Homework 3

Quantum Mechanics
February 18, 2023 C Seitz

Problem 1. *Problem 2.48*

Solution.

The polar decomposition of a matrix is $A = UJ = KU$ where U is a unitary operator and J, K are positive operators that satisfy $J = \sqrt{A^\dagger A}$ and $K = \sqrt{AA^\dagger}$. If P is itself a positive matrix we can immediately say that its polar decomposition is $P = IP = PI$. If the matrix to decompose is unitary then of course $UU^\dagger = U^\dagger U = I$ so its decomposition is itself. If the matrix to decompose is Hermitian, then $H = H^\dagger$ and

$$J = K = \sqrt{H^2} = \sqrt{\sum_i \lambda_i^2 \left|i\right\rangle \left\langle i\right|} = \sum_i \left|\lambda_i\right| \left|i\right\rangle \left\langle i\right|$$

and its polar decomposition is therefore $U \sum_i |\lambda_i| \left|i\right\rangle \left\langle i\right|$ or $\sum_i |\lambda_i| \left|i\right\rangle \left\langle i\right| U$. ■

Problem 2. *Problem 2.49*

Solution.

The polar decomposition is $A = UJ$. The spectral decomposition of J is

$$J = \sqrt{\sum_i \lambda_i \lambda_i^* \left|i\right\rangle \left\langle i\right|} = \sum_i \left|\lambda_i\right| \left|i\right\rangle \left\langle i\right|$$

For the unitary matrix U, we have

$$U = \sum_j \lambda_j \left|j\right\rangle \left\langle j\right|$$

Therefore the product UJ reads

$$
\begin{aligned}
UJ &= \left(\sum_j \lambda_j |j\rangle \langle j|\right) \left(\sum_i |\lambda_i| |i\rangle \langle i|\right) \\
&= \sum_{ij} |\lambda_i| \lambda_j |i\rangle \langle i|j\rangle \langle j|
\end{aligned}
$$

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Problem 3. *Problem 2.50*

Solution.

$$
A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}
$$

First, consider

$$
A^\dagger A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad AA^\dagger = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}
$$

$$
J = \sqrt{A^\dagger A} =
$$

■

Problem 4. *Problem 2.51*

Solution. The Hadamard gate H is unitary if $H^\dagger = H^{-1}$. It is easy to see that

$$
H^\dagger = H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
$$

It's inverse is

$$
H^{-1} = -\frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = H
$$

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Problem 5. *Problem 2.52*

Solution.
$$H^2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

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Problem 6. *Problem 2.53*

Solution. Writing out the characteristic equation gives that the eigenvalues are $\lambda = \pm\sqrt{2}$. ■

Problem 7. *Problem 2.54*

Solution. Since the two operators commute, they are simultaneously diagonalizable. Consider the following spectral decompositions

$$A = \sum_n a_n \left|n\right\rangle \left\langle n\right|$$

$$B = \sum_n b_n \left|n\right\rangle \left\langle n\right|$$

Therefore, it must be true that

$$A + B = \sum_n (a_n + b_n) \left|n\right\rangle \left\langle n\right|$$

Now these matrices are Hermitian so their eigenvectors are orthogonal, and the product of matrix exponentials is just

$$\begin{aligned}
\exp(A)\exp(B) &= \left(\sum_n \exp(a_n)\left|n\right\rangle\left\langle n\right|\right)\left(\sum_m \exp(b_m)\left|m\right\rangle\left\langle m\right|\right)\\
&= \sum_{m,n} \delta_{mn}\exp(a_n)\exp(b_m)\left|n\right\rangle\left\langle m\right|\\
&= \sum_n \exp(a_n)\exp(b_n)\left|n\right\rangle\left\langle n\right|\\
&= \sum_n \exp(a_n + b_n)\left|n\right\rangle\left\langle n\right|\\
&= \exp(A + B)
\end{aligned}$$

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Problem 8. *Problem 2.55*

Solution.

$$\begin{aligned} UU^\dagger &= \exp\left(\frac{-iH(t_2-t_1)}{\hbar}\right)\exp\left(\frac{iH(t_2-t_1)}{\hbar}\right) \\ &= \left(\sum_n \exp\!\left(\frac{-iE_n(t_2-t_1)}{\hbar}\right)|n\rangle\langle n|\right)\left(\sum_m \exp\!\left(\frac{iE_m(t_2-t_1)}{\hbar}\right)|m\rangle\langle m|\right) \\ &= \sum_{m,n}\delta_{mn}|n\rangle\langle m| \\ &= \sum_n |n\rangle\langle n| = I \end{aligned}$$

where H is a Hermitian operator.

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Problem 9. *Problem 2.56*

Solution.

U is unitary so its eigenvalues u_n have unit norm, which means

$$K = -i\log(U) = -i\sum_n \log(u_n)|n\rangle\langle n| = \sum_n \theta\,|n\rangle\langle n|$$

since

$$\log(u_n) = \log\big(|u_n|e^{i\theta}\big) = \log(|u_n|) + i\theta = i\theta$$

Therefore, $K = K^\dagger$ since $\theta\in\mathbb{R}$.

