Structured Traversals for (Multiply) Recursive Algebraic Datatypes

G. Cassian Alexandru

January 19, 2021

Presentation generated from `.lhs` sources using `lhs2TeX`

Context & Conventions

- Language: Haskell, with numerous language extensions
- Syntactic (e.g. `LambdaCase`)
- Clarifying (e.g. `TypeApplications, InstanceSigs`)
- Limited Dependent programming (e.g. `DataKinds`), for multiple recursion

Context & Conventions

- Language: Haskell, with numerous language extensions
- Syntactic (e.g. `LambdaCase`)
- Clarifying (e.g. `TypeApplications, InstanceSigs`)
- Limited Dependent programming (e.g. `DataKinds`), for multiple recursion

```
{-# LANGUAGE LambdaCase #-}
foo :: [a] → b
foo = λcase
  [] → ...
  (x:xs) → ...
```

Context & Conventions

- Language: Haskell, with numerous language extensions
- Syntactic (e.g. `LambdaCase`)
- Clarifying (e.g. `TypeApplications, InstanceSigs`)
- Limited Dependent programming (e.g. `DataKinds`), for multiple recursion

```
{-# LANGUAGE LambdaCase #-}
foo :: [a] → b
foo = λcase
  [] → ...
  (x:xs) → ...
```

- Composition in diagrammatic order: $f ; g$ reads "f, then g"
- Haskell: `f .> g`

Structure

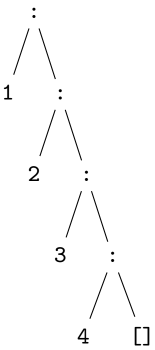
```
length :: [a] → Int
length = λcase
[] → 0
(_:xs) → 1 + length xs

filter :: (a → Bool) → [a] → [a]
filter p = go where
  go [] = []
  go (x:xs) = if p x then [x] else [] ++ go xs
```

```
length :: [a] → Int
length = λcase
[] → 0
(_:xs) → 1 + length xs

filter :: (a → Bool) → [a] → [a]
filter p = go where
  go [] = []
  go (x:xs) = if p x then [x] else [] ++ go xs
```

```haskell
length :: [a] → Int
length = λcase
[] → 0
(_:xs) → 1 + length xs

filter :: (a → Bool) → [a] → [a]
filter p = go where
  go [] = []
  go (x:xs) = if p x then [x] else [] ++ go xs
```

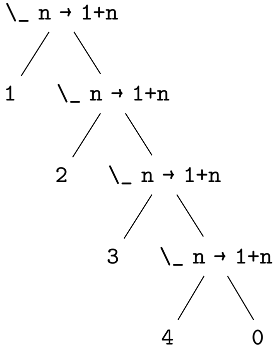
- List Design pattern?
- Design Patterns are a poor man's abstraction
- Recognize common structure & find correct abstract notion

List

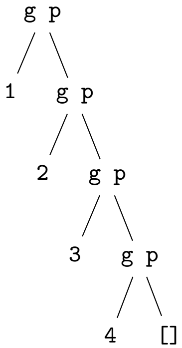


Traversals

length



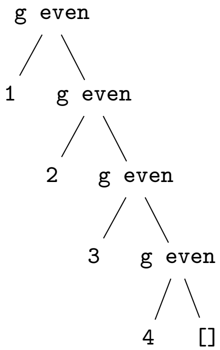
filter p




```
g p x xs =
  (if p x then [x] else [])
  ++ xs
```

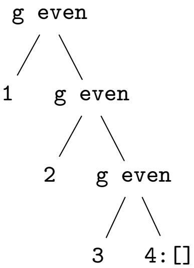
Example Evaluation of `filter even`

```
g p x xs =
  (if p x then [x] else []) ++ xs
```



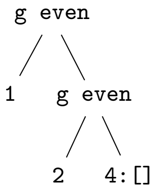
Example Evaluation of `filter even`

```
g p x xs =
  (if p x then [x] else []) ++ xs
```



Example Evaluation of `filter even`

```
g p x xs =
  (if p x then [x] else []) ++ xs
```



Example Evaluation of `filter even`

```
g p x xs =
  (if p x then [x] else []) ++ xs
```

```
g even
1    2:4:[]
```

Example Evaluation of `filter even`

```
g p x xs =
  (if p x then [x] else []) ++ xs
```

```
2:4:[]
```

Insight

Even though the functions were defined recursively, their behaviour can be understood non-recursively as simply replacing the two constructors for `[a]` by functions of the same arity.

```haskell
data List a = Nil | Cons a (List a)
```

GADT Syntax:

```
data List a where
  Nil :: List a
  Cons :: a → (List a) → (List a)
```

GADT Syntax:

```
data List a where
  Nil :: List a
  Cons :: a → (List a) → (List a)

list :: b → (a → b → b) → List a → b
list nil cons = fold where
  fold Nil = nil
  fold (x `Cons` xs) = x `cons` fold xs

length' = list 0 (\_ n → 1+n)
filter' p = list []
  (λx xs → (if p x then [x] else []) ++ xs)
```

GADT Syntax:

```
data List a where
  Nil :: List a
  Cons :: a → (List a) → (List a)

list :: b → (a → b → b) → List a → b
list nil cons = fold where
  fold Nil = nil
  fold (x `Cons` xs) = x `cons` fold xs

length' = list 0 (\_ n → 1+n)
filter' p = list []
  (λx xs → (if p x then [x] else []) ++ xs)
```

Now for Expressions

```
data Expr where
  Lit :: Int → Expr
  Plus :: Expr → Expr → Expr

expr :: (Int → b) → (b → b → b) → Expr → b
expr lit plus = fold where
  fold (Lit i) = lit i
  fold (l ‘Plus‘ r) = (fold l) ‘plus‘ (fold r)
```

- We want to define a polytypic "fold", subsuming `list`, `expr`, which encapsulates the whole "replace constructors with functions" pattern
- We need a deeper understanding of what the datatypes we are working with *are*
- $\rightsquigarrow$ Introduce a little ~~Anarchy~~Category Theory

Category

A category $\mathcal{C}$ consists of collections $\mathcal{C}_0$ of objects and $\mathcal{C}_1$ of morphisms (or arrows) between them, with the following structure:

- For every two arrows $f : A \to B$, $g : B \to C$, there is a composite arrow $f ; g : A \to C$
- For every object $A : \mathcal{C}_0$ there is an identity morphism $1_A : A \to A$
- Such that the following hold:
  - Composition is associative, that is: $f ; (g ; h) = (f ; g) ; h$.
  - Composition satisfies unit laws: For every $f : A \to B$. $id_A ; f = f$, $f ; id_B = f$.

Isomorphisms

Given a category C and two objects $A, B : \mathcal{C}_0$, we say A and B are isomorphic via $f : A \to B$, if there exists a $g : B \to A$ which is both a left- and right-inverse:

$$
\begin{tikzcd}
& B \arrow[dr, "g"] & \\
A \arrow[ur, "f"] \arrow[rr, "1_A"] & & A
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \arrow[dr, "f"] & \\
B \arrow[ur, "g"] \arrow[rr, "1_B"] & & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[loop left, "1_A"] \arrow[r, bend left, "f"] & B \arrow[l, bend left, "g"] \arrow[loop right, "1_B"]
\end{tikzcd}
$$

Functors

Let $\mathcal{C}, \mathcal{D}$ be Categories. A *Functor F* is a pair of maps (F_0, F_1), on the objects and morphisms of the category respectively, such that commuting diagrams are preserved, e.g.:

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"] & & C \end{tikzcd} \quad \overset{F}{\Rightarrow} \quad \begin{tikzcd} & F_0 B \arrow[dr, "F_1 g"] & \\ F_0 A \arrow[ur, "F_1 f"] \arrow[rr, "F_1 h"] & & F_0 C \end{tikzcd}$$

In particular, this means identities & composition are preserved.

Functors

Let $\mathcal{C}, \mathcal{D}$ be Categories. A *Functor F* is a pair of maps (F_0, F_1), on the objects and morphisms of the category respectively, such that commuting diagrams are preserved, e.g.:

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"] & & C \end{tikzcd} \quad \stackrel{F}{\Rightarrow} \quad \begin{tikzcd} & F_0 B \arrow[dr, "F_1 g"] & \\ F_0 A \arrow[ur, "F_1 f"] \arrow[rr, "F_1 h"] & & F_0 C \end{tikzcd}$$

In particular, this means identities & composition are preserved. We will often only write out the definition of a functor on objects.

