Exam 2

Quantum Mechanics

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Problem 1.

Solution.

Some of the states have the same energy, so we will need to use degenerate perturbation theory. Specifically, the subspaces spanned by $\mathcal{A} = \{\left|0^{(0)}\right\rangle, \left|1^{(0)}\right\rangle\}$ and $\mathcal{B} = \{\left|2^{(0)}\right\rangle, \left|4^{(0)}\right\rangle\}$ have a degeneracy while the lone ket $\left|3^{(0)}\right\rangle$ is nondegenerate. We assume that a perturbed ket $\alpha \in \mathcal{A}$ can be written as a linear combination of the unperturbed kets:

$$|\alpha\rangle = \sum_{n\in\mathcal{A}} \langle n|\alpha\rangle |n\rangle$$

The first order correction is given by

$$V_{\mathcal{A}} |\alpha\rangle = \sum_{n\in\mathcal{A}} \langle n|\alpha\rangle (H - H_0) |n\rangle = \Delta_{\alpha}^{(1)} |\alpha\rangle$$

We therefore need to find the eigenvectors and eigenvalues of the matrix

$$|V_{\mathcal{A}} - \Delta_\alpha I| = \det \begin{pmatrix} 2\cos\theta - \Delta_\alpha & 2\sin\theta e^{-i\phi} \\ 2\sin\theta e^{i\phi} & -2\cos\theta - \Delta_\alpha \end{pmatrix} = 0$$

which is easy to solve, and we get the first order shifts $\Delta_\alpha^{(1)} = \pm 2$. It is the same process for the $\mathcal{B}$ subspace

$$|V_{\mathcal{B}} - \Delta_\beta I| = \det \begin{pmatrix} 4\cos\theta - \Delta_\beta & 4\sin\theta e^{-i\phi} \\ 4\sin\theta e^{i\phi} & -4\cos\theta - \Delta_\beta \end{pmatrix} = 0$$

It is pretty much the same matrix, so $\Delta_\beta^{(1)} = \pm 4$. For the first order correction to the energy of $\left|3^0\right\rangle$, we have

$$\Delta_3^{(1)} = \lambda \left\langle 3\right| V \left|3\right\rangle = \lambda$$

Although to second order it is pretty much the same since $\lambda << \epsilon$

$$\begin{aligned}\Delta_3^{(1)} &= \lambda V_{33} + \lambda^2 \sum_{j\neq 3} \frac{|V_{j3}|^2}{E_3^{(0)} - E_j^{(0)}} \\ &= \lambda + \lambda^2 \left(-\frac{3}{\epsilon} - \frac{3}{\epsilon}\right) \\ &= \lambda - \frac{6\lambda^2}{\epsilon} \approx \lambda\end{aligned}$$

To get the corrections to the ground state eigenvector, we can again use nondegenerate perturbation theory

$$\begin{aligned}\left|3^{(1)}\right\rangle &= \left|3^{(0)}\right\rangle + \lambda \sum_{j\neq 3} \left|j^{(0)}\right\rangle \frac{V_{j3}}{E_3^{(0)} - E_j^{(0)}} \\ &= \left|3^{(0)}\right\rangle + \frac{\lambda}{\epsilon}\left(\left|2^{(0)}\right\rangle + \left|4^{(0)}\right\rangle\right)\end{aligned}$$

In the limit $\lambda \to 0$, the perturbed eigenvectors are the "good" linear combinations. To find them we need to find the eigenvectors of the submatrices $V_\mathcal{A}$ and $V_\mathcal{B}$, which are both just multiples of the the $\hat{S}_n$ operator. We know those eigenvectors already

$$\left|0^{(1)}\right\rangle = \cos\frac{\theta}{2}\left|0^{(0)}\right\rangle + \sin\frac{\theta}{2}e^{i\phi}\left|1^{(0)}\right\rangle$$

$$\left|1^{(1)}\right\rangle = \sin\frac{\theta}{2}\left|0^{(0)}\right\rangle - \cos\frac{\theta}{2}e^{i\phi}\left|1^{(0)}\right\rangle$$

