Homework 3

Quantum Mechanics
Sept 15th, 2022 CLAYTON SEITZ

Problem 1. *Problem 2.1 from Sakurai*

Solution. The Heisenberg equation of motion reads

$$\frac{dA}{dt} = \frac{1}{i\hbar}[A, H]$$

For the spin precession problem, we have the Hamiltonian

$$H = -\left(\frac{eB}{mc}\right) S_z = \omega S_z$$

For $A = S_x, S_y, S_z$, the time evolution is given by

$$\frac{dS_x}{dt} = \frac{\omega}{i\hbar}[S_x, S_z] = -\omega S_y$$
$$\frac{dS_y}{dt} = \frac{\omega}{i\hbar}[S_y, S_z] = \omega S_x$$
$$\frac{dS_z}{dt} = \frac{\omega}{i\hbar}[S_z, S_z] = 0$$

The above system has a straightforward solution:

$$S_x(t) = \cos(\omega t)$$
$$S_y(t) = \sin(\omega t)$$
$$S_z(t) = S_z(0)$$

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Problem 2. *Problem 2.3 from Sakurai*

Solution. We are given that $\vec{B} = B\hat{z}$ and that we are in the eigenstate $|\psi(0)\rangle = |\boldsymbol{S}\cdot\hat{\boldsymbol{n}}\rangle_+$, which reads

$$\begin{aligned}|\psi(0)\rangle &= \psi_+ |+\rangle + \psi_- |-\rangle \\ &= \cos\frac{\beta}{2}|+\rangle + \sin\frac{\beta}{2}|-\rangle\end{aligned}$$

where we have set $\alpha = 0$ since the ket is in the x-z plane. This state will evolve according to a Hamiltonian

$$H = -\left(\frac{eB}{m_e c}\right) S_z$$

Let $\omega = |e|B/m_e c$ giving $H = \omega S_z$. We have the energies

$$E_\pm = \mp\frac{e\hbar B}{2m_e c} = \mp\hbar\omega$$

$$\begin{aligned}|\psi(t)\rangle &= \psi_+(0)\exp\left(\frac{-iE_+t}{\hbar}\right)|+\rangle + \psi_-(0)\exp\left(\frac{-iE_-t}{\hbar}\right)|-\rangle \\ &= \cos\frac{\beta}{2}\exp\left(\frac{-i\omega t}{2}\right)|+\rangle + \sin\frac{\beta}{2}\exp\left(\frac{i\omega t}{2}\right)|-\rangle\end{aligned}$$

In general, the probability of measuring $|+\rangle_x = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$ is given by the inner product

$$\begin{aligned}|\langle S_x;+|\psi;t\rangle|^2 = \Bigg|&\left(\frac{1}{\sqrt{2}}\langle +| + \frac{1}{\sqrt{2}}\langle -|\right)\cdot \\ &\left(\psi_+\exp\left(\frac{-i\omega t}{2}\right)|+\rangle + \psi_-\exp\left(\frac{i\omega t}{2}\right)|-\rangle\right)\Bigg|^2 \\ =& \left|\frac{1}{\sqrt{2}}\cos\frac{\beta}{2}\exp\left(\frac{-i\omega t}{2}\right) + \frac{1}{\sqrt{2}}\sin\frac{\beta}{2}\exp\left(\frac{i\omega t}{2}\right)\right|^2\end{aligned}$$

Using the half-angle identity for $\sin\theta$ and some straightforward arithmetic gives

$$|\langle S_x;+|\psi;t\rangle|^2 = \frac{1+\sin\beta\cos\omega t}{2}$$

For the time-dependence of $\langle S_x\rangle$, we have

$$\begin{aligned}\langle S_x\rangle(t) &= \langle\psi;t|\, S_x\,|\psi;t\rangle \\ &= \left(\psi_+\exp\left(\frac{i\omega t}{2}\right)\langle +| + \psi_-\exp\left(\frac{-i\omega t}{2}\right)\langle -|\right) \\ &\cdot \frac{\hbar}{2}\left(\psi_+\exp\left(-\frac{i\omega t}{2}\right)|-\rangle + \psi_-\exp\left(\frac{i\omega t}{2}\right)|+\rangle\right)\end{aligned}$$

Subsituting ψ_+ and ψ_- with the same values as above, we get

$$\langle S_x\rangle(t) = \frac{\hbar}{2}\sin\beta\cos\omega t$$

When $\beta = \pi/2$ the probability oscillates between 0 and 1 with frequency ω and when $\beta = 0$ then the probability is always 1/2, as expected. The expectation value also makes sense because when $\beta = 0$, we can get $\pm\hbar/2$ with equal probability, giving zero on average. When $\beta = \pi/2$ the expectation value oscillates between $\hbar/2$ and $-\hbar/2$.

