

# Exam 2

Quantum Mechanics

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## Problem 1.

### Solution.

Some of the states have the same energy, so we will need to use degenerate perturbation theory. Specifically, the subspaces spanned by  $\mathcal{A} = \{|0^{(0)}\rangle, |1^{(0)}\rangle\}$  and  $\mathcal{B} = \{|2^{(0)}\rangle, |4^{(0)}\rangle\}$  have a degeneracy while the lone ket  $|3^{(0)}\rangle$  is nondegenerate. We assume that a perturbed ket  $\alpha \in \mathcal{A}$  can be written as a linear combination of the unperturbed kets:

$$|\alpha\rangle = \sum_{n \in \mathcal{A}} \langle n | \alpha \rangle |n\rangle$$

The first order correction is given by

$$V |\alpha\rangle = \sum_{n \in \mathcal{A}} \langle n | \alpha \rangle (H - H_0) |n\rangle = \Delta_\alpha^{(1)} |\alpha\rangle$$

We therefore need to find the eigenvectors and eigenvalues of the matrix

$$|V_{\mathcal{A}} - \Delta_\alpha I| = \det \begin{pmatrix} 2 \cos \theta - \Delta_\alpha & 2 \sin \theta e^{-i\phi} \\ 2 \sin \theta e^{i\phi} & -2 \cos \theta - \Delta_\alpha \end{pmatrix} = 0$$

which is easy to solve, and we get the first order shifts  $\Delta_\alpha^{(1)} = \pm 2$ . It is the same process for the  $\mathcal{B}$  subspace

$$|V_{\mathcal{B}} - \Delta_\beta I| = \det \begin{pmatrix} 4 \cos \theta - \Delta_\beta & 4 \sin \theta e^{-i\phi} \\ 4 \sin \theta e^{i\phi} & -4 \cos \theta - \Delta_\beta \end{pmatrix} = 0$$

It is pretty much the same matrix, so  $\Delta_\beta^{(1)} = \pm 4$ . For the first order correction to the nondegenerate ket  $|3^{(0)}\rangle$ , we use nondegenerate perturbation theory to first order

$$\begin{aligned}\Delta_3^{(1)} &= \lambda V_{33} + \lambda^2 \sum_{j \neq 3} \frac{|V_{j3}|^2}{E_3^{(0)} - E_j^{(0)}} \\ &= \lambda + \lambda^2 \left( -\frac{3}{\epsilon} - \frac{3}{\epsilon} \right) \\ &= \lambda - \frac{6\lambda^2}{\epsilon} \approx \lambda\end{aligned}$$

To get the corrections to the ground state eigenvector, we can again use nondegenerate perturbation theory

$$\begin{aligned}|3^{(1)}\rangle &= \lambda |3^{(0)}\rangle + \lambda^2 \sum_{j \neq 3} |j^{(0)}\rangle \frac{V_{j3}}{E_3^{(0)} - E_j^{(0)}} \\ &= \lambda |3^{(0)}\rangle + \frac{\lambda^2}{\epsilon} \left( |2^{(0)}\rangle + |4^{(0)}\rangle \right)\end{aligned}$$

In the limit  $\lambda \rightarrow 0$ , the perturbed eigenvectors are the “good” linear combinations. To find them we need to find the eigenvectors of the submatrices  $V_{\mathcal{A}}$  and  $V_{\mathcal{B}}$ , which are both just multiples of the  $\hat{S}_n$  operator. We know those eigenvectors already

$$|0^{(1)}\rangle = \cos \frac{\theta}{2} |0^{(0)}\rangle + \sin \frac{\theta}{2} e^{i\phi} |1^{(0)}\rangle$$

$$|1^{(1)}\rangle = \sin \frac{\theta}{2} |0^{(0)}\rangle - \cos \frac{\theta}{2} e^{i\phi} |1^{(0)}\rangle$$

$$|2^{(1)}\rangle = \cos \frac{\theta}{2} |2^{(0)}\rangle + \sin \frac{\theta}{2} e^{i\phi} |4^{(0)}\rangle$$

$$|4^{(1)}\rangle = \sin \frac{\theta}{2} |2^{(0)}\rangle - \cos \frac{\theta}{2} e^{i\phi} |4^{(0)}\rangle$$

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**Problem 2.**

$$|\alpha_{\pm}\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle)$$

$$|\beta_{\pm}\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm i|1\rangle)$$

