Homework 3

Quantum Mechanics
Sept 15th, 2022 CLAYTON SEITZ

Problem 1. *Problem 2.1 from Sakurai*

Solution. The Heisenberg equation of motion reads

$$\frac{dA}{dt} = \frac{1}{i\hbar}[A, H]$$

For the spin precession problem, we have the Hamiltonian

$$H = -\left(\frac{eB}{mc}\right)S_z = \omega S_z$$

For $A = S_x, S_y, S_z$, the time evolution is given by

$$\begin{aligned}
\frac{dS_x}{dt} &= \frac{\omega}{i\hbar}[S_x, S_z] = -\omega S_y \\
\frac{dS_y}{dt} &= \frac{\omega}{i\hbar}[S_y, S_z] = \omega S_x \\
\frac{dS_z}{dt} &= \frac{\omega}{i\hbar}[S_z, S_z] = 0
\end{aligned}$$

The above system has a straightforward solution:

$$\begin{aligned}
S_x(t) &= \cos(\omega t) \\
S_y(t) &= \sin(\omega t) \\
S_z(t) &= S_z(0)
\end{aligned}$$

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Problem 2. *Problem 2.3 from Sakurai*

Solution. We are given that $\vec{B} = B\hat{z}$ and that we are in the eigenstate $|\psi(0)\rangle = |\boldsymbol{S}\cdot\hat{\boldsymbol{n}}\rangle_+$, which reads

$$\begin{aligned} |\psi(0)\rangle &= \psi_+ |+\rangle + \psi_- |-\rangle \\ &= \cos\frac{\beta}{2} |+\rangle + \sin\frac{\beta}{2} |-\rangle \end{aligned}$$

where we have set $\alpha = 0$ since the ket is in the x-z plane. This state will evolve according to a Hamiltonian

$$H = -\left(\frac{eB}{m_e c}\right) S_z$$

Let $\omega = |e|B/m_e c$ giving $H = \omega S_z$. We have the energies

$$E_\pm = \mp\frac{e\hbar B}{2m_e c} = \mp\hbar\omega$$

$$\begin{aligned} |\psi(t)\rangle &= \psi_+(0)\exp\left(\frac{-iE_+t}{\hbar}\right)|+\rangle + \psi_-(0)\exp\left(\frac{-iE_-t}{\hbar}\right)|-\rangle \\ &= \cos\frac{\beta}{2}\exp\left(\frac{-i\omega t}{2}\right)|+\rangle + \sin\frac{\beta}{2}\exp\left(\frac{i\omega t}{2}\right)|-\rangle \end{aligned}$$

In general, the probability of measuring $|+\rangle_x = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$ is given by the inner product

$$\begin{aligned} |\langle S_x;+|\psi;t\rangle|^2 &= \Bigg|\left(\frac{1}{\sqrt{2}}\langle +| + \frac{1}{\sqrt{2}}\langle -|\right)\cdot \\ &\qquad \left(\psi_+\exp\left(\frac{-i\omega t}{2}\right)|+\rangle + \psi_-\exp\left(\frac{i\omega t}{2}\right)|-\rangle\right)\Bigg|^2 \\ &= \left|\frac{1}{\sqrt{2}}\cos\frac{\beta}{2}\exp\left(\frac{-i\omega t}{2}\right) + \frac{1}{\sqrt{2}}\sin\frac{\beta}{2}\exp\left(\frac{i\omega t}{2}\right)\right|^2 \end{aligned}$$

Using the half-angle identity for $\sin\theta$ and some straightforward arithmetic gives

$$| \langle S_x; + | \psi; t \rangle |^2 = \frac{1 + \sin\beta\cos\omega t}{2}$$

For the time-dependence of $\langle S_x \rangle$, we have

$$\begin{aligned}
\langle S_x \rangle (t) &= \langle \psi; t | \, S_x \, | \psi; t \rangle \\
&= \left(\psi_+ \exp\left(\frac{i\omega t}{2}\right) \langle + | + \psi_- \exp\left(\frac{-i\omega t}{2}\right) \langle - | \right) \\
&\cdot \frac{\hbar}{2} \left(\psi_+ \exp\left(-\frac{i\omega t}{2}\right) | - \rangle + \psi_- \exp\left(\frac{i\omega t}{2}\right) | + \rangle \right)
\end{aligned}$$

Subsituting ψ_+ and ψ_- with the same values as above, we get

$$\langle S_x \rangle (t) = \frac{\hbar}{2} \sin\beta\cos\omega t$$

When $\beta = \pi/2$ the probability oscillates between 0 and 1 with frequency ω and when $\beta = 0$ then the probability is always $1/2$, as expected. The expectation value also makes sense because when $\beta = 0$, we can get $\pm\hbar/2$ with equal probability, giving zero on average. When $\beta = \pi/2$ the expectation value oscillates between $\hbar/2$ and $-\hbar/2$.

