Structured Traversals for ((Multiply) Recursive) Algebraic Datatypes

G. Cassian Alexandru

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Presentation generated from `.lhs` sources using `lhs2TeX`

```haskell
length :: [a] → Int
length = λcase
[] → 0
(x:xs) → 1 + length xs

filter :: (a → Bool) → [a] → [a]
filter p = go where
  go [] = []
  go (x:xs) = if p x then [x] else [] ++ go xs
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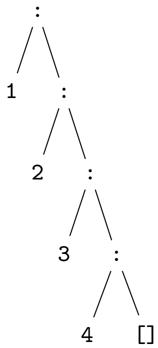
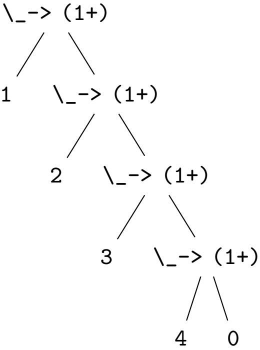
- List Design pattern?
- Design Patterns are a poor man's abstraction
- Recognize common structure & find correct abstract notion

Traversals

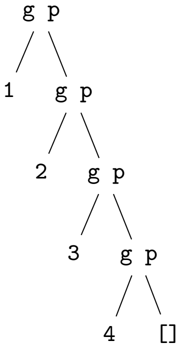
List



`length`



`filter p`




```
g p x xs =
  bool [] [x] (p x) ++ xs
```

Example Evaluation of `filter even`

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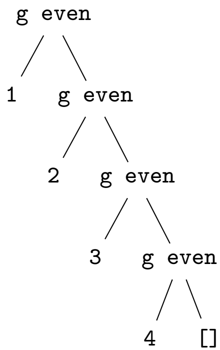

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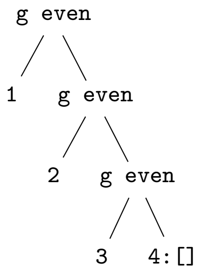

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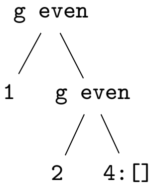

Example Evaluation of `filter even`

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```

```
  g even
   /  \
1    2:4:[]
```

Example Evaluation of filter even

```
g p x xs =
  bool [] [x] (p x) ++ xs
```

```
2:4:[]
```

```
data List a = Nil | Cons a (List a)
data BooL = TT | FF
```

GADT Syntax:

```
data List a where
  Nil :: List a
  Cons :: a → (List a) → (List a)
data Bool where
  FF :: Bool
  TT :: Bool
```

```
data List a where
  Nil :: List a
  Cons :: a → (List a) → (List a)
data BooL where
  FF :: BooL
  TT :: BooL

list :: b → (a → b → b) → List a → b
list nil cons = fold where
  fold Nil = nil
  fold (x `Cons` xs) = x `cons` fold xs

booL :: b → b → BooL → b
booL tt ff = fold where
  fold FF = ff
  fold TT = tt
```

```
data List a where
  Nil :: List a
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Structural Functors

```haskell
data ListF c x = NilF | ConsF c x -- deriving Functor
data BooLF x = TTF | FFF deriving Functor

instance Functor (ListF c) where
  fmap :: (a → b) → (((ListF c) a) → ((ListF c) b))
  fmap f = λcase
    NilF → NilF
    ConsF c a → ConsF c (f a)
```

If you are unfamiliar with functors: In our case regarding them as a computational context with holes suffices.

Structural Functors

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data ListF c x = NilF | ConsF c x -- deriving Functor
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If you are unfamiliar with functors: In our case regarding them as a computational context with holes suffices.

- The structural functors encode where the recursion should happen
- Relation to original datatype not yet wholly clear
- $\rightsquigarrow$ Introduce a little Category Theory

Structure

Endofunctors

Let $\mathcal{C}$ be a Category. An *Endofunctor* is a pair of maps, on the objects and morphisms of the category respectively: $F_0 : \mathcal{C}_0 \to \mathcal{C}_0$, $F_1 : \mathcal{C}_1 \to \mathcal{C}_1$ Such that:

- $F_1(h : A \to B) : F_0 A \to F_0 B$
- $F_1(id_A) = id_{F_0 A}$
- $F_1(h \circ g) = (F_1 h) \circ (F_1 g)$

- `fmap @f (h :: a → b) :: f a → f b`
- `fmap @f (id @a) = id @(f a)`
- `fmap @f (h . g) = fmap @f h . fmap @f g`

The category we are regarding is *Hask*, where objects are Haskell types, and morphisms are functions between them. This is usually interpreted as a *CPO* (complete partial order). We will mention what notions are specic to *CPO*s.

Algebras

Let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor $A_0 \in \mathcal{C}_0$, $\phi : A \to FA$. Then $A \xrightarrow{\phi} FA$ is an *Algebra*, and A its *Carrier*. We can phrase the business logic of the previously seen functions as such (using the transformation $A^B \times A^C \sim A^{B+C}$):

```
type Algebra f a = f a → a
boolBL :: b → b → Algebra BooLF b
boolBL tt ff = λcase
  TTF → tt
  FFF → ff
listBL :: b → (a → b → b) → Algebra (ListF a) b
listBL nil cons = λcase
  NilF → nil
  x ‘ConsF‘ b → x ‘cons‘ b
```

To conclude this exercise, we want to generalize the function for recursively applying the business logic (So a function subsuming `list` and `booL` seen earlier):
`list :: b → (a → b → b) → List a → b`
`booL :: b → b → BooL → b` We define a type family to associate the structural functors with their types.

```
type family CI (f :: * → *) :: *

type instance CI (ListF a) = List a
type instance CI (BooLF) = BooL
```

Then our that function would have type
`cata :: Algebra f b → CI f → b` . To get to its definition, we must interleave some more Cat.Th.

```
cata :: Algebra f b → (CI f) → b
cata = undefined
```

Algebra

$$\begin{tikzcd} FA \arrow[d, "\phi"] \\ A \end{tikzcd}$$

Algebra-Hom:
$(A, \phi) \to (B, \psi)$

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\phi"] & FB \arrow[d, "\psi"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Initial Algebra:
(A, κ)

$$\begin{tikzcd} FA \arrow[r, dashed, "Fh"] \arrow[d, shift left, "\kappa"] & FB \arrow[d, "\psi"] \\ A \arrow[u, shift left, "\kappa^{-1}"] \arrow[r, dashed, "h"] & B \end{tikzcd}$$

Initiality requirement: $h = \kappa^{-1}; Fh; \psi$

As Program

```
newtype Fix f = In { out :: f (Fix f) }
```