Homework 8

Quantum Mechanics
November 13, 2022 C Seitz

Problem 1. *5.27*

Solution.

$$\frac{\langle\tilde{0}|\, H\, |\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle} \geq E_0$$

The denominator is easy to compute

$$2\int_{-\infty}^{0} e^{\beta x} dx = \frac{1}{\beta}$$

The numerator

$$\begin{aligned}
\langle\tilde{0}|\, H\, |\tilde{0}\rangle &= \int_{-\infty}^{\infty} \psi^*(x) H\psi(x) dx \\
&= 2\int_{0}^{\infty} \psi^*(x) H\psi(x) dx - \frac{\hbar^2}{2m}\lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} dx e^{-\beta|x|}\frac{\partial^2}{\partial x^2}e^{-\beta|x|}
\end{aligned}$$

The first integral is relatively straighforward:

$$\begin{aligned}
\int_{0}^{\infty} \psi^*(x) H\psi(x) dx &= \int_{0}^{\infty} -e^{-\beta x}\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}e^{-\beta x} + \frac{1}{2}m\omega^2 x^2 e^{-2\beta x} dx \\
&= \int_{0}^{\infty} e^{-2\beta x}\left(\frac{1}{2}m\omega^2 x^2 - \frac{\hbar^2\beta^2}{2m}\right) dx \\
&= \Big|_{0}^{\infty} \frac{1}{2}m\omega^2\frac{e^{-2\beta x}(1+2\beta x+2\beta^2 x^2)}{4\beta^3} - e^{-2\beta x}\frac{\hbar^2\beta}{4m} \\
&= \frac{m\omega^2}{8\beta^3} - \frac{\hbar^2\beta}{4m}
\end{aligned}$$

The second is

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} dx e^{-\beta|x|} \frac{\partial^2}{\partial x^2} e^{-\beta|x|} = \Big|_{-\epsilon}^{+\epsilon} \frac{\partial}{\partial x} e^{-\beta|x|} = -2\beta$$

Putting it all together gives

$$\bar{H} = \frac{\langle \tilde{0} | H | \tilde{0} \rangle}{\langle \tilde{0} | \tilde{0} \rangle} = \frac{m\omega^2}{4\beta^2} + \frac{\hbar^2 \beta}{2m}$$

We then minimize $\bar{H}$ w.r.t. β

$$\frac{d\bar{H}}{d\beta} = -\frac{m\omega^2}{4\beta} + \frac{\hbar^2 \beta}{m} = 0$$

and $\beta^* = \sqrt{m^2\beta^2/4\hbar^2}$

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Problem 2. *5.29*

Solution.

We have the full time-dependent Hamiltonian

$$H(t) = H_0 + F_0 x \cos \omega t$$

We need to find $|\psi(t)\rangle$, which amounts to finding the expansion coefficients $c_n(t)$. In the interaction picture, we have that

$$i\hbar \dot{c_n}(t) = \sum_m V_{nm} e^{i\omega_{nm} t} c_m(t)$$

for $\omega_{nm} = (E_n - E_m)/\hbar$.

$$\begin{aligned} V_{nm} &= F_0 \cos \omega t \langle n | x | m \rangle \\ &= F_0 \cos \omega t \sqrt{\frac{\hbar}{2m\omega_0}} \left(\sqrt{n+1} \delta_{m,n-1} + \sqrt{n} \delta_{m,n+1} \right) \end{aligned}$$

But the initial condition says that $|\psi(0)\rangle = |0\rangle$, so $n = 0$ and the only term of the summation that survives has $m = 1$. Therefore,

$$\begin{aligned} i\hbar\dot{c_1}(t) &= V_{10}e^{i\omega_0 t}c_0(t) \\ &= F_0\cos\omega t\sqrt{\frac{\hbar}{2m\omega_0}}e^{i\omega_0 t}c_0(t) \end{aligned}$$

Solving for $c_1(t)$,

$$\begin{aligned} c_1(t) &= -\frac{i}{\hbar}F_0\sqrt{\frac{\hbar}{2m\omega_0}}\int_0^t e^{i\omega_0 t}\cos\omega t dt \\ &= -\frac{i}{2\hbar}F_0\sqrt{\frac{\hbar}{2m\omega_0}}\left(\frac{e^{i(\omega_0+\omega)t}-1}{\omega_0+\omega}+\frac{e^{i(\omega_0-\omega)t}-1}{\omega_0-\omega}\right) \end{aligned}$$

