Homework 8

Quantum Mechanics
November 12, 2022 C Seitz

Problem 1. *5.27*

Solution.

$$\frac{\langle\tilde{0}|\, H\, |\tilde{0}\rangle}{\langle\tilde{0}|\tilde{0}\rangle} \geq E_0$$

The denominator is easy to compute

$$2\int_{-\infty}^{0} e^{\beta x} dx = \frac{1}{\beta}$$

The numerator

$$\begin{aligned}
\langle\tilde{0}|\, H\, |\tilde{0}\rangle &= \int_{-\infty}^{\infty} \psi^*(x) H \psi(x) dx \\
&= \int_{-\infty}^{0} \psi^*(x) H \psi(x) dx + \int_{0}^{\infty} \psi^*(x) H \psi(x) dx
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{0} \psi^*(x) H \psi(x) dx &= \int_{-\infty}^{0} -e^{\beta x} \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} e^{\beta x} + \frac{1}{2} m\omega^2 x^2 e^{2\beta x} dx \\
&= \int_{-\infty}^{0} e^{2\beta x} \left(\frac{1}{2} m\omega^2 x^2 - \frac{\hbar^2 \beta^2}{2m} \right) dx \\
&= \Big|_{-\infty}^{0} \frac{1}{2} m\omega^2 \frac{e^{2\beta x}(1 - 2\beta x + 2\beta^2 x^2)}{4\beta^3} - e^{2\beta x} \frac{\hbar^2 \beta}{4m} \\
&= \frac{1}{2} m\omega^2 \frac{1}{4\beta^3} - \frac{\hbar^2 \beta}{4m}
\end{aligned}$$

$$
\begin{aligned}
\int_0^\infty \psi^*(x) H \psi(x) dx &= \int_0^\infty -e^{-\beta x} \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} e^{-\beta x} + \frac{1}{2} m\omega^2 x^2 e^{-2\beta x} dx \\
&= \int_0^\infty e^{-2\beta x} \left(\frac{1}{2} m\omega^2 x^2 - \frac{\hbar^2 \beta^2}{2m} \right) dx \\
&= \Big|_0^\infty \frac{1}{2} m\omega^2 \frac{e^{-2\beta x}(1 + 2\beta x + 2\beta^2 x^2)}{4\beta^3} - e^{-2\beta x} \frac{\hbar^2 \beta}{4m} \\
&= \frac{1}{2} m\omega^2 \frac{1}{4\beta^3} - \frac{\hbar^2 \beta}{4m}
\end{aligned}
$$

$$
\bar{H} = \frac{\langle \tilde{0} | H | \tilde{0} \rangle}{\langle \tilde{0} | \tilde{0} \rangle} = \frac{m\omega^2}{4\beta^2} - \frac{\hbar^2 \beta^2}{2m}
$$

$$
\frac{d\bar{H}}{d\beta} = -\frac{m\omega^2}{4\beta} - \frac{\hbar^2 \beta}{m} = 0
$$

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Problem 2. *5.29*

Solution.

We have the full time-dependent Hamiltonian

$$
H(t) = H_0 + F_0 x \cos \omega t
$$

We need to find $|\psi(t)\rangle$, which amounts to finding the expansion coefficients $c_n(t)$. In the interaction picture, we have that

$$
i\hbar \dot{c}_n(t) = \sum_m V_{nm} e^{i\omega_{nm} t} c_m(t)
$$

for $\omega_{nm} = (E_n - E_m)/\hbar$.

$$
\begin{aligned}
V_{nm} &= F_0 \cos \omega t \langle n | x | m \rangle \\
&= F_0 \cos \omega t \sqrt{\frac{\hbar}{2m\omega_0}} \left(\sqrt{n+1} \delta_{m,n-1} + \sqrt{n} \delta_{m,n+1} \right)
\end{aligned}
$$

But the initial condition says that $|\psi(0)\rangle = |0\rangle$, so $n = 0$ and the only term of the summation that survives has $m = 1$. Therefore,

$$\begin{aligned} i\hbar \dot{c_1}(t) &= V_{10} e^{i\omega_0 t} c_0(t) \\ &= F_0 \cos\omega t \sqrt{\frac{\hbar}{2m\omega_0}} e^{i\omega_0 t} c_0(t) \end{aligned}$$

Solving for $c_1(t)$,

$$\begin{aligned} c_1(t) &= -\frac{i}{\hbar} F_0 \sqrt{\frac{\hbar}{2m\omega_0}} \int_0^t e^{i\omega_0 t} \cos\omega t dt \\ &= -\frac{i}{2\hbar} F_0 \sqrt{\frac{\hbar}{2m\omega_0}} \left(\frac{e^{i(\omega_0+\omega)t} - 1}{\omega_0 + \omega} + \frac{e^{i(\omega_0-\omega)t} - 1}{\omega_0 - \omega} \right) \end{aligned}$$

Now, to compute $\langle x \rangle$, we can express the x operator in the interaction picture (or, equivalently, convert the $\left|\tilde{\psi(t)}\right\rangle$ back to $|\psi(t)\rangle$).

