Validated Parsing of Regular Expressions in Agda

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Abstract

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In this project, we aim to study the feasibility of constructing a certified algorithm for translating regular expressions to finite automata in Type Theory. The translation consists of several steps: regular expressions to ϵ-NFA using Thompson's construction; ϵ-NFA to NFA by removing ϵ-transitions; DFA to DFA by powerset construction; and DFA to MDFA by removing unreachable states and quotient construction. The correctness of the translation is obtained by showing that their accepting languages are equal and the translated MDFA is minimal. The above translation and its correctness proofs are formalised in Agda – a dependently-typed functional programming language based on Type Theory.

Keywords: regular expression, finite automata, type theory, agda

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All software for this project can be found at
https://codex.cs.bham.ac.uk/svn/projects/2015/wtc488/

List of Abbreviations

ϵ-NFA	Non-deterministic Finite Automaton with ϵ-transition
NFA	Non-deterministic Finite Automaton
DFA	Deterministic Finite Automaton
RDFA	Deterministic Finite Automaton in which all its states are Reachable
MDFA	Minimised Deterministic Finite Automaton

Contents

Chapter 1

Introduction

This project aims to study the feasibility of formalising Automata Theory [2] in Type Theory [14] with the aid of a dependently-typed functional programming language, Agda. Automata Theory is an extensive work; therefore, it will be unrealistic to include all the materials under the time constraints. Accordingly, this project will only focus on the theorems and proofs that are related to the translation of regular expressions to finite automata. In addition, this project also serves as an example of how complex and non-trivial proofs are formalised.

Our Agda formalisation consists of two components: 1) the translation of regular expressions to minimal DFA and 2) the correctness proofs of the translation. At this stage, we are only interested in the correctness of the translation but not the efficiency of the algorithms.

1.1 Background

Type theory was introduced by Per Martin-Löf in 1971 to provide an alternative foundation of mathematics based on the principles of mathematical constructivism where logic can be implemented within the theory. From another perspective, Type Theory is also a dependently-typed functional programming language. In order to bridge the gap between the theoretical representation of Type Theory and the requirements on a practical programming language, Norell [17] rewrote a dependently-typed language, Agda, based on Type Theory. Agda allows us to formalise mathematics and programming logic in Type Theory and to check the formalisation automatically by its type checker and termination checker. We will discuss the use of Agda in Chapter 2 in detail.

Automata Theory is the study of abstract machines and automata. Automata are abstract models of machines that perform computations on an input by moving between its states. There are several major families of automaton but we will only focus on finite state automata.

1.2 Related Work

Regular Expressions in Agda Agular and Mannaa published a similar work [1] in 2009. They constructed a decider for regular expressions which can determine whether a given string is accepted by a given regular expression. The decider was implemented by converting a regular expression into a partial automaton. Here is the type signature of the decider:

```
accept : (re : RegExp) → (as : List carrier) → Maybe (as ∈⌣ ⟦re⟧)
```

accept takes a regular expression (re) and a list of alphabets (as) as the arguments. If the string (as) is accepted by the regular expression (re), i.e. $as \in L(re)$, the decider will return its proof. However it fails to generate a proof for the opposite case, i.e. $as \notin L(re)$. As they explained in the paper, it is impossible without converting the regular expression into the entire automaton. Therefore, in our approach, we will translate a regular expression to a DFA in order to construct a full decider.

Certified Parsing of Regular Languages in Agda While in 2013, Firsov and Uustalu published another related research [8]. They translated regular expressions into NFA and proved that they are equivalent. Unlike Agular and Mannaa's decider, Firsov and Uustalu's algorithm can generate proofs for both cases. In their definition of NFA, the set of states (Q) and its subsets are represented as vectors; and the transition function (δ) takes an alphabet as the argument and returns a matrix representation of the transition table.

```
record NFA : Set where
  field
    |Q| : ℕ
    δ   : Σ → |Q| * |Q|
    I   : 1 * |Q|
    F   : |Q| * 1
```

This representation of sets allows us to iterate the sets easily but it looks unnatural compare to the actual mathematical definition of NFA. Therefore, we intend to avoid the vector representation in our approach. The comparison between these two approaches will be discussed in Chapter 6 in detail.

1.3 Motivation

My motivation on this project is to learn and apply dependent types in formalising mathematics and programming logic. At the beginning, I was new to dependent types and proof assistants; therefore, we had to choose carefully what theorems to formalise. On one hand, the theorems

should be non-trivial enough such that a substantial amount of work is required to be done. On the other hand, the theorems should not be too difficult because I am only a beginner in this area. Finally, we decided to go with the Automata Theory as its basic concepts were explained in the course *Model of Computation*.

1.4 Outline

Chapter 2 will be a brief introduction on Agda and dependent types. We will show how Agda can be used as a proof assistant by formalising logic and small programs. Experienced Agda users can skip this chapter. Following the background, Chapter 3 will be a detail description of our work. We will walk through the two components of our Agda formalisation. Note that the definitions, theorems and proofs written in this chapter are extracted from our Agda code. They may be different from their usual mathematical forms in order to adapt to the environment of Type Theory. In Chapter 4, we will describe the structure of our Agda files. Then, in Chapter 5, we will discuss two possible extensions to our project: 1) Myhill-Nerode Theorem and 2) the Pumping Lemma. After that, in Chapter 6, we will evaluate the project as a whole. Finally, the conclusions will be drawn.

Chapter 2

Agda

Agda is a dependently-typed functional programming language and a proof assistant based on Intuitionistic Type Theory [14]. The current version (Agda 2) is rewritten by Norell [17] during his doctorate study at the Chalmers University of Technology. In this chapter, we will describe the basic features of Agda and how dependent types are employed to construct programs and proofs. Most of the materials presented below can also be found in the two tutorial papers [4] and [18]. Interested readers can read the two papers in order to get a more precise idea on how to work with Agda. We will begin by showing how to do ordinary functional programming in Agda.

2.1 Simply Typed Functional Programming

Haskell is the implementation language of Agda and as shown below, Agda has borrowed many features from Haskell. In the following paragraphs, we will demonstrate how to define basic data types and functions. Let us begin with the boolean data type.

Boolean In Haskell, the boolean type can be defined as $data\ Bool = True \mid False$. While in Agda, the syntax of the declaration is slightly different.

```
data Bool : Set where
  true  : Bool
  false : Bool
```

Bool has two constructors: *true* and $false$. These two constructors are also the elements of *Bool* as they take no arguments. Furthermore, *Bool* itself is a member of the type *Set*. The type of *Set* is Set_1 and the type of Set_1 is Set_2. The type hierarchy goes on and becomes infinite. Now, let us define the negation of boolean values.

```
not : Bool → Bool
not true  = false
not false = true
```

Unlike in Haskell, a type signature must be provided explicitly for every function. Furthermore, all possible cases must be pattern matched in the function body. For instance, the function below will be rejected by the Agda compiler as the case ($false$) is missing.

```
not : Bool → Bool
not true  = false
```

Natural Number We will define the type of natural numbers in Peano style.

```
data ℕ : Set where
  zero : ℕ
  suc  : ℕ → ℕ
```

The constructor *suc* represents the successor of a given natural number. For instance, the number 1 is equivalent to (*suc zero*). Now, let us define the addition of natural numbers recursively as follow:

```
_+_ : ℕ → ℕ → ℕ
zero + m = m
(suc n) + m = suc (n + m)
```

Parameterised Types In Haskell, the type of list $[a]$ is parameterised by the type parameter a. The analogous data type in Agda is defined as follow:

```
data List (A : Set) : Set where
  []   : List A
  _::_ : A → List A → List A
```

Now, let us try to define a function that will return the first element of a given list.

```
head : {A : Set} → List A → A
head [] = {!!}
head (x :: xs) = x
```

What should be returned for case [ ]? In Haskell, the [ ] case can simply be ignored and an error will be produced by the compiler. However, as we have mentioned before, all possible cases must be pattern matched in an Agda function. One possible workaround is to return *nothing* – an element of the *Maybe* type. Another solution is to constrain the argument using dependent types such that the input list will always have at least one element.

2.2 Dependent Types

A dependent type is a type that depends on values of other types. For example, A^n is a vector that contains n elements of A. It is impossible to declare these kinds of types in simply-typed systems like Haskell[1]and Ocaml. Now, let us look at how it is declared in Agda.

```
data Vec (A : Set) : ℕ → Set where
  []    : Vec A zero
  _::_ : ∀ {n} → A → Vec A n → Vec A (suc n)
```

In the type signature, $(\mathbb{N} \to Set)$ means that Vec takes a number n from $\mathbb{N}$ and produces a type that depends on n. The inductive family Vec will produce different types with different numbers. For example, $(Vec\ A\ zero)$ is the type of empty vectors and $(Vec\ A\ 10)$ is another vector type with length ten.

Dependent types allow us to be more expressive and precise over type declaration. Let us define the *head* function for Vec.

```
head : {A : Set}{n : ℕ} → Vec A (suc n) → A
head (x :: xs) = x
```

Only the $(x :: xs)$ case needs to be pattern matched because an element of the type $(Vec\ A\ (suc\ n))$ must be in the form of $(x :: xs)$. Apart from vectors, a type of binary search tree can also be declared in which any tree of this type is guaranteed to be sorted. Interested readers can take a look at Chapter 6 in [4]. Furthermore, dependent types also allow us to encode predicate logic and program specifications as types. These two applications will be described in later part after we have discussed the idea of propositions as types.

2.3 Propositions as Types

In the 1930s, Curry identified the correspondence between propositions in propositional logic and types [6]. After that, in the 1960s, de Bruijn and Howard extended Curry's correspondence to predicate logic by introducing dependent types [5, 11]. Later on, Martin-Löf published his work, Intuitionistic Type Theory [14], which turned the correspondence into a new foundational system for constructive mathematics.

In the paragraphs below, we will show how the correspondence is formalised in Agda. Note that Intuitionistic Type Theory is based on constructive logic but not classical logic and there is a fundamental difference between them. Interested readers can take a look at [3]. Now, we will begin by showing how propositional logic is formalised in Agda.

[1]Haskell does not support dependent types by its own. However, there are several APIs that simulate dependent types, for example, Ivor [12] and GADT.

2.3.1 Propositional Logic

In general, Curry's correspondence states that a proposition can be interpreted as a set of its proofs. A proposition is true if and only if its set of proofs is inhabited, i.e. there is at least one element in the set; it is false if and only if its set of proofs is empty. Let us begin with the formalisation of $Truth$ – the proposition that is always true.

Truth For a proposition to be always true, its corresponding type must have at least one element.

```
data ⊤ : Set where
  tt : ⊤
```

Falsehood The proposition that is always false corresponds to a type having no elements at all.

```
data ⊥ : Set where
```

Conjunction Suppose A and B are propositions, then a proof of their conjunction, $A \wedge B$, should contain both a proof of A and a proof of B. In Type Theory, it corresponds to the product type.

```
data _×_ (A B : Set) : Set where
  _,_ : A → B → A × B
```

The above construction resembles the introduction rule of conjunction. The elimination rules are formalised as follow:

```
fst : {A B : Set} → A × B → A
fst (a , b) = a

snd : {A B : Set} → A × B → B
snd (a , b) = b
```

Disjunction Suppose A and B are propositions, then a proof of their disjunction, $A \vee B$, should contain either a proof of A or a proof of B. In Type Theory, it is represented by the sum type.

```
data _⊎_ (A B : Set) : Set where
  inj₁ : A → A ⊎ B
  inj₂ : B → A ⊎ B
```

The elimination rule of disjunction is defined as follow:

```
⊎-elim : {A B C : Set}
         → A ⊎ B
         → (A → C)
         → (B → C)
         → C
⊎-elim (inj₁ a) f g = f a
⊎-elim (inj₂ b) f g = g b
```

Negation Suppose A is a proposition, then its negation is defined as a function that transforms any arbitrary proof of A into the falsehood ($\bot$).

```
¬ : Set → Set
¬ A = A → ⊥
```

Implication We say that A implies B if and only if every proof of A can be transformed into a proof of B. In Type Theory, it corresponds to a function from A to B, i.e. $A \to B$.

Equivalence Two propositions A and B are equivalent if and only if A implies B and B implies A. It can be considered as a conjunction of the two implications.

```
_⟺_ : Set → Set → Set
A ⟺ B = (A → B) × (B → A)
```

Now, by using the above constructions, we can formalise theorems in propositional logic. For example, we can prove that if P implies Q and Q implies R, then P implies R. The corresponding proof in Agda is as follow:

```
prop−lem : {P Q : Set}
           → (P → Q)
           → (Q → R)
           → (P → R)
prop−lem f g = λ p → g (f p)
```

By completing the function, we have provided an element to the type $(P \to Q) \to (Q \to R) \to (P \to R)$ and thus we have also proved the theorem to be true.

