Attractor dynamics and generalization bounds of rate-distortion networks trained via spike-timing dependent plasticity

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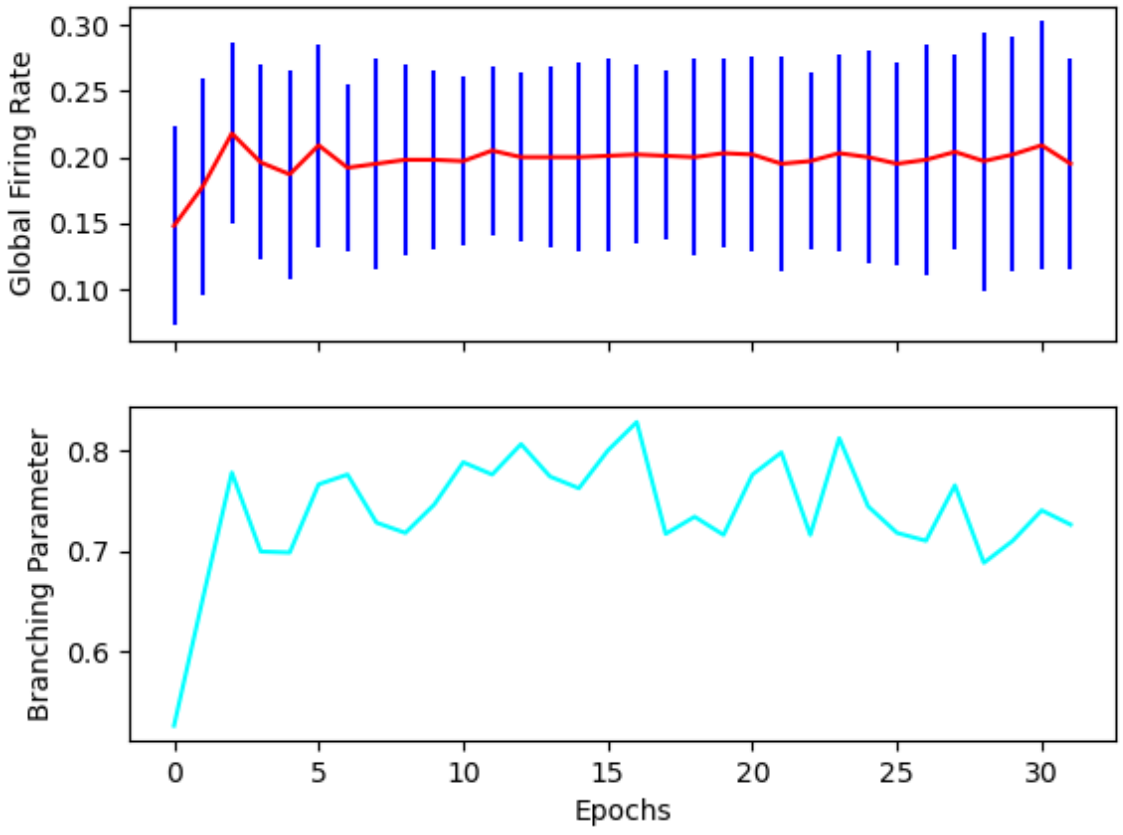
Introduction

