The Fokker-Planck Equation

C.W. Seitz

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1 The Fokker-Planck Equation

1.1 Kramers-Moyal Expansion

Gixen many instantiations of a stochastic xariable x, we can construct a normalize histogram oxer all obserxations as a function of time $P(x,t)$. Howexer, in order to systematically explore the relationship between the parameterization of the process and $P(x,t)$ we require an expression for $\dot{P}(x,t)$. If we make a fundamental assumption that the exolution of $P(x,t)$ follows a Markox process i.e. its exolution has the memoryless property, then we can write

$$P(x',t) = \int T(x',t|x,t-\tau)P(x,t-\tau)dx \tag{1}$$

which is known at the Chapman-Kolmogorox equation. The factor $T(x',t|x,t-\tau)$ is known as the *transition operator* in a Markox process and determines the exolution of $P(x,t)$ in time. We proceed by writing $T(x',t|x,t-\tau)$ in a form referred to as the Kramers-Moyal expansion

$$\begin{aligned} T(x',t|x,t-\tau) &= \int \delta(u-x')T(u,t|x,t-\tau)du \\ &= \int \delta(x+u-x'-x)T(u,t|x,t-\tau)du \end{aligned}$$

If we use the Taylor expansion of the δ-function

$$\delta(x+u-x'-x) = \sum_{n=0}^{\infty} \frac{(u-x)^n}{n!}\left(-\frac{\partial}{\partial x}\right)^n \delta(x-x')$$

Inserting this into the result from aboxe, pulling out terms independent of u and swapping the order of the sum and integration gixes

$$T(x', t|x, t-\tau) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial x}\right)^n \delta(x-x') \int (u-x)^n T(u, t|x, t-\tau) du \quad (2)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial x}\right)^n \delta(x-x') M_n(x,t) \quad (3)$$

noticing that $M_n(x,t) = \int (u-x)^n T(u, t|x, t-\tau) du$ is just the nth moment of the transition operator T. Plugging (2.6) back in to (2.4) gixes

$$P(x,t) = \int \left(1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial x}\right)^n M_n(x,t)\right) \delta(x-x') P(x, t-\tau) dx \quad (4)$$

$$= P(x', t-\tau) + \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial x}\right)^n [M_n(x,t) P(x,t)] \quad (5)$$

Approximating the derixatixe as a finite difference and taking the limit $\tau \to 0$ gixes

$$\dot{P}(x,t) = \lim_{\tau \to 0} \left(\frac{P(x,t) - P(x, t-\tau)}{\tau}\right) \quad (6)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial x}\right)^n [M_n(x,t) P(x,t)] \quad (7)$$

which is formally known as the Kramers-Moyal (KM) expansion. The Fokker-Planck equation is a special case of (2.10) where we neglect terms $n > 2$ in the *diffusion approximation.*

Consider the following Ito stochastic differential equation

$$d\vec{x} = F(\vec{x}, t) + G(\vec{x}, t) dW$$

The SDE gixen aboxe corresponds to the Kramers-Moyal expansion (KME) of a transition density $T(x', t'|x, t)$ see (Risken 1989) for a full derixation.

$$\frac{\partial P(x,t)}{\partial t} = \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial x}\right)^n [M_n(x,t) P(x,t)] \quad (8)$$

where M_n is the nth moment of the transition density. In the diffusion approximation, the KME becomes the Fokker-Planck equation (FPE) (Risken 1989). For the sake of demonstration, consider the univariate case with random variable x and the form of $T(x', t'|x, t)$ is a Gaussian with mean $\mu(t)$ and variance $\sigma^2(t)$. In this scenario, the FPE applies because $M_n = 0$ for all $n > 2$. Given that the drift $M_1(x,t) = \mu(t)$ and the diffusion $M_2(x,t) = \sigma^2(t)$, the FPE reads

$$\frac{\partial P(x,t)}{\partial t} = \left(-\frac{\partial}{\partial x}M^{(1)}(t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}M^{(2)}(t)\right)P(x,t) \tag{9}$$

