Homework 3

Quantum Mechanics

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Problem 1. *Problem 2.1 from Sakurai*

Solution.   The Heisenberg equation of motion reads

$$\frac{dA}{dt} = \frac{1}{i\hbar}\,[A, H]$$

For the spin precession problem, we have the Hamiltonian

$$H = -\left(\frac{eB}{mc}\right)S_z = \omega S_z$$

For $A = S_x, S_y, S_z$, the time evolution is given by

$$\frac{dS_x}{dt} = \frac{\omega}{i\hbar}\,[S_x, S_z] = -\omega S_y$$
$$\frac{dS_y}{dt} = \frac{\omega}{i\hbar}\,[S_y, S_z] = \omega S_x$$
$$\frac{dS_z}{dt} = \frac{\omega}{i\hbar}\,[S_z, S_z] = 0$$

The above system has a straightforward solution:

$$S_x(t) = \cos(\omega t)$$
$$S_y(t) = \sin(\omega t)$$
$$S_z(t) = S_z(0)$$

$\blacksquare$

Problem 2. *Problem 2.3 from Sakurai*

Solution. We are given that $\vec{B} = B\hat{z}$ and that we are in the eigenstate $|\psi(0)\rangle = |\boldsymbol{S} \cdot \hat{\boldsymbol{n}}\rangle_+$, which reads

$$|\psi(0)\rangle = \psi_+ |+\rangle + \psi_- |-\rangle$$
$$= \cos\frac{\beta}{2} |+\rangle + \sin\frac{\beta}{2} |-\rangle$$

where we have set $\alpha = 0$ since the ket is in the x-z plane. This state will evolve according to a Hamiltonian

$$H = -\left(\frac{eB}{m_e c}\right) S_z$$

Let $\omega = |e|B/m_e c$ giving $H = \omega S_z$. We have the energies

$$E_\pm = \mp\frac{e\hbar B}{2m_e c} = \mp\hbar\omega$$

$$|\psi(t)\rangle = \psi_+(0) \exp\left(\frac{-iE_+ t}{\hbar}\right) |+\rangle + \psi_-(0) \exp\left(\frac{-iE_- t}{\hbar}\right) |-\rangle$$
$$= \cos\frac{\beta}{2} \exp\left(\frac{-i\omega t}{2}\right) |+\rangle + \sin\frac{\beta}{2} \exp\left(\frac{i\omega t}{2}\right) |-\rangle$$

In general, the probability of measuring $|+\rangle_x = \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} |-\rangle$ is given by the inner product

$$| \langle S_x; +|\psi; t\rangle |^2 = \left| \left(\frac{1}{\sqrt{2}} \langle+| + \frac{1}{\sqrt{2}} \langle-|\right) \cdot \right.$$
$$\left. \left(\psi_+ \exp\left(\frac{-i\omega t}{2}\right) |+\rangle + \psi_- \exp\left(\frac{i\omega t}{2}\right) |-\rangle\right) \right|^2$$
$$= \left| \frac{1}{\sqrt{2}} \cos\frac{\beta}{2} \exp\left(\frac{-i\omega t}{2}\right) + \frac{1}{\sqrt{2}} \sin\frac{\beta}{2} \exp\left(\frac{i\omega t}{2}\right) \right|^2$$

Using the half-angle identity for $\sin\theta$ and some straightforward arithmetic gives

$$| \langle S_x; +|\psi; t\rangle |^2 = \frac{1 + \sin\beta \cos\omega t}{2}$$

For the time-dependence of $\langle S_x \rangle$, we have

$$\langle S_x \rangle(t) = \langle \psi; t| S_x |\psi; t\rangle$$
$$= \left(\psi_+ \exp\left(\frac{i\omega t}{2}\right) \langle +| + \psi_- \exp\left(\frac{-i\omega t}{2}\right) \langle -| \right)$$
$$\cdot \frac{\hbar}{2} \left(\psi_+ \exp\left(-\frac{i\omega t}{2}\right) |-\rangle + \psi_- \exp\left(\frac{i\omega t}{2}\right) |+\rangle \right)$$

Subsituting ψ_+ and ψ_- with the same values as above, we get

$$\langle S_x \rangle(t) = \frac{\hbar}{2} \sin\beta \cos\omega t$$

When $\beta = \pi/2$ the probability oscillates between 0 and 1 with frequency ω and when $\beta = 0$ then the probability is always $1/2$, as expected. The expectation value also makes sense because when $\beta = 0$, we can get $\pm\hbar/2$ with equal probability, giving zero on average. When $\beta = \pi/2$ the expectation value oscillates between $\hbar/2$ and $-\hbar/2$.

