Homework 3

Quantum Mechanics
February 8, 2023 C Seitz

Problem 1. *Problem 2.48*

Solution. ■

Problem 2. *Problem 2.49*

Solution. ■

Problem 3. *Problem 2.50*

Solution. ■

Problem 4. *Problem 2.51*

Solution. The Hadamard gate H is unitary if $H^\dagger = H^{-1}$. It is easy to see that

$$H^\dagger = H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

It's inverse is

$$H^{-1} = -\frac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = H$$

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Problem 5. *Problem 2.52*

Solution.

$$H^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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Problem 6. *Problem 2.53*

Solution. Writing out the characteristic equation gives that the eigenvalues are $\lambda = \pm\sqrt{2}$. ■

Problem 7. *Problem 2.54*

Solution. Since the two operators commute, they are simultaneously diagonalizable. Consider the following spectral decompositions

$$A = \sum_n a_n |n\rangle \langle n|$$

$$B = \sum_n b_n |n\rangle \langle n|$$

Therefore, it must be true that

$$A + B = \sum_n (a_n + b_n) |n\rangle \langle n|$$

Now these matrices are Hermitian so their eigenvectors are orthogonal, and the product of matrix exponentials is just

$$\begin{aligned}
\exp(A)\exp(B) &= \left(\sum_n \exp(a_n) |n\rangle \langle n|\right)\left(\sum_m \exp(b_m) |m\rangle \langle m|\right) \\
&= \sum_{m,n} \delta_{mn} \exp(a_n)\exp(b_m) |n\rangle \langle m| \\
&= \sum_n \exp(a_n)\exp(b_n) |n\rangle \langle n| \\
&= \sum_n \exp(a_n + b_n) |n\rangle \langle n| \\
&= \exp(A + B)
\end{aligned}$$

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Problem 8. *Problem 2.55*

Solution.

$$
\begin{aligned}
UU^\dagger &= \exp\left(\frac{-iH(t_2-t_1)}{\hbar}\right)\exp\left(\frac{iH(t_2-t_1)}{\hbar}\right) \\
&= \left(\sum_n \exp\!\left(\frac{-iE_n(t_2-t_1)}{\hbar}\right)|n\rangle\langle n|\right)\left(\sum_m \exp\!\left(\frac{iE_m(t_2-t_1)}{\hbar}\right)|m\rangle\langle m|\right) \\
&= \sum_{m,n}\delta_{mn}|n\rangle\langle m| \\
&= \sum_n |n\rangle\langle n| = I
\end{aligned}
$$

where H is a Hermitian operator.

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Problem 9. *Problem 2.56*

Solution.

U is unitary so its eigenvalues u_n have unit norm, which means

$$K = -i\log(U) = -i\sum_n \log(u_n)|n\rangle\langle n| = \sum_n \theta\,|n\rangle\langle n|$$

since

$$\log(u_n) = \log\!\left(|u_n|e^{i\theta}\right) = \log(|u_n|) + i\theta = i\theta$$

Therefore, $K = K^\dagger$ since $\theta \in \mathbb{R}$.

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Problem 10. *Problem 2.57*

Solution. ■

Problem 11. *Problem 2.58*

Solution. ■

Problem 12. *Problem 2.59*

Solution. ■

Problem 13. *Problem 2.60*

Solution. ■

Problem 14. *Problem 2.61*

Solution. ■