j\rangle) = i(|i\rangle + |j\rangle)$$

and $|i\rangle + |j\rangle$ is also an eigenvector of A

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Problem 5. *Problem 1.10 from Sakurai*

Solution. We will make use of the following outer-product representations of the spin operators

$$\begin{aligned} S_x &= \frac{\hbar}{2}\left(|+\rangle\langle-| + |-\rangle\langle+|\right) \\ S_y &= \frac{i\hbar}{2}\left(-|+\rangle\langle-| + |-\rangle\langle+|\right) \\ S_z &= \frac{\hbar}{2}\left(|+\rangle\langle+| - |-\rangle\langle-|\right) \end{aligned}$$

$$
\begin{aligned}
[S_x, S_y] &= \frac{i\hbar^2}{4}\left(|+\rangle\langle-| + |-\rangle\langle+|\right)\left(-|+\rangle\langle-| + |-\rangle\langle+|\right) \\
&\quad - \frac{i\hbar^2}{4}\left(-|+\rangle\langle-| + |-\rangle\langle+|\right)\left(|+\rangle\langle-| + |-\rangle\langle+|\right) \\
&= \frac{i\hbar^2}{4}\left(|+\rangle\langle+| - |-\rangle\langle-|\right) + \frac{i\hbar^2}{4}\left(|+\rangle\langle+| - |-\rangle\langle-|\right) \\
&= \frac{i\hbar^2}{2}\left(|+\rangle\langle+| - |-\rangle\langle-|\right) \\
&= i\hbar S_z
\end{aligned}
$$

Flipping the order of the commutator always flips the sign of the result i.e. $[S_i, S_j] = -[S_j, S_i]$. Thus for $[S_y, S_x]$ we would get $-i\hbar S_z$.

$$
\begin{aligned}
[S_y, S_z] &= \frac{i\hbar^2}{4}\left(-|+\rangle\langle-| + |-\rangle\langle+|\right)\left(|+\rangle\langle+| - |-\rangle\langle-|\right) \\
&\quad - \frac{i\hbar^2}{4}\left(|+\rangle\langle+| - |-\rangle\langle-|\right)\left(-|+\rangle\langle-| + |-\rangle\langle+|\right) \\
&= \frac{i\hbar^2}{4}\left(|+\rangle\langle-| + |-\rangle\langle+|\right) - \frac{i\hbar^2}{4}\left(-|+\rangle\langle-| - |-\rangle\langle+|\right) \\
&= \frac{i\hbar^2}{2}\left(|+\rangle\langle-| + |-\rangle\langle+|\right) \\
&= i\hbar S_x
\end{aligned}
$$

$$
\begin{aligned}
[S_z, S_x] &= \frac{\hbar^2}{4}\left(|+\rangle\langle+| - |-\rangle\langle-|\right)\left(|+\rangle\langle-| + |-\rangle\langle+|\right) \\
&\quad - \frac{\hbar^2}{4}\left(|+\rangle\langle-| + |-\rangle\langle+|\right)\left(|+\rangle\langle+| - |-\rangle\langle-|\right) \\
&= \frac{\hbar^2}{4}\left(-|+\rangle\langle-| + |-\rangle\langle+|\right) - \frac{\hbar^2}{4}\left(|+\rangle\langle-| - |-\rangle\langle+|\right) \\
&= -\frac{\hbar^2}{2}\left(-|+\rangle\langle-| + |-\rangle\langle+|\right) \\
&= i\hbar S_y
\end{aligned}
$$

When $i = j$ we will always have $\{S_i, S_j\} = 2S_i^2 = \frac{\hbar^2}{2}$ since $S_i^2 = I \quad \forall i$. Therefore, for the anticommutator relations, all we need to prove is that $S_i S_j = -S_j S_i$ when $i \neq j$. In fact, this is obvious from the third line of each of the above expressions. The terms are always identical up to a sign flip, which is why we always get a factor of $\frac{\hbar^2}{2}$ in the fourth line of each of them. Therefore, it is always true that $S_iS_j = -S_jS_i$ for $i \neq j$

$\blacksquare$

Problem 6. *Problem 1.11 from Sakurai*

Solution.

We would like to find a representation for the state $|\mathbf{S}\cdot\hat{n};+\rangle$ in the S_z basis. We first write the operator $\mathbf{S}\cdot\hat{n}$ explicitly in this basis

$$
\begin{aligned}
\mathbf{S}\cdot\hat{n} &= \sin\beta\cos\alpha\ S_x + \sin\beta\sin\alpha\ S_y + \cos\beta\ S_z \\
&= \frac{\hbar}{2}\begin{pmatrix} \cos\beta & \sin\beta\exp(-i\alpha) \\ \sin\beta\exp(i\alpha) & -\cos\beta \end{pmatrix}
\end{aligned}
$$

As usual, we find the eigenvalues of this operator by solving the characteristic equation:

$$
\begin{aligned}
\det\left(\mathbf{S}\cdot\hat{n} - \lambda I\right) &= \left(\frac{\hbar}{2}\cos\beta - \lambda\right)\left(-\frac{\hbar}{2}\cos\beta - \lambda\right) - \frac{\hbar^2}{4}\sin^2\beta \\
&= \lambda^2 - \frac{\hbar^2}{4} = 0
\end{aligned}
$$

Therefore $\lambda = \pm\frac{\hbar}{2}$ as expected. Let ψ_1 and ψ_2 represent the components of the eigenket $|\mathbf{S}\cdot\hat{n};+\rangle$ of this operator. We then need to solve the following system for the components ψ_1 and ψ_2

$$
\begin{aligned}
\psi_1\cos\beta + \psi_2\sin\beta\exp(-i\alpha) &= \psi_1 \\
\psi_1\sin\beta\exp(i\alpha) - \psi_2\cos\beta &= \psi_2
\end{aligned}
$$

We need to validate that $\psi_1 = \cos\frac{\beta}{2}$ and $\psi_2 = \sin\frac{\beta}{2}\exp(i\alpha)$ is a solution.

$\blacksquare$