The Quantum Fourier Transform

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Introduction

Dimension of n-qubit Hilbert space $N = 2^n$

The quantum fourier transform (QFT) transforms a quantum state $|\psi\rangle \to |\phi\rangle$ via the transformation of basis states:

$$\mathrm{QFT}\,|j\rangle = \frac{1}{2^{n/2}} \sum_{k=1}^{2^n} e^{2\pi ijk/2^n}\,|k\rangle$$

Equivalently, on the state $|\psi\rangle = \sum_j \psi_j\,|j\rangle$ reads

$$\mathrm{QFT}\,|\psi\rangle = |\phi\rangle = \frac{1}{2^{n/2}} \sum_{j=1}^{2^n} \psi_j \left(\sum_{k=1}^{2^n} e^{2\pi ijk/2^n}\,|k\rangle\right)$$

which turns out to be a unitary transformation

Product representation of the QFT

Computational basis ket $|j\rangle = |j_1 j_2 ... j_n\rangle$

Fourier basis ket $|k\rangle = |k_1 k_2 ... k_n\rangle$

Converting k to binary: $k = \sum_l k_l 2^l$

Also, note that $|k\rangle = |k_1 k_2 ... k_n\rangle = \bigotimes_{l=1}^{n} |k_l\rangle$

Product representation of the QFT

$$
\begin{aligned}
\mathrm{QFT}\,|j\rangle &= \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} e^{2\pi i jk/2^n} |k\rangle \\
&= \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} e^{2\pi i j \sum_l k_l 2^{-l}} \bigotimes_{l=1}^{n} |k_l\rangle \\
&= \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} \bigotimes_{l=1}^{n} e^{2\pi i j k_l 2^{-l}} |k_l\rangle \\
&= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \sum_{k_l=0}^{1} e^{2\pi i j k_l 2^{-l}} |k_l\rangle \\
&= \frac{1}{2^{n/2}} \bigotimes_{l=1}^{n} \left(|0\rangle + e^{2\pi i j 2^{-l}} |1\rangle \right)
\end{aligned}
$$

Phase estimation of a unitary operator

An important module in many quantum algorithms that uses QFT

Consider an eigenvector $|u\rangle$ of a Unitary operator U. Its eigenvalue can be written as $u = e^{2\pi i\theta}$

$$U|u\rangle = u|u\rangle = e^{2\pi i\theta}|u\rangle$$

Basic group theory

A **group** is a non-empty set and an operation that combines any two elements of the set to produce a third element of the set, in such a way that the operation is associative, an identity element exists and every element has an inverse
Example: the integers $\mathbb{Z}$ form a group

A **subgroup** H of a group G also satisfies the basic axioms but is a subset

A **coset** is a subset

Hidden subgroup problem (HSP)

From Nielsen and Chuang:

Let f be a function from a finitely generated group G to a finite set X such that f is constant on the cosets of a subgroup K, and distinct on each coset. Given a quantum black box for performing the unitary transform U for $g \in G$, $h \in X$, and $\oplus$ an appropriately chosen binary operation on X, find a generating set for K.

We can see how period finding is determining the subgroup over which f is constant