Training low-rate critical networks

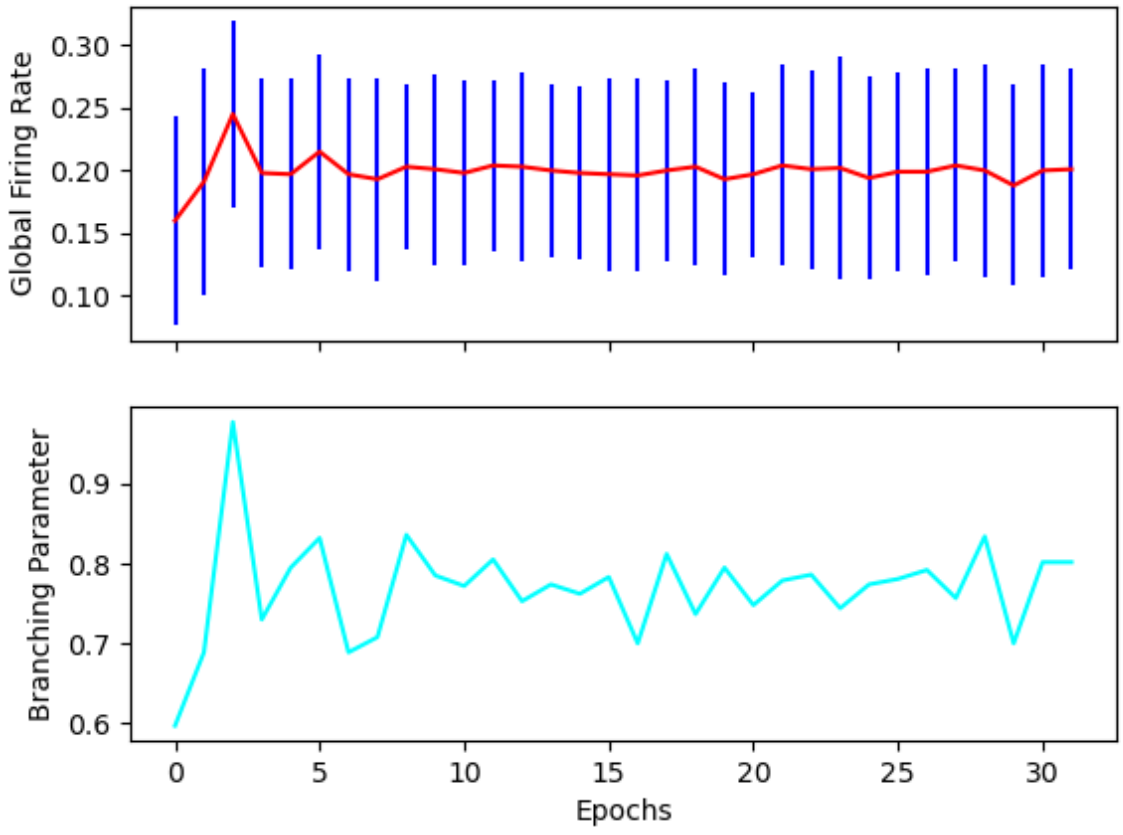
$$\mathcal{L}_{global} = \alpha(r_{global} - \hat{r})$$



Training on an average firing rate has many (perhaps undesirable) solutions

Training low-rate critical networks

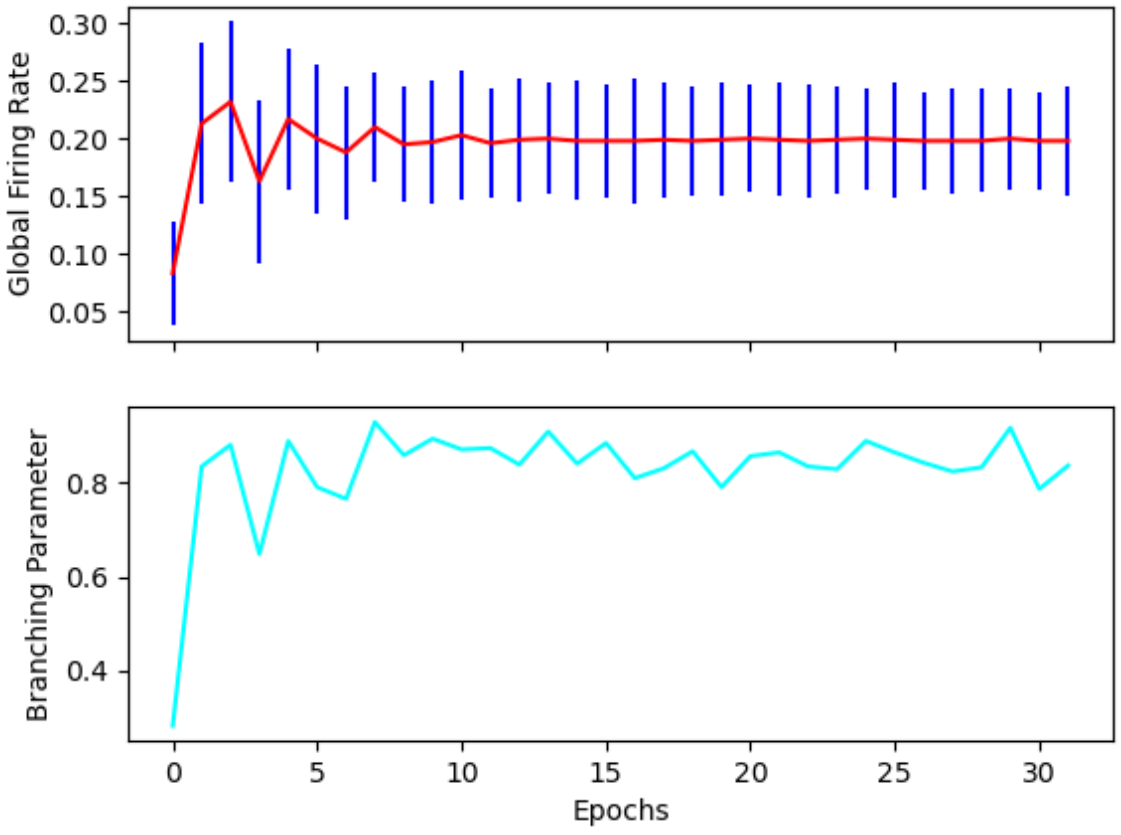
$$\mathcal{L}_{SSE} = \alpha \sum_n (r_n - \hat{r})^2$$



