Homework 4

Quantum Mechanics
February 26, 2023 C Seitz

Problem 1. *Problem 2.65*

Solution. Let us call these states $|\alpha\rangle$ and $|\beta\rangle$:

$$|\alpha\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$
$$|\beta\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

If we choose a non-orthogonal basis, such as

$$|e_1\rangle = |0\rangle \quad |e_2\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

These states have the following representation in this new basis

$$\begin{aligned} |\alpha'\rangle &= (|e_1\rangle \langle e_1| + |e_2\rangle \langle e_2|) |\alpha\rangle \\ &= \frac{1}{\sqrt{2}} |e_1\rangle + |e_2\rangle \end{aligned}$$

$$\begin{aligned} |\beta'\rangle &= (|e_1\rangle \langle e_1| + |e_2\rangle \langle e_2|) |\beta\rangle \\ &= \frac{1}{\sqrt{2}} |e_1\rangle \end{aligned}$$

The norm is not preserved, because the change of basis matrix $|e_1\rangle \langle e_1| + |e_2\rangle \langle e_2|$ was not unitary. But it is clear that these states differ neither by a global or relative phase.

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Problem 2. *Problem 2.66*

Solution.

$$\begin{aligned}\langle\alpha| X_1 Z_2 |\alpha\rangle &= \frac{1}{2}(\langle 00| + \langle 11|)X_1 Z_2(|00\rangle + |11\rangle) \\ &= \frac{1}{2}(\langle 00| + \langle 11|)(|10\rangle - |01\rangle) = 0\end{aligned}$$

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Problem 3. *Problem 2.71*

Solution.

$$\begin{aligned}\mathrm{Tr}(\rho^2) &= \sum_k \langle k| \left(\sum_i p_i |\alpha_i\rangle \langle\alpha_i|\right) \left(\sum_j p_j |\alpha_j\rangle \langle\alpha_j|\right) |k\rangle \\ &= \left(\sum_{ijk} p_i p_j \langle k|\alpha_i\rangle \langle\alpha_i|\alpha_j\rangle \langle\alpha_j|k\rangle\right) \\ &= \sum_{ij} p_i p_j |\langle\alpha_i|\alpha_j\rangle|^2 \\ &= \sum_i p_i^2 \leq 1\end{aligned}$$

if $|\alpha_i\rangle$ and $|\alpha_j\rangle$ are orthonormal.

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Problem 4. *Problem 2.72*

Solution.

The Pauli matrices form a valid basis for 2x2 matrices. The Bloch vector representation for $\rho = I/2$ is $\vec{r} = 0$.

$$\begin{aligned}\mathrm{Tr}(\rho^2) &= \mathrm{Tr}\left(\frac{I + 2(\vec{r}\cdot\sigma) + (\vec{r}\cdot\sigma)^2}{4}\right) \\ &= \frac{1}{2} + \frac{||\vec{r}||^2}{2} = 1\end{aligned}$$

which occurs when $||\vec{r}||^2 = 1$. This is just algebra once we notice that the trace of $\vec{r} \cdot \sigma$ is zero and the trace of $(\vec{r} \cdot \sigma)^2 = 2(r_x^2 + r_y^2 + r_z^2)$ (the cross terms cancel since the anticommutator $\{\sigma_i, \sigma j\} = \delta_{ij}$)

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Problem 5. *Problem 2.75*

Solution.

The bell states are

$$\begin{aligned}
\left|\phi^+\right\rangle &= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \\
\left|\phi^-\right\rangle &= \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) \\
\left|\psi^+\right\rangle &= \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) \\
\left|\psi^-\right\rangle &= \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)
\end{aligned}$$

For each state, we can trace out either the first or second qubit.

$$\begin{aligned}
\mathrm{tr}_1(\left|\phi^+\right\rangle\left\langle\phi^+\right|) &= \frac{\mathrm{tr}_1 |00\rangle\langle 00| + \mathrm{tr}_1 |00\rangle\langle 11| + \mathrm{tr}_1 |11\rangle\langle 00| + \mathrm{tr}_1 |11\rangle\langle 11|}{2} \\
&= \frac{\langle 0|0\rangle |0\rangle\langle 0| + \langle 0|1\rangle |0\rangle\langle 1| + \langle 1|0\rangle |1\rangle\langle 0| + \langle 1|1\rangle |1\rangle\langle 1|}{2} \\
&= \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2}
\end{aligned}$$

In fact we get this same result when we trace out either qubit for any of the Bell states. Applying either partial trace to the cross terms always gives zero.

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Problem 6. *Problem 2.79*

Solution.

For $|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$, we have

$$|\psi\rangle = \sum_{jk} a_{jk} |j\rangle |k\rangle \quad a = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The matrix a is already diagonal, so the unitary matrices in the SVD will be identity matrices. Therefore,

$$|i_A\rangle = |0\rangle_A, |1\rangle_A \quad |i_B\rangle = |0\rangle_B, |1\rangle_B \quad \lambda_i = \frac{1}{\sqrt{2}}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B \right)$$

For $|\psi\rangle = \frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{2}$, we have the SVD

$$a = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$|i_A\rangle = -\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle)$$

$$|i_B\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle), \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle)$$

$$|\psi\rangle = -\frac{1}{2} \left(|0\rangle + |1\rangle \right) \left(|0\rangle + |1\rangle \right)$$

For $|\psi\rangle = \frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{3}}$, we have the SVD

$$a = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

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