# Homework 5

**Quantum Mechanics**
*March 8, 2023* C Seitz

**Problem 1.** *Problem 4.4*

**Solution**.

$$\begin{aligned} H &= e^{i\alpha} R_z(\frac{\pi}{2}) R_x(\frac{\pi}{2}) \\ &= \frac{e^{i\alpha} e^{i\pi/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{aligned}$$

Therefore, $\alpha = -\pi/4$. ■

**Problem 2.** *Problem 4.5*

**Solution**.

$$\begin{aligned} (n \cdot \sigma)^2 &= (n_x \sigma_x + n_y \sigma_y + n_z \sigma_z)^2 \\ &= n_x^2 \sigma_x^2 + n_y^2 \sigma_y^2 + n_z^2 \sigma_z^2 \\ &= (n_x^2 + n_y^2 + n_z^2) I = I \end{aligned}$$

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**Problem 3.** *Problem 4.7*

**Solution**.

Simple matrix operations can confirm that $XYX = -Y$. It follows that

$$e^{\frac{i\theta}{2}(XYX)} = e^{-\frac{i\theta}{2}Y}$$

Now recall that $U^\dagger e^A U = e^{U^\dagger A U}$, which can be proven via a series expansion. Of course $X$ is both unitary and hermitian, so we get that

$$X e^{\frac{i\theta}{2}Y} X = e^{-\frac{i\theta}{2}Y}$$

which is the desired result.

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**Problem 4.** *Problem 4.16*

**Solution**. For the first circuit, the matrix representation is

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & h_{11} & h_{12} \\ 0 & 0 & h_{21} & h_{22} \end{pmatrix}$$

For the second circuit, the matrix representation is

$$A = \begin{pmatrix} h_{11} & h_{12} & 0 & 0 \\ h_{21} & h_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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**Problem 5.** *Problem 4.17*

**Solution**.

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**Problem 6.** *Problem 4.19*

**Solution**.

The density matrix is

$$\rho = \sum_i p_i |\psi_i\rangle \langle\psi_i|$$

We can see how the gate transforms the density matrix by just considering how it acts on the most general state $|\psi_i\rangle = \alpha_i |00\rangle + \beta_i |01\rangle + \gamma_i |10\rangle + \delta_i |11\rangle$.

$$\text{CNOT} = |00\rangle \langle 00| + |01\rangle \langle 01| + |11\rangle \langle 10| + |10\rangle \langle 11|$$

It is straightforward to see that

$$\begin{aligned} \text{CNOT} |\psi_i\rangle \langle\psi_i| &= (\alpha_i |00\rangle + \beta_i |01\rangle + \gamma_i |11\rangle + \delta_i |10\rangle) \\ &\quad * (\alpha_i^* \langle 00| + \beta_i^* \langle 01| + \gamma_i^* \langle 10| + \delta_i^* \langle 11|) \\ &= \begin{pmatrix} |\alpha_i|^2 & \alpha_i\beta_i^* & \alpha_i\gamma_i^* & \alpha_i\delta_i^* \\ \beta_i\alpha_i^* & |\beta_i|^2 & \beta_i\gamma_i^* & \beta_i\delta_i^* \\ \delta_i\alpha_i^* & \delta_i\beta_i^* & \delta_i\gamma_i^* & |\delta_i|^2 \\ \gamma_i\alpha_i^* & \gamma_i\beta_i^* & |\gamma_i|^2 & \gamma_i\delta_i^* \end{pmatrix} \end{aligned}$$

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**Problem 7.** *Problem 4.20*