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Problem 3. *Problem 2.9 from Sakurai*

Solution.

We were given the wavefunction

$$\langle x | \alpha \rangle = A(x-a)^2(x+a)^2 \exp(ikx)$$

We can start by visualizing the real and imaginary parts of the position representation $\langle x | \alpha \rangle$ when $A = a = k = 1$

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Plot[(x − 1)^2(x + 1)^2Cos[x], {x, −2, 2}]
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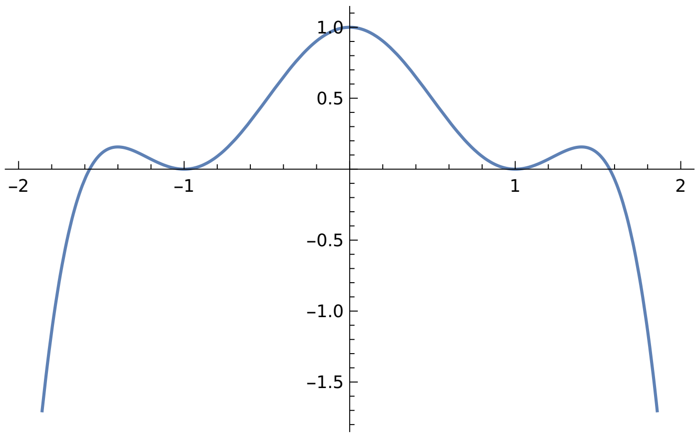
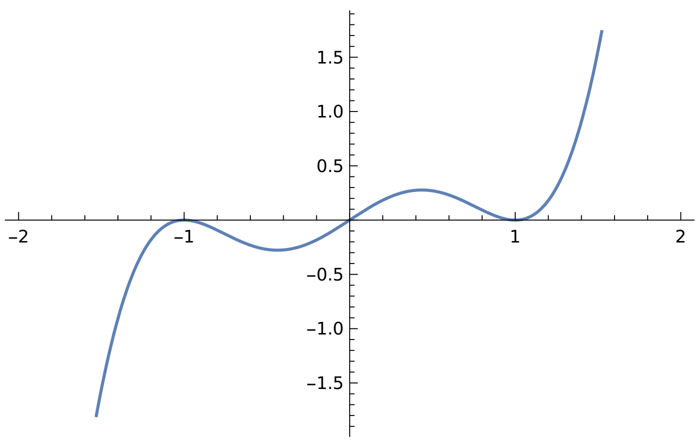


Plot $\left[(x-1)^2(x+1)^2\mathrm{Sin}[x], \{x, -2, 2\}\right]$



To find the normalization constant A, we just need to integrate $\langle x|\alpha\rangle$

$$\begin{aligned} A &= \left(\int_{-a}^{+a} \langle x|\alpha\rangle \, dx\right)^{-1} \\ &= \left(\int_{-a}^{+a} (x-a)^2(x+a)^2 \exp(ikx) dx\right)^{-1} \\ &= \left(\int_{-a}^{+a} (x-a)^2(x+a)^2 \cos(kx) dx\right)^{-1} \\ &- \left(\int_{-a}^{+a} (x-a)^2(x+a)^2 \sin(kx) dx\right)^{-1} \end{aligned}$$

We can evalulate these integrals individually:

Integrate $\left[(x-a)^2(x+a)^2\mathrm{Cos}[k*x], \{x, -a, a\}\right]$

$$-\frac{16\left(3ak\mathrm{Cos}[ak]+\left(-3+a^2k^2\right)\mathrm{Sin}[ak]\right)}{k^5}$$

Integrate $\left[(x-a)^2(x+a)^2\mathrm{Sin}[k*x],\{x,-a,a\}\right]$

0

The integral of the imaginary part is obvious since that part of the wave-function is odd.