**Solution.** We are after the ensemble expectation values  $[x^n]$  and  $[p^n]$ . In general,

$$[A] = \sum_n w_n \langle \alpha_n | A | \alpha_n \rangle$$

First, we should show

$$\begin{aligned} (a + a^\dagger)^n |0\rangle &= \sum_{k=0}^n \binom{n}{k} a^k (a^\dagger)^{n-k} |0\rangle \\ &= \sum_{k=0}^n \binom{n}{k} a^k \sqrt{(n-k)!} |n-k\rangle \\ &= \sum_{k=0}^n \binom{n}{k} \frac{(n-k)!}{\sqrt{(n-2k)!}} |n-2k\rangle \end{aligned}$$

$$\begin{aligned} (a + a^\dagger)^n |1\rangle &= \sum_{k=0}^n \binom{n}{k} a^k (a^\dagger)^{n-k} |1\rangle \\ &= \sum_{k=0}^n \binom{n}{k} a^k \sqrt{(n-k+1)!} |n-k+1\rangle \\ &= \sum_{k=0}^n \binom{n}{k} \frac{(n-k+1)!}{\sqrt{(n-2k+1)!}} |n-2k+1\rangle \end{aligned}$$

For  $(a - a^\dagger)^n$  we just replace  $(a^\dagger)^{n-k} \rightarrow (-1)^{n-k} (a^\dagger)^{n-k}$  in the sum.

$$(a + a^\dagger)^n |\alpha_+\rangle = \sum_{k=0}^n \binom{n}{k} \left( \frac{(n-k)!}{\sqrt{(n-2k)!}} |n-2k\rangle + \frac{(n-k+1)!}{\sqrt{(n-2k+1)!}} |n-2k+1\rangle \right)$$

Hitting this with a bra  $\langle 0|$  will select terms of the sum that satisfy  $k = n/2$  (which only occurs for even  $n$ ) or  $k = (n + 1)/2$  (which occurs for odd  $n$ ). On the other hand, hitting this with a bra  $\langle 1|$  will select terms of the sum that satisfy  $k = (n - 1)/2$  (odd  $n$ ) or  $k = n/2$  (even  $n$ ).

First, consider even  $n$ ,

$$\begin{aligned}\langle 0| (a + a^\dagger)^n |\alpha_+\rangle &= \binom{n}{n/2} \left(\frac{n}{2}\right)! \\ \langle 1| (a + a^\dagger)^n |\alpha_+\rangle &= \binom{n}{n/2} \left(\frac{n}{2}\right)!\end{aligned}$$

For odd  $n$ ,

$$\begin{aligned}\langle 0| (a + a^\dagger)^n |\alpha_+\rangle &= \binom{n}{(n+1)/2} \left(\frac{(n-1)}{2}\right)! \\ \langle 1| (a + a^\dagger)^n |\alpha_+\rangle &= \binom{n}{(n+1)/2} \left(\frac{(n+3)}{2}\right)!\end{aligned}$$

I will refer to these implicitly from here on.

$$\begin{aligned}[x^n] &= \langle \alpha_+ | x^n | \alpha_+ \rangle \\ &= \frac{1}{2} \left( \frac{\hbar}{2m\omega} \right)^{\frac{n}{2}} \left( \langle 0| (a + a^\dagger)^n |\alpha_+\rangle + \langle 1| (a + a^\dagger)^n |\alpha_+\rangle \right)\end{aligned}$$

For the  $|\beta_+\rangle$  ensemble, the only difference is the factor of  $i$  in front of the  $|1\rangle$  ket, but the result is the same as the  $|\alpha_+\rangle$  ensemble

$$\begin{aligned}[x^n] &= \langle \beta_+ | x^n | \beta_+ \rangle \\ &= \frac{1}{2} \left( \frac{\hbar}{2m\omega} \right)^{\frac{n}{2}} \left( \langle 0| (a + a^\dagger)^n |\beta_+\rangle + \langle 1| (a + a^\dagger)^n |\beta_+\rangle \right)\end{aligned}$$

For the mixture of  $|\alpha_+\rangle$  and  $|\alpha_-\rangle$

$$[x^n] = \frac{1}{2} (\langle \alpha_+ | x^n | \alpha_+ \rangle + \langle \alpha_- | x^n | \alpha_- \rangle)$$

To find the second term we need the odd and even cases again,

$$\begin{aligned}\langle 0 | (a + a^\dagger)^n | \alpha_- \rangle &= \binom{n}{n/2} \left(\frac{n}{2}\right)! \\ \langle 1 | (a + a^\dagger)^n | \alpha_- \rangle &= \binom{n}{n/2} \left(\frac{n}{2}\right)!\end{aligned}$$

For odd  $n$ ,

$$\begin{aligned}\langle 0 | (a + a^\dagger)^n | \alpha_- \rangle &= - \binom{n}{(n+1)/2} \left(\frac{(n-1)}{2}\right)! \\ \langle 1 | (a + a^\dagger)^n | \alpha_- \rangle &= - \binom{n}{(n+1)/2} \left(\frac{(n+3)}{2}\right)!\end{aligned}$$

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### Problem 3.

