AUTOMORPHIC FORMS AND THE LANGLANDS PROGRAM

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> Mathematics compares the most diverse phenomena and discovers the secret analogies that unite them.
>
> Joseph Fourier

The theory of automorphic forms is one of the most active areas in modern mathematics. However, it is notoriously difficult for overwhelming and technical definitions and constructions. I decided to write such a note when I was preparing my thesis. There are, of course, a lot of excellent references. I didn't mean to write down everything clearly. Instead, I just want to show some intuitions behind the complicated constructions and theorems. I will try to focus on simple examples to make everything visible. This note could be a guide for further reading, but not a rigorous reference.

The story begins with the classical *modular forms*. They are holomorphic functions on the upper half-plane satisfying certain transformation formulas with respect to an arithmetic subgroup of $SL_2(\mathbb{Z})$. The Fourier coefficients of modular forms have very good arithmetic properties and have interesting applications to classical number theory problems (sum of two squares theorem, quadratic forms, congruence relations, etc. See [BGHZ], Chapter one). An infinite sequence of mutually commutative operators, called the *Hecke operators*, act on the space of modular forms. L-functions are defined for eigenforms in terms of their Fourier coefficients. Two important properties of L-functions are their Euler product and functional equations. The modular forms can also be interpreted as sections of Hodge bundles over modular curves. This moduli interpretation makes it possible to regard Hecke operators as correspondences on modular curves.

Besides the classical modular forms, there is another class of functions called the *Maass forms* which share similar properties with the modular forms. To unify these two classes of functions, we need to consider the representations $SL_2(BR)$, in particular the regular representation $L^2(\Gamma\backslash SL_2(\mathbb{R}))$ where Γ is an arithmetic subgroup of $SL_2(\mathbb{R})$. It turns out that the classical modular forms generate discrete series, depending only on the weights of the modular forms. The Maass forms generate the principal series representations. Therefore, we could simultaneously study modular forms of all weights.

Modern automorphic form theory interprets the classical modular forms as automorphic representations. We consider the space of automorphic forms $\mathcal{A}(GL_2)$ on $GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})$, which is a $GL_2(\mathbb{A})$-representation under right translation. Classical modular forms then generate subrepresentations inside $\mathcal{A}(GL_2)$. In adelic language,

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we emphasize the local properties of modular forms [1]. In particular, Hecke operators is nothing but the convolution algebra of certain functions. Another advantage is that we could study modular forms with respect to all congruence subgroups [2].

We can easily extend the theory to general reductive algebraic groups G defined over a global field F. For example, a Dirichlet character (or more generally, a grössencharacter) is just an automorphic form of the group $GL(1)$ over $\mathbb{Q}$ (resp. a number field F).Then the automorphic form theory is essentially the harmonic analysis theory over reductive algebraic groups. An interesting problem is then to compare automorphic forms of different algebraic groups over various number fields, this is exactly the goal of Langlands program.

A standard problem in number theory is to study the decomposition of primes under field extensions. The famous Fermat's theorem asserts that a prime p is a sum of two squares if and only if $p \equiv 1 \mod 4$. Let $K := \mathbb{Q}(i)$ and we consider the field extension $K/\mathbb{Q}$. Then the theorem describes the decomposition of all primes except the only ramified prime $p = 2$: p splits completely if $p \equiv 1 \mod 4$, while p inerts if $p \equiv 3 \mod 4$. The decompositon behavior of a prime could be written in terms only of one congruence relation. This is true for any abelian extensions and is thoroughly studied in class field theory. If the field extension is not abelian, things become much more complicated.

We could also explain this in terms of Galois representations. Given a finite field extension K/L, any unramified prime $\mathfrak{P}^3$ of L defines a conjugacy class, the Fronebius element, in $\mathrm{Gal}(K/L)$. The the decomposition properties could be read off from the action of the Frobenius element on various representations. More generally, a central problem in number theory is to study the absolute Galois group Gal_F for a local or global field F. The standard method to study groups is to study their representations, i.e. Galois representations. To any Galois representation of a global field F, we may associate a L-function. If the Galois representation is motivic, the L-function controls the arithmetic of the corresponding variety or motive.

In the abelian case, the decomposition behavior is reflected in a Dirichlet character, i.e., an automorphic representation of $GL(1)$. This is because all representations of an abelian groups is just a character. For nonabelian extensions, we need to consider representations on higher dimensional spaces, and the properties of these representations should be reflected in automorphic representations of $GL(n)$ for $n \geq 2$.

The Langlands program seeks to relate Galois groups in algebraic number theory to automorphic forms and representation theory of algebraic groups over local fields and

[1]The Euler product of eigenforms suggests that we need to study modular forms locally. But this is hidden if we only discuss representations of $SL_2(\mathbb{R})$

[2]The congruence subgroups are the "real arithmetic subgroups" as a lot of arithmetic theorems only holds for congruence subgroups, say, Selberg's conjecture on the spectrum of the Laplacian on $\Gamma\backslash\mathbb{H}$

[3]Almost all factors of the Euler product are determined by the action of the corresponding Frobenius elements. Note that these Frobenius elements form a dense subgroup of the absolute Galois group by Chebotarev's density theorem.

adeles. It has been described by Edward Frenkel as "a kind of grand unified theory of mathematics." This huge bridge is so powerful that a simple known case could solve Fermat's Last Theorem.

Roughly speaking, the *Langlands correspondence* is a classification theory of automorphic representations. It asserts that (equivalent classes) Galois representations correspond to (packets of) automorphic representations in a compatible manner. Under this correspondence, the standard operators on the Galois representation side should determine natural operators on the automorphic representation side. This is the Langlands' *principle of functoriality.*

A natural construction of Galois representation is to consider the action of Gal_F on étale cohomology groups of varieties over F. In this case, the study of automorphic representations controls the properties of the varieties.

The first section is a quick review of algebraic groups. The ultimate goal is to define L-groups. The second section starts the discussion of automorphic forms. Two main points are: (1). the local decompositions of automorphic representations; (2). the decomposition of the space of automorphic forms via the Eisenstein series. The discussion of the Langlands program in the third section begins with a brief review of class field theory, which is the Langlands correspondence for $GL(1)$. After the statement of the Langlands correspondence, we briefly review known results and possible methods for the Langlands correspondence.

1. Algebraic Groups

Let F be a field. A *group scheme* G over F is a contravariant functor from the category of F-algebras to the category of groups. That is, to any F-algebra R, $G(R)$ (the set of R-points) has a natural group structure, and the group structures for different F-algebras are naturally compatible. An *affine group scheme* is a group scheme representable by an F-algebra A: $G(R) = \mathrm{Hom}_{F-\mathrm{alg}}(A, R)$ for all F-algebra R. An *affine algebraic group* is a (smooth) affine group scheme of finite type over F. In this case, the algebra A is finitely generated over F, or equivalently, A is isomorphic to the quotient of some polynomial ring over F. An affine algebraic group is automatically a linear algebraic group, i.e., an affine subgroup of $GL(N)$ for some positive N. I will simply call a linear algebraic group an algebraic group or even a group.

A connected algebraic group G over an algebraically closed field is called *semisimple* if every smooth connected solvable normal subgroup of G is trivial. A connected algebraic group G over an algebraically closed field is called *reductive* if the largest smooth connected unipotent normal subgroup of G is trivial. A group G over an arbitrary field F is called semisimple or reductive if the base change $G_{\overline{F}}$ is semisimple or reductive. We always assume that G is a reductive algebraic group over F in this note.

An algebraic group is a combination of geometry and arithmetic. The geometry is the same as that of classical Lie theory. The only thing we need to do is to translate the analytically defined objects into algebraically defined objects. The arithmetic properties make the theory of algebraic groups more interesting and useful.

1.1. **The geometry of algebraic groups.** We assume G is defined over an algebraically closed field $F = \overline{F}$ in this subsection.

1.1.1. *Root data.* The classical Lie group structure theory is based on the root decomposition of the Lie algebra. We can algebraically define Lie algebras and root decomposition for algebraic groups. The Lie algebra of the algebraic group is just the tangent space at $1 \in G$, the elements are derivatives from the function ring to F. The Lie bracket is defined to be the commutator of derivatives. In practice, we know that the Lie algebra of $GL(N)$ is $\mathfrak{gl}_n = M_{n\times n}$, with the Lie bracket defined as $[A, B] = AB - BA$. For general algebraic groups, we choose an embedding $G \hookrightarrow GL_n$ and use the induced Lie algebra. We have a natural adjoint action of G on $\mathfrak{g}$: $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$. When $G = GL_n$, it is the usual conjugation. For general G, just fix an embedding and consider the restriction.

Recall that a *root datum* is a quadruple $(X^*, X_*, \Phi, \Phi^\vee)$ consisting of a pair of free abelian groups X^* and X_* with a perfect pairing $\langle \cdot, \cdot \rangle : X^* \times X_* \to \mathbb{Z}$, together with finite subsets $\Phi \subset X^*$, $\Phi^\vee \subset X_*$ in one-to-one correspondence such that

- $\langle \alpha, \alpha^\vee \rangle = 2$;
- For each $\alpha \in \Phi$, the reflection map s_α defined by $s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$ induces an automorphism of the root system. The group generated by s_α is a finite group.

We say that a root datum is reduced if Φ does not contain 2α for any $\alpha \in \Phi$.

Now let G be a connected reductive group. We may canonically associate a root datum as follows. Let $T \leq G$ be a split maximal torus. We consider the adjoint action of T on $\mathfrak{g}$ and get a root system $\Phi(G, T) = (\Phi, V)$. The elements in Φ are the roots $\alpha \in X^*(T)$ (the group of characters of T) and $V = \langle \Phi \rangle \otimes_{\mathbb{Z}} \mathbb{R}$. We may also define the duals $\alpha^\vee \in \Phi^\vee \subset X_*(T)$ (the group of cocharacters of T) and construct the dual root system $(\Phi^\vee, V^\vee)$. The quadruple $\Psi(G, T) := (X^*(T), X_*(T), \Phi, \Phi^\vee)$ is then a root datum.

It turns out that the root datum determines the reductive group.[4]:

Theorem 1.1 (Chevalley-Demazure). *If $F = \overline{F}$, then the map from the isomorphism classes of connected reductive groups over F and the isomorphism classes of reduced root data given by mapping G to $\Psi(G, T) := (X^*(T), X_*(T), \Phi, \Phi^\vee)$ is a bijection.*

Remark 1.1. The root datum of a reductive algebraic group G depends on the choice of a maximal torus T. However, any two maximal tori are conjugate to each other. Therefore, the (isomorphic class of the) root datum is independent of the choice of T.

why we need root datum?

1.1.2. *Automorphisms of algebraic groups.* There are natural automorphisms called *inner* automorphisms which are simply defined as conjugacy by elements of $G(F)$. $\mathrm{Inn}(G)$, the group of inner automorphisms, is isomorphic to $G/Z(G)$. $\mathrm{Inn}(G_{\overline{F}})$ is a normal subgroup of $\mathrm{Aut}(G_{\overline{F}})$. The quotient group $\mathrm{Aut}(G_{\overline{F}})/\mathrm{Inn}(G_{\overline{F}})$ is called the group of *outer* automorphisms. Inner automorphisms do not change the root system and the group of outer automorphisms can be identified with the group of automorphisms of

[4]Note that root system is not enough to determine complex Lie groups: at least we may quotient by a finite group without changing the root system

the associated Dynkin diagrams D. There is a perfect theory on the classification of Dynkin diagrams, and their automorphism groups are well known. In summary, we have the following short exact sequence:

$$1 \to \mathrm{Inn}(G_{\overline{F}}) \to \mathrm{Aut}(G_{\overline{F}}) \to \mathrm{Aut}(D) \to 1 \tag{1}$$

1.2. **Forms of algebraic groups and Galois descent.** Now let G be a general field and let G be an algebraic group over F. An algebraic group H defined over F is called an F-form of G if there exists an isomorphism ϕ between G and H over $\overline{F}$, i.e., $\phi : G_{\overline{F}} \cong H_{\overline{F}}$. We want to classify all F-forms of G (up to canonical isomorphisms, of course).

An element $\phi \in \mathrm{Aut}(G_{\overline{F}})$ is uniquely determined by its action on $G_{\overline{F}}(\overline{F})$. The natural action of $\mathrm{Gal}_F := \mathrm{Gal}(\overline{F}/F)$ on $\mathrm{Aut}(G_{\overline{F}})$ is given by the formula $\psi^{\sigma}(g) = \sigma\psi\sigma^{-1}(g)$ for $\sigma \in \mathrm{Gal}_F, \psi \in \mathrm{Aut}(G_{\overline{F}})$ and $g \in G_{\overline{F}}(\overline{F})$. With this Galois action, ee can define the cohomology set $H^1(\mathrm{Gal}_F, \mathrm{Aut}(G_{\overline{F}}))$. This is only a set as $\mathrm{Aut}(G_{\overline{F}})$ is non-abelian. Now given an F-form H with an isomorphism $\phi : G_{\overline{F}} \to H_{\overline{F}}$, the map sending $\sigma \in \mathrm{Gal}_F$ to the automorphism $\phi^{-1} \circ \phi^{\sigma} = \phi^{-1} \circ \sigma \circ \phi \circ \sigma^{-1}$ (the difference of the Galois actions on the two groups) defines a 1-cocycle of Gal_F with values in $\mathrm{Aut}(G_{\overline{F}})$. This map descends to a parameterization of the F-forms of G.

Proposition 1.2. *The F-forms of G are parameterized by the cohomology set $H^1(\mathrm{Gal}_F, \mathrm{Aut}(G_{\overline{F}}))$. The neutral element corresponds to an F-form isomorphic to G over F.*

An *inner form* of G is an F-form H whose associated group cohomology element lies in $H^1(\mathrm{Gal}_F, \mathrm{Inn}(G_{\overline{F}}))$. The inner form defines an equivalence relation on the set of F-forms of G. From the long exact sequence associated with (1), we know that two F-forms are inner forms of each other if and only if the associated cocycle acts trivially on the Dynkin diagrams.

1.2.1. *Galois descent.* We give a formal derivation of the classification theorem. The reference is [GW] or [Wat].

We first consider *faithfully flat descent*. Let $S' \to S$ be a quasi-compact faithfully flat scheme, we need to study when an object X' over S' is the base change of an object X over S. Under certain mild conditions, this happens if and only if there is an isomorphism φ over the two base-changes to $S'' = S' \times S'$, and the isomorphism φ satisfies a cocycle condition when we consider its base-changes to $S' \times S' \times S'$. An example is just the gluing of schemes, morphisms, and sheaves over Zariski open sets.

