Bayesian image reconstruction

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Outline

References

Photon statistics of CMOS cameras

- Imaging noise consists of shot noise, thermal noise, and readout noise
- Shot noise is Poisson, thermal noise and readout noise are Gaussian

For a CMOS pixel n, the true signal S_n $[\mathrm{ADU}]$ is a Poisson process with rate parameter λ_n

$$S_n = \gamma g_n P_n(\lambda_n)$$

where $\gamma\ [e^-/p]$ is the quantum efficiency and $g_n\ [\mathrm{ADU}/e^-]$ is the pixel's gain

$$P(S_n) = \frac{\exp\left(-\lambda_n\right)\lambda_n^p}{p!}$$

But what is the distribution over the corrupted signal $P(\hat{S}_n)$?

Photon statistics of CMOS cameras

To find $P(\hat{S}_n)$, we first evaluate the joint density $P(S_n, \hat{S}_n)$

$$\begin{aligned} P(S_n, \hat{S}_n) &= P(\hat{S}_n | S_n = s) P(S_n = s) \\ &= \frac{1}{Z} \exp\left(-\frac{(\hat{S}_n - g_n s - \mu_n)^2}{\sigma_n^2}\right) \frac{\exp\left(-\lambda_n\right) \lambda_n^s}{s!} \end{aligned}$$

Marginalizing over S_n gives the desired distribution over $\hat{S}_n$

$$P(\hat{S}_n) = \frac{1}{Z} \sum_{s=0}^{\infty} \frac{\exp\left(-\lambda_n\right) \lambda_n^s}{s!} \exp\left(-\frac{(\hat{S}_n - g_n s - \mu_n)^2}{\sigma_n^2}\right)$$

Bayesian parameter inference for CMOS photon statistics

The parameters in our model $\theta = (\lambda_n, g_n, \mu_n, \sigma_n^2)$ are unknown apriori

$$P(\theta|\hat{S}_n) \propto P(\hat{S}_n|\theta)P(\theta)$$

We can just computed the likelihood $P(\hat{S}_n|\theta)$ on the last slide. Samples from the posterior can be found for example by MCMC or we could use MAP estimation

Either of these approaches only make sense for stationary statistics, which means the physical locations and photophysics of the sample remain unchanged in time

For example photostable fluorophores like quantum dots would be a good choice

Fisher Information and the Cramer-Rao Bound

Consider maximum likelihood estimation (MLE) where the objective is to find an optimal parameter(s) that best explains the data

$$\theta^* = \underset{\theta}{\mathrm{argmax}}\ \ell(\mathcal{D}|\theta)$$

where $\ell = \log \mathcal{L}$ is the log-likelihood function. We can define the sensitivity of ℓ with respect to θ

$$\ell'(x|\theta) = \frac{\partial}{\partial\theta}\ell(x|\theta) = \frac{\mathcal{L}'(x|\theta)}{\mathcal{L}(x|\theta)}$$

for $x \in \mathcal{D}$. Intuitively, if the likelihood is insensitive to changes in θ, then $\mathcal{D}$ does not provide very much information about θ

Fisher Information and the Cramer-Rao Bound

Since x is a continuous random variable, we have to consider the average sensitivity

That is, for each $x \sim P(x)$ we can compute $\ell'(x|\theta)$ for all θ

$$I(\theta) = \mathbb{E}\left[\frac{\partial}{\partial\theta}\left(\ell(x|\theta)\right)^2\right] = \int \frac{\partial}{\partial\theta}\left(\ell(x|\theta)\right)^2 \mathcal{L}(x|\theta)dx$$

for $x \in \mathcal{D}$. Intuitively, if the likelihood is insensitive to changes in θ, then $\mathcal{D}$ does not provide very much information about θ

The Cramer-Rao Bound places a lower bound on the variance in our parameter estimate in iterms of $I(\theta)$:

$$\mathrm{Var}(\hat{\theta}) \geq \frac{1}{I(\theta)}$$

Consequences for single molecule localization

The point spread function

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