Training on the instantaneous firing rate doesn't work either

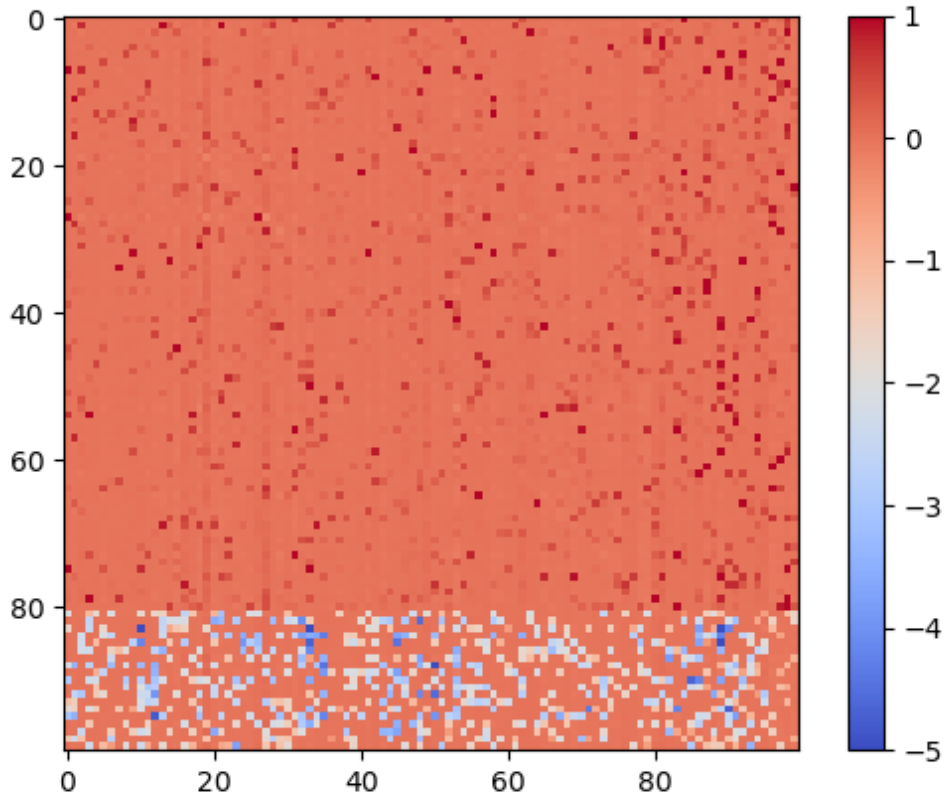
Training low-rate critical networks

$$\mathcal{L}_\alpha = \alpha \sum_n (r_n - \hat{r})^2$$

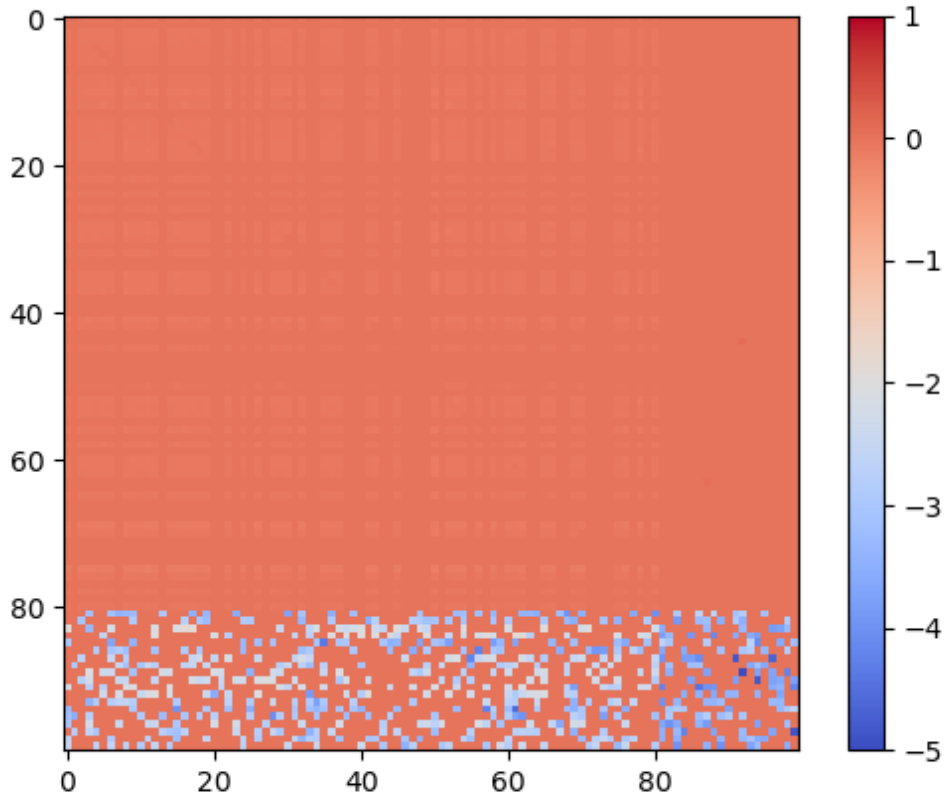


