Homework 1

Quantum Mechanics
January 22, 2023 C Seitz

Problem 1. *1.3.5 Calculations, No-cloning theorem*

Solution. Assume we have a unitary copying operator U and two quantum states $|\phi\rangle$ and $|\psi\rangle$. Suppose this unknown copying operator U could transform $|s\rangle$ to either $|\phi\rangle$ or $|\psi\rangle$.

$$|\psi\rangle \otimes |s\rangle \xrightarrow{U} |\psi\rangle \otimes |\psi\rangle$$
$$|\phi\rangle \otimes |s\rangle \xrightarrow{U} |\phi\rangle \otimes |\phi\rangle$$

If U is unitary, then it preserves inner products, so

$$(\langle\psi| \otimes \langle s|)(|\phi\rangle \otimes |s\rangle) = \langle\psi|\phi\rangle \otimes \langle s|s\rangle = \langle\psi|\phi\rangle$$

After the copying transformation, we have

$$\begin{aligned}(\langle\psi| \otimes \langle\psi|)(|\phi\rangle \otimes |\phi\rangle) &= \langle\psi|\phi\rangle \otimes \langle\psi|\phi\rangle \\ &= (\langle\psi|\phi\rangle)^2\end{aligned}$$

We demanded that the inner product be preserved, so these two results must be equivalent. However, there is only a solution when $|\psi\rangle = |\phi\rangle$ or $\langle\psi|\phi\rangle = 0$. Therefore, the copying circuit only works for orthogonal states, and not a general ket.

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Problem 2. *1.3.7 Calculations, Quantum Teleportation*

Solution.

The objective is for Alice to teleport to Bob a qubit in a state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, which can be done by using an entangled EPR pair. There three qubits in total: $|\psi\rangle$ and an entangled EPR pair $|\beta_{00}\rangle$. The first qubit in the EPR pair is kept by alice and the second is given to Bob. Since the EPR pair is entangled, the three qubits are in a state

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle\right)$$

Alice then sends this state through a CNOT gate, where the qubit $|\psi\rangle$ is the control bit and the first qubit of the EPR pair is the target bit. This of course flips the second bit for the second two terms:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle\right)$$

Then the first qubit is sent through a Hadamard gate. As a minor detour, the Hadamard gate, does

$$\begin{aligned}|0\rangle &\to (|0\rangle + |1\rangle)/\sqrt{2}\\ |1\rangle &\to (|0\rangle - |1\rangle)/\sqrt{2}\end{aligned}$$

Therefore, the effect on $|\psi_1\rangle$ is:

$$\begin{aligned}|\psi_1\rangle &= \frac{1}{2}(\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle\\ &+ \beta|010\rangle + \beta|001\rangle - \beta|110\rangle - \beta|101\rangle)\\ &= \frac{1}{2}(|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |10\rangle(\alpha|0\rangle - \beta|1\rangle)\\ &+ |01\rangle(\alpha|1\rangle + \beta|0\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle))\end{aligned}$$

Therefore, if Alice measures her two qubits, say in state $|00\rangle$, she can communicate this to bob over a classical communication channel, and Bob then knows the superposition of his qubit. Bob can then apply the necessary quantum gate to transform his qubit to $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.

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Problem 3. *Deutsch Algorithm, Deutch-Josza Algorithm*

Solution.

Suppose we have some boolean function $f : \{0,1\} \to \{0,1\}$. Deutch's algorithm can determine whether the function f is constant or balanced exponentially faster than a classical computer. If f is constant then $f(0) = f(1)$; however, if f is balanced then $f(0) \neq f(1)$. For example, $f(0) = 1$ and $f(1) = 0$ is a balanced function. A classical computer would need to evaluate $f(0)$ and $f(1)$ separately, but a quantum computer can leverage quantum parallelism to compute both $f(0)$ and $f(1)$ at the same time.