$$\left|2^{(1)}\right\rangle = \cos\frac{\theta}{2}\left|2^{(0)}\right\rangle + \sin\frac{\theta}{2}e^{i\phi}\left|4^{(0)}\right\rangle$$

$$\left|4^{(1)}\right\rangle = \sin\frac{\theta}{2}\left|2^{(0)}\right\rangle - \cos\frac{\theta}{2}e^{i\phi}\left|4^{(0)}\right\rangle$$

$$\left|3^{(1)}\right\rangle = \left|3^{(0)}\right\rangle$$

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Problem 2.

$$|\alpha_\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right)$$
$$|\beta_\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm i\,|1\rangle\right)$$

The ensemble expectation value for an operator A is

$$[A] = \sum_i w_i \langle\psi| A |\psi\rangle$$

Solution.

I will start by going through all of the moments for position. Let $c = \sqrt{\frac{\hbar}{2m\omega}}$

$$\begin{aligned}\langle\alpha_+| x^n |\alpha_+\rangle &= \frac{c^n}{2}\left(\langle 0| + \langle 1|\right)(a + a^\dagger)^n\left(|0\rangle + |1\rangle\right)\\ &= \frac{c^n}{2}(\langle 0| (a+a^\dagger)^n |0\rangle + \langle 0| (a+a^\dagger)^n |1\rangle\\ &+ \langle 1| (a+a^\dagger)^n |0\rangle + \langle 1| (a+a^\dagger)^n |1\rangle)\end{aligned}$$

for a yet unspecified constant c. We know that $(a+a^\dagger)^n$ is just a binomial expansion. The first and last terms survive only for terms in the expansion where the power of a is the same as $a^\dagger$ which happens only when n is even. The second term survives when the power of a is one less than $a^\dagger$ and vice versa for the third term. For even n, we can write

$$(a)^{n/2}(a^\dagger)^{n/2}|0\rangle = (n/2)!\,|0\rangle$$
$$(a)^{n/2}(a^\dagger)^{n/2}|1\rangle = (n/2)!\,|1\rangle$$

For odd values of n, we write

$$
\begin{aligned}
(a)^{\frac{n-1}{2}}(a^\dagger)^{\frac{n+1}{2}}\left|0\right\rangle &= \sqrt{\frac{n+1}{2}!}\sqrt{\frac{n-1}{2}!}\left|0\right\rangle \\
(a)^{\frac{n+1}{2}}(a^\dagger)^{\frac{n-1}{2}}\left|1\right\rangle &= \sqrt{\frac{n+1}{2}!}\sqrt{\frac{n-1}{2}!}\left|1\right\rangle
\end{aligned}
$$

Putting it together and using the appropriate binomial coefficients, we get

$$
\left\langle\alpha_+\right| x^n \left|\alpha_+\right\rangle = \begin{cases} c^n\binom{n}{n/2}(n/2)!, & n \text{ even} \\ c^n\binom{n}{(n+1)/2}\sqrt{\frac{n+1}{2}!}\sqrt{\frac{n-1}{2}!}, & n \text{ odd} \end{cases}
$$

For the pure ensemble of $\left|\beta_+\right\rangle$, we have

$$
\begin{aligned}
\left\langle\beta_+\right| x^n \left|\beta_+\right\rangle &= \frac{c^n}{2}\left(\left\langle 0\right| - i\left\langle 1\right|\right)(a+a^\dagger)^n\left(\left|0\right\rangle + i\left|1\right\rangle\right) \\
&= \frac{c^n}{2}(\left\langle 0\right|(a+a^\dagger)^n\left|0\right\rangle + i\left\langle 0\right|(a+a^\dagger)^n\left|1\right\rangle \\
&- i\left\langle 1\right|(a+a^\dagger)^n\left|0\right\rangle + \left\langle 1\right|(a+a^\dagger)^n\left|1\right\rangle) \\
&= \frac{c^n}{2}(\left\langle 0\right|(a+a^\dagger)^n\left|0\right\rangle + \left\langle 1\right|(a+a^\dagger)^n\left|1\right\rangle)
\end{aligned}
$$

