

Homework 4

Quantum Mechanics

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Problem 1. *Problem 2.14 from Sakurai*

Solution.

We are given that the state vector is

$$|\alpha\rangle = \exp\left(\frac{-ipa}{\hbar}\right) |0\rangle$$

The Heisenberg equation of motion reads

$$\frac{dx}{dt} = \frac{1}{i\hbar} [x, H] = 0$$

Therefore $x = x_0$ for all $t \geq t_0$

$$\begin{aligned}\langle x \rangle &= \int x_0 \langle x|\alpha \rangle \langle \alpha|x \rangle dx \\ &= \int x \exp\left(\frac{-ipa}{\hbar}\right) \langle x|0 \rangle \exp\left(\frac{ipa}{\hbar}\right) \langle 0|x \rangle dx \\ &= \int x_0 |\langle x|0 \rangle|^2 dx \\ &= \int x_0 |\langle x|0 \rangle|^2 dx\end{aligned}$$

We could write out $\langle x|0 \rangle$, its complex conjugate, and do the integral. Instead recall the general expression for the matrix element of x

$$\langle n'|x|n \rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n} \delta_{n',n-1} + \sqrt{n+1} \delta_{n',n+1} \right)$$

which is zero when $n = n'$ which means that $\langle x \rangle = 0$

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Problem 2. *Problem 2.15 from Sakurai*

Solution. We were given the state

$$|\alpha\rangle = \exp\left(\frac{-ipa}{\hbar}\right) |0\rangle$$

$$\langle x|\alpha\rangle = \pi^{-1/4} x_0^{1/2} \exp\left(\frac{-ipa}{\hbar}\right) \exp\left(-\frac{1}{2} \left(\frac{x}{x_0}\right)^2\right)$$

where $x_0 = \sqrt{\frac{\hbar}{m\omega}}$. The Hamiltonian operator \hat{H} is independent of time so we have the unitary time evolution operator

$$\mathcal{U}(t) = \exp\left(-\frac{i\hat{H}t}{\hbar}\right)$$

Assuming $|\alpha\rangle$ is expressed in the energy basis, this can be alternatively be written as the power series

$$\mathcal{U}(t) = \sum_{n=0}^{\infty} \frac{\hat{H}^n}{n!} \rightarrow \mathcal{U}(t) |\alpha\rangle = \sum_{n=0}^{\infty} \frac{\hat{H}^n}{n!} |\alpha\rangle$$

$$\sum_{n=0}^{\infty} \frac{\alpha^n}{n!} |\alpha\rangle = \sum_n \exp\left(\frac{-i\alpha_n t}{\hbar}\right) |\alpha_n\rangle$$

The probability that $|\alpha\rangle$ is measured to be in the state $|0\rangle$ is

$$\langle 0|\alpha\rangle \langle \alpha|0\rangle = \exp\left(\frac{-ipa}{\hbar}\right) \langle 0|0\rangle \exp\left(\frac{ipa}{\hbar}\right) \langle 0|0\rangle = 1$$

This probability does not change for $t > 0$. This is clear when we look at the state

$$|\alpha; t\rangle = \exp\left(-\frac{iE_0 t}{\hbar}\right) \exp\left(\frac{-ipa}{\hbar}\right) |0\rangle$$

The second exponential is just a complex number and is time independent. The first exponential is just a phase, which is not measurable directly. In other words, when we hit this state with the dual ket $\langle 0|$, the phase goes away and we are left with a time-independent probability density. ■

Problem 3. *Problem 2.16 from Sakurai*

Solution.