Binning the firing rate over 100ms (r_n) reduces its variability.

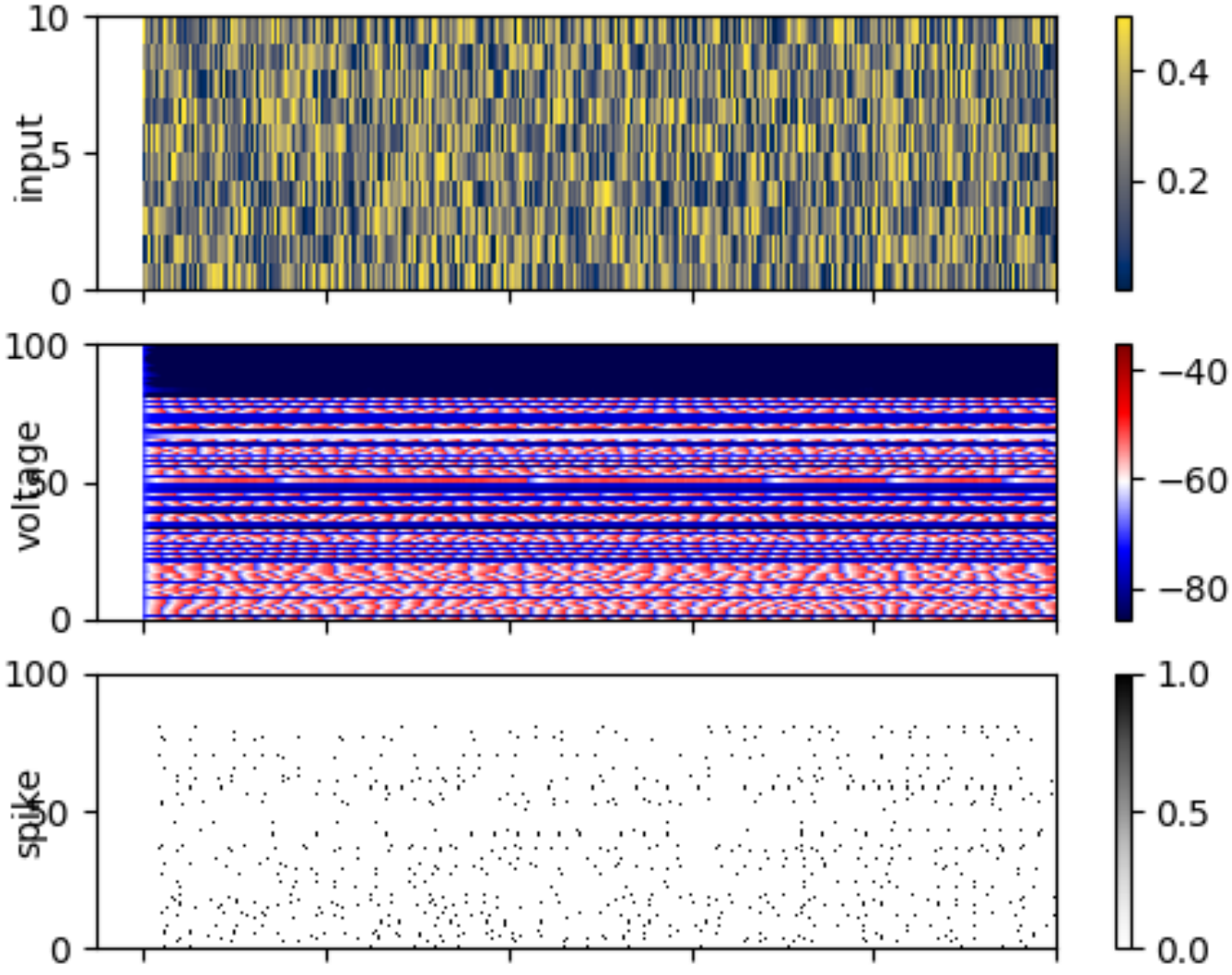
Optimization of $\mathcal{L}_{global}$ results in some strong excitatory recurrent weights