$$
\left\langle\beta_+\right| x^n \left|\beta_+\right\rangle = \begin{cases} c^n\binom{n}{n/2}(n/2)!, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}
$$

so this is only nonzero for even n. For an ensemble of $\left|\alpha_-\right\rangle$, it is a similar situation

$$
\left\langle\alpha_-\right| x^n \left|\alpha_-\right\rangle = \frac{c^n}{2}(\left\langle 0\right|(a+a^\dagger)^n\left|0\right\rangle + \left\langle 1\right|(a+a^\dagger)^n\left|1\right\rangle)
$$

$$
\left\langle\alpha_-\right| x^n \left|\alpha_-\right\rangle = \begin{cases} c^n\binom{n}{n/2}(n/2)!, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}
$$

and for $\left|\beta_-\right\rangle$, we get

$$\langle\beta_-|\, x^n\, |\beta_-\rangle = \frac{c^n}{2}(\langle 0|\,(a+a^\dagger)^n\,|0\rangle + \langle 1|\,(a+a^\dagger)^n\,|1\rangle)$$

$$\langle\beta_-|\, x^n\, |\beta_-\rangle = \begin{cases} c^n\binom{n}{n/2}(n/2)!, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

For a mixed ensemble of $|\alpha_+\rangle$ and $|\alpha_-\rangle$,

$$[x^n] = \begin{cases} c^n\binom{n}{n/2}(n/2)!, & n \text{ even} \\ \frac{1}{2}c^n\binom{n}{(n+1)/2}\sqrt{\frac{n+1}{2}!}\sqrt{\frac{n-1}{2}!}, & n \text{ odd} \end{cases}$$

and for a mixed ensemble of $|\beta_+\rangle$ and $|\beta_-\rangle$,

$$[x^n] = \begin{cases} c^n\binom{n}{n/2}(n/2)!, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

Now we go through all of the moments for momentum. Let $c = \sqrt{\frac{m\hbar\omega}{2}}$

$$\begin{aligned} \langle\alpha_+|\, p^n\, |\alpha_+\rangle &= \frac{(-i)^n c^n}{2}\left(\langle 0| + \langle 1|\right)(a-a^\dagger)^n\left(|0\rangle + |1\rangle\right) \\ &= \frac{(-i)^n c^n}{2}(\langle 0|\,(a-a^\dagger)^n\,|0\rangle + \langle 0|\,(a-a^\dagger)^n\,|1\rangle \\ &+ \langle 1|\,(a-a^\dagger)^n\,|0\rangle + \langle 1|\,(a-a^\dagger)^n\,|1\rangle) \end{aligned}$$

We can see that if n is odd, the cross terms will always cancel. So we only need to be concerned with even n. Also this will be real as long as n is even, $(-i)^n$ will alternate in sign as we ascend even values of n, but the minus signs will always occur together, making it always positive:

$$[p^n] = \begin{cases} c^n\binom{n}{n/2}(n/2)!, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

It is easy to see that the result is the same for $|\beta_+\rangle$. Now for $|\alpha_-\rangle$, we again always get zero for odd n. So just for even n,

$$\begin{aligned} \langle\alpha_-|\, p^n\, |\alpha_-\rangle &= \frac{(-i)^n c^n}{2}\left(\langle 0| - \langle 1|\right)(a-a^\dagger)^n\left(|0\rangle - |1\rangle\right) \\ &= \frac{(-i)^n c^n}{2}(\langle 0|\,(a-a^\dagger)^n\,|0\rangle + \langle 1|\,(a-a^\dagger)^n\,|1\rangle) \end{aligned}$$

$$[p^n] = \begin{cases} c^n \binom{n}{n/2}(n/2)!, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

and the result is identical for a pure ensemble of $|\beta_-\rangle$. In summary, if we have a mixed ensemble of $|\alpha_+\rangle$ and $|\alpha_-\rangle$, we will get

$$[p^n] = \begin{cases} c^n \binom{n}{n/2}(n/2)!, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

and the same for the mixed ensemble of $|\beta_+\rangle$ and $|\beta_-\rangle$.

