# Entanglement

A word that is often thrown around in popular science circles on the internet and is a favorite target for many flavors of quantum theories. There have been countless interpretations for what entanglement means, such as Many Worlds, the Copenhagen interpretation, pilot-wave theory, plus plenty more not listed here.

This is typically where I personally see much of the time around entanglement spent: on the "meaning." But the interpretation of it is entirely unnecessary to answer the question that, in my opinion, is the one that really matters:

**What even is entanglement?**

According to Wikipedia, "Quantum entanglement is the phenomenon where the quantum state of each particle in a group cannot be described independently of the state of the others, even when the particles are separated by a large distance" [1]. This is to say, when a pair of particles is entangled, they share a correlation with each other.

How correlated are they? More than classical logic allows.

Before we get into the messy questions like how something can be "more correlated than classical logic allows," we should first define what classical logic *does* allow, using the theorem laid out by John Stewart Bell, aptly named Bell's Theorem [2].

The full derivation of Bell's theorem is beyond the scope of what is needed for our discussion here, however I do encourage the reader to explore [2] if they are interested; it is fascinating stuff. To put it simply, Bell's theorem shows the boundary that any classical theory would have to obey, and how quantum mechanics violates this boundary. For our purposes, we are going to look at this inequality through the lens of the CHSH game [3].

## The Game

We will have three participants in our game: Alice, Bob, and Charlie (A, B, and C respectively). Charlie prepares a pair of input bits

$$(x, y) \in \{0, 1\}^2,$$

and sends $x$ to Alice and $y$ to Bob. In this setup, Charlie also sends $(x, y)$ to an AND gate that computes

$$x \wedge y.$$

Alice and Bob each output a single bit, $a$ and $b$ respectively, which are fed into an XOR gate that computes

$$a \oplus b.$$

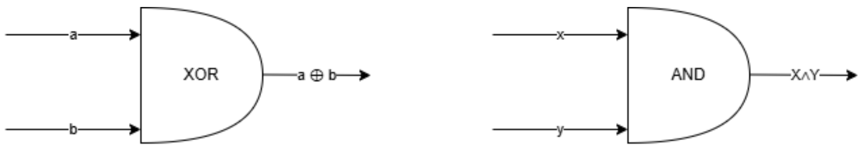
Finally, an XNOR gate checks whether these two results agree. In logical form, the win condition is

$$\neg\big((a \oplus b) \oplus (x \wedge y)\big) = 1,$$

which is equivalent to requiring

$$a \oplus b = x \wedge y.$$

If this condition holds, Alice and Bob score a point. (See Fig. 1 for a diagram of the AND and XOR gates and their truth tables.)




a, b ∈ {0, 1}

| XOR | | | |
|---|---|---|---|
| 0 | 1 | 0 | 1 |
| 0 | 0 | 1 | 1 |
| **0** | **1** | **1** | **0** |

x, y ∈ {0, 1}

| AND | | | |
|---|---|---|---|
| 0 | 1 | 0 | 1 |
| 0 | 0 | 1 | 1 |
| **0** | **0** | **0** | **1** |

Figure 1: XOR and AND gates with their truth tables.

Alice and Bob are strictly forbidden from communication during the game. Alice can only react based on $x$, and Bob only on $y$. They may meet before the game begins to discuss strategies if they would like.

There are many strategies Alice and Bob could implement, but even with the optimal classical strategy, their probability of winning is bounded. In general we can write the winning probability in the classical case as

$$P_{\text{classical}}(W = 1) = \frac{1}{4} \sum_{x,y \in \{0,1\}} P\big[(a \oplus b) = (x \wedge y)\big], \tag{1}$$

where the factor of $1/4$ comes from the four equally likely input pairs $(x, y)$.

Bell's analysis (and the CHSH form in particular) shows that, for any classical local strategy (even allowing shared randomness between Alice and Bob),

$$P^{\max}_{\text{classical}}(W = 1) = \frac{3}{4} = 0.75. \tag{2}$$

No purely classical local model can do better.