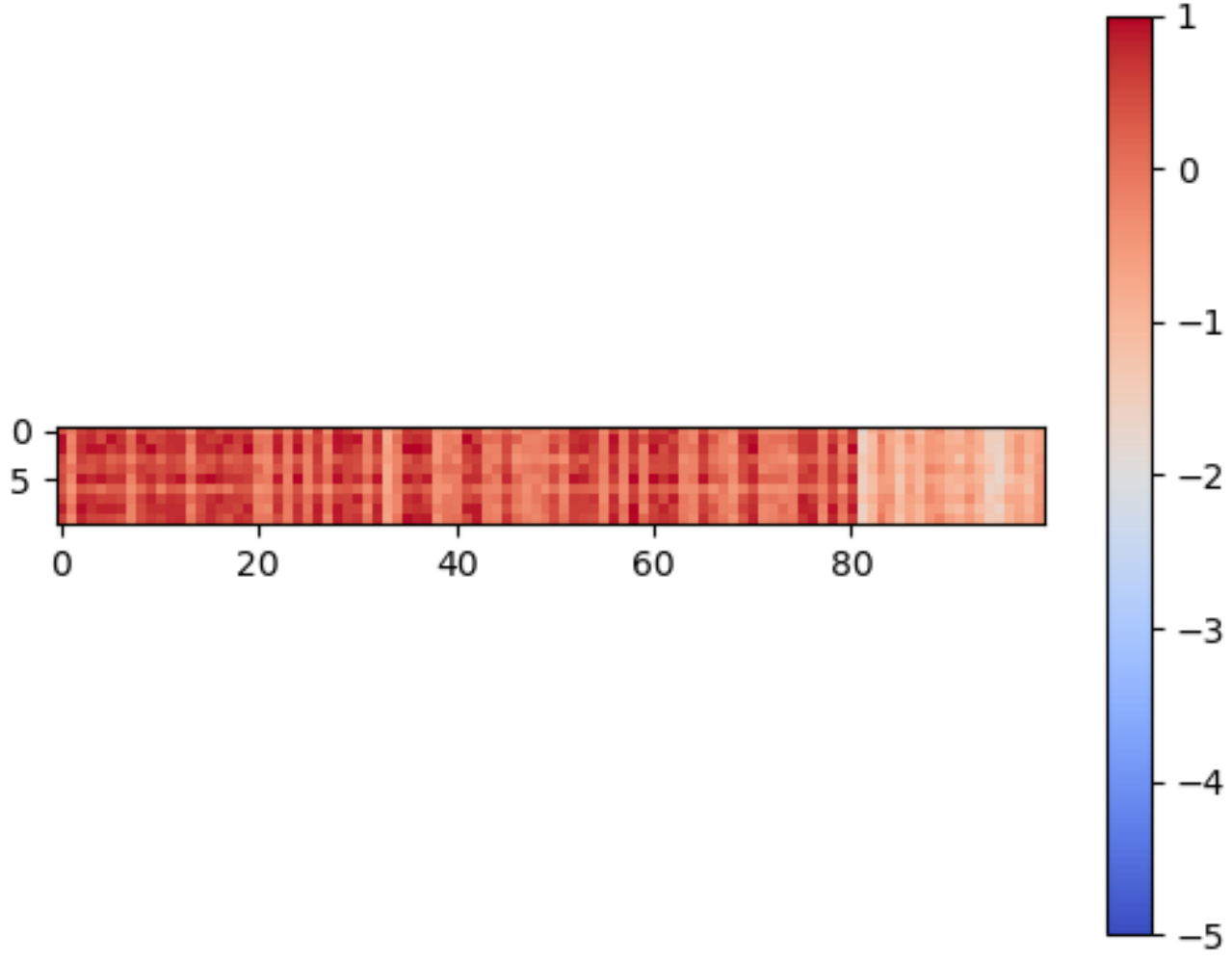
Optimization of $\mathcal{L}_\alpha$ results in few strong excitatory recurrent synapses



