Problem Set 2

Information and Coding Theory

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Problem 0.1. *Find tight upper and lower bounds on two extremely biased coins where the first coin is distributed according to*

$$P = \begin{cases} 0 & \epsilon \\ 1 & 1-\epsilon \end{cases}$$

and the second is distributed according to

$$Q = \begin{cases} 0 & 2\epsilon \\ 1 & 1-2\epsilon \end{cases}$$

Solution. I will assume that distinguishing the two coins means that, given a sequence of n flips, we can say whether it is coin P or coin Q 90 percent of the time. To start, we write out the KL-Divergence between the distributions P and Q for a sequence of n coin tosses.

$$\begin{aligned}
D(P||Q) &= \epsilon \log \frac{1}{2\epsilon} + (1-\epsilon)\log\frac{1}{1-2\epsilon} \\
&= \epsilon \log \frac{1-2\epsilon}{2\epsilon} + \epsilon \log \left(\frac{1}{1-2\epsilon}\right)^{1/\epsilon} \\
&= \epsilon \left(\log \frac{1}{2\epsilon}(1-2\epsilon)^{\frac{1-\epsilon}{\epsilon}}\right) \\
&= \frac{\epsilon}{2\ln 2}\left(\ln \frac{(1-2\epsilon)^{\frac{1-\epsilon}{\epsilon}}}{2\epsilon}\right) \\
&= \frac{\epsilon}{2\ln 2}\left(\ln\left(1 + \frac{(1-2\epsilon)^{\frac{1-\epsilon}{\epsilon}} - 2\epsilon}{2\epsilon}\right)\right) \\
&\leq \frac{1}{3\ln 2}(1-2\epsilon)^{\frac{1-\epsilon}{\epsilon}} - 2\epsilon
\end{aligned}$$

At the same time, we know that

$$n \geq \frac{1}{2\ln 2 \cdot D(P||Q)} \left(\frac{8}{5}\right)^2$$

which means that

$$n \geq \frac{3}{2} \frac{1}{(1-2\epsilon)^{\frac{1-\epsilon}{\epsilon}} - 2\epsilon} \left(\frac{8}{5}\right)^2$$

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Problem 0.2. *Show that* $0 \leq \mathbf{JSD}(P, Q) \leq 1$

Solution.

$$\mathbf{JSD}(P, Q) \;=\; \frac{1}{2}D(P||M) + \frac{1}{2}D(Q||M)$$

The lower bound must be true because $D(P||M) \geq 0$ and $Q(P||M) \geq 0$. For the upper bound, consider just one of the terms

$$\begin{aligned} D(P||M) &= \frac{1}{2}\sum_{x\sim P} P(x) \log \frac{P(x)}{M(x)} \\ &= \frac{1}{2}\sum_{x\sim P} P(x) \log \frac{2P(x)}{P(x)+Q(x)} \\ &\leq \frac{1}{2}\sum_{x\sim P} P(x) \log 2 = \frac{1}{2} \end{aligned}$$

Therefore, $\mathbf{JSD}(P, Q) \leq 1$.
Show that $\mathbf{JSD}(P, Q) \geq \frac{1}{8\ln 2} \cdot ||P - Q||_1^2$

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\begin{aligned}
\mathbf{JSD}(P,Q) &= \frac{1}{2}\left[D(P||M) + D(Q||M)\right] \\
&\geq \frac{1}{4\ln 2}\left[||P-M||_1^2 + ||Q-M||_1^2\right] \\
&= \frac{1}{4\ln 2}\left[\left(\sum |P-M|\right)^2 + \left(\sum |Q-M|\right)^2\right] \\
&= \frac{1}{8\ln 2}\left[\left(\sum |P-Q|\right)^2 + \left(\sum |Q-P|\right)^2\right] \\
&= \frac{1}{8\ln 2}\cdot ||P-Q||_1^2
\end{aligned}
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$$\mathbf{JSD}_\lambda(P_1 \,...\, P_k) = \sum_i \lambda_i D(P_i||M)$$

where $M = \sum_i \lambda_i P_i$. Show that

$$0 \leq \mathbf{JSD}_\lambda(P_1 \,...\, P_k) \leq H(\lambda)$$

As before, the lower bound must be true because $D(P_i||M) \geq 0$ and λ is non-negative. As for the upper bound,

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\begin{aligned}
\mathbf{JSD}_\lambda(P_1 \,...\, P_k) &= \sum_i \lambda_i D(P_i||M) \\
&= \sum_i \lambda_i P_i \log \frac{P_i}{M} \\
&= H(\sum_i \lambda_i P_i) - \sum_i \lambda_i H(P_i) \\
&= H(\lambda) - \sum_i \lambda_i H(P_i) \\
&\leq H(\lambda)
\end{aligned}
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Problem 0.3. *Differential entropy of the multivariate Gaussian*

$$\phi(x) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp -\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)$$

Solution.

$$
\begin{aligned}
h(x) &= -\int \phi(x)\log\phi(x)dx \\
&= \int \phi(x)\left[\frac{n}{2}\log(2\pi e) + \frac{1}{2}\log|\Sigma| + \frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right]dx \\
&= \frac{n}{2}\log(2\pi e) + \frac{1}{2}\log|\Sigma| + \mathbf{E}\left[\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right] \\
&= \frac{n}{2}\log(2\pi e) + \frac{1}{2}\log|\Sigma|
\end{aligned}
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