2.3.2 Predicate Logic

We will now move on to predicate logic and introduce the universal ($\forall$) and existential ($\exists$) quantifiers. Suppose A is a set, then a predicate on A corresponds to a dependent type in the form of

$A \to Set$. For example, we can define the predicates of even numbers and odd numbers inductively as follow:

```
mutual
  data _isEven : ℕ → Set where
    base : zero isEven
    even : ∀ n → n isOdd → (suc n) isEven

  data _isOdd : ℕ → Set where
    odd : ∀ n → n isEven → (suc n) isOdd
```

Universal Quantifier The interpretation of the universal quantifier is similar to that of implication. In order for $\forall x \in A.\ B(x)$ to be true, every proof x of A must be transformed into a proof of the predicate $B[a := x]$. In Type Theory, it is represented by the function $(x : A) \to B\ x$. For example, we can prove by induction that for every natural number, it is either even or odd.

```
lem₁ : ∀ n → n isEven ⊎ n isOdd
lem₁ zero = inj₁ base
lem₁ (suc n) with lem₁ n
... | inj₁ nIsEven = inj₂ (even n nIsEven)
... | inj₂ nIsOdd = inj₁ (odd n nIsOdd)
```

Existential Quantifier The interpretation of the existential quantifier is similar to that of conjunction. In order for $\exists x \in A.\ B(x)$ to be true, a proof x of A and a proof of the predicate $B[a := x]$ must be provided. In Type Theory, it is represented by the generalised product type Σ.

```
data Σ (A : Set) (B : A → Set) : where
  _,_ : (a : A) → B a → Σ A B
```

For simplicity, we will change the syntax of Σ type to $\exists[x \in A]\ B\ x$. As an example, let us prove that there exists an even number.

```
lem₂ : ∃[ n ∈ ℕ ] (n isEven)
lem₂ = zero , base
```

2.3.3 Decidability

A proposition is decidable if and only if there exists an algorithm that can decide whether the proposition is true or false. It is defined in Agda as follow:

```
data Dec (A : Set) : Set where
  yes : A → Dec A
  no  : ¬ A → Dec A
```

For example, we can prove that the predicate of even numbers is decidable. Interested readers can try and complete the following proofs.

```
postulate lem₃ : ∀ n → ¬ (n isEven × n isOdd)

postulate lem₄ : ∀ n → n isEven → ¬ ((suc n) isEven)

postulate lem₅ : ∀ n → ¬ (n isEven) → (suc n) isEven

even-dec : ∀ n → Dec (n isEven)
even-dec zero = yes base
even-dec (suc n) with even-dec n
... | yes nIsEven = no (lem₄ n nIsEven)
... | no ¬nIsEven = yes (lem₅ n ¬nIsEven)
```

2.3.4 Propositional Equality

One of the important features of Type Theory is to encode the equality relation of propositions as types. The equality relation is interpreted as follow:

```
data _≡_ {A : Set} (x : A) : A → Set where
  refl : x ≡ x
```

This states that for any x in A, $refl$ is an element of the type $x \equiv x$. More generally, $refl$ is a proof of $x \equiv x'$ provided that x and x' are the same after normalisation. For example, we can prove that $suc\ (suc\ zero) \equiv 1 + 1$ as follow:

```
lem₃ : suc (suc zero) ≡ (1 + 1)
lem₃ = refl
```

We can put $refl$ in the proof only because both $suc\ (suc\ zero)$ and $1 + 1$ have the same form after normalisation. Now, let us define the elimination rule of equality. The rule should allow us to substitute equivalence objects into any proposition.

```
subst : {A : Set}{x y : A} → (P : A → Set) → x ≡ y → P x → P y
subst P refl p = p
```

We can also prove the congruency of equality.

```
cong : {A B : Set}{x y : A} → (f : A → B) → x ≡ y → f x ≡ f y
cong f refl = refl
```

2.4 Program Specifications as Types

As we have mentioned before, dependent types also allow us to encode program specifications within the same platform. In order to demonstrate the idea, we will give an example on the insertion function of sorted lists. Let us begin by defining a predicate of sorted list (in ascending order). For simplicity, we will only consider the list of natural numbers.

```
All-lt : ℕ → List ℕ → Set
All-lt n [] = ⊤
All-lt n (x :: xs) = n ≤ x × All-lt n xs

Sorted-ASC : List ℕ → Set
Sorted-ASC [] = ⊤
Sorted-ASC (x :: xs) = All-lt x xs × Sorted-ASC xs
```

Note that *All-lt* defines the condition where a given number is smaller than all the numbers inside a given list. Now, let us define an insertion function that takes a natural number and a list as the arguments and returns a list of natural numbers. The insertion function is designed in a way that if the input list is already sorted, then the output list will also be sorted.

```
insert : ℕ → List ℕ → List ℕ
insert n [] = n :: []
insert n (x :: xs) with n ≤? x
... | yes _ = n :: (x :: xs)
... | no  _ = x :: insert n xs
```

Note that $\leq?$ has the type $\forall\ n\ m \rightarrow Dec\ (n \leq m)$. It is a proof of the decidability of $\leq$ and it can also be used to determine whether a given number n is less than or equal to another number m. Now, let us encode the specification of the insertion function as follow:

```
insert-sorted : ∀ {n} {as}
                  → Sorted-ASC as
                  → Sorted-ASC (insert n as)
```

In the type signature, (*Sorted-ASC as*) corresponds to the pre-condition and (*Sorted-ASC* (*insert n as*)) corresponds to the post-condition. Once we have completed the function, we will have also proved the specification to be true. Readers are recommended to finish the proof.

Chapter 3

Formalisation in Type Theory

Let us recall the two components of our formalisation: the translation of regular expressions to a minimal DFA and the correctness proofs of the translation. The translation is divided into several steps. Firstly, a regular expression is converted into an ϵ-NFA using Thompson's construction [20]. Secondly, all the ϵ-transitions are removed by computing the ϵ-closures. Thirdly, a DFA is built by using powerset construction. After that, all the unreachable states are removed. Finally, a MDFA is obtained by using quotient construction. The translation is correct if and only if 1) the accepting languages of the regular expression and its translated output are equal, i.e. $L(regex) = L(translated\ \epsilon\text{-NFA}) = L(translated\ \text{DFA}) = L(translated\ \text{MDFA})$ and 2) the translated MDFA is minimal.

In this chapter, we will walk through the formalisation of each of the above steps together with their correctness proofs. Note that all the definitions, theorems, lemmas and proofs written in this section are adapted to the environment of Agda. Now, let us begin with the representation of subsets.

3.1 Subsets, Decidable Subsets and Vector Representation

The types of subsets and decidable subsets are defined in **Subset.agda** and **Subset/Decidable-Subset.agda** respectively along with their operations such as membership ($\in$), subset ($\subseteq$), superset ($\supseteq$) and equality ($=$). To separate the operations of subsets and decidable subsets, all the operations of decidable subset are denoted by the superscript (^{d}), e.g. $\in^d$ is the membership decider for decidable subsets. Let us begin with the definition of general subsets.

Definition 1. Suppose A is a set, then its subsets are represented as unary functions on A in Type Theory, i.e. $Subset\ A = A \to Set$.

In our definition, a subset is a function from A to Set. When declaring a subset, we can write $sub = \lambda\ (x : A) \to P\ x$. $P\ x$ defines the conditions for x to be included in sub. This construction is very similar to set comprehension. For example, the above subset resembles the set $\{x \mid x \in A,\ P(x)\}$. Furthermore, sub is also a predicate on A as its type is in the form of $A \to Set$ and its decidability will remain unknown until it is either proved or disproved.

Definition 2. Another representation of subsets is $DecSubset\ A = A \to Bool$. Unlike $Subset$, its decidability is ensured by its definition.

The two representations have different roles in other parts of the formalisation. *Language* is defined using $Subset$ as not every language is decidable. For other parts in the project such as the subsets of states in an automaton, $DecSubset$ is used because the decidability is assumed.

3.1.1 Vector Representation

Recall that Firsov and Uustalu [8] represent the set of states and its subsets as vectors. However, this makes the definition of NFA becomes unnatural. Therefore, at the beginning, we intended to avoid the vector representation. However, it is impossible because we have to iterate the subset when computing ϵ-closures. The problems will be discussed in Chapter 6 in detail. The vector representation is defined in **Subset/VectorRep.agda** along with its operations and proofs.

In order to use the vector representation, several objects must be passed to the module **VecRep**. They are $(A : Set)$ – a finite set; $(dec : DecEq\ A)$ – the decidable equality of A; $(n : \mathbb{N})$ – number of elements in A minus 1; $(It : Vec\ A\ (suc\ n))$ – a vector containing elements of A with length $n+1$; $(\forall a \in It)$ – a proof that any state in A is also in the vector It; and $(unique : Unique)$ – a proof that there is no repeating elements in It. The vector representation allows us to iterate a certain set and its subsets, and to extract information regarding the set and a proposition.

Definition 3. We define *any* as a predicate of vector such that it is true if and only if there exists an element in the vector that satisfies a given proposition P.

It is defined in Agda as follow:

```
any : {A : Set}{n : ℕ}(P : A → Set) → Vec A n → Set
any P []       = ⊥
any P (a :: as) = P a ⊎ any P as
```

Lemma 1. *For a set A and any proposition P, there exists an element in It that satisfies P if and only if there exists an element in A that satisfies P.*

Proof. The proof is quite obvious. Since It contains all the elements of A, the statement must be true. However, in Type Theory, we have to prove it by induction on the vector. □

Definition 4. We define *all* as a predicate of vector such that it is true if and only if all the elements in the vector satisfy a given proposition P.

It is defined in Agda as follow:

```
all : {A : Set}{n : ℕ}(P : A → Set) → Vec A n → Set
all P []       = ⊤
all P (a :: as) = P a × all P as
```

Lemma 2. *For a set A and any proposition P, all the elements in It that satisfy P if and only if all the elements in A satisfy P.*

Proof. Again, the proof is quite obvious. Since It contains all the elements of A, the statement must be true. However, in Type Theory, we have to prove it by induction on the vector. □

Apart from the above lemmas, the represenations has other usages. For example, with the *unique* proof, we can argue that the size of a decidable subset must be less than or equal to the size of the original set when proving the computation of ϵ-closures. Furthermore, the representation is also used to prove the decidable equality of decidable subsets. The proofs are defined under the module **Decidable-$\approx$** in **Subset/DecidableSubset.agda**.

3.2 Languages

The type of languages is defined in **Language.agda** along with its operations and lemmas such as union ($\cup$), concatenation ($\bullet$) and closure ($\star$).

We represent the set of alphabets Σ as a data type in Type Theory, i.e. $\Sigma : Set$. Note that the equality relation of Σ needs to be decidable. In Agda, they are passed to every module as parameters, e.g. $module\ Language(\Sigma : Set)\ (dec : DecEq\ \Sigma)\ where$.

Definition 5. We first define Σ^* as the set of all strings over Σ. In our approach, it is expressed as a list of alphabets, i.e. $\Sigma^* = List\ \Sigma$.

For example, $(A :: g :: d :: a :: [\ ])$ is equivalent to the string 'Agda' and the empty list $[\ ]$ represents the empty string (ϵ). In this way, the first alphabet can be extracted from the input string by pattern matching in order to run a transition from a particular state to another state in an automaton.

Definition 6. A language is defined as a subset of Σ^*, i.e. $Language = Subset\ \Sigma^*$. Note that $Subset$ instead of $DecSubset$ is used because not every language is decidable.

3.2.1 Operations on Languages

Definition 7. Suppose L_1 and L_2 are languages, then the union of the two languages, $L_1 \cup L_2$, is given by the set $\{w \mid w \in L_1 \vee w \in L_2\}$. In Type Theory, we have $L_1 \cup L_2 = \lambda\, w \to w \in L_1 \uplus w \in L_2$.

Definition 8. Suppose L_1 and L_2 are languages, then the concatenation of the two languages, $L_1 \bullet L_2$, is given by the set $\{w \mid \exists u \in L_1.\ \exists v \in L_2.\ w = uv\}$. In Type Theory, we have $L_1 \bullet L_2 = \lambda\, w \to \exists[\ u \in \Sigma^*\ ]\ \exists[\ v \in \Sigma^*\ ](u \in L_1 \times v \in L_2 \times w \equiv u \mathbin{++} v)$.

Definition 9. Suppose L is a language, then we define L^n as the concatenation of L with itself over n times. In Type Theory, it is defined as a recursive function where $L^0 = \epsilon$ and $L^{k+1}) = L \bullet (L^n)$.

In Agda, it is defined as follow:

```
_∧_ : Language → ℕ → Language
L ∧ zero = [ [] ]
L ∧ (suc n) = L • L ∧ n
```

Definition 10. Suppose L is a language, then the closure of L, $L*$ is given by the set $\bigcup_{n \in N} L^n$. In Type Theory, we have $L \star = \lambda\, w \to \exists[\ n \in \mathbb{N}\ ]\ (w \in L \wedge n)$.

3.3 Regular Expressions and Regular Languages

The types of regular expressions and regular languages are defined in **RegularExpression.agda**.

Definition 11. Regular expressions over Σ are defined inductively as follow:

1. $\emptyset$ is a regular expression denoting the regular language $\emptyset$;
2. ϵ is a regular expression denoting the regular language $\{\epsilon\}$;
3. $\forall a \in \Sigma.\ a$ is a regular expression denoting the regular language $\{a\}$;
4. if e_1 and e_2 are regular expressions denoting the regular languages L_1 and L_2 respectively, then
   (a) $e_1 \mid e_2$ is a regular expression denoting the regular language $L_1 \cup L_2$;
   (b) $e_1 \cdot e_2$ is a regular expression denoting the regular language $L_1 \bullet L_2$;
   (c) ${e_1}^*$ is a regular expression denoting the regular language $L_1 \star$.

The interpretation of regular expression in Agda is as follow:

```
data RegExp : Set where
  Ø   : RegExp
  ϵ   : RegExp
  σ   : Σ → RegExp
```

$$
\begin{aligned}
&\_|\_ &&: \text{RegExp} \rightarrow \text{RegExp} \rightarrow \text{RegExp}\\
&\_\cdot\_ &&: \text{RegExp} \rightarrow \text{RegExp} \rightarrow \text{RegExp}\\
&\_^* &&: \text{RegExp} \rightarrow \text{RegExp}
\end{aligned}
$$

The accepting language of regular expressions is defined as a function from *RegExp* to *Language*.

$$
\begin{aligned}
&\mathrm{L}^R : \text{RegExp} \rightarrow \text{Language}\\
&\mathrm{L}^R\ \emptyset = \emptyset\\
&\mathrm{L}^R\ \epsilon = [\![\epsilon]\!]\\
&\mathrm{L}^R\ (\sigma\ \mathrm{a}) = [\![\ a\ ]\!]\\
&\mathrm{L}^R\ (\mathrm{e}_1\ |\ \mathrm{e}_2) = \mathrm{L}^R\ \mathrm{e}_1 \cup \mathrm{L}^R\ \mathrm{e}_2\\
&\mathrm{L}^R\ (\mathrm{e}_1 \cdot \mathrm{e}_2) = \mathrm{L}^R\ \mathrm{e}_1 \bullet \mathrm{L}^R\ \mathrm{e}_2\\
&\mathrm{L}^R\ (\mathrm{e}^*) = (\mathrm{L}^R\ \mathrm{e})\ \star
\end{aligned}
$$

3.4 ϵ-Non-deterministic Finite Automata

Recall that the set of all strings over Σ is defined as the type *List* Σ^*. However, this definition gives us no way to extract an ϵ alphabet from the input string. Therefore, we need to introduce another representation specific to this purpose. The representation in defined under the module Σ**-with-**ϵ in **Language.agda** along with its related operations and lemmas.

Definition 12. We define Σ^e as the union of Σ and $\{\epsilon\}$, i.e. $\Sigma^e = \Sigma \cup \{\epsilon\}$.

The equivalent data type is as follow:

$$
\begin{aligned}
&\text{data}\ \Sigma^e : \text{Set where}\\
&\quad \alpha : \Sigma \rightarrow \Sigma^e\\
&\quad \mathrm{E} : \Sigma^e
\end{aligned}
$$

All the alphabets in Σ are included in Σ^e by using the α constructor while the ϵ alphabet corresponds to the constructor E in the data type.

