Homework 4

Quantum Mechanics
February 23, 2023 C Seitz

Problem 1. *Problem 2.65*

Solution. Let us call these states $|\alpha\rangle$ and $|\beta\rangle$:

$$|\alpha\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$
$$|\beta\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

If we choose a non-orthogonal basis, such as

$$|e_1\rangle = |0\rangle \quad |e_2\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$

These states have the following representation in this new basis

$$\begin{aligned}|\alpha'\rangle &= (|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|)\,|\alpha\rangle \\ &= \frac{1}{\sqrt{2}}\,|e_1\rangle + |e_2\rangle\end{aligned}$$

$$\begin{aligned}|\beta'\rangle &= (|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|)\,|\beta\rangle \\ &= \frac{1}{\sqrt{2}}\,|e_1\rangle\end{aligned}$$

The norm is not preserved, because the change of basis matrix $|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|$ was not unitary. But it is clear that these states differ neither by a global or relative phase.

■

Problem 2. *Problem 2.66*

Solution.

$$\begin{aligned}\langle\alpha| X_1 Z_2 |\alpha\rangle &= \frac{1}{2}(\langle 00| + \langle 11|)X_1 Z_2(|00\rangle + |11\rangle) \\ &= \frac{1}{2}(\langle 00| + \langle 11|)(|10\rangle - |01\rangle) = 0\end{aligned}$$

■

Problem 3. *Problem 2.71*

Solution.

$$\begin{aligned}\mathrm{Tr}(\rho^2) &= \sum_k \langle k| \left(\sum_i p_i |\alpha_i\rangle \langle\alpha_i|\right)\left(\sum_j p_j |\alpha_j\rangle \langle\alpha_j|\right)|k\rangle \\ &= \left(\sum_{ijk} p_i p_j \langle k|\alpha_i\rangle \langle\alpha_i|\alpha_j\rangle \langle\alpha_j|k\rangle\right) \\ &= \sum_{ij} p_i p_j |\langle\alpha_i|\alpha_j\rangle|^2 \\ &= \sum_i p_i^2 \leq 1\end{aligned}$$

if $|\alpha_i\rangle$ and $|\alpha_j\rangle$ are orthonormal.

■

Problem 4. *Problem 2.72*

Solution.

The Pauli matrices form a valid basis for 2x2 matrices. The Bloch vector representation for $\rho = I/2$ is $\vec{r} = 0$.

$$\begin{aligned}\mathrm{Tr}(\rho^2) &= \mathrm{Tr}\left(\frac{I + 2(\vec{r}\cdot\sigma) + (\vec{r}\cdot\sigma)^2}{4}\right) \\ &= \frac{1}{2} + \frac{||\vec{r}||^2}{2} = 1\end{aligned}$$

which occurs when $||\vec{r}||^2 = 1$. This is just algebra once we notice that the trace of $\vec{r} \cdot \sigma$ is zero and the trace of $(\vec{r} \cdot \sigma)^2 = 2(r_x^2 + r_y^2 + r_z^2)$ (the cross terms cancel since the anticommutator $\{\sigma_i, \sigma j\} = \delta_{ij}$)

■

Problem 5. *Problem 2.75*

Solution. ■

Problem 6. *Problem 2.79*

Solution. ■