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Problem 3.

Solution. In this two particle system, we have $j_1 = 1$ and $j_2 = 2$. We are told the z component of the individual angular momenta $m_1 = -1$ and $m_2 = 2$. So we use the state kets $|j_1, j_2; m_1, m_2\rangle$ where m_1 and m_2 follow the usual rules. So the state is $|1, 2; -1, 2\rangle$.

$$\begin{aligned} \langle J^2 \rangle &= \langle 1, 2; -1, 2| \, J^2 \, |1, 2; -1, 2\rangle \\ &= \langle 1, 2; -1, 2| \left(J_1^2 + J_2^2 + 2J_{1z}J_{2z}\right) |1, 2; -1, 2\rangle \\ &= j_1(j_1+1)\hbar^2 + j_2(j_2+1)\hbar^2 - 4\hbar^2 \\ &= 2\hbar^2 + 6\hbar^2 - 4\hbar^2 \\ &= 4\hbar^2 \end{aligned}$$

We cannot directly compute the expectation values of J_x, J_y, J_z in this basis, because $|j_1, j_2, m_1, m_2\rangle$ are not eigenkets of J_x, J_y, J_z. But we can change basis:

$$|j_1, j_2, m_1, m_2\rangle = \sum_{m_1, m_2} |j_1, j_2; jm\rangle \langle j_1, j_2; jm | j_1, j_2; m_1, m_2\rangle$$

We need to find $\langle j_1, j_2; jm | j_1, j_2; m_1, m_2\rangle$ which are the Clebsch-Gordon coefficients. I've included the appropriate table in Figure 1. The expansion is

$$|1, 2; -1, 2\rangle = \sqrt{\frac{3}{5}} \, |1, 2; 11\rangle + \sqrt{\frac{1}{3}} \, |1, 2; 21\rangle + \sqrt{\frac{1}{15}} \, |1, 2; 31\rangle$$

$m = 1$

m_1, m_2 \ j	3	2	1
2, −1	$\sqrt{\frac{1}{15}}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{3}{5}}$
1, 0	$\sqrt{\frac{8}{15}}$	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{3}{10}}$
0, 1	$\sqrt{\frac{2}{5}}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{10}}$

Figure 1: Clebsch-Gordon coefficients for $j_1 = 1$, $j_2 = 2$, $m = 1$

The expectation value $\langle J_z \rangle$ is then

$$\begin{aligned}\langle J_z \rangle &= \langle 1,2;-1,2| \, J_z \, |1,2;-1,2\rangle \\ &= \left(\sqrt{\frac{3}{5}} \langle 1,2;11| + \sqrt{\frac{1}{3}} \langle 1,2;21| + \sqrt{\frac{1}{15}} \langle 1,2;31| \right) J_z \\ &\left(\sqrt{\frac{3}{5}} |1,2;11\rangle + \sqrt{\frac{1}{3}} |1,2;21\rangle + \sqrt{\frac{1}{15}} |1,2;31\rangle \right) \\ &= \hbar \left(\frac{3}{5} + \frac{1}{3} + \frac{1}{15} \right) = \hbar\end{aligned}$$

as we should expect. For J_x, J_y,

$$\begin{aligned}\langle J_x \rangle &= \langle 1,2;-1,2| \, J_x \, |1,2;-1,2\rangle \\ &= \left(\sqrt{\frac{3}{5}} \langle 1,2;11| + \sqrt{\frac{1}{3}} \langle 1,2;21| + \sqrt{\frac{1}{15}} \langle 1,2;31| \right) \\ &\frac{1}{2}(J_+ + J_-) \left(\sqrt{\frac{3}{5}} |1,2;11\rangle + \sqrt{\frac{1}{3}} |1,2;21\rangle + \sqrt{\frac{1}{15}} |1,2;31\rangle \right) \\ &= 0\end{aligned}$$

and $\langle J_y \rangle = 0$ since neither J_+ nor J_- connects two $|j_1, j_2; jm\rangle$ states.