Figure 2 shows the full classical CHSH game setup in logic-gate form.

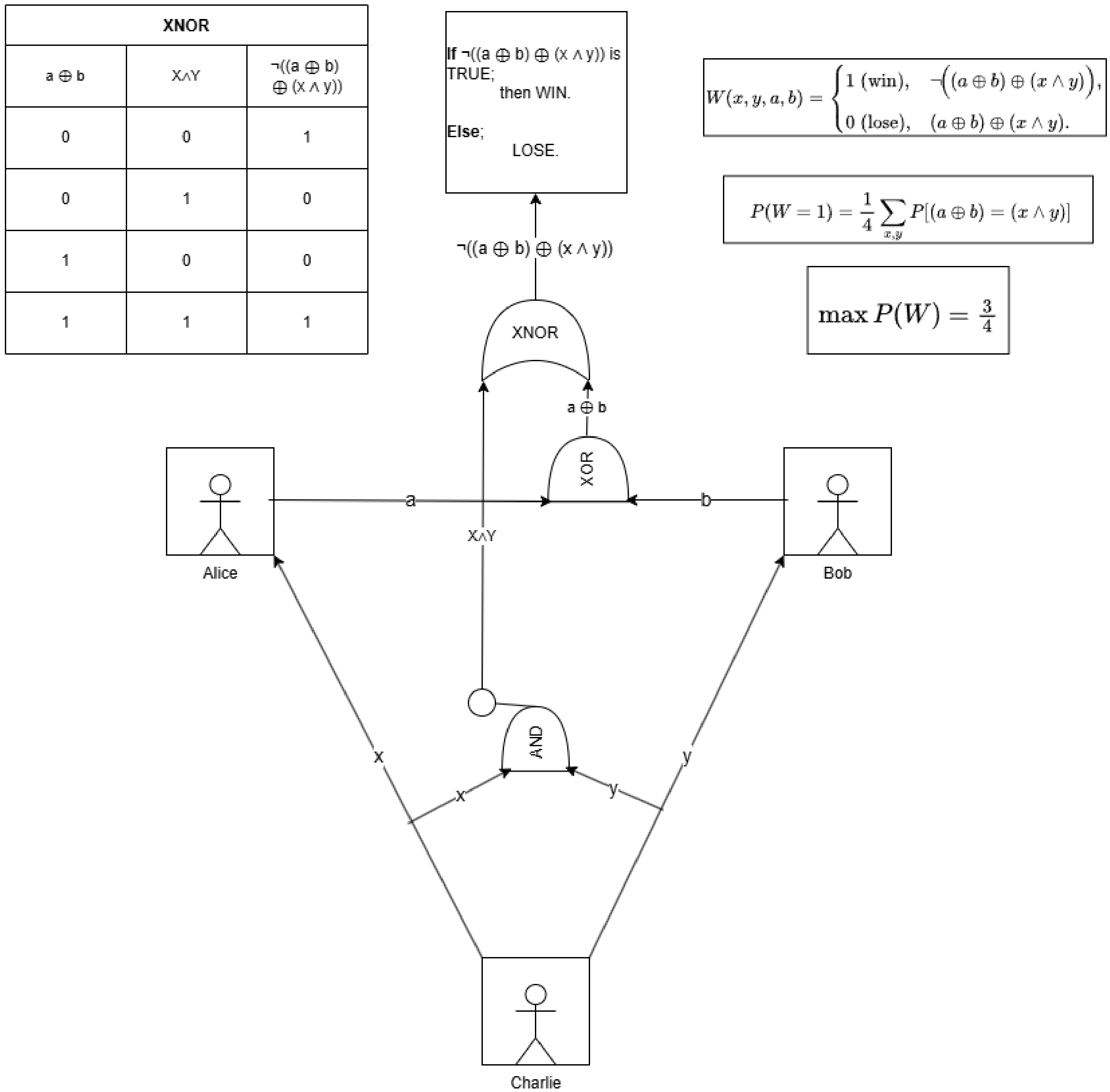



Figure 2: Classical CHSH game represented with logic gates. Alice and Bob receive inputs $x$ and $y$, output bits $a$ and $b$, and win when $a \oplus b = x \wedge y$.

