## The Security

We consider an interest paying security with an initial investment cost of \$1000 whose value at maturity depends on the performance of the S&P 500 index over the life of the security. We denote the value of the index at the birth of the security by $S_0$ and its value at maturity by $S_M$. In addition, there is an additional so-called barrier level, $B$, for the index. This is related to the interest payments described below. Here are the numerical values of $S_0$ and $B$:

$S_0$: $S_0 = 1680$
$B$: $B = 60\% \times S_0 = 1008$.

**Value at Maturity.** At maturity, the value of the security will be \$1000 if the value of the index is equal to or greater than $\frac{1}{2}S_0$; that is, if $S_M \geq 0.5S_0 = 840$. Otherwise, if $S_M < 840$, the value of the security is $\frac{S_M}{S_0} \times \$500$. For example, if at maturity the index has value 800, then the value of the security is $\frac{800}{S_M} \times \$500 = \frac{800}{840} \times \$500 = \$476.19$

From this we see that there is no guarantee that the principal will be returned. Moreover, the final payoff does not depend continuously on the value of the index: at $S_M = 840$ one gets \$1000 back; if $S_M = 839.99$, one gets only \$499.40 back.

The form of the maturity value says that this security is comprised of the following components:

(1) 500 digital call options on the index with strike 840. Each such option pays \$1 to the holder if the value of the index is equal to or greater than 840[1].
(2) a short put on the index with strike 840.
(3) other investments (1) that will guarantee that at maturity at least \$500 is available to satisfy the obligation of the short put (this could be as much as \$500 is the index fell to 0) and (2) that will generate cash flows to the owner of the security over the life of the security.

One can create, from a \$1000 investment a porfolio replicating the behavior of this security as follows:

(1) write the put option and thereby receive a certain amount of money for it.
(2) purchase the 500 digital calls. The value of a digital call is equal to the discounted probability that the final spot $S_M$ will be greater than the strike ( 840 in this case). Since this is always less than 1, we see that the cost of the digital calls will be less than \$500.
(3) invest enough money in a risk-free investment that is guarantee to be worth \$500 at maturity. As stated above, this is required since, in writing the put,

[1]An exchange traded digital option would require the final spot, $S_M$ to be strictly larger than the strike.

the investor has guaranteed to purchase the index at maturity–even if the index crashes to 0[2].

(4) Invest all the cash left over from steps $1-3$ in some high interest rate security. This money includes (1) the proceeds from the put, (2) \$500 minus the cost of the digital calls, and (3) assuming the security used in step 3 is a zero-coupon bond that pays \$500 at maturity, \$500 minus the cost of this bond.

**Cash Flow Aspects.** The security will pay "interest" quarterly (on the principal of \$1000) at the following annual rates:

(1) The fixed rate of 12% for the first year.
(2) A floating rate of $5 \times (Sw_{30} - Sw_2)$ where $Sw_{30}$ is the 30 year swap rate and $Sw_2$ is the 2 year swap rate both as observed at the start of the quarter.
(3) the interest rate is subject to the constraints that it be always larger than 0 and less than 12%.
(4) the floating rate presented above is further subject to these modifications: no interest is earned on days for which the S& P 500 falls below the level $B = 1008$.

Note that after the first year the interest rate is not fixed. However, in our analysis, we will assume a fixed rate.

## The Project

The main financial portion of the project involves the (1) evaluation of the options, the zero-coupon bond, and the cash for additional investment that form the portfolio described; and (2) the evaluation of the cash flows described above, under certain simplifying assumptions.

**Building the Portfolio.** To attempt to form the portfolio described above we must find the prices of the options it contains as well as the current price of a zero-coupon bind that matures in 15-years. We will assume continuous compounding in all of our valuations. So the main tasks here are:

(1) Use the Black-Scholes formula to calculate the value of a European put option that expires in 15 years. Assume that the dividend yield is 1.97%, and the risk free rate is 3.12%. For the volatility use both 31% and 40%.
(2) Use the Black-Scholes formula for digital calls to evaluate the cost of the digital options in the portfolio. Use the parameters given above. Make sure that you use the same volatility parameter for the digital call as for the vanilla put.

[2] In reality the amount of security required would probably depend on risk management models of the counterparty to the transaction

(3) Find the cost of a risk-free zero-coupon bond that will be worth \$500 in 15 years. Use the risk-free rate given above. (It's from CMT.)

Assuming that an investor allots \$1000 for this portfolio, after items $1-3$ have been incorporated into the portfolio, how much cash is left over for other investments?

Consider this portfolio: the 500 digital call options, the put option, the zero coupon bond. Make charts of the values of this portfolio on the following dates: date of portfolio generation, 5 years after portfolio formation, 10 years after portfolio formation. Do this for the range of S&P 500 values in the range $700 < S < 1000$.

**The Cash Flows.** Find the current value of the discounted cash flows that this security delivers under that assumptions that (1) after year 1, all interest payments are equal to the 5× (the historical average of the difference between $Sw_{30}$ and $Sw_2$), and (2) the cash flow $t$ years from now is discounted by the factor $e^{0.0544t}$. Note that the variable $t$ ranges over the set $\{0.25, 0.50, 0.75, 1.00, \ldots, 14.25, 14.50, 14.75, 15.00\}$ This valuation assumes that the S&P 500 index never falls below 1008.

Now incorporate the possibility of the index falling below 1008 by multiplying each cash flow by the probability that the index is above 1008 at the start of the quarter. In this case, note that since the probability formula involves the risk-free rate, you will have to estimate risk free rates by interpolation in the CMT curve. Linear interpolation is fine.

If you are really ambitious, you can try to replace the discount factors $e^{0.0544t}$ by $e^{r_t t}$ where $r_t$ is the rate obtained by adding 2.58% to the current CMT rate. You will need to interpolate again.

## Expectations Concerning Programming

The input for the program should come from a file which has one parameter per line. The input parameters should include $S_0$, $B$, and all parameters necessary for the option, bond, and cash flow valuations. You should create a README.txt file that describes the format of the input file (that is, the order of the parameters). The outputs for the program will consist of data to import into a spreadsheet for charting as well as output to the console for that communicates answers to the basic questions.

Your code should be separated into at least four independent .cpp files:

(1) A file called BSE.cpp which holds the implementation of the option pricing formulas used. There should also be a BSE.h which holds only the names of the functions and their signatures.
(2) A file called CashFlow.cpp which holds the code for calculating the required cash flows.
(3) A file called Main.cpp which runs the program.

(4) A file called IO.cpp[3] which holds the functions used to get input and produce output.
(5) .h files as needed.
(6) Any additional files you need. However, please explain what these are and what they do.

Your Black-Scholes function implementations should use Function Objects if possible.

You should make use of the algorithms in the <numeric> class if possible.

## What to Deliver

A hard copy of a report that describes the project and your results. Answers to all questions raised above should be in the report. You must display in a clear manner all of the option, bond, and cash flow data that you have calculated. It is desired also that you describe the methods of calculation with a formula and a short code segment if possible. Do NOT include the entire code base. Include the charts asked for and any observations you think are important and relevant. If there are any special features to your code, make sure you point them out in the report. Note that all third party code must be acknowledged.

An e-copy of the report.

An e-copy of all source code and other relevant text-based files. Do not include executables, .o files, or project files.

[3] I will give you a version of this for you to use if you wish.