Étale Fundamental Groups

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Abstract

We discuss $\pi_1^{\mathrm{et}}(X)$, following roughly Milne's Etalé Cohomology book, Chapter 1.5.

1 The classical fundamental group

There are two ways to define the topological fundamental group $\pi_1(X)$.

Definition 1. The fundamental group $\pi_1(X, x)$ is defined to be homotopy classes of loops based at x.

Definition 2. Let $\widetilde{X}$ be the universal cover of X. A map of covering spaces $\alpha : \widetilde{X} \to \widetilde{X}$ is a continuous map making the diagram

$$\begin{tikzcd} \widetilde{X} \arrow[rr, "\alpha"] \arrow[dr] & & \widetilde{X} \arrow[dl] \\ & X & \end{tikzcd}$$

commute.

Let $\mathrm{Aut}_X(\widetilde{X})$ be the group of covering maps $\widetilde{X} \to \widetilde{X}$. This is a group because of the universal property of the unversal cover $\widetilde{X}$. The Galois correspondence for covering space theory tells us that $\mathrm{Aut}_X(\widetilde{X}) \cong \pi_1(X, x)$.

The second definition of fundamental group naturally comes equipped with the following constructions:

Let $\mathrm{Cov}(X)$ be the category of covering spaces of X with finitely many connected components, with arrows being covering space maps. Let $F : \mathrm{Cov}(X) \to \mathsf{Sets}$ be the functor which sends $\pi : Y \to X$ to $\pi^{-1}(x)$. The functor F is representatble by $\widetilde{X}$. The group $\mathrm{Aut}_X(\widetilde{X})$ acts on $F(Y) \cong \mathrm{Hom}_X(\widetilde{X}, Y)$ by $\alpha \cdot f = f \circ \alpha$ where $\alpha \in \mathrm{Aut}_X(\widetilde{X})$ and $f : \widetilde{X} \to Y$. So, F is a functor $\mathrm{Cov}(X)$ to the category of $\mathrm{Aut}_X(\widetilde{X})$-sets, and is an equivalence of categories.

2 The Étale fundamental group

Let X be a connected scheme, and $\bar{x}$ a geometric point of X such that the point comes from a separable field extension of $k(x)$.

The first definition of fundamental group is clearly intractable, because loops may not exist in the category of schemes or varieties over k for general fields k. So we take the second approach, this has its own issues (namely that an algebraic universal cover does not necessarily exist).

Let FEt/X be the category of finite étale maps $\pi : Y \to X$ with arrows given by X-morphisms. We set $F : \mathsf{FEt}/X \to \mathsf{Sets}$ to be the functor sending $\pi : Y \to X$ to be $F(Y) = \mathrm{Hom}_X(\bar{x}, Y)$. If X is a variety over an algebraically closed field and $\bar{x}$ is a closed point, we have that $F(Y) = \pi^{-1}(\bar{x})$ by staring at the diagram

$$\begin{tikzcd} \bar{x} \arrow[rr] \arrow[dr] & & Y \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

As in analogy with the topological fundamental group, we want to define $\bar{X}$ to be the object representing F. But this does not exist in general.

Example 3. Let $X = \mathbb{A}^1_k \setminus \{0\}$. Let $\pi : Y \to X$ be a finite étale covering. Then, $\dim Y = \dim X = 1$. For all $x \in X$, there is an étale neighborhood U of x such that $Y \times_X U$ is a disjoint union of open subvarieties U_i of Y, where U_i is isomorphic to U. Therefore, Y and X are birational curves and hence they have the same genus. Therefore, $Y \cong X$.

The maps $Y \to X$ is therefore given by k-algebra morphisms $k[z, z^{-1}] \to k[z, z^{-1}]$ and hence the finite étale maps are given by $z \mapsto z^n$ for $n \neq 0$ and n coprime to $\mathrm{char}(k)$. However, there is clearly no universal cover since we cannot get a largest cover from $z \mapsto z^n$.

So, from Example 3 we have seen that the functor F is not representable in general.

Theorem 4. The functor F is pro-representable. That is, there is a collection $\widetilde{X} = (X_i)_{i \in I}$ of finite étale coverings of X where I is a directed set, such that

$$F(Y) = \varinjlim_{i \in I} \mathrm{Hom}(X_i, Y).$$

We call $\widetilde{X}$ the universal covering space of X, and we can pick $\widetilde{X}$ so that each X_i is Galois over X (the cover is of degree $|\mathrm{Aut}_X(X_i)|$).

Let $\phi_{ij} : X_j \to X_i$ is a finite étale map, and $f_i \in F(X_i)$. Then, we get a map $\psi_{ij} : \mathrm{Aut}_X(X_i) \to \mathrm{Aut}_X(X_j)$ defined by $\psi_{ij}(\sigma) f_i = \phi_{ij} \circ \sigma \circ f_j$.

Definition 5. The maps $X_j \to X_i$ give us a map $\mathrm{Aut}_X(X_i) \to \mathrm{Aut}_X(X_j)$ and we define the étale fundamental group to be

$$\pi_1^{\text{ét}}(X) = \varprojlim_{i \in I} \mathrm{Aut}_X(X_i).$$

For any $Y \in \mathsf{FEt}/X$, we have an action $\mathrm{Aut}_X(Y) \times F(Y) \to F(Y)$ given by $(\sigma, g) \mapsto \sigma \circ g$. The functor F is an equivalence of categories from FEt/X to the category of $\pi_1^{\text{ét}}(X, \bar{x})$-sets with continuous action.

Proposition 6. (a) Given any two geometric points $\bar{x}$ and $\bar{y}$, there is an isomorphism $\pi_1^{\text{ét}}(X, \bar{x}) \to \pi_1^{\text{ét}}(X, \bar{y})$, and this isomorphism is canonical up to an inner automorphism of $\pi_1^{\text{ét}}(X, \bar{x})$.

(b) $\pi_1^{\text{ét}}$ is a covariant functor of (geometric) pointed schemes $(X, \bar{x})$.

(c) Let L/k be fields of characteristic zero, and each algebraically closed. Then if X is a variety over k, we have an isomorphism $\pi_1^{\text{ét}}(X, \bar{x}) \cong \pi_1(X_L, \bar{x})$ for any $\bar{x}$ a geometric point of X_L. This is false for characteristic p, as we have the Artin-Schreier finite étale covers of $\mathbb{A}^1$ given by $Y^p - Y + \alpha$ where α is a constant, and as we replace k with a larger field we have more of these covers.

3 Examples

Example 7. Let $k = \bar{k}$. We compute $\pi_1^{\text{ét}}(\mathbb{P}^1, \bar{X})$. Let $f : Y \to \mathbb{P}^1$ be a connected finite étale cover and $\omega \in H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1})$. Then, $f^*\omega$ is a 1-form with associated divisor of degree $-2\deg(f)$. But then, $2g(Y) - 2 = -2\deg(f) < 0$ so $g(Y) = 0$ and so $\deg(f) = 1$. So, f is an isomorphism and therefore $\pi_1^{\text{ét}}(\mathbb{P}^1, \bar{X}) = 1$.

We discuss the case for $k = \mathbb{C}$. To do so, we relate the classical fundamental group to the étale fundamental group through the following theorem.

Theorem 8 (Riemann Existence Theorem). The functor sending finite étale covers $\pi : Y \to X$ to finite covering maps $\pi : Y(\mathbb{C}) \to X(\mathbb{C})$ is an equivalence of categories.

