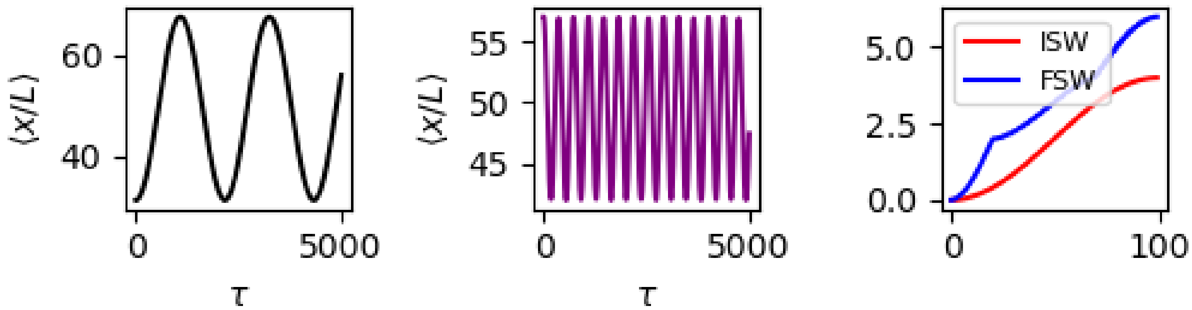

Figure 1: Expectation values of position as a function of time for the infinite (left) and finite (right) square well

Project 2

Quantum Mechanics

December 7, 2022

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Part 1

We can work in a coordinate system centered on zero, and write

$$
\begin{aligned}
\langle x \rangle &= \sum_{a'} \sum_{a''} c^*_{a'} c_{a''} \langle a' | \, x \, | a'' \rangle \exp\left(\frac{-i(E_{a''} - E_{a'})t}{\hbar} \right) \\
&= \frac{1}{2} \left(\langle 0 | \, x \, | 1 \rangle \exp\left(\frac{-i(E_1 - E_0)t}{\hbar} \right) + \langle 1 | \, x \, | 0 \rangle \exp\left(\frac{-i(E_0 - E_1)t}{\hbar} \right) \right) \\
&= \beta \cos(\omega t)
\end{aligned}
$$

where $\beta = \langle 0 | \, x \, | 1 \rangle = \langle 1 | \, x \, | 0 \rangle$ (because x is Hermitian) and $\omega = (E_1 - E_0)/\hbar$. The angular frequency is higher for the finite square well because there is a large energy gap between the first excited state and the ground state (see the differential in the eigenvalue spectrum in Figure 1c).

Part 2

We are using the time-dependent Hamiltonian

$$H(x,t) = H_0(x) + \lambda(1 - e^{-t/\tau})V(x)$$

where $V(x)$ is the finite square well potential. We are assuming that $\tau \to \infty$ so the potentially turns on exactly at $t = 0$, giving a constant perturbation. Notice that V is going to be needed in energy basis, so we will need to transform V using the unitary operator (the $|i\rangle$ basis to the $|\epsilon_n\rangle$ basis).

(H) We are after $P_n(t) = |c_n^{(1)}(t)|^2$ for $n = 1, 2, 3$. In the text, we are given

$$\begin{aligned} c_n^{(1)}(t) &= -\frac{i}{\hbar}V_{ni}\int_0^t e^{i\omega_{ni}t}dt \\ &= \frac{V_{ni}}{E_n - E_i}(1 - e^{i\omega_{ni}t}) \end{aligned}$$

where $V_{ni} = \langle n| V |i\rangle$.

$$|c_n^{(1)}(t)|^2 = \frac{4|V_{ni}|^2}{|E_n - E_i|^2}\sin^2\left(\frac{(E_n - E_i)t}{2\hbar}\right)$$

(I) To find the wavefunction $\langle i|\alpha(\tau)\rangle$ analytically, we need to solve the differential equation

$$i\hbar\dot{c_n}(t) = \sum_m V_{nm}e^{i\omega_{nm}t}c_m(t)$$

where the initial conditions are set such that $c_n(0) = \delta_{n1}$. This gives us the time-evolution of the wavefunction in the energy basis. Given all the $c_n(t)$'s we can consruction an energy superposition

$$|\beta(t)\rangle = \sum_n c_n(t)\,|E_n\rangle$$

So then we need to apply the unitary transformation

$$|\beta(t)\rangle \to U\,|\beta(t)\rangle$$

to the position basis in order to obtain the wavefunction $\langle i|\alpha(\tau)\rangle$. To ensure we have the correct normalization we can just divide expansion coefficients by $Z = \sum_n |c_n|^2$.