# Homework 6

**Quantum Mechanics**
*October 28th, 2022* C Seitz

**Problem 1.** *Problem 3.12 from Sakurai*

**Solution**.

In general the ensemble average of an operator $[A]$ is defined as

$$[A] = \mathrm{Tr}(\rho A)$$

where $\hat{\rho} = \sum_i w_i \rho_i$ and $\rho_i = |\alpha_i\rangle \langle\alpha_i|$

$$\begin{aligned}\hat{\rho} &= a\,|+\rangle\langle+| + (1-a)\,|-;y\rangle\langle-;y| \\ &= \begin{pmatrix} \frac{1-a}{2}+a & \frac{1}{2}i(1-a) \\ -\frac{1}{2}i(1-a) & \frac{1-a}{2} \end{pmatrix}\end{aligned}$$

$$\begin{aligned}[S_x] &= \mathrm{Tr}(\hat{\rho}S_x) \\ &= \frac{\hbar}{2}\mathrm{Tr}\left(\begin{pmatrix} \frac{1}{2}i(1-a) & \frac{1-a}{2}+a \\ \frac{1-a}{2} & -\frac{1}{2}i(1-a) \end{pmatrix}\right) = 0\end{aligned}$$

$$\begin{aligned}[S_y] &= \mathrm{Tr}(\hat{\rho}S_x) \\ &= \frac{\hbar}{2}\mathrm{Tr}\left(\begin{pmatrix} \frac{a-1}{2} & -i\left(\frac{1-a}{2}+a\right) \\ \frac{1}{2}i(1-a) & \frac{a-1}{2} \end{pmatrix}\right) = \frac{\hbar}{2}(a-1)\end{aligned}$$

$$\begin{aligned}[S_z] &= \mathrm{Tr}(\hat{\rho}S_z) \\ &= \frac{\hbar}{2}\mathrm{Tr}\left(\begin{pmatrix} \frac{1-a}{2}+a & -\frac{1}{2}i(1-a) \\ -\frac{1}{2}i(1-a) & \frac{a-1}{2} \end{pmatrix}\right) = \frac{\hbar}{2}a\end{aligned}$$

When $a = 1$, we get $[S_x] = 0, [S_y] = 0, [S_z] = \hbar/2$, which makes sense since it is then a pure ensemble in $|+\rangle$. When $a = 0$, we get $[S_x] = 0, [S_y] = -\hbar/2, [S_z] = 0$, which makes sense because it is a pure ensemble in $|-;y\rangle$.

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**Problem 2.** *Problem 3.13 from Sakurai*

**Solution**.

The state vector in the $S_z$ basis has the form

$$|\alpha\rangle = c_+ |+\rangle + c_- |-\rangle$$

First note that

$$\langle S_z\rangle = |c_+|^2 - |c_-|^2 \quad |c_+|^2 + |c_-|^2 = 1$$

Together, these equations tell us the magnitude of each complex component.

$$|c_+|^2 = \frac{\langle S_z\rangle + 1}{2} \quad |c_-|^2 = \frac{1 - \langle S_z\rangle}{2}$$

$$\begin{aligned}
\langle S_x\rangle &= \langle\alpha| (|+\rangle \langle -| + |-\rangle \langle +|)(c_+ |+\rangle + c_- |-\rangle) \\
&= \langle\alpha| (c_- |+\rangle + c_+ |-\rangle) \\
&= (c_+^* \langle +| + c_-^* \langle -|)(c_- |+\rangle + c_+ |-\rangle) \\
&= c_+^* c_- + c_-^* c_+ \\
&= |c_+||c_-|(e^{i(\theta-\phi)} + e^{i(\phi-\theta)}) \\
&= 2|c_+||c_-|\cos(\theta-\phi)
\end{aligned}$$

Let $\delta = \theta - \phi$, which means $\delta = \cos^{-1}\left(\frac{\langle S_x\rangle}{2|c_+||c_-|}\right)$

$$\begin{aligned}
\langle S_y\rangle &= \langle\alpha| ((i |+\rangle \langle -| - i |-\rangle \langle +|)(c_+ |+\rangle + c_- |-\rangle) \\
&= i \langle\alpha| (c_- |+\rangle - c_+ |-\rangle) \\
&= i(c_+^* \langle +| + c_-^* \langle -|)(c_- |+\rangle - c_+ |-\rangle) \\
&= c_+^* c_- - c_-^* c_+ \\
&= |c_+||c_-|(e^{i(\theta-\phi)} - e^{i(\phi-\theta)}) \\
&= 2i|c_+||c_-|\sin(\theta-\phi) \\
&= 2i|c_+||c_-|\sin(\delta)
\end{aligned}$$

So $\langle S_x \rangle$ gives us the phase difference of $c_+$ and $c_-$. Then the sign of $\langle S_y \rangle$ tells us the sign of $\delta$, since sine is odd. This is all we can hope to extract from the expectation values, since multiplying by a global phase $e^{i\delta} |\alpha\rangle$ has no effect on the expectation values. To find $\rho$ using $[S_x], [S_y], [S_z]$, first note that

$$\mathrm{Tr}(\rho) = \mathrm{Tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a + d = 0$$

$$\mathrm{Tr}\left(\rho S_x\right) = \frac{\hbar}{2}\mathrm{Tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \frac{\hbar}{2}(c + b)$$

$$\mathrm{Tr}\left(\rho S_y\right) = \frac{\hbar}{2}\mathrm{Tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right) = \frac{i\hbar}{2}(b - c)$$

$$\mathrm{Tr}\left(\rho S_z\right) = \frac{\hbar}{2}\mathrm{Tr}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) = \frac{\hbar}{2}(a - d)$$

which gives us four equations for the four unknown elements of $\rho$.