Example 9. Let X be a smooth variety over $\mathbb{C}$. Writing $\widetilde{X} = (X_i)_{i \in I}$ as before, every finite covering space of $X(\mathbb{C})$ is a quotient of some $X_i(\mathbb{C})$. We let $x \in X(\mathbb{C})$. Then,

$$\pi_1^{\text{ét}}(X, x) = \varprojlim_{i \in I} \mathrm{Aut}_X(X_i) = \varprojlim_{i \in I} \mathrm{Aut}_{X(\mathbb{C})}(X_i(\mathbb{C})) = \pi_1(X(\hat{\mathbb{C}}), x)$$

where for a group G, $\hat{G} := \varprojlim_{N \text{ finite index, normal}} G/N$ denotes its profinite completion.

Example 10. Let $X = \operatorname{Spec} k$, and $\bar{x} = \operatorname{Spec} L$ a geometric point. We let $X = (X_i)$ where $X_i = \operatorname{Spec} F_i$ with F_i an intermediate finite Galois extension of k contained in L. Then,

$$\pi_1^{\text{ét}}(X, \bar{x}) = \varprojlim \operatorname{Aut}_X(X_i) = \varprojlim \operatorname{Gal}(F_i/k) = \operatorname{Gal}(k^{\text{sep}}/k)$$

where k^{sep} is the separable closure of k in L.

Proposition 11. From Example 9, we see a general phenomenon: if we have an equivalence of categories $\mathsf{FEt}/X \leftrightarrow \mathsf{FEt}/Y$, then their étale fundamental groups are isomorphic.

Example 12. Let $X = \operatorname{Spec} A$, where A is Henselian, and $\bar{x}$ a geometric point of X. Then, the equivalence of categories $\mathsf{FEt}/X \leftrightarrow \mathsf{FEt}/\operatorname{Spec} k(x)$ gives us an isomorphism $\pi_1^{\text{ét}}(X, \bar{x}) \cong \pi_1^{\text{ét}}(\operatorname{Spec} k(x), \bar{x})$.

Example 13. Let X be a proper scheme over a Henselian local ring A such that the closed fiber X_0 is geometrically connected with point $\bar{x}$. Then, we have an equivalence of categories FEt/X and FEt/X_0 which in turn gives us isomorphisms $\pi_1^{\text{ét}}(X, x) \cong \pi_1^{\text{ét}}(X_0, \bar{x})$.

Example 14. Let X be a smooth proper scheme over $\operatorname{Spec} A$, where A is a complete DVR with residue field $k = \bar{k}$ of characteristic p. Let $K = K\bar{(A)}$ where $K(A)$ is the field of fractions of A. If X_K and the special fiber of X are connected and $\bar{x}$ is any geometric point of X_K, then $\pi_1^{\text{ét}}(X_K, \bar{x}) \to \pi_1^{\text{ét}}(X, \bar{x})$ is surjective with small kernel. That is, the kernel contained in the kernel of any homomorphism of $\pi_1^{\text{ét}}(X_K, \bar{x}) \to H$ where H is a finite group of order coprime to p.

Example 15. Let X_0 be a smooth proper curve of genus g over $k = \bar{k}$ of characteristic p. There exists a complete DVR A with residue field k and a smooth proper curve X over A such that the special fiber is X_0 with field of fractions of characteristic zero. Let K be the algebraic closure of $K(A)$. Then, $\pi_1^{\text{ét}}(X_0, \bar{x}) \cong \pi_1^{\text{ét}}(X, \bar{x})$ by Example 13. And by Example 14 we know that $\pi_1^{\text{ét}}(X_K, \bar{x}) \to \pi_1^{\text{ét}}(X, \bar{x})$ has small kernel. Furthermore, we know that $\pi_1^{\text{ét}}(X_K, \bar{x}) = \pi_1(X\hat{(\mathbb{C})}, x)$ where $X(\mathbb{C})$ is a genus g Riemann surface. In a diagram, it is given by

$$\begin{tikzcd}
\text{kernel} \arrow[r] & \pi_1^{\text{ét}}(X_K, \bar{x}) \arrow[r] \arrow[d, "\sim"'] & \pi_1^{\text{ét}}(X, \bar{x}) \arrow[d, "\sim"] \\
& \pi_1(X\hat{(\mathbb{C})}, x) & \pi_1^{\text{ét}}(X_0, \bar{x})
\end{tikzcd}$$

This lets us compute the prime-to-p part of the étale fundamental group, where the prime-to-p part is given by

$$\varprojlim_{i \in I} \operatorname{Aut}_X(X_i)$$

with (X_i) a collection of finite étale covers of degree coprime to p.

