Partitioning friends fairly

Lily Li, Evi Micha, Aleksandar Nikolov, Nisarg Shah

No website photo*

Many thanks to Evi for clarifications via email and Warut for linking us up!

Paper presentation for CS6235 Advanced Topics in Theoretical Computer Science

8 Mar 2023

Davin Choo

*Screenshot from talk: https://www.birs.ca/events/2020/5-day-workshops/20w5141/videos/watch/202010020915-Li.html

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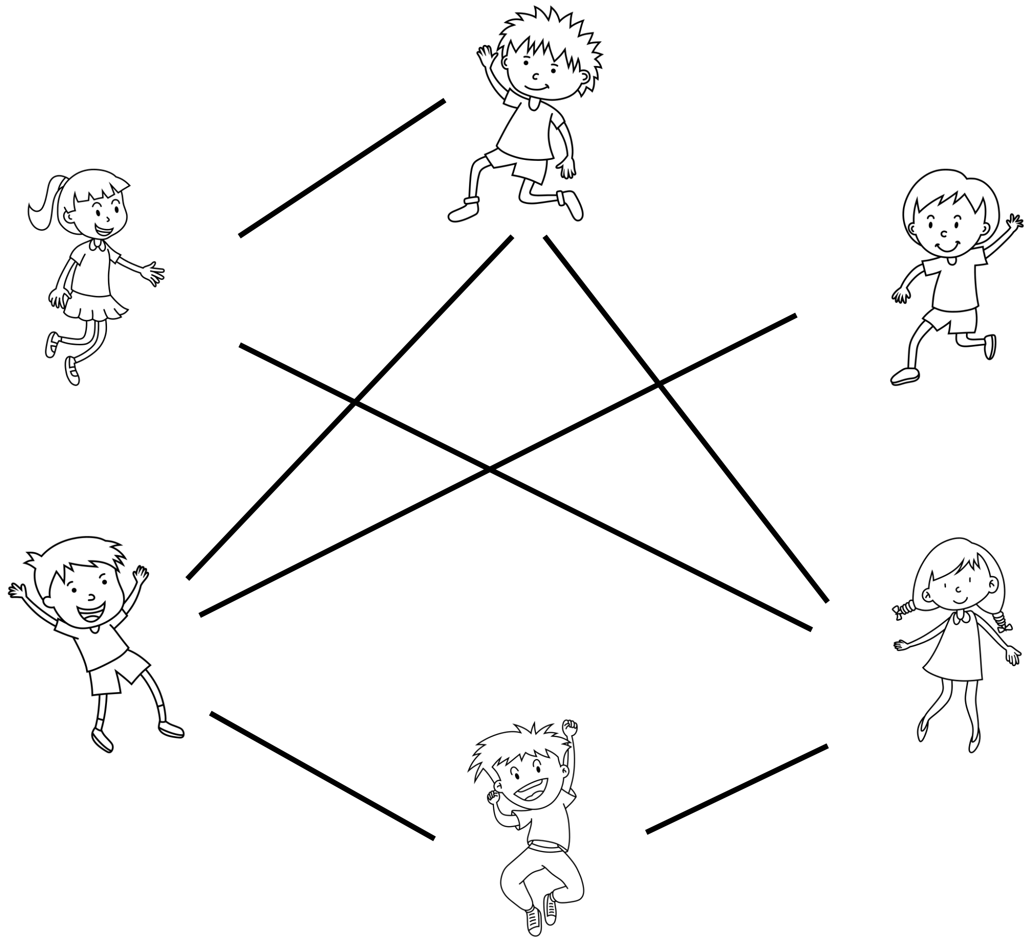
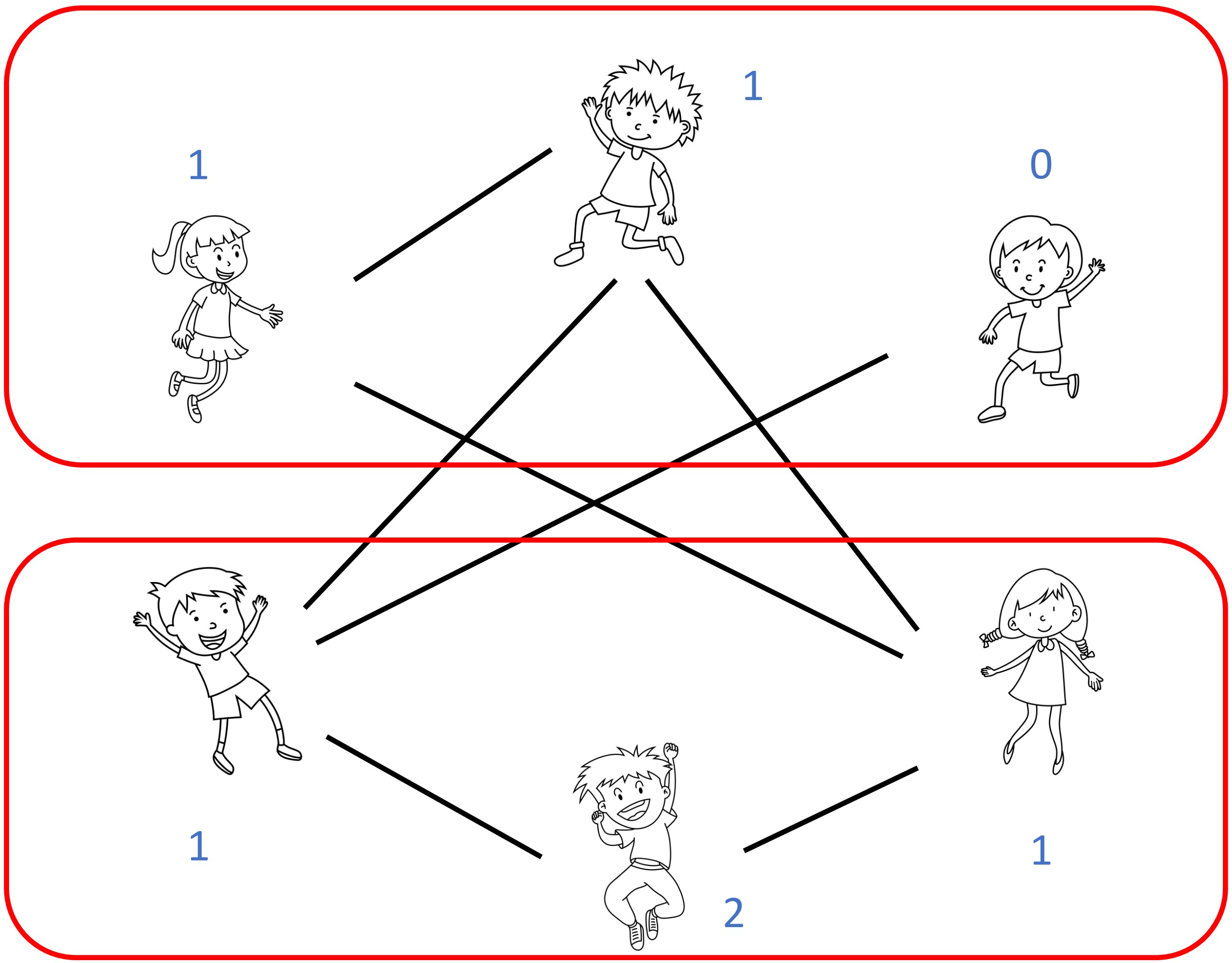
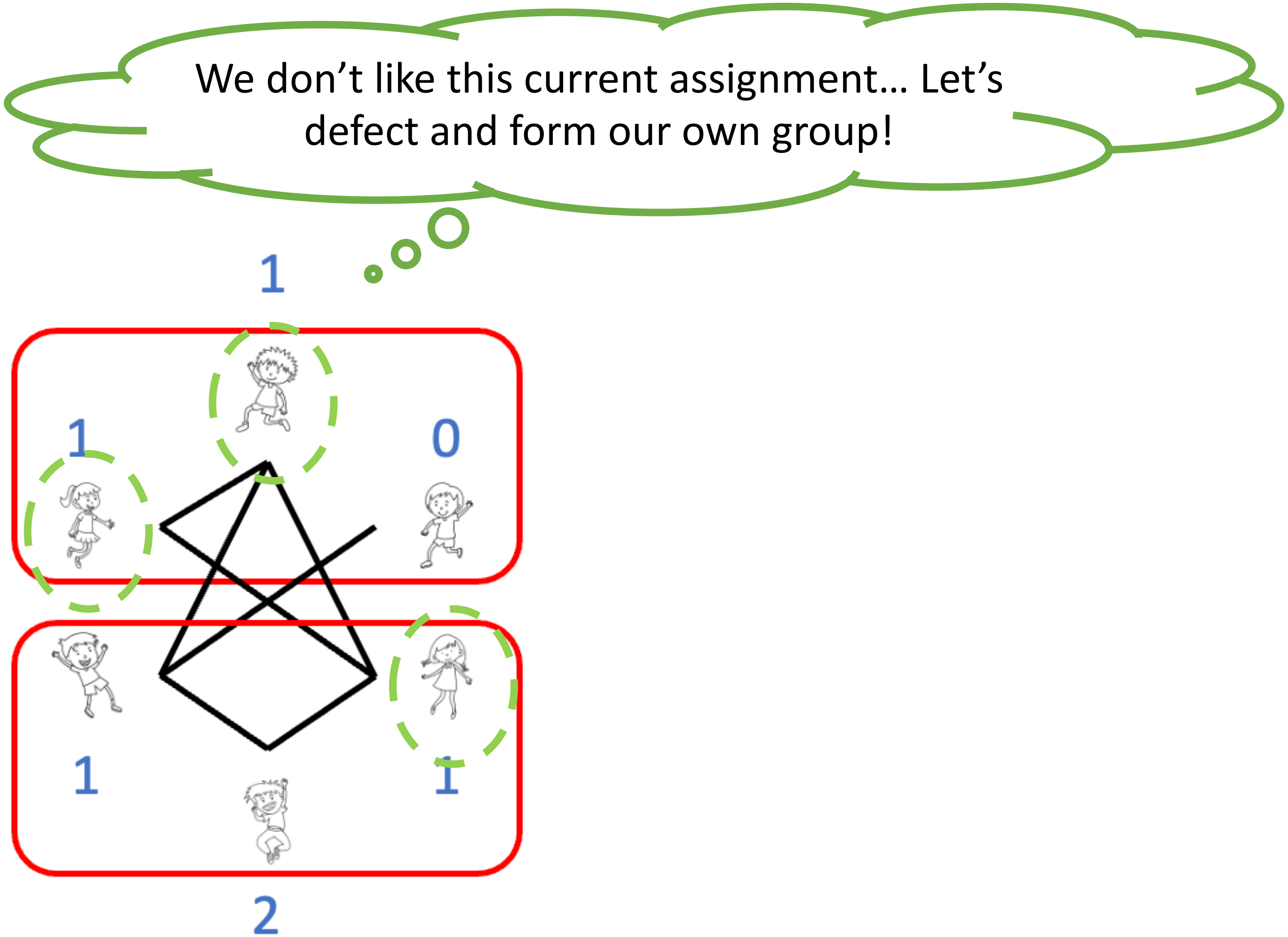


