Homework 4

Quantum Mechanics
Sept 22nd, 2022 Clayton Seitz

Problem 1. *Problem 2.14 from Sakurai*

Solution.

We are given that the state vector is

$$|\alpha\rangle = \exp\left(\frac{-ipa}{\hbar}\right)|0\rangle$$

The Heisenberg equation of motion reads

$$\frac{dx}{dt} = \frac{1}{i\hbar}[x, H] = 0$$

Therefore $x = x_0$ for all $t \geq t_0$

$$\begin{aligned}
\langle x\rangle &= \int x_0 \langle x|\alpha\rangle \langle\alpha|x\rangle\, dx \\
&= \int x \exp\left(\frac{-ipa}{\hbar}\right)\langle x|0\rangle \exp\left(\frac{ipa}{\hbar}\right)\langle 0|x\rangle\, dx \\
&= \int x_0 \,|\langle x|0\rangle|^2\, dx \\
&= \int x_0 \,|\langle x|0\rangle|^2\, dx
\end{aligned}$$

We could write out $\langle x|0\rangle$, its complex conjugate, and do the integral. Instead recall the general expression for the matrix element of x

$$\langle n'|\, x\, |n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\delta_{n',n-1} + \sqrt{n+1}\delta_{n',n+1}\right)$$

which is zero when $n = n'$ which means that $\langle x \rangle = 0$

$\blacksquare$

Problem 2. *Problem 2.15 from Sakurai*

Solution. We were given the state

$$|\alpha\rangle = \exp\left(\frac{-ipa}{\hbar}\right)|0\rangle$$

$$\langle x|\alpha\rangle = \pi^{-1/4} x_0^{1/2} \exp\left(\frac{-ipa}{\hbar}\right) \exp\left(-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right)$$

where $x_0 = \sqrt{\frac{\hbar}{m\omega}}$. The Hamiltonian operator $\hat{H}$ is independent of time so we have the unitary time evolution operator

$$\mathcal{U}(t) = \exp\left(-\frac{i\hat{H}t}{\hbar}\right)$$

Assuming $|\alpha\rangle$ is expressed in the energy basis, this can be alternatively be written as the power series

$$\mathcal{U}(t) = \sum_{n=0}^{\infty} \frac{\hat{H}^n}{n!} \rightarrow \mathcal{U}(t)|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\hat{H}^n}{n!}|\alpha\rangle$$

$$\sum_{n=0}^{\infty} \frac{\alpha^n}{n!}|\alpha\rangle = \sum_{n} \exp\left(\frac{-i\alpha_n t}{\hbar}\right)|\alpha_n\rangle$$

The probability that $|\alpha\rangle$ is measured to be in the state $|0\rangle$ is

$$\langle 0|\alpha\rangle\langle\alpha|0\rangle = \exp\left(\frac{-ipa}{\hbar}\right)\langle 0|0\rangle \exp\left(\frac{ipa}{\hbar}\right)\langle 0|0\rangle = 1$$

This probability does not change for $t > 0$. This is clear when we look at the state

$$|\alpha; t\rangle = \exp\left(-\frac{iE_0 t}{\hbar}\right) \exp\left(\frac{-ipa}{\hbar}\right) |0\rangle$$

The second exponential is just a complex number and is time independent. The first exponential is just a phase, which is not measurable directly. In other words, when we hit this state with the dual ket $\langle 0|$, the phase goes away and we are left with a time-independent probability density.

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Problem 3. *Problem 2.16 from Sakurai*

Solution.