Functors

Let $\mathcal{C}, \mathcal{D}$ be Categories. A *Functor* F is a pair of maps (F_0, F_1), on the objects and morphisms of the category respectively, such that commuting diagrams are preserved, e.g.:

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "h"] & & C \end{tikzcd} \quad \overset{F}{\Rightarrow} \quad \begin{tikzcd} & F_0 B \arrow[dr, "F_1 g"] & \\ F_0 A \arrow[ur, "F_1 f"] \arrow[rr, "F_1 h"] & & F_0 C \end{tikzcd}$$

In particular, this means identities & composition are preserved. We will often only write out the definition of a functor on objects. When $\mathcal{C} = \mathcal{D}$, we say F is an *Endofunctor*.

Building the Functor kit

- Identity ($IX := X$) is a functor.
- Constant-to-A ($K_A X := A$), for $A : C_0$, is a functor.

Categories can have products ($\times_{\mathcal{C}}$) and/or coproducts ($+_{\mathcal{C}}$). Think of coproducts as indexed unions in our case. Then if F, G are functors so are:

- $(F \times G)X := FX \times GX$
- $(F + G)X := FX + GX$

Algebra

Let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor $A : \mathcal{C}$, $\varphi : FA \to A$. Then $FA \xrightarrow{\varphi} A$ (or (A, φ)) is an *Algebra*, and A its *Carrier*.

Algebra

$$\begin{tikzcd} FA \arrow[d, "\varphi"] \\ A \end{tikzcd}$$

Algebra-Hom:
$(A, \varphi) \xrightarrow{f} (B, \psi)$

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\varphi"] & FB \arrow[d, "\psi"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Initial Algebra: (A, α)
s.t. $\forall (B, \psi)$.

$$\begin{tikzcd} FA \arrow[r, dashed, "Fh"] \arrow[d, "\alpha"] & FB \arrow[d, "\psi"] \\ A \arrow[r, dashed, "h"] & B \end{tikzcd}$$

Lambek's Lemma

If F has an initial algebra (A, α), then A is isomorphic to FA via α. Proof (We show only $h; \alpha = id_A$): Consider the algebra $(FA, F\alpha)$:

$$\begin{tikzcd} FA \arrow[d, "\alpha"] \\ A \end{tikzcd}$$

Lambek's Lemma

If F has an initial algebra (A, α), then A is isomorphic to FA via α. Proof (We show only $h; \alpha = id_A$): Consider the algebra $(FA, F\alpha)$:

$$\begin{tikzcd} FA \arrow[r, dashed, "Fh"] \arrow[d, "\alpha"] & F(FA) \arrow[d, "F\alpha"] \\ A \arrow[r, dashed, "h"] & FA \end{tikzcd}$$

Lambek's Lemma

If F has an initial algebra (A, α), then A is isomorphic to FA via α. Proof (We show only $h; \alpha = id_A$): Consider the algebra $(FA, F\alpha)$:

$$\begin{tikzcd}
FA \arrow[r, dashed, "Fh"] \arrow[d, "\alpha"] & F(FA) \arrow[r, "F\alpha"] \arrow[d, "F\alpha"] & FX \arrow[d, "\alpha"] \\
A \arrow[r, dashed, "h"] & FA \arrow[r, "\alpha"] & X
\end{tikzcd}$$

Lambek's Lemma

If F has an initial algebra (A, α), then A is isomorphic to FA via α. Proof (We show only $h; \alpha = id_A$): Consider the algebra $(FA, F\alpha)$:

$$
\begin{tikzcd}
FA \arrow[r, dashed, "Fh"] \arrow[rr, bend left, "F1_A"] \arrow[d, "\alpha"] & F(FA) \arrow[r, "F\alpha"] \arrow[d, "F\alpha"] & FX \arrow[d, "\alpha"] \\
A \arrow[r, dashed, "h"] \arrow[rr, bend right, "1_A"'] & FA \arrow[r, "\alpha"] & X
\end{tikzcd}
$$

Fixed Point

The carrier A of the initial algebra (A, α) of a functor F is a least fixed point of F. Least, that is, in that there is a morphism from it to any other algebra, by initiality.

