

Exam 2

Quantum Mechanics

November 19, 2022

C SEITZ

Problem 1.

Solution.

Some of the states have the same energy, so we will need to use degenerate perturbation theory. Specifically, the subspaces spanned by $\mathcal{A} = \{|0^{(0)}\rangle, |1^{(0)}\rangle\}$ and $\mathcal{B} = \{|2^{(0)}\rangle, |4^{(0)}\rangle\}$ have a degeneracy while the lone ket $|3^{(0)}\rangle$ is nondegenerate. We assume that a perturbed ket $\alpha \in \mathcal{A}$ can be written as a linear combination of the unperturbed kets:

$$|\alpha\rangle = \sum_{n \in \mathcal{A}} \langle n | \alpha \rangle |n\rangle$$

The first order correction is given by

$$V |\alpha\rangle = \sum_{n \in \mathcal{A}} \langle n | \alpha \rangle (H - H_0) |n\rangle = \Delta_\alpha^{(1)} |\alpha\rangle$$

We therefore need to find the eigenvectors and eigenvalues of the matrix

$$|V_{\mathcal{A}} - \Delta_\alpha I| = \det \begin{pmatrix} 2 \cos \theta - \Delta_\alpha & 2 \sin \theta e^{-i\phi} \\ 2 \sin \theta e^{i\phi} & -2 \cos \theta - \Delta_\alpha \end{pmatrix} = 0$$

which is easy to solve, and we get the first order shifts $\Delta_\alpha^{(1)} = \pm 2$. It is the same process for the \mathcal{B} subspace

$$|V_{\mathcal{B}} - \Delta_\beta I| = \det \begin{pmatrix} 4 \cos \theta - \Delta_\beta & 4 \sin \theta e^{-i\phi} \\ 4 \sin \theta e^{i\phi} & -4 \cos \theta - \Delta_\beta \end{pmatrix} = 0$$

It is pretty much the same matrix, so $\Delta_\beta^{(1)} = \pm 4$. For the first order correction to the energy of $|3^0\rangle$, we have

$$\Delta_3^{(1)} = \lambda \langle 3 | V | 3 \rangle = \lambda$$

Although to second order it is pretty much the same since $\lambda \ll \epsilon$

$$\begin{aligned} \Delta_3^{(1)} &= \lambda V_{33} + \lambda^2 \sum_{j \neq 3} \frac{|V_{j3}|^2}{E_3^{(0)} - E_j^{(0)}} \\ &= \lambda + \lambda^2 \left(-\frac{3}{\epsilon} - \frac{3}{\epsilon} \right) \\ &= \lambda - \frac{6\lambda^2}{\epsilon} \approx \lambda \end{aligned}$$

To get the corrections to the ground state eigenvector, we can again use nondegenerate perturbation theory

$$\begin{aligned} |3^{(1)}\rangle &= |3^{(0)}\rangle + \lambda \sum_{j \neq 3} |j^{(0)}\rangle \frac{V_{j3}}{E_3^{(0)} - E_j^{(0)}} \\ &= |3^{(0)}\rangle + \frac{\lambda}{\epsilon} \left(|2^{(0)}\rangle + |4^{(0)}\rangle \right) \end{aligned}$$

In the limit $\lambda \rightarrow 0$, the perturbed eigenvectors are the “good” linear combinations. To find them we need to find the eigenvectors of the submatrices $V_{\mathcal{A}}$ and $V_{\mathcal{B}}$, which are both just multiples of the \hat{S}_n operator. We know those eigenvectors already

$$|0^{(1)}\rangle = \cos \frac{\theta}{2} |0^{(0)}\rangle + \sin \frac{\theta}{2} e^{i\phi} |1^{(0)}\rangle$$

$$|1^{(1)}\rangle = \sin \frac{\theta}{2} |0^{(0)}\rangle - \cos \frac{\theta}{2} e^{i\phi} |1^{(0)}\rangle$$

$$|2^{(1)}\rangle = \cos \frac{\theta}{2} |2^{(0)}\rangle + \sin \frac{\theta}{2} e^{i\phi} |4^{(0)}\rangle$$

$$|4^{(1)}\rangle = \sin \frac{\theta}{2} |2^{(0)}\rangle - \cos \frac{\theta}{2} e^{i\phi} |4^{(0)}\rangle$$

$$|3^{(1)}\rangle = |3^{(0)}\rangle$$

■

Problem 2.