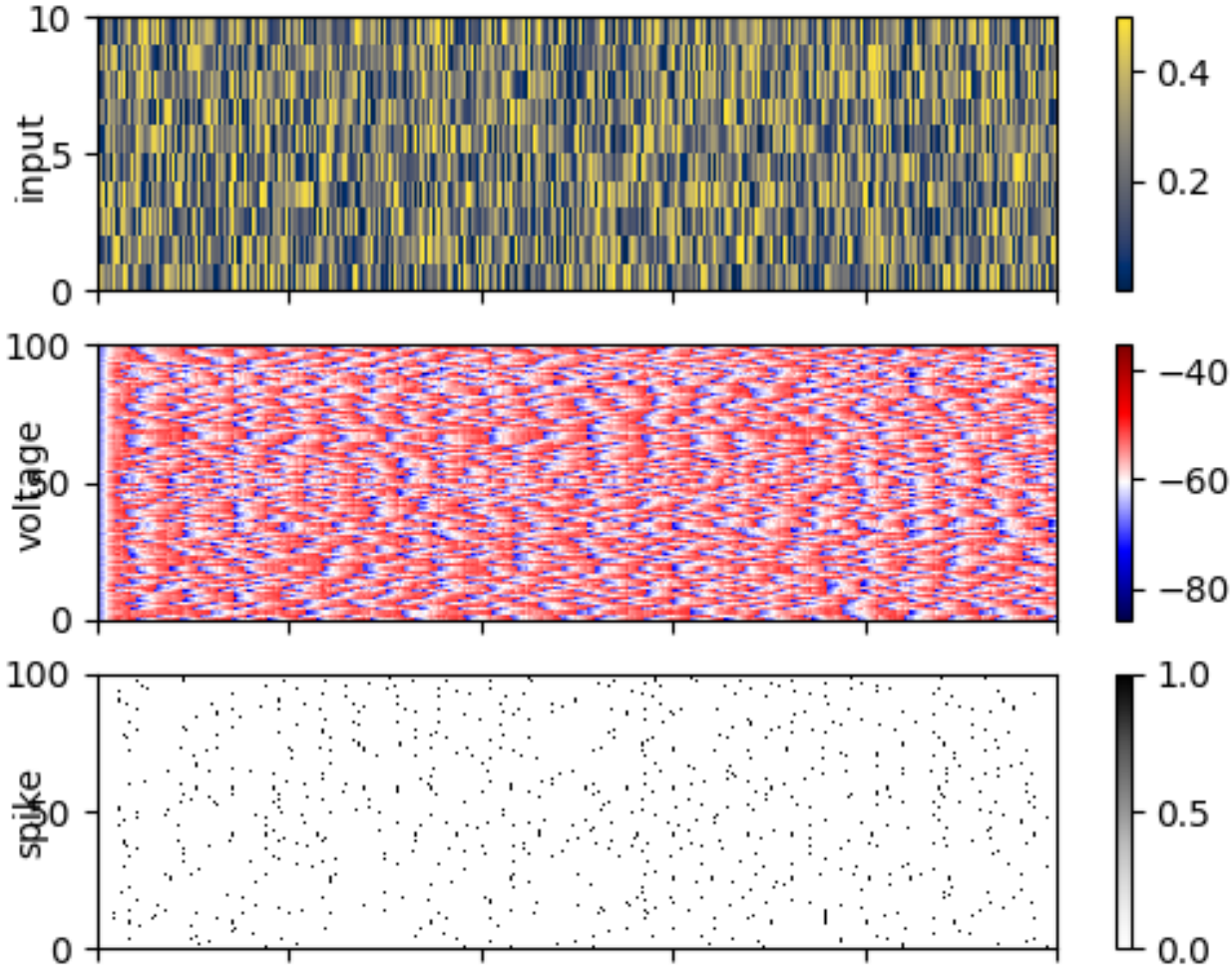
Optimization of $\mathcal{L}_\alpha$ shows inhibition of inputs