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Problem 3. *Problem 2.9 from Sakurai*

Solution.

We were given the wavefunction

$$\langle x|\alpha \rangle = A(x-a)^2(x+a)^2 \exp(ikx)$$

To find the normalization constant A, we evaluate

$$A = \left(\int_{-a}^{+a} \langle \alpha|x\rangle \langle x|\alpha \rangle \, dx \right)^{-1/2}$$
$$= \left(\int_{-a}^{+a} (x-a)^4(x+a)^4 dx \right)^{-1/2}$$
$$= \frac{\sqrt{315}}{16a^{9/2}}$$

according to Mathematica. The expectation value $\langle x \rangle$ is found by integrating

$$\langle x \rangle = \int_{-a}^{a} x \langle \alpha | x \rangle \langle x | \alpha \rangle \, dx$$
$$= \int_{-a}^{+a} x \langle \alpha | x \rangle \langle x | \alpha \rangle \, dx$$
$$= \int_{-a}^{+a} x(x-a)^4(x+a)^4 dx$$
$$= 0$$

since the function is odd. This makes physical sense, given the overall symmetry of the wavefunction. Notice that the complex exponential vanishes due to the complex conjugation. The expectation value $\langle x^2 \rangle$ is found by integrating

$$\langle x^2 \rangle = \int_{-a}^{+a} x^2(x-a)^4(x+a)^4 dx$$

This we do not expect to be zero, and indeed it is not:

Integrate $\left[x\textasciicircum 2(x-a)^4(x+a)^4, \{x, -a, a\} \right]$

$\frac{256a^{11}}{3465}$

and multiplying by A^2 gives

$$\langle x^2 \rangle = \left(\frac{\sqrt{315}}{16a^{9/2}} \right)^2 \frac{256a^{11}}{3465} = \frac{a^2}{11}$$

The expectation value $\langle p \rangle$ is found similarly

$$\langle p \rangle = \int_{-a}^{+a} \langle \alpha | x \rangle \frac{\hbar}{i} \frac{\partial}{\partial x} \langle x | \alpha \rangle \, dx$$
$$= A^2 \int_{-a}^{+a} (x-a)^2(x+a)^2 \exp(-ikx) \frac{\hbar}{i} \frac{\partial}{\partial x}(x-a)^2(x+a)^2 \exp(ikx)$$

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Leaving out the factor of $\hbar$ from p gives

$$f = D\left[(x-a)^2(x+a)^2\text{Exp}[Ikx], \{x, 1\}\right]$$

$$\text{Integrate}\left[(x-a)^2(x+a)^2\text{Exp}[-Ikx] * -I * f, \{x, -a, a\}\right]$$

$$2e^{ikx}(-a+x)^2(a+x) + 2e^{ikx}(-a+x)(a+x)^2 + ie^{ikx}k(-a+x)^2(a+x)^2$$

$$\frac{256a^9k}{315}$$

Bringing in the factor of $\hbar$ and multiplying by A^2 gives a familiar expression

$$\langle p \rangle = \hbar k$$

The expectation value $\langle p^2 \rangle$ is found similarly

$$\langle p^2 \rangle = -\int_{-a}^{+a} \langle \alpha|x\rangle \, \hbar^2 \frac{\partial^2}{\partial x^2} \langle x|\alpha\rangle \, dx$$

$$= -A^2 \int_{-a}^{+a} (x-a)^2(x+a)^2 \exp(-ikx)\hbar^2 \frac{\partial^2}{\partial x^2}(x-a)^2(x+a)^2 \exp(ikx)$$

$$f = -D\left[(x-a)^2(x+a)^2\text{Exp}[Ikx], \{x, 2\}\right]$$

$$\text{Integrate}\left[(x-a)^2(x+a)^2\text{Exp}[-Ikx] * f, \{x, -a, a\}\right]$$