$$|\alpha_{\pm}\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle)$$

$$|\beta_{\pm}\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm i|1\rangle)$$

Solution.

I will start by going through all of the moments for position. Let $c = \sqrt{\frac{\hbar}{2m\omega}}$

$$\begin{aligned} \langle \alpha_+ | x^n | \alpha_+ \rangle &= \frac{c^n}{2} (\langle 0 | + \langle 1 |) (a + a^\dagger)^n (|0\rangle + |1\rangle) \\ &= \frac{c^n}{2} (\langle 0 | (a + a^\dagger)^n |0\rangle + \langle 0 | (a + a^\dagger)^n |1\rangle \\ &\quad + \langle 1 | (a + a^\dagger)^n |0\rangle + \langle 1 | (a + a^\dagger)^n |1\rangle) \end{aligned}$$

for a yet unspecified constant c . We know that $(a + a^\dagger)^n$ is just a binomial expansion. The first and last terms survive only for terms in the expansion where the power of a is the same as a^\dagger which happens only when n is even. The second term survives when the power of a is one less than a^\dagger and vice versa for the third term. For even n , we can write

$$(a)^{n/2} (a^\dagger)^{n/2} |0\rangle = (n/2)! |0\rangle$$

$$(a)^{n/2} (a^\dagger)^{n/2} |1\rangle = (n/2)! |1\rangle$$

For odd values of n , we write

$$(a)^{\frac{n-1}{2}} (a^\dagger)^{\frac{n+1}{2}} |0\rangle = \sqrt{\frac{n+1}{2}}! \sqrt{\frac{n-1}{2}}! |0\rangle$$

$$(a)^{\frac{n+1}{2}} (a^\dagger)^{\frac{n-1}{2}} |1\rangle = \sqrt{\frac{n+1}{2}}! \sqrt{\frac{n-1}{2}}! |1\rangle$$

Putting it together and using the appropriate binomial coefficients, we get

$$\langle \alpha_+ | x^n | \alpha_+ \rangle = \begin{cases} c^n \binom{n}{n/2} (n/2)!, & n \text{ even} \\ c^n \binom{n}{(n+1)/2} \sqrt{\frac{n+1}{2}}! \sqrt{\frac{n-1}{2}}!, & n \text{ odd} \end{cases}$$

For the pure ensemble of $|\beta_+\rangle$, we have

$$\begin{aligned} \langle \beta_+ | x^n | \beta_+ \rangle &= \frac{c^n}{2} (\langle 0 | - i \langle 1 |) (a + a^\dagger)^n (|0\rangle + i |1\rangle) \\ &= \frac{c^n}{2} (\langle 0 | (a + a^\dagger)^n |0\rangle + i \langle 0 | (a + a^\dagger)^n |1\rangle \\ &\quad - i \langle 1 | (a + a^\dagger)^n |0\rangle + \langle 1 | (a + a^\dagger)^n |1\rangle) \\ &= \frac{c^n}{2} (\langle 0 | (a + a^\dagger)^n |0\rangle + \langle 1 | (a + a^\dagger)^n |1\rangle) \end{aligned}$$

$$\langle \beta_+ | x^n | \beta_+ \rangle = \begin{cases} c^n \binom{n}{n/2} (n/2)!, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

so this is only nonzero for even n . For an ensemble of $|\alpha_-\rangle$, it is a similar situation