Definition 13. Now we define Σ^{e*}, the set of all strings over Σ^e in a way similar to Σ^*, i.e. $\Sigma^{e*} = \mathit{List}\ \Sigma^e$.

For example, the string 'Agda' can be represented by $(\alpha\ A :: \alpha\ g :: E :: \alpha\ d :: E :: \alpha\ a :: [\,])$ or $(E :: \alpha\ A :: E :: E :: \alpha\ g :: \alpha\ d :: E :: \alpha\ a :: [\,])$. We call these two lists as the ϵ-strings of the string 'Agda'.

Definition 14. Now we define $to\Sigma^*(w^e)$ as a function that takes an ϵ-string of w, w^e and returns w.

It is define in Agda as follow:

```
to Σ* : Σ^e* → Σ*
to Σ* [] = []
to Σ* (α a :: w) = a :: to Σ* w
to Σ* (E :: w) = to Σ* w
```

Now, let us define ϵ-NFA using Σ^{e*}. The type of ϵ-NFA is defined in **eNFA.agda** along with its operations and properties.

Definition 15. An ϵ-NFA is a 5-tuple $M = (Q,\ \Sigma^e,\ \delta,\ q_0,\ F)$, where

1. Q is a finite set of states;
2. Σ^e is the union of Σ and $\{\epsilon\}$;
3. δ is a mapping from $Q \times \Sigma^e$ to $\mathcal{P}(Q)$ that defines the behaviour of the automata;
4. q_0 in Q is the initial state;
5. $F \subseteq Q$ is the set of accepting states.

It is formalised as a record in Agda as shown below:

```
record ϵ-NFA : Set₁ where
  field
    Q       : Set
    δ       : Q → Σ^e → DecSubset Q
    q₀      : Q
    F       : DecSubset Q
    ∀qEq    : ∀ q → q ∈^d δ q E
    Q?      : DecEq Q
    |Q|-1   : ℕ
    It      : Vec Q (suc |Q|-1)
    ∀q∈It   : (q : Q) → (q ∈^V It)
    unique  : Unique It
```

The set of alphabets Σ is passed into the module as a parameter and Σ^e is constructed using Σ. Together with Q, δ, q_0 and F, these five fields correspond to the 5-tuple ϵ-NFA. The other extra fields are used when computing ϵ-closures. They are $\forall qEq$ – a proof that any state in Q can reach itself by an ϵ-transition; $Q?$ – the decidable equality of Q; $|Q| - 1$ – the number of states minus 1; It – a vector containing all the states in Q; $\forall q \in It$ – a proof that every state in Q is also in the vector It; and *unique* – a proof that there is no repeating elements in It.

Now, before we can define the accepting language of a given ϵ-NFA, we need to define several operations of ϵ-NFA.

Definition 16. A configuration is composed of a state and an alphabet from Σ^e, i.e. $C = Q \times \Sigma^e$.

Definition 17. A move in an ϵ-NFA is represented by a binary function ($\vdash$) on two configurations. We say that for all $w \in \Sigma^{e*}$ and $a \in \Sigma^e$, $(q, aw) \vdash (q', w)$ if and only if $q' \in \delta(q, a)$.

The binary function is defined in Agda as follow:

```
_⊢_ : (Q × Σᵉ × Σᵉ*) → (Q × Σᵉ*) → Set
(q , a , w) ⊢ (q' , w') = w ≡ w' × q' ∈ᵈ δ q a
```

Definition 18. Suppose C and C' are two configurations. We say that $C \vdash^0 C'$ if and only if $C = C'$; and $C_0 \vdash^k C_k$ for any $k \geq 1$ if and only if there exists a chain of configurations $C_1, C_2, ..., C_{k-1}$ such that $C_i \vdash C_{i+1}$ for all $0 \leq i < k$.

It is defined as a recursive function in Agda as follow:

```
_⊢ᵏ_−_ : (Q × Σᵉ*) → ℕ → (Q × Σᵉ*) → Set
(q , wᵉ) ⊢ᵏ zero − (q' , wᵉ')
  = q ≡ q' × wᵉ ≡ wᵉ'
(q , wᵉ) ⊢ᵏ suc n − (q' , wᵉ')
  = ∃[ p ∈ Q ] ∃[ aᵉ ∈ Σᵉ ] ∃[ uᵉ ∈ Σᵉ* ]
     (wᵉ ≡ aᵉ :: uᵉ × (q , aᵉ , uᵉ) ⊢ (p , uᵉ)
      × (p , uᵉ) ⊢ᵏ n − (q' , wᵉ'))
```

Definition 19. We say that $C \vdash^* C'$ if and only if there exists a number of chains n such that $C \vdash^n C'$.

Its corresponding type is defined as follow:

```
_⊢*_ : (Q × Σᵉ*) → (Q × Σᵉ*) → Set
(q , wᵉ) ⊢* (q' , wᵉ') = ∃[ n ∈ ℕ ] (q , wᵉ) ⊢ᵏ n − (q' , wᵉ')
```

Definition 20. For any string w, it is accepted by an ϵ-NFA if and only if there exists an ϵ-string of w that can take q_0 to an accepting state q. Therefore, the accepting language of an ϵ-NFA is given by the set $\{w \mid \exists w^e \in \Sigma^{e*}.\ w = to\Sigma^*(w^e) \wedge \exists q \in F.\ (q_0, w^e) \vdash^* (q, \epsilon)\}$.

The corresponding formalisation in Agda is as follow:

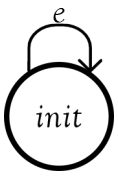
```
Lᵉᴺ : ϵ-NFA → Language
Lᵉᴺ nfa = λ w →
          ∃[ wᵉ ∈ Σᵉ* ] (w ≡ toΣ* wᵉ ×
          (∃[ q ∈ Q ] (q ∈ᵈ F × (q₀ , wᵉ) ⊢* (q , [])))))
```

3.5 Thompson's Construction

Now, let us look at the translation of regular expressions to ϵ-NFA. The translation is defined as the function **regexToeNFA** in **Translation/RegExp-eNFA.agda** while the constructions of states are defined in **State.agda**.

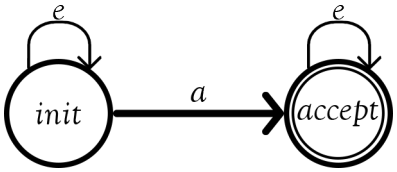
Definition 21. The translation of regular expressions to ϵ-NFA is defined inductively as follow:

1. for $\emptyset$, we have $M = (\{init\},\ \Sigma^e,\ \delta,\ init,\ \emptyset)$ and graphically,

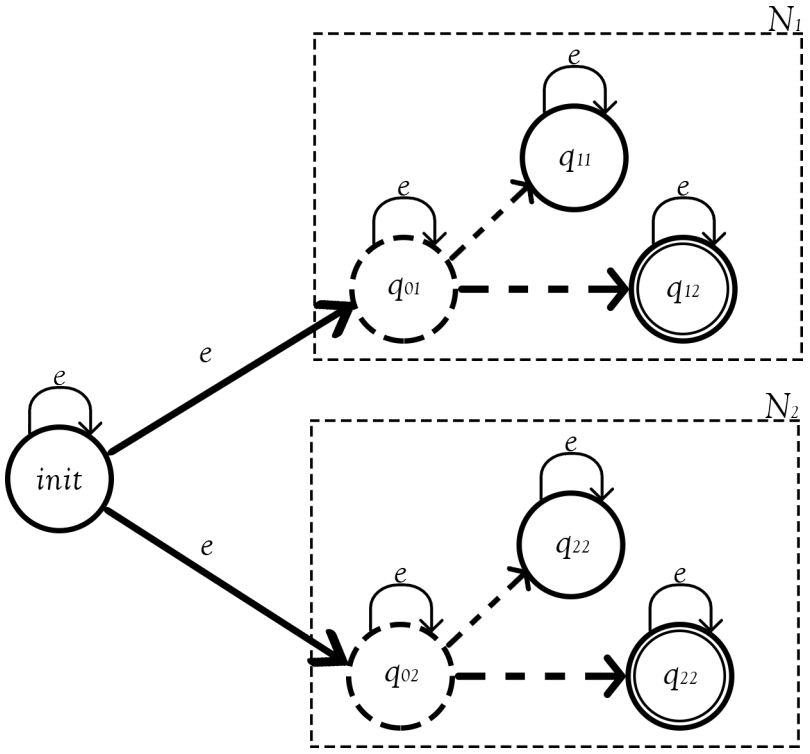


2. for ϵ, we have $M = (\{init\},\ \Sigma^e,\ \delta,\ init,\ \{init\})$ and graphically,

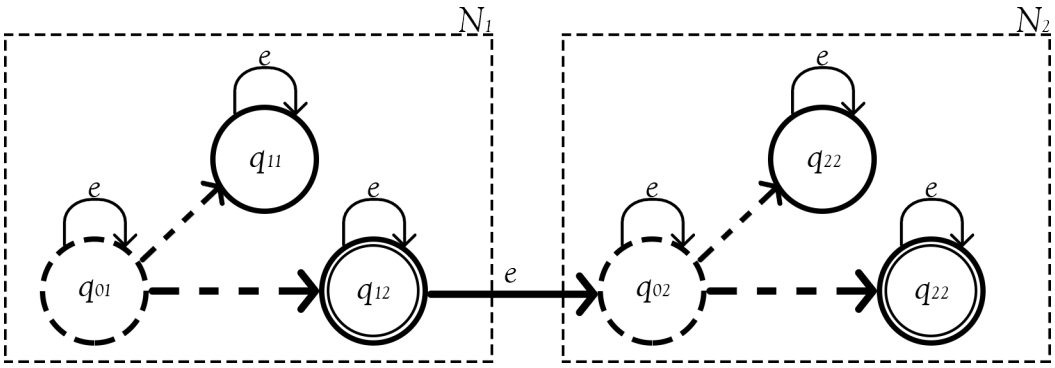
3. for a, we have $M = (\{init, accept\},\ \Sigma^e,\ \delta,\ init,\ \{accept\})$ and graphically,



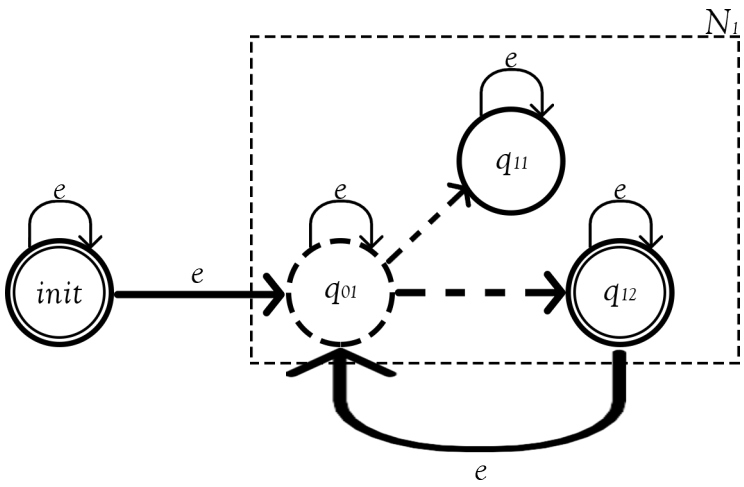
4. suppose $N_1 = (Q_1,\ \delta_1,\ q_{01},\ F_1)$ and $N_2 = (Q_2,\ \delta_2,\ q_{02},\ F_2)$ are ϵ-NFAs translated from the regular expressions e_1 and e_2 respectively, then
   (a) for $(e_1 \mid e_2)$, we have $M = (\{init\} \cup Q_1 \cup Q_2,\ \Sigma^e,\ \delta,\ init,\ F_1 \cup F_2)$ and graphically,



(b) for $e_1 \cdot e_2$, we have $M = (Q_1 \cup \{mid\} \cup Q_2,\ \Sigma^e,\ \delta,\ init,\ F_2)$ and graphically,



(c) for ${e_1}^*$, we have $M = (\{init\} \cup Q_1,\ \Sigma^e,\ \delta,\ init,\ \{init\} \cup F_1)$ and graphically,



Apart from the five fields, the other fields in the record *ϵ-NFA* are also constructed by the function. Now, let us prove the correctness of the above translation by proving that their accepting languages are equal. The correctness proof is defined as the function $\boldsymbol{L^R \approx L^{eN}}$ in **Correctness.agda** while the detail proofs are defined in **Correctness/RegExp-eNFA.agda**.

Theorem 1. *For any given regular expression, e, its accepting language is equal to the language accepted by the ϵ-NFA translated from e using Thompson's Construction, i.e.* $L(e) = L(\textit{translated } \epsilon\textit{-NFA})$.

Proof. We can prove the theorem by induction on regular expressions.
Base cases. By Definition (21), it is obvious that the statement holds for $\emptyset$, ϵ and a.
Induction hypothesis 1. For any two regular expressions e_1 and e_2, let $N_1 = (Q_1,\ \delta_1,\ q_{01},\ F_1)$ and $N_2 = (Q_2,\ \delta_2,\ q_{02},\ F_2)$ be their translated ϵ-NFA respectively, we assume that $L(e_1) = L(N_1)$ and $L(e_2) = L(N_2)$.
Inductive steps. There are three cases: 1) $e_1 \mid e_2$, 2) $e_1 \cdot e_2$ and 3) ${e_1}^*$.
1) *Case* $(e_1 \mid e_2)$: Let $M = (Q,\ \delta,\ q_0,\ F) = (\{init\} \cup Q_1 \cup Q_2,\ \delta,\ init,\ F_1 \cup F_2)$ be its translated ϵ-NFA. Then for any string w,

1.1) if $(e_1 \mid e_2)$ accepts w, then by Definition (11) and Definition (7), either i) e_1 accepts w or ii) e_2 accepts w. Assuming case i), then by induction hypothesis, N_1 also accepts w. Therefore, there must exist an ϵ-string of w, w^e, that can take q_{01} to an accepting state q in N_1. Now, consider another ϵ-string of w, ϵw^e, it can take *init* to q in M because ϵ can take *init* to q_{01}. Recall that q is an accepting state in N_1; therefore, q is also an accepting state in M and thus, by Definition (20), M accepts w. The same argument also applies to the case when e_2 accepts w; therefore, $L(e_1 \mid e_2) \subseteq L(M)$ is true;

1.2) if M accepts w, then by Definition (20), there must exist an ϵ-string of w, w^e, that can take *init* to an accepting state q in M. q must be different from *init* because q is an accepting

state but *init* is not. Now, by Definition (21), there are only two possible ways for *init* to reach q in M, i) via q_{01} or ii) via q_{02}. Assuming case i), then w^e must be in the form of aw'^e because ϵ is the only alphabet that can take *init* to q_{01} and thus w'^e can take q_{01} to q. Furthermore, q is an accepting state in M; therefore, q is also an accepting state in N_1 and thus N_1 accepts w. By induction hypothesis, e_1 also accepts w; therefore, we have $w \in L(e_1 \mid e_2)$. The same argument also applies to case ii) and thus $L(e_1 \mid e_2) \supseteq L(M)$ is true;

1.3) combining 1.1 and 1.2, we have $L(e_1 \mid e_2) = L(M)$.