Now, if we measure the total angular momentum and obtain the largest possible value, then we are in the state $|1,2;31\rangle$ in the $|j_1, j_2; jm\rangle$ basis.

However, to compute J_{1z} and J_{2z} we need to transform this back to the $|j_1, j_2, m_1, m_2\rangle$ basis. The coefficients are in the first column of the table and we get

$$|1,2;31\rangle = \left(\sqrt{\frac{1}{15}}\,|1,2;-12\rangle + \sqrt{\frac{2}{5}}\,|1,2;10\rangle + \sqrt{\frac{8}{15}}\,|1,2;01\rangle\right)$$

$$\begin{aligned}
\langle J_{1z}\rangle &= \langle 1,2;31|\,J_{1z}\,|1,2;31\rangle \\
&= \left(\sqrt{\frac{1}{15}}\,\langle 1,2;-12| + \sqrt{\frac{2}{5}}\,\langle 1,2;10| + \sqrt{\frac{8}{15}}\,\langle 1,2;01|\right) \\
&J_{1z}\left(\sqrt{\frac{1}{15}}\,|1,2;-12\rangle + \sqrt{\frac{2}{5}}\,|1,2;10\rangle + \sqrt{\frac{8}{15}}\,|1,2;01\rangle\right) \\
&= -\hbar\frac{1}{15} + \hbar\frac{2}{5} = \frac{\hbar}{3}
\end{aligned}$$

$$\begin{aligned}
\langle J_{2z}\rangle &= \langle 1,2;31|\,J_{2z}\,|1,2;31\rangle \\
&= \left(\sqrt{\frac{1}{15}}\,\langle 1,2;-12| + \sqrt{\frac{2}{5}}\,\langle 1,2;10| + \sqrt{\frac{8}{15}}\,\langle 1,2;01|\right) \\
&J_{2z}\left(\sqrt{\frac{1}{15}}\,|1,2;-12\rangle + \sqrt{\frac{2}{5}}\,|1,2;10\rangle + \sqrt{\frac{8}{15}}\,|1,2;01\rangle\right) \\
&= 2\hbar\frac{1}{15} + \hbar\frac{8}{15} = \frac{2\hbar}{3}
\end{aligned}$$

The probability that J_{1z} and J_{2z} never change from their original values is given by

$$|\,\langle 1,2;-12|1,2;31\rangle\,|^2 = 1/15$$

If we instead measure the smallest possible value, we are in state $|1,2;11\rangle$. The third particle being added has $j_3 = 1$ and $m_3 = -1$. We can consider the first two particles as a single composite particle in state $|jm\rangle = |11\rangle$. We now have two particles with $j = 1$. Taking $m_1 = 1$ and $m_2 = -1$ and reading off the table in Figure 2, the probabilities are

$m = 0$

m_1, m_2 \ j	2	1	0
1, −1	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$
0, 0	$\sqrt{\frac{2}{3}}$	0	$-\sqrt{\frac{1}{3}}$
−1, 1	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$

Figure 2: Clebsch-Gordon coefficients for $j_1 = 1$, $j_2 = 1$ and $m = 0$

$$\mathbf{Pr}(j) = \begin{cases} \frac{1}{3}, j = 0 \\ \frac{1}{2}, j = 1 \\ \frac{1}{6}, j = 2 \end{cases}$$

Finally, the expectation value of J^2 for this three particle system is

$$\begin{aligned} \langle J^2 \rangle &= \langle 1, 1; 1, -1 | \, J^2 \, | 1, 1; 1, -1 \rangle \\ &= \langle 1, 1; 1, -1 | \left(J_1^2 + J_2^2 + 2J_{1z}J_{2z} \right) | 1, 1; 1, -1 \rangle \\ &= j_1(j_1 + 1)\hbar^2 + j_2(j_2 + 1)\hbar^2 - 2\hbar^2 \\ &= 2\hbar^2 \end{aligned}$$

$\blacksquare$