We will assume the form of the annihilation and creation operators

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(x + \frac{ip}{m\omega}\right)$$
$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(x - \frac{ip}{m\omega}\right)$$

Adding these equations gives and rearranging we can express x as

$$x = \sqrt{\frac{\hbar}{2m\omega}} \left(a + a^\dagger\right)$$

$$\begin{aligned}
\langle m| x |n\rangle &= \langle m| \sqrt{\frac{\hbar}{2m\omega}} \left(a + a^\dagger\right) |n\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}} \left(\langle m| a |n\rangle + \langle m| a^\dagger |n\rangle\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n}\delta_{m,n-1} + \sqrt{n+1}\delta_{m,n+1}\right)
\end{aligned}$$

Subtracting the creation operator from the annihalation operator allows us to write the momentum operator as

$$p = i\sqrt{\frac{m\hbar\omega}{2}} \left(a^\dagger - a\right)$$

$$\begin{aligned}
\langle m| p |n\rangle &= \langle m| \left(i\sqrt{\frac{m\hbar\omega}{2}} \left(a^\dagger - a\right) \right) |n\rangle \\
&= \left(i\sqrt{\frac{m\hbar\omega}{2}} \left(\langle m| a^\dagger |n\rangle - \langle m| a |n\rangle \right) \right) \\
&= i\sqrt{\frac{m\hbar\omega}{2}} \left(\sqrt{n+1}\delta_{m,n+1} - \sqrt{n}\delta_{m,n-1} \right)
\end{aligned}$$

$$\begin{aligned}
\langle m| \{x,p\} |n\rangle &= \langle m| xp |n\rangle + \langle m| px |n\rangle \\
&= \frac{i\hbar}{2} \langle m| \left((a^\dagger)^2 - a^2 \right) |n\rangle + \frac{i\hbar}{2} \langle m| \left((a^\dagger)^2 + a^\dagger a - a a^\dagger - a^2 \right) |n\rangle \\
&= \frac{i\hbar}{2} \left(\sqrt{n+1}\sqrt{n+2}\delta_{m,n+2} - \sqrt{n}\sqrt{n-1}\delta_{m,n-2} \right) \\
&+ \frac{i\hbar}{2} (\sqrt{n+1}\sqrt{n+2}\delta_{m,n+2} + \sqrt{n}\sqrt{n-1}\delta_{m,n-2})
\end{aligned}$$

$$\langle m| x^2 |n\rangle = \frac{\hbar}{2m\omega} \langle m| \left(a^2 + a a^\dagger + a^\dagger a + (a^\dagger)^2 \right) |n\rangle$$

$$\langle m| p^2 |n\rangle = -\frac{m\hbar\omega}{2} \langle m| \left((a^\dagger)^2 + a^\dagger a - a a^\dagger - a^2 \right) |n\rangle$$

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Problem 4. *Problem 2.28 from Sakurai*

Solution.

First of all, the solution is not trivial since x does not commute with the Hamiltonian since $[x,p^2] \neq 0$. At $t = t_0$ we are in the position eigenstate

$$\langle x|\alpha; t_0\rangle = \delta\left(x - \frac{L}{2}\right)$$

Since this is the infinite square well, we have the following energy eigenstates, in the position representation

$$\langle x|\alpha\rangle = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$$

Of course $|\alpha; t_0\rangle$ is not an eigenstate of H, so this state will measurably evolve in time. The state $|\alpha; t_0\rangle$ in the energy basis is

$$\begin{aligned}
|\beta\rangle &= \sum_n |\epsilon_n\rangle \langle\epsilon_n|\alpha; t_0\rangle \\
&= \sqrt{\frac{2}{L}}\sum_n \sin\left(\frac{n\pi}{2}\right)|\epsilon_n\rangle
\end{aligned}$$

From this, we can show the probability of measuring the particle in energy eigenstate $|\epsilon_n\rangle$

$$\begin{aligned}
\langle\epsilon_m|\beta\rangle &= \sqrt{\frac{2}{L}}\sum_n \sin\left(\frac{n\pi}{2}\right)\langle\epsilon_m|\epsilon_n\rangle \\
&= \sqrt{\frac{2}{L}}\sum_n \sin\left(\frac{n\pi}{2}\right)\delta_{mn} \\
&= \sqrt{\frac{2}{L}}\sin\left(\frac{m\pi}{2}\right)
\end{aligned}$$

and therefore $|\langle\epsilon_m|\beta\rangle|^2 = \sqrt{\frac{2}{L}}\sin^2\left(\frac{m\pi}{2}\right)$. The relative probabilities with respect to the ground state are then given by

$$r_{m+1} = \sin^2\left(\frac{(m+1)\pi}{2}\right)\csc^2\left(\frac{m\pi}{2}\right)$$

