

Homework 8

Quantum Mechanics

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Problem 1. 5.27

Solution.

$$\frac{\langle \tilde{0} | H | \tilde{0} \rangle}{\langle \tilde{0} | \tilde{0} \rangle} \geq E_0$$

The denominator is easy to compute

$$2 \int_{-\infty}^0 e^{\beta x} dx = \frac{1}{\beta}$$

The numerator

$$\begin{aligned} \langle \tilde{0} | H | \tilde{0} \rangle &= \int_{-\infty}^{\infty} \psi^*(x) H \psi(x) dx \\ &= \int_{-\infty}^0 \psi^*(x) H \psi(x) dx + \int_0^{\infty} \psi^*(x) H \psi(x) dx \end{aligned}$$

$$\begin{aligned} \int_{-\infty}^0 \psi^*(x) H \psi(x) dx &= \int_{-\infty}^0 -e^{\beta x} \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} e^{\beta x} + \frac{1}{2} m \omega^2 x^2 e^{2\beta x} dx \\ &= \int_{-\infty}^0 e^{2\beta x} \left(\frac{1}{2} m \omega^2 x^2 - \frac{\hbar^2 \beta^2}{2m} \right) dx \\ &= \left|_{-\infty}^0 \frac{1}{2} m \omega^2 \frac{e^{2\beta x} (1 - 2\beta x + 2\beta^2 x^2)}{4\beta^3} - e^{2\beta x} \frac{\hbar^2 \beta}{4m} \right. \\ &= \frac{1}{2} m \omega^2 \frac{1}{4\beta^3} - \frac{\hbar^2 \beta}{4m} \end{aligned}$$

$$\begin{aligned}
\int_0^\infty \psi^*(x) H \psi(x) dx &= \int_0^\infty -e^{-\beta x} \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} e^{-\beta x} + \frac{1}{2} m \omega^2 x^2 e^{-2\beta x} dx \\
&= \int_0^\infty e^{-2\beta x} \left(\frac{1}{2} m \omega^2 x^2 - \frac{\hbar^2 \beta^2}{2m} \right) dx \\
&= \left|_0^\infty \frac{1}{2} m \omega^2 \frac{e^{-2\beta x} (1 + 2\beta x + 2\beta^2 x^2)}{4\beta^3} - e^{-2\beta x} \frac{\hbar^2 \beta}{4m} \right. \\
&= \frac{1}{2} m \omega^2 \frac{1}{4\beta^3} - \frac{\hbar^2 \beta}{4m}
\end{aligned}$$

$$\bar{H} = \frac{\langle \tilde{0} | H | \tilde{0} \rangle}{\langle \tilde{0} | \tilde{0} \rangle} = \frac{m \omega^2}{4\beta^2} - \frac{\hbar^2 \beta^2}{2m}$$

$$\frac{d\bar{H}}{d\beta} = -\frac{m \omega^2}{4\beta} - \frac{\hbar^2 \beta}{m} = 0$$

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Problem 2. 5.29

Solution.

We have the full time-dependent Hamiltonian

$$H(t) = H_0 + F_0 x \cos \omega t$$

We need to find $|\psi(t)\rangle$, which amounts to finding the expansion coefficients $c_n(t)$. In the interaction picture, we have that

$$i\hbar \dot{c}_n(t) = \sum_m V_{nm} e^{i\omega_{nm}t} c_m(t)$$

for $\omega_{nm} = (E_n - E_m)/\hbar$.

$$\begin{aligned}
V_{nm} &= F_0 \cos \omega t \langle n | x | m \rangle \\
&= F_0 \cos \omega t \sqrt{\frac{\hbar}{2m\omega_0}} \left(\sqrt{n+1} \delta_{m,n-1} + \sqrt{n} \delta_{m,n+1} \right)
\end{aligned}$$

But the initial condition says that $|\psi(0)\rangle = |0\rangle$, so $n = 0$ and the only term of the summation that survives has $m = 1$. Therefore,

$$\begin{aligned} i\hbar\dot{c}_1(t) &= V_{10}e^{i\omega_0 t}c_0(t) \\ &= F_0 \cos \omega t \sqrt{\frac{\hbar}{2m\omega_0}} e^{i\omega_0 t} c_0(t) \end{aligned}$$

