

# The Fokker-Planck Equation

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## 1 The Fokker-Planck Equation

### 1.1 Kramers-Moyal Expansion

Given many instantiations of a stochastic variable  $x$ , we can construct a normalized histogram over all observations as a function of time  $P(x, t)$ . However, in order to systematically explore the relationship between the parameterization of the process and  $P(x, t)$  we require an expression for  $\dot{P}(x, t)$ . If we make a fundamental assumption that the evolution of  $P(x, t)$  follows a Markov process i.e. its evolution has the memoryless property, then we can write

$$P(x', t) = \int T(x', t|x, t - \tau)P(x, t - \tau)dx \quad (1)$$

which is known as the Chapman-Kolmogorov equation. The factor  $T(x', t|x, t - \tau)$  is known as the *transition operator* in a Markov process and determines the evolution of  $P(x, t)$  in time. We proceed by writing  $T(x', t|x, t - \tau)$  in a form referred to as the Kramers-Moyal expansion

$$\begin{aligned} T(x', t|x, t - \tau) &= \int \delta(u - x')T(u, t|x, t - \tau)du \\ &= \int \delta(x + u - x' - x)T(u, t|x, t - \tau)du \end{aligned}$$

If we use the Taylor expansion of the  $\delta$ -function

$$\delta(x + u - x' - x) = \sum_{n=0}^{\infty} \frac{(u - x)^n}{n!} \left( -\frac{\partial}{\partial x} \right)^n \delta(x - x')$$

Inserting this into the result from above, pulling out terms independent of  $u$  and swapping the order of the sum and integration gives

$$\begin{aligned}
T(x', t|x, t - \tau) &= \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{\partial}{\partial x} \right)^n \delta(x - x') \int (u - x)^n T(u, t|x, t - \tau) du \quad (2) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{\partial}{\partial x} \right)^n \delta(x - x') M_n(x, t) \quad (3)
\end{aligned}$$

noticing that  $M_n(x, t) = \int (u - x)^n T(u, t|x, t - \tau) du$  is just the  $n$ th moment of the transition operator  $T$ . Plugging (2.6) back in to (2.4) gives

$$P(x, t) = \int \left( 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left( -\frac{\partial}{\partial x} \right)^n M_n(x, t) \right) \delta(x - x') P(x, t - \tau) dx \quad (4)$$

$$= P(x', t - \tau) + \sum_{n=1}^{\infty} \frac{1}{n!} \left( -\frac{\partial}{\partial x} \right)^n [M_n(x, t) P(x, t)] \quad (5)$$

Approximating the derivative as a finite difference and taking the limit  $\tau \rightarrow 0$  gives

$$\dot{P}(x, t) = \lim_{\tau \rightarrow 0} \left( \frac{P(x, t) - P(x, t - \tau)}{\tau} \right) \quad (6)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!} \left( -\frac{\partial}{\partial x} \right)^n [M_n(x, t) P(x, t)] \quad (7)$$

which is formally known as the Kramers-Moyal (KM) expansion. The Fokker-Planck equation is a special case of (2.10) where we neglect terms  $n > 2$  in the *diffusion approximation*.

Consider the following Ito stochastic differential equation

$$d\vec{x} = F(\vec{x}, t) + G(\vec{x}, t) dW$$

The SDE given above corresponds to the Kramers-Moyal expansion (KME) of a transition density  $T(x', t'|x, t)$  see (Risken 1989) for a full derivation.

$$\frac{\partial P(x, t)}{\partial t} = \sum_{n=1}^{\infty} \frac{1}{n!} \left( -\frac{\partial}{\partial x} \right)^n [M_n(x, t) P(x, t)] \quad (8)$$

where  $M_n$  is the  $n$ th moment of the transition density. In the diffusion approximation, the KME becomes the Fokker-Planck equation (FPE) (Risken 1989). For the sake of demonstration, consider the univariate case with random variable  $x$  and the form of  $T(x', t'|x, t)$  is a Gaussian with mean  $\mu(t)$  and variance  $\sigma^2(t)$ . In this scenario, the FPE applies because  $M_n = 0$  for all  $n > 2$ . Given that the drift  $M_1(x, t) = \mu(t)$  and the diffusion  $M_2(x, t) = \sigma^2(t)$ , the FPE reads

$$\frac{\partial P(x, t)}{\partial t} = \left( -\frac{\partial}{\partial x} M^{(1)}(t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} M^{(2)}(t) \right) P(x, t) \quad (9)$$