Since we know a representation for $|\alpha; t_0\rangle$ in the energy basis, we can determine the time evolution of the wavefunction $\langle x|\alpha\rangle$

$$\begin{aligned}
|\alpha; t\rangle &= \mathcal{U}(t)|\beta\rangle \\
&= \sqrt{\frac{2}{L}}\sum_n \exp\left(\frac{-i\epsilon_n t}{\hbar}\right)\sin\left(\frac{n\pi}{2}\right)|\epsilon_n\rangle
\end{aligned}$$

which has the position representation (wave function)

$$\begin{aligned}
\langle x|\alpha; t\rangle &= \psi(x,t) \\
&= \sqrt{\frac{2}{L}} \sum_n \exp\left(\frac{-i\epsilon_n t}{\hbar}\right) \sin\left(\frac{n\pi}{2}\right) \langle x|\epsilon_n\rangle \\
&= \sqrt{\frac{2}{L}} \sum_n \exp\left(\frac{-i\epsilon_n t}{\hbar}\right) \sin\left(\frac{n\pi}{2}\right) \psi_n(x)
\end{aligned}$$

where $\psi_n(x)$ are the energy eigenstates given above.

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Problem 5. *Problem 2.29 from Sakurai*

Solution.

We are free to choose our zero of potential so we can solve an alternative system where $V(x) = \nu_0 - \delta(x)\nu_0$. We then decompose $\psi(x)$ into two regions $\psi_I(x), \psi_{II}(x)$ where

$$\begin{cases} I : x < 0, \\ II : x > 0, \end{cases}$$

We then solve Schrodingers equation in each region. Both regions have constant potential $|\nu_0|$

$$\begin{aligned}
\frac{d\psi_I^2}{dx^2} &= \frac{2m(E-\nu_0)}{\hbar^2}\psi_I(x) \\
\frac{d\psi_{II}^2}{dx^2} &= \frac{2m(E-\nu_0)}{\hbar^2}\psi_{II}(x)
\end{aligned}$$

$$\begin{aligned}
\psi_I(x) &= A\exp(\kappa x) + B\exp(-\kappa x) \\
\psi_{II}(x) &= C\exp(\kappa x) + D\exp(-\kappa x)
\end{aligned}$$

Taking limits shows that $B = 0$ and $C = 0$. From the symmetry of the potential we should have that $\psi_I' = -\psi_{II}'$ which is satisfied when $A = D$. This reduces the solution to

$$\psi_I(x) = A\exp(\kappa x)$$
$$\psi_{II}(x) = A\exp(-\kappa x)$$

Looking back at Schrodinger's equation, we can see that when $E > \nu_0$ the solutions are complex exponentials and are unbound but when $E < \nu_0$, the solutions are real exponentials and bound. There are an infinite number of bound states since E and therefore κ are continuous parameters. When E hits ν_0, we have the solutions for a free particle.