Galois descent is a special case of faithfully flat descent. Let K/F be a finite Galois field extension with Galois group $\mathrm{Gal}_{K/F}$. Then $\mathrm{Gal}_{K/F}$ has an action on objects over K. The descent data can be expressed in terms of the Galois group $\mathrm{Gal}_{K/F}$. And the Galois descent asserts that $\mathrm{Gal}_{K/F}$-equivariant objects over K are the same thing as objects over F. In particular, $\mathrm{Gal}_{K/F}$-invariant K-vectors spaces are F-vector spaces.

Let G be an algebraic group over F. We assume the F-form H is isomorphic to G over some finite Galois field K. This means that we want to study the descent data of G_K. The original form G-defines a descent data, and all other data can be obtained from an automorphism of $G_{K \times K}$ with cocycle conditions. So we may write descent data in terms of group cohomology. The Galois cohomology set is just the descent data associated with the F-form H: the family of $\mathrm{Aut}(G)$ is a homomorphism, the cocycle condition is the $\mathrm{Gal}_{K/F}$-equivariant condition, elements in the coboundary determines the isomorphic F-forms. Taking a limit over K, we get the classification theorem.

Remark 1.2. Galois descent could also be considered a special case of descent along torsors. Let K/F be a finite Galois field extension with Galois group $\mathrm{Gal}_{K/F}$. Then K is a $\mathrm{Gal}_{K/F}$-torsor over F. The descent along torsors says that G-equivariant objects along a torsor are the same thing as objects over the base scheme.

1.3. **Rationality.**

1.3.1. *The hierarchy of forms.* We say that G is *split* if G contains a maximal torus that is split. Of course, G is split if G is defined over an algebraically closed field, and G_K is always split for some field extension K/F. We call G is *quasisplit* if G contains a Borel subgroup B over F. A quasi-split group is determined by a homomorphism from the Galois group Gal_F to the symmetric group of the diagram.

(Galois actions!)

Start with a F-split G_s, we first take twists on Dynkin diagrams to get quasisplit forms, then we could take inner forms of quasisplit forms. From the short exact sequence (1), any F-form can be obtained in such a two-step twisting.

1.3.2. *Localizations and unramified groups.* A global field is a number field or a function field of curves defined over a finite field. A local field is just a completion of a global field.

Now let G be defined over a nonarchimedean local field F. G is called *unramified* if G is quasisplit and splits over some finite unramified extension of F. These are groups defined by base change from a smooth reductive group scheme over $\mathcal{O}$. They are in bijection with the reductive groups defined over the residue field k. Note that every reductive group defined over a finite field is quasi-split.

Let G be an algebraic group defined over a number field F. We consider the base changes of G over various completions F_v. G is unramified at a place v if G_{F_v} is unramified over F_v. It is obvious that G is unramified at all but finitely many places v.

1.4. **Some examples.**

1.4.1. *Torus.* Recall that a torus T over F is an algebraic group whose base change $T_{\overline{F}}$ is isomorphic to a finite product of $\mathbb{G}_{m,\overline{F}}$. A split torus T is a product of $\mathbb{G}_{m,F}$. Now all forms are outer as G is abelian. Let G_0 be the split torus of rank n then the root datum associated with G_0 is the free abelian group of rank n with trivial Galois action. Therefore, the set of rank n tori is parameterized by the group $\mathrm{Hom}(\mathrm{Gal}_F, GL_n(\mathbb{Z}))$.

Real tori of rank one: $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \{\pm 1\}$ with -1 representing the conjugation map. $GL_1(\mathbb{Z}) = \{\pm 1\}$. So there are two real tori of rank one: the split torus $\mathbb{G}_{m,\mathbb{R}}$ and the compact torus $U(1)$.

Rational tori of rank one: We need to consider the set $\mathrm{Hom}(\mathrm{Gal}_{\mathbb{Q}}, \{\pm 1\})$. The trivial map corresponds to the split torus $\mathbb{G}_m$. Given a nontrivial homomorphism $\varphi : \mathrm{Gal}_{\mathbb{Q}} \to \{\pm 1\}$, the kernel of φ determines a quadratic extension K by Galois theory. We may construct a rational tori T_K whose rational points are exactly the norm one element in K. All rational tori are constructed as above.

Real tori of higher rank: Each matrix $A \in GL_2(\mathbb{Z})$ satisfying $A^2 = I$ defines a real torus of rank n. The identity matrix I corresponds to the split tori. Since $A^2 = I$, its eigenvalues are 1 or -1. (???) What is the meaning of numbers of eigenvalue 1? products of $\mathbb{R}^\times$ and $U(1)$? Up to conjugate, A is diagonal with $\{\pm 1\}$???

Rational tori of higher rank: This is Galois representation.

1.4.2. *Real forms of classical algebraic groups.* Let G be the split form of a simple classical group. Then we are required to study $H^1(\mathbb{Z}/2\mathbb{Z}, \mathrm{Aut}(G_\mathbb{C}))$. I will just consider the real forms of SL_n in this part.

The real forms of simple complex Lie algebras are classified in terms of Vogan diagrams. The classification theorem and the full list of real forms of classical and exceptional types can be find in [Kna4]. The real forms of type $A_n (n \geq 2)$ depends on the parity of n. They have a split form $SL(n = 1)$, and a unique quasi-split form because the automorphism group of the Dynkin diagram is generated by the reflection. Let A be the antidiagonal matrix in $GL_n(\mathbb{C})$ with alternating 1s and -1s going from the bottom left to the top right. Then $\rho(g) = Ag^{-t}A^{-1}$ represents the nontrivial automorphism.

- If $n = 2r$ is even, we have the split form SL_{n+1}, and unitary forms $SU_{p,n+1-p} (0 \leq p \leq [\frac{n}{2}])$. The inner forms of SL_{n+1} is just itself. All unitary forms are outer forms of the split form, and they are inner forms of each other. The quasi-split form is the compact form SU_{n+1}.
- If $n = 2r - 1 > 1$ is odd, we have the split form SL_{n+1}, a quaternion form $SL_r(\mathbb{H})$, and unitary forms $SU_{p,n+1-p} (0 \leq p \leq [\frac{n}{2}])$. The inner forms of the split form is SL_{n+1} and $SL_r(\mathbb{H})$. All unitary forms are outer forms of the split form, and they are inner forms of each other. The quasi-split form is the unitary group $SU_{r,r}$.[5]

We may use the Galois cohomology to study the classification. Here we study the forms of the split form. The Galois action is just the action on coefficients in this case. The short exact sequence of automorphisms for $SL(n+1, \mathbb{C})$ reads

$$1 \to PSL_{n+1}(\mathbb{C}) \to \mathrm{Aut}(SL_{n+1}(\mathbb{C})) \to \{\pm 1\} \to 0.$$

We first study the inner forms, that is, $H^1(\mathrm{Gal}_\mathbb{R}, PSL_{n+1}(\mathbb{C})) = H^1(\mathrm{Gal}_\mathbb{R}, PGL_{n+1}(\mathbb{C}))$. From the short exact sequence

$$1 \mapsto \mathbb{C}^\times \to GL_{n+1}(\mathbb{C}) \to PGL_{n+1}(\mathbb{C}) \to 1,$$

we have the exact sequence of cohomology groups[6]

[5]If $r = 1$, we have only a split form $SL(2)$ and a compact form $SU(2)$. $SU(1,1)$ is isomorphic to $SL(2)$ by the cayley transform from the upper half plane to the unit disc. The quaternion group $SL_1(\mathbb{H})$ is isomorphic to $SU(1)$ by the conjugation on the norm one quaternions. $SU(2)$ is an inner form of $SL(2)$ since there is no outer automorphism for $SL_2(\mathbb{C})$.

[6]We could only define H^0 and H^1 for non-abelian G-modules, and the long exact sequence ends at $H^1(G, C)$ for a short exact sequence $1 \to A \to B \to C \to 1$ of non-abelian G-modules. However, if A is central in B (thus abelian), we may extend the long exact sequence to $H^2(G, A)$. See [Ser2].

$$H^1(\mathrm{Gal}_{\mathbb{R}}, GL_{n+1}(\mathbb{C})) \to H^1(\mathrm{Gal}_{\mathbb{R}}, PGL_{n+1}(\mathbb{C})) \to H^2(\mathrm{Gal}_{\mathbb{R}}, \mathbb{C}^{\times}).$$

The first term is zero by the (generalized) Hilbert 90, the second is isomorphic to $\mathbb{R}^{\times}/\mathbb{R}^{+}$ from the cohomology of finite cyclic groups. Therefore, $H^1(\mathrm{Gal}_{\mathbb{R}}, PGL_{n+1}(\mathbb{C}))$ is of order either one or two.

Writing everything explicitly, we need to decide whether given $D \in GL_{n+1}(\mathbb{C}))$ with $D\overline{D}$ a scalar, we can find B and λ such that $D = \lambda B^{-1}\overline{B}$. If $n+1$ is even, $D = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ represents a nontrivial class because it has a negative determinant. If $n+1$ is odd, $D/\sqrt{\lambda}$ has determinant one, and is of the form $B^{-1}\overline{B}$ by the vanishing of $H^1(\mathrm{Gal}_{\mathbb{R}}, GL_{n+1}(\mathbb{C}))$. This recovers the classification of inner forms of type A_n.

Remark 1.3. The long exact sequence associated with the $\mathrm{Aut}(G)$-exact sequence asserts that $H^1(\mathrm{Gal}_{\mathbb{R}}, PSL(\mathbb{C}))$ is a subset of $H^1(\mathrm{Gal}_{\mathbb{R}}, \mathrm{Aut}(G(\mathbb{C})))$. Therefore the inner forms are parameterized by $H^1(\mathrm{Gal}_{\mathbb{R}}, PSL(\mathbb{C}))$. I believe this is true for general F-forms of algebraic groups, but I do not know proof.

Now we consider the outer forms [7]. We just need to specify an outer automorphism $O(x) := g \mapsto xg^{-t}x^{-1}(x \in PGL_{n+1}(\mathbb{C}))$. We need to find $x \in GL_{n+1}(\mathbb{C})$ with $x\bar{x}^{-t} = \lambda \in \mathbb{C}^{\times}$ subject to the equivalence relation defined by $x \sim \mu x$, $x \sim Bx\bar{B}^t$ and $x \sim x^{-t}$. This is exactly the equivalence classes of Hermitian forms, and we recover the inner forms as various unitary group.

All real forms of type B_n and C_n are inner. The quasisplit form of type D_n is the group $SO(2n-1,1)$.

1.4.3. *inner forms of GL_n.* Let G be an inner form of G' and choose an isomorphism ψ. Then the base changes over various completions are also inner forms. The associated cohomology cycle is just the natural map $H^1(\mathrm{Gal}_F, \mathrm{Aut}(G(\overline{F}))) \to H^1(\mathrm{Gal}_{F_v}, H^1(\mathrm{Aut}(G(\overline{F}_v))))$ induced from the inclusion $\mathrm{Gal}_{F_v} \hookrightarrow \mathrm{Gal}_F$ and $\overline{F} \to \overline{F}_v$.

Now let $G' = GL(n)$. The general classification of reductive groups over local and global fields assigns a family of invariants

$$\{\mathrm{inv}_v = \mathrm{inv}_v(G,\psi)\}$$

to (G,ψ) parameterized by the valuations v of F. The local invariant inv_v is attached to the localization of (G,ψ) at F_v, and takes values in the cyclic group $\mathbb{Z}/n\mathbb{Z}$. It can assume any value if v is nonarchimedean, but satisfies the constraints $2\ \mathrm{inv}_v = 0$ if $F_v \cong \mathbb{R}$ and $\mathrm{inv}_v = 0$ if $F_v \cong \mathbb{C}$. The elements in the family $\{\mathrm{inv}_v\}$ vanish for almost all v, and satisfy the global constraint $\sum \mathrm{inv}_v = 0$. Conversely, given $G' = GL(n)$ and any set of invariants $\{\mathrm{inv}_v\}$ in $\mathbb{Z}/n\mathbb{Z}$ with these constraints, there is an essentially unique twist (G,ψ) of G^* with the given invariants.

If $F = \mathbb{Q}$. Then G is a central simple algebra of degree n. Let $\mathbb{Q}_p$ be a nonarchimedean local field, let $c \in \mathbb{Z}/n\mathbb{Z}$, if $(c,n) = 1$, we get $\varphi(n)$-non-isomorphic central division algebras of rank n over F. If $(c,n) \neq 1$, G_v is a matrix over a central division algebra of a smaller rank.

If $n = 2$. There are two possibilities of central simple algebras over a local field. The split algebra $M_{2\times 2}(\mathbb{Q}_v)$ and the unique quaternion algebra D. From the above

[7] We have to explicitly compute the outer forms since we do not have a good long exact sequence.

discussion, an inner twist of $GL(2)$ is uniquely determined by an even number of places, above which the group G ramifies.

1.5. **Dual group and L-group.** Given an algebraic group G with root datum $\Psi(G,T) := (X^*(T), X_*(T), \Phi, \Phi^\vee)$, the dual root datum is defined as $\Psi^\vee(G,T) := (X_*(T), X^*(T), \Phi^\vee, \Phi)$. By Chevalley-Demazure theorem, $\Psi^\vee$ gives rise to a complex reductive connected algebraic group $\hat{G}$, called the *complex dual* of G.

In practice, this duality preserves the types A_n and D_n and interchanges the types B_n and C_n. In addition, it interchanges the adjoint and simply connected forms of the relevant group.

Remark 1.4. The dual groups distinguish F-forms up to inner forms. It determines completely a quasisplit form: each inner class has a unique quasi-split F-form.

The *L-group* ${}^LG$ of a reductive algebraic group G is a group introduced by Langlands that controls the representation theory of G. We first explain a Galois action on the complex dual group $\hat{G}$. For simplicity we first assume that G is quasisplit, we can always find a Borel subgroup B. This Borel subgroup defines a set of positive root system $\Psi_0(G,B,T)$. The Galois action on this based root datum induces a Galois action on its dual root datum, and then lifts to an action on $\hat{G}$. If G is split, Gal_F acts trivially. In general, we have a based root datum over $\overline{F}$. If $\gamma \in \mathrm{Gal}_F$, it preserves $T(\overline{F})$ and Δ, but not preserve Δ^+. However, we could find $g_\gamma \in G(\overline{F})$ such that $g_\gamma \gamma$ carries Δ^+ to itself. The resulting action on Δ^+ is independent of the choice of g_γ. In any case, we have a natural action of Gal_F on $\hat{G}$. The L-group ${}^LG$ is defined as $\hat{G} \rtimes \mathrm{Gal}_F$.

Remark 1.5. There is a modification of the L-group. Choose a finite extension K/F such that G_K is split, the Galois action of Gal_F on $\hat{G}$ factors through the finite quotient $\mathrm{Gal}_F/\mathrm{Gal}_K = \mathrm{Gal}_{K/F}$. We simply define the L-group as the semi-diret product $\hat{G} \rtimes \mathrm{Gal}_{K/F}$. This definition does not lose information since we are just collapsing the components with the same structure. The L-groups then have only finitely many components. This variant is used in Arthur's classification theory.