How do we split them into 2 groups of equal size?
Desiderata: Everyone wants to be in a group with as many of their friends as possible

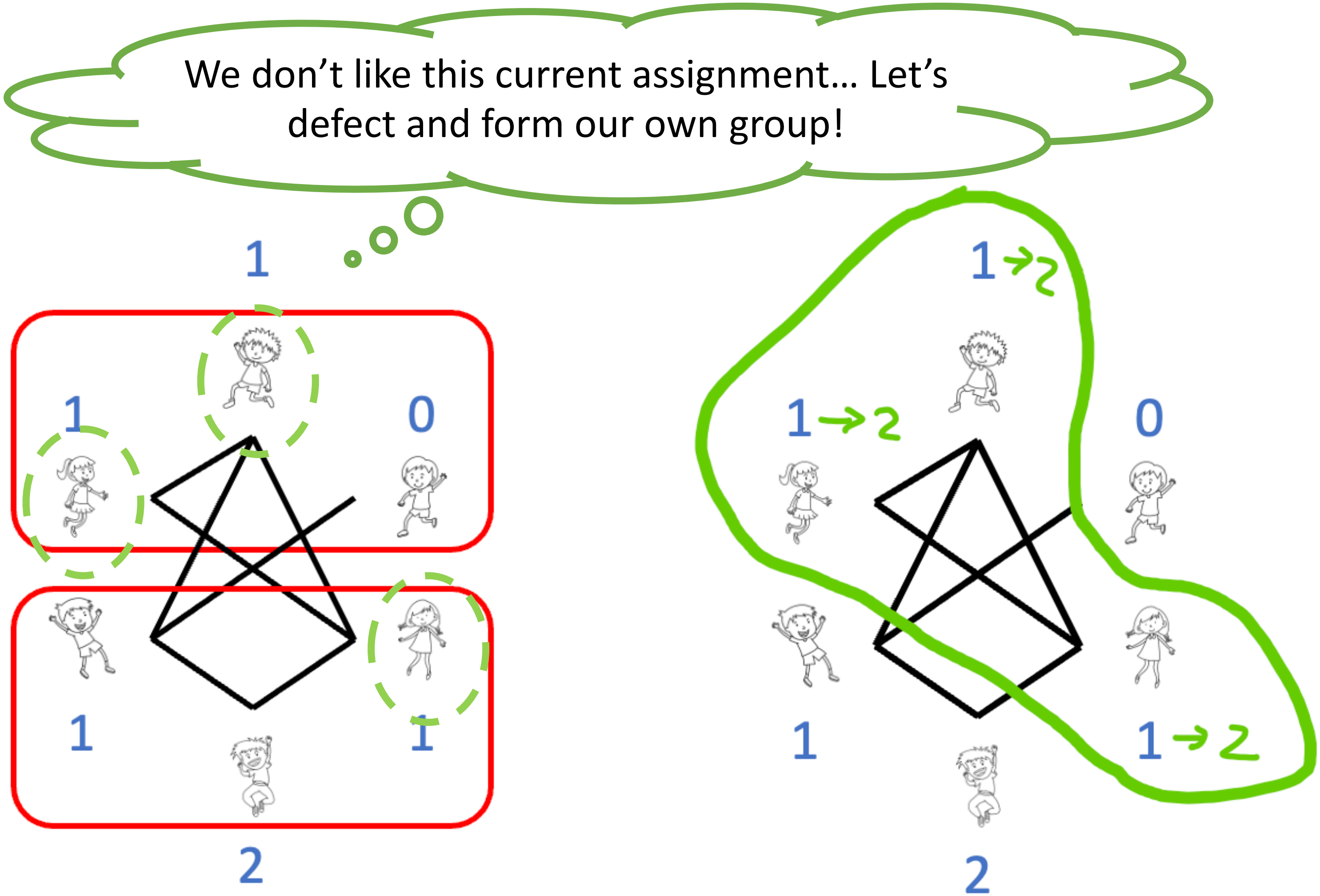


Is this a “good” partitioning?



Notion 1: **Core**

(Related to “stability” of an assignments in cooperative game theory)