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Problem 6. *Problem 2.32 from Sakurai*

Solution. Let us define

$$\begin{aligned}
\psi_I &= A\exp(\alpha x) \\
\psi_{II} &= B\exp(ikx) + C\exp(-ikx) \\
\psi_{III} &= D\exp(-\alpha x)
\end{aligned}$$

Here α, k are constants. We can enforce continuity in the wavefunction itself at $x = -a$ and $x = +a$

$$\begin{aligned}
A\exp(-\alpha a) &= B\exp(-ika) + C\exp(ika) \\
D\exp(-\alpha a) &= B\exp(ika) + C\exp(-ika)
\end{aligned}$$

And we can also enforce continuity in the first-order derivative at these points

$$\begin{aligned}
\alpha A\exp(-\alpha a) &= ikB\exp(-ika) - ikC\exp(ika) \\
-\alpha D\exp(-\alpha a) &= ikB\exp(ika) - ikC\exp(-ika)
\end{aligned}$$

This system of four equations can be written in matrix form

$$\begin{pmatrix} e^{-\alpha a} & e^{-ika} & e^{ika} & 0 \\ 0 & e^{ika} & e^{-ika} & e^{-\alpha a} \\ \alpha e^{-\alpha a} & ike^{-ika} & ike^{ika} & 0 \\ 0 & ike^{ika} & -ike^{-ika} & -\alpha e^{-\alpha a} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = 0$$

According to Mathematica, the determinant is

$$\mathcal{D} = \exp(-2a(ik+\alpha)) \left(-\exp(4iak)(k-i\alpha)^2 + (k+i\alpha)^2\right)$$

If the determinant is zero, then a solution exists. The determinant $\mathcal{D}$ is zero when

$$\exp(-2iak)(k+i\alpha)^2 = \exp(2iak)(k-i\alpha)^2$$

Notice that we have just distributed the $\exp(-2ika)$ from the prefactor. If we let $z = \exp(-iak)(k+i\alpha)$ then the above equation just reads $z^2 = (z^*)^2$ or $z = \pm z^*$.

Considering the purely real solution first, we make the substitutions

$$\exp(-iak) \to \frac{\sqrt{k^2+\alpha^2}}{k+i\alpha}$$
$$\exp(iak) \to \frac{\sqrt{k^2+\alpha^2}}{k-i\alpha}$$

$$\begin{pmatrix} e^{-\alpha a} & \frac{-i\sqrt{k^2+\alpha^2}}{k+i\alpha} & \frac{\sqrt{k^2+\alpha^2}}{k-i\alpha} & 0 \\ 0 & \frac{\sqrt{k^2+\alpha^2}}{k-i\alpha} & \frac{-i\sqrt{k^2+\alpha^2}}{k+i\alpha} & e^{-\alpha a} \\ \alpha e^{-\alpha a} & ik\frac{-i\sqrt{k^2+\alpha^2}}{k+i\alpha} & ik\frac{\sqrt{k^2+\alpha^2}}{k-i\alpha} & 0 \\ 0 & ik\frac{\sqrt{k^2+\alpha^2}}{k-i\alpha} & -ik\frac{-i\sqrt{k^2+\alpha^2}}{k+i\alpha} & -\alpha e^{-\alpha a} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = 0$$

Now considering the purely imaginary solution, we make the substitutions

$$\exp(-iak) \to \frac{i\sqrt{k^2+\alpha^2}}{k+i\alpha}$$
$$\exp(iak) \to \frac{-i\sqrt{k^2+\alpha^2}}{k-i\alpha}$$

$$\begin{pmatrix} e^{-\alpha a} & \frac{i\sqrt{k^2+\alpha^2}}{k+i\alpha} & \frac{-i\sqrt{k^2+\alpha^2}}{k-i\alpha} & 0 \\ 0 & \frac{-i\sqrt{k^2+\alpha^2}}{k-i\alpha} & \frac{i\sqrt{k^2+\alpha^2}}{k+i\alpha} & e^{-\alpha a} \\ \alpha e^{-\alpha a} & ik\frac{i\sqrt{k^2+\alpha^2}}{k+i\alpha} & ik\frac{-i\sqrt{k^2+\alpha^2}}{k-i\alpha} & 0 \\ 0 & ik\frac{-i\sqrt{k^2+\alpha^2}}{k-i\alpha} & -ik\frac{i\sqrt{k^2+\alpha^2}}{k+i\alpha} & -\alpha e^{-\alpha a} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = 0$$

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