Modeling Complex Systems (CS/CSYS 302), Fall 2019

Assignment #3 of 4
Done in groups of 3. Due on Blackboard by midnight on Friday, November 1st. Your write-up should contain everything but the codes, such as any necessary figures as well as your answers to the questions. Make sure to justify your answers to each question. Codes should be submitted separately as readable text files (e.g. .py, .cpp, .m, .R).

Random Walks

A *random walk* is a sequence $(S_n)_{n\geq 0}$ over $\mathbb{Z}^d$, $d \geq 1$, where

$$S_0 = z,$$
$$S_n = S_{n-1} + X_n, \quad n \geq 1$$

and $X_1, X_2 \ldots,$ is a sequence of i.i.d random variables in $\mathbb{Z}^d$.

1. (Central limit theorem) A *symmetric simple random walk* over $\mathbb{Z}^d$ satisfies the property $X_n \sim \text{Unif}\{i = 1, \ldots d : e_i\}$, where e_i is the ith basis vector in $\mathbb{Z}^d$. A symmetric SRW steps only a single neighbor away from its current position and selects from each neighbor uniformly. In class, we briefly covered the phenomenon that $\Pr(\tau < \infty) = 1$ when $d \in \{1, 2\}$, and $\Pr(\tau < \infty) < 1$ for $d \geq 3$, where τ is the time for a symmetric SRW random walk to revisit its origin point, i.e. $\tau = \min_{n\geq 1}\{S_n = z\}$.

   Implement a symmetric SRW over $\mathbb{Z}$. For $N \in \mathbb{N}$ and $z \in \mathbb{Z}$, use your code to draw M samples $S_N^{(1)}, \ldots, S_N^{(M)}$ and plot the resulting histogram for the points

   $$\frac{S_N^{(1)}}{\sqrt{N}}, \ldots, \frac{S_N^{(M)}}{\sqrt{N}}.$$

   What are the effects of N and M on the plot? What do you think the mean and variance of the resulting distribution is?

2. (Gambler's ruin theorem) Suppose you and your friend are playing the following game: in each round, your friend flips a biased coin, which is heads with probability p. If the coin is heads, you gain a dollar, and if tails, you lose a dollar. The game ends when you run out of money or you reach a fixed winning value W, which you get to keep if you win. Suppose you have an initial $0 < z < W$ dollars to begin the game with.

   Let S_n denote your money in round n and $T = \min\{n : S_n = 0, S_n = W\}$ be the time it takes for the game to end. What is the probability that you eventually win? Prove your conclusion with rigorous arguments.

3. Prove $\Pr(T = \infty) = 0$. Together with your answer to the previous question, what does this say about your "friend"?

Networks (Part 1)

4. The *cascade model* is a simple toy model to help us think about energy flow in directed acyclic graph such as food webs and some power grids. Imagine N nodes with indices $i = 1, \ldots, N$ and place an undirected edge between each unique pair with probability p just as in the Erdős-Rényi random graph seen in class. Now add a direction to every edge such that each edge goes from node i to node $i' < i$. The index i then informs us of the position of a node in a hierarchy (e.g. trophic level in food webs, or distance from source in power grids) and the model can be used to help us understand the distribution of energy flow through the hierarchy.

   a) What is the average in-degree (edges going in) of vertex i? What is the average out-degree (edges going out) of vertex i?

   b) Show that the expected number of edges that run from nodes $i' > i$ to nodes $i' \leq i$ is $(ni - i^2)p$.

   c) Assuming N is even, what are the largest and smallest values of the quantity calculated in *c)* and where do they occur (in i).

5. While Poisson or Erdős-Rényi networks correspond almost exactly to well-mixed systems, *degree-regular networks* are the simplest generalization of cellular automata to networks. Degree-regular networks are essentially a version of the configuration model seen in class but where every node has the same degree k. Hence, if we randomized the connections between every site in an infinite square grid, we would end up with an infinite degree-regular network of degree 4.

   a) What is the degree distribution $\{p_k\}$? What are the generating functions $G_0(x)$ and $G_1(x)$ for the degree and excess degree distributions?

   b) Find the minimal value of k that is still large enough to expect that the entire network will be connected in one single, giant, component.

   c) Explain what happens for the different values of k smaller than the value calculated in *b)*.

6. Pick a network of your choice on the Colorado Index of Complex Networks. Play around with network visualization, and take note of what information you can glean from visualizing the network. Describe the network: What are the nodes? What are the edges? What is the average degree? What are some other static network measures, and why are they significant to this system? (You will have the chance to use this network again in the next assignment).