

# Homework 6

Quantum Mechanics

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C SEITZ

**Problem 1.** *Problem 3.12 from Sakurai*

**Solution.**

In general the ensemble average of an operator  $[A]$  is defined as

$$[A] = \sum_i w_i \langle \alpha_i | A | \alpha_i \rangle$$

where  $\sum_i w_i = 1$

$$\begin{aligned} [\sigma_x] &= a \langle + | \sigma_x | + \rangle + (1 - a) \langle -; y | \sigma_x | -; y \rangle \\ &= a \langle + | (| + \rangle \langle - | + | - \rangle \langle + |) | + \rangle + (1 - a) \langle -; y | (| + \rangle \langle - | + | - \rangle \langle + |) | -; y \rangle \\ &= 0 \end{aligned}$$

$$\begin{aligned} [\sigma_y] &= a \langle + | \sigma_y | + \rangle + (1 - a) \langle -; y | \sigma_y | -; y \rangle \\ &= ai \langle + | (| + \rangle \langle - | - | - \rangle \langle + |) | + \rangle + i(1 - a) \langle -; y | (| + \rangle \langle - | - | - \rangle \langle + |) | -; y \rangle \\ &= i(1 - a) \langle -; y | \left( -\frac{i}{\sqrt{2}} | + \rangle - \frac{1}{\sqrt{2}} | - \rangle \right) \\ &= -i(1 - a) \langle -; y | +; y \rangle = 0 \end{aligned}$$

$$\begin{aligned} [\sigma_z] &= a \langle + | (| - \rangle \langle - | - | + \rangle \langle + |) | + \rangle + i(1 - a) \langle -; y | (| - \rangle \langle - | - | + \rangle \langle + |) | -; y \rangle \\ &= -a + i(1 - a) \langle -; y | \left( -\frac{i}{\sqrt{2}} | + \rangle - \frac{1}{\sqrt{2}} | - \rangle \right) \end{aligned}$$

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**Problem 2.** *Problem 3.13 from Sakurai*

**Solution.**

The state vector in the  $S_z$  basis has the form

$$|\alpha\rangle = c_+ |+\rangle + c_- |-\rangle$$

First note that

$$\langle S_z \rangle = |c_+|^2 - |c_-|^2 \quad |c_+|^2 + |c_-|^2 = 1$$

Together, these equations tell us the magnitude of each complex component.

$$|c_+|^2 = \frac{\langle S_z \rangle + 1}{2} \quad |c_-|^2 = \frac{1 - \langle S_z \rangle}{2}$$

$$\begin{aligned} \langle S_x \rangle &= \langle \alpha | (|+\rangle \langle -| + |-\rangle \langle +|) (c_+ |+\rangle + c_- |-\rangle) \\ &= \langle \alpha | (c_- |+\rangle + c_+ |-\rangle) \\ &= (c_+^* \langle +| + c_-^* \langle -|) (c_- |+\rangle + c_+ |-\rangle) \\ &= c_+^* c_- + c_-^* c_+ \\ &= |c_+| |c_-| (e^{i(\theta-\phi)} + e^{i(\phi-\theta)}) \\ &= 2|c_+| |c_-| \cos(\theta - \phi) \end{aligned}$$

$$\begin{aligned} \langle S_y \rangle &= \langle \alpha | ((i |+\rangle \langle -| - i |-\rangle \langle +|) (c_+ |+\rangle + c_- |-\rangle) \\ &= i \langle \alpha | (c_- |+\rangle - c_+ |-\rangle) \\ &= i(c_+^* \langle +| + c_-^* \langle -|) (c_- |+\rangle - c_+ |-\rangle) \\ &= c_+^* c_- - c_-^* c_+ \\ &= |c_+| |c_-| (e^{i(\theta-\phi)} - e^{i(\phi-\theta)}) \\ &= 2i|c_+| |c_-| \sin(\theta - \phi) \end{aligned}$$

So  $\langle S_x \rangle$  gives us the phase difference of  $c_+$  and  $c_-$ . Then the sign of  $\langle S_y \rangle$  tells us whether  $\theta$  or  $\phi$  is larger, since sine is odd. This is all we can hope to extract from the expectation values, since multiplying by a global phase  $e^{i\delta} |\alpha\rangle$  has no effect on the expectation values.

