Homework 10

Quantum Mechanics
December 6, 2022 C Seitz

Problem 1. *4.7*

Solution.
The wave function in three dimensions for a free particle ($V = 0$), is

$$\psi(\boldsymbol{x}, t) = u(\boldsymbol{x})e^{-iE_n t/\hbar}$$
$$\psi^*(\boldsymbol{x}, -t) = u^*(\boldsymbol{x})e^{-iE_n t/\hbar}$$

where $u(\boldsymbol{x}) = e^{i\vec{p}\cdot\vec{k}}$. Note that the phase remains unchanged under complex conjugation and time reversal. Now if we reverse the direction of momentum i.e. $|p\rangle \to |p'\rangle$ for $\vec{p} \cdot \vec{p'} = -1$,

$$\psi'(\boldsymbol{x}, t) = u'(\boldsymbol{x})e^{-iE_n t/\hbar}$$

Notice that $u'(\boldsymbol{x}) = e^{-i\vec{p'}\cdot\vec{k}} = u^*(\boldsymbol{x})$. Therefore $\psi'(\boldsymbol{x}, t) = \psi^*(\boldsymbol{x}, -t)$

$$\chi_+(\hat{n}) = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\gamma} \end{pmatrix}$$

It is also known that

$$\chi_-(\hat{n}) = \begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\gamma} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

So we just need to show that the given transformation gives this result:

$$-i\sigma_2\chi^*(\hat{n}) = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{-i\gamma} \end{pmatrix} = \begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\gamma} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

■

Problem 2. *4.8*

Solution.

First note that

$$H\Theta\left|n\right\rangle = \Theta H\left|n\right\rangle = E_n\Theta\left|n\right\rangle$$

so $\left|n\right\rangle$ and $\Theta\left|n\right\rangle$ have the same energy. If the states are nondegenerate then $\left|n\right\rangle$ and $\Theta\left|n\right\rangle$ represent the same state. Their wavefunctions are then the same:

$$\langle x'|n\rangle = \langle n|x'\rangle^*$$

which occurs if they are real, or have a phase difference independent of x. For this reason the wavefunction $\psi = e^{ip\cdot x/\hbar}$ does not violate time reversal invariance, because it is degenerate with $e^{ip\cdot x/\hbar}$.

■

Problem 3. *4.9*

Solution.

$$\begin{aligned}
\Theta\left|\alpha\right\rangle &= \int d^3\boldsymbol{p}\;\Theta\left|\boldsymbol{p}\right\rangle\langle\boldsymbol{p}|\alpha\rangle^* \\
&= \int d^3\boldsymbol{p}\;\left|-\boldsymbol{p}\right\rangle\langle\boldsymbol{p}|\alpha\rangle^* \\
&= \int d^3\boldsymbol{p}\;\left|\boldsymbol{p}\right\rangle\langle-\boldsymbol{p}|\alpha\rangle^* \\
&= \phi^*(-p)
\end{aligned}$$

■

Problem 4. *4.10*

Solution. This is because, for the spherical harmonics, $(Y_l^m(\theta,\phi))^* = (-1)^m Y_l^{-m}(\theta,\phi)$. Therefore,

$$\Theta\,|j,m\rangle = e^{i\delta}(-1)^m\,|j,m\rangle$$

■

Problem 5. *4.11*

Solution.

Since the Hamiltonian is time reversal invariant, then energy eigenkets transform as $\Theta\,|\alpha\rangle = e^{i\delta}\,|\alpha\rangle$,

$$\begin{aligned}\langle \boldsymbol{L}\rangle &= \langle\alpha|\,\boldsymbol{L}\,|\alpha\rangle \\ &= e^{i\delta}e^{-i\delta}\,\langle\alpha|\,\Theta\boldsymbol{L}\Theta^{-1}\,|\alpha\rangle \\ &= -\,\langle\alpha|\,\boldsymbol{L}\,|\alpha\rangle\end{aligned}$$

This is only satisfied when $\langle \boldsymbol{L}\rangle = 0$. If the wavefunction is expanded as

$$\sum_l\sum_m F_{lm}(r)Y_l^m(\theta,\phi)$$

We know that when the Hamiltonian is invariant under time-reversal, the eigenkets must be real. Therefore, the phase restriction must satisfy the equality $F_{lm}(r)Y_l^m(\theta,\phi) = F_{lm}^*(r)(Y_l^m(\theta,\phi))^*$.

■

Problem 6. *4.12*

Solution.

We were given the Hamiltonian:

$$H = AS_z^2 + B(S_x^2 - S_y^2)$$

This Hamiltonian must be invariant under time reversal because these are all scalar values which are invariant e.g, $\Theta S_x^2\Theta^{-1} = S_x^2\Theta\Theta^{-1} = S_x^2$ and A and B are real. The explicit matrix representation is

$$H = \hbar^2 \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix}$$

which according to Mathematica has the following eigenvectors

$$\begin{aligned} |E_1\rangle &= (|{+1}\rangle + |{-1}\rangle)/\sqrt{2} \\ |E_{-1}\rangle &= (|{+1}\rangle - |{-1}\rangle)/\sqrt{2} \\ |E_0\rangle &= |0\rangle \end{aligned}$$

with eigenvalues $\hbar^2(A+B), \hbar^2(A-B), 0$ respectively. These eigenvectors transform in the following way under time reversal:

$$\begin{aligned} \Theta\,|E_1\rangle &= (\Theta\,|{+1}\rangle + \Theta\,|{-1}\rangle)/\sqrt{2} = -(|{+1}\rangle + |{-1}\rangle)/\sqrt{2} \\ \Theta\,|E_{-1}\rangle &= (\Theta\,|{+1}\rangle - \Theta\,|{-1}\rangle)/\sqrt{2} = -(|{+1}\rangle - |{-1}\rangle)/\sqrt{2} \\ \Theta\,|E_0\rangle &= (-1)^0\,|E_0\rangle = |E_0\rangle \end{aligned}$$

■