2) *Case* $(e_1 \cdot e_2)$: Let $M = (Q,\ \delta,\ q_0,\ F) = (Q_1 \cup \{mid\} \cup Q_2,\ \delta,\ q_{01},\ F_2)$ be its translated ϵ-NFA. Then for any string w,

2.1) if $(e_1 \cdot e_2)$ accepts w, then by Definition (11) and Definition (8), there must exist a string $u \in L(e_1)$ and a string $v \in L(e_2)$ such that $w = uv$. By induction hypothesis, N_1 accepts u and N_2 accepts v. Therefore, there must exist i) an ϵ-string of u, u^e, that can take q_{01} to an accepting state q_1 in N_1 and ii) an ϵ-string of v, v^e, that can take q_{02} to an accepting state q_2 in N_2. Now, let us consider another ϵ-string of w, $u^e\epsilon\epsilon v^e$, it can take q_{01} to q_2 in M because u^e takes q_{01} to q_1, ϵ takes q_1 to *mid*, another ϵ takes *mid* to q_{02} and v^e takes q_{02} to q_2. Furthermore, q_2 is also an accepting state in M because q_2 is an accepting state in N_2. Therefore, M accepts w and thus $L(e_1 \cdot e_2) \subseteq L(M)$ is true;

2.2) if M accepts w, then by Definition (20), there must exist an ϵ-string of w, w^e, which can take q_{01} to an accepting state q in M. Since q is an accepting state in M; therefore, q must be in Q_2. The only possible way for q_{01} to reach q is by going through the state *mid*. Therefore, there must exist i) an ϵ-string, u^e, that can take q_{01} to an accepting state q_1 in N_1 and ii) an ϵ-string v^e that can take q_{02} to q_2 in N_1 and $w^e = u^e\epsilon\epsilon v^e$. Let u and v be the normal strings of u^e and v^e respectively, then we have $u \in L(N_1)$, $v \in L(N_2)$ and $w = uv$. Now, by induction hypothesis, e_1 accepts u and e_2 accepts v; and thus $e_1 \cdot e_2$ accepts w. Therefore $L(e_1 \cdot e_2) \supseteq L(M)$ is true;

2.3) combining 2.1 and 2.2, we have $L(e_1 \cdot e_2) = L(M)$.

3) *Case* ${e_1}^*$: Let $M = (Q,\ \delta,\ q_0,\ F) = (Q_1 \cup \{mid\} \cup Q_2,\ \delta,\ q_{01},\ F_2)$ be its translated ϵ-NFA. Then for any string w,

3.1) if $({e_1}^*)$ accepts w, then by Definition (11) and Definition (10), there must exist a number n such that $w \in L(e_1)^n$. Now, let us do induction on n.

Base case. When $n = 0$, then the language L^0 can only accept the empty string ϵ; and thus $w = \epsilon$. From Definition (21), it is obvious that M accepts ϵ.

Induction hypothesis 2. Suppose there exists a number k such that $w \in L(e_1)^k$, then w is also accepted by M.

Induction steps. When $n = k + 1$, by Definition (8) and Definition (9), there must exist a string $u \in L(e_1)$ and a string $v \in L(e_1)^k$ such that $w = uv$. By induction hypothesis (1), we have N_1 accepts u. Therefore there must exist an ϵ-string u, u^e, that can take q_{01} to an accepting state q in N_1. Since q is an accepting state; an ϵ alphabet can take q back to q_{01}. Furthermore,

by induction hypothesis (2), M also accepts v which implies that there exists an ϵ-string of v, v^e, that can take *init* to an accepting state p. Since the only alphabet that can take *init* to q_{01} is ϵ; therefore, v^e must be in the form of $\epsilon v'^e$. Now, we have proved that there exists an ϵ string of w, $\epsilon u^e \epsilon v'^e$, that can take *init* to an accepting state p in M; and thus M accepts w. Therefore $L({e_1}^*) \subseteq L(M)$ is true;

3.2) if M accepts w, then by Definition (20), there must exist an ϵ-string w, w^e, that can take *init* to an accepting state q with n transitions in M. If $init = q$, then w must be an empty string. By Definition (21), it is obvious that the empty string is accepted by ${e_1}^*$. If $init \neq q$, then there are only two possible ways for *init* to reach q: 1) from *init* to q without going back to q_{01} from an accepting state with an ϵ, we say that this path has no loops and 2) from *init* to q with at least one loop.

Case 1: Since $q \neq init$, then w^e must be in the form of $\epsilon w'^e$. Recall that the path has no loops, it is obvious that N_1 accepts w. Therefore by induction hypothesis (1), e_1 accepts w and thus ${e_1}^*$ also accepts w.

Case 2: Since $q \neq init$, then w^e must be in the form of $\epsilon w'^e$. Recall that the path has loops, we can separate w'^e into two parts: 1) an ϵ-string u^e that takes *init* to an accepting state p without loops and 2) an ϵ-string ϵv^e that takes p to q_{01} to q with loops. Let u and v be the normal string of u^e and ϵv^e respectively, then it is obvious that $w = uv$. By *case 1*, e_1 accepts u. Now, consider the path from q_{01} to q. The path must have less than n transitions; therefore, we can prove by induction the there must exist a number k such that $L(e_1)^k$ accepts v. Combining the above proofs, we have $w \in L({e_1}^*)$.

Combining *Case 1* and *Case 2*, we have $w \in L({e_1}^*) \supseteq L(M)$;

3.3) combining 3.1 and 3.2, we have $L({e_1}^*) = L(M)$.

Therefore, by induction, $L(e) = L$(translated ϵ-NFA) is true for all any regular expression e. □

Below is a code snippet of our formalisation of $L^R \subseteq L^{eN}$.

```
{- ∀e∈RegExp. L(e) ⊆ L(regexTo𝜀-NFA e) -}
L^R⊆L^eN : ∀ e → L^R e ⊆ L^eN (regexTo𝜀-NFA e)
-- null
L^R⊆L^eN ∅ _ ()
-- ε
L^R⊆L^eN ε = L^R⊆L^N.lem₁
  where open import Correctness.RegExp-eNFA.Epsilon-lemmas Σ dec
-- singleton
L^R⊆L^eN (σ a) = L^R⊆L^N.lem₁
  where open import Correctness.RegExp-eNFA.Singleton-lemmas Σ dec a
-- union
L^R⊆L^eN (e₁ | e₂) w (inj₁ w∈L^R) = L^R⊆L^N.lem₁ (L^R⊆L^eN e₁ w w∈L^R)
  where open import Correctness.RegExp-eNFA.Union-lemmas Σ dec e₁ e₂
L^R⊆L^eN (e₁ | e₂) w (inj₂ w∈L^R) = L^R⊆L^N.lem₄ (L^R⊆L^eN e₂ w w∈L^R)
  where open import Correctness.RegExp-eNFA.Union-lemmas Σ dec e₁ e₂
-- concatenation
L^R⊆L^eN (e₁ · e₂) w (u , v , u∈L^Re₁ , v∈L^Re₂ , w≡uv)
  = L^R⊆L^N.lem₁ w≡uv (L^R⊆L^eN e₁ u u∈L^Re₁) (L^R⊆L^eN e₂ v v∈L^Re₂)
  where open import Correctness.RegExp-eNFA.Concatenation-lemmas Σ dec e₁ e₂
-- kleen star
L^R⊆L^eN (e * ) .[] (zero , refl)
  = [] , refl , init , refl , 0 , refl , refl
L^R⊆L^eN (e * )  w  (suc n , u , v , u∈L^Re , v∈L^Re^n+1 , w≡uv)
  = lem₁ w u v n w≡uv (L^R⊆L^eN e u u∈L^Re) v∈L^Re^n+1
```

Pattern matching the regular expression corresponds to the case analysis in the proof. As shown above, there are six cases: $\emptyset$, *epsilon*, σ a, union, concatenation and kleen star. The first three corresponds to the three base cases. Notice that the recursive calls in last three cases is equivalent to the induction hypothesis of the proof. These cases are equivalent to the induction steps.

3.6 Non-deterministic Finite Automata

Although the definition of NFA is very similar to that of ϵ-NFA, we will still define NFA separately. The type of NFA is defined in **NFA.agda** along with its operations and properties.

Definition 22. A NFA is a 5-tuple $M = (Q,\ \Sigma,\ \delta,\ q_0,\ F)$, where

1. Q is a finite set of states;
2. Σ is the set of alphabets;
3. δ is a mapping from $Q \times \Sigma$ to $\mathcal{P}(Q)$ that defines the behaviour of the automata;
4. q_0 in Q is the initial state;
5. $F \subseteq Q$ is the set of accepting states.

It is formalised as a record in Agda as shown below:

```
record NFA : Set₁ where
  field
    Q       : Set
    δ       : Q → Σ → DecSubset Q
    q₀      : Q
    F       : DecSubset Q
    Q?      : DecEq Q
    |Q|−1   : ℕ
    It      : Vec Q (suc |Q|−1)
    ∀q∈It   : (q : Q) → (q ∈ᵛ It)
    unique  : Unique It
```

The set of alphabets Σ is passed into the module as a parameter. Together with Q, δ, q_0 and F, these five fields correspond to the 5-tuple NFA. The other extra fields are used in powerset construction. They are $Q?$ – the decidable equality of Q; $|Q|-1$ – the number of states minus 1; It – a vector containing every state in Q; $\forall q \in It$ – a proof that every state in Q is also in the vector It; and *unique* – a proof that there is no repeating elements in It.

Now, before we can define the accepting language of a given NFA, we need to define several operations of NFA.

Definition 23. A configuration is composed of a state and an alphabet from Σ, i.e. $C = Q \times \Sigma$.

Definition 24. A move in an NFA is represented by a binary function $(\vdash)$ on two configurations. We say that for all $w \in \Sigma^*$ and $a \in \Sigma$, $(q, aw) \vdash (q', w)$ if and only if $q' \in \delta(q, a)$.

The binary function is defined in Agda as follow:

```
_⊢_ : (Q × Σ × Σ*) → (Q × Σ*) → Set
(q , a , w) ⊢ (q' , w') = w ≡ w' × q' ∈ᵈ δ q a
```

Definition 25. Suppose C and C' are configurations. We say that $C \vdash^0 C'$ if and only if $C = C'$; and $C_0 \vdash^k C_k$ for any $k \geq 1$ if and only if there exists a chain of configurations $C_1, C_2, ..., C_{k-1}$ such that $C_i \vdash C_{i+1}$ for all $0 \leq i < k$.

It is defined as a recursive function in Agda as follow:

```
_⊢ᵏ_−_ : (Q × Σ*) → ℕ → (Q × Σ*) → Set
(q , w) ⊢ᵏ zero − (q' , w')
  = q ≡ q' × w ≡ w'
(q , w) ⊢ᵏ suc n − (q' , w')
  = ∃[ p ∈ Q ] ∃[ a ∈ Σ ] ∃[ u ∈ Σ* ]
```

```
(w ≡ a :: u × (q , a , u) ⊢ (p , u)
 × (p , u) ⊢^k n − (q' , w'))
```

Definition 26. We say that $C \vdash^* C'$ if and only if there exists a number of chains n such that $C \vdash^n C'$.

Its corresponding type is defined as follow:

```
_⊢*_ : (Q × Σ*) → (Q × Σ*) → Set
(q , w) ⊢* (q' , w') = ∃[ n ∈ ℕ ] (q,w) ⊢^k n − (q',w')
```

Definition 27. For any string w, it is accepted by an NFA if and only w can take q_0 to an accepting state q. Therefore, the accepting language of an NFA is given by the set $\{w \mid \exists q \in F.\ (q_0, w) \vdash^* (q, \epsilon)\}$.

The corresponding formalisation in Agda is as follow:

```
L^N : NFA → Language
L^N nfa = λ w →
          ∃[ q ∈ Q ] (q ∈^d F × (q0 , w) ⊢* (q , []))
```

3.7 Removing ϵ-transitions

In order to remove all the ϵ-transitions, we need to know whether a state q can reach another state q' with only ϵ-transitions. Let us begin by defining such relation. Note that the algorithm and proofs regarding the computation of ϵ-closures can be found in the module **Compute-ϵ-closure** in **Translation/eNFA-NFA.agda**.

Definition 28. We say that $q \to_\epsilon^0 q'$ if and only if q is equal to q'; and $q \to_\epsilon^k q'$ for $k \geq 1$ if and only if q can be transited to q' with k ϵ-transitions. We call this an ϵ-path from q to q'.

It is defined as a recursive function in Agda as follow:

```
_→ε^k_−_ : Q → ℕ → Q → Set
q →ε^k zero − q' = q ≡ q'
q →ε^k suc n − q' = ∃[ p ∈ Q ] ( p ∈^d δ q E × p →ε^k n − q' )
```

Definition 29. We say that $q \to_\epsilon^* q'$ if and only if there exists an ϵ-path from q to q' with n transitions, i.e. $\exists n.\ q \to_\epsilon^n q'$.

The corresponding type is as follow:

```
_→*ϵ_ : Q → Q → Set
q →*ϵ q' = ∃[ n ∈ ℕ ] q →kϵ n − q'
```

Now we have to prove that for any two states q and q', $q \rightarrow_\epsilon^* q'$ is decidable. However, it is not possible to prove it directly because the set of natural numbers is infinite. Therefore, we will introduce an algorithm that computes the ϵ-closure of a state. The ϵ-closure of a state q, $\epsilon\text{-}closure(q)$ should contain all the states that are reachable from q with only ϵ-transitions. We will prove that for any two states q and q', $q \rightarrow_\epsilon^* q'$ if and only if $q' \in \epsilon\text{-}closure(q)$. By proving that they are equivalent, we will have proved the decidability of $q \rightarrow_\epsilon^* q'$.

