Homework 3

Quantum Mechanics
February 9, 2023 C Seitz

Problem 1. *Problem 2.48*

Solution. ■

Problem 2. *Problem 2.49*

Solution. ■

Problem 3. *Problem 2.50*

Solution. ■

Problem 4. *Problem 2.51*

Solution. The Hadamard gate H is unitary if $H^\dagger = H^{-1}$. It is easy to see that

$$H^\dagger = H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

It's inverse is

$$H^{-1} = -\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & -1\\ -1 & 1\end{pmatrix} = H$$

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Problem 5. *Problem 2.52*

Solution.

$$H^2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

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Problem 6. *Problem 2.53*

Solution. Writing out the characteristic equation gives that the eigenvalues are $\lambda = \pm\sqrt{2}$. ■

Problem 7. *Problem 2.54*

Solution. Since the two operators commute, they are simultaneously diagonalizable. Consider the following spectral decompositions

$$A = \sum_n a_n |n\rangle \langle n|$$

$$B = \sum_n b_n |n\rangle \langle n|$$

Therefore, it must be true that

$$A + B = \sum_n (a_n + b_n) |n\rangle \langle n|$$

Now these matrices are Hermitian so their eigenvectors are orthogonal, and the product of matrix exponentials is just

$$\begin{aligned}
\exp(A)\exp(B) &= \left(\sum_n \exp(a_n) |n\rangle \langle n|\right) \left(\sum_m \exp(b_m) |m\rangle \langle m|\right) \\
&= \sum_{m,n} \delta_{mn} \exp(a_n)\exp(b_m) |n\rangle \langle m| \\
&= \sum_n \exp(a_n)\exp(b_n) |n\rangle \langle n| \\
&= \sum_n \exp(a_n + b_n) |n\rangle \langle n| \\
&= \exp(A + B)
\end{aligned}$$

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Problem 8. *Problem 2.55*

Solution.

$$
\begin{aligned}
UU^\dagger &= \exp\left(\frac{-iH(t_2-t_1)}{\hbar}\right)\exp\left(\frac{iH(t_2-t_1)}{\hbar}\right) \\
&= \left(\sum_n \exp\!\left(\frac{-iE_n(t_2-t_1)}{\hbar}\right)|n\rangle\langle n|\right)\left(\sum_m \exp\!\left(\frac{iE_m(t_2-t_1)}{\hbar}\right)|m\rangle\langle m|\right) \\
&= \sum_{m,n}\delta_{mn}|n\rangle\langle m| \\
&= \sum_n |n\rangle\langle n| = I
\end{aligned}
$$

where H is a Hermitian operator.

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Problem 9. *Problem 2.56*

Solution.

U is unitary so its eigenvalues u_n have unit norm, which means

$$K = -i\log(U) = -i\sum_n \log(u_n)|n\rangle\langle n| = \sum_n \theta|n\rangle\langle n|$$

since

$$\log(u_n) = \log\big(|u_n|e^{i\theta}\big) = \log(|u_n|) + i\theta = i\theta$$

Therefore, $K = K^\dagger$ since $\theta \in \mathbb{R}$.

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Problem 10. *Problem 2.57*

Solution.

$$L_l|\alpha\rangle = \frac{\ell|l\rangle}{|\ell|}$$

$$M_m\frac{\ell|l\rangle}{|\ell|} = \frac{m\ell}{|m||\ell|}|m\rangle$$

which is equivalent to

$$
\begin{aligned}
N_{m\ell}\,|\alpha\rangle &= M_m L_\ell\,|\alpha\rangle \\
&= \frac{|m\rangle\,\langle m|\ell\rangle\,\langle\ell|}{|m||\ell|}\,|\alpha\rangle \\
&= \frac{\ell\,|m\rangle\,\langle m|}{|m||\ell|}\,|\ell\rangle \\
&= \frac{m\ell}{|m||\ell|}\,|m\rangle
\end{aligned}
$$

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Problem 11. *Problem 2.58*

Solution.

Since the system is in an eigenstate of M with eigenvalue m, the average will be m

$$
\langle M\rangle = \langle m|\,M\,|m\rangle = \langle m|\,m\,|m\rangle = m
$$

The variance must then be zero

$$
\begin{aligned}
(\Delta M)^2 &= \langle M^2\rangle - \langle M\rangle^2 \\
&= m^2 - m^2 = 0
\end{aligned}
$$

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Problem 12. *Problem 2.59*

Solution.

$$
\langle 0|\,X\,|0\rangle = \langle 0|1\rangle = 0
$$

$$
\begin{aligned}
(\Delta X)^2 &= \langle X^2\rangle - \langle X\rangle^2 \\
&= \langle X^2\rangle \\
&= \langle 0|\,X^2\,|0\rangle \\
&= 1
\end{aligned}
$$

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Problem 13. *Problem 2.60*

Solution.

$$\vec{v} \cdot \sigma = \begin{pmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z \end{pmatrix}$$

The corresponding characteristic equation is

$$\lambda^2 - (v_z^2 + v_y^2 + v_x^2) = 0$$

If $\vec{v}$ is normalized then $\lambda = \pm 1$.

$\blacksquare$

Problem 14. *Problem 2.61*

Solution. $\blacksquare$