We can additionally define the term in parentheses as a differential operator acting on $P(x,t)$

$$\hat{\mathcal{L}}_{FP} = \left(-\frac{\partial}{\partial x}M^{(1)}(t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}M^{(2)}(t)\right) \tag{10}$$

It is common to additionally define the probability current $J(x,t)$ as

$$J(x,t) = \left(M^{(1)}(t) - \frac{1}{2}\frac{\partial}{\partial x}M^{(2)}(t)\right)P(x,t) \tag{11}$$

This definition provides some useful intuition. The value of $J(x,t)$ is the net probability flux into the interval between x and $x+dx$ at at time t. This also allows us to write the FPE as a continuity equation

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial x} \tag{12}$$

1.2 Solving the FPE: Heat (Diffusion) Equation

The well-known heat equation (it has several names: diffusion equation, heat equation, Brownian motion, Wiener process) is a special case of the FPE where $M^{(1)}(t) = 0$ and $M^{(2)}(t) = \sigma^2 = \text{const.}$

$$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)}{\partial x^2} \tag{13}$$

with $D = \sigma^2/2$. We would like to solve the above equation, but it is a PDE which usually require some tricks to solve e.g., integral transforms. Generally a transform can reduce a differential equation to a simpler form, like an ODE. Upon Fourier tranformation, spatial derivatives turn into factors of ik. That is,

$$\frac{\partial}{\partial x}\psi(x,t) \to ik\tilde{\psi}(k,t) \qquad \frac{\partial^2}{\partial x^2}\psi(x,t) \to -k^2\tilde{\psi}(k,t)$$

1.2.1 Fourier Transform of the Heat Equation

Recall the general definition of a Fourier pair

$$\tilde{\psi}(k) = \mathcal{F}[\psi] = \int_{-\infty}^{\infty} \psi(x) e^{-2\pi i k x} dx$$
$$\psi(x) = \mathcal{F}^{-1}[\tilde{\psi}] = \int_{-\infty}^{\infty} \tilde{\psi}(k) e^{2\pi i k x} dk$$

Let's see the Fourier transformation of Eq. (6)

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} P(x,t) e^{-2\pi i k x} dx = D \int_{-\infty}^{\infty} \frac{\partial^2 P(x,t)}{\partial x^2} e^{-2\pi i k x} dx \tag{14}$$

As mentioned above, $\mathcal{F}[\partial_x \psi] = ik\mathcal{F}[\psi]$ and $\mathcal{F}[\partial_x^2 \psi] = -k^2 \mathcal{F}[\psi]$ which allows us to write the heat equation as a first order equation

$$\frac{\partial \tilde{P}(k,t)}{\partial t} = -Dk^2 \tilde{P}(k,t) \tag{15}$$

which suggests the solution $\tilde{p}_0(k) \exp\left(-Dk^2 t\right)$, which is Gaussian in k-space. Let's say our initial condition satisfies $\tilde{P}(x,t_0) = \delta(x - x_0)$ which in the Fourier domain is $P(k,t_0) = \exp(-ikx_0)$. The inverse transform is

$$\int_{-\infty}^{\infty} \tilde{p}_0(k) \exp\left(ikx - Dk^2 t\right) dk = \int_{-\infty}^{\infty} \exp\left(ik(x - x_0) - Dk^2 t\right) dk \tag{16}$$

which we can rewrite as

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\left(Dk^2 t - ik(x - x_0)\right)\right) dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-Dt\left(k^2 - \frac{ik(x-x_0)}{Dt}\right)\right) dk$$

Now we would like to complete the square in the exponential, since we know how to do Gaussian integrals. This can be done as follows:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-Dt\left(k^2 - \frac{ik(x-x_0)}{Dt} + \frac{(x-x_0)^2}{4D^2t^2} - \frac{(x-x_0)^2}{4D^2t^2}\right)\right) dk$$

We are then left to simplify,

$$\begin{aligned}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-Dt\left(k - \frac{i(x-x_0)}{2Dt}\right)^2\right) dk &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-x_0)^2}{4Dt}\right) \int_{-\infty}^{\infty} \exp\left(-Dtk'^2\right) dk' \\ &= \frac{1}{\sqrt{2Dt}} \exp\left(-\frac{(x-x_0)^2}{4Dt}\right)\end{aligned}$$

which is a Gaussian distribution will time-dependent variance $\sigma = 4Dt$, given originally by Einstein in his famous paper on Brownian motion in 1905.