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Problem 3. *Problem 2.9 from Sakurai*

Solution.

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Problem 4. *Problem 2.10 from Sakurai*

Solution. Let $|\psi\rangle = \alpha\,|a'\rangle + \beta\,|a''\rangle$ be an eigenvector of the Hamiltonian. Note that this must be real for the eigenvalue to be real. That means that

$$\begin{aligned}H\,|\psi\rangle &= (|a'\rangle\,\delta\,\langle a''| + |a''\rangle\,\delta\,\langle a'|)\,(\alpha\,|a'\rangle + \beta\,|a''\rangle) \\ &= \delta\,(\alpha\,|a''\rangle + \beta\,|a'\rangle)\end{aligned}$$

Therefore $\alpha = \beta = \frac{1}{\sqrt{2}}$ or $\alpha = \frac{1}{\sqrt{2}}$ and $\beta = -\frac{1}{\sqrt{2}}$. Giving eigenvalues $\pm\delta$. To get the time evolution of the state, we need to express these in the basis of H. Just based on inspection of the the two bases, we can tell that

$$|a'\rangle = \frac{1}{\sqrt{2}}\left(|\psi_1\rangle - |\psi_2\rangle\right)$$
$$|a''\rangle = \frac{1}{\sqrt{2}}\left(|\psi_1\rangle + |\psi_2\rangle\right)$$

and, since the Hamiltonian is time-independent, a state prepared in $|a'\rangle$ will evolve according to

$$|\alpha(t)\rangle = \frac{1}{\sqrt{2}}\exp\left(\frac{-i\delta t}{\hbar}\right)|\psi_1\rangle - \frac{1}{\sqrt{2}}\exp\left(\frac{i\delta t}{\hbar}\right)|\psi_2\rangle$$

The probability of finding the system in the state $|a''\rangle$ at a later time is

$$\begin{aligned}
|\langle a''|\alpha(t)\rangle|^2 &= \left|\frac{1}{\sqrt{2}}\left(\langle\psi_1| + \langle\psi_2|\right)\right. \\
&\quad \left.\cdot\left(\frac{1}{\sqrt{2}}\exp\left(\frac{-i\delta t}{\hbar}\right)|\psi_1\rangle - \frac{1}{\sqrt{2}}\exp\left(\frac{i\delta t}{\hbar}\right)|\psi_2\rangle\right)\right|^2 \\
&= \frac{1}{4}\sin^2\frac{\delta t}{\hbar}
\end{aligned}$$

This could describe a system in which the eigenvectors of the Hamiltonian are simultaneous with the eigenvectors of S_x, however the states $|a'\rangle$ and $|a''\rangle$ are expressed in the S_z basis.

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Problem 5. *Problem 2.12 from Sakurai*

Solution. The state is prepared in

$$|\alpha; t=0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{\exp(i\delta)}{\sqrt{2}}|1\rangle$$

In general, the energies of $|n\rangle$ are $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$. Therefore, the time dependence of the state can be evalulated as

$$\begin{aligned}
|\alpha; t\rangle &= \exp\left(-\frac{iHt}{\hbar}\right)|\alpha; t\rangle \\
&= \frac{1}{\sqrt{2}}\exp\frac{-i\omega t}{2}|0\rangle + \frac{1}{\sqrt{2}}\exp(i\delta)\exp\frac{-3i\omega t}{2}|1\rangle
\end{aligned}$$

$$\langle x'|\alpha;t\rangle = \frac{1}{\sqrt{2}}\exp\frac{-i\omega t}{2}\langle x'|0\rangle + \frac{1}{\sqrt{2}}\exp(i\delta)\exp\frac{-3i\omega t}{2}\langle x'|1\rangle$$

