

Homework 3

Quantum Mechanics

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Problem 1. *Problem 2.1 from Sakurai*

Solution. The Heisenberg equation of motion reads

$$\frac{dA}{dt} = \frac{1}{i\hbar} [A, H]$$

For the spin precession problem, we have the Hamiltonian

$$H = - \left(\frac{eB}{mc} \right) S_z = \omega S_z$$

For $A = S_x, S_y, S_z$, the time evolution is given by

$$\begin{aligned} \frac{dS_x}{dt} &= \frac{\omega}{i\hbar} [S_x, S_z] = -\omega S_y \\ \frac{dS_y}{dt} &= \frac{\omega}{i\hbar} [S_y, S_z] = \omega S_x \\ \frac{dS_z}{dt} &= \frac{\omega}{i\hbar} [S_z, S_z] = 0 \end{aligned}$$

The above system has a straightforward solution:

$$\begin{aligned} S_x(t) &= S_x(0) \cos(\omega t) \\ S_y(t) &= S_y(0) \sin(\omega t) \\ S_z(t) &= S_z(0) \end{aligned}$$

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Problem 2. *Problem 2.3 from Sakurai*

Solution. We are given that $\vec{B} = B\hat{z}$ and that we are in the eigenstate $|\psi(0)\rangle = |\mathbf{S} \cdot \hat{\mathbf{n}}\rangle_+$, which reads

$$\begin{aligned} |\psi(0)\rangle &= \psi_+ |+\rangle + \psi_- |-\rangle \\ &= \cos \frac{\beta}{2} |+\rangle + \sin \frac{\beta}{2} |-\rangle \end{aligned}$$

where we have set $\alpha = 0$ since the ket is in the x-z plane. This state will evolve according to a Hamiltonian

$$H = - \left(\frac{eB}{m_e c} \right) S_z$$

Let $\omega = |e|B/m_e c$ giving $H = \omega S_z$. We have the energies

$$E_{\pm} = \mp \frac{e\hbar B}{2m_e c} = \mp \hbar \omega$$

$$\begin{aligned} |\psi(t)\rangle &= \psi_+(0) \exp \left(\frac{-iE_+ t}{\hbar} \right) |+\rangle + \psi_-(0) \exp \left(\frac{-iE_- t}{\hbar} \right) |-\rangle \\ &= \cos \frac{\beta}{2} \exp \left(\frac{-i\omega t}{2} \right) |+\rangle + \sin \frac{\beta}{2} \exp \left(\frac{i\omega t}{2} \right) |-\rangle \end{aligned}$$

In general, the probability of measuring $|+\rangle_x = \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} |-\rangle$ is given by the inner product

$$\begin{aligned} |\langle S_x; + | \psi; t \rangle|^2 &= \left| \left(\frac{1}{\sqrt{2}} \langle + | + \frac{1}{\sqrt{2}} \langle - | \right) \cdot \right. \\ &\quad \left. \left(\psi_+ \exp \left(\frac{-i\omega t}{2} \right) |+\rangle + \psi_- \exp \left(\frac{i\omega t}{2} \right) |-\rangle \right) \right|^2 \\ &= \left| \frac{1}{\sqrt{2}} \cos \frac{\beta}{2} \exp \left(\frac{-i\omega t}{2} \right) + \frac{1}{\sqrt{2}} \sin \frac{\beta}{2} \exp \left(\frac{i\omega t}{2} \right) \right|^2 \end{aligned}$$

Using the half-angle identity for $\sin \theta$ and some straightforward arithmetic gives

$$|\langle S_x; + | \psi; t \rangle|^2 = \frac{1 + \sin \beta \cos \omega t}{2}$$

For the time-dependence of $\langle S_x \rangle$, we have

$$\begin{aligned} \langle S_x \rangle(t) &= \langle \psi; t | S_x | \psi; t \rangle \\ &= \left(\psi_+ \exp\left(\frac{i\omega t}{2}\right) \langle + | + \psi_- \exp\left(\frac{-i\omega t}{2}\right) \langle - | \right) \\ &\quad \cdot \frac{\hbar}{2} \left(\psi_+ \exp\left(-\frac{i\omega t}{2}\right) | - \rangle + \psi_- \exp\left(\frac{i\omega t}{2}\right) | + \rangle \right) \end{aligned}$$

Substituting ψ_+ and ψ_- with the same values as above, we get

$$\langle S_x \rangle(t) = \frac{\hbar}{2} \sin \beta \cos \omega t$$

When $\beta = \pi/2$ the probability oscillates between 0 and 1 with frequency ω and when $\beta = 0$ then the probability is always 1/2, as expected. The expectation value also makes sense because when $\beta = 0$, we can get $\pm\hbar/2$ with equal probability, giving zero on average. When $\beta = \pi/2$ the expectation value oscillates between $\hbar/2$ and $-\hbar/2$. ■

Problem 3. *Problem 2.9 from Sakurai*

Solution.

