Homework 5

Quantum Mechanics
March 6, 2023 C Seitz

Problem 1. *Problem 4.4*

Solution.

$$\begin{aligned} H &= e^{i\alpha} R_z(\frac{\pi}{2}) R_x(\frac{\pi}{2}) \\ &= \frac{e^{i\alpha} e^{i\pi/4}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{aligned}$$

Therefore, $\alpha = -\pi/4$. ∎

Problem 2. *Problem 4.5*

Solution.

$$\begin{aligned} (n \cdot \sigma)^2 &= (n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)^2 \\ &= n_x^2\sigma_x^2 + n_y^2\sigma_y^2 + n_z^2\sigma_z^2 \\ &= (n_x^2 + n_y^2 + n_z^2)I = I \end{aligned}$$

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Problem 3. *Problem 4.7*

Solution.

Simple matrix operations can confirm that $XYX = -Y$. It follows that

$$e^{\frac{i\theta}{2}(XYX)} = e^{-\frac{i\theta}{2}Y}$$

Now recall that $U^\dagger e^A U = e^{U^\dagger A U}$, which can be proven via a series expansion. Of course X is both unitary and hermitian, so we get that

$$X e^{\frac{i\theta}{2}Y} X = e^{-\frac{i\theta}{2}Y}$$

which is the desired result.

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Problem 4. *Problem 4.16*

Solution. For the first circuit, the matrix representation is

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & h_{11} & h_{12} \\ 0 & 0 & h_{21} & h_{22} \end{pmatrix}$$

For the second circuit, the matrix representation is

$$A = \begin{pmatrix} h_{11} & h_{12} & 0 & 0 \\ h_{21} & h_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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Problem 5. *Problem 4.17*

Solution.

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Problem 6. *Problem 4.18*

Solution. The controlled-Z gate acts in the following way on the basis kets

$$\begin{aligned} CZ\,|00\rangle &= |00\rangle \\ CZ\,|01\rangle &= |01\rangle \\ CZ\,|10\rangle &= |10\rangle \\ CZ\,|11\rangle &= -\,|11\rangle \end{aligned}$$

For $|01\rangle$ and $|10\rangle$, the result is the same, so the controlled-Z operator shown is the same at least with respect to those states. Then we see that the result on $|11\rangle$ is just a global phase, so we can safely conclude the same operator works, regardless of which qubit is the control.

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Problem 7. *Problem 4.19*

Solution.

The density matrix is

$$\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|$$

We can see how the gate transforms the density matrix by just considering how it acts on the most general state $\left|\psi_i\right\rangle = \alpha_i \left|00\right\rangle + \beta_i \left|01\right\rangle + \gamma_i \left|10\right\rangle + \delta_i \left|11\right\rangle$.

$$\text{CNOT} = \left|00\right\rangle \left\langle 00\right| + \left|01\right\rangle \left\langle 01\right| + \left|11\right\rangle \left\langle 10\right| + \left|10\right\rangle \left\langle 11\right|$$

It is straightforward to see that

$$\begin{aligned} \text{CNOT} \left|\psi_i\right\rangle \left\langle\psi_i\right| &= \left(\alpha_i \left|00\right\rangle + \beta_i \left|01\right\rangle + \gamma_i \left|11\right\rangle + \delta_i \left|10\right\rangle\right) \\ &\quad * \left(\alpha_i^* \left\langle 00\right| + \beta_i^* \left\langle 01\right| + \gamma_i^* \left\langle 10\right| + \delta_i^* \left\langle 11\right|\right) \\ &= \begin{pmatrix} |\alpha_i|^2 & \alpha_i\beta_i^* & \alpha_i\gamma_i^* & \alpha_i\delta_i^* \\ \beta_i\alpha_i^* & |\beta_i|^2 & \beta_i\gamma_i^* & \beta_i\delta_i^* \\ \delta_i\alpha_i^* & \delta_i\beta_i^* & \delta_i\gamma_i^* & |\delta_i|^2 \\ \gamma_i\alpha_i^* & \gamma_i\beta_i^* & |\gamma_i|^2 & \gamma_i\delta_i^* \end{pmatrix} \end{aligned}$$

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