Solving for $c_1(t)$,

$$\begin{aligned} c_1(t) &= -\frac{i}{\hbar} F_0 \sqrt{\frac{\hbar}{2m\omega_0}} \int_0^t e^{i\omega_0 t} \cos \omega t dt \\ &= -\frac{i}{2\hbar} F_0 \sqrt{\frac{\hbar}{2m\omega_0}} \left(\frac{e^{i(\omega_0+\omega)t} - 1}{\omega_0 + \omega} + \frac{e^{i(\omega_0-\omega)t} - 1}{\omega_0 - \omega} \right) \end{aligned}$$

Now, to compute $\langle x \rangle$, we can express the x operator in the interaction picture (or, equivalently, convert the $|\psi(t)\rangle$ back to $|\psi(t)\rangle$).

$$\begin{aligned} \langle x \rangle &= \langle \psi(t) | x | \psi(t) \rangle \\ &= \left\langle \psi(t) \left| e^{iH_0 t/\hbar} x e^{-iH_0 t/\hbar} \right| \psi(t) \right\rangle \\ &= \sqrt{\frac{\hbar}{2m\omega_0}} \left(\langle 0 | c_0^* e^{i\omega_0 t/2} + \langle 1 | e^{3i\omega_0 t/2} c_1^*(t) \right) (a + a^\dagger) \left(e^{-i\omega_0 t/2} c_0 | 0 \rangle + e^{-3i\omega_0 t/2} c_1(t) | 1 \rangle \right) \\ &= \sqrt{\frac{\hbar}{2m\omega_0}} (c_1(t) e^{-i\omega_0 t} + c_1^*(t) e^{i\omega_0 t}) \end{aligned}$$

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Problem 3. 5.30

Solution. The potential is

$$V(x, t) = x F_0 e^{-t/\tau}$$

This is very similar to the previous problem, just with a different time-dependence to the potential. Write,

$$\begin{aligned} c_1(t) &= -\frac{i}{\hbar} F_0 \sqrt{\frac{\hbar}{2m\omega_0}} \int_0^t e^{i\omega_0 t} e^{-t/\tau} dt \\ &= -\frac{i}{2\hbar} F_0 \sqrt{\frac{\hbar}{2m\omega_0}} \frac{(e^{(i\omega_0 - 1/\tau)t} - 1)}{(i\omega_0 - 1/\tau)t} \end{aligned}$$

The probability of finding the particle in the first excited state is

$$|c_1(t)|^2 =$$

which is clearly independent of time. This is expected since the force is transient. We cannot find higher order states because, as was shown in the previous problem, $c_n(0) = 0$ and $\dot{c}_n(t) = 0$ for all $n > 1$. ■

Problem 4. 5.32

Solution. $c_n(t)$

We need to find $|\psi(t)\rangle$, which amounts to finding the expansion coefficients $c_0(t)$ and $c_1(t)$. In the interaction picture, we have that

$$i\hbar\dot{c}_n(t) = \sum_m V_{nm} e^{i\omega_{nm}t} c_m(t)$$

We were given $V(t)$, so we already know the V_{nm} . The differential equations for $c_0(t)$ and $c_1(t)$ are

$$i\hbar\dot{c}_0(t) = \sum_m V_{nm} e^{i\omega_{nm}t} c_m(t) = \lambda e^{i\omega_{01}t} \cos \omega t c_1(t)$$

$$i\hbar\dot{c}_1(t) = \sum_m V_{nm} e^{i\omega_{nm}t} c_m(t) = \lambda e^{-i\omega_{10}t} \cos \omega t c_0(t)$$

So we have the coupled set of differential equations

$$\begin{aligned}\dot{c}_0(t) &= -\frac{i\lambda}{\hbar} c_1(t) e^{i\omega_{10}t} \cos \omega t \\ \dot{c}_1(t) &= -\frac{i\lambda}{\hbar} c_0(t) e^{-i\omega_{10}t} \cos \omega t\end{aligned}$$

subject to the initial conditions $c_0(0) = 1$ and $c_1(0) = 0$. The probability the system is found in the state $|1\rangle$ is

$$|\langle\psi(t)|1\rangle|^2 = |c_1(t)|^2$$
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Problem 5. 5.35**Solution.**

