The variational autoencoder

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Variational Bayes

The variable $\mathbf{x}$ has a latent representation or code $\mathbf{z}$. We often say that $\mathbf{z}$ is the *causal source* of $\mathbf{x}$. The distribution we are after is the *evidence* in Bayes rule $P(\mathbf{x})$

$$P(\mathbf{x}) = \frac{P_{\Phi}(\mathbf{x}|\mathbf{z})P_{\Omega}(\mathbf{z})}{Q_{\Psi}(\mathbf{z}|\mathbf{x})}$$

It is common to take $P_{\Omega}(\mathbf{z})$ to be Gaussian. We then try find the model parameters $\Theta = (\Phi, \Psi)$ that maximize the likelihood of the observed data:

$$\Theta^{*} = \underset{\Theta}{\mathrm{argmin}} - \log P(\mathbf{x}_{\mathrm{obs}})$$

Computing the evidence

We can alternatively write the evidence as

$$\begin{aligned} P(\mathbf{x}) &= \int P_\Omega(\mathbf{z}) P_\Phi(\mathbf{x}|\mathbf{z}) d\mathbf{z} \\ &= \int P_\Omega(\mathbf{z}) P_\Phi(\mathbf{x}|\mathbf{z}) \frac{P_\Psi(\mathbf{z}|\mathbf{x})}{P_\Psi(\mathbf{z}|\mathbf{x})} d\mathbf{z} \\ &= \mathbb{E}_{\mathbf{z}\sim P_\Psi(\mathbf{z}|\mathbf{x})} \frac{P_\Omega(\mathbf{z}) P_\Phi(\mathbf{x}|\mathbf{z})}{P_\Psi(\mathbf{z}|\mathbf{x})} \end{aligned}$$

We call Ψ and Φ the encoder and decoder, respectively

The evidence lower bound (ELBO)

$$\log P(\mathbf{x}) = \log \ \mathbb{E}_{\mathbf{z}\sim P_\Psi(\mathbf{z}|\mathbf{x})} \frac{P_\Omega(\mathbf{z})P_\Phi(\mathbf{x}|\mathbf{z})}{P_\Psi(\mathbf{z}|\mathbf{x})}$$

$$\geq \mathbb{E}_{\mathbf{z}\sim P_\Psi(\mathbf{z}|\mathbf{x})} \log \ \frac{P_\Omega(\mathbf{z})P_\Phi(\mathbf{x}|\mathbf{z})}{P_\Psi(\mathbf{z}|\mathbf{x})}$$

$$-\log P(\mathbf{x}) \leq \mathbb{E}_{\mathbf{z}\sim P_\phi(\mathbf{z}|\mathbf{x})} \log \frac{P_\Psi(\mathbf{z}|\mathbf{x})}{P_\Omega(\mathbf{z})} - \log P_\Phi(\mathbf{x}|\mathbf{z})$$

The ELBO objective

$$\Theta^* = \underset{\Phi,\Psi}{\text{argmin}} \;\; \mathbb{E}_{\mathbf{x}\sim\text{Pop},\, \mathbf{z}\sim P_\Psi(\mathbf{z}|\mathbf{x})} \log \frac{P_\Psi(\mathbf{z}|\mathbf{x})}{P_\Omega(\mathbf{z})} - \log P_\Phi(\mathbf{x}|\mathbf{z})$$

$$= \underset{\Phi,\Psi}{\text{argmin}} \;\; D_{\text{KL}}\left(P_\Psi || P_\Omega\right) - \mathbb{E}_{\mathbf{x}\sim\text{Pop},\, \mathbf{z}\sim P_\Psi(\mathbf{z}|\mathbf{x})} \log P_\Phi(\mathbf{x}|\mathbf{z})$$

The first term is a **rate term** to be minimized and the second a **reconstruction term** to be maximized

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