Let G be an algebraic group over a number field F. Then its localizations G_{F_v} has the same root datum and therefore the same complex dual $\hat{G}$. The Galois action of Gal_{F_v} is just the restriction of the action of Gal_F. Therefore, we have canonical morphisms ${}^LG_{F_v} \to^L G_F$.

2. Automorphic representations

2.1. **Automorphic Forms.** $L^2(G(F)\backslash G(\mathbb{A}))$: quotient of a discrete group, like $\Gamma\backslash D$. harmonic analysis

2.1.1. *The automorphic space.* Let F be a number field[8]. If we choose an embedding $F \to \mathbb{C}$, the induced topology is not locally compact. Actually, there is no canonical

[8] We may also consider another type of global field, function fields over finite groups. The only difference is that they do not have archimedean completions

embedding. We need a locally compact group associated with F. The various completions are locally compact, and we take the restricted product of these local fields to get a locally compact group. This is the ring of adeles $\mathbb{A}_F$.

There are two types of places of F: (1). The non-archimedean ones. Each prime $\mathfrak{P}$ of $\mathcal{O}_F$ defines a non-archimedean place, and the corresponding completion is $F_{\mathfrak{P}}$. (2). The archimedean places are defined by embeddings $i : F \hookrightarrow \mathbb{C}$. We use the symbol v to denote a place, and F_v to denote the corresponding completion of F. If $F = \mathbb{Q}$, the non-archimedean places are the primes p, with completions $\mathbb{Q}_p$'s. The only archimedean place is defined by the usual absolute value, and the completion is $\mathbb{R}$.

$\mathbb{A}$ is the restricted direct product of $\mathbb{Q}_v$'s. With the restricted direct product topology, $\mathbb{A}$ is a locally compact group. $\mathbb{Q}$ embeds into $\mathbb{A}$ as a discrete subgroup via the diagonal embedding. The coset space $\mathbb{Q}\backslash\mathbb{A}$ is compact.

Let G be an algebraic group defined over a number field F. We may explicitly describe $G(\mathbb{A}_F)$, the groups of $\mathbb{A}_F$-valued points of G, in coordinates. Fix an embedding $i : G \to GL(V)$ form F-vector space and pick a lattice $\Lambda \in V$. For any nonarchimedean place v, let K_v (*hyperspecial maximal compact subgroup*) be the stabilizer of $\Lambda \otimes F_v$ in $G(F_v)$. We get a family of compact open subgroups. The sequence (K_p) depends on i, Λ, but different choices give equivalent sequences in the sense that only finitely many K_p's could be changed under different choices. Elements in $G(\mathbb{A}_f)$ are sequences $(g_v) \in \prod G(F_v)$ such that $g_v \in K_v$ for almost all places. This description is independent of the choice of the sequence (K_p).

We have the inclusion $G(F) \subset G(\mathbb{A}_F)$ defined by diagonal embedding. $G(F)$ is a discrete subgroup of $G(\mathbb{A}_F)$. The coset space $G(F)\backslash G(\mathbb{A}_F)$ has a fundamental domain which can be covered by a sufficiently large Siegel set. $G(F)\backslash G(\mathbb{A}_F)$ has finite volume if G is semisimple; it is compact if G is anisotropic.

To study the geometry of $G(F)\backslash G(\mathbb{A}_F)$, we need the strong approximation theorem. Recall that if S is a finite set of places. them $G(\mathbb{A}_{F,S})$ means $\prod\limits_{v\in S} G(F_v)$, and $G(\mathbb{A}^S)$ means the elements supported outside S.

Proposition 2.1 (Strong approximation theorem)**.** *Assume that G is simply-connected and S is a finite set of places of F such that $G(\mathbb{A}_{F,S})$ is not compact, then $G(F)$ is dense in $G(\mathbb{A}^S)$.*

Remark 2.1. The weak approximation asks whether $G(F)$ is dense in $G(\mathbb{A}_{F,S})$. If F is an algebraic number field then any group G satisfies weak approximation with respect to the set of infinite places.

The main theorem of strong approximation states that a non-solvable linear algebraic group G over a global field F has strong approximation for the finite set S if and only if its radical N is unipotent, G/N is simply connected, and each almost simple component H of G/N has a non-compact component H_s for some s in S.

Let U be an open compact subgroup of $G(\mathbb{A}_f)$, then the subgroup $\Gamma_U := G(F) \cap U$ is called a *congruence subgroup* of $G(F)$ [9]. The congruence subgroups of split classical

[9] Congruence subgroups should be considered as "real arithmetic subgroups". For example, any curve of genus $g \geq 2$ is an arithmetic quotient of the upper half plane, but only the quotient by congruence subgroups have dee arithmetic properties.

groups are exactly the classical congruence subgroups. If Γ is congruence, and G satisfies strong approximation, we have

$$\Gamma_U \backslash G(F_\infty) \cong G(F)\backslash/U.$$

Remark 2.2. If G is defined over $\mathbb{Q}$, choose an embedding $G \to GL(N)$, we may define congruence subgroups of G as those containing a principal congruence subgroup. The definition is independent of the embedding. To see that they are the same as those defined abouve, note that a basis of $G(\mathbb{A}_f)$ is defined by a finite product of local open subgroups over finite places and local open subgroups are exactly defined by congruence subgroups.

For general reductive G, the quotient is a finite union of arithmetic quotients of $G(F_\infty)$.

Example 2.2. Take $G = SL_2$ over $\mathbb{Q}$ and $S = \{\infty\}$. Fix a positive number N. Let I_p be the subgroup of $SL_2(\mathbb{Q}_p)$ consisting of elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $c \equiv 0 \mod p$. For $(p, N) = 1$, $I_p = SL_2(\mathbb{Z}_p)$. Define $K_0(N) = \prod I_p$, it is compact open in $G(\mathbb{A}_f)$. And the corresponding congruence subgroup is $\Gamma_0(N)$. The open modular curve $X_0(N) = \Gamma_0(N)\backslash SL_2(\mathbb{R})$ is therefore isomorphic to $SL_2(\mathbb{Q})\backslash SL_2(\mathbb{A})/K_0(N)$.

$G(F)\backslash G(\mathbb{A}_F)$ maps canonically to the congruence subgroup quotients, and defines a map $\varprojlim_U G(F)\backslash G(\mathbb{A}_F)/U \cong \varprojlim_\Gamma \Gamma\backslash G(F_\infty)$. When the strong approximation theorem holds for G, it is surjective. The kernel is always trivial: the intersection of all open compact subgroup of $G(\mathbb{A}_F)$ is the trivial group. Therefore, the automorphic space $G(F)\backslash G(\mathbb{A}_F)$ can be considered as a universal object of the arithmetic quotients of $G(F)$.

This seems to be a standard procedure. Given a bad object, the inverse limit of its "finite quotients" can be described in adelic language, therefore locally compact and have rich arithmetic properties. Let me give some examples.

- The ring of integers $\mathbb{Z}$ is a discrete set, we take quotients by its ideals, their inverse limit is $\hat{Z}$ which is the maximal compact subgroup of $\mathbb{A}_{\mathbb{Q},f}$. Actually, $\mathbb{A}_f = \hat{\mathbb{Z}} \otimes \mathbb{Q}$.
- If G is defined over $\mathbb{Q}$, $G(\mathbb{R})$ is a just Lie group. The inverse limit of congruence quotients is the automorphic space. It is then an arithnetic object.
- Let D be a Hermitian symmetric domain, which is just an analytic object. Its congruence quotients are algebrac objects, and their inverse limit is the connect Shimura variety.
- The roots of unity are torsion points in $\mathbb{C}$. The set of n-th root of unity is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. Let n-vary, these finite quotients form an inverse system with respect to divisibility, and the inverse limit is $\hat{\mathbb{Z}}$. This is the maximal compact subgroup of $\mathbb{A}_f$. If we consider torsion points on an elliptic curve, the same construction yields the Tate-module.

2.1.2. *Automorphic forms and automorphic representations.* Let G be a reductive algebraic group over F.

Definition 2.3. *A function f on $G(F)\backslash G(\mathbb{A}_F)$ is called an automorphic form if*

- *f is smooth. This means that the nonarchimedean component f_∞ is smooth, and locally constant in $G(\mathbb{A}_{F,f})$.*
- *f is right K-finite. Here $K = (K_p)$ is in the definition of $G(\mathbb{A}_F)$, and the condition means that the right K-translations of f span a finite dimensional vector space. Equivelently, f is K_∞-finite and is right invariant under an open compact subgroup of $G(\mathbb{A}_{F,f})$.*
- *f is of moderate growth. This means that $f(g)$ can be controlled by some powers of the matrix elements of g and $i(g)$ (after choose an embedding $G \to GL_N$).*
- *f is $Z(\mathfrak{g})$-finite. Here $Z(\mathfrak{g})$ is the center of the universal enveloping algebra. The $Z(\mathfrak{g})$-finiteness is equivalent to a system of differential equations for f_∞.*

We let $\mathcal{A}(G)$ denote the space of automorphic forms on G.

Definition 2.4. *An automorphic form f on G is called a cusp form if, for any parabolic F-subgroup $P = MN$ of G, the constant term*

$$f_N(g) = \int_{N(F)\backslash N(\mathbb{A}_F)} f(ng)dn$$

is zero as function on $G(\mathbb{A}_F)$.

Remark 2.3. In certain literature, the automorphic form is defined with a character $\omega : Z(\mathbb{A}_F) \to \mathbb{C}$, so the function f are functions on $G(F)\backslash G(A)$ such that $f(zg) = \phi(z)f(g)$. Our space is just the sum over all characters.

We let $\mathcal{A}_0(G)$ be the space of cusp forms on G.

2.2. **Automorphic representations.** Recall that $\mathcal{A}(G)$ denotes the space of automorphic forms on G. As usual , we define right translation r by $r(g)f(h) = f(hg^{-1})$. But we cannot regard $\mathcal{A}(G)$ as a $G(\mathbb{A}_F)$-module. At archimedean places, the K-finite property is not preserved by all $G(F_\infty)$-translations, so we can only get a $(\mathfrak{g}_\infty, K_\infty)$-module at archimedean places. The space $\mathcal{A}(G)$ is a $G(\mathbb{A}_{F,f}) \times (\mathfrak{g}_\infty, K_\infty)$-module. By abuse of language, we still say $\mathcal{A}(G)$ is a $G(\mathbb{A}_F)$-representation. An irreducible $G(\mathbb{A}_F)$-representation is called an *automorphic representation* if it is isomorphic to a subquotient of $\mathcal{A}(G)$. We study the properties of automorphic representations in this section.

2.2.1. *Harish-Chandra modules.* First, we need to define the Harish-Chandra module. We assume we are studying representations of Lie groups[10]. If $G(\mathbb{R})$ is compact, the Peter-Weyl theorem asserts that every irreducible representation (π, V) is finite-dimensional. The topology on finite-dimensional spaces is naive, and all vectors $v \in V$ are smooth. Th

To study the representations of noncompact Lie groups, we have to study infinite dimension representations. The first problem in infinite dimensional representation is the topology, we need to equip appropriate topology over the space so that the operator has continuous properties. Some functional analysis must be involved in this step. The second problem is that not all vectors are smooth vectors. All these problems come from

[10] $(\mathfrak{g}_\infty, K_\infty)$-module is just a finite product of Harish-Chandra modules over archimedean places, we also regard a complex Lie group as a real Lie group to study representations.

analysis. Harish-Chandra modules, or $(\mathfrak{g}, K)$-modules, is an algebraic modification of these representations. We could use it to study representations using algebraic methods to avoid subtle analysis issues, while at the same time no "representation structure" is lost in this modification. Let K be a fixed maximal compact subgroup of G. A $(\mathfrak{g}, K)$-module (π, V) is vector space which is at the same time a $\mathfrak{g}$-module and a K-module, the two-modules structures are compatible: (1). The infinitesimal representation of K is the restriction of $\mathfrak{g}$-representation. (2). the adjoint action of K on $\mathfrak{g}$ becomes the conjugation of matrices as operators on V. A G-representation defines canonically a $(\mathfrak{g}, K)$-module as follows. First to define smooth vectors, so we have to restrict to smooth vectors to get a $\mathfrak{g}$-module. Then we consider the space of K-finite vectors.

Two representations are called *infinitesimal equivalence* if their associated $(\mathfrak{g}, K)$-modules are isomorphic. this means the vector spaces are the same, with different completions. See [Sch2].

What representations are we interested in? The simplest generalization is unitary representations. The underlying spaces should be Hilbert spaces and $G(\mathbb{R})$ acts as unitary operators. Except for the trivial representation, all unitary representations are infinite-dimensional. We also need to consider a larger class of representations, called *admissible representations.* An irreducible $(\mathfrak{g}, K)$-representation V is called admissible if, for any K-representation π, the multiplicity of π (considered as a K-representation by restriction) in V is finite. Unitary representations are admissible. One reason why we need admissible representation is that the category of unitary representation is not good as it is not stable under standard representation theory operators, say, parabolic inductions. Another reason is that the classification theory for admissible representations is much simpler, while it is really hard to determine which admissible representation is actually unitary.

2.2.2. *Hecke algebras.* According to Grothendieck's philosophy, structure sheaves over spaces (or functions over spaces) are more fundamental objects than the spaces themselves. If G is a group, we define Hecke algebra $\mathcal{H}(G)$ as the space of certain functions over G, with the algebra structure defined by convolution. Therefore a G-representation (π, V) can be identified as a $\mathcal{H}(G)$-representation as follows: let $f \in \mathcal{H}(G)$, then we define $\pi(f)$ as $\int_G f(g)\pi(g)dg$ where dg is a Haar measure. Conversely, the Hecke algebra representation could recover the original representation: just approximate Dirac operators by functions. The advantage of the Hecke algebra interpretation is that it is simpler to study the structure of space of functions instead of single points.

The Hecke algebra is defined as follows:

- *Nonarchimedean Hecke algebras.* Let S be a set of nonarchimedean places of F. A function of f on $G(F_S)$ is called *smooth* if it is locally constant. In particular, if S is just a single point v, we get the smooth functions over $G(F_v)$. Similarly, if S contains all archimedean places of F, then a function on $G(\mathbb{A}_F^S)$ is smooth if it is locally constant. We also define the space $C_c^\infty(G(F_S))$ (resp. $C_c^\infty(G(\mathbb{A}_F^S))$) of smooth, compactly supported functions on $G(F_S)$ (resp. $G(\mathbb{A}_F^S)$). They are algebras under conolution of functions:

$$f * h(g) := \int f(x)h(x^{-1}g)dx.$$

In particular, if F is a number field, and $S=\infty$ is the set of archimedean places, then we hace $\mathcal{H}^\infty := C_c^\infty(G(\mathbb{A}_F^\infty))$.