We will assume the form of the annihilation and creation operators

$$\begin{aligned} a &= \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{ip}{m\omega} \right) \\ a^\dagger &= \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{ip}{m\omega} \right) \end{aligned}$$

Adding these equations gives and rearranging we can express x as

$$\begin{aligned} x &= \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger) \\ \langle m | x | n \rangle &= \langle m | \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger) | n \rangle \\ &= \sqrt{\frac{\hbar}{2m\omega}} (\langle m | a | n \rangle + \langle m | a^\dagger | n \rangle) \\ &= \sqrt{\frac{\hbar}{2m\omega}} (\sqrt{n}\delta_{m,n-1} + \sqrt{n+1}\delta_{m,n+1}) \end{aligned}$$

Subtracting the creation operator from the annihilation operator allows us to write the momentum operator as

$$p = i\sqrt{\frac{m\hbar\omega}{2}} (a^\dagger - a)$$

$$\begin{aligned}
\langle m|p|n\rangle &= \langle m|\left(i\sqrt{\frac{m\hbar\omega}{2}}(a^\dagger - a)\right)|n\rangle \\
&= \left(i\sqrt{\frac{m\hbar\omega}{2}}(\langle m|a^\dagger|n\rangle - \langle m|a|n\rangle)\right) \\
&= i\sqrt{\frac{m\hbar\omega}{2}}(\sqrt{n+1}\delta_{m,n+1} - \sqrt{n}\delta_{m,n-1})
\end{aligned}$$

$$\begin{aligned}
\langle m|\{x,p\}|n\rangle &= \langle m|xp|n\rangle + \langle m|px|n\rangle \\
&= \frac{i\hbar}{2}\langle m|((a^\dagger)^2 - a^2)|n\rangle + \frac{i\hbar}{2}\langle m|((a^\dagger)^2 + a^\dagger a - aa^\dagger - a^2)|n\rangle \\
&= \frac{i\hbar}{2}(\sqrt{n+1}\sqrt{n+2}\delta_{m,n+2} - \sqrt{n}\sqrt{n-1}\delta_{m,n-2}) \\
&\quad + \frac{i\hbar}{2}(\sqrt{n+1}\sqrt{n+2}\delta_{m,n+2} + \sqrt{n}\sqrt{n-1}\delta_{m,n-2})
\end{aligned}$$

$$\langle m|x^2|n\rangle = \frac{\hbar}{2m\omega}\langle m|(a^2 + aa^\dagger + a^\dagger a + (a^\dagger)^2)|n\rangle$$

$$\langle m|p^2|n\rangle = -\frac{m\hbar\omega}{2}\langle m|((a^\dagger)^2 + a^\dagger a - aa^\dagger - a^2)|n\rangle$$

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Problem 4. *Problem 2.28 from Sakurai*

Solution.

First of all, the solution is not trivial since x does not commute with the Hamiltonian since $[x, p^2] \neq 0$. At $t = t_0$ we are in the position eigenstate

$$\langle x|\alpha; t_0\rangle = \delta\left(x - \frac{L}{2}\right)$$

Since this is the infinite square well, we have the following energy eigenstates, in the position representation

$$\langle x|\alpha\rangle = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

Of course $|\alpha; t_0\rangle$ is not an eigenstate of H , so this state will measurably evolve in time. The state $|\alpha; t_0\rangle$ in the energy basis is

$$\begin{aligned} |\beta\rangle &= \sum_n |\epsilon_n\rangle \langle \epsilon_n|\alpha; t_0\rangle \\ &= \sqrt{\frac{2}{L}} \sum_n \sin\left(\frac{n\pi}{2}\right) |\epsilon_n\rangle \end{aligned}$$

From this, we can show the probability of measuring the particle in energy eigenstate $|\epsilon_n\rangle$

$$\begin{aligned} \langle \epsilon_m|\beta\rangle &= \sqrt{\frac{2}{L}} \sum_n \sin\left(\frac{n\pi}{2}\right) \langle \epsilon_m|\epsilon_n\rangle \\ &= \sqrt{\frac{2}{L}} \sum_n \sin\left(\frac{n\pi}{2}\right) \delta_{mn} \\ &= \sqrt{\frac{2}{L}} \sin\left(\frac{m\pi}{2}\right) \end{aligned}$$

and therefore $|\langle \epsilon_m|\beta\rangle|^2 = \frac{2}{L} \sin^2\left(\frac{m\pi}{2}\right)$. The relative probabilities with respect to the ground state are then given by

$$r_{m+1} = \sin^2\left(\frac{(m+1)\pi}{2}\right) \csc^2\left(\frac{m\pi}{2}\right)$$

