Problem Set 3

Information and Coding Theory

February 26, 2021 CLAYTON SEITZ

Problem 0.1. *A single dice is rolled and we gain a dollar if the outcome is 2,3,4,5 and lose a dollar if the outcome is 1 or 6. Find the expected gain and the maximum entropy distribution over the possible outcomes of a roll.*

Solution.

Let P be the uniform distribution over the dice universe χ where an outcome of a roll is $x \in \chi$. Furthermore, let $\phi(x)$ be the gain given the outcome of a roll x according the problem definition

$$\phi = \begin{cases} 1 & 2,3,4,5 \\ -1 & 1,6 \end{cases}$$

and $\bar{x} \sim P^n$ be a draw of a sequence of n rolls from the product distribution P^n. We can then calculate the expected gain over n rolls as

$$\begin{aligned} \mathop{\mathbf{E}}_{\bar{x}\sim P^n}[\phi(\bar{x})] &= \sum_n \left(\sum_i \phi(x_n) \cdot p(x_n) \right) \\ &= \sum_n \left(\frac{1}{6} \sum_i \phi(x_n) \right) \\ &= \frac{n}{3} \end{aligned}$$

Now, we would like to find the maximum entropy distribution P^* over χ in the set of distributions Π such that

$$\mathop{\mathbf{E}}_{\bar{x}\sim (P^*)^n}[\phi(\bar{x})] > \frac{n}{3} \tag{1}$$

We can find such a distribution P^* by defining the linear family of distributions that satisfy this constraint on the expected gain

$$\mathcal{L} = \left\{ P : \mathop{\mathbf{E}}_{\bar{x} \sim P^n} [\phi(\bar{x})] = \sum_{x \in \chi} p(x) \cdot \phi(x) > \alpha \right\}$$

We would like to find the distribution P^* such that $P^* = \mathbf{Proj}_{\mathcal{L}}(Q)$ and we now compute this projection by using the Lagrangian

$$\mathbf{\Lambda}(P, \lambda_0, \lambda_1) = D(P||Q) + \lambda_0 \left(\sum p(x) - 1 \right) + \lambda_1 \xi_\alpha(x) \tag{2}$$

where

$$\xi_\alpha = \begin{cases} -x & x < \alpha \\ 0 & x \geq \alpha \end{cases}$$

We find a solution by setting the derivative of this Lagrangian to zero

$$\nabla \Lambda = \log \left(\frac{p^*(x)}{q(x)} \right) + \frac{1}{2 \ln 2} + \lambda_0 + \nabla \xi_\alpha$$

$$\nabla \xi_\alpha = \begin{cases} -\lambda_1 & x < \alpha \\ 0 & x > \alpha \end{cases}$$

Ultimately, we have the solution

$$p^*(x) = q(x) \cdot 2^{\lambda_0 - \lambda_1 \cdot \phi(x)}$$

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