$\mathcal{H}^\infty$ is the direct limit of Hecke algebras over finite places:

$$C_c^\infty(G(\mathbb{A}_F^\infty)) = \varinjlim_S (C_c^\infty(G(F_S)) \otimes_{v\notin S} \mathbb{1}_{G(\mathcal{O})_v}),$$

where S is a finite set of nonarchimedean places, v is in the complementary of S in the set of nonarchimedean places.

- Archimedean Hecke algebras. Let F be a number field. Then $G(F\otimes\mathbb{R})$ is a real reductive group. The hecke algebra is just the convolution algebra of compactly supported smooth functions. In this case, a smooth function is in the usual sense, that is, the partial derivatives of any order exist and are continuous.
- The global Hecke algebra is just the tensor product of the nonarchimedean hecke algebra and the archimedean Hecke algebra.
- If $G(F_v)$ is unramified, we also define the *spherical Hecke algebra* $\mathcal{H}(G_v//K_v)$ as the space of K_v-biinvariant compactly supported functions, with the algebra structure defined by convolution. It plays a central role in studying spherical representations.

2.2.3. *The tensor product theorem.* We start by defining a *restricted tensor product of vector spaces.* Let Ξ be a finite subset, and Ξ_0 be a finite subset. Let $\{W_v\}_{v\in\Xi}$ be a family of $\mathbb{C}$-vector spaces and choose ϕ_{0v}[11] of local representations. for each $v\in\Xi-\Xi_0$. For all sets $\Xi_0\subset S\subset\Xi$ of finite cardinality set $W_S := \prod_{v\in S} W_v$. If $S\subset S'$, there is a map $W_S\to W_{S'}$ defined by:

$$\otimes_{v\in S} w_v \mapsto \otimes(\otimes_{v\in S'-S}\phi_{0v}).$$

The vector space

$$W := \otimes' \otimes W_v := \varinjlim_S W_S$$

is the restricted tensor product of the W_v with respect to the ϕ_{0v}. Thus W is the set of sequences $(W_v)_{v\in\Xi}\subset \otimes_v W_v$ such that $w_v=\phi_{0v}$ for all but finitely many $v\in\Xi$.

Remark 2.4. The isomorphism classes of W in general depend on the choice of ϕ_{0v}. However, if we replace $\phi_{0,v}$ by nonzero scalar multiples we obtain isomorphism vector spaces.

Example 2.5. One has

$$C_c^\infty(G(\mathbb{A}_F^\infty)) \cong \otimes' C_c^\infty(G((F_v)))$$

with respect to the idempotents $e_{K_v} := \frac{1}{\mathrm{vol}(K_v)} 1_{K_v}$ where K_v is a hyperspecial subgroup.

Theorem 2.6 (Flath's theorem)**.** *Every admissible irreducible representation W of $C_c^\infty(G(\mathbb{A}_F))$ can be written as*

$$W \cong \otimes_v' W_v$$

[11]Unlike the direct limit of a family of abelian groups, we have to specify nonzero vectors for almost all indices. This is because we cannot tensor with the "canonical element" $0\in W_v$.

where the restricted tensor product is with respect to elements $\phi_{v_0} \in W_v^{K_v}$, $\dim W_v^{K_v} = 1$*, and the isomorphism interwins the action of* $C_c^\infty(\mathbb{A}_F)$ *with the action of* $\otimes_v'(G(F_v)$*, the restricted tensor product being with respect to the idempotents* e_{K_v}*.*

In particular, automorphic representations are restricted tensor products.

Let v be a nonarchimedean place, W_v be an irreducible smooth admissible representation of $G(F_v)$, then $\dim W_v^{K_v} \leq 1$. If $\dim V^K = 1$, the representation is called a *spherical* representation. One implication of Flath's theorem is that for almost all nonarchimedean places v, the representation W_v is spherical.

2.3. **Classical modular forms.** The interpretation of modular forms as automorphic representations: The point is that all operators can be interpreted as the right translations.

We only consider elliptic modular forms with respect to the full modular group $SL_2(\mathbb{Z})$. Elliptic modular forms for general congruence subgroups, or modular forms associated with more complicated groups (say, Siegel modular forms, Hilbert modular forms, Picard modular forms) will be omitted.

A modular form of weight k is just a holomorphic function $f(z)$ on the upper half plane $\mathbb{H}$ satisfies the functional equations

$$f(\gamma z) = \frac{1}{(cz+d)^k} f\left(\frac{az+b}{cz+d}\right), \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

$f(z)$ then has a Fourier expansion $f(z) = a_n q^n$ where $q = e^{2\pi i z}$. $f(z)$ is called a cusp form if $a_0 = 0$. Let S_k be the space of cusp form s of weight k. There is a family of linear operators T_n, called the Hecke operators, on S_k. They can be defined in terms of their action on the Fourier coefficients of cuap forms, see [Bum]. An eigenform f is a cusp form that is simitaneously eigenvaluea for all T_n.

Let f be an eigenform. We may associate an automorphic representation of π_f on $PGL_2(\mathbb{Q})\backslash PGL_2(\mathbb{A})$ in two steps. First, we have the isomorphism $PGL_2(\mathbb{R})/PGO_2 = \mathbb{H}$ where g is mapped to gi. f could be considered as a function on $PGL_2(\mathbb{R})$ that is right PGO_2-invariant but has a good transition formula with respect to the left $PGL_2(\mathbb{Z})$. We define a function ϕ_f on $PGL_2(\mathbb{Z})\backslash PGL_2(\mathbb{R})$ by the formula

$$\phi_f(g) = f(gi).$$

Then ϕ_f is left $PGL_2(\mathbb{Z})$-invariant but is an eigenfunction with eigenvalue $e^{2\pi i k}$ under the right translation of the circle $SO(2)$. From the isomorphism $PGL_2(\mathbb{Z})\backslash PGL_2(\mathbb{R}) = GL_2(\mathbb{Q})\backslash PGL_2(\mathbb{A})/U$ where U is a compact subgroup of $PGL_2(\mathbb{A}_f)$, we get a function on $GL_2(\mathbb{Q})\backslash PGL_2(\mathbb{A})$. The subspace generated by this vector is the automorphic representation π_f.

The archimedean component of π_f is the holomorphic discrete series D_{k-1}^+. The nonarchimedean components are all spherical representations. Let p be a prime, the Hecke eigenvalue of f is exactly the action of an element (?) in the local spherical Hecke algebra.

The multiplicity one theorem says that the association $f \to \pi_f$ is injective. Note that only the archimedean component is not enough to distinguish eigenforms with the same weight.

Remark 2.5. There are several other ways to define Hecke algebras and Hecke operators. But they are defined on automorphic forms over $\Gamma\backslash D$, instead of representations of G.

Hecke algebra can be defined as the algebra of double cosets with the algebra structure defined in terms of the multiplication of double cosets. For elliptic modular forms, the Hecke operators can be explicitly written down in terms of the actions on Fourier coefficients. See [Sch].

In most cases, the space $\Gamma\backslash D$ is a moduli space for certain algebraic structures (say, curves). The automorphic forms are then global sections of certain line bundles over $\Gamma\backslash D$. With the moduli interpretation, the Hecke operators can be defined as a correspondence defined in terms of operators on the geometric structures. The correspondence induces operators on global sections of line bundles. See [RS].

2.4. **Satake isomorphism.** We want to classify spherical representations of G over non-archimedean places. Almost all components of an automorphic representation is spherical. In many cases, these spherical components dertemines the automorphic representations via multiplicity one theorems.

The classification of spherical representation is equivalent to the structure theorem of the spherical Hecke algebra. This is why we prefer to consider representations as over hecke algebras rather than over the groups: function theory becomes a powerful tool then to study the representations.

In this subsection, F is a non-archimedean local field with a ring of integers $\mathcal{O}_F$. Let us assume for simplicity that G is a split reductive group over F and let K be a hyperspecial subgroup. Let $B = TN$ be a Borel subgroup of G, with maximal torus T. So $T \cong (GL_1)^r$ and $T(F) \cong (F^\times)^r$. Let $W := N_G(T)/T$ be the Weyl group of G. The action of the Weyl group W on the set of characters of $T(F)$ is defined

$$(w\chi)(t) = \chi(w^{-1}tw).$$

Let $\mathcal{H}(G(F)//K)$ be the spherical Hecke algebra of G. The Satake isomorphism tells us the structure of $\mathcal{H}(G(F)//K)$.

Theorem 2.7 (Satake isomorphism)**.** *Assume that G is split. There is an isomorphism of algebras*

$$S : \mathcal{H}(G(F)//K) \cong \mathbb{C}[\hat{T}]^{W(\hat{G},\hat{T})(\mathbb{C})}$$

where $\hat{G}$ is the complex dual of G and $\hat{T}$ is a maximal torus.

A sketch of the construction: Choose a Borel subgroup $B = TN$ containing T, normalize Haar measure dn so that $N \cap K$ gets total measure 1, and let $\delta(t)$ be the positive function on T defined by $\Delta(t) = d(tnt^{-1})/dn$ (modulus). For $f \in \mathcal{G}(\mathcal{F}), \mathcal{K}$, define

$$Sf(t) = \delta(t)^{1/2} \int_N f(tn)dn, \ t \in T.$$

The mapping S defines an isomorphism of $\mathcal{H}(G(F), K)$ onto the subalgebra $\mathcal{H}(T, T \cap K)^W$. See [Kna1] (P.294).

Remark 2.6. The RHS can be interpreted as the algebra of virtual finite-dimensional complex algebraic representations of $\hat{G}$ under tensor product, by taking the character [12]. So we can also write the above isomorphism as

$$\mathcal{H}(G(F)//K) \otimes \mathbb{C} \cong K_0(\mathrm{Rep}(\hat{G})) \otimes \mathbb{C}$$

One application of Satake isomorphism is to classify spherical representations of $G(F)$. First, recall that

Proposition 2.8. *If V is a smooth representation of a locally profinite group G and $K \subset G$ is an open compact subgroup, then the map $V \mapsto V^K$ defines a functor from the category of smooth representations of G to the category of modules for the Hecke algebra $\mathcal{H}(G//K)$. The restriction to the representations V of G with nonzero V^K defines an equivalence of categories.*

Now $\mathcal{H}(G(F)//K)$ is commutative, so the map $V \mapsto V^K$ induces a bijection of spherical representations with characters of $\mathcal{H}(G(F)//K)$. The character associated with a spherical representation V is exactly the action of $\mathcal{H}(G(F)/K)$ on the one-dimensional space V^K. From the Satake isomorphism,

$$\mathrm{Hom}_{\mathbb{C}-alg}(\mathbb{C}[\hat{T}]^{W(\hat{G},\hat{T})(\mathbb{C})}, \mathbb{C}) = \hat{T}/W(\hat{G},\hat{T})(\mathbb{C}).$$

On the other hand, every semisimple conjugacy class in $\hat{G}(\mathbb{C})$ intersects $\hat{T}(\mathbb{C})$ and two elements of $\hat{T}(\mathbb{C})$ are conjugate in G if and only if they are in the same $W(\hat{G},\hat{T})$-orbit. In conclusion,

Theorem 2.9. *Irreducible spherical representations of $G(F)$ are parameterized by semisimple conjugacy classes in $\hat{G}$. The semisimple class associated with an unramified representation π is called the Satake parameter of π.*

Example 2.10 (GL_n)**.** Let $G = GL_n$ over $\mathbb{Q}_p$. Its compact dual is $GL_n(\mathbb{C})$. The Weyl group is isomorphic to the group S_n. The maximal torus is the subgroup of diagonal matrices, isomorphic to $(\mathbb{G}_m)^n$ An element $w \in W \cong S_n$ is represented by the corresponding permutation matrix.

The Hecke algebra is $\mathcal{H}(G(F)//K) = \mathbb{C}[t_i^{\pm}, \cdots, t_n^{\pm}]^{S_n}$. From the theory of elementary divisors, we have

$$GL(n)(\mathbb{Q}_p) = GL_n(\mathbb{Z}_p)T(\mathbb{Q}_p)GL_n(\mathbb{Z}_p).$$

Each double coset can be written as the double coset associated with a diagonal matrix whose entries are powers of p. A permutation of the entries gives the same double coset since permutation matrices are in the subring $GL_n(\mathbb{Z}_p)$.

So the irreducible spherical representations are parameterized by the S_n-orbits of the n-tuples of complex numbers.

The Hecke algebra $C_c^\infty(GL_n(F)//GL_n(\mathcal{O}_F))$, as a $\mathbb{C}$-module, has a basis given by

$$\mathbb{1}_\lambda := GL_n(\mathcal{O}_F)\mathrm{diag}(\varpi^{\lambda_1}, \cdots, \varpi^{\lambda_n})GL_n(\mathcal{O}_F)$$

[12]The RHS in Satake isomorphism is the center of $Z(\mathfrak{g})$ by the Harish-Chandra isomorphism. But infinite dimensional representations are classified by its infinitesimal character by the theorem of highest weight.

with $\lambda := (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n$ and $\lambda_i \geq \lambda_{i+1}$. As a $\mathbb{C}$-algebra, it is generated by $\mathbb{1}_{\lambda(r)}$ with $\lambda(r) = (1, \cdots, 1, 0, \cdots, 0)(1 \leq r \leq n)$ and $\lambda = (-1, \cdots, -1)$.

On the generating set above the Satake isomorphism is given by $S(\mathbb{1}_{\lambda(r)}) = q^{r(n-r)/2}\mathrm{tr}(\wedge^r \mathbb{C}^n)$.

Now we explain the construction of the spherical representation associated with a semisimple conjugacy class. Choose a diagonal element $(t_1, \cdots, t_r)$ in the conjugacy class, then it defines an unramified character $\chi : T(F) \to \mathbb{C}^\times$. And we can construct the representation associated with this unramified character. Recall that χ is an *unramified* character if χ is trivial when restricted to $T(\mathcal{O}_F) \cong (\mathcal{O}_F)^r$. In this case, χ is of the form

$$\chi(a_1, \cdots, a_r) = t_1^{ord_p(a_1)} \cdots t_r^{ord_p(a_r)}, a_i \in \mathbb{Q}_p^\times$$

for some $t_i \in \mathbb{C}^\times$.

We may regard χ as a character of $B(F)$ using the projection $B(F) \to N(F)\backslash B(F) \cong T(F)$. Then we may form the induced representation

$$I_B(\chi) := \mathrm{Ind}_{B(F)}^{G(F)} \delta_B^{1/2} \cdot \chi^{13}.$$

Here, δ_B is the modulus character of B, defined by:

$$\delta_B(b) = |\det(Ad(b)|_{Lie(N)})|_{v(F)}.$$

Explicitly, the space of $I_B(\chi)$ is the subspace of $C^\infty(G(F))$ satisfying:

(a). $f(bg) = \delta(b)^{1/2}\chi(b) \cdot f(g)$ for any $b \in B(F)$ and $g \in G(F)$.
(b). f is right-invariant under some open compact subgroup U_f of $G(F)$.

Then $I_B(\chi)$ is an admissible representation of $G(F)$. These representations $I_B(\chi)$ are called the *principal series representations.*

Because of the Iwasawa decomposition $G(F) = B(F) \cdot K$, an element f of $I_B(\chi)$ is completely determined by its restriction to K. The constant function 1 on K lifts to a function

$$f_0(bk) = \delta_B^{1/2} \cdot \chi(b).$$

This function spans the one-dimensional space of K-invariant functions of $I_B(\chi)$.

Thus $I_B(\chi)$ has a unique irreducible subquotient π_χ with the property that $\pi_\chi^K \neq 0$. This is the spherical representation associated with the unramified character χ.

Remark 2.7. Nonsplit case, see Getz's book. This is why we need Galois twist.

2.5. **Eisenstein series and Langlands spectral decomposition.** In general, the regular representation R of $G(\mathbb{A})$ on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ does not decompose discretely. Eisenstein series describe the continuous part of the spectrum.

The Eisenstein series gives embedding of automorphic representations in the space of automorphic forms???

[13]This is the normalized induction. When we are studying representations over $G(\mathbb{R})$, the factor $\delta^{1/2}$ is used so that the induction of a unitary representation is again unitary

2.5.1. *Preliminaries.* Let G be an algebraic group over $\mathbb{Q}$. We write A_G for the largest central subgroup of G over $\mathbb{Q}$ that is a $\mathbb{Q}$-split torus. The rank of A_G is called the rank of G. We write $X(G)_{\mathbb{Q}}$ for the additive group of homomorphisms $\chi : g \to g^{\chi}$ from G to $GL(1)$ that are defined over $\mathbb{Q}$. Then $X(G)_{\mathbb{Q}}$ is a free abelian group of rank k. WE also form the real vector space

$$\mathfrak{a}_G = \mathrm{Hom}_{\mathbb{Z}}(X(G)_{\mathbb{Q}}, \mathbb{R})$$

of dimension k. There is then a surjective homomorphism $H_G : G(\mathbb{A}) \to \mathfrak{a}_G$, defined by

$$\langle H_G(x), \chi\rangle = |\log(x^{\chi})|, \quad x \in G(\mathbb{A}), \chi \in X(G)_{\mathbb{Q}}.$$

Sometimes we consider a smaller space. Let G^1 be the kernel of H_G, and let $X_G = G(\mathbb{Q})\backslash G^1 = A_G G(\mathbb{Q})\backslash G(\mathbb{A})$. X_G is not compact in general but it is of finite volume With respect to the G-invariant measure. We want to study the right regular representation $L^2(X_G)$ of G. All components in the decomposition are unitary from the G-invariance of the measure. X_G is finite volume, so there are sufficiently many interesting functions over X_G, say, the constant function is L^2. This is why we study X_G instead of $G(\mathbb{Q})\backslash G(\mathbb{A})$. $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ is the direct sum of twists of $L^2(X_G)$. It is, strictly speaking, not L^2!

We also need to consider parabolic subgroups of G. A *parabolic subgroup* of G is a $\mathbb{Q}$-algebraic subgroup P such that $P(\mathbb{C})\backslash G(\mathbb{C})$ is compact. Any such P has a Levi decomposition $P = MN_P$, which is a semidirect product of a reductive subgroup M of G over $\mathbb{Q}$ with a normal unipotent subgroup N_P of G over $\mathbb{Q}$. The *unipotent radical* N_P is uniquely determined by P, while the *Levi component* M is uniquely determined up to conjugacy by $P(\mathbb{Q})$.

Let P_0 be a fixed parabolic subgroup of G with a fixed Levi decomposition $P_0 = M_0N_0$. Any subgroup P that contains P_0 is called a *standard parabolic subgroup.* The set of standard parabolic subgroup is finite, and is a set of representatives of the set of $G(\mathbb{Q})$-conjugacy classes of parabolic subgroups over $\mathbb{Q}$. A standard parabolic subgroup has a canonical Levi decomposition $P = M_PN_P$, where M_P is the unique Levi component of P that contains P_0. From M_P, we can form the central subgroup $A_P = A_{M_P}$, the real vector space $\mathfrak{a}_P = a_{M_P}$, and the surjective homomorphism $H_P = H_{M_P}$. When $P = G$, we recover the original definition of A_G, $\mathfrak{a}_G$, and H_G. If $P = P_0$, we use the notations A_0, $\mathfrak{a}_0$, and H_0. We extend H_P to a fucntion from $G(\mathbb{A})$ by setting $H_P(nmk) = H_{M_P}(m)$.

We have a variant of the regular representation R for any standard parabolic subgroup P. It is the regular representation R_P of $G(\mathbb{A})$ on $L^2(N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A}))$. It is the induced representation

$$R_P = \mathrm{Ind}_{N_P(\mathbb{A})M_P(\mathbb{Q})}^{G(\mathbb{A})}(1_{N_P(\mathbb{A})M_P(\mathbb{Q})}) \cong \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}(1_{N_P(\mathbb{A})} \otimes R_{M_P}).$$

$R_P(f)$ is an integral operator with kernel

$$K_P(x,y) = \int_{N_P(\mathbb{A})} \sum_{\gamma \in M_P(\mathbb{Q})} f(x^{-1}\gamma ny)dn, x, y \in N_P(\mathbb{A})M_P(\mathbb{Q})\backslash G(\mathbb{A}).$$

We need to compare different parabolic subgroups. Two standard parabolic subgroups P and P' are associated with each other if the restrictions of elements in the

Weyl group induce linear isomorphisms of corresponding subspaces $\mathfrak{a}_P$ and $\mathfrak{a}_{P'}$. Let $W(\mathfrak{a}_P, \mathfrak{a}_{P'})$ be the group of such elements. Take GL_n as an example. We fix the minimal parabolic subgroup as the subgroup of upper triangular matrices. Then any standard parabolic subgroup is a subgroup of quasi-upper triangular matrices. They are parameterized by partitions $n = n_1 + \cdots + n_r$. Two such standard parabolic subgroups are associated with each other if and only if when their corresponding partitions are the same up to a permutation.

2.5.2. *Eisenstein series and the Langlands decomposition theorem.* Let P be a standard parabolic subgroup of G, and that λ lies in $\mathfrak{a}^*_{P,\mathbb{C}}$. We have a family of representations $\mathcal{I}_P(\lambda)$ parameterized by λ.

The representation space $\mathcal{I}_P(\lambda)$[14] is the Hilbert space $\mathcal{H}_P$ of measurable functions

$$\phi : N_P(\mathbb{A})M_P(\mathbb{Q})A_P(\mathbb{R})^0\backslash G(\mathbb{A}) \to \mathbb{C}$$

such that the function

$$\phi_x : m \to \phi(mx), \quad m \in M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1,$$

belongs to $L^2_{disc}(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1)$ for any $x \in G(\mathbb{A})$, and such that

$$||\phi||^2 = \int_K \int_{M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1} |\phi(mk)|^2 dmdk < \infty.$$

Any $y \in G(\mathbb{A})$, acts on $\mathcal{I}_P(\lambda)$ via the operator $\mathcal{I}_P(\lambda, y)$ which maps a function $\phi \in \mathcal{H}_P$ to the function

$$(\mathcal{I}_P(\lambda, y)\phi)(x) = \phi(xy)e^{\lambda+\rho_P}(H_P(xy))e^{-(\lambda+\rho_P)(H_P(x))}.$$

Eisenstein series associates a function on $G(\mathbb{A})$ to any function ϕ in $\mathcal{I}_P(\lambda)$. The formal definition apply to any elements $x \in G(\mathbb{A})$, $\phi \in \mathcal{H}_P$, and $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$. The associated Eisenstein series is

$$E(x, \phi, \lambda) = \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \phi(\delta x)e^{(\lambda+\rho_P)(H_P(\delta x))}.$$

We have the formal equations

$$E(x, \mathcal{I}_P(\lambda, y), \lambda) = E(xy, \phi, \lambda).$$

Therefore, Eisenstein series is an intertwining operator from $\mathcal{I}_P(\lambda)$ to the space of functions on $G(\mathbb{A})$ with right regular representation. The Eisenstein series need not to be in $L^2(G(\mathbb{Q})\backslash G(BA))$ in general. If $\lambda \in i\mathfrak{a}_P$, the Eisenstein series are in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. The Langlands decomposition theorem asserts that these automorphic forms generate the full space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. However, associated parabolic groups produces the same Eisenstein series. So the intwetwing operator is actually defined on associated classes of parabolic subgroups, and we have to compare parabolic subgroups.

We need to compare different parabolic subgroups. Two standard parabolic subgroups P and P' are associated with each other if the restrictions of elements in the Weyl group induce linear isomorphisms of corresponding subspaces $\mathfrak{a}_P$ and $\mathfrak{a}_{P'}$. Let

[14] The representation space is independent of λ. This is essential in the Langlands spectral decomposition theorem.

$W(\mathfrak{a}_P, \mathfrak{a}_{P'})$ be the group of such elements. Take GL_n as an example. We fix the minimal parabolic subgroup as the subgroup of upper triangular matrices. Then any standard parabolic subgroup is a subgroup of quasi-upper triangular matrices. They are parameterized by partitions $n = n_1 + \cdots + n_r$. Two such standard parabolic subgroups are associated with each other if and only if when their corresponding partitions are the same up to a permutation.

If $s \in W(\mathfrak{a}_P, \mathfrak{a}_{P'})$, we may define the operator ([Art])

$$M(s, \lambda) : \mathcal{H}_P \to \mathcal{H}_{P'}$$

that interwines $\mathcal{I}_P(\lambda)$ with $\mathcal{I}_{P'}(s\lambda)$:

$$M(s, \lambda)\mathcal{I}_P(\lambda, y) = \mathcal{I}_{P'}(s\lambda, y)M(s, \lambda).$$

The Eisenstein series $E(x, \phi, \lambda)$ and intertwining operators $M(s, \lambda)$ are *formally* defined for all $\lambda \in i\mathfrak{a}^*_{P,\mathbb{C}}$. They actually converge absolutely for K-finite ϕ and sufficiently regular λ. They can be analytically continued to meromorphic functions of $\lambda \in \mathfrak{a}^*_{P,\mathbb{C}}$ satisfy the functional equations

$$E(x, M(s, \lambda)\phi, s\lambda) = E(x, \phi, \lambda)$$

and

$$M(ts, \lambda) = M(t, s\lambda)M(s, \lambda).$$

Theorem 2.11 (Langlands decomposition theorem)**.** *Given an associated class $\mathcal{P} = \{P\}$, define $\hat{L}_{\mathcal{P}}$ to be the Hilbert space of families of measurable functions*

$$F = \{F_P : i\mathfrak{a}_p^{*15} \to \mathcal{H}_P, P \in \mathcal{P}\}$$

that satisfy the symmetry condition

$$F_{P'}(s\lambda) = M(s, \lambda)F_P(\lambda)$$

and a finiteness condition. Then the mapping that sends F to the function

$$\sum_{P \in \mathcal{P}} n_P^{-1} \int_{i\mathfrak{a}_P^*} E(x, F_P(\lambda), \lambda)d\lambda,$$

where $n_P = \sum_{P' \in \mathcal{P}} |W(\mathfrak{a}_P, \mathfrak{a}_{P'})|$, defined whenever $F_P(\lambda)$ is a smooth, compactly supported function of λ with values in a finite-dimensional subspace of $\mathcal{H}_P^0$, extends to a unitary mapping from $\hat{L}_{\mathcal{P}}$ onto a closed $G(\mathbb{A})$-invariant subspace $L^2_{\mathcal{P}}(G(\mathbb{Q})\backslash G(\mathbb{A}))$. Moreover, the original space $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ has an orthogonal direct sum decomposition

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})) = \bigoplus_{\mathcal{P}} L^2_{\mathcal{P}}(G(\mathbb{Q})\backslash G(\mathbb{A})).$$

[15]These λ represents unitary characters, therefore the induced representations $\mathcal{I}_P(\lambda)$ are unitary

2.5.3. *An example: $GL(2)$.* .

Let $G = GL_2$ over $\mathbb{Q}$, and G^1 be the norm one element of $GL_2(\mathbb{A})$. Let $\Gamma = GL_2(\mathbb{Z})$. We are interested in the spectral decomposition $L^2(\Gamma\backslash G^1)$.

The discrete part $L^2_{\text{disc}}(\Gamma\backslash G^1)$ is the direct sum of the cuspidal spectrum $L^2_{\text{cusp}}(\Gamma\backslash G^1)$ and a residual spectrum $L^2_{\text{res}}(\Gamma\backslash G)$. The residual spectrum admits a very explicit description

$$L^2_{\text{res}}(\Gamma\backslash G^1) = \bigoplus_{\chi} \chi \circ \det$$

where χ belongs to the group of continuous characters of $\mathbb{Q}\backslash\mathbb{A}^1$.

Now we consider the continuous spectrum. We have only one rational parabolic subgroup P. It consists of upper triangular matrices. $\dim \mathfrak{a}_P = 1$ and $W(\mathfrak{a}_P, \mathfrak{a}_P) \cong S_2$.

Let A_∞ be the group of diagonal matrices (a, a^{-1}) with $a \in \mathbb{R}^\times_+$, T^1 be the subgroup of T of diagonal matrices (a, b) with a and b in $\mathbb{A}^1$, N be the group of unipotent matrices. Then $\mathcal{H} := \mathcal{H}_P$ is the space of measurable functions

$$\phi : N(T \cap \Gamma)A_\infty\backslash G^1 \to \mathbb{C}$$

such that for $x \in G^1$ the function

$$t \in T^1 \mapsto \phi(tx)$$

belongs to $L^2(()T \cap \Gamma)\backslash T^1$ and such that

$$||\phi||^2 = \int_K \int_{(T\cap\Gamma)\backslash T^1} |\phi(tk)|^2 dtdk < \infty$$

is a Hilbert space.

For any $s \in \mathbb{C} = \mathfrak{a}_{P,\mathbb{C}}$, $y \in G^1$ acts on $\mathcal{I}_P(s)$ by the formula

$$\mathcal{I}_P(s, y)\phi(x) = \exp(-(s+1)H(x))\phi_s(xy)$$

where

$$\phi_s(x) = \phi(x)\exp((s+1)H(x))$$

for any $x, y \in G^1$.

We define the Eisenstein series $E(x, \phi, s)$ and intertwing operators $M(s)$ from $\mathcal{I}_P(s)$ to $\mathcal{I}_P(-s)$.

Let $\mathcal{F}$ be the space of functions $F : i\mathbb{R} \to \mathcal{H}$ which satisfy

$$F(-s) = M(s)F(s)$$

and

$$||F||^2 = \frac{1}{2}\int_{i\mathbb{R}} ||F(s)||^2 ds < \infty.$$

The map

$$F \mapsto \frac{1}{2}\int_{i\mathbb{R}} E(x, F(s), s)ds.$$

defines an isometry from $\mathcal{F}$ onto the the continuous spectrum $L^2_{\text{cont}}(\Gamma\backslash G^1)$.

We consider the spectral decomposition of $L^2(\Gamma\backslash\mathbb{H})$ where $\Gamma = SL_2(\mathbb{Z})$. This is equivalent to studying the function theory of a quotient of $G(\mathbb{Q})\backslash G(\mathbb{Q})$.

Define

$$E(z;s) = \sum_{(m,n)=1} \frac{y^{s+\frac{1}{2}}}{|mz+n|^{2s+1}} = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y(\gamma z)^{s+\frac{1}{2}},$$

where Γ_∞ is the group of integral unipotent matrices. The convergence and continuation are well-known.

The function equation

$$E(z,s) = \phi(s)E(z,-s), \phi(s) = \frac{\sqrt{\pi}\Gamma(s)\zeta(2s)}{\Gamma(s+\frac{1}{2})\zeta(2s+1)}$$

comes from the group $W(\mathfrak{a}_P, \mathfrak{a}_P) \cong S_2$.

$E(z,s)$ has a simple pole at $s = \frac{1}{2}$ with a constant residue. This is the residual spectrum inside the discrete spectrum.

The map $L^2(\mathbb{R})^{even} \to L^2(\Gamma\backslash\mathbb{H})$ given by

$$F \mapsto Ef = \int f(t)E(z,it)dt$$

is an isometry onto $L^2_{cont}(\Gamma\backslash\mathbb{H})$ and

$$\Delta(Ef) = E\left(\left(\frac{1}{4}+t^2\right)f\right).$$

Any $f \in L^2(\Gamma\backslash\mathbb{H})$ admits a decomposition

$$f(z) = \sum (f,u_j)u_j(z) + \frac{1}{4\pi}\int_{-\infty}^{\infty} (f, E(\cdot;it))E(z,it)dt.$$

Example 2.12. Kim, Eisenstein series.

2.5.4. *Another decomposition theorem.* See Langlands' paper [Lan1].

The residues of $E(x.\phi,\lambda)$ at the poles produce certain representations in the discrete spectrum. These representations are orthogonal to the cuspidal spectrum and are called the *residual spectrum*.

There is another Langlands spectral decomposition based on automorphic data.

Given a pair (P,σ), $\mathcal{H}_{P,cusp,\sigma}$ is the subspace of $\phi \in \mathcal{H}_P$ whose slices $\phi_x(m) = \phi(mx), x \in G(\mathbb{A})$ lie in the space $L^2_{cusp,\sigma}(M_P(\mathbb{Q})\backslash M_P(\mathbb{A})^1)$. Let $\Psi(lambda, x)$ be a holomorphic family of functions in $\mathcal{H}_{P,cusp,\sigma}$, $\psi(x)$ is the Fourier transform of Ψ over a real plane, and we then form the Eisenstein series $E\psi$. If $\mathcal{X}$ is the class in $\mathcal{X}$ represented by a pair (P,σ), let $L^2_\chi(G(\mathbb{Q})\backslash G(\mathbb{A}))$ be the closed, $G(\mathbb{A})$-invariant subspace generated by the Eisenstein series $E\psi$ associated to certain functions ψ.

Theorem 2.13 (Langlands)**.** *There is an orthogonal decomposition*

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A})) = \bigoplus_{\chi\in\mathcal{X}} L^2_\chi(G(\mathbb{Q})\backslash G(\mathbb{A})).$$

Remark 2.8. If $\chi = (P,\sigma)$, we cannot have $L^2_\chi \subset L^2_{\mathcal{P}}$.

Remark 2.9. Be aware of the two decompositions.

All constituents of cuspidal inductions are automorphic, but may produce discrete components, say the residual. Parabolic induction is simple in representation theory. The point is to relate the representation with Eisenstein series.

However, when considering parabolic ones, we only use a subspace of functions.

What is the relation with Eisenstein series? Let (M, π) be a cuspidal automorphic representation, we may use parabolic induction to get an automorphic representation of G. The Eisenstein series is an explicit realization. But we could do more, we twist the modulus factor by a number s. So to each element $f \in I_P(\sigma, s)$, we get a function $E(f, s, \cdot)$. The leading terms with respect to s define intertwining operators [16] from $I_P(\sigma, s_0) \to \mathcal{A}(G)$.

Fix s, we get an embedding of $I_P(\sigma, s)$. But to get spectral decomposition, we only need to integrate with respect to some real line with a fixed real part. Different choices of such lines differ from (finitely many) residues, and these contribute to the residual spectrum.

we are considering $L^2(G(F)\backslash G(\mathbb{A}))$, so we need to fix the line. Intuitively, the line should be the line with real part zero. So that the resulting Eisenstein series is L^2 (unitary).

But if we consider all cuspidal representations, those arbitrary s induce the Eisenstein series that is not L^2. The lowest term is a subquotient of $E(\sigma, s)$, but there are more subquotients, that is, the higher coefficients. These subquotients exhaust all automorphic representations.

Each (M, π) has both a continuous part and a discrete part.

Just consider the spectral decomposition of $L^2(\mathbb{R})$. Each exponential function e^{tx} is a function (not L^2), the representation is $x \to e^{tx}$ (check right translation). Then integration over these functions (Fourier transform) is the spectral decomposition.

Integral over vertical lines, different lines induce residues.

3. Langlands correspondence and Langlands functoriality

Sataka isomorphism says that certain representations of a (split) algebraic group G over a nonarchimedean local field are classified by something related to its compact dual. Langlands correspondence is a generalization of Satake isomorphism. It aims to find a classification of admissible representations over $G(F_v)$ or automorphic representations over $G(\mathbb{A}_F)$. The parameters are some Galois representations, or the modified version of Galois representations, the L-parameters.

We always assume G is a quasi-split reductive algebraic group over F.

3.1. **Class field theory.** Let F be a local or global field, $\overline{F}$ be its separable closure, and $\mathrm{Gal}_F := \mathrm{Gal}(\overline{F}/F)$ be the absolute Galois group of F. The finite Galois extensions E of F determine normal subgroups Gal_E of finite index and an isomorphism $\mathrm{Gal}_F/\mathrm{Gal}_E \cong \mathrm{Gal}_{E/F}$. Equipped with the profinite topology, Gal_F is a locally compact topological group and there is an isomorphism of topological groups

$$\mathrm{Gal}_F = \varprojlim \mathrm{Gal}_{E/F} = \varprojlim \mathrm{Gal}_F/\mathrm{Gal}_E.$$

The central problem in number theory is to understand the Galois group Gal_F. The standard method to study a group is to consider its representations. Class field theory is the theory of characters of Gal_F, or equivalently, classification of abelian extensions in terms of an invariant of F itself.

[16] They are not embedding, $I_P(\sigma)$ is not irreducible in general

3.1.1. *Local class field theory.* Let F be a nonarchimedean local field[17]. For simplicity, we assume F is the completion of a number field with respect to a nonarchimedean valuation. F, equipped with the metric topology, is locally compact. Let $\mathcal{O}$ be the subring of F consisting of elements of nonnegative valuation, π be a uniformizer, and $\mathfrak{P}$ be the unique maximal ideal generated by π.

We first consider a finite Galois extension K/F. Then we have a *local reciprocity map* $\theta_{K/F} : \mathrm{Gal}^{ab}_{K/F} \cong F^\times/N_{K/F}K^\times$. If $x \in F^\times$, then we write $\theta_{K/F}(\bar{x}) = (x, K/F)$. If K/F is unramified, then $(x, K/F) = \mathrm{Fr}^{v(x)}$ where Fr is the Frobenius element in $G_{K/F}$. The inverse of the local reciprocity map is exactly the isomorphism of Tate groups $\check{H}^{-1}(\mathrm{Gal}_{K/F}, K^\times) = G^{ab}_{K/F} \cong \check{H}^1(\mathrm{Gal}_{K/F}, K^\times) = F^\times/N_{K/F}K^\times$ given by cup product of a canonical generator $u_{K/F}$ of the cyclic group $H^2(\mathrm{Gal}_{K/F}, K^\times)$.

Let F^{ab} be the maximal abelian extension of F, the Galois group $\mathrm{Gal}_{F^{ab}/F}$ is Gal^{ab}_F, the abelianization of G_F. If K is abelian, then $\theta_{K/F}(x, K/F) \in \mathrm{Gal}_{K/F}$. Let K varies over finite abelian extensions, we get a homomorphism $\theta_F : F^\times \to \varprojlim \mathrm{Gal}_{K/F} = \mathrm{Gal}_{F^{ab}/F} = \mathrm{Gal}^{ab}_F$. The local reciprocity map defines an isomorphism of $F^\times$ onto W^{ab}_F, where W is the Weil group to be defined later.

3.1.2. *Global class field theory.* The main object in class field theory for global fields is the *idele class group* $C_F = \mathbb{A}^\times_F/F^\times$. The group $\mathbb{A}^\times_F$ has a natural map to the group of fractional ideals by "ignoring local units". This descends to a map form C_F to the ideal class group. Therefore, a map from the ideal class group to some group G naturally lifts to a map from the idele class group. But they are not so good because the ideal class group is just a finite group. We may construct more homomorphisms of C_F from some admissible homomorphisms. Let S be a finite set and I^S the group of fractional ideals outside S. An admissible homomorphism $\varphi : I^S \to G$ is a homomorphism satisfying a continuous property with respect to $F^\times$. The lifting is simple: we may multiply a sequence $a_n \in F^\times$ to twist an element $x \in \mathbb{A}^\times_F$ so that the S-component $(a_n x)_S$ tends to zero, and the image of the class $\bar{x}$ is therefore determined by the value of $(a_n x)^S$, the component outside of S. The admissibility of φ is exactly the condition that guarantees the convergence of $\varphi(a_n x)^S$ [18].

Remark 3.1. A Dirichlet character. and Grossencharacter.

One important property is that, given an admissible homomorphism from the group of fractional ideals outside a finite set S

Now we can study the global class field theory. Let K/F be an abelian extension. Let $\mathfrak{P}$ be a prime of F that is unramified in K, we may define a unique Frobenius element $\left[\frac{K/F}{\mathfrak{P}}\right] \in G_{K/F}$. We may extend linearly to fractional ideals of F whose prime factors are unramified in K, called the *Artin symbol* of K/F. The Artin symbol is admissible, thus lifts uniquely to a continuous homomorphism of the *idele class group*

[17]The Galois group of $\mathbb{C}$ is trivial. The Galois group $\mathrm{Gal}_{\mathbb{R}} = \mathrm{Gal}_{\mathbb{C}/\mathbb{R}} \cong \{\pm 1\}$ is isomorphic to $\mathbb{R}^\times/N(\mathbb{C}^\times) = \mathbb{R}^\times/\mathbb{R}^+ = \{\pm 1\}$.

[18]This is common sense in number theory: an object defined over a global field is determined by its behavior at unramified places, which is an all but finitely many set. Another example is the Galois group, the Frobenius defined over unramified places is a dense subset.

$C_F = \mathbb{A}_F^\times / F^\times$. We call the homomorphism the *Artin map* of K/F and denote it by $\theta_{K/F} : C_F \to G_{K/F}$. The Artin map is naturally compatible with the local Artin reciprocity maps.

Now let K vary over finite abelian extension, the compatible finite Artin maps lift to map $\theta_F : C_F \to \mathrm{Gal}_F^{ab}$. θ_F is surjective and the kernel is the identity component $(C_F)_0$ of C_F.

Example 3.1. We consider the field $\mathbb{Q}$. By Kronecker-Weber theorem, any abelian extension is a subfield of a cyclotomic field $\mathbb{Q}(\zeta_n)$, so $\mathbb{Q}^{ab}$ is just the union of all cyclotomic fields. Now the Galois group of $\mathbb{Q}(\zeta_n)$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\times$. Let n vary, the Galois group $\mathrm{Gal}_{\mathbb{Q}}^{ab} = \mathrm{Gal}_{\mathbb{Q}^{ab}/\mathbb{Q}}$ is the inverse limit

$$\varprojlim (\mathbb{Z}/n\mathbb{Z})^\times = \hat{\mathbb{Z}}^\times.$$

The finite adele $\mathbb{A}_f$ is $\hat{\mathbb{Z}} \otimes \mathbb{Q}$, and therefore $\mathbb{A}^\times = \mathbb{A}_f^\times \oplus \mathbb{R}^\times (\hat{\mathbb{Z}}^\times \otimes \mathbb{Q}) \oplus \mathbb{R}^\times$.

The isomorphism $x \mapsto \overline{x \oplus 1}$ induces the isomorphism

$$\hat{\mathbb{Z}}^\times \cong (\mathbb{A}_f^\times \times \mathbb{R}^\times)/(\mathbb{Q}^\times \mathbb{R}^+)^{19}.$$

We thus establish the class field theory for $\mathbb{Q}$.

The Frobenius $\mathrm{Fr}(p)$ maps to $(p \cdots, 1, \cdots, p)$ where 1 is in the p-th place, and the geometric Frobenius $\mathrm{Fr}^{-1}(p)$ maps to $(1, \cdots, p, \cdots, 1)$. This rigidity condition actually determines the isomorphism since the all the Frobenius generates a dense subgroup by Chebotarev density theorem.

3.1.3. *Class field theory as Langlands correspondence.* Let F be a number field, a *grössencharacter* is just a quasiquaracter of C_F. In particular, if K/F is a finite abelian extension of number fields and ω is a character of $G_{K/F}$, then we may naturally lift to a grössencharacter $\varphi := \omega \circ \theta_{K/F}$. This correspondence has very nice properties. We may canonically define Artin L-functions for characters ω and Hecke L-functions for Grossencharacters. The Artin map has the property that the $L(s, \omega)$ and $L(s, \theta\omega)$ coincide. Roughly, this means that Frobenius on the Galois side corresponds to the primes on the C_F side.

In modern language, a grössencharacter is just an automorphic representation of G_1 over F. Now class field says that the Artin map gives a correspondence of characters (one-dimensional representation) of the Galois group G_F corresponding to automorphic forms of GL_1. The Langlands correspondence seeks a parameterization of automorphic representation over reductive groups in terms of Galois representations.

3.1.4. *The Weil group.* We need modifications of the Galois groups Gal_F to study representations. The *Weil group* for F is a tuple $(W_F, \phi, \{r_E\})$ where W_F is a topological group, $\phi : W_F \to \mathrm{Gal}_F$ is a continuous homomorphism with dense image. For each finite extension E/F set $W_E := \phi^{-1}(\mathrm{Gal}_E)$, $r_E : C_E \to W_E^{ab}$ is required to be an isomorphism, where $C_E = E^\times$ (when F is local) or $E^\times \backslash \mathbb{A}_E$ (when F is global). These data are required to satisfy some compatibility conditions (See [Tat]) and the isomorphism

$$W_F \to \varprojlim W_{E/F}$$

[19]Note that the quotient group $\mathbb{R}^\times/\mathbb{R}^+ \cong \mathbb{Z}/2\mathbb{Z}$ is essential to distinguish $\{\pm\} \in \hat{\mathbb{Z}}^\times$

is an isomorphism, where $W_{E/F} := W_F/\overline{W}_E^{ab}$. In short, W_F modifies the finite level structures without changing (absolute) class field theory.

- If $F = \mathbb{C}$, then $W_F = \mathbb{C}^\times$, ϕ is the trivial map and $r_F = \mathrm{id}$.
- If $F = \mathbb{R}$, then $W_F = \mathbb{C}^\times \cup \mathbb{C}^\times$ where $j^2 = 1$ and $jcj^{-1} = \bar{c}$. Here ϕ takes $\mathbb{C}^\times$ to 1 and $j\mathbb{C}^\times$ to the conjugacy automorphism.
- If F is a nonarchimedean local field with residue field extension, W_F has a simple description. Let F^{ur} be the maximal unramified extension of F in $\overline{F}$. Then we have a short exact sequence

$$1 \to I_F \to \mathrm{Gal}_F \xrightarrow{\pi} \mathrm{Gal}_k \to 1$$

  where $I_F = \mathrm{Gal}(\overline{F}/F^{ur})$ is the inertia group of F, $\mathrm{Gal}(F^{ur}/F) \cong \mathrm{Gal}_k$. The group $\mathrm{Gal}_k = \varprojlim \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}$ is a profinite group. The Frobenius Fr is mapped to $1 \in \hat{\mathbb{Z}}$, and is a topological generator of Gal_k. As a set, W_F is the inverse image of $\mathbb{Z} \subset \hat{\mathbb{Z}}$ under π. But W_F is equipped with a different topology with the subset topology induced from Gal_F: we require that $\mathbb{Z}$ is equipped with the discrete topology. In other words, $W_F = I_F \rtimes \langle \mathrm{Fr} \rangle$.
- For global fields, there is no intrinsic description. Formally, the Weil group W_F is the extension of C_F by $\mathrm{Gal}_{E/F}$ defined by the fundamental class $u_{E/F}$ and the absolute Weil group W_F is the inverse limit of $W_{E/F}$.

The Weil form of the L-group. Given a reductive group G over F, the L-group is the semidirect product of $\hat{G}$ and Gal_F induced by a Galois action on $\hat{G}$. Now we have a continuous map $\phi : W_F \to \mathrm{Gal}_F$, the Weil form of the L-group is simply defined as the semidirect product of $\hat{G}$ and W_F where the Weil group action is the obvious composition. The map ϕ has a dense image, so no information is lost in such a variation.

3.2. **The local and global Langlands correspondence.**

3.2.1. *The Langlands group L_F.* The local Langlands group for a local field is W_F if F is archimedean, and is $L_F = W_F \times SU(2)$ if F is nonarchimedean.

If F is a number field, the hypothetical Langlands group L_F is supposed to be characterized in terms of automorphic representations of general linear groups. It was predicted by Langlands on the assumption that for any N, there is a bijective correspondence between irreducible N-dimensional representations of L_F, and cuspidal automorphic representations of the group $GL(N)$.

The group L_F is expected to be a locally compact extension

$$1 \to K_F \to L_F \to W_F \to 1$$

of the global Weil group W_F by a compact connected group K_F. It should come with a conjugacy class of local embeddings over the corresponding Weil groups, for any valuation v of F.

3.2.2. *The local Langlands correspondence.*

Definition 3.2. *An L-parameter is a representation of W_F into ${}^LG$ that commutes wuth the projection to* Gal_F. *Two L-parameters are equivalent if they are conjugate by an element of* $\hat{G}(\mathbb{C})$.

Remark 3.2. Not all L-parameters are of interest in Langlands program. We only need to consider a subset called the relevant L-parameters. This is a board condition that is satisfied in many interesting cases. If G is quasisplit, all L-parameters are relevant.

Let v be a place of F. We have a natural embedding $\mathrm{Gal}_{F_v} \to \mathrm{Gal}_F$. This induces a natural morphism from the local L-group to the global L-group. Given an automorphism representation π, the composition defines a map $\mathrm{Gal}_{F_v} \to^L G$ which factors through ${}^L G_v$. So we may get a local L-parameter. This is compatible with the natural decomposition.

Conjecture 1 (Local Langlands conjecture)**.** *There is a bijection between L-packets and equivalent classes of L-parameters satisfying various conditions. See* [BCSGKK]*, chapter 11.*

Axioms, rigidity

This is proven. Geometric representation for a survey.

3.2.3. *The global Langlands correspondence.* Now we consider the global case. Let G be a reductive algebraic group defined over a number field F. It is not known what group (if any) parameterizes L-packets of automorphic representations.

3.2.4. *L-functions.* A natural question is: Which automorphic representation corresponds to which Galois representation?

We need to describe the correspondence is to consider L-functions. They are meromorphic functions defined over $\mathbb{C}$ associated with certain mathematical objects. The two most significant properties of L-functions are: (1), Euler product. This is the local-global principle. (2), functional equation. For example, the functional equations of automorphic forms for cusp eigenforms is a reflection of the automorohy conditions of the cusp forms with respect to the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The converse theorem says that a Fourier series $\sum a_n q^n$ is a cusp form for $SL(2, \mathbb{Z})$ if and only if the associated L-function satisfies a functional equation. The point is that, we can define L-functions for Galois representations and automorphic representations, so we may expect that an automorphic representation is associated with a Galois representation if their L-functions are the same.

There are two ways of associating an L-function to an automorphic representation. One goes back to Hecke, and was generalized to GL_n by Jacquet, Shalika and Piatetskii-Shapiro. We study the eigenvalues of Hecke operators in this case. The simplest examples are Hecke L-function for Grossencharacter (GL_1), and the L-function for cusp eigenforms defined by the Fourier coefficients (GL_2) (Mellin transform). There is another way to associate L-functions to automorphic representations, using the local Langlands correspondence. The local factors are L-factors associated with L-parameters, a modification of Galois representations. They essentially study the traces of the Frobenius. Examples are Artin L-functions. Part of the content of the local Langlands conjecture is that these two constructions give the same answer.

See [BCSGKK].

Finite group, Artin L-function. Galois side. infinite group? Galois representation?

One-dimensional case.

The motivic L-function: easier to get Euler product. The automorphic L-function.

3.3. Representation theory over local fields.

3.3.1. *Generalities.* One interesting point in the representation theory of p-adic groups is that, besides the subgroups defined geometrically (Borel subgroups can be considered as stabilizers of certain flags), there are several arithmetically defined subgroups (Iwahori subgroups).

General picture: three classes of representations:

The main method of producing new representations from old is the process of parabolic induction[20]. Let $P = MN$ be a parabolic subgroup with M the Levi factor and N the unipotent factor. Given a representation of M, we may construct a representation of G. The Jacquet-functor J_N is the adjoint of this functor:

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G\sigma) \cong \mathrm{Hom}_M(J_N(\pi), \sigma)$$

where π is a smooth representation of G and σ is a smooth representation of M. Explicitly, $J_N(\pi)$ is the spacce of N-coinvarisnts of π with the action of M as the natural action twisted by $\delta_m^{-1/2}$.

Remark 3.3. If We have a finite dimensional representation of P, we get a vector bundle over the flag variety. G acts by translation on the bundles, thus on global sections. paranolic induction? geometric representation theory, Borel-Weil-Bott theorem?

An irreducible algebraic representation π is *supercuspidal* if $r_N(\pi) = 0$ for all proper standard parabolic subgroups $P = MN$ of G. equivalently, supercuspidal representations are those that do not occur as constituents of any $\mathrm{Ind}_P^G(\sigma)$. We may consider them as "primitive" representations of G. They can also be characterized by the fact that their matrix coefficients are compactly supported modulo the center Z of G [21].

An important way to construct supercuspidal representation is the Weil representation. We form good reductive pairs, so that: (1). we do not need to lift to the metaplectic group. (2),$GO(V)$ is simple. Say, it is a quadratic extension. Then using theta correspondence, we get representations of $SL(2, F)$ from characters of $E^\times$. Relation with Jacquet-Langlands???

Some examples: from number theory. This is not surprising. From class field theory, characters of the F^* (local) or C_F (global) are parameterized by abelian extensions. If E is a quadratic extension of F, the quardratic character of $F^\times$ is simple, just look at if an element is the norm of $E^\times$.

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[20]Geometrically, G/P is a flag variety, a P-representation is a vector bundle over G/P. Induction? at least the morphisms of induced representations are differential operators. Falting, Chai, P.227. Representations are determined by characters, but uniponent elements have zero traces, so we may ignore them?

[21]Casselman's subrepresentation theorem asserts that for a reductive Lie group G, any irreducible admissible representation is a subrepresentation of a parabolically induced representation. So there are no supercuspidal representations in the real case.

3.3.2. *Representations of $GL(2)$.* See Bump's book for a detailed description.

Remark 3.4. Why do we need the factor $SU(2)$? Consider $GL(2)$ over a nonarchimedean local field. The non-decomposable representations are used to parameterize **special representations**. See[Lan1] for a discussion.

3.3.3. *The archimedean Case.* Local Langlands classification is already a theorem (Langlands classification).

Roughly speaking, admissible representations are Langlands quotients of parabolic induced representation of tempered representations.

Example 3.3 ($SL_2(\mathbb{R})$)**.** Wiki: representation theory of $SL_2(\mathbb{R})$.

- The discrete series, limit of discrete series, and unitary principal series representations $I_{\varepsilon,\mu}$ with μ purely imaginary are already tempered, so in the cases the parabolic subgroup P is $SL(2,\mathbb{R})$ itself.
- The finite-dimensional representations and the representations $I_{\varepsilon,\mu}$ are the irreducible quotients of the principal series representations. The finite dimensional representations are quotients of special representations.

We may explain this using L-parameters and L-packets.

Every admissible representation of $G(\mathbb{R})$ is the Langlands quotient of a parabolic induced representation.

Using L-parameters, discrete parameters, if not, then is in a parabolic subgroup. see [Gol].

Example 3.4 (Langlands classification for $GL_n(\mathbb{R})$)**.** The building blocks for irreducible admisssible representations of the representations $GL_n(\mathbb{R})$ are the following three types representations of $GL_1(\mathbb{R})$ and $GL_2(\mathbb{R})$: (1). $1 \otimes |\cdot|_{\mathbb{R}}^t$, where 1 is the representation of $SL_1^{\pm}(\mathbb{R}) = \{\pm\}$. $|\cdot|_{\mathbb{R}}^t (t \in \mathbb{C})$ is a character on $\mathbb{R}^+$. (2). $\mathrm{sgn} \otimes |\cdot|_{\mathbb{R}}^t$, here sgn is the signature map on $\{\pm 1\}$. (3), $D_l \otimes |\det(\cdot)|_{\mathbb{R}}^t$. here D_l is the discrete series on $SL_2^{\pm}$ defined as the $\mathrm{Ind}_{SL_2(\mathbb{R})}^{SL_2^{\pm}(\mathbb{R})} D_l^+ = \mathrm{Ind}_{SL_2}^{SL_2^{\pm}}(\mathbb{R}) D_l^-$. $|\det(\cdot)|_{\mathbb{R}}^t$ is defined for positive scalar matrices. To any partition $n = \sum_{i=1}^{r} n_i$ such that $n_i = 1$ or 2, we associate the block diagonal subgroup $D = \prod GL_{n_i}$. For each i, let σ_i be a representation of the above three types, and write t_j for the parameter t of σ_j. If the parameters satisfies the condition $n_i^{-1}\mathrm{Re}\, t_i \geq n_{i+1}^{-1}\mathrm{Re}\, t_{i+1} (\forall i)$, then the Langlands quotient $J(\sigma_1, \cdots, \sigma_r)$ is an irreducible admissible representation of $GL_n(\mathbb{R})$. They exhaust all the irreducible admissible representations up to infinitesimal equivalence. Two parameters (σ_i) and (σ_j) define the same representation if and only if they are the same up to a permutation.

Now the real Weil group $W_{\mathbb{R}} = \mathbb{C}^\times \cup j\mathbb{C}^\times$ is already defined. Irreducible representations of $W_{\mathbb{R}}$ have dimension one or two. They are listed as follows: (1). One-dimensional representations (+,t): $\varphi(z) = |z|_{\mathbb{R}}^t$, $\varphi(j) = 1$. (2). One-dimensional representations (-,t): $\varphi(z) = |z|_{\mathbb{R}}^t$, $\varphi(j) = -1$. (3). Two-dimensional representations (l,t) where $l \in \mathbb{Z}, l \geq 1$. Let (e_1, e_2) be a basis, then $\varphi(re^{i\theta}) = \begin{pmatrix} r^{2t}e^{il\theta} & 0 \\ 0 & r^{2t}e^{-il\theta} \end{pmatrix}$, and $\varphi(j) = \begin{pmatrix} 0 & (-1)^l \\ 1 & 0 \end{pmatrix}$.

Every finite-dimensional semisimple representation of $W_{\mathbb{R}}$ is absolutely irreducible. So an n-dimensional representation of $W_{\mathbb{R}}$ defines a partition of n into 1's and 2's. From the above description, there is an obvious map from an n-dimensional representation to a representation of some block diagonal matrix D. Up to a permutation, we may assume the decreasing condition for t_i is satisfied, and we get a admissible representation of GL_n. After modulo further permutations, we get the Langlands correspondence for $GL_n(\mathbb{R})$. Note that the multiplicity one theorem holds for $GL_n(\mathbb{R})$, that is, each L-packet consists of one representation.

The calculation is even simpler for $GL_n(\mathbb{C})$. See [Kna2] for details.

Note that in these cases, a packet consists of only one representation.

There are two approaches to define L-functions and ε-factors: the Tate method using Fourier analysis (studying K-finite matrix coefficients) or the Langlands method using classification theorem. They are also discussed in [Kna2] and they get the same results.

3.4. **Langlands conjecture for classical groups.**

3.4.1. *Langlangds conjecture for $GL(N)$.* They are already verified.

Some strong theorems.

Theorem 3.5 (multiplicity one theorem)**.**

converse theorem.

superspecial

[Art2] global case: suggested by the existence of the Langlands group L_F, parameters (μ, ν), with μ a unitary, cuspidal automorphic representation of $GL(m)$, ν the irreducible representation of $SU(2)$ of dimension n.

simple: discrete; general, all automorphic representations. (Eisenstein series)

Theorem 3.6 (Moeglin-Waldspurger)**.** *There is a canonical bijection*

$$\psi \to \pi_\psi, \quad \psi \in \Psi_{sim}(N)$$

from Ψ_{sim} onto the set of irreducible unitary representations of $GL(N, \mathbb{A})$ that occur in the automorphic, relative discrete spectrum $L^2_{disc}(GL(N,F)\backslash GL(N,\mathbb{A}))$ of $GL(N)$. Moreover, for any ψ, π_ψ occurs in the relative discrete spectrum with multiplicity one.

If $\psi = (\mu, \nu)$ belongs to $\Psi_{sim}(N)$, the representation π_ψ is a Langlands quotient. It is by definition the unique irreducible quotient of the representation of $GL(n, \mathbb{A})$ obtained by parabolic induction from the unitary representation

$$x \to \mu(x_1)|\det x_1|^{\frac{n-1}{2}} \otimes \mu(x_2)|\det x_2|^{\frac{n-3}{2}} \otimes \cdot \otimes \mu(x_n)|\det x_n|^{-\frac{n-1}{2}}$$

of the standard Levi subgroup

$$M_P(\mathbb{A}) = \{x = (x_1, \cdot, x_n) : x_i \in GL(m_i,)\}.$$

Corollary 3.7. *There is a canonical bijection*

$$\psi \to \pi_\psi, \quad \psi \in \Psi(N),$$

from $\Psi(N)$ onto the irreducible constituents of the full automorphic spectrum $L^2(GL(N,F)\backslash GL(N,\mathbb{A}))$.

Theorem 3.8 (Jacquet-Shalika)**.** *The mapping*

$$\psi \to C(\psi), \quad \psi \in \Psi(N)$$

is a bijection from $\Psi(N)$ *to* $\mathcal{C}(N)$*.*

Theorem 3.9 (Local Langlands correspondence)**.** *compatible with L-functions, tensor product to Rankin-Selberg products*

The Rankin-Selberg product is the automorphic form side of the tensor product.

3.5. **Langlangds conjecture for classical groups.** Just a survey.

3.6. **Global Langlands correspondence and geometry.** Langlands program is a link between automorphic representations and Galois representations. Suppose we could associate a Galois representation with a modular form, one natural problem is to seek the geometric realization of this Galois representation. If such a correspondence could be found, the information of the geometric object X is encoded in automorphic forms. Conversely, numerous constructions and theorems in algebraic geometry can be translated into results in automorphic forms. One example is Deligne's proof of Fourier coefficient estimation(?) using Weil conjecture in algebraic geometry.

We may define L-functions in this case to study the correspondence. If X is an algebraic variety defined over a number field, the classical *Hasse-Weil* L-function essentially counts the points of X after modulo primes. The generalization for motives is the *motivic L-function.*

There is a class of Galois representations called *geometric Galois representations* using p-adic Hodge theory. It is conjectured that geometric Galois representations are *motivic*, that is, they are subquotients of étale cohomology groups of some variety X over F.

3.6.1. *Shimura varieties.* Maybe the simplest example of Galois representations is étale cohomology groups. Given a variety X over F, the ℓ-adic cohomology groups are natural Galois representations. The automorphic representations? Deligne's paper. congruence relations. If we assume further that there is also a Hecke action. We may compare the two actions and get global Langlands correspondence. Shimura varieties are examples of such good varieties.

Let $\mathbb{S} = \mathbf{Res}_{\mathbb{C}/\mathbb{R}}$ be the Deligne torus. A *Shimura datum* is a pair (G, X) consisting of a (connected) reductive algebraic group G defined over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class X of homomorphism $h : \mathbb{S} \to G_R$ satisfying certain axioms (See [Mil1]). Then X has a unique structure of a complex manifold such that for every representation $\rho : G_{\mathbb{R}} \to GL(V)$, the family $(V, \rho \circ h)$ is a holomorphic family of Hodge structures. Moreover, it forms a variation of Hodge structure, and X is a disjoint union of hermitian symmetric domains.

For any sufficiently small open subgroup K of $G(\mathbb{A}_f)$, the double coset space

$$Sh_k(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K^{22}$$

[22] Compare with the isomorphism in automorphic form: the archimedean component is replaced with X.

is a finite union of locally symmetric varieties. The varieties $Sh_K(G,X)$ are complex algebraic varieties and they form an inverse system over all sufficiently small open subgroups K. The Shimura variety $Sh(G,X)$ associated with the Shimura datum (G,X) is defined to be the inverse limit of such $Sh_K(G,X)$. It admits a natural right action of $G(\mathbb{A}_f)$. This action induces a natural Hecke algebra action. Note that this is only defined over the inverse limit of varieties.

Though Shimura varieties are constructed as (inverse limits of) complex algebraic varieties, they turn out to be algebraic. They can be defined canonically over a number field F, called the reflex field of the Shimura datum. The construction of the canonical model relies on the moduli interpretation of the Shimura varieties, and is characterized by the Galois action on CM points, i.e., those representing varieties with the most exceptional Hodge classes. In particular, the étale cohomology groups are naturally Galois representations.

Remark 3.5. One reason we consider Shimura varieties instead of connected Shimura varieties is that they are defined over a common number field. If we consider only connected Shimura varieties, each congruence quotient is again defined over a number field, but the number field depends on the congruence subgroup. One example is the Siegel Shimura variety: The principal congruence subgroup $\Gamma(N)$ defines $X(N)$ parameterizing abelian varieties with the principal level structure of level N, which is defined over the cyclotomic field $\mathbb{Q}(\zeta_N)$.

Another important property is that Shimura varieties have moduli interpretations: they are moduli space of abelian varietis with structures, motives, or other special algebraic varieties ([Mil3]). Therefore, Hecke algebra action. Hecke correspondence.

3.6.2. *Some examples. Deligne-Serre theory*

Theorem 3.10 (Deligne, Deligne-Serre). *Suppose $f = \sum_{\geq 1} a_n q^n$ is a normalized, cuspidal Hecke eigenform of weight $k \geq 1$ and level $\Gamma_0(N)$. Then there exists a unique semisimple Galois representation*

$$R_\ell(f) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\overline{\mathbb{Q}}_\ell)$$

whose trace of Frobenius is the Fourier coefficient a_p for almost all primes p.

The construction relies on a realization of cusp forms as cohomology over modular curves.

The *Eichler-Shimura congruence relation* expresses the local Lfunction of a modular curve at a prime p in terms of the eigenvalues of Hecke operators. It identifies a part of the Hasse-Weil zeta function of a modular curve with the product of Mellin transforms of weight 2 modular forms. For precise formulation and proof, see [RS].

The *modularity theorem* (or *Tanayama-Shimura conjecture*) states that any elliptic curve over $\mathbb{Q}$ is a modular curve. One consequence of this theorem is *Fermat's Last Theorem.*

3.7. **Langlands' principle of functoriality.** One amazing consequence of the Langlands correspondence is Langlands' principle of functoriality. The automorphic representations are parameterized by some representations, and there are some natural maps of representations between different groups. So there should be some natural correspondence between automorphic representations of different reductive groups. This is not obvious at all. Just think why (???). On the other hand, Langlands functoriality may also be viewed as a tool for proving cases of the Langlands correspondence. In fact, W_F can be considered as the L-group of the trivial group, then functoriality recovers the Langlands correspondence.

Here are some classical examples. See [BCSGKK] or [Bum].

3.7.1. *Real representations.* We consider the simplest case. Consider the inclusion $SL_2(\mathbb{R}) \to GL_2(\mathbb{R})$. But we first study the inclusion $SL_2(\mathbb{R}) \to SL_2^{\pm}(\mathbb{R})$ and the induced representations. The finite-dimensional representations and principal series representation split into two inequivalent pieces, and the discrete series yield irreducible representations on $S_2^{\pm}(\mathbb{R})$ whose restriction to $SL_2(\mathbb{R})$ is the direct sum of the discrete series and its conjugate. This construction gives us the representations of $SL_2^{\pm}(\mathbb{R})$. The irreducible representations of $GL_2(\mathbb{R})$ are just representations of $SL_2(\mathbb{R})$ twisted by a character of $\mathbb{R}$. See [Kna5].

Their L-groups are $PGL_2(\mathbb{C})$ and $GL_2(\mathbb{C})$. The natural projection of L-groups induces the Langlands functoriality $\Pi(SL_2(\mathbb{R})) \to \Pi(GL_2(\mathbb{R}))$. The finite-dimensional representations and principal series representations corresponds to packets of twists of two representations. The L-packets of discrete series corresponds to packets of twists of a discrete series.

What is the L-group of $SL_2^{\pm}(\mathbb{R})$? It should be a double cover of $PGL_2(\mathbb{R})$. I am not even sure that $SL_2^{\pm}$ is algebraic...

3.7.2. *Inner forms.* Let G be connected, reductive, and quasisplit over F and let H be an inner form of G. Then ${}^LH ={}^L G$, the identity map is an L-homomorphism, and we should have a corresponding lifting. In the case $G = GL_2$ and $H = D^{\times}$, the multiplicative group of a rank-2 division algebra over F, we get the Jacquet-Langlands correspondence.

3.7.3. *Base change and induction.* Suppose that K is a finite extension of F. Let $I : \mathrm{Gal}_K \to \mathrm{Gal}_F$ be the natural inclusion. Suppose we have a split group H over F, then $G = \mathbf{Res}_{K/F} H_K$ is an algebraic group and we have a canonical L-morphism $u :{}^L H \to{}^L G$, where ${}^LG$ is $[K;F]$-copies of ${}^LH^0$ parameterized by $\mathrm{Gal}(K/F)$, the Galois group acts on ${}^LG$ through the quotient $\mathrm{Gal}(K/F)$. Given a L-parameter φ for H, $u \circ \varphi \circ i$ is a L-morphism for G.(maybe we could define G just as the base change and the restriction is defined only for the natural comparison of L-groups???) This is called base-change or automorphic restriction.

The L-functions.

Example 3.11 (Zeta functions)**.** Let $F/\mathbb{Q}$ be a finite abelian extension and let $N : F^{\times} \to \mathbb{Q}^{\times}$ be the norm map. A Dirichlet character χ induces a Hecke character $psi := \chi \circ N$. Their L-functions are related by:

$$L(\psi, s) = \prod_{\rho \in \hat{G}} L(\rho\chi, s)(?)$$

In particular, if we choose χ to be the trivial character, then we get the Dirichlet series for F.

If F is a quadratic field, the zeta function $\zeta_F(s)$ tells you how the primes decompose, and the functional equation is equivalent with quadratic reciprocity.

If F is $\mathbb{Q}(\zeta_n)$, both zeta functions have a simple pole at $s = 1$. Therefor $L(\chi, 1) \neq 0$ whenever $\chi \neq 1$. This is an essentical step in proving Dirichlet's theorem on arithmetic progressions. See [Ser] for details.

Example 3.12 (Maass forms)**.** Maass form. real quadratic extension. The characters of the quadratic field should correspond to elliptic modular forms. These are, Maass forms.

We may also define automorphic induction.

3.7.4. *Group theoretic examples.* There are standard algebraic operators in representations, this defines natural L-homomorphisms between algebraic groups. The tensor product map $\otimes : GL(m) \otimes GL(n) \to GL(mn)$ is a L-homomorphism of L-groups from $GL_m \times GL_n$ to GL_{mn}[23].

Symmetric powers: the standard group homomorphism: $GL(2) \to GL(r+1)$: consider the symmetric powser $\mathrm{Sym}^r V$ where V is the standard representation of $GL(2)$.

3.7.5. *Methods.* Some methods to study functoriality (Gelbart):

- theta correspondence
- L-function, converse theorem
- trace formula

I will focus on the trace formula method.

3.8. **Geometric Langlands correspondence.** By definition, global fields are either number fields or function fields of algebraic curves defined over finite groups. In the function field case, everything should be interpreted as geometric objects, which makes the proof simpler because we may use geometrical methods. The Langlands correspondence in this case is now a theorem proved by V.Drinfeld and L.Lafforgue. Another advantage is that all completions are non-archimedean: points of F are irreducible polynomials. Let P be an irreducible polynomial of degree d, the residue field is $\mathbb{F}_{q^d}[[t]]$. see [Fre].

But we can do more. Since the geometric constructions can be defined for curves over arbitrary fields, we get a correspondence of these geometric objects for curves over an arbitrary field. This is the geometric Langlands correspondence. We briefly discuss the geometric Langlands for GL_n defined over a curve X[24] defined over finite fields or $\mathbb{C}$.

Now let's recall some definitions. First, assume X is over a finite field. One side is simple, the Galois group $\mathrm{Gal}(\overline{F}/F)$ can be considered as the fundamental group of the curve X, and its n-dimensional representations can be interpreted as local systems

[23]Let (μ, U) (resp. (ν, V)) be a m(resp, n)-dimensional representation of a group G. Fix basis for U and V, we get matrices $\mu(g) \in GL(m)$ and $\nu(g) \in GL_n$. Then the matrix for the tensor product $(\mu \otimes \nu, U \otimes V)$ is $\mu(g) \otimes \nu(g)$. So this is the universal tensor product formula for representations.

[24]Here a curve is the analogue of number fields in number theory

of rank n over X. The other side is more complicated. An automorphic representation unramified at all points of X defines a function over $GL_n(F)\backslash GL_n(\mathbb{A})/GL_n(\mathcal{O})$ whose value at the point x is the vector v_x stable under $\mathcal{O}_x$[25]. It is well known that $GL_n(F)\backslash GL_n(\mathbb{A})/GL_n(\mathcal{O})$ is the set of isomorphism classes of rank n vector bundles on X. Now we want to extend these constructions to curves over $\mathbb{C}$. $GL_n(F)\backslash GL_n(\mathbb{A})/GL_n(\mathcal{O})$ should be replaced with Bun_n, the *moduli stack* of rank n vector bundles. Functions over Bun_n should be replaced with perverse sheaves (or, $\mathcal{D}$-modules): we have an action of local Galois groups, the alternating sum of traces of the geometric Frobenius on H^i defines a function. There are Hecke correspondences $Hecke_i$ with projections to Bun_n and $X \times \mathrm{Bun}_n$, whose points are vector bundles extended by i-copies of skyscraper sheaf at one point. This Hecke correspondence defines operators on the category of perverse sheaves: $H_i : \mathcal{P}(\mathrm{Bun}_n) \to \mathcal{P}(\mathrm{Bun}_n)$. Then a perverse sheaf K is called a Hecke eigensheaf with eigenvalue E if $K \neq 0$ and we have the isomorphisms

$$\iota_i : H_i(K) \cong \wedge^i E \boxtimes K[-i(n-i)], \quad i = 1, \cdots, n.$$

The geometric interpretations of Galois representations are *irreducible rank n local systems on X*, and the automorphic representations are *Hecke eigensheaves on* Bun_n. Then the geometric Langlands correspondence for GL_n is an equivalence of the two categories associating a local system E with the Hecke eigensheaf Aut_E, a Hecke eigensheaf with respect to E.

The geometric Langlands correspondence is also related to conformal field theory. But I have to skip the discussion. See [Fre].

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