1.3 Solving the FPE: Ornstein-Uhlenbeck

The Ornstein-Uhlenbeck process is another special case of the FPE where $M^{(1)}(t) = -\gamma$ and $M^{(2)}(t) = \sigma^2 = \text{const}$. It is a stationary Gauss–Markov process, which means that it is a Gaussian process, a Markov process, and is temporally homogeneous. The Ito SDE for this process reads

$$dx = -\gamma x dt + \sigma dW \tag{17}$$

which of course has a corresponding Fokker-Planck equation

$$\frac{\partial P(x,t)}{\partial t} = -\gamma \frac{\partial}{\partial x} x P(x,t) + D \frac{\partial^2 P(x,t)}{\partial x^2} \tag{18}$$

In this form, the solution is slightly complicated by the presence of the first order spatial derivative. However, we can still find a solution via a Fourier transform:

$$\frac{\partial \tilde{P}(k,t)}{\partial t} = -\gamma k \frac{\partial \tilde{P}(k,t)}{\partial k} - k^2 D \tilde{P}(k,t) \tag{19}$$

Notice that this is a partial differential equation with the general form

$$a(\tilde{P},k,t)\partial_k \tilde{P} + b(\tilde{P},k,t)\partial_t \tilde{P} - c(\tilde{P},k,t) = 0 \tag{20}$$

Therefore can solve the above equation using the method of characteristics. As a brief review, suppose we know a solution surface $\tilde{P}$. A vector normal to this surface has the form $\vec{u} = \langle \partial_k \tilde{P}, \partial_t \tilde{P}, -1 \rangle$. If this vector is normal to the surface, then the vector field

$$\vec{v} = \langle a(\tilde{P},k,t), b(\tilde{P},k,t), c(\tilde{P},k,t) \rangle \tag{21}$$

is tangent to the surface at every point. In other words, we would like to find a surface $\tilde{P}(k,t)$ for which the vector field above lies in the tangent plane to $\tilde{P}(k,t)$ and therefore $\vec{u} \cdot \vec{v} = 0$. The task that remains then is to find a $\tilde{P}(k,t)$ s.t. the vector $\vec{u}$ is orthogonal to $\vec{v}$. Now, if we construct a curve $\mathcal{C}$ which is an integral curve of $\vec{v}$, then this curve lies on the solution surface $\tilde{P}(k,t)$. Such a curve satisfies the ODEs

$$\begin{aligned}
\frac{dk}{ds} &= \gamma k \\
\frac{dt}{ds} &= 1 \\
\frac{d\tilde{P}}{ds} &= -k^2 D \tilde{P}
\end{aligned}$$

since the vector field given by the Fokker-Planck equation we have is $\vec{v} = \langle \gamma k, 1, -k^2 D \rangle$. Clearly $t = s$ and $k = k_0 \exp(\gamma t)$ and thus

$$\frac{d\tilde{P}}{dt} = -k^2 D \tilde{P} \tag{22}$$
$$= -D k_0^2 \exp(2\gamma t) \tilde{P} \tag{23}$$

and we have the solution in the Fourier domain

$$\tilde{P}(k,t) = \tilde{P}(k,0) \exp\left(-\frac{D k_0^2}{2\gamma}(\exp(2\gamma t) - 1)\right) \tag{24}$$
$$= \exp\left(-i k_0 x_0 - \frac{D k_0^2}{2\gamma}(\exp(2\gamma t) - 1)\right) \tag{25}$$
$$= \exp\left(-i k e^{-\gamma t} x_0 - \frac{D k^2}{2\gamma}(1 - \exp(-2\gamma t))\right) \tag{26}$$

Let $\mu(t) = x_0 \exp(-\gamma t)$ and $\sigma^2(t) = \frac{D}{\gamma}(1 - e^{-2\gamma t})$

$$\tilde{P}(k,t) = \exp\left(-ik\mu(t) - \frac{k^2}{2}\sigma^2(t)\right) \tag{27}$$

