Homework 10

Quantum Mechanics
December 5, 2022 C Seitz

Problem 1. *4.7*

Solution.
The wave function in three dimensions for a free particle ($V = 0$), is

$$\psi(\boldsymbol{x}, t) = u(\boldsymbol{x})e^{-iE_n t/\hbar}$$
$$\psi^*(\boldsymbol{x}, -t) = u^*(\boldsymbol{x})e^{-iE_n t/\hbar}$$

where $u(\boldsymbol{x}) = e^{i\vec{p}\cdot\vec{k}}$. Note that the phase remains unchanged under complex conjugation and time reversal. Now if we reverse the direction of momentum i.e. $|p\rangle \to |p'\rangle$ for $\vec{p} \cdot \vec{p'} = -1$,

$$\psi'(\boldsymbol{x}, t) = u'(\boldsymbol{x})e^{-iE_n t/\hbar}$$

Notice that $u'(\boldsymbol{x}) = e^{-i\vec{p'}\cdot\vec{k}} = u^*(\boldsymbol{x})$. Therefore $\psi'(\boldsymbol{x}, t) = \psi^*(\boldsymbol{x}, -t)$

$$\chi_+(\hat{n}) = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\gamma} \end{pmatrix}$$

It is also known that

$$\chi_-(\hat{n}) = \begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\gamma} \\ \cos\frac{\theta}{2} \end{pmatrix}$$

So we just need to prove that the given transformation gives this result, which it does

$$-i\sigma_2\chi^*(\hat{n}) = -i\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{-i\gamma}\end{pmatrix} = \begin{pmatrix}-\sin\frac{\theta}{2}e^{-i\gamma}\\ \cos\frac{\theta}{2}\end{pmatrix}$$

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Problem 2. *4.8*

Solution.

First note that

$$H\Theta\left|n\right\rangle = \Theta H\left|n\right\rangle = E_n\Theta\left|n\right\rangle$$

so $|n\rangle$ and $\Theta|n\rangle$ have the same energy. If the states are nondegenerate then $|n\rangle$ and $\Theta|n\rangle$ represent the same state. Their wavefunctions are then the same:

$$\langle x'|n\rangle = \langle n|x'\rangle^*$$

which occurs if they are real, or have a phase difference independent of x. For this reason the wavefunction $\psi = e^{ip\cdot x/\hbar}$ does not violate time reversal invariance, because it is degenerate with $e^{ip\cdot x/\hbar}$.

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Problem 3. *4.9*

Solution.

$$\begin{aligned}
\Theta\left|\alpha\right\rangle &= \int d^3\boldsymbol{p}\ \Theta\left|\boldsymbol{p}\right\rangle\langle\boldsymbol{p}|\alpha\rangle^* \\
&= \int d^3\boldsymbol{p}\ \left|-\boldsymbol{p}\right\rangle\langle\boldsymbol{p}|\alpha\rangle^* \\
&= \int d^3\boldsymbol{p}\ \left|\boldsymbol{p}\right\rangle\langle-\boldsymbol{p}|\alpha\rangle^* \\
&= \phi^*(-p)
\end{aligned}$$

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Problem 4. *4.10*

Solution. ∎

Problem 5. *4.11*

Solution.

For energy eigenkets $|\alpha\rangle$,

$$\begin{aligned}\langle \boldsymbol{L} \rangle &= \langle \alpha | \boldsymbol{L} | \alpha \rangle \\ &= \langle \tilde{\alpha} | \Theta \boldsymbol{L} \Theta^{-1} | \tilde{\alpha} \rangle \\ &= - \langle \tilde{\alpha} | \boldsymbol{L} | \tilde{\alpha} \rangle\end{aligned}$$

But the Hamiltonian is invariant under time-reversal, so this is only satisfied when $\langle \boldsymbol{L} \rangle = 0$. If the wavefunction is expanded as

$$\sum_l \sum_m F_{lm}(r) Y_l^m(\theta, \phi)$$

We know that when the Hamiltonian is invariant under time-reversal, the eigenkets must be real. Therefore, the phase restriction must satisfy the equality $F_{lm}(r)Y_l^m(\theta, \phi) = F_{lm}^*(r)(Y_l^m(\theta, \phi))^*$.

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Problem 6. *4.12*

Solution. ∎