$$A=-\frac{k^5}{16\left(3ka\cos(ka)+\left(a^2k^2-3\right)\sin(ka)\right)}$$

The expectation value $\langle x\rangle$ is found by integrating

$$\begin{aligned}\langle x\rangle &= \int_{-a}^{a} x\,\langle\alpha|x\rangle\,\langle x|\alpha\rangle\,dx\\ &= \int_{-a}^{+a} x\,\langle\alpha|x\rangle\,\langle x|\alpha\rangle\,dx\\ &= \int_{-a}^{+a} x(x-a)^2(x+a)^2dx\end{aligned}$$

where the complex exponential vanishes due to the complex conjugation. The expectation value $\langle x^2\rangle$ is found by integrating

$$\langle x^2\rangle = \int_{-a}^{+a} x^2(x-a)^2(x+a)^2dx$$

The expectation value $\langle p\rangle$ is found similarly

$$\begin{aligned}\langle p\rangle &= \int_{-a}^{+a}\langle\alpha|x\rangle\,\frac{\hbar}{i}\frac{\partial}{\partial x}\,\langle x|\alpha\rangle\,dx\\ &= A^2\int_{-a}^{+a}(x-a)^2(x+a)^2\exp(-ikx)\frac{\hbar}{i}\frac{\partial}{\partial x}(x-a)^2(x+a)^2\exp(ikx)\end{aligned}$$

The expectation value $\langle p^2\rangle$ is found by integrating

$$\begin{aligned}\langle p^2\rangle &= -\int_{-a}^{+a}\langle\alpha|x\rangle\,\hbar^2\frac{\partial^2}{\partial x^2}\,\langle x|\alpha\rangle\,dx\\ &= -A^2\int_{-a}^{+a}(x-a)^2(x+a)^2\exp(-ikx)\hbar^2\frac{\partial^2}{\partial x^2}(x-a)^2(x+a)^2\exp(ikx)\end{aligned}$$

The variance $\langle(\Delta x)^2\rangle$ is just

$$\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2$$

The variance $\langle(\Delta p)^2\rangle$ is just

$$\langle(\Delta p)^2\rangle = \langle p^2\rangle - \langle p\rangle^2$$

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Problem 4. *Problem 2.10 from Sakurai*

Solution. Let $|\psi\rangle = \alpha\,|a'\rangle + \beta\,|a''\rangle$ be an eigenvector of the Hamiltonian. Note that this must be real for the eigenvalue to be real. That means that

$$\begin{aligned} H\,|\psi\rangle &= (|a'\rangle\,\delta\,\langle a''| + |a''\rangle\,\delta\,\langle a'|)\,(\alpha\,|a'\rangle + \beta\,|a''\rangle) \\ &= \delta\,(\alpha\,|a''\rangle + \beta\,|a'\rangle) \end{aligned}$$

Therefore $\alpha = \beta = \frac{1}{\sqrt{2}}$ or $\alpha = \frac{1}{\sqrt{2}}$ and $\beta = -\frac{1}{\sqrt{2}}$. Giving eigenvalues $\pm\delta$. To get the time evolution of the state, we need to express these in the basis of H. Just based on inspection of the the two bases, we can tell that

$$\begin{aligned} |a'\rangle &= \frac{1}{\sqrt{2}}\,(|\psi_1\rangle - |\psi_2\rangle) \\ |a''\rangle &= \frac{1}{\sqrt{2}}\,(|\psi_1\rangle + |\psi_2\rangle) \end{aligned}$$

and, since the Hamiltonian is time-independent, a state prepared in $|a'\rangle$ will evolve according to

$$|\alpha(t)\rangle = \frac{1}{\sqrt{2}}\exp\left(\frac{-i\delta t}{\hbar}\right)|\psi_1\rangle - \frac{1}{\sqrt{2}}\exp\left(\frac{i\delta t}{\hbar}\right)|\psi_2\rangle$$

The probability of finding the system in the state $|a''\rangle$ at a later time is

$$\begin{aligned} |\,\langle a''|\alpha(t)\rangle\,|^2 &= \left|\frac{1}{\sqrt{2}}\,(\langle\psi_1| + \langle\psi_2|)\right. \\ &\quad \left.\cdot\left(\frac{1}{\sqrt{2}}\exp\left(\frac{-i\delta t}{\hbar}\right)|\psi_1\rangle - \frac{1}{\sqrt{2}}\exp\left(\frac{i\delta t}{\hbar}\right)|\psi_2\rangle\right)\right|^2 \\ &= \frac{1}{4}\sin^2\frac{\delta t}{\hbar} \end{aligned}$$

This could describe a system in which the eigenvectors of the Hamiltonian are simultaneous with the eigenvectors of S_x, however the states $|a'\rangle$ and $|a''\rangle$ are expressed in the S_z basis.