$$\langle \alpha_- | x^n | \alpha_- \rangle = \frac{c^n}{2} (\langle 0 | (a + a^\dagger)^n |0\rangle + \langle 1 | (a + a^\dagger)^n |1\rangle)$$

$$\langle \alpha_- | x^n | \alpha_- \rangle = \begin{cases} c^n \binom{n}{n/2} (n/2)!, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

and for $|\beta_-\rangle$, we get

$$\langle \beta_- | x^n | \beta_- \rangle = \frac{c^n}{2} (\langle 0 | (a + a^\dagger)^n |0\rangle + \langle 1 | (a + a^\dagger)^n |1\rangle)$$

$$\langle \beta_- | x^n | \beta_- \rangle = \begin{cases} c^n \binom{n}{n/2} (n/2)!, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

For a mixed ensemble of $|\alpha_+\rangle$ and $|\alpha_-\rangle$,

$$[x^n] = \begin{cases} c^n \binom{n}{n/2} (n/2)!, & n \text{ even} \\ \frac{1}{2} c^n \binom{n}{(n+1)/2} \sqrt{\frac{n+1}{2}}! \sqrt{\frac{n-1}{2}}!, & n \text{ odd} \end{cases}$$

and for a mixed ensemble of $|\beta_+\rangle$ and $|\beta_-\rangle$,

$$[x^n] = \begin{cases} c^n \binom{n}{n/2} (n/2)!, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

Now we go through all of the moments for momentum. Let $c = \sqrt{\frac{m\hbar\omega}{2}}$

$$\begin{aligned} \langle \alpha_+ | p^n | \alpha_+ \rangle &= \frac{(-i)^n c^n}{2} (\langle 0 | + \langle 1 |) (a - a^\dagger)^n (|0\rangle + |1\rangle) \\ &= \frac{(-i)^n c^n}{2} (\langle 0 | (a - a^\dagger)^n |0\rangle + \langle 0 | (a - a^\dagger)^n |1\rangle \\ &\quad + \langle 1 | (a - a^\dagger)^n |0\rangle + \langle 1 | (a - a^\dagger)^n |1\rangle) \end{aligned}$$

We can see that if n is odd, the cross terms will always cancel. So we only need to be concerned with even n . Also this will be real as long as n is even, $(-i)^n$ will alternate in sign as we ascend even values of n , but the minus signs will always occur together, making it always positive:

$$[p^n] = \begin{cases} c^n \binom{n}{n/2} (n/2)!, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

It is easy to see that the result is the same for $|\beta_+\rangle$. Now for $|\alpha_-\rangle$, we again always get zero for odd n . So just for even n ,

$$\begin{aligned} \langle \alpha_- | p^n | \alpha_- \rangle &= \frac{(-i)^n c^n}{2} (\langle 0 | - \langle 1 |) (a - a^\dagger)^n (|0\rangle - |1\rangle) \\ &= \frac{(-i)^n c^n}{2} (\langle 0 | (a - a^\dagger)^n |0\rangle + \langle 1 | (a - a^\dagger)^n |1\rangle) \end{aligned}$$

$$[p^n] = \begin{cases} c^n \binom{n}{n/2} (n/2)!, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

and the result is identical for a pure ensemble of $|\beta_-\rangle$. In summary, if we have a mixed ensemble of $|\alpha_+\rangle$ and $|\alpha_-\rangle$, we will get

$$[p^n] = \begin{cases} c^n \binom{n}{n/2} (n/2)!, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

and the same for the mixed ensemble of $|\beta_+\rangle$ and $|\beta_-\rangle$. ■

$m = 1$			
$m_1, m_2 \backslash j$	3	2	1
$2, -1$	$\sqrt{\frac{1}{15}}$	$\sqrt{\frac{1}{3}}$	$\sqrt{\frac{3}{5}}$
$1, 0$	$\sqrt{\frac{8}{15}}$	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{3}{10}}$
$0, 1$	$\sqrt{\frac{2}{5}}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{10}}$

Figure 1: Clebsch-Gordon coefficients for $j_1 = 1, j_2 = 2, m = 1$

Problem 3.