Definition 30. For any given state q, $\epsilon\text{-}closure(q)$ is obtained by:

1. put q into a subset S, i.e. $S = \{q\}$
2. loop for $|Q| - 1$ times:
   (a) for every state p in S, if ϵ can take p to another state r, i.e. $r \in \delta(p, \epsilon)$, then put r into S.
3. the result subset S is the ϵ-closure of q

Now, we can prove that the two representations are equivalent. The theorem is defined as the function $\rightarrow_\epsilon^*$**-lem**$_1$.

Lemma 3. *For any two states q and q', $q \rightarrow_\epsilon^* q'$ if and only if $q' \in \epsilon\text{-}closure(q)$.*

Proof. We have to prove for both directions.
1) If $q \rightarrow_\epsilon^* q'$, then there must exist a number n such that ϵ^n can take q to q'. If $n < |Q|$, then it is obvious that $q' \in \epsilon\text{-}closure(q)$ is true. If $n \geq |Q|$, the ϵ-path from q to q' must have loop(s) inside. By removing the loop(s), the equivalent ϵ-path must have at most $|Q| - 1$ ϵ-transitions. Therefore, $q' \in \epsilon\text{-}closure(q)$ is true.
2) If $q' \in \epsilon\text{-}closure(q)$, it is obvious that $q \rightarrow_\epsilon^{|Q|-1} q'$ must be true and thus $q \rightarrow_\epsilon^* q'$ is true. □

Since we have proved that they are equivalent; therefore the decidability of $q \rightarrow_\epsilon^* q'$ follows. Now, let us define the translation of ϵ-NFA to NFA using $q \rightarrow_\epsilon^* q'$. The translation is defined as the function **remove-ϵ-step** in the bottom of **Translation/eNFA-NFA.agda**.

Definition 31. For a given ϵ-NFA, $(Q,\ \delta,\ q_0,\ F)$, its translated NFA will be $(Q,\ \delta',\ q_0, F')$ where

- $\delta'(q, a) = \delta(q, a) \cup \{q' \mid \exists p.\ q \rightarrow_\epsilon^* p \wedge q' \in \delta(p, a)\}$;
- $F' = F \cup \{q \mid \exists p \in F.\ q \rightarrow_\epsilon^* p\}$.

Now, let us prove the correctness of the above translation by proving their accepting languages are equal. The correctness proofs is defined as the function $\boldsymbol{L^{eN} \approx L^N}$ in **Correctness.agda** while the detail proofs can be found in **Correctness/eNFA-NFA.agda**.

Theorem 2. *For any ϵ-NFA, its accepting language is equal to the language accepted by its translated NFA, i.e. $L(\epsilon\text{-NFA}) = L(\text{translated NFA})$.*

Proof. Let the ϵ-NFA be $\epsilon N = (Q,\ \delta,\ q_0,\ F)$, and its translated NFA be $N = (Q,\ \delta',\ q_0,\ F')$ according to Definition (31). To prove the theorem, we have to prove that $L(\epsilon N) \subseteq L(N)$ and $L(\epsilon N) \supseteq L(N)$. Then for any string w,
1) if ϵN accepts w, then there must exist an ϵ-string of w, w^e, that can take q_0 to an accepting state q with n transitions. There are three possibilities: a) the last transition in the path is not an ϵ-transition, b) the path is divided into three parts, the first part from q_0 to a state p with less than n transitions; the second part from p to a state p_1 with an alphabet a and the third part from p_1 to q with only ϵ-transitions and c) the path from q_0 to q consists of only ϵ-transitions.

Case a: There must exist a state p, an ϵ-string u^e, and an alphabet a such that u^e can take q_0 to p in less than n transitions, $q \in \delta(p, a)$ and $w^e = u^e a$. Since that path from q_0 to p is less than n transitions; therefore, we can prove by induction that the normal string of u^e, u, can take q_0 to p in N. Furthermore, $w = ua$; therefore, w can take q_0 to q in N and thus N accepts w.

Case b: There must exist two states p and p_1, an ϵ-string of w, u^e, and an alphabet a such that u^e can take q_0 to p in less than n transitions, $p_1 \in \delta(p, a)$, $p_1 \to_\epsilon^* q$ and $w = ua$. Since that path from q_0 to p is less than n transitions; therefore, we can prove by induction that the normal string of u^e, u, can take q_0 to p in N. Furthermore, p_1 must be an accepting state in N because it can be transited to an accepting state q with only ϵ-transitions. Therefore, w can take q_0 to an accepting state p_1 in N and thus N accepts w.

Case c: If the path from q_0 to q consists of only ϵ-transitions, then $q_0 \to_\epsilon^* q$ is true; therefore, q_0 is also an accepting state in N. Furthermore, the accepted string consists of ϵ only; therefore, $w = \epsilon$. It is obvious that N accepts ϵ.
2) if N accepts w, then w must be able to take q_0 to an accepting state q. Since q is an accepting state, then q is also an accepting state in ϵN or there exists a state p such that $q \to_\epsilon^* p$ and $p \in F$. For the former case, since w is also an ϵ-string of itself; therefore, it is obvious that ϵN accepts w. For the latter case, suppose the path from q to p has n ϵ-transitions, the consider another ϵ-string of w, $w\epsilon^n$, it can take q_0 to p in ϵN. Therefore ϵN accepts w. □

3.8 Deterministic Finite Automata

The type of DFA is defined in **DFA.agda** along with its operations and properties.

Definition 32. A DFA is a 5-tuple $M = (Q,\ \Sigma,\ \delta,\ q_0,\ F)$, where

1. Q is a finite set of states;
2. Σ is the set of alphabets;
3. δ is a mapping from $Q \times \Sigma$ to Q that defines the behaviour of the automata;

4. q_0 in Q is the initial state;
5. $F \subseteq Q$ is the set of accepting states.

It is formalised as a record in Agda as shown below:

```
record DFA : Set₁ where
  field
    Q         : Set
    δ         : Q → Σ → DecSubset Q
    q₀        : Q
    F         : DecSubset Q
    _≈_       : Q → Q → Set
    ≈-isEquiv : IsEquivalence _≈_
    δ-lem     : ∀ {q} {p} a → q ≈ p → δ q a ≈ δ p a
    F-lem     : ∀ {q} {p}   → q ≈ p → q ∈ᵈ F → p ∈ᵈ F
```

The set of alphabets Σ is passed into the module as a parameter. Together with Q, δ, q_0 and F, these five fields correspond to the 5-tuple DFA. The other extra fields are used in proving its decidability. They are $\_ \approx \_$ – an equivalence relation on states; $\approx$-$isEquiv$ – a proof that the relation $\approx$ is an equivalence relation; δ-lem – a proof that for any alphabet a and any two states q and p, if $q \approx p$ then $\delta(q, a) \approx \delta(p, a)$; and F-lem – a proof that for any two states q and p, if $q \approx p$ and q is an accepting state, then p is also an accepting state.

Now, before we can define the accepting language of a given DFA, we need to define several operations of DFA.

Definition 33. A configuration is composed of a state and an alphabet from Σ, i.e. $C = Q \times \Sigma$.

Definition 34. A move in an DFA is represented by a binary function ($\vdash$) on two configurations. We say that for all $w \in \Sigma^*$ and $a \in \Sigma$, $(q, aw) \vdash (q', w)$ if and only if $q' = \delta(q, a)$.

The binary function is defined in Agda as follow:

```
_⊢_ : (Q × Σ × Σ*) → (Q × Σ*) → Set
(q , a , w) ⊢ (q' , w') = w ≡ w' × q' ≈ δ q a
```

Definition 35. Suppose C and C' are configurations. We say that $C \vdash^0 C'$ if and only if $C = C'$; and $C_0 \vdash^k C_k$ for any $k \geq 1$ if and only if there exists a chain of configurations $C_1, C_2, ..., C_{k-1}$ such that $C_i \vdash C_{i+1}$ for all $0 \leq i < k$.

It is defined as a recursive function in Agda as follow:

```
_⊢^_-_ : (Q × Σ*) → ℕ → (Q × Σ*) → Set
```

```
(q , w) ⊢^k zero − (q' , w')
  = q ≡ q' × w ≡ w'
(q , w) ⊢^k suc n − (q' , w')
  = ∃[ p ∈ Q ] ∃[ a ∈ Σ ] ∃[ u ∈ Σ* ]
     (w ≡ a :: u × (q , a , u) ⊢ (p , u)
      × (p , u) ⊢^k n − (q' , w'))
```

Definition 36. We say that $C \vdash^* C'$ if and only if there exists a number of chains n such that $C \vdash^n C'$.

Its corresponding type is defined as follow:

```
_⊢*_ : (Q × Σ*) → (Q × Σ*) → Set
(q , w) ⊢* (q' , w') = ∃[ n ∈ ℕ ] (q,w) ⊢^k n − (q',w')
```

Definition 37. For any string w, it is accepted by an DFA if and only w can take q_0 to an accepting state q. Therefore, the accepting language of an DFA is given by the set $\{w \mid \exists q \in F.\ (q_0, w) \vdash^* (q, \epsilon)\}$.

The corresponding formalisation in Agda is as follow:

```
L^D : DFA → Language
L^D dfa = λ w →
            ∃[ q ∈ Q ] (q ∈^d F × (q₀ , w) ⊢* (q , [])))
```

3.9 Powerset Construction

The translation of NFA to DFA is defined as the function **powerset-construction** in **Translation/NFA-DFA.agda**.

Definition 38. For any given NFA, $(Q,\ \delta,\ q_0,\ F)$, its translated DFA will be $(\mathcal{P}(Q),\ \delta',\ \{q_0\},\ F')$ where

- $\delta'(qs, a) = \{q' \mid \exists q \in qs.\ q' \in \delta(q, a)\}$;
- $F' = \{qs \mid \exists q \in F.\ q \in qs\}$.

In Agda, the set $\mathcal{P}(Q)$ is defined as the set of decidable subsets, i.e. $Q' = DecSubsetQ$. Therefore, the decidability of the set $\delta'(qs, a)$ and F' must be proved using the vector representation of Q. The corresponding proofs are defined in the module **Powerset-Construction** in the same file.

Now, before proving that their accepting languages are equal, we need to prove the following lemmas which can be found in **Correctness/NFA-DFA.agda**.

Lemma 4. *Let a NFA be* $N = (Q,\ \delta,\ q_0,\ F)$ *and its translated DFA be* $D = (\mathcal{P}(Q),\ \delta',\ q_0, F')$ *according to Definition (38). For any string* w*, if* w *can take* q_0 *to a state* q *with* n *transitions in* N*, then there must exist a subset* qs *such that* $q \in qs$ *and* w *can take* $\{q_0\}$ *to* qs *in* D*, i.e.* $\forall w.\exists q.\exists n.\ (q_0, w) \vdash^n (q, w) \Rightarrow \exists qs.\ q \in qs \wedge (q_0, w) \vdash^* (qs, \epsilon)$.

The proof is defined as the function $\boldsymbol{lem_1}$ under the module $\boldsymbol{L^N \subseteq L^D}$.

Proof. We can prove the lemma by induction on n.
Base case. If $n = 0$, then $q_0 = q$ and $w = \epsilon$. It is obvious that the statement holds.
Induction hypothesis. For any string w, if w can take q_0 to a state q with k transitions in N, then there exists a subset qs such that $q \in qs$ and w can take $\{q_0\}$ to qs in D.
Induction step. If $n = k + 1$, then w can take q_0 to a state q by $k + 1$ transitions. Let $w = w'a$ where a is an alphabet, then w' can take q_0 to a state p by k transitions and $q \in \delta(p, a)$. By induction hypothesis, there must exist a subset ps such that $p \in ps$ and w' can take $\{q_0\}$ to ps in D. Furthermore, since $q \in \delta(p, a)$, then a must be able to take ps to a subset qs where $q \in qs$. Therefore, there exists a subset qs such that $q \in qs$ and w can take $\{q_0\}$ to qs; and thus the statement is true. □

Lemma 5. *Let a NFA be* $N = (Q,\ \delta,\ q_0,\ F)$ *and its translated DFA be* $D = (\mathcal{P}(Q),\ \delta',\ q_0, F')$ *according to Definition (38). For any string* w*, any number* n *and any two states* qs *and* ps *in* $\mathcal{P}(Q)$*, if* $(qs, w) \vdash^n (ps, \epsilon)$ *then* $ps = \{p \mid \exists q \in qs.\ (q, w) \vdash^n (p, \epsilon)\}$.

The proof is defined as the function $\boldsymbol{lem_2}$ under the module $\boldsymbol{L^N \supseteq L^D}$.

Proof. We can prove the lemma by induction on n.
Base case. If $n = 0$, then $qs = ps$ and $w = \epsilon$. Then for any state p in Q,
1) if $p \in ps$, then p is also in qs. It is obvious that $(p, \epsilon) \vdash^0 (p, \epsilon)$ is true; therefore, $p \in \{p \mid \exists q \in qs.\ (q, w) \vdash^0 (p, \epsilon)\}$;
2) if $p \in \{p \mid \exists q \in qs.\ (q, w) \vdash^0 (p, \epsilon)\}$, then p must be in qs and thus $p \in ps$.
Induction hypothesis. For any string w and any two states qs and ps in $\mathcal{P}(Q)$, if $(qs, w) \vdash^k (ps, \epsilon)$ then $ps = \{p \mid \exists q \in qs.\ (q, w) \vdash^k (p, \epsilon)\}$.
Induction step. If $n = k + 1$, then w must be able to take qs to ps with $k + 1$ transitions in D. Therefore there must exist an alphabet a that can take qs to a subset rs, i.e. $rs = \delta'(qs, a)$; and a string u that can take rs to ps with k transitions. By induction hypothesis, we have $ps = \{p \mid \exists r \in rs.\ (r, u) \vdash^k (p, \epsilon)\}$. Then for any state p in Q,
1) if $p \in ps$, then there must exist a state $r \in rs$ such that $(r, u) \vdash^k (p, \epsilon)$. Since $rs = \delta'(qs, a)$; therefore, $r \in \delta'(qs, a)$ and thus there exists a state $q \in qs$ such that $r \in \delta(q, a)$. Therefore, $(q, w) \vdash^{k+1} (p, \epsilon)$ is true and thus $p \in \{p \mid \exists q \in qs.\ (q, w) \vdash^{k+1} (p, \epsilon)\}$.
2) if $p \in \{p \mid \exists q \in qs.\ (q, w) \vdash^{k+1} (p, \epsilon)\}$, then there exists a state $q \in qs$ such that $(q, w) \vdash^{k+1} (p, \epsilon)$. Also, there must exist a state $r \in Q$ such that $r \in \delta(q, a)$ and the string u can take r to p in N.