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Problem 5. *Problem 2.12 from Sakurai*

Solution. The state is prepared in

$$|\alpha; t=0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{\exp(i\delta)}{\sqrt{2}}|1\rangle$$

In general, the energies of $|n\rangle$ are $E_n = \left(n+\frac{1}{2}\right)\hbar\omega$. Therefore, the time dependence of the state can be evalulated as

$$\begin{aligned}|\alpha; t\rangle &= \exp\left(-\frac{iHt}{\hbar}\right)|\alpha; t\rangle \\ &= \frac{1}{\sqrt{2}}\exp\frac{-i\omega t}{2}|0\rangle + \frac{1}{\sqrt{2}}\exp(i\delta)\exp\frac{-3i\omega t}{2}|1\rangle\end{aligned}$$

$$\langle x|\alpha; t\rangle = \frac{1}{\sqrt{2}}\exp\frac{-i\omega t}{2}\langle x|0\rangle + \frac{1}{\sqrt{2}}\exp(i\delta)\exp\frac{-3i\omega t}{2}\langle x|1\rangle$$

and we know in general that the position representation of $|n\rangle$ i.e., $\langle x|n\rangle$ are

$$\langle x|n\rangle = \psi_n(x) = \frac{1}{\sqrt{2^n\, n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$$

$$\begin{aligned}\langle x\rangle(t) &= \langle\alpha; t|\, x\, |\alpha; t\rangle \\ &= \left(\frac{1}{\sqrt{2}}\exp\frac{i\omega t}{2}\langle 0| + \frac{1}{\sqrt{2}}\exp(-i\delta)\exp\frac{3i\omega t}{2}\langle 1|\right) \\ &\quad x\left(\frac{1}{\sqrt{2}}\exp\frac{-i\omega t}{2}|0\rangle + \frac{1}{\sqrt{2}}\exp(i\delta)\exp\frac{-3i\omega t}{2}|1\rangle\right) \\ &= \frac{1}{2}\langle 0|\, x\, |0\rangle + \frac{1}{2}\langle 1|\, x\, |1\rangle \\ &\quad + \frac{1}{2}\exp(i\delta)\exp(-i\omega t)\langle 0|\, x\, |1\rangle + \frac{1}{2}\exp(-i\delta)\exp(i\omega t)\langle 1|\, x\, |0\rangle\end{aligned}$$

Now recall the general expression for the matrix element of x

$$\langle n'| x |n\rangle = \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n}\delta_{n',n-1} + \sqrt{n+1}\delta_{n',n+1}\right)$$

which means that the above expression simplifies to

$$\begin{aligned}
\langle x\rangle(t) &= \frac{1}{2}\exp(i\delta)\exp(-i\omega t)\langle 0| x |1\rangle + \frac{1}{2}\exp(-i\delta)\exp(i\omega t)\langle 1| x |0\rangle \\
&= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(\exp(i\delta)\exp(-i\omega t) + \exp(-i\delta)\exp(i\omega t)\right) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\cos(\delta - \omega t) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t - \delta)
\end{aligned}$$

For momentum, we can just replace the operator x with p in the expressions above:

$$\begin{aligned}
\langle p\rangle(t) &= \langle \alpha; t| x |\alpha; t\rangle \\
&= \left(\frac{1}{\sqrt{2}}\exp\frac{i\omega t}{2}\langle 0| + \frac{1}{\sqrt{2}}\exp(-i\delta)\exp\frac{3i\omega t}{2}\langle 1|\right) \\
&\hat{p}\left(\frac{1}{\sqrt{2}}\exp\frac{-i\omega t}{2}|0\rangle + \frac{1}{\sqrt{2}}\exp(i\delta)\exp\frac{-3i\omega t}{2}|1\rangle\right) \\
&= \frac{1}{2}\langle 0| p |0\rangle + \frac{1}{2}\langle 1| p |1\rangle \\
&+ \frac{1}{2}\exp(i\delta)\exp(-i\omega t)\langle 0| p |1\rangle + \frac{1}{2}\exp(-i\delta)\exp(i\omega t)\langle 1| p |0\rangle
\end{aligned}$$

and we have another general expression for the matrix element of p

$$\langle n'| p |n\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\left(-\sqrt{n}\delta_{n',n-1} + \sqrt{n+1}\delta_{n',n+1}\right)$$

which again means that the above expression simplifies to

$$
\begin{aligned}
\langle p\rangle(t) &= \frac{1}{2}\exp(i\delta)\exp(-i\omega t)\langle 0|\,p\,|1\rangle + \frac{1}{2}\exp(-i\delta)\exp(i\omega t)\langle 1|\,p\,|0\rangle \\
&= \frac{i}{2}\sqrt{\frac{m\hbar\omega}{2}}\left(-\exp(i\delta)\exp(-i\omega t)+\exp(-i\delta)\exp(i\omega t)\right) \\
&= -\sqrt{\frac{m\hbar\omega}{2}}\sin(\omega t-\delta)
\end{aligned}
$$

