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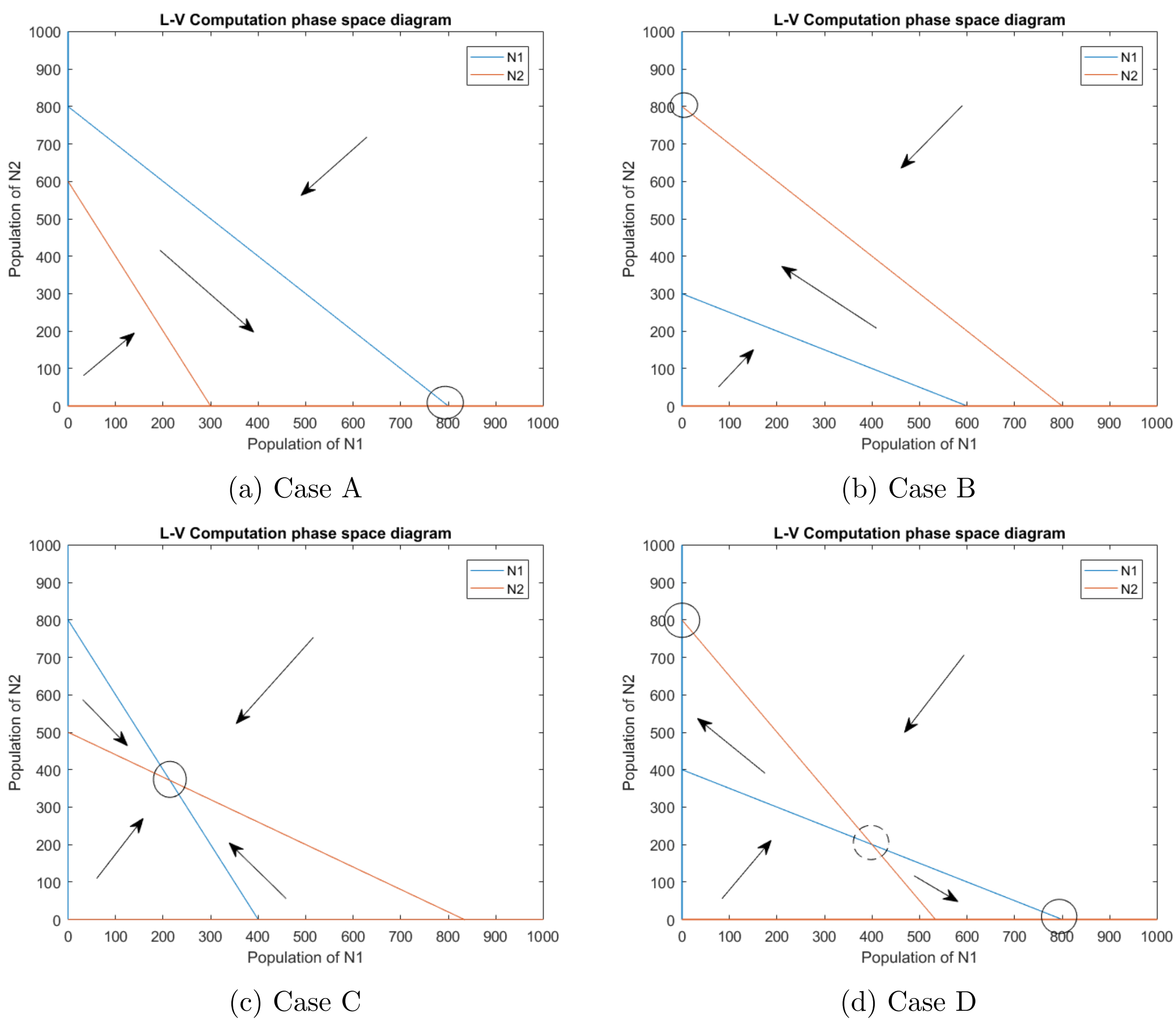
Homework 1

Problem 1

The Lotka-Volterra (L-V) competition model with two species can be expressed as:

$$N_1' = r_1 N_1 \left(\frac{k_1 - N_1 - \alpha_2 N_2}{k_1}\right)$$
$$N_2' = r_2 N_2 \left(\frac{k_2 - N_2 - \alpha_1 N_1}{k_2}\right)$$

where r_1 and r_2 represent the intrinsic rates of population increase for species 1 and 2, and k_1 and k_2 are the capacities of species 1 and 2, α_1 and α_2 indicate the effects of competition pressure from the other species, respectively.