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Problem 10. *Problem 2.57*

Solution.

$$L_l\,|\alpha\rangle = \frac{\ell\,|l\rangle}{|\ell|}$$

$$M_m\frac{\ell\,|l\rangle}{|\ell|} = \frac{m\ell}{|m||\ell|}\,|m\rangle$$

which is equivalent to

$$\begin{aligned} N_{m\ell}\,|\alpha\rangle &= M_m L_\ell\,|\alpha\rangle \\ &= \frac{|m\rangle\,\langle m|\ell\rangle\,\langle \ell|}{|m||\ell|}\,|\alpha\rangle \\ &= \frac{\ell\,|m\rangle\,\langle m|}{|m||\ell|}\,|\ell\rangle \\ &= \frac{m\ell}{|m||\ell|}\,|m\rangle \end{aligned}$$

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Problem 11. *Problem 2.58*

Solution.

Since the system is in an eigenstate of M with eigenvalue m, the average will be m

$$\langle M\rangle = \langle m|\,M\,|m\rangle = \langle m|\,m\,|m\rangle = m$$

The variance must then be zero

$$\begin{aligned} (\Delta M)^2 &= \langle M^2\rangle - \langle M\rangle^2 \\ &= m^2 - m^2 = 0 \end{aligned}$$

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Problem 12. *Problem 2.59*

Solution.

$$\langle 0|\,X\,|0\rangle = \langle 0|1\rangle = 0$$

$$\begin{aligned} (\Delta X)^2 &= \langle X^2\rangle - \langle X\rangle^2 \\ &= \langle X^2\rangle \\ &= \langle 0|\,X^2\,|0\rangle \\ &= 1 \end{aligned}$$

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Problem 13. *Problem 2.60*

Solution.

$$\begin{aligned}
\vec{v} \cdot \sigma &= \begin{pmatrix} v_z & v_x - iv_y \\ v_x + iv_y & -v_z \end{pmatrix} \\
&= v_z \left(|0\rangle \langle 0| - |1\rangle \langle 1| \right) + (v_1 - iv_2) |0\rangle \langle 1| + (v_1 + iv_2) |1\rangle \langle 0|
\end{aligned}$$

from the outer product representations of $\sigma_x, \sigma_y, \sigma_z$. The corresponding characteristic equation is

$$\lambda^2 - (v_z^2 + v_y^2 + v_x^2) = 0$$

If $\vec{v}$ is normalized then $\lambda = \pm 1$. We now show that the projectors onto the respective eigenspaces are $P_\pm = (I \pm \vec{v} \cdot \sigma)/2$. Let $|\pm\rangle$ be the eigenvectors of $\vec{v} \cdot \sigma$ with eigenvalues ± 1, respectively.

$$\begin{aligned}
P_+ &= |+\rangle \langle +| \\
&= \frac{|+\rangle \langle +| + |-\rangle \langle -| + |+\rangle \langle +| - |-\rangle \langle -|}{2} \\
&= \frac{I + \vec{v} \cdot \sigma}{2}
\end{aligned}$$

since by spectral decomposition we know that $\vec{v} \cdot \sigma = |+\rangle \langle +| - |-\rangle \langle -|$. Of course, we also have that

$$\begin{aligned}
P_- &= |-\rangle \langle -| \\
&= \frac{|+\rangle \langle +| + |-\rangle \langle -| - |+\rangle \langle +| + |-\rangle \langle -|}{2} \\
&= \frac{I - \vec{v} \cdot \sigma}{2}
\end{aligned}$$

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Problem 14. *Problem 2.61*

Solution.
Let $|0\rangle$ and $|1\rangle$ be the eigenvectors of σ_z.

$$
\begin{aligned}
p(+) &= |c_+|^2 \\
&= \langle 0| P_+ |0\rangle \\
&= \langle 0| \frac{I + \vec{v}\cdot\sigma}{2} |0\rangle \\
&= \frac{1}{2}\left(1 + \langle 0| \left(v_3 \left(|0\rangle\langle 0| - |1\rangle\langle 1|\right) + (v_1 - iv_2)|0\rangle\langle 1| + (v_1 + iv_2)|1\rangle\langle 0|\right)|0\rangle\right) \\
&= \frac{1}{2}(1 + v_3)
\end{aligned}
$$

The state of the system must be then in the eigenvector $|+\rangle$ of $\vec{v}\cdot\sigma$ with eigenvalue $+1$. This can be conveniently obtained by applying the measurement operator P_+, which was obtained in the last problem. Consider,

$$
\begin{aligned}
P_+|0\rangle &= \frac{I + \vec{v}\cdot\sigma}{2}|0\rangle \\
&= \frac{1}{2}\left((1 + v_3)|0\rangle + (v_1 + iv_2)|1\rangle\right)
\end{aligned}
$$

Then applying the appropriate normalization, we get

$$
|+\rangle = \frac{1}{2\sqrt{(1+v_3)/2}}\left((1 + v_3)|0\rangle + (v_1 + iv_2)|1\rangle\right)
$$

$\blacksquare$