**Solution.** In this two particle system, we have  $j_1 = 1$  and  $j_2 = 2$ . We are told the  $z$  component of the individual angular momenta  $m_1 = -1$  and  $m_2 = 2$ . So we use the state kets  $|j_1, j_2; m_1, m_2\rangle$  where  $m_1$  and  $m_2$  follow the usual rules. So the state is  $|1, 2; -1, 2\rangle$ .

$$\begin{aligned}\langle J^2 \rangle &= \langle 1, 2; -1, 2 | J^2 | 1, 2; -1, 2 \rangle \\ &= \langle 1, 2; -1, 2 | (J_1^2 + J_2^2 + 2J_{1z}J_{2z}) | 1, 2; -1, 2 \rangle \\ &= j_1(j_1 + 1)\hbar^2 + j_2(j_2 + 1)\hbar^2 - 4\hbar^2 \\ &= 2\hbar^2 + 6\hbar^2 - 4\hbar^2 \\ &= 4\hbar^2\end{aligned}$$

We cannot directly compute the expectation values of  $J_x, J_y, J_z$  in this basis, because  $|j_1, j_2, m_1, m_2\rangle$  are not eigenkets of  $J_x, J_y, J_z$ . But can change basis:

$$|j_1, j_2, m_1, m_2\rangle = \sum_{m_1, m_2} |j_1, j_2; jm\rangle \langle j_1, j_2; jm | j_1, j_2; m_1, m_2 \rangle$$

We need to determine  $\langle j_1, j_2; jm | j_1, j_2; m_1, m_2 \rangle$  which are the Clebsch-Gordon coefficients.

$m = 1$			
$m_1, m_2 \backslash j$	<b>3</b>	<b>2</b>	<b>1</b>
<b>2, -1</b>	$\sqrt{\frac{1}{15}}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{3}{5}}$
<b>1, 0</b>	$\sqrt{\frac{8}{15}}$	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{3}{10}}$
<b>0, 1</b>	$\sqrt{\frac{2}{5}}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{10}}$

Figure 1: Clebsch-Gordon coefficients for  $j_1 = 1$ ,  $j_2 = 2$ ,  $m = 1$

Noting that  $m = m_1 + m_2 = 1$ , we find

$$|1, 2; -1, 2\rangle = \sqrt{\frac{3}{5}} |1, 2; 11\rangle + \sqrt{\frac{1}{3}} |1, 2; 21\rangle + \sqrt{\frac{1}{15}} |1, 2; 31\rangle$$

The expectation value  $\langle J_z \rangle$  is then

$$\begin{aligned}
\langle J_z \rangle &= \langle 1, 2; -1, 2 | J_z | 1, 2; -1, 2 \rangle \\
&= \left( \sqrt{\frac{3}{5}} \langle 1, 2; 11 | + \sqrt{\frac{1}{3}} \langle 1, 2; 21 | + \sqrt{\frac{1}{15}} \langle 1, 2; 31 | \right) J_z \\
&\quad \left( \sqrt{\frac{3}{5}} |1, 2; 11\rangle + \sqrt{\frac{1}{3}} |1, 2; 21\rangle + \sqrt{\frac{1}{15}} |1, 2; 31\rangle \right) \\
&= \hbar \left( \frac{3}{5} + \frac{1}{3} + \frac{1}{15} \right) = \hbar
\end{aligned}$$

as we should expect. For  $J_x, J_y$ ,

$$\begin{aligned}
\langle J_x \rangle &= \langle 1, 2; -1, 2 | J_x | 1, 2; -1, 2 \rangle \\
&= \left( \sqrt{\frac{3}{5}} \langle 1, 2; 11 | + \sqrt{\frac{1}{3}} \langle 1, 2; 21 | + \sqrt{\frac{1}{15}} \langle 1, 2; 31 | \right) \\
&\quad \frac{1}{2} (J_+ + J_-) \left( \sqrt{\frac{3}{5}} |1, 2; 11\rangle + \sqrt{\frac{1}{3}} |1, 2; 21\rangle + \sqrt{\frac{1}{15}} |1, 2; 31\rangle \right) \\
&= 0
\end{aligned}$$

and  $\langle J_y \rangle = 0$  since neither  $J_+$  nor  $J_-$  connects two  $|j_1, j_2; jm\rangle$  states.