We were given the wavefunction

$$\langle x | \alpha \rangle = A(x - a)^2(x + a)^2 \exp(ikx)$$

To find the normalization constant A , we evaluate

$$\begin{aligned} A &= \left(\int_{-a}^{+a} \langle \alpha | x \rangle \langle x | \alpha \rangle dx \right)^{-1/2} \\ &= \left(\int_{-a}^{+a} (x - a)^4 (x + a)^4 dx \right)^{-1/2} \\ &= \frac{\sqrt{315}}{16a^{9/2}} \end{aligned}$$

according to Mathematica. The expectation value $\langle x \rangle$ is found by integrating

$$\begin{aligned}\langle x \rangle &= \int_{-a}^a x \langle \alpha | x \rangle \langle x | \alpha \rangle dx \\ &= \int_{-a}^{+a} x \langle \alpha | x \rangle \langle x | \alpha \rangle dx \\ &= \int_{-a}^{+a} x(x-a)^4(x+a)^4 dx \\ &= 0\end{aligned}$$

since the function is odd. This makes physical sense, given the overall symmetry of the wavefunction. Notice that the complex exponential vanishes due to the complex conjugation. The expectation value $\langle x^2 \rangle$ is found by integrating

$$\langle x^2 \rangle = \int_{-a}^{+a} x^2(x-a)^4(x+a)^4 dx$$

This we do not expect to be zero, and indeed it is not:

Integrate $[x^2(x-a)^4(x+a)^4, \{x, -a, a\}]$

$$\frac{256a^{11}}{3465}$$

and multiplying by A^2 gives

$$\langle x^2 \rangle = \left(\frac{\sqrt{315}}{16a^{9/2}} \right)^2 \frac{256a^{11}}{3465} = \frac{a^2}{11}$$

The expectation value $\langle p \rangle$ is found similarly

$$\begin{aligned}\langle p \rangle &= \int_{-a}^{+a} \langle \alpha | x \rangle \frac{\hbar}{i} \frac{\partial}{\partial x} \langle x | \alpha \rangle dx \\ &= A^2 \int_{-a}^{+a} (x-a)^2(x+a)^2 \exp(-ikx) \frac{\hbar}{i} \frac{\partial}{\partial x} (x-a)^2(x+a)^2 \exp(ikx) dx\end{aligned}$$

Leaving out the factor of \hbar from p gives

$$f = D [(x - a)^2(x + a)^2 \text{Exp}[Ikx], \{x, 1\}]$$

$$\text{Integrate} [(x - a)^2(x + a)^2 \text{Exp}[-Ikx] * -I * f, \{x, -a, a\}]$$

$$2e^{ikx}(-a + x)^2(a + x) + 2e^{ikx}(-a + x)(a + x)^2 + ie^{ikx}k(-a + x)^2(a + x)^2$$

$$\frac{256a^9k}{315}$$

Bringing in the factor of \hbar and multiplying by A^2 gives a familiar expression

$$\langle p \rangle = \hbar k$$

The expectation value $\langle p^2 \rangle$ is found similarly

$$\begin{aligned} \langle p^2 \rangle &= - \int_{-a}^{+a} \langle \alpha | x \rangle \hbar^2 \frac{\partial^2}{\partial x^2} \langle x | \alpha \rangle dx \\ &= -A^2 \int_{-a}^{+a} (x - a)^2(x + a)^2 \exp(-ikx) \hbar^2 \frac{\partial^2}{\partial x^2} (x - a)^2(x + a)^2 \exp(ikx) \end{aligned}$$

