

The Kramers-Moyal Expansion

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1 The Kramers-Moyal Expansion

Given many instantiations of a stochastic variable V , we can construct a normalized histogram over all observations as a function of time $P(V, t)$. However, in order to systematically explore the relationship between the parameterization of the process and $P(V, t)$ we require an expression for $\dot{P}(V, t)$. If we make a fundamental assumption that the evolution of $P(V, t)$ follows a Markov process i.e. its evolution has the memoryless property, then we can write

$$P(V', t) = \int T(V', t|V, t - \tau) P(V, t - \tau) dV \quad (1)$$

which is known as the Chapman-Kolmogorov equation. The factor $T(V', t|V, t - \tau)$ is known as the *transition operator* in a Markov process and determines the evolution of $P(V, t)$ in time. We proceed by writing $T(V', t|V, t - \tau)$ in a form referred to as the Kramers-Moyal expansion

$$\begin{aligned} T(V', t|V, t - \tau) &= \int \delta(u - V') T(u, t|V, t - \tau) du \\ &= \int \delta(V + u - V' - V) T(u, t|V, t - \tau) du \end{aligned}$$

If we use the Taylor expansion of the δ -function

$$\delta(V + u - V' - V) = \sum_{n=0}^{\infty} \frac{(u - V)^n}{n!} \left(-\frac{\partial}{\partial V} \right)^n \delta(V - V')$$

Inserting this into the result from above, pulling out terms independent of u and swapping the order of the sum and integration gives

$$T(V', t|V, t - \tau) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial V} \right)^n \delta(V - V') \int (u - V)^n T(u, t|V, t - \tau) du \quad (2)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial V} \right)^n \delta(V - V') M_n(V, t) \quad (3)$$

noticing that $M_n(V, t) = \int (u - V)^n T(u, t|V, t - \tau) du$ is just the n th moment of the transition operator T . Plugging (2.6) back in to (2.4) gives

$$P(V, t) = \int \left(1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial V} \right)^n M_n(V, t) \right) \delta(V - V') P(V, t - \tau) dV \quad (4)$$

$$= P(V', t - \tau) + \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial V} \right)^n [M_n(V, t) P(V, t)] \quad (5)$$

Approximating the derivative as a finite difference and taking the limit $\tau \rightarrow 0$ gives

$$\dot{P}(V, t) = \lim_{\tau \rightarrow 0} \left(\frac{P(V, t) - P(V, t - \tau)}{\tau} \right) \quad (6)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{\partial}{\partial V} \right)^n [M_n(V, t) P(V, t)] \quad (7)$$

which is formally known as the Kramers-Moyal (KM) expansion. The Fokker-Planck equation is a special case of (2.10) where we neglect terms $n > 2$ in the *diffusion approximation*.