**Solution**. Writing the circuit identity out in symbols,

$$H^{\otimes 2} * \text{CNOT} * H^{\otimes 2} \left( |\psi\rangle \otimes |\phi\rangle \right) = \text{CNOT} \left( |\psi\rangle \otimes |\phi\rangle \right)$$

where $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$ and $|\phi\rangle = \gamma |0\rangle + \delta |1\rangle$. We can start with the left hand side. After the first Hadamard gate, the qubits are

$$|\psi'\rangle = \frac{\alpha + \beta}{\sqrt{2}} |0\rangle + \frac{\alpha - \beta}{\sqrt{2}} |1\rangle$$

$$|\phi'\rangle = \frac{\gamma + \delta}{\sqrt{2}} |0\rangle + \frac{\gamma - \delta}{\sqrt{2}} |1\rangle$$

Now, we write

$$\begin{aligned} |\psi'\rangle \otimes |\phi'\rangle = & \frac{(\alpha+\beta)(\gamma+\delta)}{2} |00\rangle + \frac{(\alpha+\beta)(\gamma-\delta)}{2} |01\rangle \\ & + \frac{(\alpha-\beta)(\gamma+\delta)}{2} |10\rangle + \frac{(\alpha-\beta)(\gamma-\delta)}{2} |11\rangle \end{aligned}$$

$$\begin{aligned} \text{CNOT}(|\psi'\rangle \otimes |\phi'\rangle) &= \left( |00\rangle \langle 00| + |01\rangle \langle 01| + |11\rangle \langle 10| + |10\rangle \langle 11| \right) \left( |\psi'\rangle \otimes |\phi'\rangle \right) \\ &= \frac{(\alpha+\beta)(\gamma+\delta)}{2} |00\rangle + \frac{(\alpha+\beta)(\gamma-\delta)}{2} |01\rangle \\ &+ \frac{(\alpha-\beta)(\gamma+\delta)}{2} |11\rangle + \frac{(\alpha-\beta)(\gamma-\delta)}{2} |10\rangle \end{aligned}$$

For the right hand side

$$\begin{aligned} \text{CNOT} \left( |\psi\rangle \otimes |\phi\rangle \right) &= \text{CNOT} \left( \alpha\gamma |00\rangle + \alpha\delta |01\rangle + \beta\gamma |10\rangle + \beta\delta |11\rangle \right) \\ &= \alpha\gamma |00\rangle + \alpha\delta |01\rangle + \beta\gamma |11\rangle + \beta\delta |10\rangle \end{aligned}$$

Using the circuit identity, when the first qubit is control and the second qubit is target, we see that

$$\begin{aligned} \text{CNOT} |{++}\rangle &= H^{\otimes 2} * \text{CNOT} * H^{\otimes 2} |{++}\rangle \\ &= H^{\otimes 2} * \text{CNOT} |00\rangle \\ &= H^{\otimes 2} |00\rangle \\ &= |{++}\rangle \end{aligned}$$

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\begin{aligned}
\text{CNOT}\,|{+}{-}\rangle &= H^{\otimes 2} * \text{CNOT} * H^{\otimes 2}\,|{+}{-}\rangle \\
&= H^{\otimes 2} * \text{CNOT}\,|01\rangle \\
&= H^{\otimes 2}\,|01\rangle \\
&= |{+}{-}\rangle
\end{aligned}
$$

$$
\begin{aligned}
\text{CNOT}\,|{-}{+}\rangle &= H^{\otimes 2} * \text{CNOT} * H^{\otimes 2}\,|{-}{+}\rangle \\
&= H^{\otimes 2} * \text{CNOT}\,|10\rangle \\
&= H^{\otimes 2}\,|10\rangle \\
&= |{+}{+}\rangle
\end{aligned}
$$

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\begin{aligned}
\text{CNOT}\,|{-}{-}\rangle &= H^{\otimes 2} * \text{CNOT} * H^{\otimes 2}\,|{-}{-}\rangle \\
&= H^{\otimes 2} * \text{CNOT}\,|11\rangle \\
&= H^{\otimes 2}\,|10\rangle \\
&= |{+}{-}\rangle
\end{aligned}
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So in this case, we can see that target qubit is unchanged while the control qubit is flipped in the last two cases. Therefore, what we call control and target are dependent on the basis we think of the device as operating in.

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