Remark: Value of everyone in coalition strictly increases

Notion 1: **Core**

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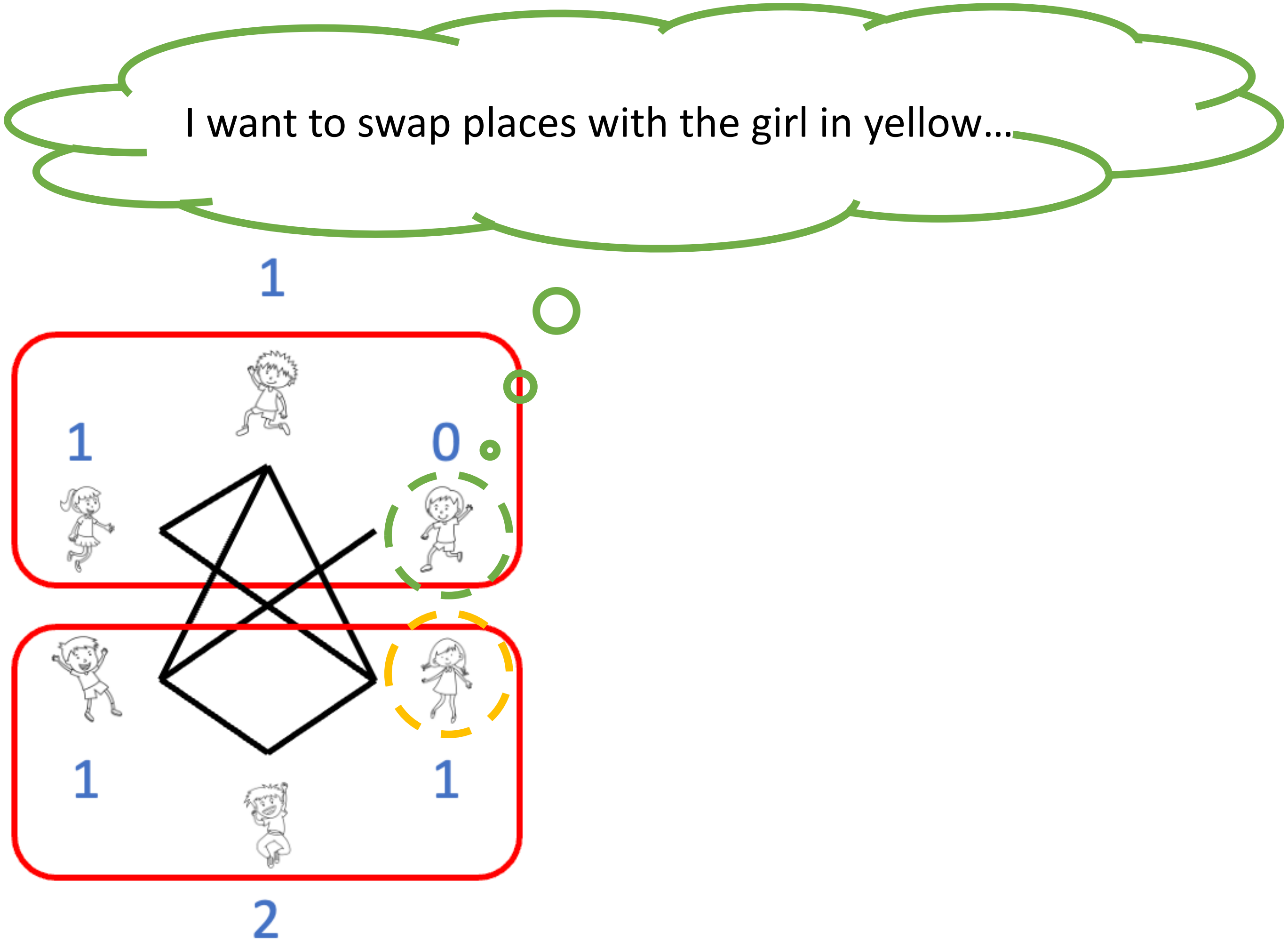
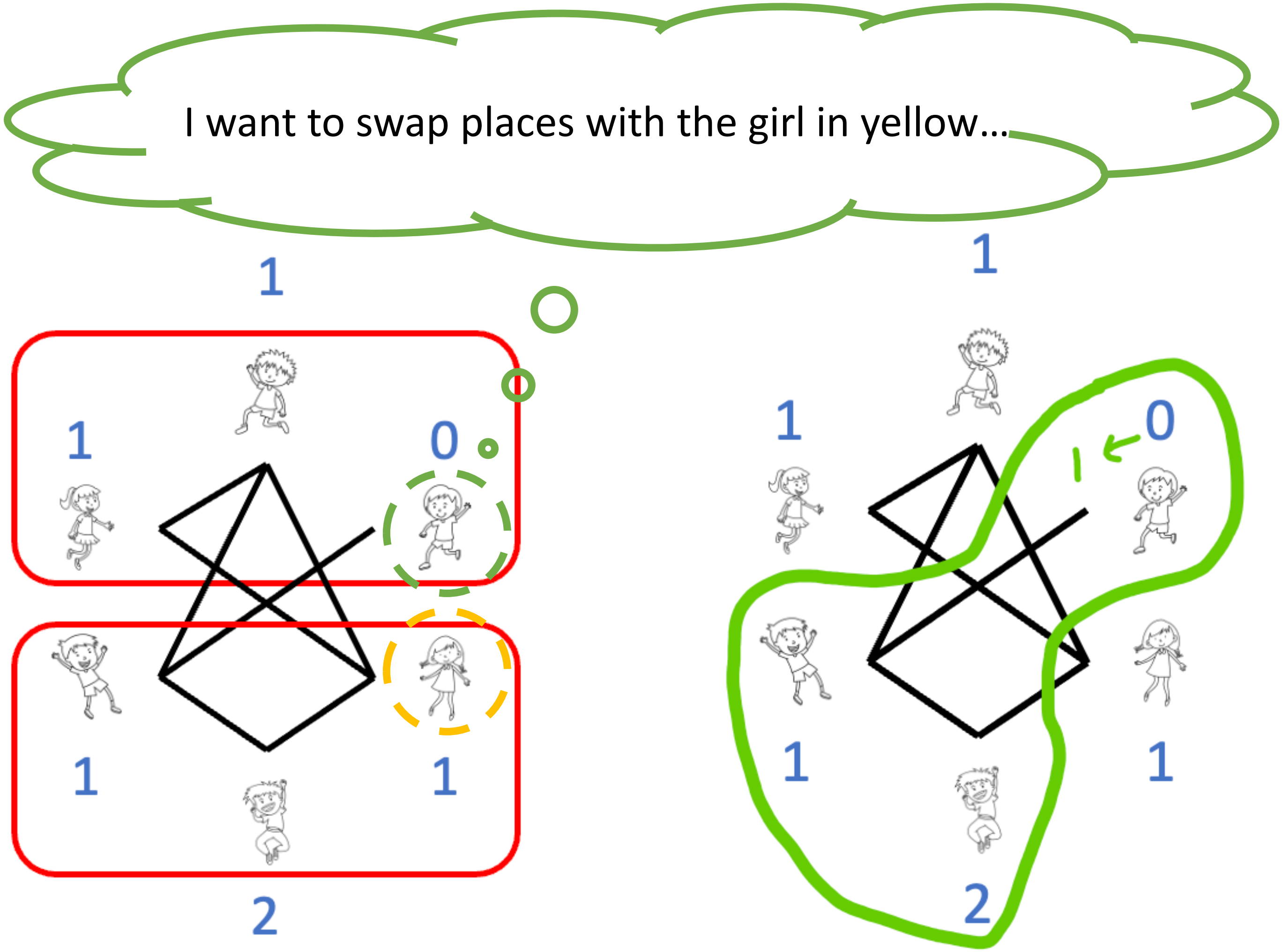