$$-2e^{ikx}(-a+x)^2 - 4(a+x)\left(2e^{ikx}(-a+x) + ie^{ikx}k(-a+x)^2\right) -$$

$$(a+x)^2\left(2e^{ikx} + 4ie^{ikx}k(-a+x) - e^{ikx}k^2(-a+x)^2\right)$$

$$\frac{256a^7}{105} + \frac{256a^9k^2}{315}$$

Bringing in a factor of $\hbar^2$ and multiplying by A^2 gives

$$\langle p^2 \rangle = \frac{3\hbar^2}{a^2} + \hbar^2 k^2$$

The variance $\langle (\Delta x)^2 \rangle$ is just

$$\langle (\Delta x)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 = \frac{a^2}{11}$$

The variance $\langle (\Delta p)^2 \rangle$ is just

$$\langle (\Delta p)^2 \rangle = \langle p^2 \rangle - \langle p \rangle^2$$
$$= \frac{3\hbar^2}{a^2}$$

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Problem 4. *Problem 2.10 from Sakurai*

Solution. Let $|\psi\rangle = \alpha |a'\rangle + \beta |a''\rangle$ be an eigenvector of the Hamiltonian. Note that this must be real for the eigenvalue to be real. That means that

$$H |\psi\rangle = (|a'\rangle \delta \langle a''| + |a''\rangle \delta \langle a'|) (\alpha |a'\rangle + \beta |a''\rangle)$$
$$= \delta (\alpha |a''\rangle + \beta |a'\rangle)$$

Therefore $\alpha = \beta = \frac{1}{\sqrt{2}}$ or $\alpha = \frac{1}{\sqrt{2}}$ and $\beta = -\frac{1}{\sqrt{2}}$. Giving eigenvalues $\pm \delta$. To get the time evolution of the state, we need to express these in the basis of H. Just based on inspection of the the two bases, we can tell that

$$|a'\rangle = \frac{1}{\sqrt{2}} (|\psi_1\rangle - |\psi_2\rangle)$$
$$|a''\rangle = \frac{1}{\sqrt{2}} (|\psi_1\rangle + |\psi_2\rangle)$$

and, since the Hamiltonian is time-independent, a state prepared in $|a'\rangle$ will evolve according to

$$|\alpha(t)\rangle = \frac{1}{\sqrt{2}} \exp\left(\frac{-i\delta t}{\hbar}\right) |\psi_1\rangle - \frac{1}{\sqrt{2}} \exp\left(\frac{i\delta t}{\hbar}\right) |\psi_2\rangle$$

The probability of finding the system in the state $|a''\rangle$ at a later time is

$$| \langle a''|\alpha(t) \rangle |^2 = \left| \frac{1}{\sqrt{2}} \left(\langle \psi_1 | + \langle \psi_2 | \right) \right.$$

$$\left. \cdot \left(\frac{1}{\sqrt{2}} \exp \left(\frac{-i\delta t}{\hbar} \right) |\psi_1\rangle - \frac{1}{\sqrt{2}} \exp \left(\frac{i\delta t}{\hbar} \right) |\psi_2\rangle \right) \right|^2$$

$$= \frac{1}{4} \sin^2 \frac{\delta t}{\hbar}$$

This could describe a system in which the eigenvectors of the Hamiltonian are simultaneous with the eigenvectors of S_x, however the states $|a'\rangle$ and $|a''\rangle$ are expressed in the S_z basis.

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Problem 5. *Problem 2.12 from Sakurai*

Solution. The state is prepared in

$$|\alpha; t = 0\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{\exp(i\delta)}{\sqrt{2}} |1\rangle$$