Since we know a representation for $|\alpha; t_0\rangle$ in the energy basis, we can determine the time evolution of the wavefunction $\langle x|\alpha\rangle$

$$\begin{aligned} |\alpha; t\rangle &= \mathcal{U}(t) |\beta\rangle \\ &= \sqrt{\frac{2}{L}} \sum_n \exp\left(\frac{-i\epsilon_n t}{\hbar}\right) \sin\left(\frac{n\pi}{2}\right) |\epsilon_n\rangle \end{aligned}$$

which has the position representation (wave function)

$$\begin{aligned}
\langle x|\alpha;t\rangle &= \psi(x,t) \\
&= \sqrt{\frac{2}{L}} \sum_n \exp\left(\frac{-i\epsilon_n t}{\hbar}\right) \sin\left(\frac{n\pi}{2}\right) \langle x|\epsilon_n\rangle \\
&= \sqrt{\frac{2}{L}} \sum_n \exp\left(\frac{-i\epsilon_n t}{\hbar}\right) \sin\left(\frac{n\pi}{2}\right) \psi_n(x)
\end{aligned}$$

where $\psi_n(x)$ are the energy eigenstates given above. ■

Problem 5. *Problem 2.29 from Sakurai*

Solution.

We are free to choose our zero of potential so we can solve an alternative system where $V(x) = \nu_0 - \delta(x)\nu_0$. We then decompose $\psi(x)$ into two regions $\psi_I(x), \psi_{II}(x)$ where

$$\begin{cases} I : x < 0, \\ II : x > 0, \end{cases}$$

We then solve Schrodingers equation in each region. Both regions have constant potential $|\nu_0|$

$$\begin{aligned}
\frac{d\psi_I^2}{dx^2} &= \frac{2m(E - \nu_0)}{\hbar^2} \psi_I(x) \\
\frac{d\psi_{II}^2}{dx^2} &= \frac{2m(E - \nu_0)}{\hbar^2} \psi_{II}(x)
\end{aligned}$$

$$\begin{aligned}
\psi_I(x) &= A \exp(\kappa x) + B \exp(-\kappa x) \\
\psi_{II}(x) &= C \exp(\kappa x) + D \exp(-\kappa x)
\end{aligned}$$

Taking limits shows that $B = 0$ and $C = 0$. From the symmetry of the potential we should have that $\psi'_I = -\psi'_{II}$ which is satisfied when $A = D$. This reduces the solution to

$$\begin{aligned}\psi_I(x) &= A \exp(\kappa x) \\ \psi_{II}(x) &= A \exp(-\kappa x)\end{aligned}$$

Looking back at Schrodinger's equation, we can see that when $E > \nu_0$ the solutions are complex exponentials and are unbound but when $E < \nu_0$, the solutions are real exponentials and bound. There are an infinite number of bound states since E and therefore κ are continuous parameters. When E hits ν_0 , we have the solutions for a free particle. ■

Problem 6. *Problem 2.32 from Sakurai*

Solution. Let us define

$$\begin{aligned}\psi_I &= A \exp(\alpha x) \\ \psi_{II} &= B \exp(ikx) + C \exp(-ikx) \\ \psi_{III} &= D \exp(-\alpha x)\end{aligned}$$

Here α, k are constants. We can enforce continuity in the wavefunction itself at $x = -a$ and $x = +a$

$$\begin{aligned}A \exp(-\alpha a) &= B \exp(-ika) + C \exp(ika) \\ D \exp(-\alpha a) &= B \exp(ika) + C \exp(-ika)\end{aligned}$$

And we can also enforce continuity in the first-order derivative at these points

$$\begin{aligned}\alpha A \exp(-\alpha a) &= ikB \exp(-ika) - ikC \exp(ika) \\ -\alpha D \exp(-\alpha a) &= ikB \exp(ika) - ikC \exp(-ika)\end{aligned}$$

Define $z = \exp(ika)(k - i\alpha)$ and we must have α, k such that $z = \pm z^*$ ■