In the Heisenberg picture, we have the Heisenberg equations of motion

$$
\begin{aligned}
\frac{dp}{dt} &= -m\omega^2 x \\
\frac{dx}{dt} &= \frac{p}{m}
\end{aligned}
$$

It is has been shown in the text how to uncouple these in terms of the ladder operators and solve the system for the time dependent operators $x(t)$ and $p(t)$

$$
\begin{aligned}
x(t) &= x(0)\cos(\omega t) + \left(\frac{p(0)}{m\omega}\right)\sin(\omega t) \\
p(t) &= -m\omega x(0)\sin(\omega t) + p(0)\cos(\omega t)
\end{aligned}
$$

To get $\langle x\rangle(t)$ we have

$$
\begin{aligned}
\langle x\rangle(t) &= \langle\alpha|\,x(0)\cos(\omega t) + \left(\frac{p(0)}{m\omega}\right)\sin(\omega t)\,|\alpha\rangle \\
&= \langle\alpha|\,x(0)\,|\alpha\rangle\cos(\omega t) + \langle\alpha|\,p(0)\,|\alpha\rangle\,\frac{\sin(\omega t)}{m\omega}
\end{aligned}
$$

$$
\begin{aligned}
\langle\alpha|\,x(0)\,|\alpha\rangle &= \frac{1}{2}\langle 0|\,x(0)\,|0\rangle + \frac{1}{2}\langle 1|\,x(0)\,|1\rangle \\
&\quad + \frac{1}{2}\exp(i\delta)\langle 0|\,x(0)\,|1\rangle + \frac{1}{2}\exp(i\delta)\langle 1|\,x(0)\,|0\rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}}\exp(i\delta)
\end{aligned}
$$

The factor of one-half disappeared in the $\langle\alpha|\,x(0)\,|\alpha\rangle$ term since $\langle n'|\,x\,|n\rangle$ is real and therefore equal to its complex conjugate

$$\begin{aligned}
\langle x\rangle(t) &= \sqrt{\frac{\hbar}{2m\omega}}\exp(i\delta)\cos(\omega t) + \frac{i}{2}\exp(i\delta)\sqrt{\frac{m\hbar\omega}{2}}\frac{\sin(\omega t)}{m\omega} \\
&= \sqrt{\frac{\hbar}{2m\omega}}\exp(i\delta)\exp(i\omega t) \\
&= \sqrt{\frac{\hbar}{2m\omega}}\exp(i(\omega t + \delta))
\end{aligned}$$

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Problem 6. *Problem 2.13 from Sakurai*

Solution.

The Heisenberg equations of motion are

$$\begin{aligned}
\frac{dp}{dt} &= F \\
\frac{dx}{dt} &= \frac{p_0 + Ft}{m}
\end{aligned}$$

The solution is simply

$$\begin{aligned}
p(t) &= p(0) + Ft \\
x(t) &= x(0) + \frac{p(0)}{m}t + \frac{1}{2}\frac{Ft^2}{m}
\end{aligned}$$

$$\begin{aligned}
\langle x\rangle(t) &= \langle\alpha|\left(x(0) + \frac{p_0}{m}t + \frac{1}{2}\frac{Ft^2}{m}\right)|\alpha\rangle \\
&= x_0 + \frac{p_0 t}{m} + \frac{Ft^2}{2m}
\end{aligned}$$

and for $\langle p\rangle$ we have

$$\begin{aligned}
\langle p\rangle(t) &= \langle\alpha|\, p(0) + Ft\,|\alpha\rangle \\
&= p_0 + Ft
\end{aligned}$$

In the Schrodinger picture, this Hamiltonian is not a constant

$$H(t) = \frac{p(t)^2}{2m} + V(x,t) = \frac{p(t)^2}{2m} + Fx(t)$$

although, using the canonical commutation relations, we can see that $H(t_0)$ commutes with $H(t)$. So we should be able to define

$$\mathcal{U}(t,t_0) = \exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} H(t)dt\right)$$

such that $|\alpha;t\rangle = \mathcal{U}(t,t_0)\,|\alpha;t_0\rangle$. If we evaluated that expression, then we could write $|\alpha;t\rangle$ explicitly.

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