## The Quantum Strategy

As established above, with an optimal classical strategy Alice and Bob are only able to win the game at most 75% of the time. But what about a quantum strategy? This is where things get interesting.

In the quantum version, Charlie still prepares inputs $(x, y)$ just as before, but he also prepares an entangled pair of qubits in the Bell state

$$|\psi_{ab}\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle_a |1\rangle_b - |1\rangle_a |0\rangle_b\big), \tag{3}$$

sending one particle to Alice and the other to Bob along with $x$ and $y$, respectively.

We also add two measurement devices (shown as boxes with diagonal lines in Fig. 3) that measure the state of the incoming qubit. Alice's detector angle is determined by her input $x$ and Bob's by his input $y$. We can denote these measurement settings by

$$\varphi_{A,x} \quad \text{and} \quad \varphi_{B,y},$$

and the detector outputs define their bits $a$ and $b$:

$$a = f(x, \varphi_{A,x}, |\psi_{ab}\rangle), \tag{4}$$

$$b = f(y, \varphi_{B,y}, |\psi_{ab}\rangle). \tag{5}$$

The rest of the rules of the game are exactly the same as they were in the classical version: they still win only when $a \oplus b = x \wedge y$.

Given these changes, the probability that Alice and Bob win a round in the quantum case can be written as

$$P_q(W=1) = \frac{1}{8}\sum_{x=0}^{1}\sum_{y=0}^{1}\left[1 + (-1)^{x\wedge y}\cos\big(\varphi_{A,x} - \varphi_{B,y}\big)\right]. \tag{6}$$

For the usual optimal choice of measurement angles,

$$\varphi_{A,0} = 0^\circ, \quad \varphi_{A,1} = 90^\circ, \quad \varphi_{B,0} = 45^\circ, \quad \varphi_{B,1} = -45^\circ,$$

this achieves the maximum quantum value, known as Tsirelson's bound [4]:

$$P_q^{\max}(W=1) = \frac{2+\sqrt{2}}{4} \approx 0.8536. \tag{7}$$

This decisively violates the limit set by the optimal classical strategy in Eq. (2).

Figure 3 shows the quantum version of the CHSH game, with the entangled source and measurement settings made explicit.