$$f = -D [(x - a)^2(x + a)^2 \text{Exp}[Ikx], \{x, 2\}]$$

$$\text{Integrate} [(x - a)^2(x + a)^2 \text{Exp}[-Ikx] * f, \{x, -a, a\}]$$

$$\begin{aligned} &-2e^{ikx}(-a + x)^2 - 4(a + x) (2e^{ikx}(-a + x) + ie^{ikx}k(-a + x)^2) - \\ &(a + x)^2 (2e^{ikx} + 4ie^{ikx}k(-a + x) - e^{ikx}k^2(-a + x)^2) \end{aligned}$$

$$\frac{256a^7}{105} + \frac{256a^9k^2}{315}$$

Bringing in a factor of \hbar^2 and multiplying by A^2 gives

$$\langle p^2 \rangle = \frac{3\hbar^2}{a^2} + \hbar^2 k^2$$

The variance $\langle(\Delta x)^2\rangle$ is just

$$\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2 = \frac{a^2}{11}$$

The variance $\langle(\Delta p)^2\rangle$ is just

$$\begin{aligned}\langle(\Delta p)^2\rangle &= \langle p^2\rangle - \langle p\rangle^2 \\ &= \frac{3\hbar^2}{a^2}\end{aligned}$$

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Problem 4. *Problem 2.10 from Sakurai*

Solution. Let $|\psi\rangle = \alpha|a'\rangle + \beta|a''\rangle$ be an eigenvector of the Hamiltonian. Note that this must be real for the eigenvalue to be real. That means that

$$\begin{aligned}H|\psi\rangle &= (|a'\rangle\delta\langle a''| + |a''\rangle\delta\langle a'|)(\alpha|a'\rangle + \beta|a''\rangle) \\ &= \delta(\alpha|a''\rangle + \beta|a'\rangle)\end{aligned}$$

Therefore $\alpha = \beta = \frac{1}{\sqrt{2}}$ or $\alpha = \frac{1}{\sqrt{2}}$ and $\beta = -\frac{1}{\sqrt{2}}$. Giving eigenvalues $\pm\delta$. To get the time evolution of the state, we need to express these in the basis of H . Just based on inspection of the the two bases, we can tell that

$$\begin{aligned}|a'\rangle &= \frac{1}{\sqrt{2}}(|\psi_1\rangle - |\psi_2\rangle) \\ |a''\rangle &= \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_2\rangle)\end{aligned}$$

and, since the Hamiltonian is time-independent, a state prepared in $|a'\rangle$ will evolve according to

$$|\alpha(t)\rangle = \frac{1}{\sqrt{2}}\exp\left(\frac{-i\delta t}{\hbar}\right)|\psi_1\rangle - \frac{1}{\sqrt{2}}\exp\left(\frac{i\delta t}{\hbar}\right)|\psi_2\rangle$$

The probability of finding the system in the state $|a''\rangle$ at a later time is

$$\begin{aligned}
|\langle a''|\alpha(t)\rangle|^2 &= \left| \frac{1}{\sqrt{2}} (\langle\psi_1| + \langle\psi_2|) \right. \\
&\quad \cdot \left(\frac{1}{\sqrt{2}} \exp\left(\frac{-i\delta t}{\hbar}\right) |\psi_1\rangle - \frac{1}{\sqrt{2}} \exp\left(\frac{i\delta t}{\hbar}\right) |\psi_2\rangle \right) \left. \right|^2 \\
&= \frac{1}{4} \sin^2 \frac{\delta t}{\hbar}
\end{aligned}$$

This could describe a system in which the eigenvectors of the Hamiltonian are simultaneous with the eigenvectors of S_x , however the states $|a'\rangle$ and $|a''\rangle$ are expressed in the S_z basis. ■

Problem 5. *Problem 2.12 from Sakurai*

Solution. The state is prepared in

$$|\alpha; t=0\rangle = \frac{1}{\sqrt{2}} |0\rangle + \frac{\exp(i\delta)}{\sqrt{2}} |1\rangle$$

