

Homework 6

Quantum Mechanics

October 28th, 2022

C SEITZ

Problem 1. *Problem 3.12 from Sakurai*

Solution.

In general the ensemble average of an operator $[A]$ is defined as

$$[A] = \sum_i w_i \langle \alpha_i | A | \alpha_i \rangle$$

where $\sum_i w_i = 1$

$$\begin{aligned} [\sigma_x] &= a \langle + | \sigma_x | + \rangle + (1 - a) \langle -; y | \sigma_x | -; y \rangle \\ &= a \langle + | (| + \rangle \langle - | + | - \rangle \langle + |) | + \rangle + (1 - a) \langle -; y | (| + \rangle \langle - | + | - \rangle \langle + |) | -; y \rangle \\ &= 0 \end{aligned}$$

$$\begin{aligned} [\sigma_y] &= a \langle + | \sigma_y | + \rangle + (1 - a) \langle -; y | \sigma_y | -; y \rangle \\ &= ai \langle + | (| + \rangle \langle - | - | - \rangle \langle + |) | + \rangle + i(1 - a) \langle -; y | (| + \rangle \langle - | - | - \rangle \langle + |) | -; y \rangle \\ &= i(1 - a) \langle -; y | \left(-\frac{i}{\sqrt{2}} | + \rangle - \frac{1}{\sqrt{2}} | - \rangle \right) \\ &= -i(1 - a) \langle -; y | +; y \rangle = 0 \end{aligned}$$

$$\begin{aligned} [\sigma_z] &= a \langle + | (| - \rangle \langle - | - | + \rangle \langle + |) | + \rangle + i(1 - a) \langle -; y | (| - \rangle \langle - | - | + \rangle \langle + |) | -; y \rangle \\ &= -a + i(1 - a) \langle -; y | \left(-\frac{i}{\sqrt{2}} | + \rangle - \frac{1}{\sqrt{2}} | - \rangle \right) \end{aligned}$$

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Problem 2. *Problem 3.13 from Sakurai*

Solution.

The state vector in the S_z basis has the form

$$|\alpha\rangle = c_+ |+\rangle + c_- |-\rangle$$

First note that

$$\langle S_z \rangle = |c_+|^2 - |c_-|^2 \quad |c_+|^2 + |c_-|^2 = 1$$

Together, these equations tell us the magnitude of each complex component.

$$|c_+|^2 = \frac{\langle S_z \rangle + 1}{2} \quad |c_-|^2 = \frac{1 - \langle S_z \rangle}{2}$$

$$\begin{aligned} \langle S_x \rangle &= \langle \alpha | (|+\rangle \langle -| + |-\rangle \langle +|) (c_+ |+\rangle + c_- |-\rangle) \\ &= \langle \alpha | (c_- |+\rangle + c_+ |-\rangle) \\ &= (c_+^* \langle +| + c_-^* \langle -|) (c_- |+\rangle + c_+ |-\rangle) \\ &= c_+^* c_- + c_-^* c_+ \\ &= |c_+| |c_-| (e^{i(\theta-\phi)} + e^{i(\phi-\theta)}) \\ &= 2|c_+| |c_-| \cos(\theta - \phi) \end{aligned}$$

$$\begin{aligned} \langle S_y \rangle &= \langle \alpha | ((i |+\rangle \langle -| - i |-\rangle \langle +|) (c_+ |+\rangle + c_- |-\rangle) \\ &= i \langle \alpha | (c_- |+\rangle - c_+ |-\rangle) \\ &= i(c_+^* \langle +| + c_-^* \langle -|) (c_- |+\rangle - c_+ |-\rangle) \\ &= c_+^* c_- - c_-^* c_+ \\ &= |c_+| |c_-| (e^{i(\theta-\phi)} - e^{i(\phi-\theta)}) \\ &= 2i|c_+| |c_-| \sin(\theta - \phi) \end{aligned}$$

So $\langle S_x \rangle$ gives us the phase difference of c_+ and c_- . Then the sign of $\langle S_y \rangle$ tells us whether θ or ϕ is larger, since sine is odd. This is all we can hope to extract from the expectation values, since multiplying by a global phase $e^{i\delta} |\alpha\rangle$ has no effect on the expectation values.

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Problem 3. *Problem 3.14 from Sakurai*

Solution.

$$\begin{aligned}\rho &= \sum_i w_i |\psi_i\rangle \langle \psi_i| \\ &= \frac{1}{3} (|\alpha\rangle \langle \alpha| + |\beta\rangle \langle \beta| + |2\rangle \langle 2|)\end{aligned}$$

We can write this out explicitly in the subspace spanned by $|0, 1, 2\rangle$

$$|\alpha\rangle \langle \alpha| = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad |\beta\rangle \langle \beta| = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad |2\rangle \langle 2| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rho = \frac{1}{6} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Now recall that $H = \hbar\omega(N + \frac{1}{2})$ which is

$$H = \left(\frac{\hbar\omega}{2}\right) \mathbb{I}_{3 \times 3} + \frac{\hbar\omega}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$[H] = \text{Tr}(\rho H) = \frac{\hbar\omega}{2} \text{Tr}(\rho N + \rho) = \frac{\hbar\omega}{2} (\text{Tr}(\rho N) + \text{Tr}(\rho))$$

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Problem 4. *Problem 3.15 from Sakurai*

Solution.

$$\rho(t_0) = \sum_i w_i |\psi_i; t_0\rangle \langle \psi_i; t_0|$$

In the Schrodinger picture, the coefficients of the state vectors evolve. Therefore,

$$\begin{aligned}
\rho(t) &= \sum_i w_i \mathcal{U}(t, t_0) |\psi_i; t_0\rangle \langle \psi_i; t_0| \mathcal{U}^\dagger(t, t_0) \\
&= \mathcal{U}(t, t_0) \left(\sum_i w_i |\psi_i; t_0\rangle \langle \psi_i; t_0| \right) \mathcal{U}^\dagger(t, t_0) \\
&= \mathcal{U}(t, t_0) \rho(t_0) \mathcal{U}^\dagger(t, t_0)
\end{aligned}$$

For a pure ensemble in state $|\psi_i\rangle$, the density matrix is

$$\rho(t_0) = |\psi_i; t_0\rangle \langle \psi_i; t_0|$$

At a later time, the density matrix is

$$\begin{aligned}
\rho(t) &= \mathcal{U}(t, t_0) \rho(t_0) \mathcal{U}^\dagger(t, t_0) \\
&= \mathcal{U}(t, t_0) |\psi_i; t_0\rangle \langle \psi_i; t_0| \mathcal{U}^\dagger(t, t_0) \\
&= |\psi_i; t_0\rangle \langle \psi_i; t_0|
\end{aligned}$$

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Problem 5. *Problem 3.16 from Sakurai*

Solution. A 3x3 matrix has 9 parameters, but we only need

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Problem 6. *Problem 3.40 from Sakurai*

Solution. The singlet state is

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)$$

The probability is 1/2 to obtain $s_{1z} = \hbar/2$. The probability is 1/2 to obtain $s_{1x} = \hbar/2$ since obtaining $s_{1x} = \hbar/2$ is equiprobable for the two states in the singlet superposition. If observer B has determined that $s_{2z} = \hbar/2$, then observer A must observe $s_{1z} = -\hbar/2$ since the measurement made by B collapses $|\psi\rangle$ to $|-+\rangle$. Furthermore, if observer B has measured $s_{2z} = \hbar/2$, then particle 1 must be in the $|+\rangle$ state which means $s_{1x} = \pm\hbar/2$ with equal probability.

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