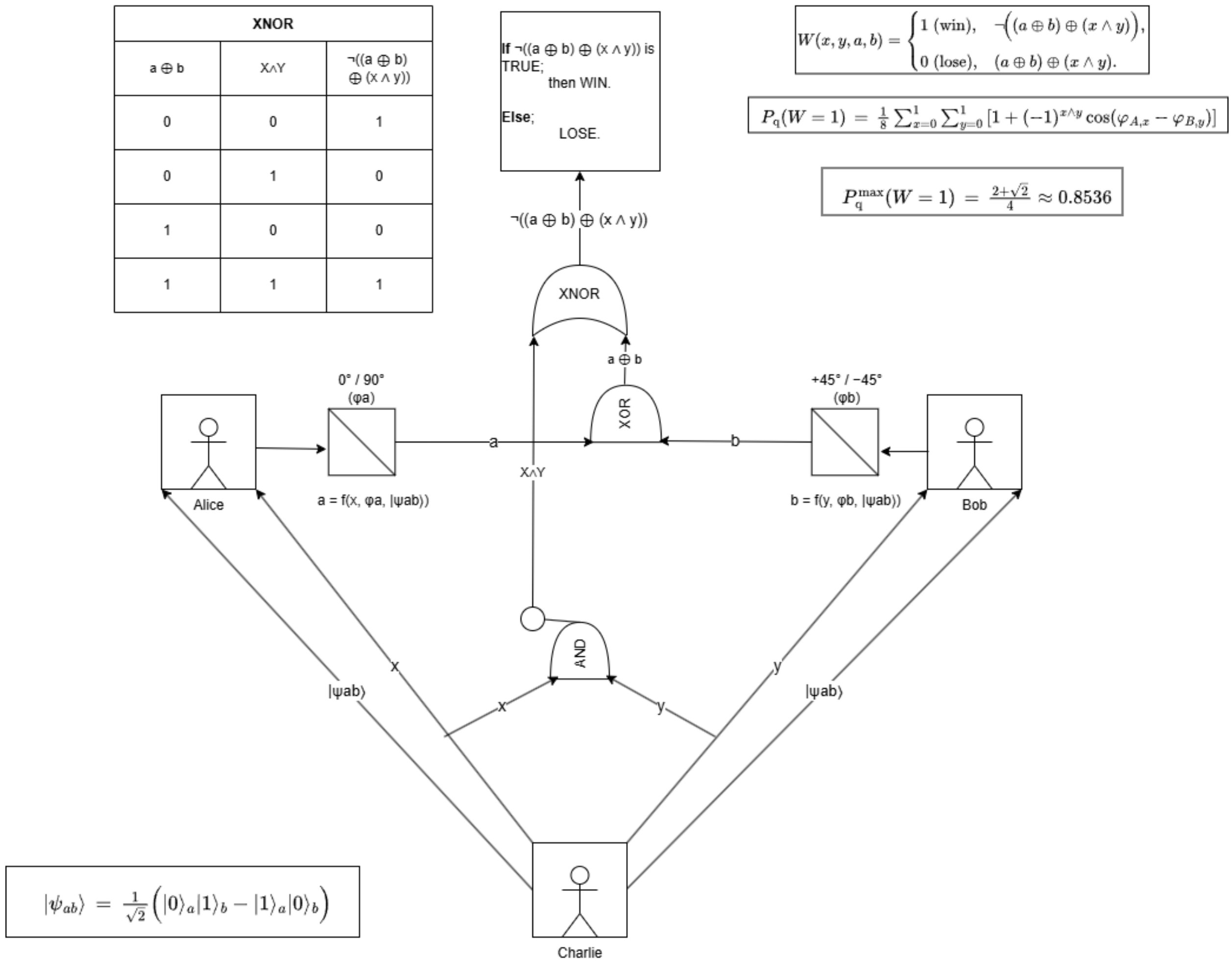



Figure 3: Quantum CHSH game. Alice and Bob share an entangled state $|\psi_{ab}\rangle$ and perform measurements at angles determined by $x$ and $y$. The win condition remains $a \oplus b = x \wedge y$, but the achievable success probability is higher than in any classical local model.

## Conclusion

The roughly ten percent gap between the classical limit (75%) and the quantum bound (about 85%) proves that quantum entanglement creates fundamentally stronger correlations than any classical system can achieve. This has been experimentally confirmed in many tests [5], with the 2022 Nobel Prize in Physics being awarded for experimental verifications of violations of Bell's inequality [6].

A link to this PDF and a simulation that lets you see the game played out in real time are included below. Go try it out at:

`https://just-some-vibe-physics.netlify.app/`

## References

[1] "Quantum entanglement, " *Wikipedia*, https://en.wikipedia.org/wiki/Quantum_entanglement.

[2] "Bell's theorem," *Wikipedia*, https://en.wikipedia.org/wiki/Bell%27s_theorem.

[3] "CHSH inequality," *Wikipedia*, https://en.wikipedia.org/wiki/CHSH_inequality.

[4] "Tsirelson's bound," *Wikipedia*, https://en.wikipedia.org/wiki/Tsirelson%27s_bound.

[5] "Bell test," *Wikipedia*, https://en.wikipedia.org/wiki/Bell_test.

[6] "The Nobel Prize in Physics 2022," *NobelPrize.org*, https://www.nobelprize.org/prizes/physics/2022/summary/.