We start with the two qubits prepared in state $|\psi_0\rangle = |0\rangle|1\rangle$. Each qubit is Hadamard transformed to give

$$\begin{aligned}
|\psi_1\rangle &= \frac{1}{2}(|0\rangle + |1\rangle)(|0\rangle - |1\rangle) \\
&= \frac{1}{2}(|0\rangle|0\rangle - |0\rangle|1\rangle + |1\rangle|0\rangle - |1\rangle|1\rangle)
\end{aligned}$$

The state then goes through an oracle, which implements the unitary transformation $|\alpha\rangle|\beta\rangle \to |\alpha\rangle|\beta \oplus f(\alpha)\rangle$. Note that $\oplus$ is addition modulo two, which is essentially an XOR operation. Thus, after transformation, the state is

$$\begin{aligned}
|\psi_2\rangle &= \frac{1}{2}(|0\rangle|0 \oplus f(0)\rangle - |0\rangle|1 \oplus f(0)\rangle + |1\rangle|0 \oplus f(1)\rangle - |1\rangle|1 \oplus f(1)\rangle) \\
&= \frac{1}{2}(|0\rangle(|0 \oplus f(0)\rangle - |1 \oplus f(0)\rangle) + |1\rangle(|0 \oplus f(1)\rangle - |1 \oplus f(1)\rangle)) \\
&= \frac{1}{2}\Big((-1)^{f(0)}|0\rangle(|0\rangle - |1\rangle) + (-1)^{f(1)}|1\rangle(|0\rangle - |1\rangle)\Big)
\end{aligned}$$

Then we Hadamard transform the first qubit:

$$\begin{aligned}
|\psi_3\rangle &= \frac{1}{2}\Big((-1)^{f(0)}(|0\rangle + |1\rangle)(|0\rangle - |1\rangle) + (-1)^{f(1)}(|0\rangle - |1\rangle)(|0\rangle - |1\rangle)\Big) \\
&= ((-1)^{f(0)} + (-1)^{f(1)})(|00\rangle - |01\rangle) + ((-1)^{f(0)} - (-1)^{f(1)})(|10\rangle - |11\rangle)
\end{aligned}$$

Writing it in this way makes it clear how we determine if the function is constant or balanced. If it is constant($f(0) = f(1)$) then the the second term vanishes. So, if we measure the first qubit, it will be in state $|0\rangle$. However, if it is balanced ($f(0) \neq f(1)$), then the first term vanishes and the first qubit will be measured in state $|1\rangle$.

This algorithm generalizes to the case where the function $f : \{0,1\}^n \rightarrow \{0,1\}$. This is where the power of quantum computing really shines. Clasically, (if f can be constant or balanced and nothing else), we would need $2^{n-1} + 1$ function calls. But a quantum computer can do this in a single function call. First, note that the Hadamard transform on a bit string $|\alpha\rangle$ of length n is

$$H|\alpha\rangle = \left(\frac{|0\rangle + |1\rangle}{2}\right)^{\otimes n}$$

For brevity, I will now also adopt the notation where $H|0\rangle = |+\rangle$ and $H|1\rangle = |-\rangle$. We have,

$$|\psi_1\rangle = \left(\frac{|0\rangle + |1\rangle}{2}\right)^{\otimes n} \otimes |-\rangle = \frac{1}{2^{n/2}} \sum_{\alpha \in \{0,1\}^n} |\alpha\rangle \otimes |-\rangle$$

Looking at the last line from $|\psi_2\rangle$ in Deutsch's algorithm above, we can see that the effect of the oracle U is efficiently summarized as:

$$U|\alpha\rangle|1\rangle = \frac{1}{2^{n/2}} \sum_{\alpha \in \{0,1\}} (-1)^{f(\alpha)} |\alpha\rangle |-\rangle$$

Therefore, for the n-bit case, we get

$$|\psi_2\rangle = \frac{1}{2^{n/2}} \sum_{\alpha \in \{0,1\}^n} (-1)^{f(\alpha)} |\alpha\rangle \otimes |-\rangle$$

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