In general, the energies of $|n\rangle$ are $E_n = (n + \frac{1}{2}) \hbar\omega$. Therefore, the time dependence of the state can be evaluated as

$$\begin{aligned}
|\alpha; t\rangle &= \exp\left(-\frac{iHt}{\hbar}\right) |\alpha; t=0\rangle \\
&= \frac{1}{\sqrt{2}} \exp\frac{-i\omega t}{2} |0\rangle + \frac{1}{\sqrt{2}} \exp(i\delta) \exp\frac{-3i\omega t}{2} |1\rangle
\end{aligned}$$

$$\langle x|\alpha; t\rangle = \frac{1}{\sqrt{2}} \exp\frac{-i\omega t}{2} \langle x|0\rangle + \frac{1}{\sqrt{2}} \exp(i\delta) \exp\frac{-3i\omega t}{2} \langle x|1\rangle$$

and we know in general that the position representation of $|n\rangle$ i.e., $\langle x|n\rangle$ are

$$\langle x|n\rangle = \psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}} x\right)$$

$$\begin{aligned}
\langle x \rangle(t) &= \langle \alpha; t | x | \alpha; t \rangle \\
&= \left(\frac{1}{\sqrt{2}} \exp \frac{i\omega t}{2} \langle 0 | + \frac{1}{\sqrt{2}} \exp(-i\delta) \exp \frac{3i\omega t}{2} \langle 1 | \right) \\
&\quad x \left(\frac{1}{\sqrt{2}} \exp \frac{-i\omega t}{2} | 0 \rangle + \frac{1}{\sqrt{2}} \exp(i\delta) \exp \frac{-3i\omega t}{2} | 1 \rangle \right) \\
&= \frac{1}{2} \langle 0 | x | 0 \rangle + \frac{1}{2} \langle 1 | x | 1 \rangle \\
&\quad + \frac{1}{2} \exp(i\delta) \exp(-i\omega t) \langle 0 | x | 1 \rangle + \frac{1}{2} \exp(-i\delta) \exp(i\omega t) \langle 1 | x | 0 \rangle
\end{aligned}$$

Now recall the general expression for the matrix element of x

$$\langle n' | x | n \rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n} \delta_{n', n-1} + \sqrt{n+1} \delta_{n', n+1} \right)$$

which means that the above expression simplifies to

$$\begin{aligned}
\langle x \rangle(t) &= \frac{1}{2} \exp(i\delta) \exp(-i\omega t) \langle 0 | x | 1 \rangle + \frac{1}{2} \exp(-i\delta) \exp(i\omega t) \langle 1 | x | 0 \rangle \\
&= \frac{1}{2} \sqrt{\frac{\hbar}{2m\omega}} (\exp(i\delta) \exp(-i\omega t) + \exp(-i\delta) \exp(i\omega t)) \\
&= \sqrt{\frac{\hbar}{2m\omega}} \cos(\delta - \omega t) \\
&= \sqrt{\frac{\hbar}{2m\omega}} \cos(\omega t - \delta)
\end{aligned}$$

For momentum, we can just replace the operator x with p in the expressions above:

$$\begin{aligned}
\langle p \rangle(t) &= \langle \alpha; t | p | \alpha; t \rangle \\
&= \left(\frac{1}{\sqrt{2}} \exp \frac{i\omega t}{2} \langle 0 | + \frac{1}{\sqrt{2}} \exp(-i\delta) \exp \frac{3i\omega t}{2} \langle 1 | \right) \\
&\quad \hat{p} \left(\frac{1}{\sqrt{2}} \exp \frac{-i\omega t}{2} | 0 \rangle + \frac{1}{\sqrt{2}} \exp(i\delta) \exp \frac{-3i\omega t}{2} | 1 \rangle \right) \\
&= \frac{1}{2} \langle 0 | p | 0 \rangle + \frac{1}{2} \langle 1 | p | 1 \rangle \\
&\quad + \frac{1}{2} \exp(i\delta) \exp(-i\omega t) \langle 0 | p | 1 \rangle + \frac{1}{2} \exp(-i\delta) \exp(i\omega t) \langle 1 | p | 0 \rangle
\end{aligned}$$

and we have another general expression for the matrix element of p

$$\langle n' | p | n \rangle = i \sqrt{\frac{m\hbar\omega}{2}} \left(-\sqrt{n} \delta_{n',n-1} + \sqrt{n+1} \delta_{n',n+1} \right)$$

which again means that the above expression simplifies to

$$\begin{aligned} \langle p \rangle(t) &= \frac{1}{2} \exp(i\delta) \exp(-i\omega t) \langle 0 | p | 1 \rangle + \frac{1}{2} \exp(-i\delta) \exp(i\omega t) \langle 1 | p | 0 \rangle \\ &= \frac{i}{2} \sqrt{\frac{m\hbar\omega}{2}} \left(-\exp(i\delta) \exp(-i\omega t) + \exp(-i\delta) \exp(i\omega t) \right) \\ &= -\sqrt{\frac{m\hbar\omega}{2}} \sin(\omega t - \delta) \end{aligned}$$

