Bell's Inequality

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Bell's Inequality

Alice: Q, R Bob: S, T

Classical observables distributed according to $P(Q, R, S, T)$. Combination of correlations between Alice and Bobs measurements are bounded according to the CHSH inequality

$$|E(QS) + E(RS) + E(RT) - E(QT)| \leq 2$$

For the quantum version, define 4 spin operators along arbitrary directions $Q = \vec{q} \cdot \sigma, R = \vec{r} \cdot \sigma, S = \vec{s} \cdot \sigma, T = \vec{t} \cdot \sigma$.
Let $\vec{q} = (0, 0, 1), \vec{r} = (1, 0, 0), \vec{s} = (-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}), \vec{t} = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$

$$|\langle Q \otimes S \rangle + \langle R \otimes S \rangle + \langle R \otimes T \rangle - \langle Q \otimes T \rangle| \leq 2\sqrt{2}$$

Calculating expectations

$$\vec{q}\cdot\sigma\otimes\vec{s}\cdot\sigma = \begin{pmatrix} \vec{s}\cdot\sigma & 0 \\ 0 & -\vec{s}\cdot\sigma \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\vec{r}\cdot\sigma\otimes\vec{s}\cdot\sigma = \begin{pmatrix} 0 & \vec{s}\cdot\sigma \\ \vec{s}\cdot\sigma & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 1 \\ -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}$$

$$\vec{r}\cdot\sigma\otimes\vec{t}\cdot\sigma = \begin{pmatrix} 0 & \vec{t}\cdot\sigma \\ \vec{t}\cdot\sigma & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix}$$

Calculating expectations

$$\vec{q} \cdot \sigma \otimes \vec{t} \cdot \sigma = \begin{pmatrix} \vec{t} \cdot \sigma & 0 \\ 0 & -\vec{t} \cdot \sigma \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\langle \vec{q} \cdot \sigma \otimes \vec{s} \cdot \sigma \rangle = \frac{1}{\sqrt{2}} (-\alpha^*(\alpha + \beta) + \beta^*(\beta - \alpha) + \gamma^*(\gamma + \delta) + \delta^*(\gamma - \delta))$$
$$\langle \vec{r} \cdot \sigma \otimes \vec{s} \cdot \sigma \rangle = \frac{1}{\sqrt{2}} (-\alpha^*(\gamma + \delta) + \beta^*(\delta - \gamma) - \gamma^*(\alpha + \beta) + \delta^*(\beta - \alpha))$$
$$\langle \vec{r} \cdot \sigma \otimes \vec{t} \cdot \sigma \rangle = \frac{1}{\sqrt{2}} (\alpha^*(\gamma - \delta) - \beta^*(\delta + \gamma) + \gamma^*(\alpha - \beta) - \delta^*(\beta + \alpha))$$
$$\langle \vec{q} \cdot \sigma \otimes \vec{t} \cdot \sigma \rangle = \frac{1}{\sqrt{2}} (\alpha^*(\alpha - \beta) - \beta^*(\beta + \alpha) + \gamma^*(\delta - \delta) + \delta^*(\gamma + \delta))$$

Full density matrix

$$\rho_{AB} = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* & \alpha\gamma^* & \alpha\delta^* \\ \beta\alpha^* & |\beta|^2 & \beta\gamma^* & \beta\delta^* \\ \gamma\alpha^* & \gamma\beta^* & |\gamma|^2 & \gamma\delta^* \\ \delta\alpha^* & \delta\beta^* & \delta\gamma^* & |\delta|^2 \end{pmatrix}$$

Partial traces

$$\begin{aligned}\mathrm{Tr}_A(\rho_{AB}) &= \sum_{ijkl} \rho_{ij}^{kl} \mathrm{Tr}_A(|i\rangle\langle k|) \otimes |j\rangle\langle l| \\ &= \sum_i \left(\sum_{jl} \rho_{ij}^{il} |j\rangle\langle l| \right) \\ &= (\rho_{00}^{00} + \rho_{10}^{10})|0\rangle\langle 0| + (\rho_{00}^{01} + \rho_{10}^{11})|0\rangle\langle 1| + (\rho_{01}^{00} + \rho_{11}^{10})|1\rangle\langle 0| + (\rho_{01}^{01} + \rho_{11}^{11})|1\rangle\langle 1|\end{aligned}$$

$$\begin{aligned}\mathrm{Tr}_B(\rho_{AB}) &= \sum_{ijkl} \rho_{ij}^{kl} |i\rangle\langle k| \otimes \mathrm{Tr}_B(|j\rangle\langle l|) \\ &= \sum_j \left(\sum_{ik} \rho_{ij}^{kj} |i\rangle\langle k| \right) \\ &= (\rho_{00}^{00} + \rho_{01}^{01})|0\rangle\langle 0| + (\rho_{00}^{10} + \rho_{01}^{11})|0\rangle\langle 1| + (\rho_{10}^{00} + \rho_{11}^{01})|1\rangle\langle 0| + (\rho_{10}^{10} + \rho_{11}^{11})|1\rangle\langle 1|\end{aligned}$$