and we know in general that the position representation of $|n\rangle$ i.e., $\langle x'|n\rangle$ are

$$\langle x'|n\rangle = \psi_n(x) = \frac{1}{\sqrt{2^n\,n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$$

$$\begin{aligned}
\langle x\rangle(t) &= \langle\alpha;t|\,x\,|\alpha;t\rangle \\
&= \left(\frac{1}{\sqrt{2}}\exp\frac{i\omega t}{2}\langle 0| + \frac{1}{\sqrt{2}}\exp(-i\delta)\exp\frac{3i\omega t}{2}\langle 1|\right) \\
x&\left(\frac{1}{\sqrt{2}}\exp\frac{-i\omega t}{2}|0\rangle + \frac{1}{\sqrt{2}}\exp(i\delta)\exp\frac{-3i\omega t}{2}|1\rangle\right) \\
&= \frac{1}{2}\langle 0|\,x\,|0\rangle + \frac{1}{2}\langle 1|\,x\,|1\rangle \\
&+ \frac{1}{2}\exp(i\delta)\exp(-i\omega t)\langle 0|\,x\,|1\rangle + \frac{1}{2}\exp(-i\delta)\exp(i\omega t)\langle 1|\,x\,|0\rangle
\end{aligned}$$

Now recall the general expression for the matrix element of x

$$\langle n'|\,x\,|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\delta_{n',n-1} + \sqrt{n+1}\delta_{n',n+1}\right)$$

which means that the above expression simplifies to

$$\begin{aligned}
\langle x\rangle(t) &= \frac{1}{2}\exp(i\delta)\exp(-i\omega t)\langle 0|\,x\,|1\rangle + \frac{1}{2}\exp(-i\delta)\exp(i\omega t)\langle 1|\,x\,|0\rangle \\
&= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(\exp(i\delta)\exp(-i\omega t) + \exp(-i\delta)\exp(i\omega t)\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\cos(\delta - \omega t) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t - \delta)
\end{aligned}$$

For momentum, we can just replace the operator x with p in the expressions above:

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\begin{aligned}
\langle p\rangle(t) &= \langle\alpha;t|\, x\, |\alpha;t\rangle \\
&= \left(\frac{1}{\sqrt{2}}\exp\frac{i\omega t}{2}\langle 0| + \frac{1}{\sqrt{2}}\exp(-i\delta)\exp\frac{3i\omega t}{2}\langle 1|\right) \\
&\quad \hat{p}\left(\frac{1}{\sqrt{2}}\exp\frac{-i\omega t}{2}|0\rangle + \frac{1}{\sqrt{2}}\exp(i\delta)\exp\frac{-3i\omega t}{2}|1\rangle\right) \\
&= \frac{1}{2}\langle 0|\, p\, |0\rangle + \frac{1}{2}\langle 1|\, p\, |1\rangle \\
&\quad + \frac{1}{2}\exp(i\delta)\exp(-i\omega t)\langle 0|\, p\, |1\rangle + \frac{1}{2}\exp(-i\delta)\exp(i\omega t)\langle 1|\, p\, |0\rangle
\end{aligned}
$$

and we have another general expression for the matrix element of p

$$
\langle n'|\, p\, |n\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(-\sqrt{n}\delta_{n',n-1} + \sqrt{n+1}\delta_{n',n+1}\right)
$$

which again means that the above expression simplifies to

$$
\begin{aligned}
\langle p\rangle(t) &= \frac{1}{2}\exp(i\delta)\exp(-i\omega t)\langle 0|\, p\, |1\rangle + \frac{1}{2}\exp(-i\delta)\exp(i\omega t)\langle 1|\, p\, |0\rangle \\
&= \frac{i}{2}\sqrt{\frac{m\hbar\omega}{2}}\left(-\exp(i\delta)\exp(-i\omega t) + \exp(-i\delta)\exp(i\omega t)\right) \\
&= -\sqrt{\frac{m\hbar\omega}{2}}\sin(\omega t - \delta)
\end{aligned}
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Problem 6. *Problem 2.13 from Sakurai*

Solution.

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