TTIC 31230, Fundamentals of Deep Learning

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Some Information Theory

Why Information Theory?

The fundamental equation involves cross-entropy.

Cross-entropy is an information-theoretic concept.

Information theory arises in many places and many forms in deep learning.

Entropy of a Distribution

The entropy of a distribution P is defined by

$$H(P) = E_{y \sim \mathrm{Pop}} \; -\ln P(y) \;\; \text{in units of “nats”}$$

$$H_2(P) = E_{y \sim \mathrm{Pop}} \; -\log_2 P(y) \text{ in units of bits}$$

Why Bits?

Why is $-\log_2\ P(y)$ a number of bits?

Example: Let P be a uniform distribution on 256 values.

$$E_{y\sim P}\ -\log_2 P(y) = -\log_2 \frac{1}{256} = \log_2 256 = 8 \text{ bits} = 1 \text{ byte}$$

$$1 \text{ nat} = \frac{1}{\ln 2} \text{ bits} \approx 1.44 \text{ bits}$$

Shannon's Source Coding Theorem

Why is $-\log_2 P(y)$ a number of bits?

A prefix-free code for $\mathcal{Y}$ assigns a bit string $c(y)$ to each $y \in \mathcal{Y}$ such that no code string is prefix of any other code string.

For a probability distribution P on $\mathcal{Y}$ we consider the average code length $E_{y \sim P} \, |c(y)|$.

Theorem: For any c we have $E_{y \sim P} \, |c(y)| \geq H_2(P)$.

Theorem: There exists c with $E_{y \sim P} \, |c(y)| \leq H_2(P) + 1$.

Cross Entropy

Let P and Q be two distribution on the same set.

$$H(P,Q) = E_{y\sim P}\;\; -\ln\; Q(y)$$

$$\Phi^* = \operatorname*{argmin}_{\Phi}\;\; H(\mathrm{Pop}, P_\Phi)$$

H(P,Q) also has a data compression interpretation.

$H(P,Q)$ can be interpreted as 1.44 times the number of bits used to code draws from P when using the imperfect code defined by Q.

Entropy, Cross Entropy and KL Divergence

Let P and Q be two distribution on the same set.

$$\begin{aligned}
\text{Entropy :} \qquad\qquad\qquad H(P) &= E_{y\sim P}\; -\ln\; P(y) \\
\text{CrossEntropy :} \qquad H(P,Q) &= E_{y\sim P}\; -\ln\; Q(y) \\
\text{KL Divergence :}\; KL(P,Q) &= H(P,Q) - H(P) \\
&= E_{y\sim P}\quad \ln\; \frac{P(y)}{Q(y)}
\end{aligned}$$

We have $H(P,Q) \geq H(P)$ or equivalently $KL(P,Q) \geq 0$.

The Universality Assumption

$$\Phi^* = \operatorname*{argmin}_\Phi \; H(\mathrm{Pop}, P_\Phi) = \operatorname*{argmin}_\Phi \; H(\mathrm{Pop}) + KL(\mathrm{Pop}, P_\Phi)$$

Universality assumption: P_Φ can represent any distribution and Φ can be fully optimized.

This is clearly false for deep networks. But it gives important insights like:

$$P_{\Phi^*} = \mathrm{Pop}$$

This is the motivatation for the fundamental equation.

Asymmetry of Cross Entropy

Consider

$$\Phi^* = \operatorname*{argmin}_{\Phi} \; H(P, Q_\Phi) \qquad (1)$$

$$\Phi^* = \operatorname*{argmin}_{\Phi} \; H(Q_\Phi, P) \qquad (2)$$

For (1) Q_Φ must cover all of the support of P.

For (2) Q_Φ concentrates all mass on the point maximizing P.

Asymmetry of KL Divergence

Consider

$$\begin{aligned}
\Phi^* &= \operatorname*{argmin}_{\Phi}\; KL(P, Q_\Phi) \\
&= \operatorname*{argmin}_{\Phi}\; H(P, Q_\Phi) \qquad (1)
\end{aligned}$$

$$\begin{aligned}
\Phi^* &= \operatorname*{argmin}_{\Phi}\; KL(Q_\Phi, P) \\
&= \operatorname*{argmin}_{\Phi} H(Q_\Phi, P) - H(Q_\Phi) \qquad (2)
\end{aligned}$$

If Q_Φ is not universally expressive we have that (1) still forces Q_Φ to cover all of P (or else the KL divergence is infinite) while (2) allows Q_Φ to be restricted to a single mode of P (a common outcome).

Proving $KL(P, Q) \geq 0$: Jensen's Inequality

For f convex (upward curving) we have

$$E[f(x)] \geq f(E[x])$$

Proving $KL(P, Q) \geq 0$

$$\begin{aligned} KL(P, Q) &= E_{y \sim P} \; -\ln \frac{Q(y)}{P(y)} \\ &\geq -\ln E_{y \sim P} \; \frac{Q(y)}{P(y)} \\ &= -\ln \sum_{y} \; P(y) \frac{Q(y)}{P(y)} \\ &= -\ln \sum_{y} Q(y) \\ &= 0 \end{aligned}$$

Appendix: The Rearrangement Trick

$$
\begin{aligned}
KL(P,Q) &= E_{x\sim P} \ln \frac{P(x)}{Q(x)} \\
&= E_{x\sim P} \left[(-\ln Q(x)) - (-\ln P(x))\right] \\
&= (E_{x\sim P} \; -\ln Q(x)) - (E_{x\sim P} \; -\ln P(x)) \\
&= H(P,Q) - H(P)
\end{aligned}
$$

In general $E_{x\sim P} \; \ln\left(\prod_i A_i\right) = E_{x\sim P} \; \sum_i \ln A_i$

Summary

$\Phi^* = \mathrm{argmin}_\Phi \;\; H(\mathrm{Pop}, P_\Phi)$ unconditional

$\Phi^* = \mathrm{argmin}_\Phi \;\; E_{x\sim \mathrm{Pop}}\; H(\mathrm{Pop}(y|x), P_\Phi(y|x))$ conditional

$$\begin{array}{lrcl}
\text{Entropy :} & H(P) & = & E_{y\sim P}\; -\ln\; P(y) \\[2ex]
\text{CrossEntropy :} & H(P,Q) & = & E_{y\sim P}\; -\ln\; Q(y) \\[2ex]
\text{KL Divergence :} & KL(P,Q) & = & H(P,Q) - H(P) \\[2ex]
 & & = & E_{y\sim P}\;\; \ln\; \frac{P(y)}{Q(y)}
\end{array}$$

$H(P,Q) \geq H(P), \quad KL(P,Q) \geq 0, \quad \mathrm{argmin}_Q\; H(P,Q) = P$

END