Homework 2

Quantum Mechanics
February 2, 2023 C Seitz

Problem 1. *2.2*

Solution.
In general, the matrix representation of A in a basis $|i\rangle, |j\rangle$ is such that the matrix element is $A_{ij} = \langle i| A |j\rangle$. Therefore, in the input basis, the matrix representation of A is

$$A = \begin{pmatrix} \langle 0| A |0\rangle & \langle 0| A |1\rangle \\ \langle 1| A |0\rangle & \langle 1| A |1\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In the output basis

$$A = \begin{pmatrix} \langle 0| A |0\rangle & \langle 0| A |1\rangle \\ \langle 1| A |0\rangle & \langle 1| A |1\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We can choose a different basis, say $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

In this basis A takes the form:

$$A' = UA = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

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Problem 2. *2.9*

Solution.

$$\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$$
$$\sigma_x = |1\rangle\langle 0| + |0\rangle\langle 1|$$
$$\sigma_y = i|0\rangle\langle 1| - i|1\rangle\langle 0|$$

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Problem 3. *2.12*

Solution. A matrix is diagonalizable if and only if the algebraic multiplicity equals the geometric multiplicity of each eigenvalue. It is easy to show that the characteristic equation here is $(1-\lambda)^2 = 0$ which only has one solution.

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Problem 4. *2.17*

Solution.

If H is normal, it must be diagonalizable and has the eigendecomposition

$$H = U\Lambda U^\dagger$$

where U is some unitary matrix. The conjugate transpose is

$$H^\dagger = U^\dagger \Lambda^\dagger U$$

If $H = H^\dagger$, and Λ is diagonal, then

$$U^\dagger \Lambda^\dagger U = U\Lambda U^\dagger$$

which means $\Lambda = \Lambda^\dagger$ i.e. the eigenvalues are real. Furthermore, if Λ is diagonal and purely real, then clearly $H = H^\dagger$.

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Problem 5. *2.18*

Solution. For a unitary matrix $U^\dagger U = I$, so for an eigenvector $|\alpha\rangle$,

$$\langle\alpha| U^\dagger U |\alpha\rangle = \langle\alpha| I |\alpha\rangle = 1$$

and $\langle\alpha| U^\dagger U |\alpha\rangle = \lambda^*\lambda$, so $\lambda^*\lambda = 1$. ■

Problem 6. *2.24*

Solution. ■

Problem 7. *Grahm-Schmidt*

Solution. It suffices to show that the following matrix has nonzero determinant:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

And it is straightforward to show that

$$\det(A) = 1$$

Therefore these vectors are indeed linearly independent, but not orthogonal. We can make them orthogonal using the Gram-Schmidt procedure. Let $|0\rangle, |1\rangle, |2\rangle$ be our non-orthogonal basis vectors.

$$|v_{k+1}\rangle \propto |w_{k+1}\rangle - \sum_{i=1}^{k} \langle v_i|w_{k+1}\rangle |v_i\rangle$$

$$\begin{aligned} |0'\rangle &= |0\rangle \\ |1'\rangle &= |1\rangle - \langle 0'|1\rangle |0'\rangle \\ |2'\rangle &= |2\rangle - \langle 0'|2\rangle |0'\rangle - \langle 1'|2\rangle |1'\rangle \end{aligned}$$

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Problem 8. *Normal matrix parameterization*

Solution.

Consider first

$$\begin{aligned} AA^\dagger &= (a_0\mathbb{I} + \mathbf{a}\cdot\sigma)(a_0^*\mathbb{I} + \mathbf{a}^*\cdot\sigma^\dagger) \\ &= |a_0|^2 + a_0(\mathbf{a}^*\cdot\sigma^\dagger) + a_0^*(\mathbf{a}\cdot\sigma) + \end{aligned}$$

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Problem 9. *2.26*

Solution. Writing out $|\psi\rangle^{\otimes 2}$ explicitly, we have

$$|\psi\rangle^{\otimes 2} = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$$

or in terms of tensor products we have

$$|\psi\rangle^{\otimes 2} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

Writing out $|\psi\rangle^{\otimes 3}$ explicitly, we have

$$|\psi\rangle^{\otimes 3} = \frac{1}{2^{3/2}}\left(|000\rangle + |001\rangle + |100\rangle + |010\rangle + |101\rangle + |111\rangle + |110\rangle + |011\rangle\right)$$

or in terms of tensor products we have

$$|\psi\rangle^{\otimes 3} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2^{3/2}}\begin{pmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{pmatrix}$$

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Problem 10. *2.27*

Solution.

$$X \otimes Z = \begin{pmatrix}0 & 0 & 1 & 0\\0 & 0 & 0 & -1\\1 & 0 & 0 & 0\\0 & -1 & 0 & 0\end{pmatrix}$$

$$I \otimes X = \begin{pmatrix}0 & 1 & 0 & 0\\1 & 0 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\end{pmatrix}$$

$$X \otimes I = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Clearly from this last result, the tensor product does not necessarily commute.

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Problem 11. *2.33*

Solution.

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Problem 12. *2.34*

Solution.

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Problem 13. *2.35*

Solution.

First recall that since the Pauli matrices are involutory $(\vec{a} \cdot \vec{\sigma})^k = (\vec{a} \cdot \vec{\sigma})$

$$\begin{aligned}
\exp(i\theta\vec{a} \cdot \vec{\sigma}) &= \sum_{k=0}^{\infty} \frac{1}{k!} (i\theta\vec{a} \cdot \vec{\sigma})^k \\
&= \sum_{k=0}^{\infty} \frac{(i\theta)^k}{k!} (\vec{a} \cdot \vec{\sigma})^k \\
&= \sum_{k \text{ odd}}^{\infty} \frac{(-1)^k\theta^k}{k!} (\vec{a} \cdot \vec{\sigma})^k + i \sum_{k \text{ even}}^{\infty} \frac{(-1)^k(\theta)^k}{k!} (\vec{a} \cdot \vec{\sigma})^k \\
&= \cos\theta I + i\sin\theta(\vec{a} \cdot \vec{\sigma})
\end{aligned}$$

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Problem 14. *2.39*

Solution.

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