We can additionally define the term in parentheses as a differential operator acting on  $P(x, t)$

$$\hat{\mathcal{L}}_{FP} = \left( -\frac{\partial}{\partial x} M^{(1)}(t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} M^{(2)}(t) \right) \quad (10)$$

It is common to additionally define the probability current  $J(x, t)$  as

$$J(x, t) = \left( M^{(1)}(t) - \frac{1}{2} \frac{\partial}{\partial x} M^{(2)}(t) \right) P(x, t) \quad (11)$$

This definition provides some useful intuition. The value of  $J(x, t)$  is the net probability flux into the interval between  $x$  and  $x + dx$  at time  $t$ . This also allows us to write the FPE as a continuity equation

$$\frac{\partial P(x, t)}{\partial t} = -\frac{\partial J(x, t)}{\partial x} \quad (12)$$

## 1.2 Solving the FPE: Heat (Diffusion) Equation

The well-known heat equation (it has several names: diffusion equation, heat equation, Brownian motion, Wiener process) is a special case of the FPE where  $M^{(1)}(t) = 0$  and  $M^{(2)}(t) = \sigma^2 = \text{const.}$

$$\frac{\partial P(x, t)}{\partial t} = D \frac{\partial^2 P(x, t)}{\partial x^2} \quad (13)$$

with  $D = \sigma^2/2$ . We would like to solve the above equation, but it is a PDE which usually require some tricks to solve e.g., integral transforms. Generally a transform can reduce a differential equation to a simpler form, like an ODE. Upon Fourier transformation, spatial derivatives turn into factors of  $ik$ . That is,

$$\frac{\partial}{\partial x} \psi(x, t) \rightarrow ik \tilde{\psi}(k, t) \quad \frac{\partial^2}{\partial x^2} \psi(x, t) \rightarrow -k^2 \tilde{\psi}(k, t)$$

### 1.2.1 Fourier Transform of the Heat Equation

Recall the general definition of a Fourier pair

$$\begin{aligned}\tilde{\psi}(k) &= \mathcal{F}[\psi] = \int_{-\infty}^{\infty} \psi(x) e^{-2\pi i k x} dx \\ \psi(x) &= \mathcal{F}^{-1}[\tilde{\psi}] = \int_{-\infty}^{\infty} \tilde{\psi}(k) e^{2\pi i k x} dk\end{aligned}$$

Let's see the Fourier transformation of Eq. (6)

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} P(x, t) e^{-2\pi i k x} dx = D \int_{-\infty}^{\infty} \frac{\partial^2 P(x, t)}{\partial x^2} e^{-2\pi i k x} dx \quad (14)$$

As mentioned above,  $\mathcal{F}[\partial_x \psi] = ik\mathcal{F}[\psi]$  and  $\mathcal{F}[\partial_x^2 \psi] = -k^2\mathcal{F}[\psi]$  which allows us to write the heat equation as a first order equation

$$\frac{\partial \tilde{P}(k, t)}{\partial t} = -Dk^2 \tilde{P}(k, t) \quad (15)$$

which suggests the solution  $\tilde{p}_0(k) \exp(-Dk^2 t)$ , which is Gaussian in  $k$ -space. Let's say our initial condition satisfies  $\tilde{P}(x, t_0) = \delta(x - x_0)$  which in the Fourier domain is  $P(k, t_0) = \exp(-ikx_0)$ . The inverse transform is

$$\int_{-\infty}^{\infty} \tilde{p}_0(k) \exp(ikx - Dk^2 t) dk = \int_{-\infty}^{\infty} \exp(ik(x - x_0) - Dk^2 t) dk \quad (16)$$

which we can rewrite as

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-(Dk^2 t - ik(x - x_0))) dk = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-Dt \left(k^2 - \frac{ik(x - x_0)}{Dt}\right)\right) dk$$

Now we would like to complete the square in the exponential, since we know how to do Gaussian integrals. This can be done as follows:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-Dt \left(k^2 - \frac{ik(x - x_0)}{Dt} + \frac{(x - x_0)^2}{4D^2 t^2} - \frac{(x - x_0)^2}{4D^2 t^2}\right)\right) dk$$

We are then left to simplify,

$$\begin{aligned}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-Dt \left(k - \frac{i(x - x_0)}{2Dt}\right)^2\right) dk &= \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x - x_0)^2}{4Dt}\right) \int_{-\infty}^{\infty} \exp(-Dtk'^2) dk' \\ &= \frac{1}{\sqrt{2Dt}} \exp\left(-\frac{(x - x_0)^2}{4Dt}\right)\end{aligned}$$

which is a Gaussian distribution with time-dependent variance  $\sigma = 4Dt$ , given originally by Einstein in his famous paper on Brownian motion in 1905.