In the Heisenberg picture, we have the Heisenberg equations of motion

$$\begin{aligned} \frac{dp}{dt} &= -m\omega^2 x \\ \frac{dx}{dt} &= \frac{p}{m} \end{aligned}$$

It has been shown in the text how to uncouple these in terms of the ladder operators and solve the system for the time dependent operators $x(t)$ and $p(t)$

$$\begin{aligned} x(t) &= x(0) \cos(\omega t) + \left(\frac{p(0)}{m\omega} \right) \sin(\omega t) \\ p(t) &= -m\omega x(0) \sin(\omega t) + p(0) \cos(\omega t) \end{aligned}$$

To get $\langle x \rangle(t)$ we have

$$\begin{aligned} \langle x \rangle(t) &= \langle \alpha | x(0) \cos(\omega t) + \left(\frac{p(0)}{m\omega} \right) \sin(\omega t) | \alpha \rangle \\ &= \langle \alpha | x(0) | \alpha \rangle \cos(\omega t) + \langle \alpha | p(0) | \alpha \rangle \frac{\sin(\omega t)}{m\omega} \end{aligned}$$

$$\begin{aligned}
\langle \alpha | x(0) | \alpha \rangle &= \frac{1}{2} \langle 0 | x(0) | 0 \rangle + \frac{1}{2} \langle 1 | x(0) | 1 \rangle \\
&+ \frac{1}{2} \exp(i\delta) \langle 0 | x(0) | 1 \rangle + \frac{1}{2} \exp(i\delta) \langle 1 | x(0) | 0 \rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}} \exp(i\delta)
\end{aligned}$$

The factor of one-half disappeared in the $\langle \alpha | x(0) | \alpha \rangle$ term since $\langle n' | x | n \rangle$ is real and therefore equal to its complex conjugate

$$\begin{aligned}
\langle x \rangle(t) &= \sqrt{\frac{\hbar}{2m\omega}} \exp(i\delta) \cos(\omega t) + \frac{i}{2} \exp(i\delta) \sqrt{\frac{m\hbar\omega}{2}} \frac{\sin(\omega t)}{m\omega} \\
&= \sqrt{\frac{\hbar}{2m\omega}} \exp(i\delta) \exp(i\omega t) \\
&= \sqrt{\frac{\hbar}{2m\omega}} \exp(i(\omega t + \delta))
\end{aligned}$$

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Problem 6. *Problem 2.13 from Sakurai*

Solution.

The Heisenberg equations of motion are

$$\begin{aligned}
\frac{dp}{dt} &= F \\
\frac{dx}{dt} &= \frac{p_0 + Ft}{m}
\end{aligned}$$

The solution is simply

$$\begin{aligned}
p(t) &= p(0) + Ft \\
x(t) &= x(0) + \frac{p(0)}{m}t + \frac{1}{2} \frac{Ft^2}{m} \\
\langle x \rangle(t) &= \langle \alpha | \left(x(0) + \frac{p_0}{m}t + \frac{1}{2} \frac{Ft^2}{m} \right) | \alpha \rangle \\
&= x_0 + \frac{p_0 t}{m} + \frac{Ft^2}{2m}
\end{aligned}$$

and for $\langle p \rangle$ we have

$$\begin{aligned}\langle p \rangle(t) &= \langle \alpha | p(0) + Ft | \alpha \rangle \\ &= p_0 + Ft\end{aligned}$$

In the Schrodinger picture, this Hamiltonian is not a constant

$$H(t) = \frac{p(t)^2}{2m} + V(x, t) = \frac{p(t)^2}{2m} + Fx(t)$$

although, using the canonical commutation relations, we can see that $H(t_0)$ commutes with $H(t)$. So we should be able to define

$$\mathcal{U}(t, t_0) = \exp \left(-\frac{i}{\hbar} \int_{t_0}^t H(t) dt \right)$$

such that $|\alpha; t\rangle = \mathcal{U}(t, t_0) |\alpha; t_0\rangle$. If we evaluated that expression, then we could write $|\alpha; t\rangle$ explicitly.

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