(a) Case A

(b) Case B

(c) Case C

(d) Case D

Figure 1: Different scenarios of L-V Computation

A phase space diagram could be used to describe the equilibrium state of the L-V model, the lines represent the steady state of each species ($N_1' = 0$ or $N_2' = 0$), the arrows demonstrate the population change direction in different areas. Fig.1 shows 4 possible outcome cases:

Case A: 1 stable equilibrium point. N_1 dominate N_2.

Case B: 1 stable equilibrium point. N_2 dominate N_1.

Case C: 1 stable equilibrium point. No extinction.

Case D: 2 stable equilibrium point, and 1 unstable equilibrium point.

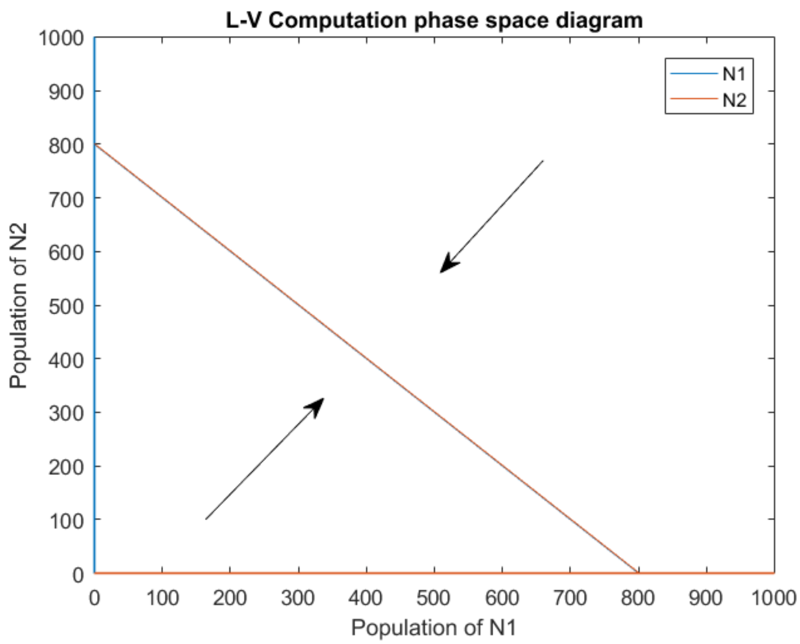


Figure 2: the final case E

There is an special case when two species have exactly same population capacity ($k_1 = k_2$) and same competition pressure effect ($\alpha_1 = \alpha_2$), the two species will be at the stable equilibrium point in anywhere on the isocline (Fig.2).

Problem 2

We have three solutions for this problem depending on the way we interpret the problem. From the problem, we know that N_2 dominates N_1. The situation is shown in Fig. 1b.

Solution 1

Let us define the healthy levels of both N_1 and N_2 as the region $\{ (N_1, N_2) \mid \dot{N}_1 > 0, \dot{N}_2 > 0\}$. Then suppose the current state at time t of the system is in $(N_1(t), N_2(t))$. After we apply the intervention at time t, the new state is $(\rho N_1(t), \rho N_2(t))$ and the dynamic equations at t becomes (substitute $(\rho N_1(t), \rho N_2(t))$ in the ODEs):

$$N_1' = r_1 N_1 (\frac{k_1 - \rho N_1 - \alpha_2 \rho N_2}{k_1}) \tag{1}$$
$$N_2' = r_2 N_2 (\frac{k_2 - \rho N_2 - \alpha_1 \rho N_1}{k_2}) \tag{2}$$

By our definition, to keep healthy levels of both N_1 and N_2, we need $N_1' > 0$ and $N_2' > 0$, i.e.

$$k_1 - \rho N_1 - \alpha_2 \rho N_2 > 0 \tag{3}$$
$$k_2 - \rho N_2 - \alpha_1 \rho N_1 > 0 \tag{4}$$

From equation (3) and (4), we have

$$\rho < \frac{k_1}{N_1 + \alpha_2 N_2}$$

Then look at the function of $f(N_1, N_2) = \frac{k_1}{N_1 + \alpha_2 N_2}$, where (N_1, N_2) is in the lower left triangle in Fig. 1b. Our goal is to find the range of ρ such that for all (N_1, N_2) in the triangle region, $\rho < f(N_1, N_2)$. The equivalent problem is a convex optimization problem defined as follow:

maximize: $N_1 + \alpha_2 N_2$
subject to (N_1, N_2) in the triangle region

The theory of convex optimization tells us that the optimal points occur at the vertex when the decision variables are in the region of a polyhedron, in this case, a triangle.

Thus the solution is

$$0 < \rho < \min(\frac{k_1}{\alpha_2 K_2}, \frac{\alpha_1 K_1}{K_2})$$

The conclusion is that if the ρ is in the range specified above, the colleagues suggestion can work.

Solution 2

Since we can apply the intervention any time we like, it does not matter what the value of ρ is. There are only two cases that need to be considered:

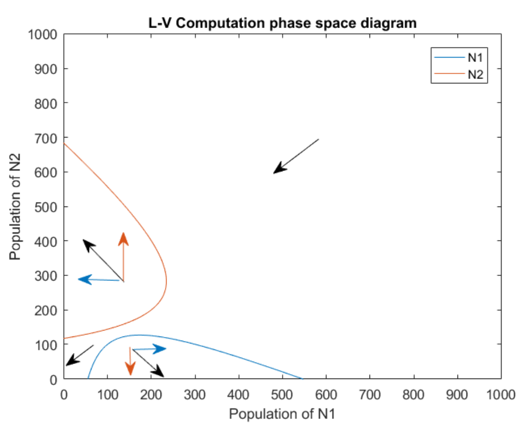
- when (N_1, N_2) is not in the lower triangle, then we can use any $0 < \rho < 1$ to bring the state variables into the lower triangle (the healthy levels for bot N_1 and N_2) by applying this intervention certain times.
- when (N_1, N_2) is in the lower triangle, we can wait until the (N_1, N_2) goes out of the lower triangle and then apply the intervention to bring (N_1, N_2) back to the lower triangle.

Thus the range of ρ is $(0, 1)$.

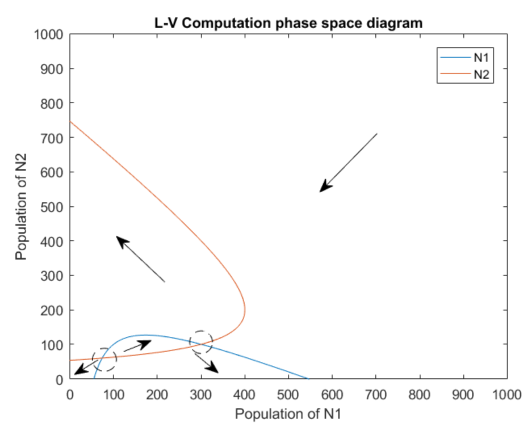
Solution 3

If N_1 and N_2 were both reduced by a constant ρ each time, the new L-V computation could be derived as:

$$N_1' = r_1 N_1 (\frac{k_1 - N_1 - \alpha_2 N_2}{k_1}) - \rho$$
$$N_2' = r_2 N_2 (\frac{k_2 - N_2 - \alpha_1 N_1}{k_2}) - \rho$$



(a) Case A



(b) Case B

Figure 3: Phase space diagram of L-V computation with intervention

The colleague's suggestion will not help preserve healthy levels of both N1 and N2. With the intervention method, there will be two possible outcomes (fig. 3). In the case A, there is no equilibrium point, which means ether one or both of N_1 and N_2 will die out. In the case B, there are two unstable equilibrium points (could be one), the population changes are not direct to the equilibrium points. Unless the populations are at the equilibrium initially, ether one or both ofN_1 and N_2 will be extinct eventually.

Problem 3

To solve a differential equation, the known parameters should include: the objective equation $y' = f(y, t)$, the initial state $y(t_0)$ and t_0, and the end time t_e. The following algorithms show the functions in Euler's method and Heun's method in Python.

Algorithm 1: Euler's method

```
def euler(f,y0,t0,te,h):
        t,y = t0,y0
        while t < te:
         y = y+h * f(t,y)
         t = t+h
```

Algorithm 2: Heun's method

```
def heun(f,y0,t0,te,h):
        t,y = t0,y0
        while t < te:
         y_hat = y+h * f(t,y)
         y = y + (f(t,y)+f(t+h,y_hat))*h/2
         t = t+h
```

The relationship between Euler's Method and a discrete time model:

Euler's Method is equivalent to a discrete time model in the sense that after the step size used in Euler's method is fixed, the differential equations for a model become difference equations.

$$\frac{dx}{dt} \approx \frac{\Delta x}{\Delta t}$$

Problem 4

The simulation results of the infected individuals I change over time are shown in fig. 5. We can observe that:

1. For some combination pairs of (β, h), the plots produced by the two methods do not have noticeable difference, such as the first column of the 9 plots ($h = 0.01$). When the time step h is small enough, both methods are able to calculate the differential model.

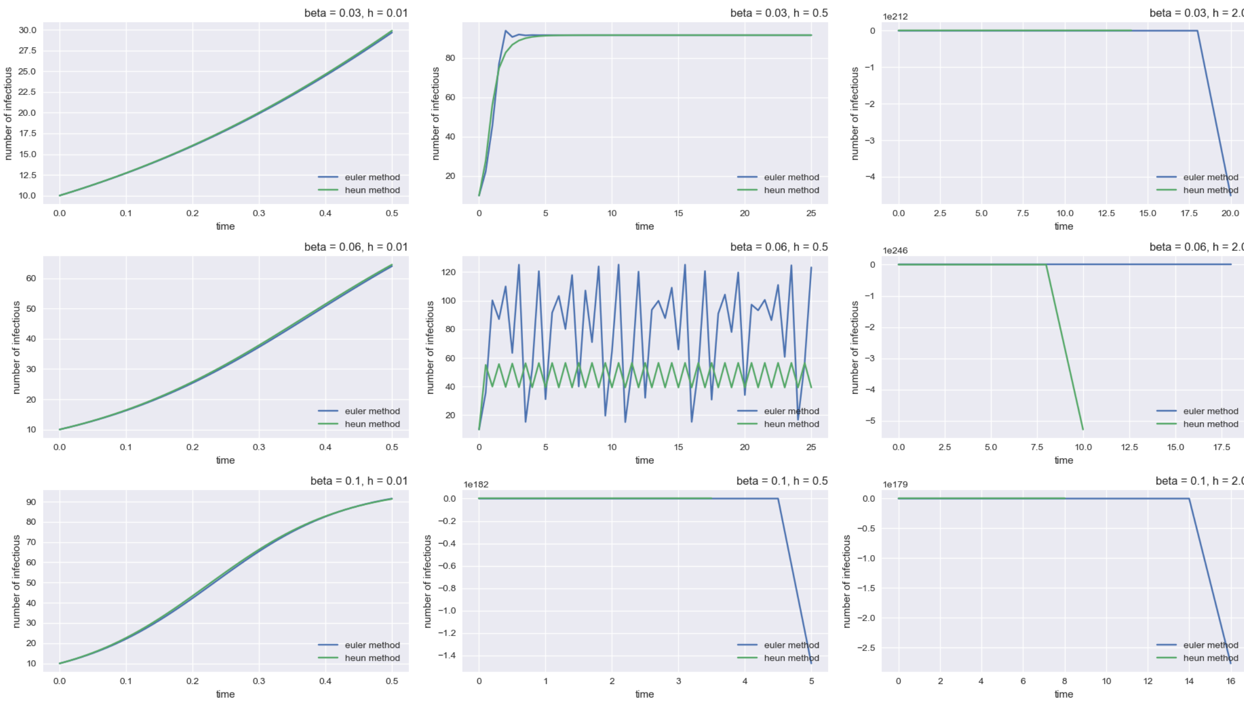


Figure 4: Simulation with different step sizes and β values

2. A noticeable difference appears in the second plot ($\beta = 0.03, h = 0.5$). The plot of Euler's method shows some bouncing near equilibrium point, which is incorrect since the SIS model is always monotonic. The plot of Heun's method shows a more accurate expedition. In both methods, the infected individuals approaching to the equilibrium point $I^\star \approx 92$(the precise value should be $100 - \gamma/\beta = 95.8$).

3. When the step size larger, like $h = 2.0$, the simulations of both methods shown negative results. Which means the changes of the infected individuals cannot be observed with this large time interval.

4. In the middle plot when $\beta = 0.06, h = 0.5$, the plots of both methods interestingly shown some oscillation. Still, the variation of the infected individuals is unable to be observed.

Problem 5

Theorem (Global Precision of Heun's Method)**.** *The global precision of Heun's Method for solving an Initial Value Problem is in* $\mathcal{O}(h^2)$

Proof.

- To begin with, let us restate the Heun's Method.

$$\begin{aligned}\hat{x}(t_{i+1}) &= x(t_i) + hf(x(t_i))\\ x(t_{i+1}) &= x(t_i) + h/2(f(x(t_i)) + f(\hat{x}(t_{i+1})))\\ t_{i+1} &= t_i + h\end{aligned}$$

where i goes from 0 to N - 1, h is the step size.

- Next, we will use Taylor's Theorem to $x(t)$ at $t = t_i$ and then compute $x(t)$ at $t = t_{i+1}$:

$$x(t_{i+1}) = x(t_i) + hx^{(1)}(t_i) + \frac{h^2}{2}x^{(2)}(t_i) + \frac{h^3}{6}x^{(3)}(\tau_i) \text{ where } \tau_i \in [t_i, t_{i+1}] \tag{5}$$

Since in Heun's Method there is no second order derivative for x, we can apply Taylor's Theorem again to eliminate that term in equation (1):

$$x^{(2)}(t_i) = \frac{x^{(1)}(t_{i+1}) - x^{(1)}(t_i)}{h} - \frac{x^{(3)}(\tau_i)h}{2} \text{ where } \tau_i \in [t_i, t_{i+1}] \tag{6}$$

Using equation (2), substitute $x^{(2)}(t_i)$ in equation (1) we get:

$$x(t_{i+1}) = x(t_i) + \frac{h}{2}x^{(1)}(t_i) + \frac{h}{2}x^{(1)}(t_{i+1}) + \mathcal{O}(h^3) \tag{7}$$

Again, comparing equation (3) with Heun's Method, we need to deal with the term $x^{(1)}(t_{i+1})$. What's that? The trick is again to apply Taylor's Theorem to the argument of the function f.

$$x^{(1)}(t_{i+1}) = f(x(t_{i+1})) = f(x(t_i) + hx^{(1)}(t_i) + \frac{h^2}{2}x^{(2)}(\tau_i)) = f(\hat{x}(t_{i+1})) + \mathcal{O}(h^2) \tag{8}$$

in the last step, we apply Taylor's Theorem to the function f at $t = t_{i+1}$.

Next, we will substitute equation (4) into equation (3) and get the local error:

$$x(t_{i+1})) = x(t_i) + h/2(f(x(t_i)) + f(\hat{x}(t_{i+1}))) + \mathcal{O}(h^3) \tag{9}$$

Thus the global error of Heun's Method at time step $t = t_N$ is $\mathcal{O}(h^3) \times \frac{b-a}{h} = \mathcal{O}(h^2)$

$\square$

Problem 6

- Why Lotka-Volterra predator-prey model has cycles but SIS model does not?
  Looking at the L-V predator-prey model equation:

  $$F' = aF - bFS$$
  $$S' = -cS + dFS$$

  we can see that, on the one hand, if "Fish" gets more, then "Sharks" will get more and if "Sharks" get more, "Fish" will become less; On the other hand, when "Fish" becomes less, "Sharks" will also become less, which will lead to more "Fish". From a system point of view, The two ODEs in the L-V predator-prey model kind of describe a negative feedback system. The majority will be always in disadvantage and the minority will be in advantage. Thus, cycles can be observed in this model.

- With the SIS model:

  $$S' = \gamma I - \beta SI$$
  $$I' = \beta SI - \gamma I$$
  $$N = S + I$$

  Since S can be expressed as $N - I$. There is actually one parameter in this model. And thus the one dimensional model is always monotonic and no oscillation will occurs.

- How about SIR and SIRS system? The SIR model should not show cycles because there is no feedback that make each variable oscillate. The final state of the system should contain only recovered or susceptible people.

  The SIRS model should show cycles since the 3 states of S, I and R interact with each other in the form of $S \implies I \implies R \implies S$.

- Should we expect chaos in classic epidemic models?
  Chaos is not possible in classic epidemic models since the classic epidemic models only have at most 2-dimensions. A 2-dimensional system will not show chaos.

Problem 7

Three Body Problem

The "three body problem" is described in a sci-fi book by Chinese author Liu Cixin. In the book, he describes an alien civilization called "Trisolarians" living on a planet surrounded by 3 suns. It is unlike our solar system in which only one sun exists and therefore our solar system is stable. Unlucky for them, due to the unpredictable motion of the 3 suns, they might have long winters or long summers. Their climate is in a chaos state.

Mathematical Model for Three Body Problem

Assumptions of our three body system:

- 3 suns with comparable mass: m_1, m_2 and m_3.
- collision is ignored, i.e. when the suns collide, they simple pass through each other.
- the shapes of the suns are ignored, and therefore they are all considered as points with mass.
- only gravitation is considered and it follows the law of Newton's Gravitation
- initial speed should not be too large so that the suns don't escape from each other.
- the 3 suns move in 3-D space.

Based on Newton's Law of Gravitation and Motion, the mathematical model of the three body problem is as follows.

$$
\begin{aligned}
\ddot{\mathbf{r}}_{\mathbf{1}} &= -Gm_2 \frac{\mathbf{r_1} - \mathbf{r_2}}{|\mathbf{r_1} - \mathbf{r_2}|^3} - Gm_3 \frac{\mathbf{r_1} - \mathbf{r_3}}{|\mathbf{r_1} - \mathbf{r_3}|^3}, \\
\ddot{\mathbf{r}}_{\mathbf{2}} &= -Gm_3 \frac{\mathbf{r_2} - \mathbf{r_3}}{|\mathbf{r_2} - \mathbf{r_3}|^3} - Gm_1 \frac{\mathbf{r_2} - \mathbf{r_1}}{|\mathbf{r_2} - \mathbf{r_1}|^3}, \\
\ddot{\mathbf{r}}_{\mathbf{3}} &= -Gm_1 \frac{\mathbf{r_3} - \mathbf{r_1}}{|\mathbf{r_3} - \mathbf{r_1}|^3} - Gm_2 \frac{\mathbf{r_3} - \mathbf{r_2}}{|\mathbf{r_3} - \mathbf{r_2}|^3}.
\end{aligned}
$$