Now, if we measure the total angular momentum and obtain the largest possible value, then we are in the state  $|1, 2; 31\rangle$  in the  $|j_1, j_2; jm\rangle$  basis. However, to compute  $J_{1z}$  and  $J_{2z}$  we need to transform this back to the  $|j_1, j_2, m_1, m_2\rangle$  basis. Looking up the coefficients, we get

$$|1, 2; 31\rangle = \left( \sqrt{\frac{1}{15}} |1, 2; -12\rangle + \sqrt{\frac{2}{5}} |1, 2; 10\rangle + \sqrt{\frac{8}{15}} |1, 2; 01\rangle \right)$$

$$\begin{aligned} \langle J_{1z} \rangle &= \langle 1, 2; 31 | J_{1z} | 1, 2; 31 \rangle \\ &= \left( \sqrt{\frac{1}{15}} \langle 1, 2; -12 | + \sqrt{\frac{2}{5}} \langle 1, 2; 10 | + \sqrt{\frac{8}{15}} \langle 1, 2; 01 | \right) \\ &\quad J_{1z} \left( \sqrt{\frac{1}{15}} |1, 2; -12\rangle + \sqrt{\frac{2}{5}} |1, 2; 10\rangle + \sqrt{\frac{8}{15}} |1, 2; 01\rangle \right) \\ &= -\hbar \frac{1}{15} + \hbar \frac{2}{5} = \frac{\hbar}{3} \end{aligned}$$

$$\begin{aligned} \langle J_{2z} \rangle &= \langle 1, 2; 31 | J_{2z} | 1, 2; 31 \rangle \\ &= \left( \sqrt{\frac{1}{15}} \langle 1, 2; -12 | + \sqrt{\frac{2}{5}} \langle 1, 2; 10 | + \sqrt{\frac{8}{15}} \langle 1, 2; 01 | \right) \\ &\quad J_{2z} \left( \sqrt{\frac{1}{15}} |1, 2; -12\rangle + \sqrt{\frac{2}{5}} |1, 2; 10\rangle + \sqrt{\frac{8}{15}} |1, 2; 01\rangle \right) \\ &= 2\hbar \frac{1}{15} + \hbar \frac{8}{15} = \frac{2\hbar}{3} \end{aligned}$$

The probability that  $J_{1z}$  and  $J_{2z}$  never change from their original values is given by

$$|\langle 1, 2; -12 | 1, 2; 31 \rangle|^2 = 1/15$$

If we instead measure the smallest possible value, we are in state  $|1, 2; 11\rangle$ . The third particle being added has  $j_3 = 1$  and  $m_3 = -1$ . We can consider the first two particles as a single composite particle in state  $|jm\rangle = |11\rangle$ . We

$m = 0$			
$m_1, m_2 \backslash j$	2	1	0
1, -1	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$
0, 0	$\sqrt{\frac{2}{3}}$	0	$-\sqrt{\frac{1}{3}}$
-1, 1	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$

Figure 2: Clebsch-Gordon coefficients for  $j_1 = 1$ ,  $j_2 = 1$  and  $m = 0$

now have two particles with  $j = 1$ . Taking  $m_1 = 1$  and  $m_2 = -1$  and reading off the table above, the probabilities are

$$\mathbf{Pr}(j) = \begin{cases} \frac{1}{3}, j = 0 \\ \frac{1}{2}, j = 1 \\ \frac{1}{6}, j = 2 \end{cases}$$

Finally, the expectation value of  $J^2$  for this three particle system is

$$\begin{aligned}
\langle J^2 \rangle &= \langle 1, 1; 1, -1 | J^2 | 1, 1; 1, -1 \rangle \\
&= \langle 1, 1; 1, -1 | (J_1^2 + J_2^2 + 2J_{1z}J_{2z}) | 1, 1; 1, -1 \rangle \\
&= j_1(j_1 + 1)\hbar^2 + j_2(j_2 + 1)\hbar^2 - 2\hbar^2 \\
&= 2\hbar^2
\end{aligned}$$

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