Homework 2

Quantum Mechanics

August 29th, 2022 CLAYTON SEITZ

Problem 1. *Problem 1.12 from Sakurai*

Solution.

If we choose the representation such that $|1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $|2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ then we can use the definition of the outer product to show that

$$H = a\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}$$

The energy eigenvalues are then found by

$$\begin{aligned}\det(H-\lambda I) &= \det\begin{pmatrix}a-\lambda & a\\ a & -a-\lambda\end{pmatrix}\\ &= (a-\lambda)(-a-\lambda)-a^2\\ &= \lambda^2-2a^2=0\end{aligned}$$

therefore $E_\pm = \pm\sqrt{2a}$. The + eigenvector $|\psi_1\rangle$ is given by the system

$$(\psi_1^1+\psi_1^2) = \sqrt{\frac{2}{a}}\psi_1^1$$
$$(\psi_1^1-\psi_1^2) = \sqrt{\frac{2}{a}}\psi_1^2$$

The − eigenvector $|\psi_2\rangle$ is given by the system

$$(\psi_2^1+\psi_2^2) = -\sqrt{\frac{2}{a}}\psi_2^1$$
$$(\psi_2^1-\psi_2^2) = -\sqrt{\frac{2}{a}}\psi_2^2$$

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Problem 2. *Problem 1.13 from Sakurai*

Solution.

Writing H out in matrix form gives

$$\begin{aligned}H &= H_{11}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} + H_{12}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} + H_{22}\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix} + \begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}\\ &= \begin{pmatrix}H_{11}+H_{12}+H_{22}+1 & H_{11}-H_{12}-H_{22}+1\\ H_{11}-H_{12}+H_{22}-1 & H_{11}+H_{12}-H_{22}-1\end{pmatrix}\end{aligned}$$

$$\det(H-\lambda I) = \det\begin{pmatrix}H_{11}+H_{12}+H_{22}+1-\lambda & H_{11}-H_{12}-H_{22}+1\\ H_{11}-H_{12}+H_{22}-1 & H_{11}+H_{12}-H_{22}-1-\lambda\end{pmatrix}$$

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Problem 3. *Problem 1.15 from Sakurai*

Solution. After the first measurement along $+\hat{z}$, all of our atoms are prepared in the $|+\rangle$ state in the S_z basis. At the next apparatus oriented along $\hat{n}$, more atoms will be filtered out since $|+\rangle$ is not an eigenket of the $\mathbf{S}\cdot\hat{n}$ operator. Recall that $|+\rangle_n$ is

$$|+\rangle_n = \cos\frac{\beta}{2}|+\rangle + \sin\frac{\beta}{2}|-\rangle$$

The probability the state $|+\rangle$ survives is given by the inner product

$$\begin{aligned}|\langle+|+\rangle_n|^2 &= |\langle+|\cos\frac{\beta}{2}|+\rangle + \langle+|\sin\frac{\beta}{2}|-\rangle|^2\\ &= \cos^2\frac{\beta}{2}\end{aligned}$$

After this, all atoms are in the $|+\rangle_n$ state. We then filter the atoms one more time with an apparatus along $-\hat{z}$. The fraction that survive this one is given by

$$
\begin{aligned}
|\langle -|+\rangle_n|^2 &= |\langle -|\cos\frac{\beta}{2}|+\rangle + \langle -|\sin\frac{\beta}{2}|-\rangle|^2 \\
&= \sin^2\frac{\beta}{2}
\end{aligned}
$$

Therefore the fraction output is $\cos^2\frac{\beta}{2}\sin^2\frac{\beta}{2}$. We can maximize this function by setting $\beta = \pi/2$

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Problem 4. *Problem 1.16 from Sakurai*

Solution.

We have the observable

$$
O = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
$$

$$
\begin{aligned}
\det(O - \lambda I) &= \det\begin{pmatrix} -\lambda & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\lambda & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\lambda \end{pmatrix} \\
&= -\lambda\left(\lambda^2 - \frac{1}{2}\right) - \frac{1}{\sqrt{2}}\left(-\frac{\lambda}{\sqrt{2}}\right) \\
&= -\lambda^3 + \lambda = 0
\end{aligned}
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Clearly our eigenvalues are $\lambda = \pm 1$

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Problem 5. *Problem 1.23 from Sakurai*

Solution.

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Problem 6. *Problem 1.24 from Sakurai*

Solution.

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