Notion 2: **Envy** (with respect to partition swapping)



Remark: We only care about a single individual's value

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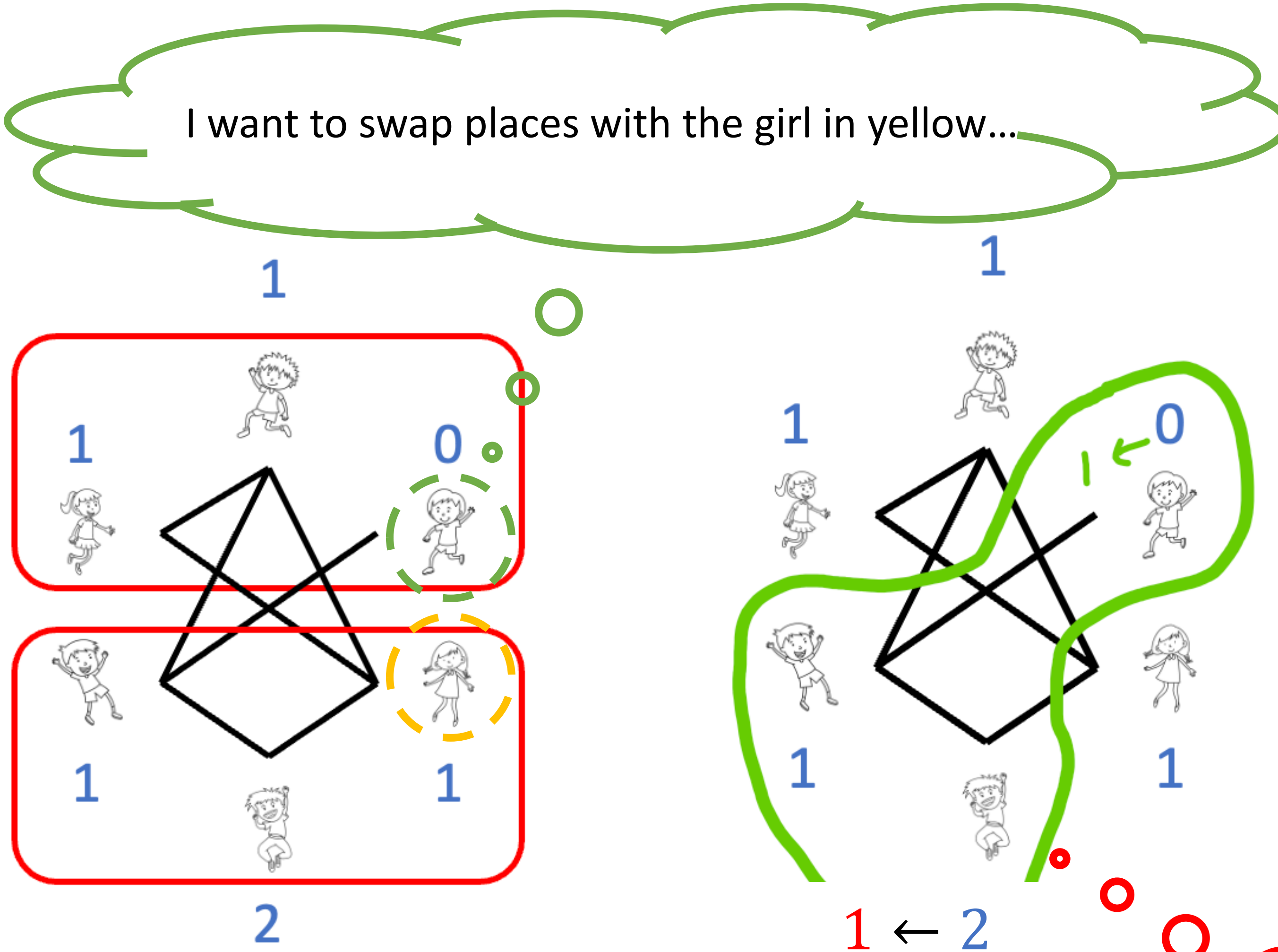