Solution. In this two particle system, we have $j_1 = 1$ and $j_2 = 2$. We are told the z component of the individual angular momenta $m_1 = -1$ and $m_2 = 2$. So we use the state kets $|j_1, j_2; m_1, m_2\rangle$ where m_1 and m_2 follow the usual rules. So the state is $|1, 2; -1, 2\rangle$.

$$\begin{aligned}
\langle J^2 \rangle &= \langle 1, 2; -1, 2 | J^2 | 1, 2; -1, 2 \rangle \\
&= \langle 1, 2; -1, 2 | (J_1^2 + J_2^2 + 2J_{1z}J_{2z}) | 1, 2; -1, 2 \rangle \\
&= j_1(j_1 + 1)\hbar^2 + j_2(j_2 + 1)\hbar^2 - 4\hbar^2 \\
&= 2\hbar^2 + 6\hbar^2 - 4\hbar^2 \\
&= 4\hbar^2
\end{aligned}$$

We cannot directly compute the expectation values of J_x, J_y, J_z in this basis, because $|j_1, j_2, m_1, m_2\rangle$ are not eigenkets of J_x, J_y, J_z . But can change basis:

$$|j_1, j_2, m_1, m_2\rangle = \sum_{m_1, m_2} |j_1, j_2; jm\rangle \langle j_1, j_2; jm | j_1, j_2; m_1, m_2 \rangle$$

We need to determine $\langle j_1, j_2; jm | j_1, j_2; m_1, m_2 \rangle$ which are the Clebsch-Gordon coefficients.

Noting that $m = m_1 + m_2 = 1$, we find

$$|1, 2; -1, 2\rangle = \sqrt{\frac{3}{5}} |1, 2; 11\rangle + \sqrt{\frac{1}{3}} |1, 2; 21\rangle + \sqrt{\frac{1}{15}} |1, 2; 31\rangle$$

The expectation value $\langle J_z \rangle$ is then

$$\begin{aligned}
\langle J_z \rangle &= \langle 1, 2; -1, 2 | J_z | 1, 2; -1, 2 \rangle \\
&= \left(\sqrt{\frac{3}{5}} \langle 1, 2; 11 | + \sqrt{\frac{1}{3}} \langle 1, 2; 21 | + \sqrt{\frac{1}{15}} \langle 1, 2; 31 | \right) J_z \\
&\quad \left(\sqrt{\frac{3}{5}} | 1, 2; 11 \rangle + \sqrt{\frac{1}{3}} | 1, 2; 21 \rangle + \sqrt{\frac{1}{15}} | 1, 2; 31 \rangle \right) \\
&= \hbar \left(\frac{3}{5} + \frac{1}{3} + \frac{1}{15} \right) = \hbar
\end{aligned}$$

as we should expect. For J_x, J_y ,

$$\begin{aligned}
\langle J_x \rangle &= \langle 1, 2; -1, 2 | J_x | 1, 2; -1, 2 \rangle \\
&= \left(\sqrt{\frac{3}{5}} \langle 1, 2; 11 | + \sqrt{\frac{1}{3}} \langle 1, 2; 21 | + \sqrt{\frac{1}{15}} \langle 1, 2; 31 | \right) \\
&\quad \frac{1}{2} (J_+ + J_-) \left(\sqrt{\frac{3}{5}} | 1, 2; 11 \rangle + \sqrt{\frac{1}{3}} | 1, 2; 21 \rangle + \sqrt{\frac{1}{15}} | 1, 2; 31 \rangle \right) \\
&= 0
\end{aligned}$$

and $\langle J_y \rangle = 0$ since neither J_+ nor J_- connects two $|j_1, j_2; jm\rangle$ states.

