The Finite State Projection Algorithm

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July 13, 2022

Master equations

- Master equations describe the time-evolution of a discrete state Markov process in continous time
- We define a probability T_{ij} of transitioning to the arbitrary state ω_j from ω_i where $\omega_i, \omega_j \in \Omega$
- These probabilities are efficiently described by a matrix $\mathbf{T} \in \mathbb{R}^{N \times N}$ where $N = |\Omega|$
- $\mathbf{T}[(I, t + dt), (j, t)] = \mathbf{Pr}\left((I, t + dt), (j, t)\right)$ is a conditional distribution, given that we are in a state j at time t

The forward equation

The time evolution of $P(\Omega, t) \in \mathbb{R}^{N\times 1}$ is determined by the net probability flux into and out of each state:

$$
\begin{aligned}
P(\omega_i, t+dt) &= \overbrace{T_{ii}P(\omega_i)dt + \sum_{j\neq i} T_{ij}P(\omega_j, t)dt}^{j\to i} + \overbrace{\sum_{j\neq i} T_{ji}P(\omega_i, t)dt}^{i\to j} \\
&= \overbrace{\sum_{j\neq i} T_{ij}P(\omega_j, t)dt}^{j\to i} + \overbrace{P(\omega_i, t)\sum_{j} T_{ji}dt}^{i\to j} \\
&= \overbrace{\sum_{j\neq i} T_{ij}P(\omega_j, t)dt}^{j\to i} + \overbrace{P(\omega_i, t)\left(1 - \sum_{j} T_{ij}dt\right)}^{i\to j}
\end{aligned}
$$

$$P(\omega_i, t + dt) = \sum_{j \neq i} T_{ij} P(\omega_j, t) dt + P(\omega_i, t) \left(1 - \sum_j T_{ij} dt \right)$$

$$\lim_{dt \to 0} \frac{P(\omega_i, t + dt) - P(\omega_i, t)}{dt} = \sum_{j \neq i} T_{ij} P(\omega_j, t) - P(\omega_i, t) \sum_j T_{ij}$$

It is common to then define a matrix $\mathbf{W}$ s.t. $W_{ij} = T_{ij}$ and $W_{ii} = -\sum_j T_{ij}$

$$\frac{dP(\omega_i)}{dt} = \sum_j W_{ij} P(\omega_j) \rightarrow \frac{dP(\omega)}{dt} = \mathbf{W} P(\omega)$$

We have the following simplified form of a general master equation

$$\frac{dP(\omega)}{dt} = \mathbf{W}P(\omega)$$

which suggests a solution in terms of a matrix exponential

$$P(\omega, t) = \exp\left(\mathbf{W}P(\omega)\right)$$

However, the computation of this exponential is often intractable given the size of $|\Omega|$