In general, the energies of $|n\rangle$ are $E_n = \left(n + \frac{1}{2} \right) \hbar \omega$. Therefore, the time dependence of the state can be evaluated as

$$|\alpha; t\rangle = \exp \left(-\frac{iHt}{\hbar} \right) |\alpha; t\rangle$$

$$= \frac{1}{\sqrt{2}} \exp \frac{-i\omega t}{2} |0\rangle + \frac{1}{\sqrt{2}} \exp(i\delta) \exp \frac{-3i\omega t}{2} |1\rangle$$

$$\langle x|\alpha; t\rangle = \frac{1}{\sqrt{2}} \exp \frac{-i\omega t}{2} \langle x|0\rangle + \frac{1}{\sqrt{2}} \exp(i\delta) \exp \frac{-3i\omega t}{2} \langle x|1\rangle$$

and we know in general that the position representation of $|n\rangle$ i.e., $\langle x|n\rangle$ are

$$\langle x|n\rangle = \psi_n(x) = \frac{1}{\sqrt{2^n \, n!}} \left(\frac{m\omega}{\pi \hbar} \right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} H_n \left(\sqrt{\frac{m\omega}{\hbar}} x \right)$$

$$\langle x \rangle(t) = \langle \alpha; t | \, x \, | \alpha; t \rangle$$

$$= \left(\frac{1}{\sqrt{2}} \exp \frac{i\omega t}{2} \langle 0 | + \frac{1}{\sqrt{2}} \exp(-i\delta) \exp \frac{3i\omega t}{2} \langle 1 | \right)$$

$$x \left(\frac{1}{\sqrt{2}} \exp \frac{-i\omega t}{2} | 0 \rangle + \frac{1}{\sqrt{2}} \exp(i\delta) \exp \frac{-3i\omega t}{2} | 1 \rangle \right)$$

$$= \frac{1}{2} \langle 0 | \, x \, | 0 \rangle + \frac{1}{2} \langle 1 | \, x \, | 1 \rangle$$

$$+ \frac{1}{2} \exp(i\delta) \exp(-i\omega t) \langle 0 | \, x \, | 1 \rangle + \frac{1}{2} \exp(-i\delta) \exp(i\omega t) \langle 1 | \, x \, | 0 \rangle$$

Now recall the general expression for the matrix element of x

$$\langle n' | \, x \, | n \rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n} \delta_{n',n-1} + \sqrt{n+1} \delta_{n',n+1} \right)$$

which means that the above expression simplifies to

$$\langle x \rangle(t) = \frac{1}{2} \exp(i\delta) \exp(-i\omega t) \langle 0 | \, x \, | 1 \rangle + \frac{1}{2} \exp(-i\delta) \exp(i\omega t) \langle 1 | \, x \, | 0 \rangle$$

$$= \frac{1}{2} \sqrt{\frac{\hbar}{2m\omega}} \left(\exp(i\delta) \exp(-i\omega t) + \exp(-i\delta) \exp(i\omega t) \right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \cos(\delta - \omega t)$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \cos(\omega t - \delta)$$

For momentum, we can just replace the operator x with p in the expressions above:

$$\langle p \rangle(t) = \langle \alpha; t | \, x \, | \alpha; t \rangle$$

$$= \left(\frac{1}{\sqrt{2}} \exp \frac{i\omega t}{2} \langle 0 | + \frac{1}{\sqrt{2}} \exp(-i\delta) \exp \frac{3i\omega t}{2} \langle 1 | \right)$$

$$\hat{p} \left(\frac{1}{\sqrt{2}} \exp \frac{-i\omega t}{2} | 0 \rangle + \frac{1}{\sqrt{2}} \exp(i\delta) \exp \frac{-3i\omega t}{2} | 1 \rangle \right)$$

$$= \frac{1}{2} \langle 0 | \, p \, | 0 \rangle + \frac{1}{2} \langle 1 | \, p \, | 1 \rangle$$

$$+ \frac{1}{2} \exp(i\delta) \exp(-i\omega t) \langle 0 | \, p \, | 1 \rangle + \frac{1}{2} \exp(-i\delta) \exp(i\omega t) \langle 1 | \, p \, | 0 \rangle$$

and we have another general expression for the matrix element of p

$$\langle n' | p | n \rangle = i \sqrt{\frac{m\hbar\omega}{2}} \left(-\sqrt{n}\delta_{n',n-1} + \sqrt{n+1}\delta_{n',n+1} \right)$$