Since, $q \in qs$ and $r \in \delta(q, a)$; therefore, $r \in \delta'(qs, a) = rs$ and thus $p \in \{p \mid \exists r \in rs.\ (r, u) \vdash^k (p, \epsilon)\} = ps$. □

Now, by using the above lemmas, we can prove the correctness of the translation by proving that their accepting languages are equal. The correctness proof is defined as the function $\boldsymbol{L^N \approx L^D}$ in **Correctness.agda** while the detail proofs are defined in **Correctness/NFA-DFA.agda**.

Theorem 3. *For any NFA, its accepting language is equal to the language accepted by its translated DFA, i.e. L(NFA) = L(translated DFA).*

Proof. For a given NFA, $N = (Q,\ \delta,\ q_0,\ F)$, let its translated DFA be $D = (\mathcal{P}(Q),\ \delta',\ q_0,\ F')$. To prove the theorem, we have to prove that $L(N) \subseteq L(D)$ and $L(N) \supseteq L(D)$. For any string w,
1) if N accepts w, then w can take q_0 to an accepting state q with n transitions in N. By Lemma (4), there must exist a subset qs such that $q \in qs$ and w can take $\{q_0\}$ to qs in D. Since q is an accepting state; therefore, qs is also an accepting state in D and thus D accepts w.
2) if D accepts w, then w can take $\{q_0\}$ to an accepting state qs in D with n transitions. Since qs is an accepting state, therefore, there must exist a state q in Q such that $q \in qs$ and q is also an accepting state in N. Assuming w cannot take q_0 to q in N for any number of transitions, then by Lemma (5), $qs = \emptyset$ and thus $q \notin qs$ which contradicts the assumption that $q \in qs$. Therefore, w must be able to take q_0 to q in N and thus N accepts w. □

3.10 Decidability of DFA and Regular Expressions

Recall that the accepting language of a DFA is given by the set $\{w \mid \exists q \in F.\ (q_0, w) \vdash^* (q, \epsilon)\}$, which is equivalent to the set $\{w \mid \exists q \in F.\ \exists n.\ (q_0, w) \vdash^n (q, \epsilon)\}$. Its decidability cannot be proved directly because the set of natural numbers is infinite. Therefore, we have to introduce another representation and prove that they are equivalent. The representation and the related lemmas are defined under the module **DFA-Operations** in **DFA.agda**.

Definition 39. We define a function $\delta^*(q, w)$ that takes a state q and a string w as the arguments and returns a state p after running the DFA. It is defined recursively as follow:

```
δ* : Q → Σ* → Q
δ* q [] = q
δ* (a :: w) = δ* (δ q a) w
```

Definition 40. We define $\delta_0(w)$ as the function that runs the DFA from q_0 with a string w.

```
δ₀ : Σ* → Q
δ₀ w = δ* q₀ w
```

Now, before proving that the two representations are equivalent, we have to prove the following lemmas.

Lemma 6. *For any state q and any string w, $\delta^*(q, w) \in F$ if and only if $\exists q' \in F.\ \exists n.\ (q, w) \vdash^n (q', \epsilon)$.*

Proof. We have to prove for both directions.
1) If $\delta^*(q, w) \in F$, we do induction on w.

Base case. If $w = [\,]$, then $\delta^*(q, [\,]) = q \in F$. Therefore, the statement holds.

Induction hypothesis. For any q and w', if $\delta^*(q, w') \in F$, then $\exists q' \in F.\ \exists n.\ (q, w') \vdash^n (q', \epsilon)$.

Induction step. If $w = aw'$, then $\delta^*(q, aw') = \delta^*(\delta(q, a), w')$. By induction hypothesis, there exists a state $q' \in F$ and a number n such that $(\delta(q, a), w') \vdash^n (q', \epsilon)$. It is equivalent to $(q, aw') \vdash^{n+1} (q', \epsilon)$ and thus, the statement holds.
2) If $\exists q' \in F.\ \exists n.\ (q, w') \vdash^n (q', \epsilon)$, then we do induction on n.

Base case. If $n = 0$, then $q' = q$ and $w = [\,]$. Therefore, $\delta^*(q, [\,]) = q = q' \in F$.

Induction hypothesis. For any state q and string w, if there exists another state q' and a number k such that $q' \in F \wedge (q, w) \vdash^k (q', \epsilon)$, then $\delta^*(q, w) \in F$.

Induction step. If $n = k + 1$, then there exists a state $q' \in F$ and a number k such that $(q, aw) \vdash^{k+1} (q', \epsilon)$. It is equivalent to $(\delta(q, a), w) \vdash^k (q', \epsilon)$. By induction hypothesis, we have $\delta^*(\delta(q, a), w) \in F$ and thus $\delta^*(q, aw) \in F$. □

Now, we can prove that the two representations are equivalent.

Lemma 7. *For any string w, $\delta_0(w) \in F$ if and only if $\exists q \in F.\ (q_0, w) \vdash^* (q, \epsilon)$.*

The proof is defined as the function $\delta_0 - lem_1$.

Proof. Since $\delta_0(w) = \delta^*(q_0, w)$; therefore, by Lemma (6), $\delta_0(w) \in F$ if and only if $\exists q \in F.\ (q_0, w) \vdash^* (q, \epsilon)$. □

Now, we can prove that the accepting language of a given DFA is decidable by using the above theorems. The proof is defined as the function **Dec-L**^{D} in **DFA.agda**.

Theorem 4. *For any DFA, its accepting language is decidable, i.e. $\forall w.\ w \in L(DFA)$ is decidable.*

Proof. Since the language of DFA is given by the set $\{w \mid \exists q \in F.\ (q_0, w) \vdash^* (q, \epsilon)\}$, which, by Lemma (7), is equal to the set $\{w \mid \delta_0(w) \in F\}$. Since F is a decidable subset; therefore, the set is also decidable. □

Since we have also proved that the accepting language of regular expressions and DFA are equal; therefore, the accepting language of regular expression must also be decidable. The proof is defined as the function **Dec-L**^{R} in **RegExp-Decidability.agda**.

Theorem 5. *For any given regular expression, e, its accepting language is decidable, i.e.* $\forall w.\ w \in L(e)$ *is decidable.*

Proof. Since $L(e) = L(translated\ \text{DFA})$ and the language accepted by a DFA is decidable; therefore, $L(e)$ is also decidable. □

3.11 Minimising DFA

There are two procedures in minimising a DFA: 1) removing all the unreachable states to construct a RDFA and 2) perform quotient construction on the RDFA to build a MDFA.

3.11.1 Removing unreachable states

Let us begin by defining the reachability of a state. The following definitions and proofs are defined under the module **Remove-Unreachable-States** in **Translation/DFA-MDFA.agda**.

Definition 41. For a given DFA, $(Q,\ \delta,\ q_0,\ F)$, we say that a state q is reachable if and only if there exists a string w that can take q_0 to q,

It is defined in Agda as follow:

```
Reachable : Q → Set
Reachable q = ∃[ w ∈ Σ* ] (q₀ , w) ⊢* (q , [])
```

Definition 42. For a given DFA, $(Q,\ \delta,\ q_0,\ F)$, we define Q^R as a subset of Q such that Q^R contains all and only the reachable states in Q.

The set Q^R is defined in Agda as follow:

```
data Q^R : Set where
  reach : ∀ q → Reachable q → Q^R
```

There are some problems regarding this formalisation of Q^R, they will be discussed in Chapter 6 in details. Now, we can construc the translation of DFA to RDFA which is defined as the function **remove-unreachable-states** in the same file.

Definition 43. For any given DFA, $(Q,\ \delta,\ q_0,\ F)$, its translated RDFA will be $(Q^R,\ \delta,\ q_0,\ Q^R \cap F)$.

Now, let us prove the correctness of the translation by proving that their accepting languages are equal. The correctness proof is defined as the function $\boldsymbol{L^D \approx L^R}$ under the module **Remove-Unreachable-States-Proof** in **Correctness/DFA-MDFA.agda**.

Theorem 6. *For any DFA, its accepting language is equal to the language accepted by its translated RDFA, i.e.* $L(DFA) = L(translated\ RDFA)$.

Proof. Let the DFA be $D = (Q,\ \delta,\ q_0,\ F)$ and is translated RDFA be $R = (Q^R,\ \delta,\ q_0,\ Q^R \cap F)$ according to Definition (43). To prove the theorem, we have to prove that $L(D) \subseteq L(R)$ and $L(D) \supseteq L(R)$. For any string w,
1) if D accepts w, then w must be able to take q_0 to an accepting state q. This implies that all the states in the path must be reachable; therefore, this path is also valid in R and thus R accepts w;
2) if R accepts w, then the path from q_0 to the accepting state q must be also valid in D; therefore, D also accepts w. □

3.11.2 Quotient construction

Now, we need to perform quotient construction on the newly constructed RDFA. The following definitions and proofs are defined under the module **Quotient-Construct** in **Translation/DFA-MDFA.agda**. Now, let us begin by defining a binary relation on states that will be used to construct the quotient set.

Definition 44. Suppose we have a DFA $(Q,\ \delta,\ q_0,\ F)$, then for any two states p and q, we say the $p \sim q$ if and only if for any string w, w cannot distinguish p and q, i.e. $p \sim q = \forall w.\ \delta^*(p, w) \in F \Leftrightarrow \delta^*(q, w) \in F$

It is easy to show that the relation is an equivalence relation. Now, we have to prove that $p \sim q$ is decidable. One possible method is to use the Table-filling algorithm. However, the formalisation of this algorithm and its correctness proofs has not been completed. Therefore, in the following parts, we will assume that $p \sim q$ is decidable. Now, let us construct the quotient set by using the above relation of states. The quotient set is defined in the **Quotient.agda**.

Definition 45. For a state p of a given DFA, its equivalence class is a subset of all indistinguishable states of p, i.e. $\langle\!\langle p \rangle\!\rangle = \{q \mid p \sim q\}$.

From the above definition, we observe that a equivalence class is a subset of the set of states of a given DFA. The corresponding formalisation in Agda is as follow, note that $Dec\text{-}\sim$ is the decidability of $\sim$:

```
⟪_⟫ : Q → DecSubset Q
⟪ p ⟫ q with Dec-∼
... | yes _ = true
... | no  _ = false
```

Definition 46. For a given DFA, $(Q,\ \delta,\ q_0,\ F)$, we define $Q/\sim$ as the set of all equivalence classes of $\sim$ on Q.

It is defined in Agda as follow:

```
data Q/∼ : Set where
  class : ∀ qs → ∃[ q ∈ Q ] (qs =ᵈ ⟪q⟫) → Quot−Set
```

Now, we can define the translation of RDFA to MDFA which is defined as the function **quotient-construction** at the bottom of **Translation/DFA-MDFA.agda**.

Definition 47. For any given RDFA, $(Q,\ \delta,\ q_0,\ F)$, its translated MDFA will be $(Q/\sim,\ \delta',\ \langle\!\langle q_0 \rangle\!\rangle,\ F')$ where

- $\delta'(q, a) = \langle\!\langle \delta(q, a) \rangle\!\rangle$; and
- $F' = \{\langle\!\langle q \rangle\!\rangle \mid q \in F\}$.

Now, before proving the translation is correct, we have to first prove the following lemmas. The theorems and proofs below can be found in the module **Quotient-Construction-Proof** in **Correctness/DFA-MDFA.agda**.

Lemma 8. *Let a RDFA be $R = (Q,\ \delta,\ q_0,\ F)$ and its translated MDFA be $M = (Q/\sim,\ \delta',\ \langle\!\langle q_0 \rangle\!\rangle,\ F')$ according to Definition (47). For any state q in Q, if a string w can take q to another state q' with n transitions in R, then w can take $\langle\!\langle q \rangle\!\rangle$ to $\langle\!\langle q' \rangle\!\rangle$ with n transitions in M.*

The proof is defined as the function $\mathbf{lem}_1$.

Proof. We can prove the lemma by induction on n.
Base case. If $n = 0$, then $q = q'$ and $w = \epsilon$. It is obvious that the statement holds.
Induction hypothesis. For any state q in Q, if a string w can take q to another state q' with k transitions in R, then w can take $\langle\!\langle q \rangle\!\rangle$ to $\langle\!\langle q' \rangle\!\rangle$ with k transitions in M.
Induction step. If $n = k + 1$, then there must exist a string in the form of aw that can take q to q' with $k + 1$ transitions. This implies that there exists a state p such that $p = \delta(q, a)$ and w can take p to q' with k transitions. By induction hypothesis, w can also take $\langle\!\langle p \rangle\!\rangle$ to $\langle\!\langle q' \rangle\!\rangle$ with k transitions in M. Furthermore, $p = \delta(q, a)$ implies that $\langle\!\langle p \rangle\!\rangle = \langle\!\langle \delta(q, a) \rangle\!\rangle = \delta'(\langle\!\langle q \rangle\!\rangle, a)$; Therefore, aw can take $\langle\!\langle q \rangle\!\rangle$ to $\langle\!\langle q' \rangle\!\rangle$ with $k + 1$ transitions in M and thus the statement is true. □

Lemma 9. *Let a RDFA be $R = (Q,\ \delta,\ q_0,\ F)$ and its translated MDFA be $M = (Q/\sim,\ \delta',\ \langle\!\langle q_0 \rangle\!\rangle,\ F')$ according to Definition (47). For any string w and any state q in Q, $\delta'^*(\langle\!\langle q \rangle\!\rangle, w) = \langle\!\langle \delta^*(q, w) \rangle\!\rangle$.*

The proof is defined as the function $\mathbf{lem}_2$.

Proof. We can prove the lemma by induction on w.
Base case. If $w = \epsilon$, $\delta'^*(\langle\!\langle q \rangle\!\rangle, \epsilon) = \langle\!\langle q \rangle\!\rangle = \langle\!\langle \delta^*(q, \epsilon) \rangle\!\rangle$ and thus the statement is true.
Induction hypothesis. For a string w' and any state q in Q, $\delta'^*(\langle\!\langle q \rangle\!\rangle, w') = \langle\!\langle \delta^*(q, w') \rangle\!\rangle$.
Induction step. If $w = aw'$, then $\delta'^*(\langle\!\langle q \rangle\!\rangle, aw') = \delta'^*(\delta'(\langle\!\langle q \rangle\!\rangle, a), w')$. Since, $\delta'(\langle\!\langle q \rangle\!\rangle, a) = \langle\!\langle \delta(q, a) \rangle\!\rangle$; therefore, $\delta'^*(\delta'(\langle\!\langle q \rangle\!\rangle, a), w') = \delta'^*(\langle\!\langle \delta(q, a) \rangle\!\rangle, w')$. By induction hypothesis, $\delta'^*(\langle\!\langle \delta(q, a) \rangle\!\rangle\, w') = \langle\!\langle \delta^*(\delta(q, a), w) \rangle\!\rangle = \langle\!\langle \delta^*(q, aw) \rangle\!\rangle$ and thus the statement is true. □

Now, we can prove the correctness of the translation by proving that their accepting languages are equal. The correctness proof is defined as the function $\boldsymbol{L^R \approx L^M}$ in **Correctness.agda** while the detail proofs are defined in **Correctness/DFA-MDFA.agda**

Theorem 7. *For any RDFA, its accepting language is equal to the language accepted by its translated MDFA, i.e.* $L(RDFA) = L(\textit{translated MDFA})$.

