The Quantum Fourier Transform

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April 18, 2023

Introduction

Classical discrete Fourier transform maps a vector $\vec{x} \in \mathbb{C}^N$ to another vector $\vec{y} \in \mathbb{C}^N$, with elements

$$y_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x_n \omega_N^{-nk}$$

where $\omega_N = e^{2\pi i/N}$. $\vec{x}$ is expanded in a basis for $\mathbb{C}^N$

The quantum fourier transform (QFT) does exactly the same thing but the vector is now interpreted as a quantum state $|\psi\rangle = \sum_n \psi_n |n\rangle$ in a Hilbert space $\mathcal{H}$.

$$\mathrm{QFT} : |c_n\rangle \rightarrow \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} c_n \omega_N^{-nk}$$

The QFT as a unitary transformation

Suppose we have the N qubit state

$$|\psi\rangle = \sum_{x \in \{0,1\}^N} \alpha_x |x\rangle$$

where $|x\rangle$ are basis states. By definition,

$$\mathrm{QFT}\,|x\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{2\pi ijk/N} |k\rangle$$