The electric potential in the capacitor is

$$V(z, t) = \int_0^z E_0 e^{-t/\tau} dz = z E_0 e^{-t/\tau}$$

We are asked to find the probability of finding the hydrogen atom in the states $|2, 1, \pm 1\rangle, |2, 1, 0\rangle$, given the initial state $|1, 0, 0\rangle$. As usual, we make use of the interaction picture

$$i\hbar \dot{c}_n(t) = \sum_m V_{nm} e^{i\omega_{nm}t} c_m(t)$$

But need to find the matrix element $V_{nm} = \langle nlm | V | n'l'm' \rangle$. These matrix elements are constrained by the selection rules for transitions of the hydrogen atom. It is well-known that the matrix element $\langle nlm | z | n'l'm' \rangle$ is nonzero only when $m' = m$ and $l' = l \pm 1$. Therefore, we don't need to calculate the probability of a transition to $|2, 1, \pm 1\rangle$, because it is impossible. However, a transition to state $|2, 1, 0\rangle$ is possible and we write

$$i\hbar \dot{c}_{210}(t) = E_0 e^{-t/\tau} \langle 1, 0, 0 | z | 2, 1, 0 \rangle e^{i\omega t} c_{100}(t)$$

where $\omega = (E_{210} - E_{1,0,0})/\hbar$.

For the $2s$ state $|2, 0, 0\rangle$, the same selection rules apply, so we only need to find

$$i\hbar \dot{c}_{210}(t) = E_0 e^{-t/\tau} \langle 2, 0, 0 | z | 2, 1, 0 \rangle c_{200}(t)$$

where the exponential is 1 because the two states are degenerate in energy thus $\omega = (E_{210} - E_{2,0,0})/\hbar = 0$. ■

Problem 6. 5.36**Solution.**

We were given the Hamiltonian

$$H = \beta (S_1 \cdot S_2)$$

where $\beta = 4\Delta/\hbar^2$. We would like to use Schrodinger's equation to write out the time evolution; however, it isn't immediately clear what the eigenkets of H are. Recall,

$$S_1 \cdot S_2 = \frac{1}{2} (J^2 - S_1^2 - S_2^2)$$

where $J = S_1 + S_2$. Consider a test ket $|\alpha\rangle$

$$H |\alpha\rangle = \beta (S_1 \cdot S_2) |\alpha\rangle = \frac{\beta}{2} (J^2 - S_1^2 - S_2^2) |\alpha\rangle$$

There must exist simultaneous eigenkets of all three operators since S_1 and S_2 commute. The Hilbert space dimension is 4, so we need 4 kets. Those kets are the three triplet states and the singlet state:

$$\begin{aligned} |1, 1\rangle &= |++\rangle \\ |1, 0\rangle &= \frac{1}{\sqrt{2}} (|+-\rangle + |-+\rangle) \\ |1, -1\rangle &= |--\rangle \\ |0, 0\rangle &= \frac{1}{\sqrt{2}} (|+-\rangle - |-+\rangle) \end{aligned}$$

Also note that the energies of the triplet states and singlet state are $\hbar^2/4$ and $-3\hbar^2/4$, respectively (which gives energy eigenvalues of Δ and -3Δ). Now, to find the time evolution of the system prepared in $|\psi(0)\rangle = |+-\rangle$, we can write this state in that basis

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} (|1, 0\rangle + |0, 0\rangle)$$

$$\begin{aligned} |\psi(t)\rangle &= e^{-i\hat{H}t/\hbar} |\psi(0)\rangle \\ &= \frac{1}{\sqrt{2}} \left(e^{-i\Delta t/\hbar} |1, 0\rangle + e^{i3\Delta t/\hbar} |0, 0\rangle \right) \end{aligned}$$

The probabilities immediately follow from this result:

