Kalman Filter and State-Space Models

- state-space models:
  - time-varying parameter models;
  - models with unobservable variables;
  - basic tool: Kalman filter;
  - implementation is task-specific.

Time-Varying Parameters

$$y_t = \mathbf{x}_t \boldsymbol{\beta}_t + e_t \tag{1}$$

$$\boldsymbol{\beta}_t = \boldsymbol{\mu} + \mathbf{F}\boldsymbol{\beta}_{t-1} + \mathbf{v}_t, \tag{2}$$

$$e_t \sim \text{i.i.d. N}(0, R);\ \mathbf{v}_t \sim \text{i.i.d. N}(\mathbf{0}, \mathbf{Q});\ \text{E}(e_t \mathbf{v}_t) = \mathbf{0}. \tag{3}$$

Possible expected values of time-t-beta: $\boldsymbol{\beta}_{t|t-1}$, $\boldsymbol{\beta}_{t|t}$, $\boldsymbol{\beta}_{t|T}$.

Notation:

$$\begin{aligned}
\mathbf{P}_{t|t-1} &= \text{E}[(\boldsymbol{\beta}_t - \boldsymbol{\beta}_{t|t-1})(\boldsymbol{\beta}_t - \boldsymbol{\beta}_{t|t-1})'], \\
\mathbf{P}_{t|t} &= \text{E}[(\boldsymbol{\beta}_t - \boldsymbol{\beta}_{t|t})(\boldsymbol{\beta}_t - \boldsymbol{\beta}_{t|t})'], \\
\mathbf{P}_{t|T} &= \text{E}[(\boldsymbol{\beta}_t - \boldsymbol{\beta}_{t|T})(\boldsymbol{\beta}_t - \boldsymbol{\beta}_{t|T})'], \\
y_{t|t-1} &= \mathbf{x}_t \boldsymbol{\beta}_{t|t-1}, \\
\eta_{t|t-1} &= y_t - y_{t|t-1}, \\
f_{t|t-1} &= \text{E}(\eta^2_{t|t-1}).
\end{aligned}$$

Time-Varying Parameters and Kalman Filter

Prediction

$$\begin{aligned}
\boldsymbol{\beta}_{t|t-1} &= \boldsymbol{\mu} + \mathbf{F}\boldsymbol{\beta}_{t-1|t-1}, \\
\mathbf{P}_{t|t-1} &= \mathbf{F}\mathbf{P}_{t-1|t-1}\mathbf{F}' + \mathbf{Q}, \\
\eta_{t|t-1} &= y_t - \mathbf{x}_t\boldsymbol{\beta}_{t|t-1}, \\
f_{t|t-1} &= \mathbf{x}_t\mathbf{P}_{t|t-1}\mathbf{x}_t' + R.
\end{aligned}$$

Updating

$$\begin{aligned}
\boldsymbol{\beta}_{t|t} &= \boldsymbol{\beta}_{t|t-1} + \mathbf{K}_t\eta_{t|t-1}, \\
\mathbf{P}_{t|t} &= \mathbf{P}_{t|t-1} - \mathbf{K}_t\mathbf{x}_t\mathbf{P}_{t|t-1},
\end{aligned}$$

where $\mathbf{K}_t = \mathbf{P}_{t|t-1}\mathbf{x}_t' f_{t|t-1}^{-1}$ is the "Kalman gain." It is a weight, that depends on the prediction uncertainty and the variance of the error.

This process can be depicted graphically.

Time-Varying Parameters and Kalman Filter (continued)

- If we are given the following:
  - values of the parameters $\boldsymbol{\mu}$, $\mathbf{F}$, R, $\mathbf{Q}$, and
  - the starting values, $\boldsymbol{\beta}_{0|0}$ and $\mathbf{P}_{0|0}$,

  then we can iterate the Prediciton and Updating equations (in the order listed on the previous slide) over all observations, $t \in [0, T]$, so as to compute the estimates of the time-varying parameters, $\boldsymbol{\beta}_{t|t}$.

- If we are not given the starting values, but the time-varying parameters are known to be stationary, then we can calculate them:

$$\begin{aligned}
\boldsymbol{\beta}_{0|0} &= (\mathbf{I} - \mathbf{F})^{-1}\boldsymbol{\mu}, \text{ and} \\
\text{vec}(\mathbf{P}_{0|0}) &= (\mathbf{I} - \mathbf{F}\otimes\mathbf{F})^{-1}\text{vec}(\mathbf{Q}).
\end{aligned}$$

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Time-Varying Parameters and Kalman Filter (continued)

- If we are <u>not</u> given the statring values, and the time-varying parameters may <u>not</u> be stationary, then:
  - set $\boldsymbol{\beta}_{0|0}$ to any arbitrary values and set diagonal elements of $\mathbf{P}_{0|0}$ to large values, or
  - set $\mathbf{P}_{0|0} = \mathbf{0}$ and estimate $\boldsymbol{\beta}_{0|0}$ using maximum likelihood method, then calculate $\mathbf{P}_{0|0} = \text{cov}(\boldsymbol{\beta}_{0|0}^{MLE})$ and then rerun the Kalman filter with these estimates as the starting values.

- If we are <u>not</u> given the values of the parameters $\boldsymbol{\mu}$, $\mathbf{F}$, R, $\mathbf{Q}$, then estimate them using the maximum likelihood method.

Time-Varying Parameters and Kalman Filter: ML estimation

Given the normality assumptions in equation (3), the log-likelihood function for the model is:

$$\mathcal{L} = -\frac{1}{2}\sum_{t=1}^{T} \ln(2\pi f_{t|t-1}) - \frac{1}{2}\sum_{t=1}^{T} \eta'_{t|t-1} f_{t|t-1}^{-1} \eta_{t|t-1}. \tag{4}$$

If the initial values, $\boldsymbol{\beta}_{0|0}$ and $\mathbf{P}_{0|0}$ are known or at least the time-varying parameters are stationary, then maximize the function in equation (4) with respect to the unknown parameters.

Time-Varying Parameters and Kalman Filter: ML estimation

If the initial values, $\boldsymbol{\beta}_{0|0}$ and $\mathbf{P}_{0|0}$ are unknown and the time-varying parameters are not stationary, then proceed in one of two ways:

- estimate the initial values:
  - maximize the function in equation (4) with respect to both the unknown parameters and the initial values by setting $\mathbf{P}_{0|0} = \mathbf{0}$,
  - calculate $\mathbf{P}_{0|0} = \text{cov}(\beta_{0|0}^{MLE})$,
  - then rerun the Kalman filter (i.e., the Prediction and Updating equations) with these estimates as the starting values; or
- use random initial values:
  - set $\boldsymbol{\beta}_{0|0}$ to any arbitrary values;
  - set the diagonal elements of $\mathbf{P}_{0|0}$ to very large values;
  - maximize the following log-likelihood funciton:

$$\mathcal{L} = -\frac{1}{2}\sum_{t=\tau}^{T} \ln(2\pi f_{t|t-1}) - \frac{1}{2}\sum_{t=\tau}^{T} \eta'_{t|t-1} f_{t|t-1}^{-1} \eta_{t|t-1}, \tag{5}$$

    where τ is large enough as to dampen the effect of the random initial values (i.e., the period $[0, \tau - 1]$ is the "burn-in" period).

Time-Varying Parameters and Kalman Filter: Smoothing

The forecast values of time-varying parameters, $\boldsymbol{\beta}_{t|t-1}$, and their updated values, $\boldsymbol{\beta}_{t|t}$ are used in the basic Kalman filter.
Using more information provides more accurate estimates of the time-varying parameters. How about calculating $\boldsymbol{\beta}_{t|T}$?
After running the basic Kalman filter (Prediction + Updating equations), run backwards (i.e., from time T to time 0) the following equations:

Smoothing

$$\begin{aligned} \boldsymbol{\beta}_{t|T} &= \boldsymbol{\beta}_{t|t} + \mathbf{P}_{t|t}\mathbf{F}'\mathbf{P}_{t+1|t}^{-1}(\boldsymbol{\beta}_{t+1|T} - \mathbf{F}\boldsymbol{\beta}_{t|t} - \boldsymbol{\mu}), \text{ and} \\ \mathbf{P}_{t|T} &= \mathbf{P}_{t|t} + \mathbf{P}_{t|t}\mathbf{F}'\mathbf{P}_{t+1|t}^{-1}(\mathbf{P}_{t+1|T} - \mathbf{P}_{t+1|t})(\mathbf{P}_{t+1|t}^{-1})'\mathbf{F}\mathbf{P}'_{t|t}. \end{aligned}$$

The starting values for the smoothing equations, $\boldsymbol{\beta}_{T|T}$ and $\mathbf{P}_{T|T}$, come from where?

Time-Varying Parameters
Smoothed Kalman Filter

Prediction

$$\begin{aligned}\boldsymbol{\beta}_{t|t-1} &= \boldsymbol{\mu} + \mathbf{F}\boldsymbol{\beta}_{t-1|t-1},\\ \mathbf{P}_{t|t-1} &= \mathbf{F}\mathbf{P}_{t-1|t-1}\mathbf{F}' + \mathbf{Q},\\ \eta_{t|t-1} &= y_t - \mathbf{x}_t\boldsymbol{\beta}_{t|t-1},\\ f_{t|t-1} &= \mathbf{x}_t\mathbf{P}_{t|t-1}\mathbf{x}_t' + R.\end{aligned}$$

Updating

$$\begin{aligned}\boldsymbol{\beta}_{t|t} &= \boldsymbol{\beta}_{t|t-1} + \mathbf{K}_t\eta_{t|t-1},\\ \mathbf{P}_{t|t} &= \mathbf{P}_{t|t-1} - \mathbf{K}_t\mathbf{x}_t\mathbf{P}_{t|t-1},\end{aligned}$$

Smoothing

$$\begin{aligned}\boldsymbol{\beta}_{t|T} &= \boldsymbol{\beta}_{t|t} + \mathbf{P}_{t|t}\mathbf{F}'\mathbf{P}^{-1}_{t+1|t}(\boldsymbol{\beta}_{t+1|T} - \mathbf{F}\boldsymbol{\beta}_{t|t} - \boldsymbol{\mu}), \text{ and}\\ \mathbf{P}_{t|T} &= \mathbf{P}_{t|t} + \mathbf{P}_{t|t}\mathbf{F}'\mathbf{P}^{-1}_{t+1|t}(\mathbf{P}_{t+1|T} - \mathbf{P}_{t+1|t})(\mathbf{P}^{-1}_{t+1|t})'\mathbf{F}\mathbf{P}'_{t|t}.\end{aligned}$$

Unobservable State Variables

Measurement Equation

$$\mathbf{y}_t = \mathbf{B}_t\mathbf{x}_t + \mathbf{A}\mathbf{z}_t + \mathbf{e}_t; \tag{6}$$

Transition Equation

$$\mathbf{x}_t = \boldsymbol{\mu} + \mathbf{F}\mathbf{x}_{t-1} + \mathbf{v}_t, \tag{7}$$

$$e_t \sim \text{i.i.d. N}(\mathbf{0}, \mathbf{R});\ \mathbf{v}_t \sim \text{i.i.d. N}(\mathbf{0}, \mathbf{Q});\ \mathrm{E}(\mathbf{e}_t\mathbf{v}_t') = 0. \tag{8}$$

This model, including the parameters and the unobservable components, can be estimated using the Kalman filter. Notation:

$$\begin{aligned}P_{t|t-1} &= \mathrm{E}[(\mathbf{x}_t - \mathbf{x}_{t|t-1})(\mathbf{x}_t - \mathbf{x}_{t|t-1})'],\\ P_{t|t} &= \mathrm{E}[(\mathbf{x}_t - \mathbf{x}_{t|t})(\mathbf{x}_t - \mathbf{x}_{t|t})'],\\ P_{t|T} &= \mathrm{E}[(\mathbf{x}_t - \mathbf{x}_{t|T})(\mathbf{x}_t - \mathbf{x}_{t|T})'],\\ \mathbf{y}_{t|t-1} &= \mathbf{B}_t\mathbf{x}_{t|t-1} + \mathbf{A}\mathbf{z}_t,\\ \boldsymbol{\eta}_{t|t-1} &= \mathbf{y}_t - \mathbf{y}_{t|t-1},\\ \mathbf{f}_{t|t-1} &= \mathrm{E}(\boldsymbol{\eta}_{t|t-1}\boldsymbol{\eta}'_{t|t-1}).\end{aligned}$$

State-Space Models
and Kalman Filter

Prediction

$$\begin{aligned}
\mathbf{x}_{t|t-1} &= \boldsymbol{\mu} + \mathbf{F}\mathbf{x}_{t-1|t-1}, \\
\mathbf{P}_{t|t-1} &= \mathbf{F}\mathbf{P}_{t-1|t-1}\mathbf{F}' + \mathbf{Q}, \\
\boldsymbol{\eta}_{t|t-1} &= \mathbf{y}_t - \mathbf{B}_t\mathbf{x}_{t|t-1} - \mathbf{A}\mathbf{z}_t, \\
\mathbf{f}_{t|t-1} &= \mathbf{B}_t\mathbf{P}_{t|t-1}\mathbf{B}_t' + \mathbf{R}.
\end{aligned}$$

Updating

$$\begin{aligned}
\mathbf{x}_{t|t} &= \mathbf{x}_{t|t-1} + \mathbf{K}_t\boldsymbol{\eta}_{t|t-1}, \\
\mathbf{P}_{t|t} &= \mathbf{P}_{t|t-1} - \mathbf{K}_t\mathbf{B}_t\mathbf{P}_{t|t-1},
\end{aligned}$$

where $\mathbf{K}_t = \mathbf{P}_{t|t-1}\mathbf{B}_t'\mathbf{f}_{t|t-1}^{-1}$.

Smoothing

$$\begin{aligned}
\mathbf{x}_{t|T} &= \mathbf{x}_{t|t} + \mathbf{P}_{t|t}\mathbf{F}'\mathbf{P}_{t+1|t}^{-1}(\mathbf{x}_{t+1|T} - \mathbf{F}\mathbf{x}_{t|t} - \boldsymbol{\mu}), \text{ and} \\
\mathbf{P}_{t|T} &= \mathbf{P}_{t|t} + \mathbf{P}_{t|t}\mathbf{F}'\mathbf{P}_{t+1|t}^{-1}(\mathbf{P}_{t+1|T} - \mathbf{P}_{t+1|t})(\mathbf{P}_{t+1|t}^{-1})'\mathbf{F}\mathbf{P}_{t|t}'.
\end{aligned}$$

State-Space Models
Examples

Many familiar models can be cast in the form of the State-Space Model. Some examples.

AR(2) Model:

- $y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t,\ w_t \sim$ i.i.d.N$(0, \sigma^2)$;
- measurement equation:

$$y_t = \delta^* + [1 \quad 0]\begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix};$$

- transition equation:

$$\begin{bmatrix} x_t \\ x_{t-1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_{t-1} \\ x_{t-2} \end{bmatrix} + \begin{bmatrix} w_t \\ 0 \end{bmatrix},$$

  where $\delta^* = \delta/(1 - \phi_1 - \phi_2)$.

State-Space Models
Examples (continued)

ARMA(2,1) Model:

- $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t + \theta w_{t-1},\ w_t \sim$ i.i.d.N$(0, \sigma^2)$;
- measurement equation:

$$y_t = [1 \quad \theta] \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix};$$

- transition equation:

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{1,t-1} \\ x_{2,t-1} \end{bmatrix} + \begin{bmatrix} w_t \\ 0 \end{bmatrix}.$$

State-Space Models
Examples (continued)

Time-varying Parameter Model:

- discussed above;
- measurement equation is equation (1) above;
- transition equation is equation (2) above.
- Kim and Nelson's (1989) model for money supply growth:

$$\begin{aligned} \Delta M_t &= \mathbf{x}_{t-1}\boldsymbol{\beta}_t + e_t, \\ \mathbf{x}_t &= \begin{bmatrix} 1 & \Delta i_t & INF_t & SURP_t & \Delta M_t \end{bmatrix}. \end{aligned} \tag{9}$$

- generates ARCH-like effects via $\mathbf{f}_{t|t-1} = \mathbf{x}_t \mathbf{P}_{t|t-1} \mathbf{x}_t' + \mathbf{R}$.

State-Space Models
Examples (continued)

Unobserved-Components Model:

- $$\begin{aligned} y_t &= y_{1t} + y_{2t}, \\ y_{1t} &= \delta + y_{1,t-1} + e_{1t}, \\ y_{2t} &= \phi_1 y_{2,t-1} + \phi_2 y_{2,t-2} + e_{2t}, \\ e_{it} &\sim \text{i.i.d.N}(0, \sigma_i^2),\ i = 1, 2,\ \text{E}(e_{1t}, e_{2s}) = 0 \text{ for all } t \text{ and } s. \end{aligned}$$
- measurement equation:
$$y_t = [1\ 1\ 0] \begin{bmatrix} y_{1t} \\ y_{2t} \\ y_{2,t-1} \end{bmatrix};$$
- transition equation:
$$\begin{bmatrix} y_{1t} \\ y_{2t} \\ y_{2,t-1} \end{bmatrix} = \begin{bmatrix} \delta \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \phi_1 & \phi_2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \\ y_{2,t-2} \end{bmatrix} + \begin{bmatrix} e_{1t} \\ e_{2t} \\ 0 \end{bmatrix}.$$

State-Space Models
Examples (continued)

Unobserved-Components Model (continued):
A model for log-real GDP:

$$\begin{aligned} y_t &= n_t + x_t, \\ n_t &= g_{t-1} + n_{t-1} + v_t,\ v_t \sim \text{i.i.d.N}(0, \sigma_v^2), \\ g_t &= g_{t-1} + w_t,\ w_t \sim \text{i.i.d.N}(0, \sigma_w^2), \\ x_t &= \phi_1 x_{t-1} + \phi_2 x_{t-2} + e_t,\ e_t \sim \text{i.i.d.N}(0, \sigma_e^2), \\ U_t &= L_t + C_t, \\ L_t &= L_{t-1} + \psi_t,\ \psi_t \sim \text{i.i.d.N}(0, \sigma_\psi^2), \\ C_t &= \alpha_0 x_t + \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + \varepsilon_t,\ \varepsilon_t \sim \text{i.i.d.N}(0, \sigma_\varepsilon^2), \\ &\ \text{where all errors are independent.} \end{aligned}$$

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State-Space Models
Examples (continued)

Unobserved-Components Model (continued):

A model for log-real GDP (continued):

- measurement equation:

$$\begin{bmatrix} y_t \\ U_t \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_0 & \alpha_1 & \alpha_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_t \\ x_t \\ x_{t-1} \\ x_{t-2} \\ g_t \\ L_t \end{bmatrix} + \begin{bmatrix} 0 \\ \varepsilon_t \end{bmatrix};$$

- transition equation:

$$\begin{bmatrix} n_t \\ x_t \\ x_{t-1} \\ x_{t-2} \\ g_t \\ L_t \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & \phi_1 & \phi_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_{t-1} \\ x_{t-1} \\ x_{t-2} \\ x_{t-3} \\ g_{t-1} \\ L_{t-1} \end{bmatrix} + \begin{bmatrix} v_t \\ e_t \\ 0 \\ 0 \\ w_t \\ \psi_t \end{bmatrix}.$$

State-Space Models
Examples (continued)

Dynamic Factor Model:

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$$\begin{aligned} y_{1t} &= \gamma_1 c_t + z_{1t}, \\ y_{2t} &= \gamma_2 c_t + z_{2t}, \\ c_t &= \phi_1 c_{t-1} + v_t,\ v_t \sim \text{i.i.d.N}(0,1), \\ z_{it} &= \alpha_i z_{i,t-1} + e_{it},\ e_{it} \sim \text{i.i.d.N}(0,\sigma_i^2), \text{ and} \\ \mathrm{E}(e_{1t}, e_{2t}) &= \mathrm{E}(e_{1t}, v_t) = \mathrm{E}(e_{2t}, v_t) = 0. \end{aligned}$$

- measurement equation:

$$\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} \gamma_1 & 1 & 0 \\ \gamma_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_t \\ z_{1t} \\ z_{2t} \end{bmatrix};$$

- transition equation:

$$\begin{bmatrix} c_t \\ z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} \phi_1 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_2 \end{bmatrix} \begin{bmatrix} c_{t-1} \\ z_{1,t-1} \\ z_{2,t-1} \end{bmatrix} + \begin{bmatrix} v_t \\ e_{1t} \\ e_{2t} \end{bmatrix}.$$

State-Space Models
Examples (continued)

Dynamic Factor Model (continued):
A model for coincident economic indicators (see Stock and Watson (1991)):

$$\begin{aligned}
\Delta y_{it} &= \gamma_i \Delta c_t + e_{it}, \quad i = 1,2,3,4, \\
\Delta c_t &= \phi_1 \Delta c_{t-1} + \phi_2 \Delta c_{t-2} + w_t,\ w_t \sim \text{i.i.d.} N(0,1), \\
e_{it} &= \psi_{i1} e_{i,t-1} + \psi_{i2} e_{i,t-2} + \epsilon_{it},\ \epsilon_{it} \sim \text{i.i.d.} N(0,\sigma_i^2),\ i = 1,2,3,4
\end{aligned}$$

State-Space Models
Examples (continued)

Dynamic Factor Model (continued):
A model for coincident economic indicators (continued):

- measurement equation:

$$\begin{bmatrix} \Delta y_{1t} \\ \Delta y_{2t} \\ \Delta y_{3t} \\ \Delta y_{4t} \end{bmatrix} = \begin{bmatrix} \gamma_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma_2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \gamma_3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \gamma_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta c_t \\ \Delta c_{t-1} \\ e_{1t} \\ e_{1,t-1} \\ e_{2t} \\ e_{2,t-1} \\ e_{3t} \\ e_{3,t-1} \\ e_{4t} \\ e_{4,t-1} \end{bmatrix};$$

State-Space Models
Examples (continued)

Dynamic Factor Model (continued):

A model for coincident economic indicators (continued):

- transition equation:

$$\begin{bmatrix} \Delta c_t \\ \Delta c_{t-1} \\ e_{1t} \\ e_{1,t-1} \\ e_{2t} \\ e_{2,t-1} \\ e_{3t} \\ e_{3,t-1} \\ e_{4t} \\ e_{4,t-1} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \psi_{11} & \psi_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \psi_{21} & \psi_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \psi_{31} & \psi_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \psi_{41} & \psi_{42} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta c_{t-1} \\ \Delta c_{t-2} \\ e_{1,t-1} \\ e_{1,t-2} \\ e_{2,t-1} \\ e_{2,t-2} \\ e_{3,t-1} \\ e_{3,t-2} \\ e_{4,t-1} \\ e_{4,t-2} \end{bmatrix}$$

$$+ \begin{bmatrix} w_t & 0 & \epsilon_{it} & 0 & \epsilon_{2t} & 0 & \epsilon_{3t} & 0 & \epsilon_{4t} & 0 \end{bmatrix}'.$$

State-Space Models
Examples (continued)

Common Stochastic Trend Model:

- $$\begin{aligned} y_{1t} &= z_t + x_{1t}, \\ y_{2t} &= z_t + x_{2t}, \\ z_t &= z_{t-1} + \epsilon_t,\ \epsilon_t \sim \text{i.i.d.N}(0, \sigma_\epsilon^2), \\ x_{1t} &= \mu_1 + \phi_{11} x_{1,t-1} + \phi_{12} x_{2,t-1} + e_{1t} + \theta_{11} e_{1,t-1} + \theta_{12} e_{2,t-1}, \\ x_{2t} &= \mu_2 + \phi_{21} x_{1,t-1} + \phi_{22} x_{2,t-1} + e_{2t} + \theta_{21} e_{1,t-1} + \theta_{22} e_{2,t-1}, \\ \begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix} &\sim \text{i.i.d.N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} \right). \end{aligned}$$
- measurement equation:

$$\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_t \\ x_{1t} \\ x_{2t} \\ e_{1t} \\ e_{2t} \end{bmatrix};$$

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State-Space Models
Examples (continued)

Common Stochastic Trend Model (continued):

- transition equation:

$$\begin{bmatrix} z_t \\ x_{1t} \\ x_{2t} \\ e_{1t} \\ e_{2t} \end{bmatrix} = \begin{bmatrix} 0 \\ \mu_1 \\ \mu_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \phi_{11} & \phi_{12} & \theta_{11} & \theta_{12} \\ 0 & \phi 21 & \phi_{22} & \theta_{21} & \theta_{22} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_{t-1} \\ x_{1,t-1} \\ x_{2,t-1} \\ e_{1,t-1} \\ e_{2,t-1} \end{bmatrix} + \\ + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_t \\ e_{1t} \\ e_{2t} \end{bmatrix}.$$