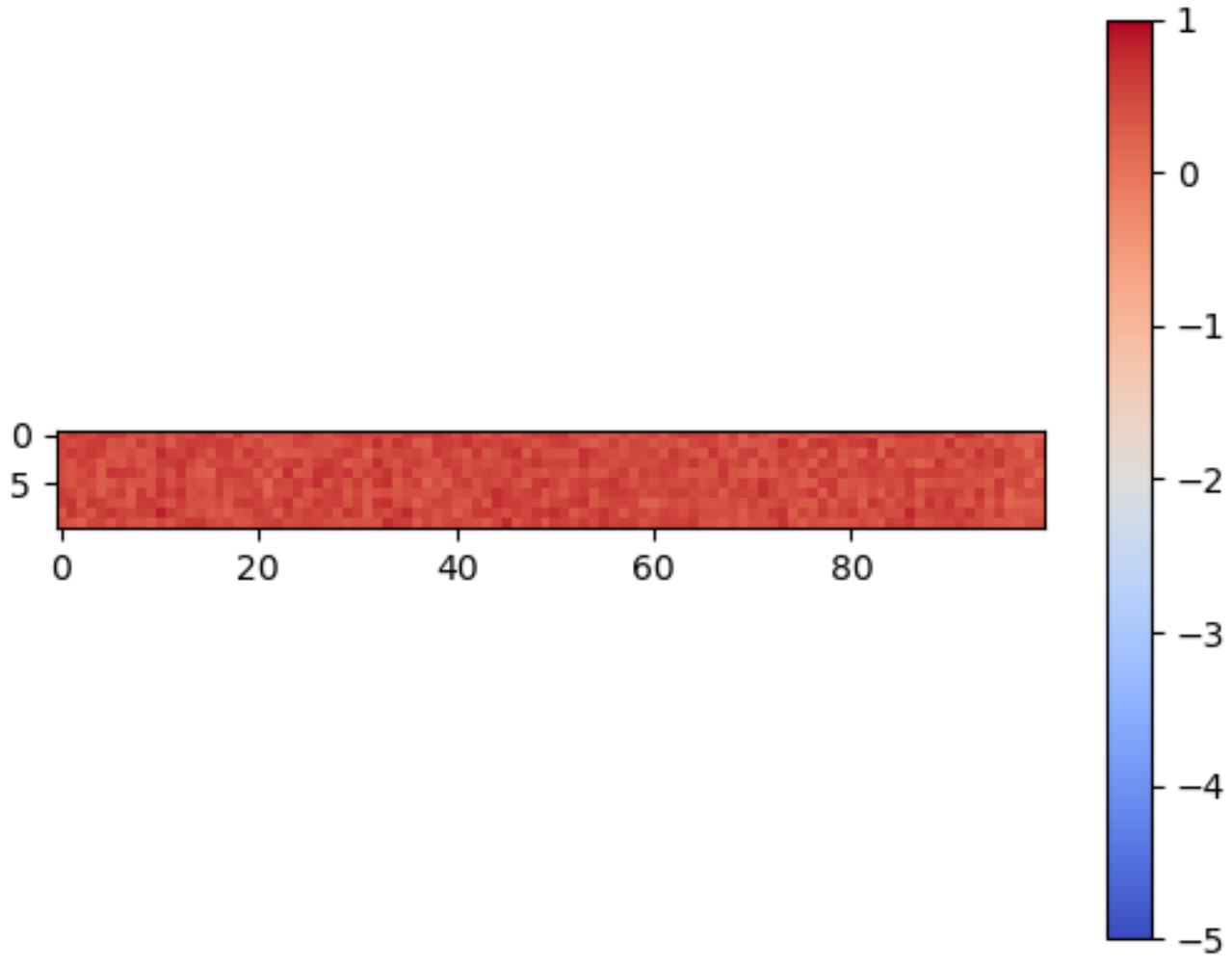
Optimization of $\mathcal{L}_\alpha$ shows inhibition of inputs



Optimization of $\mathcal{L}_{global}$ doesn't



Optimization of $\mathcal{L}_{global}$ doesn't



To me the above results suggest that the network is finding a balance between recurrence and input to maintain a stable firing rate as shown in Cramer et al. 2020

Channel coding for neural networks

Networks of neurons can be viewed as a communication channel
Except this communication channel *learns* the transformation F based on the statistical structure of its input X. Visual cortex has learned an encoding for visual scenes (that perhaps maximizes information)

RNN Gradients

Say we have a model $\Phi = (W^0, W^1)$ and want to use gradient descent to train a network to have a target rate or a target branching parameter. The rate and its associated loss for a single unit is

$$r(t) = \frac{1}{\Delta t} \int_t^{t+\Delta t} d\tau \langle \rho(\tau) \rangle \qquad \mathcal{L} = \alpha (r - r_0)^2$$

We would like the standard update

$$\Delta W_{ij} = -\eta \frac{\partial \mathcal{L}}{\partial W_{ij}}$$

But it is intractable to compute $\frac{\partial \mathcal{L}}{\partial W_{ij}}$ since $\rho(t)$ depends on other neurons through space and time.

