Homework 9

Quantum Mechanics
November 27, 2022 C Seitz

Problem 1. *4.1*

Solution.

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Problem 2. *4.2*

Solution.
Two unique translation operators $\mathcal{T}$ and $\mathcal{T}'$ commute because

$$\mathcal{T}\mathcal{T}' = \exp\left(-\frac{i}{\hbar}(d \cdot p + d' \cdot p)\right)$$

which is clearly the same if we swap the order, due to the exponential property. It is similar for rotations:

$$\mathcal{D}\mathcal{D}' = \exp\left(-\frac{i}{\hbar}(\phi n \cdot J + \phi' n' \cdot J')\right)$$

For the translation operator and parity operator

$$\mathcal{T}\pi \left|\boldsymbol{x}\right\rangle = \left|-\boldsymbol{x} + d\right\rangle$$
$$\pi\mathcal{T} \left|\boldsymbol{x}\right\rangle = \left|-\boldsymbol{x} - d\right\rangle$$

so they do not commute. However, parity and rotations commute

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Problem 3. *4.3*

Solution. If two operators A and B anticommute, then

$$AB\,|\psi\rangle = -BA\,|\psi\rangle$$

We are told they have common eigenvectors ket $|\psi\rangle$, so

$$ab\,|\psi\rangle = -ba\,|\psi\rangle$$

In words, the product of eigenvalues changes sign when the order of operators flips. If we take one of the operators to be the parity operator and the other to be the momentum operator. Let $A = \pi$ and $B = \hat{p}$:

$$\begin{aligned}
\pi\hat{p}\,|\psi\rangle &= -\hat{p}\pi\,|\psi\rangle \\
p\pi\,|\psi\rangle &= -\hat{p}\,|-\psi\rangle \\
p\,|-\psi\rangle &= p\,|-\psi\rangle
\end{aligned}$$

because the momentum operator is odd under parity. The product of eigenvalues changed sign, but the eigenvalue of the parity operator is unity.

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Problem 4. *4.4*

Solution.

$$\mathcal{Y}_{l=0}^{j=1/2,m=1/2} = Y_0^0(\theta,\phi)\chi_+$$

Rotating the spin gives

$$\begin{aligned}
(\sigma\cdot r)\mathcal{Y}_{l=0}^{j=1/2,m=1/2} &= Y_0^0(\theta,\phi)(\sigma\cdot r)\chi_+ \\
&= Y_0^0(\theta,\phi)r(\cos\theta\chi_+ + \sin\theta e^{i\phi}\chi_-) \\
&= r\left(\sqrt{3}Y_1^0\chi_+ - \sqrt{\frac{2}{3}}Y_1^1\chi_-\right) \\
&= -r\mathcal{Y}_{l=1}^{j=1/2,m=1/2}
\end{aligned}$$

which makes sense because, under space inversion,

$$\pi^{-1}(\sigma\cdot r)\pi = -(\sigma\cdot r)$$

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Problem 5. *4.5*

Solution.

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Problem 6. *4.6*

Solution. The symmetric state $|S\rangle$ is a sum of sines and cosines in the classically allowed region and $\cosh(x)$ in the classically forbidden region. The antisymmetric state $|A\rangle$ is similar, but has $\sinh(x)$ in the classically forbidden region.

$$\psi_S = \begin{cases} B\sin(kx) & a < |x| < a+b \\ C\cosh(x) & |x| < a \end{cases}$$

Continuity of the symmetric wavefunction at $x = a$ requires

$$B\sin(kx) = C\cosh(\kappa x)$$

and continuity of the first derivative of the symmetric wavefunction at $x = a$ requires

$$kB\cos(kx) = \kappa C\sinh(\kappa x)$$

$$\psi_A = \begin{cases} B\sin(kx) & a < |x| < a+b \\ C\sinh(x) & |x| < a \end{cases}$$

$$B\sin(kx) = C\sinh(\kappa x)$$

and continuity of the first derivative of the symmetric wavefunction at $x = a$ requires

$$kB\cos(kx) = \kappa C\cosh(\kappa x)$$

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