Now, if we measure the total angular momentum and obtain the largest possible value, then we are in the state $|1, 2; 31\rangle$ in the $|j_1, j_2; jm\rangle$ basis. However, to compute J_{1z} and J_{2z} we need to transform this back to the $|j_1, j_2, m_1, m_2\rangle$ basis. Looking up the coefficients, we get

$$|1, 2; 31\rangle = \left(\sqrt{\frac{1}{15}} |1, 2; -12\rangle + \sqrt{\frac{2}{5}} |1, 2; 10\rangle + \sqrt{\frac{8}{15}} |1, 2; 01\rangle \right)$$

$$\begin{aligned}
\langle J_{1z} \rangle &= \langle 1, 2; 31 | J_{1z} | 1, 2; 31 \rangle \\
&= \left(\sqrt{\frac{1}{15}} \langle 1, 2; -12 | + \sqrt{\frac{2}{5}} \langle 1, 2; 10 | + \sqrt{\frac{8}{15}} \langle 1, 2; 01 | \right) \\
&\quad J_{1z} \left(\sqrt{\frac{1}{15}} | 1, 2; -12 \rangle + \sqrt{\frac{2}{5}} | 1, 2; 10 \rangle + \sqrt{\frac{8}{15}} | 1, 2; 01 \rangle \right) \\
&= -\hbar \frac{1}{15} + \hbar \frac{2}{5} = \frac{\hbar}{3}
\end{aligned}$$

$$\begin{aligned}
\langle J_{2z} \rangle &= \langle 1, 2; 31 | J_{2z} | 1, 2; 31 \rangle \\
&= \left(\sqrt{\frac{1}{15}} \langle 1, 2; -12 | + \sqrt{\frac{2}{5}} \langle 1, 2; 10 | + \sqrt{\frac{8}{15}} \langle 1, 2; 01 | \right) \\
&\quad J_{2z} \left(\sqrt{\frac{1}{15}} | 1, 2; -12 \rangle + \sqrt{\frac{2}{5}} | 1, 2; 10 \rangle + \sqrt{\frac{8}{15}} | 1, 2; 01 \rangle \right) \\
&= 2\hbar \frac{1}{15} + \hbar \frac{8}{15} = \frac{2\hbar}{3}
\end{aligned}$$

The probability that J_{1z} and J_{2z} never change from their original values is given by

$$| \langle 1, 2; -12 | 1, 2; 31 \rangle |^2 = 1/15$$

If we instead measure the smallest possible value, we are in state $|1, 2; 11\rangle$. The third particle being added has $j_3 = 1$ and $m_3 = -1$. We can consider the first two particles as a single composite particle in state $|jm\rangle = |11\rangle$. We now have two particles with $j = 1$. Taking $m_1 = 1$ and $m_2 = -1$ and reading off the table above, the probabilities are

$$\mathbf{Pr}(j) = \begin{cases} \frac{1}{3}, j = 0 \\ \frac{1}{2}, j = 1 \\ \frac{1}{6}, j = 2 \end{cases}$$

Finally, the expectation value of J^2 for this three particle system is

$m = 0$			
$m_1, m_2 \backslash j$	2	1	0
1, -1	$\sqrt{\frac{1}{6}}$	$\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$
0, 0	$\sqrt{\frac{2}{3}}$	0	$-\sqrt{\frac{1}{3}}$
-1, 1	$\sqrt{\frac{1}{6}}$	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{1}{3}}$

Figure 2: Clebsch-Gordon coefficients for $j_1 = 1$, $j_2 = 1$ and $m = 0$

$$\begin{aligned}
\langle J^2 \rangle &= \langle 1, 1; 1, -1 | J^2 | 1, 1; 1, -1 \rangle \\
&= \langle 1, 1; 1, -1 | (J_1^2 + J_2^2 + 2J_{1z}J_{2z}) | 1, 1; 1, -1 \rangle \\
&= j_1(j_1 + 1)\hbar^2 + j_2(j_2 + 1)\hbar^2 - 2\hbar^2 \\
&= 2\hbar^2
\end{aligned}$$

■