Taking the inverse Fourier transform of this equation gives

$$P(x,t) = \frac{1}{\sqrt{2\sigma^2(t)}} \exp\left(-\frac{(x - \mu(t))^2}{2\sigma^2(t)}\right) \tag{28}$$

1.4 The Multivariate Case

If we now generalize the above equation to a case where we are faced with many variables $\boldsymbol{x} = (x_1, x_2, ..., x_n)$. The continuity equation becomes

$$\frac{\partial P(\vec{x},t)}{\partial t} = -\vec{\nabla} \cdot J(\vec{x},t) \tag{29}$$

where the multivariate probability current now has the interpretation of the net flux into or out of a volume dx^n centered around $\boldsymbol{x}$. If we consider each dimension,

$$J(x_i,t) = \left(M_i^{(1)}(t) - \sum_j \frac{\partial}{\partial x_j} M_{ij}^{(2)}(t) \right) P(\vec{x},t) \tag{30}$$

The full Fokker-Planck equation then reads

$$\frac{\partial P(\vec{x},t)}{\partial t} = \vec{\nabla} \cdot J(\vec{x},t) \tag{31}$$

$$= \sum_{i=1}^{N} \left(-\frac{\partial}{\partial x_i} M_i^{(1)}(t) + \sum_{j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} M_{ij}^{(2)}(t) \right) P(\vec{x},t) \tag{32}$$

It proves quite useful in this form because we can see that the Fokker-Planck equation represents a differentiation operator acting on the distribution $P(\vec{x},t)$

$$\hat{\mathcal{L}}_{FP} = \sum_{i=1}^{N} \left(-\frac{\partial}{\partial x_i} M_i^{(1)}(t) + \sum_{j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j} M_{ij}^{(2)}(t) \right) \tag{33}$$

1.5 Ornstein-Uhlenbeck Process

If the transition density is Gaussian then the density is fully specified by the first two moments $M^{(1)}(t) = \vec{\mu}(t)$ and $M^{(2)}(\vec{x},t) = \Sigma(t)$. The moments can also be functions of $\vec{x}$. Both of these possibilities are evident in the Ornstein-Uhlenbeck (OU) process. Let the drift vector be a linear function of the state $\vec{x}$ and the diffusion matrix the square of the Gaussian covariances

$$M^{(1)}(t) = \Gamma\vec{x} \qquad M^{(2)}(t) = 2D$$

with $D = \Sigma\Sigma^T$ and Γ are assumed to be independent of time (non-volatile).

The Fourier transform of (34) is fairly simple, since the FT is linear. We will switch to matrix notation to drop the summations

$$\frac{\partial \tilde{P}(\vec{x},t)}{\partial t} = -\mathbf{\Gamma}\boldsymbol{k}\frac{\partial}{\partial \boldsymbol{k}}\tilde{P}(\vec{k},t) + \mathbf{D}\boldsymbol{k}\boldsymbol{k}^T\tilde{P}(\vec{k},t) \tag{34}$$

This is in a very similar form to the single variable case (18). We then make the ansatz in analogy with (27)

$$\tilde{P}(\boldsymbol{k},t) = \exp\left(-i\boldsymbol{k}^T\boldsymbol{\mu}(t) - \frac{\boldsymbol{k}\boldsymbol{k}^T}{2}\mathbf{\Sigma}(t)\right) \tag{35}$$

Plugging this into (34) we have

$$\frac{\partial \tilde{P}(\vec{x},t)}{\partial t} + \mathbf{\Gamma}\boldsymbol{k}\frac{\partial}{\partial \boldsymbol{k}}\tilde{P}(\vec{k},t) + \mathbf{D}\boldsymbol{k}\boldsymbol{k}^T\tilde{P}(\vec{k},t) \tag{36}$$

$$= -i\boldsymbol{k}\dot{\boldsymbol{\mu}}(t) - \frac{1}{2}\boldsymbol{k}\boldsymbol{k}^T\dot{\mathbf{\Sigma}} - i\mathbf{\Gamma}\boldsymbol{\mu}^T - \mathbf{\Gamma} \tag{37}$$