Proof. Let the RDFA be $R = (Q,\ \delta,\ q_0,\ F)$ and its translated MDFA be $M = (Q/\sim,\ \delta',\ \langle\!\langle q_0 \rangle\!\rangle,\ F')$ according to Definition (47). To prove the theorem, we have to prove that $L(R) \subseteq L(M)$ and $L(R) \supseteq L(M)$. For any string w,
1) if R accepts w, then w must be able to take q_0 to an accepting state q in R. By Lemma (8), w must be able to take $\langle\!\langle q_0 \rangle\!\rangle$ to $\langle\!\langle q \rangle\!\rangle$ in M. Since $q \in F$; therefore, $\langle\!\langle q \rangle\!\rangle \in F'$ and thus M accepts w.
2) if M accepts w, then w must be able to take $\langle\!\langle q_0 \rangle\!\rangle$ to an accepting state $\langle\!\langle q \rangle\!\rangle$. Then by Lemma (6), $\delta' * (\langle\!\langle q_0 \rangle\!\rangle, w) \in F'$. Furthermore, by Lemma (9), $\delta' * (\langle\!\langle q_0 \rangle\!\rangle, w) = \langle\!\langle \delta(q_0, w) \rangle\!\rangle)$; therefore, $\langle\!\langle \delta(q_0, w) \rangle\!\rangle \in F'$ and thus $\textit{delta}(q_0, w) \in F$. By Lemma (7), w can take q_0 to an accepting state q in R; therefore, R accepts w. □

We also need to prove that the translated MDFA is minimal, but first, we have to define what is a minimal DFA.

3.12 Minimal DFA

The definition of minimal DFA can be found in **MinimalDFA.agda**. In order for a DFA to be minimal, it must satisfy two criteria: 1) every state must be reachable and 2) the states cannot be further reduced.

Criteria (1) is defined as the function **All-Reachable-States**. It is defined as follow:

```
All-Reachable-States : DFA → Set
All-Reachable-States dfa = ∀ q → Reachable q
```

For criteria 2), we have to first introduce a binary relation of states.

Definition 48. Suppose we have a DFA, $(Q,\ \delta,\ q_0,\ F)$, for any two states p and q, we say that a string w can distinguish p and q if and only if exactly one of $\delta^*(p, w)$ and $\delta^*(q, w)$ is an accepting state, i.e. $\delta^*(p, w) \in F \wedge \delta^*(q, w) \notin F \vee \delta^*(p, w) \notin F \wedge \delta^*(q, w) \in F$.

Definition 49. For any two states p and q in a given DFA, we say that $p \nsim q$ if and only if there exists a string w that can distinguish p and q.

It is defined in Agda as follow:

```
_≁_ : Q → Q → Set
p ≁ q = ∃[ w ∈ Σ* ]
      (δ* p w ∈d F × δ* q w ∉d F ⊎ δ* p w ∉d F × δ* q w ∈d F)
```

Definition 50. For a given DFA, it is irreducible if and only if for any two states p and q, if p is not equal to q, then $p \not\sim q$.

It is defined in Agda as follow:

```
Irreducible : DFA → Set
Irreducible dfa = ∀ p q → ¬ p ≈ q → p ≁ q
```

Now, we can define what is a minimal DFA.

Definition 51. For a given DFA, it is minimal if and only if all its states are reachable and it is irreducible.

It is defined in Agda as follow:

```
Minimal : DFA → Set
Minimal dfa = All−Reachable−States dfa × Irreducible dfa
```

Now, let us prove that any MDFA is minimal by proving that all the states in MDFA are reachable and the MDFA is irreducible. The proofs are defined under the module **Reachable-Proof** and **Minimal-Proof** in **Correctness/DFA-MDFA.agda**.

Theorem 8. *For any MDFA, all its states are reachable.*

Proof. Since for any state in a RDFA is reachable, then for any state q in RDFA, there must exist a string w that can take q_0 to q. By Lemma (8), w can also take $\langle\langle q_0 \rangle\rangle$ to $\langle\langle q \rangle\rangle$ and thus the statement is true. □

To prove that MDFA is irreducible, we have to first prove that for any two states p and q, $\neg(p \sim q)$ if and only if $p \not\sim q$.

Lemma 10. *For any two states p and q in a DFA and a string w, $\delta^*(p, w) \in F \Leftrightarrow \delta^*(q, w)$ if and only if not exactly one of $\delta^*(p, w)$ and $\delta^*(q, w)$ is an accepting state, i.e. $(\delta^*(p, w) \in F \Leftrightarrow \delta^*(q, w)) \Leftrightarrow \neg(\delta^*(p, w) \in F \wedge \delta^*(q, w) \notin F \vee \delta^*(p, w) \notin F \wedge \delta^*(q, w) \in F)$.*

Proof. We have to prove for both directions.
1) Case $(\delta^*(p, w) \in F \Leftrightarrow \delta^*(q, w) \in F) \Rightarrow not exactly one of \delta^*(p, w) and \delta^*(q, w) is an accepting state$: suppose exactly one of $\delta^*(p, w)$ and $\delta^*(q, w)$ is an accepting state, for example, $\delta^*(p, w) \in F \wedge \delta^*(q, w) \notin F$. This contradicts to the assumption that $\delta^*(p, w) \in F \Leftrightarrow \delta^*(q, w) \in F$. Therefore, not

exactly one of $\delta^*(p, w)$ and $\delta^*(q, w)$ is an accepting state. The same argument also applies for the case when $\delta^*(p, w) \notin F \wedge \delta^*(q, w) \in F$.
2) Case *not exactly one of* $\delta^*(p, w)$ *and* $\delta^*(q, w)$ *is an accepting state* $\Rightarrow (\delta^*(p, w) \in F \Leftrightarrow \delta^*(q, w) \in F)$: if $\delta^*(p, w) \in F$ and$\delta^*(q, w) \notin F$, then it contradicts to the assumption that *not exactly one of* $\delta^*(p, w)$ *and* $\delta^*(q, w)$ *is an accepting state.* Therefore, $\delta^*(q, w) \in F$ is true; the same argument applies for the case when $\delta^*(p, w) \notin F$ and $\delta^*(q, w) \in F$. □

Lemma 11. *For any two states p and q in a given DFA, $\neg(p \sim q)$ if and only if $p \nsim q$.*

Proof. We have to prove for both directions.
1) Case $\neg(p \sim q) \Rightarrow p \nsim q$: suppose $p \nsim q$ is false, then there does not exist a string w that can distinguish p and q. This implies that for any string w, w cannot distinguish p and q, i.e. $\forall w.\ \neg(\delta^*(p, w) \in F \wedge \delta^*(q, w) \notin F \vee \delta^*(p, w) \notin F \wedge \delta^*(q, w) \in F)$. By Lemma (10), we have $\forall w.\ \delta^*(p, w) \in F \Leftrightarrow \delta^*(q, w)$ which is equivalent to $p \sim q$. This contradicts to the assumption that $\neg(p \sim q)$; therefore, $p \nsim q$ must be true.
2) Case $p \nsim q \Rightarrow \neg(p \sim q)$: suppose $p \sim q$, then for any string w, $\delta^*(p, w) \in F \Leftrightarrow \delta^*(q, w)$. Since $p \nsim q$, then there must exist a string w that can distinguish p and q, i.e. exactly one of $\delta^*(p, w)$ and $\delta^*(q, w)$ is an accepting state. This contradicts to the assumption that $\delta^*(p, w) \in F \Leftrightarrow \delta^*(q, w)$; therefore, $p \sim q$ must be true. □

Theorem 9. *For any MDFA, it is irreducible, i.e. for any two states p and q in the MDFA, if p is not equal to q, then $p \nsim q$.*

Proof. For any two states $\langle\!\langle p \rangle\!\rangle$ and $\langle\!\langle q \rangle\!\rangle$ in MDFA, if $\langle\!\langle p \rangle\!\rangle$ is not equal to $\langle\!\langle q \rangle\!\rangle$, then $p \sim q$ is false. By Lemma (11), $p \nsim q$ is true. □

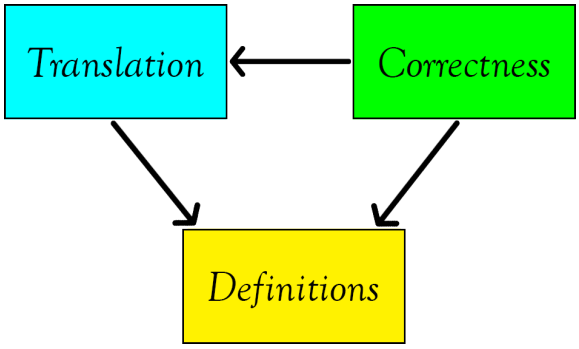
Theorem 10. *For any MDFA, it is minimal, i.e. all the states in the MDFA are reachable and the MDFA is irreducible.*

Proof. The statement follows from Theorem (8) and Theorem (9). □

Chapter 4

Structure of our Agda files

We have written 32 agda files with a total of 5785 lines of code including comments and blank lines. These files can be divided into three categories: 1) definitions, 2) translation algorithms and 3) correctness proofs. All the translation algorithms are inside the **Translation** directory while the correctness proofs are inside the **Correctness** directory. The other files belong to the first category. Here is a simplified dependency graph of the files:



Readers are recommended to start with **Parsing-Regular-Expressions-in-Agda.agda** ,which is the index file of our formalisation. Now, let us look into each of the agda files.

Parsing-Regular-Expressions-in-Agda.agda	is the index file of the Agda formalisation which imports all the other agda files. Once this file has been compiled, all the other files will have been compiled as well.
Subset.agda	contains the definition and operations of subset.

../DecidableSubset.agda	contains the definition and operations of decidable subset.
../VectorRep.agda	contains the vector representation of sets.
Language.agda	contains the definition and operations of language.
RegularExpression.agda	contains the definition of regular expression and regular language.
eNFA.agda	contains the definition, operations and properties of ϵ-NFA.
NFA.agda	contains the definition, operations and properties of NFA.
DFA.agda	contains the definition, operations and properties of DFA.
MinimalDFA.agda	contains the definition of minimal DFA.
State.agda	contains the state construction algorithm for translating regular expressions to ϵ-NFA.
Quotient.agda	contains the quotient set construction function for translating DFA to MDFA.
RelationTable.agda	contains the matrix representation of a binary relation. This file is not completed and thus it is not used in any other files.
Translation.agda	is the index file of all the translation algorithms.
../RegExp-eNFA.agda	contains the translation function **regexToϵ-NFA** from regular expressions to ϵ-NFA.
../eNFA-NFA.agda	contains the translation function **remove-ϵ-step** from ϵ-NFA to NFA.
../NFA-DFA.agda	contains the translation function **powerset-construction** from NFA to DFA.
../DFA-MDFA.agda	contains the translation function **minimise** from DFA to MDFA.
../TableFillingAlgorithm.agda	contains the algorithm that computes the relation of distinguishable states. This file is not completed and thus it is not used in any other files.
Correctness.agda	is the index file of all the correctness proofs.
../RegExp-eNFA.agda	contains the proof that $\forall e \in RegExp.\ L(e) = L(regexTo\epsilon\text{-NFA}\ e)$.

../../Epsilon-lemmas.agda	contains the ϵ case of the above proof.
../../Singleton-lemmas.agda	contains the $\sigma\ a$ case of the above proof.
../../Union-lemmas.agda	contains the union case of the above proof.
../../Concatenation-lemmas.agda	contains the concatenation case of the above proof.
../../KleenStar-lemmas.agda	contains the kleen star case of the above proof.
../eNFA-NFA.agda	contains the proof that $\forall enfa \in \epsilon$-NFA. $L(enfa) = L(remove\text{-}\epsilon\text{-}step\ enfa)$.
../NFA-DFA.agda	contains the proof that $\forall nfa \in$NFA. $L(nfa) = L(powerset\text{-}construction\ nfa)$.
../DFA-MDFA.agda	contains the proof that $\forall dfa \in$DFA. $L(dfa) = L(minimise\ dfa)$.
RegExp-Decidability.agda	contains the proof of the decidability of regular expressions.
Example.agda	contains an example on the recognition of a string by a regular expression.
Util.agda	contains miscellaneous definitions and proofs such as decidable equality.

We have also generated html files from the agda files. They are located in the **web** directory. Readers are recommended to read the html files as it is much more convenient. Similarly, readers can start with the **Parsing-Regular-Expressions-in-Agda.html** which contains all the links to other files.

Chapter 5

Further Extensions

In this chapter, we will briefly describe two possible extensions to our project: 1) the Myhill-Nerode Theorem and 2) the Pumping Lemma. These two theorems both use the translation of regular expressions to DFA in their arguments. Therefore, they can be built on top of our formalisation.

5.1 Myhill-Nerode Theorem

Given a language L, and a pair of strings u and v, we define a relation R_L on strings such that uR_Lv if and only if for any string z, L accepts uz if and only if L accepts vz, i.e. $uz \in L \Leftrightarrow vz \in L$. It is easy to show that the relation R_L is an equivalence relation and thus R_L divides the set of strings into some equivalence classes.

In general, the Myhill-Nerode Theorem states that a language L is regular if and only if R_L has a finite number of equivalence classes, and furthermore the number of states in the minimal DFA translated from L is equal to the number of equivalence classes in R_L. The theorem is often used to prove that a language is not regular, such as the language of well-bracketed expressions.

In the proof, if L is a regular language, then L is translated to a DFA in order to prove that R_L has finite equivalence classes. For the opposite direction, suppose R_L has finite equivalence classes, then a DFA can be constructed from the relation. By proving the DFA recognises L, L is proved to be regular. Therefore, the theorem can be proved under the environment of our formalisation as we have already formalised the translation from regular languages to DFA together with the correctness proofs.