Now, to compute $\langle x\rangle$, we can express the x operator in the interaction picture (or, equivalently, convert the $\left|\tilde{\psi(t)}\right\rangle$ back to $|\psi(t)\rangle$).

$$\begin{aligned} \langle x\rangle &= \langle\psi(t)|\, x\, |\psi(t)\rangle \\ &= \left\langle\tilde{\psi(t)}\right| e^{iH_0 t/\hbar}xe^{-iH_0 t/\hbar}\left|\tilde{\psi(t)}\right\rangle \\ &= \sqrt{\frac{\hbar}{2m\omega_0}}\left(\langle 0|\, c_0^* e^{i\omega_0 t/2}+\langle 1|\, e^{3i\omega_0 t/2}c_1^*(t)\right)(a+a^\dagger)\left(e^{-i\omega_0 t/2}c_0\,|0\rangle+e^{-3i\omega_0 t/2}c_1(t)\,|1\rangle\right) \\ &= \sqrt{\frac{\hbar}{2m\omega_0}}\left(c_1(t)e^{-i\omega_0 t}+c_1^*(t)e^{i\omega_0 t}\right) \end{aligned}$$

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Problem 3. *5.30*

Solution. The potential is

$$V(x,t) = xF_0e^{-t/\tau}$$

This is very similar to the previous problem, just with a different time-dependence to the potential. Write,

$$\begin{aligned} c_1(t) &= -\frac{i}{\hbar}F_0\sqrt{\frac{\hbar}{2m\omega_0}}\int_0^t e^{i\omega_0 t}e^{-t/\tau}dt \\ &= -\frac{i}{2\hbar}F_0\sqrt{\frac{\hbar}{2m\omega_0}}\frac{\left(e^{(i\omega_0-1/\tau)t}-1\right)}{(i\omega_0-1/\tau)t} \end{aligned}$$

The probability of finding the particle in the first excited state is

$$|c_1(t)|^2 =$$

which is clearly independent of time. This is expected since the force is transient. We cannot find higher order states because, as was shown in the previous problem, $c_n(0) = 0$ and $\dot{c_n}(t) = 0$ for all $n > 1$.

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Problem 4. *5.32*

Solution. $c_n(t)$

We need to find $|\psi(t)\rangle$, which amounts to finding the expansion coefficients $c_0(t)$ and $c_1(t)$. In the interaction picture, we have that

$$i\hbar\dot{c_n}(t) = \sum_m V_{nm}e^{i\omega_{nm}t}c_m(t)$$

We were given $V(t)$, so we already know the V_{nm}. The differential equations for $c_0(t)$ and $c_1(t)$ are

$$i\hbar\dot{c_0}(t) = \sum_m V_{nm}e^{i\omega_{nm}t}c_m(t) = \lambda e^{i\omega_{01}t}\cos\omega t c_1(t)$$

$$i\hbar\dot{c_1}(t) = \sum_m V_{nm}e^{i\omega_{nm}t}c_m(t) = \lambda e^{-i\omega_{10}t}\cos\omega t c_0(t)$$

So we have the coupled set of differential equations

$$\dot{c_0}(t) = -\frac{i\lambda}{\hbar}c_1(t)e^{i\omega_{10}t}\cos\omega t$$
$$\dot{c_1}(t) = -\frac{i\lambda}{\hbar}c_0(t)e^{-i\omega_{10}t}\cos\omega t$$

subject to the initial conditions $c_0(0) = 1$ and $c_1(0) = 0$. The probability the system is found in the state $|1\rangle$ is

$$|\langle\psi(t)|1\rangle|^2 = |c_1(t)|^2$$

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Problem 5. *5.35*

Solution.

The electric potential in the capacitor is

$$V(z,t) = \int_0^z E_0 e^{-t/\tau} dz = z E_0 e^{-t/\tau}$$

We are asked to find the probability of finding the hydrogen atom in the states $|2,1,\pm1\rangle, |2,1,0\rangle$, given the initial state $|1,0,0\rangle$. As usual, we make use of the interaction picture

$$i\hbar \dot{c_n}(t) = \sum_m V_{nm} e^{i\omega_{nm}t} c_m(t)$$

But need to find the matrix element $V_{nm} = \langle nlm| V |n'l'm'\rangle$. These matrix elements are constrained by the selection rules for transitions of the hydrogen atom. It is well-known that the matrix element $\langle nlm| z |n'l'm'\rangle$ is nonzero only when $m' = m$ and $l' = l \pm 1$. Therefore, we don't need to calculate the probability of a transition to $|2,1,\pm1\rangle$, because it is impossible. However, a transition to state $|2,1,0\rangle$ is possible and we write

$$i\hbar \dot{c_{210}}(t) = E_0 e^{-t/\tau} \langle 1,0,0| z |2,1,0\rangle e^{i\omega t} c_{100}(t)$$

where $\omega = (E_{210} - E_{1,0,0})/\hbar$.

