

Homework 5

Quantum Mechanics

October 17th, 2022

C SEITZ

Problem 1. *Problem 3.10 from Sakurai*

Solution.

$$\begin{aligned}\exp(i(\boldsymbol{\sigma} \cdot \hat{\mathbf{n}})\theta) &= \begin{pmatrix} \cos \theta + i n_z \sin \theta & (-i n_x + n_y) \sin \theta \\ (i n_x + n_y) \sin \theta & \cos \theta - i n_z \sin \theta \end{pmatrix} \\ &= \begin{pmatrix} e^{-(i\alpha+\gamma)/2} \cos \frac{\beta}{2} & -e^{-(i\alpha-\gamma)/2} \sin \frac{\beta}{2} \\ e^{-(i\alpha-\gamma)/2} \sin \frac{\beta}{2} & e^{(i\alpha+\gamma)/2} \cos \frac{\beta}{2} \end{pmatrix}\end{aligned}$$

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Equating the trace of these matrices gives

$$2 \cos \theta = 2 \cos \left(\frac{\alpha + \gamma}{2} \right) \cos \frac{\beta}{2}$$

$$\text{So } \theta = \cos^{-1} \left(\cos \left(\frac{\alpha + \gamma}{2} \right) \cos \frac{\beta}{2} \right)$$

Problem 2. *Problem 3.20 from Sakurai*

Solution.

Recall that

$$J_{\pm} = J_x \pm iJ_y$$

and thus $J_x = (J_+ + J_-)/2$ and $J_y = \frac{J_+ - J_-}{2i}$. We know that the matrix elements of J_{\pm} are

$$\langle j', m' | J_{\pm} | j, m \rangle = \sqrt{(j \mp m)(j \pm m + 1)} \hbar \delta_{jj'} \delta_{m, m' \pm 1}$$

where j is our usual shorthand for $\hbar^2 j(j+1)$ (the eigenvalue of J^2) and m is short for $m\hbar$ (the eigenvalue of J_z). For a spin-1 system, $j = 1$ and $m = -1, 0, 1$ which gives the eigenkets $|1, -1\rangle, |1, 0\rangle, |1, 1\rangle$

$$J_+ = \begin{pmatrix} 0 & \sqrt{2}\hbar & 0 \\ 0 & 0 & \sqrt{2}\hbar \\ 0 & 0 & 0 \end{pmatrix} \quad J_- = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2}\hbar & 0 & 0 \\ 0 & \sqrt{2}\hbar & 0 \end{pmatrix}$$

$$J_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad J_y = \frac{\hbar}{\sqrt{2}i} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

We can use Mathematica to find the eigenvectors of these two matrices

$$|J_x; +\rangle = \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix} \quad |J_x; 0\rangle = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \quad |J_x; -1\rangle = \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix}$$

$$|J_y; +\rangle = \begin{pmatrix} -1/2 \\ -i\sqrt{2} \\ 1/2 \end{pmatrix} \quad |J_y; 0\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \quad |J_y; -1\rangle = \begin{pmatrix} -1/2 \\ i\sqrt{2} \\ 1/2 \end{pmatrix}$$

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Problem 3. *Problem 3.22 from Sakurai*

Solution.

We are asked to derive

$$\langle x | L_z | \alpha \rangle = -i\hbar \frac{\partial}{\partial \phi} \langle x | \alpha \rangle$$

$$\begin{aligned}
\langle x | L_z | \alpha \rangle &= \langle x | (xp_y - yp_x) | \alpha \rangle \\
&= -\langle x | i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) | \alpha \rangle \\
&= \left(r \cos \phi \sin \theta \left(\sin \phi \sin \theta \frac{\partial}{\partial r} \right) - r \sin \phi \sin \theta \left(\cos \phi \sin \theta \frac{\partial}{\partial r} \right) \right) \langle x | \alpha \rangle \\
&+ \left(r \cos \phi \sin \theta \left(\frac{1}{r} \cos \theta \sin \phi \right) - r \sin \phi \sin \theta \left(\frac{1}{r} \sin \theta \sin \phi \right) \right) \langle x | \alpha \rangle \\
&+ \left(r \cos \phi \sin \theta \left(-\frac{\sin \phi}{r \sin \theta} \frac{\partial}{\partial \theta} \right) - r \sin \phi \sin \theta \left(\cos \phi \sin \theta \frac{\partial}{\partial r} \right) \right) \langle x | \alpha \rangle
\end{aligned}$$

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Problem 4. *Problem 3.23 from Sakurai*

Solution.

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Problem 5. *Problem 3.24 from Sakurai*

Solution.

$$\begin{aligned}
\langle l, m | L_x | l, m \rangle &= \frac{1}{2} \langle l, m | (L_+ + L_-) | l, m \rangle = 0 \\
\langle l, m | L_y | l, m \rangle &= \frac{1}{2i} \langle l, m | (L_+ - L_-) | l, m \rangle = 0
\end{aligned}$$

$$\begin{aligned}
\langle l, m | L_x^2 | l, m \rangle &= \frac{1}{4} \langle l, m | (L_+^2 + L_+ L_- + L_- L_+ + L_-^2) | l, m \rangle \\
&= \frac{1}{4} \langle l, m | (L_+ L_- + L_- L_+) | l, m \rangle \\
&= \frac{1}{4} (\hbar^2 l(l+1) - m^2 \hbar^2) + \frac{1}{4} (\hbar^2 l(l+1) - m^2 \hbar^2) \\
&= \frac{1}{2} (\hbar^2 l(l+1) - m^2 \hbar^2)
\end{aligned}$$

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Problem 6. *Problem 3.38 from Sakurai*

Solution.

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