which again means that the above expression simplifies to

$$\langle p \rangle(t) = \frac{1}{2}\exp(i\delta)\exp(-i\omega t)\langle 0 | p | 1 \rangle + \frac{1}{2}\exp(-i\delta)\exp(i\omega t)\langle 1 | p | 0 \rangle$$

$$= \frac{i}{2}\sqrt{\frac{m\hbar\omega}{2}}\left(-\exp(i\delta)\exp(-i\omega t) + \exp(-i\delta)\exp(i\omega t) \right)$$

$$= -\sqrt{\frac{m\hbar\omega}{2}}\sin(\omega t - \delta)$$

In the Heisenberg picture, we have the Heisenberg equations of motion

$$\frac{dp}{dt} = -m\omega^2 x$$

$$\frac{dx}{dt} = \frac{p}{m}$$

It is has been shown in the text how to uncouple these in terms of the ladder operators and solve the system for the time dependent operators $x(t)$ and $p(t)$

$$x(t) = x(0)\cos(\omega t) + \left(\frac{p(0)}{m\omega} \right)\sin(\omega t)$$

$$p(t) = -m\omega x(0)\sin(\omega t) + p(0)\cos(\omega t)$$

To get $\langle x \rangle(t)$ we have

$$\langle x \rangle(t) = \langle \alpha | x(0)\cos(\omega t) + \left(\frac{p(0)}{m\omega} \right)\sin(\omega t) | \alpha \rangle$$

$$= \langle \alpha | x(0) | \alpha \rangle \cos(\omega t) + \langle \alpha | p(0) | \alpha \rangle \frac{\sin(\omega t)}{m\omega}$$

$$\langle\alpha|\,x(0)\,|\alpha\rangle = \frac{1}{2}\langle 0|\,x(0)\,|0\rangle + \frac{1}{2}\langle 1|\,x(0)\,|1\rangle$$
$$+ \frac{1}{2}\exp(i\delta)\langle 0|\,x(0)\,|1\rangle + \frac{1}{2}\exp(i\delta)\langle 1|\,x(0)\,|0\rangle$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\,\exp(i\delta)$$

The factor of one-half disappeared in the $\langle\alpha|\,x(0)\,|\alpha\rangle$ term since $\langle n'|\,x\,|n\rangle$ is real and therefore equal to its complex conjugate

$$\langle x\rangle(t) = \sqrt{\frac{\hbar}{2m\omega}}\,\exp(i\delta)\cos(\omega t) + \frac{i}{2}\exp(i\delta)\sqrt{\frac{m\hbar\omega}{2}}\frac{\sin(\omega t)}{m\omega}$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\,\exp(i\delta)\exp(i\omega t)$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\,\exp(i(\omega t + \delta))$$

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Problem 6. *Problem 2.13 from Sakurai*

Solution.

The Heisenberg equations of motion are

$$\frac{dp}{dt} = F$$
$$\frac{dx}{dt} = \frac{p_0 + Ft}{m}$$

The solution is simply

$$p(t) = p(0) + Ft$$
$$x(t) = x(0) + \frac{p(0)}{m}t + \frac{1}{2}\frac{Ft^2}{m}$$

$$\langle x\rangle(t) = \langle\alpha|\left(x(0) + \frac{p_0}{m}t + \frac{1}{2}\frac{Ft^2}{m}\right)|\alpha\rangle$$
$$= x_0 + \frac{p_0 t}{m} + \frac{Ft^2}{2m}$$

and for $\langle p \rangle$ we have

$$\langle p \rangle (t) = \langle \alpha | \, p(0) + Ft \, | \alpha \rangle$$
$$= p_0 + Ft$$

In the Schrodinger picture, this Hamiltonian is not a constant

$$H(t) = \frac{p(t)^2}{2m} + V(x, t) = \frac{p(t)^2}{2m} + Fx(t)$$

although, using the canonical commutation relations, we can see that $H(t_0)$ commutes with $H(t)$. So we should be able to define

$$\mathcal{U}(t, t_0) = \exp \left(-\frac{i}{\hbar} \int_{t_0}^{t} H(t) dt \right)$$

such that $|\alpha; t \rangle = \mathcal{U}(t, t_0) \, |\alpha; t_0 \rangle$. If we evaluated that expression, then we could write $|\alpha; t \rangle$ explicitly.

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