Exam 1

Quantum Mechanics
October 14th, 2022 C Seitz

Problem 1.

Solution.
I will use Mathematica to find the eigenvectors and eigenvalues of the operators A and B. Omitting the factor of λ temporarily

$A = \{\{2, I/\text{Sqrt}[2], 0\}, \{-I/\text{Sqrt}[2], 2, I/\text{Sqrt}[2]\}, \{0, -I/\text{Sqrt}[2], 2\}\}$

$\{\text{vals}, \text{vecs}\} = \text{Eigensystem}[A];$

vecs

vals

$$\left\{\left\{2, \frac{i}{\sqrt{2}}, 0\right\}, \left\{-\frac{i}{\sqrt{2}}, 2, \frac{i}{\sqrt{2}}\right\}, \left\{0, -\frac{i}{\sqrt{2}}, 2\right\}\right\}$$

$$\left\{\left\{-1, i\sqrt{2}, 1\right\}, \{1, 0, 1\}, \left\{-1, -i\sqrt{2}, 1\right\}\right\}$$

$$\{3, 2, 1\}$$

Including the appropriate normalization so $\langle a_n | a_n \rangle = 1$, the eigenvectors are

$$|a_1\rangle = \frac{1}{2}\begin{pmatrix} -1 \\ i\sqrt{2} \\ 1 \end{pmatrix} \quad |a_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad |a_3\rangle = \frac{1}{2}\begin{pmatrix} -1 \\ -i\sqrt{2} \\ 1 \end{pmatrix}$$

with eigenvalues $3\lambda, 2\lambda, \lambda$, repspectively. As a side note, see that $|a_1\rangle$ and $|a_3\rangle$ are not orthogonal in this basis. The eigenvectors of operator B are

$B = \{\{3, \mathrm{Sqrt}[2], 0\}, \{\mathrm{Sqrt}[2], 3, \mathrm{Sqrt}[2]\}, \{0, \mathrm{Sqrt}[2], 3\}\}$

$\{\mathrm{vals}, \mathrm{vecs}\} = \mathrm{Eigensystem}[B];$

vecs

vals

$\{\{3, \sqrt{2}, 0\}, \{\sqrt{2}, 3, \sqrt{2}\}, \{0, \sqrt{2}, 3\}\}$

$\{\{1, \sqrt{2}, 1\}, \{-1, 0, 1\}, \{1, -\sqrt{2}, 1\}\}$

$\{5, 3, 1\}$

Again, including appropriate normalization factors

$$|b_1\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \quad |b_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad |b_3\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

with eigenvalues $5\lambda, 3\lambda, \lambda$, in that order.

If the physicist sends a pure ensemble in state $|1\rangle$ and we measure A, the probability we observe the particle to be in the state $|a_1\rangle$, $|a_2\rangle$, and $|a_3\rangle$ can be found by using the expressions for eigenkets of A above. These probabilities are

$$\begin{aligned}
|\langle a_1|1\rangle|^2 &= \frac{1}{4} \\
|\langle a_2|1\rangle|^2 &= \frac{1}{2} \\
|\langle a_3|1\rangle|^2 &= \frac{1}{4}
\end{aligned}$$

The values of A that correspond with each beam are given by the eigenvalues of A written above. The relative intensities are just given by the relative probabilities. After the coffee spill, measuring A filters out only the beam corresponding to state $|a_2\rangle$. Therefore, these are the only particles that enter the apparatus measuring B. So when we measure B, the number of beams we get depends on the values of

$$
\begin{aligned}
|\langle b_1|a_2\rangle|^2 &= \frac{1}{2}\\
|\langle b_2|a_2\rangle|^2 &= 0\\
|\langle b_3|a_2\rangle|^2 &= \frac{1}{2}
\end{aligned}
$$

so there are only two beams. The intensities are equal for these two beams. The values of B that correspond with these two beams are b_1 and b_3. ■

Problem 2.