### 1.3 Solving the FPE: Ornstein-Uhlenbeck

The Ornstein-Uhlenbeck process is another special case of the FPE where  $M^{(1)}(t) = -\gamma$  and  $M^{(2)}(t) = \sigma^2 = \text{const.}$  It is a stationary Gauss–Markov process, which means that it is a Gaussian process, a Markov process, and is temporally homogeneous. The Ito SDE for this process reads

$$dx = -\gamma x dt + \sigma dW \quad (17)$$

which of course has a corresponding Fokker-Planck equation

$$\frac{\partial P(x, t)}{\partial t} = -\gamma \frac{\partial}{\partial x} x P(x, t) + D \frac{\partial^2 P(x, t)}{\partial x^2} \quad (18)$$

In this form, the solution is slightly complicated by the presence of the first order spatial derivative. However, we can still find a solution via a Fourier transform:

$$\frac{\partial \tilde{P}(k, t)}{\partial t} = -\gamma k \frac{\partial \tilde{P}(k, t)}{\partial k} - k^2 D \tilde{P}(k, t) \quad (19)$$

Notice that this is a partial differential equation with the general form

$$a(\tilde{P}, k, t) \partial_k \tilde{P} + b(\tilde{P}, k, t) \partial_t \tilde{P} - c(\tilde{P}, k, t) = 0 \quad (20)$$

Therefore can solve the above equation using the method of characteristics. As a brief review, suppose we know a solution surface  $\tilde{P}$ . A vector normal to this surface has the form  $\vec{u} = \langle \partial_k \tilde{P}, \partial_t \tilde{P}, -1 \rangle$ . If this vector is normal to the surface, then the vector field

$$\vec{v} = \langle a(\tilde{P}, k, t), b(\tilde{P}, k, t), c(\tilde{P}, k, t) \rangle \quad (21)$$

is tangent to the surface at every point. In other words, we would like to find a surface  $\tilde{P}(k, t)$  for which the vector field above lies in the tangent plane to  $\tilde{P}(k, t)$  and therefore  $\vec{u} \cdot \vec{v} = 0$ . The task that remains then is to find a  $\tilde{P}(k, t)$  s.t. the vector  $\vec{u}$  is orthogonal to  $\vec{v}$ . Now, if we construct a curve  $\mathcal{C}$  which is an integral curve of  $\vec{v}$ , then this curve lies on the solution surface  $\tilde{P}(k, t)$ . Such a curve satisfies the ODEs

$$\begin{aligned} \frac{dk}{ds} &= \gamma k \\ \frac{dt}{ds} &= 1 \\ \frac{d\tilde{P}}{ds} &= -k^2 D \tilde{P} \end{aligned}$$

since the vector field given by the Fokker-Planck equation we have is  $\vec{v} = \langle \gamma k, 1, -k^2 D \rangle$ . Clearly  $t = s$  and  $k = k_0 \exp(\gamma t)$  and thus

$$\frac{d\tilde{P}}{dt} = -k^2 D \tilde{P} \quad (22)$$

$$= -D k_0^2 \exp(2\gamma t) \tilde{P} \quad (23)$$

and we have the solution in the Fourier domain

$$\tilde{P}(k, t) = \tilde{P}(k, 0) \exp\left(-\frac{D k_0^2}{2\gamma} (\exp(2\gamma t) - 1)\right) \quad (24)$$

$$= \exp\left(-i k_0 x_0 - \frac{D k_0^2}{2\gamma} (\exp(2\gamma t) - 1)\right) \quad (25)$$

$$= \exp\left(-i k e^{-\gamma t} x_0 - \frac{D k^2}{2\gamma} (1 - \exp(-2\gamma t))\right) \quad (26)$$

