

Bell's Inequality

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March 27, 2023

Basically I want to relate the expansion coefficients of the two-qubit pure state to the degree of entanglement using entanglement entropy. Then, I want to demonstrate that entangled states can violate Bell's inequality (but I'm not sure if this is exactly correct or under what conditions). Finally, I want to show maximally entangled states that saturate the Tsirelson bound.

CHSH Inequality

Define 4 spin operators along arbitrary directions

$$Q = \vec{q} \cdot \sigma, R = \vec{r} \cdot \sigma, S = \vec{s} \cdot \sigma, T = \vec{t} \cdot \sigma.$$

Alice: Q, R Bob: S, T

Combination of correlations between Alice and Bobs measurements are bounded according to the CHSH inequality

$$E(Q \otimes S) + E(R \otimes S) + E(R \otimes T) - E(Q \otimes T) \leq 2$$

$$\text{Let } \vec{q} = (0, 0, 1), \vec{r} = (1, 0, 0), \vec{s} = (-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}), \vec{t} = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$$

Calculating expectations

$$\vec{q} \cdot \sigma \otimes \vec{s} \cdot \sigma = \begin{pmatrix} \vec{s} \cdot \sigma & 0 \\ 0 & -\vec{s} \cdot \sigma \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\vec{r} \cdot \sigma \otimes \vec{s} \cdot \sigma = \begin{pmatrix} 0 & \vec{s} \cdot \sigma \\ \vec{s} \cdot \sigma & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & 1 \\ -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}$$

$$\vec{r} \cdot \sigma \otimes \vec{t} \cdot \sigma = \begin{pmatrix} 0 & \vec{t} \cdot \sigma \\ \vec{t} \cdot \sigma & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix}$$

Calculating expectations

$$\vec{q} \cdot \sigma \otimes \vec{t} \cdot \sigma = \begin{pmatrix} \vec{t} \cdot \sigma & 0 \\ 0 & -\vec{t} \cdot \sigma \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\langle \vec{q} \cdot \sigma \otimes \vec{s} \cdot \sigma \rangle = \frac{1}{\sqrt{2}} (-\alpha^*(\alpha + \beta) + \beta^*(\beta - \alpha) + \gamma^*(\gamma + \delta) + \delta^*(\gamma - \delta))$$

$$\langle \vec{r} \cdot \sigma \otimes \vec{s} \cdot \sigma \rangle = \frac{1}{\sqrt{2}} (-\alpha^*(\gamma + \delta) + \beta^*(\delta - \gamma) - \gamma^*(\alpha + \beta) + \delta^*(\beta - \alpha))$$

$$\langle \vec{r} \cdot \sigma \otimes \vec{t} \cdot \sigma \rangle = \frac{1}{\sqrt{2}} (\alpha^*(\gamma - \delta) - \beta^*(\delta + \gamma) + \gamma^*(\alpha - \beta) - \delta^*(\beta + \alpha))$$

$$\langle \vec{q} \cdot \sigma \otimes \vec{t} \cdot \sigma \rangle = \frac{1}{\sqrt{2}} (\alpha^*(\alpha - \beta) - \beta^*(\beta + \alpha) + \gamma^*(\delta - \delta) + \delta^*(\gamma + \delta))$$

Entanglement entropy of a two-qubit system

Von Neumann entropy of a pure state is zero and cannot distinguish between entangled and separable pure states