It is given by

$$\pi_1^{\text{ét}}(X_0, \bar{x})^{(p)} \cong \varprojlim_{[\pi_1(\Sigma_g)]:N] \text{ coprime to } p} \pi_1(\Sigma_g)/N$$

where $\pi_1(\Sigma_g) \cong \langle a_1, b_1, \ldots, a_g, b_g | [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$.

Example 16. The following example and argument are from Jacob Tismerman's notes on étale cohomology.

Let X be a connected and geometrically connected variety over k. Then, we have a short exact sequence

$$1 \longrightarrow \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \xrightarrow{i} \pi_1^{\text{ét}}(X, \bar{x}) \xrightarrow{j} \operatorname{Gal}(\bar{k}, k) \longrightarrow 1$$

where the map $i : \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to \pi_1^{\text{ét}}(X, \bar{x})$ is obtained by functoriality and the map $(X_{\bar{k}}, \bar{x}) \to (X, \bar{x})$.

1. We know that $j \circ i$ is trivial as the map $X_{\bar{k}} \to \operatorname{Spec} k$ factors through $\operatorname{Spec} \bar{k}$ which has trivial fundamental group.

2. We show injectivity of i.Let $g \in \ker(i)$ be a non-identity element. Then, there is some cover $Y_{\bar{k}}$ of $X_{\bar{k}}$ upon which g acts nontrivially. If we can find some cover Z_k of X whose base change $Z_{\bar{k}}$ contains $Y_{\bar{k}}$, then we have a contradiction since g must act trivially on Z_k because $i(g) = 1$.

   To construct Z_k, we note that Y is finite over $X_{\bar{k}}$ and is finite type over $\bar{k}$, so we can find a finite étale cover Y_L over X_L where L/k is a finite Galois extension of k, and whose base change via $\operatorname{Spec} \bar{k} \to \operatorname{Spec} L$ is $Y_{\bar{k}}$.

   Then for $\sigma \in \operatorname{Gal}(L/k)$, let Y_L^σ be the base change of Y_L along the map $\sigma : \operatorname{Spec} L \to \operatorname{Spec} L$. We define

   $$Z_L = \bigcup_{\sigma \in \operatorname{Gal}(L/k)} Y_L^\sigma$$

   and take $Z = Z_L / \operatorname{Gal}(L/k)$.

3. Surjectivity of j comes from the fact that X_L is a Galois cover of X_k and $\operatorname{Aut}_{X_k}(X_L) = \operatorname{Gal}(L/k)$. Hence, j surjects onto all finite quotients of $\operatorname{Gal}(\bar{k}/k)$ and therefore surjects onto $\operatorname{Gal}(\bar{k}/k)$ since $\operatorname{Gal}(\bar{k}/k)$ is a profinite group.

Example 17. Let $X = \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, \infty\}$. Then, $\pi_1^{\text{ét}}(X_{\overline{\mathbb{Q}}}) = \pi_1^{\text{ét}}(X_{\mathbb{C}}) = \hat{F}_2$ where F_2 is the free group on two elements. Then, by Example 16 we have the short exact sequence

$$1 \to \hat{F}_2 \to \pi_1^{\text{ét}}(X) \to \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1.$$

If we can understand $\pi_1^{\text{ét}}(X_{\mathbb{Q}})$ then we may be able to understand the absolute Galois group.