Reduced density matrices for an arbitrary state

$$\mathrm{Tr}_A(\rho_{AB}) = \begin{pmatrix} \rho_{00}^{00} + \rho_{10}^{10} & \rho_{00}^{01} + \rho_{10}^{11} \\ \rho_{01}^{00} + \rho_{11}^{10} & \rho_{01}^{01} + \rho_{11}^{11} \end{pmatrix} = \begin{pmatrix} |\alpha|^2 + |\gamma|^2 & \alpha\beta^* + \gamma\delta^* \\ \beta\alpha^* + \delta\gamma^* & |\beta|^2 + |\delta|^2 \end{pmatrix}$$

$$\mathrm{Tr}_B(\rho_{AB}) = \begin{pmatrix} \rho_{00}^{00} + \rho_{01}^{01} & \rho_{00}^{10} + \rho_{01}^{11} \\ \rho_{10}^{00} + \rho_{11}^{01} & \rho_{10}^{10} + \rho_{11}^{11} \end{pmatrix} = \begin{pmatrix} |\alpha|^2 + |\beta|^2 & \alpha\gamma^* + \beta\delta^* \\ \gamma\alpha^* + \delta\beta^* & |\gamma|^2 + |\delta|^2 \end{pmatrix}$$

Definition of entanglement entropy

The Von Neumann entropy is

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho) = -\sum_x \lambda_x \log \lambda_x$$

for eigenvalues λ_x of ρ. This tells us: do the reduced states ρ_A and ρ_B contain all the information in ρ_{AB}? Maybe analagous to the mutual information

$$I(X;Y) = H(X) + H(Y) - H(X,Y) \geq 0$$

So a possible measurement of entanglement is

$$\Delta S = S(\rho) - S(\rho_A) - S(\rho_B)$$

Entanglement entropy of $|\phi^+\rangle$

$$S(\rho) = -\mathrm{Tr}(\rho \log \rho) = -\sum_x \lambda_x \log \lambda_x$$

where λ_x are the eigenvalues of ρ. Let $|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

$$\rho_{AB} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

which only has one nonzero eigenvalue $\lambda = 2$. Therefore $S(\rho) = 1$.

$$\rho_A = \rho_B = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad S(\rho_A) = S(\rho_B) = 0 \;\; \Delta S = 1$$

Entanglement entropy of $|00\rangle$

Let $\psi = |00\rangle$

$$\rho_{AB} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which only has one nonzero eigenvalue $\lambda = 1$. Therefore $S(\rho) = 0$.

$$\rho_A = \rho_B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \lambda = 1$$

Therefore,

$$S(\rho_A) = S(\rho_B) = 0 \;\; \Delta S = 0$$

Correlation functions for $|\phi^+\rangle$

$|\phi^+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$

$$\langle \vec{q}\cdot\sigma \otimes \vec{s}\cdot\sigma\rangle = \frac{1}{\sqrt{2}}\left(-\alpha^*(\alpha+\beta) + \beta^*(\beta-\alpha) + \gamma^*(\gamma+\delta) + \delta^*(\gamma-\delta)\right) = -\frac{1}{\sqrt{2}}$$
$$\langle \vec{r}\cdot\sigma \otimes \vec{s}\cdot\sigma\rangle = \frac{1}{\sqrt{2}}\left(-\alpha^*(\gamma+\delta) + \beta^*(\delta-\gamma) - \gamma^*(\alpha+\beta) + \delta^*(\beta-\alpha)\right) = -\frac{1}{\sqrt{2}}$$
$$\langle \vec{r}\cdot\sigma \otimes \vec{t}\cdot\sigma\rangle = \frac{1}{\sqrt{2}}\left(\alpha^*(\gamma-\delta) - \beta^*(\delta+\gamma) + \gamma^*(\alpha-\beta) - \delta^*(\beta+\alpha)\right) = -\frac{1}{\sqrt{2}}$$
$$\langle \vec{q}\cdot\sigma \otimes \vec{t}\cdot\sigma\rangle = \frac{1}{\sqrt{2}}\left(\alpha^*(\alpha-\beta) - \beta^*(\beta+\alpha) + \gamma^*(\delta-\delta) + \delta^*(\gamma+\delta)\right) = \frac{1}{\sqrt{2}}$$

$$|\langle Q\otimes S\rangle + \langle R\otimes S\rangle + \langle R\otimes T\rangle - \langle Q\otimes T\rangle| = 2\sqrt{2}$$