Solution. We are given the commutation relations between the operators C and D and the Hamiltonian, which suggests we should use the Heisenberg picture. The Heisenberg equations of motion are

$$
\begin{aligned}
\frac{dC}{dt} &= -\frac{1}{i\hbar}[H, C] = \alpha D - \beta C\\
\frac{dD}{dt} &= -\frac{1}{i\hbar}[H, D] = -\alpha C - \beta D
\end{aligned}
$$

When $\beta = 0$, the system becomes

$$
\begin{aligned}
\frac{dC}{dt} &= -\frac{1}{i\hbar}[H, C] = \alpha D\\
\frac{dD}{dt} &= -\frac{1}{i\hbar}[H, D] = -\alpha C
\end{aligned}
$$

which has the solution

$$
\begin{aligned}
C(t) &= -c_0\cos(\alpha t)\\
D(t) &= d_0\sin(\alpha t)
\end{aligned}
$$

$$
\begin{aligned}
\langle C(t)\rangle &= \langle\alpha|\, C(t)\,|\alpha\rangle\\
&= -\langle\alpha|\, c_0\cos(\alpha t)\,|\alpha\rangle\\
&= -c_0\cos(\alpha t)
\end{aligned}
$$

since $|\alpha\rangle$ is presumed to be normalized.

$$\begin{aligned}\langle D(t)\rangle &= \langle\alpha|\, D(t)\, |\alpha\rangle \\ &= -\langle\alpha|\, d_0 \sin(\alpha t)\, |\alpha\rangle \\ &= d_0 \sin(\alpha t)\end{aligned}$$

since $|\alpha\rangle$ is again presumed to be normalized. The constants α and β must then relate to the angular frequency of $C(t)$ and $D(t)$. Of course, when $\beta = 0$, only α determines the angular frequency of $C(t)$ and $D(t)$.

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Problem 3.

Solution.
We are given the wavefunction

$$\psi(r,\phi) = Ae^{-br^2}$$

We know that probability current is related to the spatial gradient of the phase of the wavefunction. The exponential is real if b is real, so the phase of $\psi(r,\phi)$ is the same for all (r,ϕ) and therefore the probability current is zero everywhere. For the wavefunction

$$\psi(r,\phi) = Ae^{-br^2}e^{-im\phi}$$

The phase of the wavefunction is clearly dependent on ϕ, so there is a probability current. Recall that we can always write the wavefunction in the form:

$$\begin{aligned}\psi(r,\phi) &= \sqrt{\rho(r,\phi)}\exp\left(\frac{iS(r,\phi)}{\hbar}\right) \\ &= Ae^{-br^2}e^{-im\phi}\end{aligned}$$

so $-im\phi = iS/\hbar$. Expanding S gives

$$S(r,\phi) = -m\hbar\phi$$

The probability flux is related to $S(r,\phi)$ by

$$
\begin{aligned}
\boldsymbol{j}(r,\phi) &= \frac{\rho(r,\phi)\nabla S}{m} \\
&= \frac{\rho}{m}\left(\frac{\partial S}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial S}{\partial \phi}\hat{\phi}\right) \\
&= -\frac{\hbar\rho}{r}\hat{\phi}
\end{aligned}
$$

In the limit $r \to \infty$, we can see that

$$
\lim_{r\to\infty} \boldsymbol{j}(r,\phi) = \lim_{r\to\infty} \; -\frac{\hbar\rho}{r}\hat{\phi} = 0
$$

which makes sense if the particle is localized to some region of space. However, $\rho(r,\phi)$ is static because

$$
\nabla \cdot \boldsymbol{j} = \frac{j_r}{r} + \frac{\partial j_r}{\partial r}\frac{1}{r} + \frac{\partial j_\phi}{\partial \phi} = 0
$$

This situation might correspond to a particle with spin. This might be analagous to the the case of a eigenstate of linear momentum, where we associate the probability flux with a "velocity". Presumably this is an eigenfunction of some Hamiltonian, so when we apply time evolution, a probability flux going around a loop makes the real and imaginary parts of the wave function oscillate. Yet, ρ is preserved.

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