For the $2s$ state $|2,0,0\rangle$, the same selection rules apply, so we only need to find

$$i\hbar \dot{c_{210}}(t) = E_0 e^{-t/\tau} \langle 2,0,0| z |2,1,0\rangle c_{200}(t)$$

where the exponential is 1 because the two states are degenerate in energy thus $\omega = (E_{210} - E_{2,0,0})/\hbar = 0$. ■

Problem 6. *5.36*

Solution.

We were given the Hamiltonian

$$H = \beta\left(S_1 \cdot S_2\right)$$

where $\beta = 4\Delta/\hbar^2$. We would like to use Schrodinger's equation to write out the time evolution; however, it isn't immediately clear what the eigenkets of H are. Recall,

$$S_1 \cdot S_2 = \frac{1}{2}\left(J^2 - S_1^2 - S_2^2\right)$$

where $J = S_1 + S_2$. Consider a test ket $|\alpha\rangle$

$$H|\alpha\rangle = \beta\left(S_1 \cdot S_2\right)|\alpha\rangle = \frac{\beta}{2}\left(J^2 - S_1^2 - S_2^2\right)|\alpha\rangle$$

There must exist simultaneous eigenkets of all three operators since S_1 and S_2 commute. The Hilbert space dimension is 4, so we need 4 kets. Those kets are the three triplet states and the singlet state:

$$\begin{aligned}
|1,1\rangle &= |{+}{+}\rangle \\
|1,0\rangle &= \frac{1}{\sqrt{2}}\left(|{+}{-}\rangle + |{-}{+}\rangle\right) \\
|1,-1\rangle &= |{-}{-}\rangle \\
|0,0\rangle &= \frac{1}{\sqrt{2}}\left(|{+}{-}\rangle - |{-}{+}\rangle\right)
\end{aligned}$$

Also note that the energies of the triplet states and singlet state are $\hbar^2/4$ and $-3\hbar^2/4$, respectively (which gives energy eigenvalues of Δ and -3Δ). Now, to find the time evolution of the system prepared in $|\psi(0)\rangle = |{+}{-}\rangle$, we can write this state in that basis

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,0\rangle\right)$$

$$\begin{aligned}
|\psi(t)\rangle &= e^{-i\hat{H}t/\hbar}|\psi(0)\rangle \\
&= \frac{1}{\sqrt{2}}\left(e^{-i\Delta t/\hbar}|1,0\rangle + e^{i3\Delta t/\hbar}|0,0\rangle\right)
\end{aligned}$$

The probabilities immediately follow from this result. Those corresponding to $|{-}{-}\rangle$ and $|{+}{+}\rangle$ are zero. The others are:

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\begin{aligned}
|\langle +-|\psi(t)\rangle|^2 &= \frac{1}{4}\left(e^{-i\Delta t/\hbar} + e^{i3\Delta t/\hbar}\right)\left(e^{i\Delta t/\hbar} + e^{-i3\Delta t/\hbar}\right) \\
&= \frac{1 + e^{-4i\Delta t/\hbar} + e^{4i\Delta t/\hbar}}{2} \\
&= \frac{1 + \cos\frac{4\Delta t}{\hbar}}{2}
\end{aligned}
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\begin{aligned}
|\langle -+|\psi(t)\rangle|^2 &= \frac{1}{4}\left(e^{-i\Delta t/\hbar} - e^{i3\Delta t/\hbar}\right)\left(e^{i\Delta t/\hbar} - e^{-i3\Delta t/\hbar}\right) \\
&= \frac{1 - e^{-4i\Delta t/\hbar} - e^{4i\Delta t/\hbar}}{2} \\
&= \frac{1 - \cos\frac{4\Delta t}{\hbar}}{2}
\end{aligned}
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To solve this problem using time-dependent perturbation theory, we will need the matrix element for our Hamiltonian. We should use the

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