5.2 Pumping Lemma (for regular languages)

In general, the Pumping Lemma states that if L is a regular language, then there must exist an integer $n \geq 1$ depending on L such that every string $w \in L$ of length at least n can be divided into three sub-strings, i.e. $w = xyz$ where

1. $|y| \geq 1$;
2. $|xy| \leq n$;
3. for all $i \geq 0$, $xy^iz \in L$.

y is the sub-string that can be pumped (removed or repeated any number of times) and the result string is always in L. The lemma is again often used to prove a language non-regular. In the proof, a DFA is constructed from L to prove that there must exist some loops in the path of w. Therefore, it can be proved by using our formalisation.

Chapter 6

Evaluation

Throughout the project, we have made several decisions over the representations of mathematical objects such as subsets. We will discuss their consequences in this chapter. Furthermore, we will also review the whole development process. At the end of this section, we will also discuss the feasibility of formalising mathematics and programming logic in practice.

6.1 Different choices of representations

In computer proofs, an abstract mathematical object usually requires a concrete representation. Different representations will lead to different formalisations and thus contribute to the easiness or difficulty in completing the proofs. In the following paragraphs, we will discuss the representations we have chosen and their consequences.

The set of states (Q) and its subsets As we have mentioned before, Firsov and Uustalu [8] represent the set of states (Q) and its subsets as column vectors. However, this representation looks unnatural compare to the actual mathematical definition. Therefore, at the beginning, we intended to avoid the vector representation and to represent the sets in abstract forms. In our approach, the set of states is represented as a data type in Agda, i.e. $Q : Set$, and its subsets are represented as unary functions on Q, i.e. $DecSubset\ Q = Q \to Bool$.

Our definition allows us to finish the proofs in **Correctness/RegExp-eNFA.agda** without having to manipulate matrices. The proofs also look much more natural compare to that in [8]. However, problems arise when the sets have to be iterated during the computation of ϵ-closures because it is impossible to iterate the sets under this representation. In order to solve the problems, several extra fields are added into the definition of automata such as It – a vector containing all the states in Q. With It, a subset of Q can be iterated by applying it own function, $Q \to Bool$, to

all the elements in It. Note that the vector It is equivalent to the vector representation of Q.

The accepting languages of regular expressions and finite automata At first, the accepting language of regular expression was defined as a decidable subset, i.e. $L^R : RegExp \to DecSubset\ \Sigma^*$. The decision was reasonable because the language has been proved decidable for many years. However, L^R is also a boolean decider for regular expressions and this definition added a great amount of difficulties in writing the proofs because the proofs had to be built on top of the decider. For example, in the concatenation case, an extra recursive function was needed to iterate different combinations of input string and the correctness the algorithm was difficult to prove. More importantly, the decidability of the languages is actually not necessary in proving their equality. Therefore, in our current version, the language is defined using the general subset, i.e. $L^R : RegExp \to Subset\ \Sigma^*$. This definition allows us to prove the equality of the languages in an abstract level.

The same situation also applies to the accepting language of finite automata. At first, the accepting language of NFA was obtained by running an algorithm that produces all the reachable states from q_0 using the input string. The algorithm was also a decider for NFA. Once again, this definition mixes the decider and the proposition together. Therefore, the accepting language of NFA is now defined in an abstract form.

However, this does not mean that we can ignore all the algorithms. In fact, we still need to design algorithms for computing ϵ-closures and powerset construction. Here is our approach: 1) separate the algorithm from the mathematical definition and 2) prove that they are equivalent. In this way, the decidability of the mathematical definition will follow from that of the algorithm. The advantage is that it is easier to use the abstract mathematical definition in other proofs than using the algorithm.

The set of reachable states from q_0 Let us recall the definition of reachable states from q_0.

```
-- Reachable from q₀
Reachable : Q → Set
Reachable q = Σ[ w ∈ Σ* ] (q₀ , w) ⊢* (q , [])

data Qᴿ : Set where
  reach : ∀ q → Reachable q → Qᴿ
```

We say that a state q is reachable if and only if there exists a string w that can take q_0 to q. Therefore, the set Q^R should contain all and only the reachable states in Q. However, there may exist more than one reachability proofs for the same state. Therefore, for a state in Q, there may be more than one elements in Q^R and thus Q^R may be larger than the original set Q or even worse, it may be infinite. This leads to a problem when a DFA is constructed using Q^R as the set of states. If Q^R is infinite, then it is impossible to iterate the set during quotient construction. Even if the set

Q^R is finite, it contradicts our original intention to minimise the set of states. This is also one of the reasons why our formalisation of quotient construction cannot be completed. However, this has no effects to the proof of $L(\text{DFA}) = L(\text{MDFA})$ because we can provide an equality relation of states in the DFA. In the translation from DFA to MDFA, we defined the equality relation as follow: any two elements in Q^R are equal if and only if their states are equal. Therefore, two elements with same state but different reachability proofs are still considered the same in the new DFA.

One possible way to solve the problem is to re-define the reachability of a state such that any reachable state will have a unique reachability proof. For example, the representative proof can be selected by choosing the shortest string (w) sorted in alphabetical order that can take q_0 to q. Another possible solution is to use Homotopy Type to declare the set Q^R. This type allows us to group different reachability proofs into a single element such that every state will only appear once in Q^R. However, the most straight-forward solution is to re-define the type of Q^R to decidable subset, i.e. $Q^R : DecSubsetQ$, because the set is a subset of Q. We can compute the set by the following algorithm:

1. put q_0 into a subset S, i.e. $S = \{q_0\}$
2. loop for $|Q| - 1$ times:
   (a) for every state p in S, for every alphabet a in Σ, if a can take p to another state r, i.e. $r = \delta(p, a)$, then put r into S.
3. the result subset S should contain all and only the reachable states.

This algorithm will require a vector representation of the set of alphabets and a another formalisation of RDFA. Since the new Q^R has the type $DecSubset$, then it can be represented by a vector and thus it is possible to iterate.

6.2 Development Process

As we have mentioned before, the parts related to table filling algorithm were not completed. The major reason is that there was only very limited time left when we started it. In the following paragraphs, we will evaluate the whole development process and discuss what could be improved.

In the first 6 weeks, I was struggling to find the most suitable representations for regular expressions, finite automata and their accepting languages. During the time, I was rushing in coding without thinking about the whole picture of the theory. As a result, many bad decisions had been made; for example, omitting the ϵ-transitions when converting regular expressions to NFA and trying to prove the decidability of regular expressions directly. After taking the advice from my supervisor, I followed the definitions in the book [2] and started to write a framework of the theory. After that, in just one week, the Agda codes that formed the basis of the final version were developed by using the framework. One could argue that the work done in the first 6 weeks also

contributed to those Agda codes but there is no doubt the written framework highly influenced the development. Therefore, even though it is convenient to write proofs using the interactive features in Agda, it would still be better to start with a framework in early stages.

After that, the development went smooth until the first month of the second semester. During the time, I started to prove several properties of ϵ-closures. The plan was to finish this part within 2 weeks. However, 4 weeks were spent on it and very little progress was made. These 4 weeks were crucial to the development and the time should have been spent more wisely on other parts of the project such as powerset construction or the report.

6.3 Computer-aided verification in practice

In order to evaluate the feasibility of computer-aided verification in practice, we will discuss: 1) the difference between computer proofs and written proofs, 2) the easiness or difficulty in formalising mathematics and programming logic and 3) the difference between computer-aided verification and testing.

6.3.1 Computer proofs and written proofs

According to Geuvers [9], a proof has two major roles: 1) to convince the readers that a statement is correct and 2) to explain why a statement is correct. The first role requires the proof to be convincing. The second role requires the proof to give an intuition of why the statement is correct.

Correctness Traditionally, when mathematicians submit the proof of their concepts, a group of other mathematicians will evaluate and check the correctness of the proof. Alternatively, if the proof is formalised in a proof assistant, it will be checked automatically by the compiler. The only difference is that we are now relying on the compiler and the machine that runs the compile rather than the group of mathematicians. Therefore, if the compiler and the machine work properly, then any formalised proof that can be compiled without errors is said to be correct. Furthermore, a proof can been seen as a group of smaller reasoning steps. We can say that the a proof is correct if and only if all the reasoning steps within the proof are correct. When writing proofs in paper, the proofs of some obvious lemmas are usually omitted and this sometimes leads to mistakes. However, in most proof assistants, the proofs of every lemma must be provided explicitly. Therefore, the correctness of a computer proof always depends on the correctness of the smaller reasoning steps within it.

Readability The second purpose of a proof is to explain why a certain statement is correct. Let us consider the following proof extracted from **Correctness/RegExp-eNFA.agda**.

```
lem₃ : ∀ wᵉ n q₁
      → (q₀ , wᵉ)  ⊢ᵏ suc n − (inj q₁ , [])
      → Σ[ n₁ ∈ ℕ ] Σ[ uᵉ ∈ Σᵉ* ] (toΣ* wᵉ ≡ toΣ* uᵉ × (inj q₀₁ , uᵉ) ⊢ᵏ n₁ − (inj q₁ , []))
lem₃ ._ zero    q₁  (inj .q₁  , α _ , .[] , refl , (refl ,   ()) , (refl , refl))
lem₃ ._ zero    q₁  (inj .q₁  , E   , .[] , refl , (refl , prf₁) , (refl , refl)) with Q₁? q₁ q₀₁
lem₃ ._ zero   .q₀₁ (inj .q₀₁ , E   , .[] , refl , (refl , prf₁) , (refl , refl)) | yes refl  = zero , [] , (refl , (refl , refl))
lem₃ ._ zero    q₁  (inj .q₁  , E   , .[] , refl , (refl ,   ()) , (refl , refl)) | no  p≠q₀₁
lem₃ ._ (suc n) q₁  (init     , α _ ,  uᵉ , refl , (refl ,   ()) , prf₂)
lem₃ ._ (suc n) q₁  (init     , E   ,  uᵉ , refl , (refl , prf₁) , prf₂) = lem₃ uᵉ n q₁ prf₂
lem₃ ._ (suc n) q₁  (inj  p   , α _ ,  uᵉ , refl , (refl ,   ()) , prf₂)
lem₃ ._ (suc n) q₁  (inj  p   , E   ,  uᵉ , refl , (refl , prf₁) , prf₂) with Q₁? p q₀₁
lem₃ ._ (suc n) q₁  (inj .q₀₁ , E   ,  uᵉ , refl , (refl , prf₁) , prf₂) | yes refl  = suc n , uᵉ , refl , prf₂
lem₃ ._ (suc n) q₁  (inj  p   , E   ,  uᵉ , refl , (refl ,   ()) , prf₂) | no  p≠q₀₁
```

There are several techniques used in the above proof; for example, induction on natural numbers and case analysis on state comparison. However, by just looking at the function body, the proving process can hardly be understood. Therefore, in general, a computer proof is very inadequate on this purpose and thus an outline of the proof in natural language is still necessary.

Although computer proofs seem to be unreadable, the advantages are still impressive. Some large proofs are nearly impossible to validate by human beings, for example, the 4-colour theorem. Gonthier [10] formalised the proof in Coq which consists of a reduction to the problem that includes 633 cases. Furthermore, the formalised theories can be re-used in other formalisations within the same platform. Therefore, we can build a repository of theories and formalise other theories using the repository.

6.3.2 Easiness or difficulty in the process of formalisation

As a computer science student without a strong mathematical background, I find it not very difficult to formalise proofs in computer as there are many similarities between writing codes and writing proofs; for example, pattern matching compare to case analysis and recursive function compare to mathematical induction. Furthermore, Agda is convenient to use for its interactive features. The features allow us to know what is happening inside the proof body by showing the goal and all the elements in the proof. Also, many theorems can be automatically proved by Agda.

On the other hand, most of the proof assistants based on dependent types support only primitive or structural recursion. Many algorithms may be very difficult to implement under this limitation. For example, the usual algorithm that is used to compute the ϵ-closure for a state q is as follow:

1. put q into a subset S, i.e. $S = \{q\}$
2. for every state p in S, if another state r is reachable from p with an ϵ, i.e. $r \in \delta(p,\ \epsilon)$, then put r into S.
3. loop (2) until no new elements is added to S
4. the resulting subset S is the ϵ-closure of q

6.3.3 Computer-aided verification and testing

The most common way to verify a program is via testing. However, no matter how sophisticated the design of the test cases is, the program still cannot be proved to be 100% correct. On the other hand, total correctness can be achieved by proving the specifications of a program. Already in 1997, Necula [15] has raised the notion of *Proof Carrying code*. The idea is to accompany a program with several proofs that proves the program specifications within the same platform. In fact, there is already an extraction mechanism [13] in Coq that allows us to extract proofs and functions in Coq into Ocaml, Haskell or Scheme programs. Furthermore, testing are sometimes inapplicable in certain situations. For example, it is very difficult to design test cases for concurrent programs. On the other hand, it is relatively easier to prove the specifications of such programs. This also applies to programs in distributive systems and cloud computing. The question remains – how to formalise such programs in Type Theory.

On the other hand, most proof assistants are not Turing-complete which means that there might exist some useful functions that are impossible to express.

Chapter 7

Conclusion

We have formalised the he translation from regular expressions to DFA and its correctness proofs in Agda. We have fully also formalized the conversion from DFA to MDFA. Further work includes several lemmas regarding the Table-filling algorithm, which could be completed without major difficulties given three or four more weeks, based on the development discussed in the report.

Throughout the project, we have made several decisions on the representation of mathematical objects, for example, the set of states (Q) as a data type, accepting language of regular expressions as general subset. These decisions have influenced the easiness or difficulties in completing the proofs. Furthermore, there are many similarites between writing proofs and writing codes. Therefore, in conclusion, as a computer science student without a strong mathematical background, I find it quite feasible to formalise mathematics and programming logic in Type Theory.

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Appendix A

File description

There are two directories in **CHEUNG_1465388.zip**: 1) **Code** and 2) **web**. The **Code** directory contains all the **.agda** files. In order to compile these files, first install Agda by following the instructions stated in The Agda Wiki. Then open **Parsing-Regular-Expressions-in-Agda.agda** with the IDE recommended by the website. Press Ctrl-c Ctrl-l to compile the file. The **Parsing-Regular-Expressions-in-Agda.agda** file is an index file that imports all other agda files. Once this file has been compiled, the other Agda files will have been compiled as well.

The **web** directory contains all the html files generated from the agda files. Readers are recommended to read the html files instead of the agda files especially for those who do not have Agda installed. The **Parsing-Regular-Expressions-in-Agda.html** is the index file that contains the links to all other html files.