Bell's Inequality

Clayton W. Seitz

March 26, 2023

Basically I want to relate the expansion coefficients of the two-qubit pure state to the degree of entanglement using entanglement entropy. Then, I want to demonstrate that entangled states can violate Bell's inequality (but I'm not sure if this is exactly correct or under what conditions). Finally, I want to show maximally entangled states that saturate the Tsirelson bound.

CHSH Inequality

Define 4 spin operators along arbitrary directions
$Q = \vec{q} \cdot \sigma, R = \vec{r} \cdot \sigma, S = \vec{s} \cdot \sigma, T = \vec{t} \cdot \sigma.$

Alice: $Q, R = \pm 1$ Bob: $S, T = \pm 1$

Combination of correlations between Alice and Bobs measurements are bounded according to the CHSH inequality

$$E(Q \otimes S) + E(R \otimes S) + E(R \otimes T) - E(Q \otimes T) \leq 2$$

Let $\vec{q} = (0, 0, 1), \vec{r} = (1, 0, 0), \vec{s} = (-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}), \vec{t} = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$

Calculating expectations

$$\vec{q} \cdot \sigma \otimes \vec{s} \cdot \sigma = \begin{pmatrix} \vec{s} \cdot \sigma & 0 \\ 0 & -\vec{s} \cdot \sigma \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\langle \vec{q} \cdot \sigma \otimes \vec{s} \cdot \sigma \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \alpha^* & \beta^* & \gamma^* & \delta^* \end{pmatrix} \begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \left(-\alpha^*(\alpha + \beta) + \beta^*(\beta - \alpha) + \gamma^*(\gamma + \delta) + \delta^*(\gamma - \delta) \right)$$

Entanglement entropy of a two-qubit system

Suppose the system is in a pure state

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle$$

$$\rho_{AB} = |\psi\rangle\langle\psi| =$$

$$\rho_A = \mathrm{Tr}_B(\rho_{AB}) =$$