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**Problem 3.** *Problem 3.14 from Sakurai*

**Solution.**

$$\begin{aligned}\rho &= \sum_i w_i |\psi_i\rangle \langle \psi_i| \\ &= \frac{1}{3} (|\alpha\rangle \langle \alpha| + |\beta\rangle \langle \beta| + |2\rangle \langle 2|)\end{aligned}$$

We can write this out explicitly in the subspace spanned by  $|0, 1, 2\rangle$

$$|\alpha\rangle \langle \alpha| = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad |\beta\rangle \langle \beta| = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad |2\rangle \langle 2| = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rho = \frac{1}{6} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Now recall that  $H = \hbar\omega(N + \frac{1}{2})$  which is

$$H = \left(\frac{\hbar\omega}{2}\right) \mathbb{I}_{3 \times 3} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$[H] = \text{Tr}(\rho H) = \hbar\omega \text{Tr}(\rho N + \rho/2) = \hbar\omega (\text{Tr}(\rho N) + \text{Tr}(\rho/2)) = \frac{17\hbar\omega}{12}$$

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**Problem 4.** *Problem 3.15 from Sakurai*

**Solution.**

$$\rho(t_0) = \sum_i w_i |\psi_i; t_0\rangle \langle \psi_i; t_0|$$

In the Schrodinger picture, the coefficients of the state vectors evolve. Therefore,

$$\begin{aligned}
\rho(t) &= \sum_i w_i \mathcal{U}(t, t_0) |\psi_i; t_0\rangle \langle \psi_i; t_0| \mathcal{U}^\dagger(t, t_0) \\
&= \mathcal{U}(t, t_0) \left( \sum_i w_i |\psi_i; t_0\rangle \langle \psi_i; t_0| \right) \mathcal{U}^\dagger(t, t_0) \\
&= \mathcal{U}(t, t_0) \rho(t_0) \mathcal{U}^\dagger(t, t_0)
\end{aligned}$$

For a pure ensemble in state  $|\psi_i\rangle$ , the density matrix is

$$\rho(t_0) = |\psi_i; t_0\rangle \langle \psi_i; t_0|$$

At a later time, the density matrix is

$$\begin{aligned}
\rho(t) &= \mathcal{U}(t, t_0) \rho(t_0) \mathcal{U}^\dagger(t, t_0) \\
&= \mathcal{U}(t, t_0) |\psi_i; t_0\rangle \langle \psi_i; t_0| \mathcal{U}^\dagger(t, t_0) \\
&= |\psi_i; t_0\rangle \langle \psi_i; t_0|
\end{aligned}$$

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**Problem 5.** *Problem 3.16 from Sakurai*

**Solution.** A 3x3 matrix has 9 parameters, but we only need

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**Problem 6.** *Problem 3.40 from Sakurai*

**Solution.** The singlet state is

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)$$

The probability is 1/2 to obtain  $s_{1z} = \hbar/2$ . The probability is 1/2 to obtain  $s_{1x} = \hbar/2$  since obtaining  $s_{1x} = \hbar/2$  is equiprobable for the two states in the singlet superposition. If observer  $B$  has determined that  $s_{2z} = \hbar/2$ , then observer  $A$  must observe  $s_{1z} = -\hbar/2$  since the measurement made by  $B$  collapses  $|\psi\rangle$  to  $|-+\rangle$ . Furthermore, if observer  $B$  has measured  $s_{2z} = \hbar/2$ , then particle 1 must be in the  $|+\rangle$  state which means  $s_{1x} = \pm\hbar/2$  with equal probability.

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