Factorizing loss gradients for BPTT

BPTT involves unrolling an RNN into a large feedforward network where each layer is a time step.

$$\frac{\partial \mathcal{L}}{\partial W_{ij}^t} = \frac{\partial \mathcal{L}}{\partial h_j^t} \frac{\partial h_j^t}{\partial W_{ij}^t}$$

and the total gradient is a sum over the layers (time)

$$\frac{\partial \mathcal{L}}{\partial W_{ij}^t} = \sum_t \frac{\partial \mathcal{L}}{\partial h_j^t} \frac{\partial h_j^t}{\partial W_{ij}^t}$$

Deriving e-prop from BPTT

Consider the first term above. The hidden state is computed by some function $h_j^t = F(z_j^t, h_j^{t-1}, W)$. Backpropagating through time is then

$$\frac{\partial \mathcal{L}}{\partial h_j^t} = \frac{\partial \mathcal{L}}{\partial z_j^t}\frac{\partial z_j^t}{\partial h_j^t} + \frac{\partial \mathcal{L}}{\partial h_j^{t+1}}\frac{\partial h_j^{t+1}}{\partial h_j^t}$$

which must be expressed recursively

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial h_j^t} &= \frac{\partial \mathcal{L}}{\partial z_j^t}\frac{\partial z_j^t}{\partial h_j^t} + \left(\frac{\partial \mathcal{L}}{\partial z_j^{t+1}}\frac{\partial z_j^{t+1}}{\partial h_j^{t+1}} + (\ldots)\frac{\partial h_j^{t+2}}{\partial h_j^{t+1}}\right)\frac{\partial h_j^{t+1}}{\partial h_j^t} \\
&= L_j^t\frac{\partial z_j^t}{\partial h_j^t} + \left(L_j^{t+1}\frac{\partial z_j^{t+1}}{\partial h_j^{t+1}} + (\ldots)\frac{\partial h_j^{t+2}}{\partial h_j^{t+1}}\right)\frac{\partial h_j^{t+1}}{\partial h_j^t} \\
&= L_j^t\frac{\partial z_j^t}{\partial h_j^t} + \left(L_j^{t+1}\frac{\partial z_j^{t+1}}{\partial h_j^{t+1}} + (\ldots)\frac{\partial h_j^{t+2}}{\partial h_j^{t+1}}\right)\frac{\partial h_j^{t+1}}{\partial h_j^t}
\end{aligned}$$

Deriving e-prop from BPTT

Plugging into the original factorization gives

$$\frac{\partial \mathcal{L}}{\partial W_{ij}} = \left(\sum_t L_j^t \frac{\partial z_j^t}{\partial h_j^t} + \left(L_j^{t+1} \frac{\partial z_j^{t+1}}{\partial h_j^{t+1}} + (\ldots) \frac{\partial h_j^{t+2}}{\partial h_j^{t+1}} \right) \frac{\partial h_j^{t+1}}{\partial h_j^t} \right) \frac{\partial h_j^{t'}}{\partial W_{ij}}$$

You can then collect terms that are multiplied L_j^t

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial W_{ij}} &= \sum_t L_j^t \frac{\partial z_j^t}{\partial h_j^t} \left(\sum_{t' \leq t} \left(\prod_{t'} \frac{\partial h_j^{t'+1}}{\partial h_j^{t'}} \right) \frac{\partial h_j^{t'}}{\partial W_{ij}} \right) \\ &= \sum_t L_j^t \frac{\partial z_j^t}{\partial h_j^t} \epsilon_{\mathrm{ij}}^{\mathrm{t}} = \sum_t L_j^t e_{ij}^t \end{aligned}$$

Constraining the global firing rate distribution

We can define a constraint on the variance of the global firing rate (which simultaneously constrains the mean)

$$\mathcal{L} = \beta(\sigma - \sigma_r)^2 \qquad \sigma = \frac{1}{T}\sum_t (r - \mu_r)^2$$

where we constrain branching by constraining the variance s of the global firing rate where branching $\to 1$ as $s \to 0$.

$$L_j^t = \frac{\partial \mathcal{L}}{\partial z_j^t} = \frac{\partial \mathcal{L}}{\partial \sigma}\frac{\partial \sigma}{\partial n}\frac{\partial n}{\partial z_j^t} = \pm\beta(\sigma - \sigma_r)\cdot(r - \mu_r)$$

Think push-pull. Some variation is necessary for refractoriness.

Receptive fields of neurons in a low-rate network

Multivariate information theory

Adaptation of the transfer function

How do neuron transfer functions adapt to stimuli in an unsupervised manner?

Adaptation defines an energy function over phase space

Generalization bounds

What is the distance of a code defined by a particular energy function E

The energy function defines a dynamical system

The energy function is a generative model

Application to natural image statistics