Let  $\mu(t) = x_0 \exp(-\gamma t)$  and  $\sigma^2(t) = \frac{D}{\gamma} (1 - e^{-2\gamma t})$

$$\tilde{P}(k, t) = \exp\left(-i k \mu(t) - \frac{k^2}{2} \sigma^2(t)\right) \quad (27)$$

Taking the inverse Fourier transform of this equation gives

$$P(x, t) = \frac{1}{\sqrt{2\sigma^2(t)}} \exp\left(-\frac{(x - \mu(t))^2}{2\sigma^2(t)}\right) \quad (28)$$

## 1.4 The Multivariate Case

If we now generalize the above equation to a case where we are faced with many variables  $\mathbf{x} = (x_1, x_2, \dots, x_n)$ . The continuity equation becomes

$$\frac{\partial P(\vec{x}, t)}{\partial t} = -\vec{\nabla} \cdot \mathbf{J}(\vec{x}, t) \quad (29)$$

where the multivariate probability current now has the interpretation of the net flux into or out of a volume  $dx^n$  centered around  $\mathbf{x}$ . If we consider each dimension,

$$J(x_i, t) = \left( M_i^{(1)}(t) - \sum_j \frac{\partial}{\partial x_j} M_{ij}^{(2)}(t) \right) P(\vec{x}, t) \quad (30)$$

The full Fokker-Planck equation then reads

$$\frac{\partial P(\vec{x}, t)}{\partial t} = \vec{\nabla} \cdot J(\vec{x}, t) \quad (31)$$

$$= \sum_{i=1}^N \left( -\frac{\partial}{\partial x_i} M_i^{(1)}(t) + \sum_{j=1}^N \frac{\partial^2}{\partial x_i \partial x_j} M_{ij}^{(2)}(t) \right) P(\vec{x}, t) \quad (32)$$

It proves quite useful in this form because we can see that the Fokker-Planck equation represents a differentiation operator acting on the distribution  $P(\vec{x}, t)$

$$\hat{\mathcal{L}}_{FP} = \sum_{i=1}^N \left( -\frac{\partial}{\partial x_i} M_i^{(1)}(t) + \sum_{j=1}^N \frac{\partial^2}{\partial x_i \partial x_j} M_{ij}^{(2)}(t) \right) \quad (33)$$

## 1.5 Ornstein-Uhlenbeck Process

If the transition density is Gaussian then the density is fully specified by the first two moments  $M^{(1)}(t) = \vec{\mu}(t)$  and  $M^{(2)}(\vec{x}, t) = \Sigma(t)$ . The moments can also be functions of  $\vec{x}$ . Both of these possibilities are evident in the Ornstein-Uhlenbeck (OU) process. Let the drift vector be a linear function of the state  $\vec{x}$  and the diffusion matrix the square of the Gaussian covariances

$$M^{(1)}(t) = \Gamma \vec{x} \quad M^{(2)}(t) = 2D$$

with  $D = \Sigma \Sigma^T$  and  $\Gamma$  are assumed to be independent of time (non-volatile).

The Fourier transform of (34) is fairly simple, since the FT is linear. We will switch to matrix notation to drop the summations

$$\frac{\partial \tilde{P}(\vec{k}, t)}{\partial t} = -\Gamma \mathbf{k} \frac{\partial}{\partial \mathbf{k}} \tilde{P}(\vec{k}, t) + \mathbf{D} \mathbf{k} \mathbf{k}^T \tilde{P}(\vec{k}, t) \quad (34)$$

This is in a very similar form to the single variable case (18). We then make the ansatz in analogy with (27)

$$\tilde{P}(\mathbf{k}, t) = \exp \left( -i \mathbf{k}^T \boldsymbol{\mu}(t) - \frac{\mathbf{k} \mathbf{k}^T}{2} \Sigma(t) \right) \quad (35)$$

Plugging this into (34) we have

$$\frac{\partial \tilde{P}(\vec{k}, t)}{\partial t} + \Gamma \mathbf{k} \frac{\partial}{\partial \mathbf{k}} \tilde{P}(\vec{k}, t) + \mathbf{D} \mathbf{k} \mathbf{k}^T \tilde{P}(\vec{k}, t) \quad (36)$$

$$= -i \mathbf{k} \dot{\boldsymbol{\mu}}(t) - \frac{1}{2} \mathbf{k} \mathbf{k}^T \dot{\Sigma} - i \Gamma \boldsymbol{\mu}^T - \Gamma \quad (37)$$