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**Problem 3.** *Problem 3.14 from Sakurai*

**Solution**.

$$\begin{aligned} \hat{\rho} &= \sum_i w_i |\psi_i\rangle \langle\psi_i| \\ &= \frac{1}{3}\left(|\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta| + |2\rangle\langle 2|\right) \end{aligned}$$

We can write this out explicitly in the subspace spanned by $|0, 1, 2\rangle$

$$|\alpha\rangle\langle\alpha| = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad |\beta\rangle\langle\beta| = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad |2\rangle\langle 2| = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\hat{\rho} = \frac{1}{6}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

Now recall that $H = \hbar\omega(N + \frac{1}{2})$ which reads

$$H = \hbar\omega \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \left(\frac{\hbar\omega}{2}\right) \mathbb{I}_{3\times 3}$$

$$\begin{aligned} [H] = \mathrm{Tr}(\rho H) &= \hbar\omega \mathrm{Tr}(\rho N + \rho/2) \\ &= \hbar\omega \left(\mathrm{Tr}(\rho N) + \mathrm{Tr}(\rho/2)\right) \\ &= \frac{11}{6}\hbar\omega \end{aligned}$$

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**Problem 4.** *Problem 3.15 from Sakurai*

**Solution**.

$$\rho(t_0) = \sum_i w_i \left|\psi_i; t_0\right\rangle \left\langle\psi_i; t_0\right|$$

(a) In the Schrodinger picture, the coefficients of the state vectors evolve. Therefore,

$$\begin{aligned} \rho(t) &= \sum_i w_i\, \mathcal{U}(t, t_0) \left|\psi_i; t_0\right\rangle \left\langle\psi_i; t_0\right| \mathcal{U}^\dagger(t, t_0) \\ &= \mathcal{U}(t, t_0) \left(\sum_i w_i \left|\psi_i; t_0\right\rangle \left\langle\psi_i; t_0\right|\right) \mathcal{U}^\dagger(t, t_0) \\ &= \mathcal{U}(t, t_0)\rho(t_0)\mathcal{U}^\dagger(t, t_0) \end{aligned}$$

(b) Note that this does not mean that the states do not evolve in time. Rather, they must evolve in the same way so that the ensemble remains pure. For example, a magnetic field along $z$ can cause precession of a state prepared in $|S_n; +\rangle$, but every member of the ensemble evolves identically. We therefore need the more general property that $\rho^2 = 1$. So we write

$$\begin{aligned} \rho^2(t) &= \mathcal{U}(t, t_0)\rho(t_0)\mathcal{U}^\dagger(t, t_0)\mathcal{U}(t, t_0)\rho(t_0)\mathcal{U}^\dagger(t, t_0) \\ &= \mathcal{U}(t, t_0)\rho^2(t_0)\mathcal{U}^\dagger(t, t_0) \\ &= \mathcal{U}(t, t_0)\mathcal{U}^\dagger(t, t_0) = 1 \end{aligned}$$

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**Problem 5.** *Problem 3.16 from Sakurai*

**Solution**.

We can write a general form of the density matrix

$$\hat{\rho} = \begin{pmatrix} \alpha & a & b \\ a* & \beta & c \\ b* & c* & 1-\alpha-\beta \end{pmatrix}$$

which is due to the fact that the density matrix must have zero trace and must be Hermitian. This matrix has eight real parameters (two along the diagonal and 6 from the three complex numbers).

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**Problem 6.** *Problem 3.40 from Sakurai*

**Solution**. The singlet state is

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)$$

If $B$ doesn't make a measurement, $B$ will have no effect on $A$'s measurement. So the probability for $A$ to obtain $s_{1z} = \hbar/2$ is of course $1/2$.

The probability that $A$ measures $s_{1x} = \hbar/2$ in this state is also $1/2$. This is because obtaining $s_{1x} = \hbar/2$ is equiprobable for the two states in the singlet superposition.

If observer $B$ has determined that $s_{2z} = \hbar/2$, then observer $A$ must observe $s_{1z} = -\hbar/2$ since the measurement made by $B$ collapses $|\psi\rangle$ to $|-+\rangle$. Furthermore, if observer $B$ has measured $s_{2z} = \hbar/2$, then particle 1 must be in the $|+\rangle$ state (as stated before) which means $s_{1x} = \pm\hbar/2$ with equal probability.

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