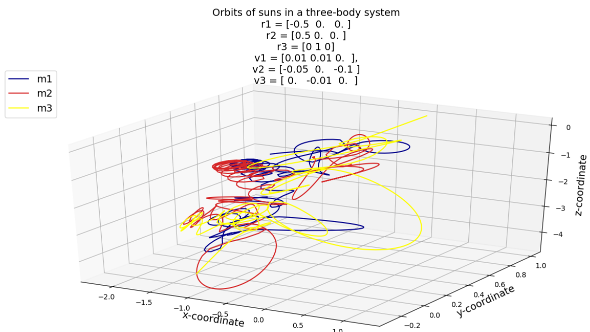
Figure 5: Mathematical Description of Three Body Problem

We can change the model to only include first order ODEs by introducing the speed variables. Define $\dot{r_i} = v_i$ for $i = 1, 2, 3$. Then the model becomes first order ODEs.

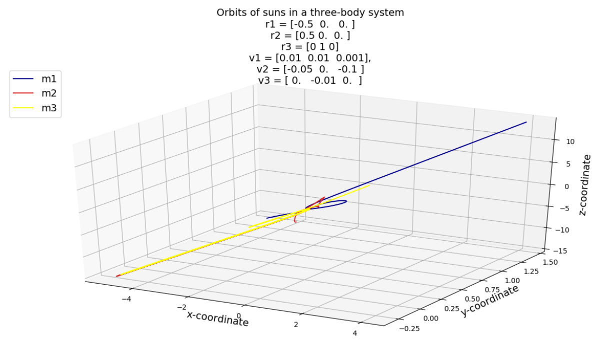
In summary,

- the state variables of our model are $r_1, r_2, r_3, v_1, v_2, v_3$, where they are all 3-vector.
- the parameters are the mass m_1, m_2, m_3 and the initial state, i.e. their initial positions and velocities.

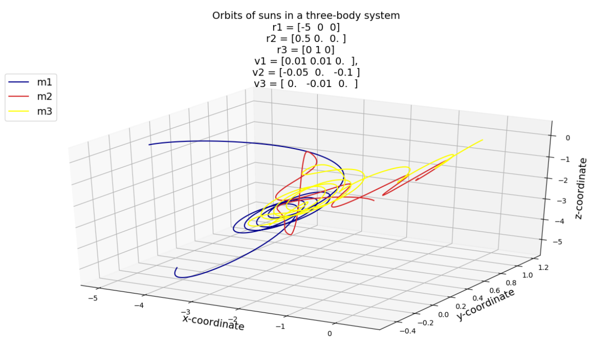
Simulation Result Using Scipy's odeint Solver



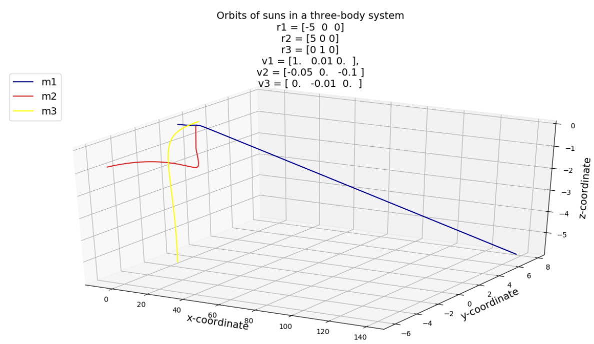
(a) Simulation 1



(b) Simulation 2



(c) Simulation 3



(d) Simulation 4

Discussion

- The system looks very chaos. The only different between Fig.6 and Fig.7 is that the initial velocity of m1 in Fig.7 is $v_1 = (0.01, 0.01, 0)$ and the initial velocity of m1 in Fig. 6 is $v_1 = (0.01, 0.01, 0.001)$. Little difference in initial condition leads to completely different trajectories.
- There seems to be no cycles or stable states in the four simulations above.
- The model could have cycles if we carefully set the initial conditions and the mass of the three suns.
- If there do exist a three suns system, the situation might be really like what the book describes and it will be very hard for civilization to exist and prosper.