$$\begin{aligned} \langle x \rangle &= \langle \psi(t)| \, x \, |\psi(t)\rangle \\ &= \left\langle \tilde{\psi(t)} \right| e^{iH_0 t/\hbar} x e^{-iH_0 t/\hbar} \left| \tilde{\psi(t)} \right\rangle \\ &= \sqrt{\frac{\hbar}{2m\omega_0}} \left(\langle 0| \, c_0^* e^{i\omega_0 t/2} + \langle 1| \, e^{3i\omega_0 t/2} c_1^*(t) \right) (a + a^\dagger) \left(e^{-i\omega_0 t/2} c_0 \, |0\rangle + e^{-3i\omega_0 t/2} c_1(t) \, |1\rangle \right) \\ &= \sqrt{\frac{\hbar}{2m\omega_0}} \left(c_1(t) e^{-i\omega_0 t} + c_1^*(t) e^{i\omega_0 t} \right) \end{aligned}$$

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Problem 3. *5.30*

Solution. The potential is

$$V(x,t) = xF_0 e^{-t/\tau}$$

This is very similar to the previous problem, just with a different time-dependence to the potential. Write,

$$\begin{aligned} c_1(t) &= -\frac{i}{\hbar} F_0 \sqrt{\frac{\hbar}{2m\omega_0}} \int_0^t e^{i\omega_0 t} e^{-t/\tau} dt \\ &= -\frac{i}{2\hbar} F_0 \sqrt{\frac{\hbar}{2m\omega_0}} \frac{\left(e^{(i\omega_0 - 1/\tau)t} - 1\right)}{(i\omega_0 - 1/\tau)t} \end{aligned}$$

The probability of finding the particle in the first excited state is

$$|c_1(t)|^2 =$$

which is clearly independent of time. This is expected since the force is transient. We cannot find higher order states because, as was shown in the previous problem, $c_n(0) = 0$ and $\dot{c_n}(t) = 0$ for all $n > 1$.

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Problem 4. *5.32*

Solution. $c_n(t)$

We need to find $|\psi(t)\rangle$, which amounts to finding the expansion coefficients $c_0(t)$ and $c_1(t)$. In the interaction picture, we have that

$$i\hbar\dot{c_n}(t) = \sum_m V_{nm}e^{i\omega_{nm}t}c_m(t)$$

We were given $V(t)$, so we already know the V_{nm}. The differential equations for $c_0(t)$ and $c_1(t)$ are

$$i\hbar\dot{c_0}(t) = \sum_m V_{nm}e^{i\omega_{nm}t}c_m(t) = \lambda e^{i\omega_{01}t}\cos\omega t c_1(t)$$

$$i\hbar\dot{c_1}(t) = \sum_m V_{nm}e^{i\omega_{nm}t}c_m(t) = \lambda e^{-i\omega_{10}t}\cos\omega t c_0(t)$$

So we have the coupled set of differential equations

$$\dot{c_0}(t) = -\frac{i\lambda}{\hbar}c_1(t)e^{i\omega_{10}t}\cos\omega t$$
$$\dot{c_1}(t) = -\frac{i\lambda}{\hbar}c_0(t)e^{-i\omega_{10}t}\cos\omega t$$

subject to the initial conditions $c_0(0) = 1$ and $c_1(0) = 0$. The probability the system is found in the state $|1\rangle$ is

$$|\langle\psi(t)|1\rangle|^2 = |c_1(t)|^2$$

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Problem 5. *5.35*

Solution.

The electric potential in the capacitor is

$$V(z,t) = \int_0^z E_0 e^{-t/\tau} dz = z E_0 e^{-t/\tau}$$

We are asked to find the probability of finding the hydrogen atom in the states $|2,1,\pm 1\rangle, |2,1,0\rangle$, given the initial state $|1,0,0\rangle$. As usual, we make use of the interaction picture

$$i\hbar \dot{c_n}(t) = \sum_m V_{nm} e^{i\omega_{nm} t} c_m(t)$$

But need to find the matrix element $V_{nm} = \langle nlm| V |n'l'm'\rangle$. These matrix elements are constrained by the selection rules for transitions of the hydrogen atom. It is well-known that the matrix element $\langle nlm| z |n'l'm'\rangle$ is nonzero only when $m' = m$ and $l' = l \pm 1$. Therefore, we don't need to calculate the probability of a transition to $|2,1,\pm 1\rangle$, because it is impossible. However, a transition to state $|2,1,0\rangle$ is possible and we write

$$i\hbar \dot{c_{210}}(t) = E_0 e^{-t/\tau} \langle 1,0,0| z |2,1,0\rangle e^{i\omega t} c_{100}(t)$$

where $\omega = (E_{210} - E_{1,0,0})/\hbar$.

For the $2s$ state $|2,0,0\rangle$, the same selection rules apply, so we only need to find

$$i\hbar \dot{c_{210}}(t) = E_0 e^{-t/\tau} \langle 2,0,0| z |2,1,0\rangle c_{200}(t)$$

where the exponential is 1 because the two states are degenerate in energy thus $\omega = (E_{210} - E_{2,0,0})/\hbar = 0$.

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Problem 6. *5.36*

Solution. ■