Homework 8

Quantum Mechanics
November 10, 2022 C Seitz

Problem 1. *5.27*

Solution.

$$\frac{\langle\tilde{0}|\, H \,|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle} \geq E_0$$

The denominator is easy to compute

$$2\int_{-\infty}^{0} e^{\beta x} dx = \frac{1}{\beta}$$

The numerator

$$\begin{aligned}
\langle\tilde{0}|\, H \,|\tilde{0}\rangle &= \int_{-\infty}^{\infty} \psi^*(x) H \psi(x) dx \\
&= \int_{-\infty}^{0} \psi^*(x) H \psi(x) dx + \int_{0}^{\infty} \psi^*(x) H \psi(x) dx
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{0} \psi^*(x) H \psi(x) dx &= \int_{-\infty}^{0} -e^{\beta x} \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} e^{\beta x} + \frac{1}{2} m\omega^2 x^2 e^{2\beta x} dx \\
&= \int_{-\infty}^{0} e^{2\beta x} \left(\frac{1}{2} m\omega^2 x^2 - \frac{\hbar^2 \beta^2}{2m} \right) dx \\
&= \Big|_{-\infty}^{0} \frac{1}{2} m\omega^2 \frac{e^{2\beta x}(1 - 2\beta x + 2\beta^2 x^2)}{4\beta^3} - e^{2\beta x} \frac{\hbar^2 \beta}{4m} \\
&= \frac{1}{2} m\omega^2 \frac{1}{4\beta^3} - \frac{\hbar^2 \beta}{4m}
\end{aligned}$$

$$\int_0^\infty \psi^*(x)H\psi(x)dx = \int_0^\infty -e^{-\beta x}\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}e^{-\beta x} + \frac{1}{2}m\omega^2 x^2 e^{-2\beta x}dx$$
$$= \int_0^\infty e^{-2\beta x}\left(\frac{1}{2}m\omega^2 x^2 - \frac{\hbar^2\beta^2}{2m}\right)dx$$
$$= \Big|_0^\infty \frac{1}{2}m\omega^2\frac{e^{-2\beta x}(1+2\beta x+2\beta^2x^2)}{4\beta^3} - e^{-2\beta x}\frac{\hbar^2\beta}{4m}$$
$$= \frac{1}{2}m\omega^2\frac{1}{4\beta^3} - \frac{\hbar^2\beta}{4m}$$

$$\bar{H} = \frac{\langle\tilde{0}|\,H\,|\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle} = \frac{m\omega^2}{4\beta^2} - \frac{\hbar^2\beta^2}{2m}$$

$$\frac{d\bar{H}}{d\beta} = -\frac{m\omega^2}{4\beta} - \frac{\hbar^2\beta}{m} = 0$$

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Problem 2. *5.29*

Solution.

We have the full time-dependent Hamiltonian

$$H(t) = H_0 + F_0 x\cos\omega t$$

We need to find $|\psi(t)\rangle$, which amounts to finding the expansion coefficients $c_n(t)$. In the interaction picture, we have that

$$i\hbar\dot{c}_n(t) = \sum_m V_{nm}e^{i\omega_{nm}t}c_m(t)$$

for $\omega_{nm} = (E_n - E_m)/\hbar$.

$$V_{nm} = F_0\cos\omega t\,\langle n|\,x\,|m\rangle$$
$$= F_0\cos\omega t\sqrt{\frac{\hbar}{2m\omega_0}}\left(\sqrt{n+1}\delta_{m,n-1} + \sqrt{n}\delta_{m,n+1}\right)$$

But the initial condition says that $|\psi(0)\rangle = |0\rangle$, so $n = 0$ and the only term of the summation that survives has $m = 1$. Therefore,

$$\begin{aligned}
i\hbar\dot{c_1}(t) &= V_{10}e^{i\omega_0 t}c_0(t) \\
&= F_0 \cos\omega t \sqrt{\frac{\hbar}{2m\omega_0}} e^{i\omega_0 t}c_0(t)
\end{aligned}$$

Solving for $c_1(t)$,

$$\begin{aligned}
c_1(t) &= -\frac{i}{\hbar}F_0\sqrt{\frac{\hbar}{2m\omega_0}}\int_0^t e^{i\omega_0 t}\cos\omega t dt \\
&= -\frac{i}{2\hbar}F_0\sqrt{\frac{\hbar}{2m\omega_0}}\left(\frac{e^{i(\omega_0+\omega)t}-1}{\omega_0+\omega}+\frac{e^{i(\omega_0-\omega)t}-1}{\omega_0-\omega}\right)
\end{aligned}$$

Now, to compute $\langle x\rangle$, we can express the x operator in the interaction picture (or, equivalently, convert the $\left|\tilde{\psi(t)}\right\rangle$ back to $|\psi(t)\rangle$).

$$\begin{aligned}
\langle x\rangle &= \langle\psi(t)|\, x\, |\psi(t)\rangle \\
&= \left\langle\tilde{\psi(t)}\right| e^{iH_0 t/\hbar} x e^{-iH_0 t/\hbar} \left|\tilde{\psi(t)}\right\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega_0}}\left(\langle 0|\, c_0^* e^{i\omega_0 t/2} + \langle 1|\, e^{3i\omega_0 t/2}c_1^*(t)\right)(a+a^\dagger)\left(e^{-i\omega_0 t/2}c_0\,|0\rangle + e^{-3i\omega_0 t/2}c_1(t)\,|1\rangle\right) \\
&= \sqrt{\frac{\hbar}{2m\omega_0}}\left(c_1(t)e^{-i\omega_0 t} + c_1^*(t)e^{i\omega_0 t}\right)
\end{aligned}$$

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Problem 3. *5.30*

Solution. The potential is

$$V(x,t) = xF_0 e^{-t/\tau}$$

This is very similar to the previous problem, just with a different time-dependence to the potential. Write,

$$\begin{aligned}
c_1(t) &= -\frac{i}{\hbar}F_0\sqrt{\frac{\hbar}{2m\omega_0}}\int_0^t e^{i\omega_0 t}e^{-t/\tau}dt \\
&= -\frac{i}{2\hbar}F_0\sqrt{\frac{\hbar}{2m\omega_0}}\frac{\left(e^{(i\omega_0-1/\tau)t}-1\right)}{(i\omega_0-1/\tau)t}
\end{aligned}$$

The probability of finding the particle in the first excited state is

$$|c_1(t)|^2 =$$

which is clearly independent of time. This is expected since the force is transient. We cannot find higher order states because, as was shown in the previous problem, $c_n(0) = 0$ and $\dot{c_n}(t) = 0$ for all $n > 1$.

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Problem 4. *5.32*

Solution. ■

Problem 5. *5.35*

Solution. ■

Problem 6. *5.36*

Solution. ■