Fixed Point

The carrier A of the initial algebra (A, α) of a functor F is a least fixed point of F. Least, that is, in that there is a morphism from it to any other algebra, by initiality.

$$\begin{tikzcd} FA \arrow[r, dashed, "Fh"] \arrow[d, shift left, "\alpha"] \arrow[dr, phantom, "\circlearrowleft"] & FB \arrow[d, "\psi"] \\ A \arrow[u, shift left, "\alpha^{-1}"] \arrow[r, dashed, "h"] & B \end{tikzcd}$$

We get a recursive definition for h: $h = \alpha^{-1}; Fh; \psi$

Back Again

We will now return to the motivating problem and see how what we have just learned is applicable to its solution. In particular, we will see that:

- Our recursive datatypes correspond to fixpoints of associated "structural/base" functors, and are carriers of their initial algebras
- The non-recursive business logic of the traversals, that is, the functions to replace the constructors, correspond to algebras for this functor
- The morphism h we get given (A, ψ) is the polytypic *fold* we were looking for

Functors in Haskell

- Haskell can be seen as a category, where the objects are types and the arrows functions between them.
- Endofunctors in Haskell can be implemented as type constructors $(* \to *)$
- Definition on arrows `a → b` via typeclass *Functor*, defining function `fmap`
- $F_1(h : A \to B) : F_0 A \to F_0 B \quad \sim$ `fmap @F (h :: A → B) :: F A → F B`

Structural Functors

- We obtain the structural functors for our datatypes by factoring the recursion out of their definition, then adding it back in via a fixed-point operator.
- We compare with how this can be done on the value level:

```
type Endo a = a → a
-- Recursive Definition:
fac :: Endo Int
fac n = n * fac (n-1)
-- Fixpoint operator:
fix :: (a → a) → a
fix f = f (fix f)
-- Factoring Fac:
fac' :: Endo Int
fac' = fix g where
  g :: Endo Int → Endo Int
  g f n = n * f (n-1)
```

Recall our list datatype:

```
data List a :: * =
  Nil | Cons a (List a)

-- Fixpoint operator (typelevel)
type Fix :: (* → *) → *
newtype Fix f = In (f (Fix f))
-- Factoring List
type List' a = Fix (ListF a)
data ListF a l =
  NilF | ConsF a l
```

Structural Functors

- We obtain the structural functors for our datatypes by factoring the recursion out of their definition, then adding it back in via a fixed-point operator.
- We compare with how this can be done on the value level:

```
type Endo a = a → a
-- Recursive Definition:
fac :: Endo Int
fac n = n * fac (n-1)
-- Fixpoint operator:
fix :: (a → a) → a
fix f = f (fix f)
-- Factoring Fac:
fac' :: Endo Int
fac' = fix g where
  g :: Endo Int → Endo Int
  g f n = n * f (n-1)
```

Recall our list datatype:

```
data List a :: * =
  Nil | Cons a (List a)

-- Fixpoint operator (typelevel)
type Fix :: (* → *) → *
newtype Fix f = In (f (Fix f))
-- Factoring List
type List' a = Fix (ListF a)
data ListF a l =
  NilF | ConsF a l
```

Business Logic as Algebra

We can store the functions meant to replace the constructors of a type as an algebra (using the transformation $A^B \times A^C \sim A^{B+C}$):

```
type Algebra f a = f a → a
listBL :: b → (a → b → b) → Algebra (ListF a) b
listBL nil cons = λcase
  NilF → nil
  x `ConsF` b → x `cons` b
```

Defining a Functor instance

- Do we have to manually define Functor instances?
- Not if working with *polynomial functors*
- Recall the functor kit from theory

```
type ListF' a l = (K () :+: K a :×: I) l
inG :: ListF a l → ListF' a l
inG = λcase
  NilF → InL $ K ()
  a `ConsF` l → InR (K a :×: I l)
outG :: ListF' a l → ListF a l
outG = λcase
  InL _ → NilF
  InR (K a :×: I l) → a `ConsF` l
instance Functor (ListF a) where
  fmap f = inG .> fmap f .> outG
```

Generic/Polytypic Programming

- `fmap` is a polytypic function
- Our iso to the generic representation is still boilerplate, though
- Some essentially polytypic functions derivable in Haskell, (e.g. `(≡)`, via **`deriving`** *Eq*), `fmap` using
  {-# LANGUAGE DeriveFunctor #-}
- Whole topic on its own

Cigar!

Recall the recursive definition $h : A \to B = \alpha^{-1}; Fh; \psi$. To implement it, we only still lack $\alpha^{-1} : A \to FA$. Recall that a carrier of the initial algebra for functor F is $Fix\ F$. `unFix :: Fix f → f (Fix f)` is easily defined:

```
unFix (In f) = f
```

- Finally, all parts are asembled for our polytypic *fold* function!
- Also called a *catamorphism*, like "cataclysm": Collapses a structure into a value (which of course can also again be a structure)

```
cata :: Functor f => Algebra f b → (Fix f) → b
cata ψ = unFix .> fmap (cata ψ) .> ψ
```