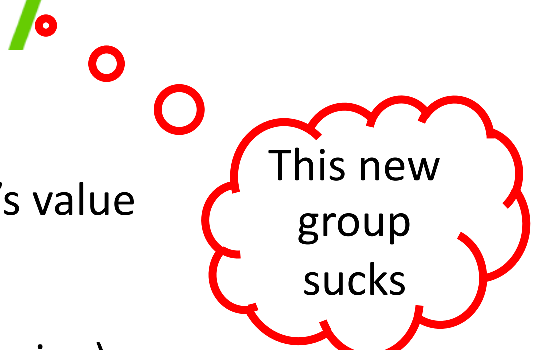

Remark: We only care about a single individual's value

Notion 2: **Envy** (with respect to partition swapping)

Problem setup

- Given a graph $G = (V, E)$
  - Vertices are agents: $[n] = \{1, \ldots, n\}$
  - Edges denote symmetric friendship between agents
  - Binary utility $u_i(j) = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$

No self-loops: $u_i(i) = 0$

Problem setup

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- Output a partitioning of agents $X = (X_1, \dots, X_k)$ of V
  - $X(i) \in X$ denotes partition which agent i is assigned to
  - (Additive) utility gained by agent i with respect to a set $\mathrm{S} \subseteq V$

$$u_i(S) = \sum_{j \in S} u_i(j) = |S \cap N(i)|$$

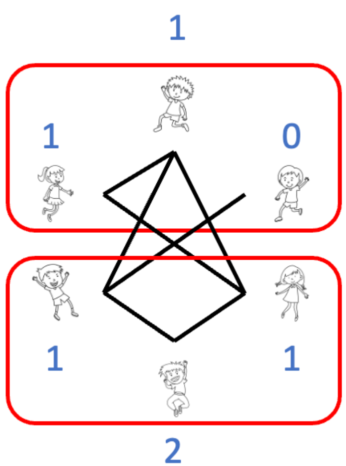
  - Balanced partitioning when $\left\lfloor \frac{n}{k} \right\rfloor \leq |\mathrm{X_i}| \leq \left\lceil \frac{n}{k} \right\rceil$ for all partitions

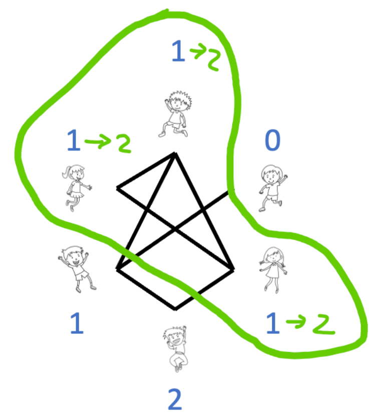
Fairness notion 1: Core

- No subset of agents can benefit from deviating and forming their own coalition/group
  - A coalition $S \subseteq V$ is a blocking core for k-partition X if

$$u_i(S) > u_i\big(X(i)\big)$$

  - Size of coalition matters. For balanced, $\left\lfloor \frac{n}{k} \right\rfloor \leq |S| \leq \left\lceil \frac{n}{k} \right\rceil$





Blocking coalition S

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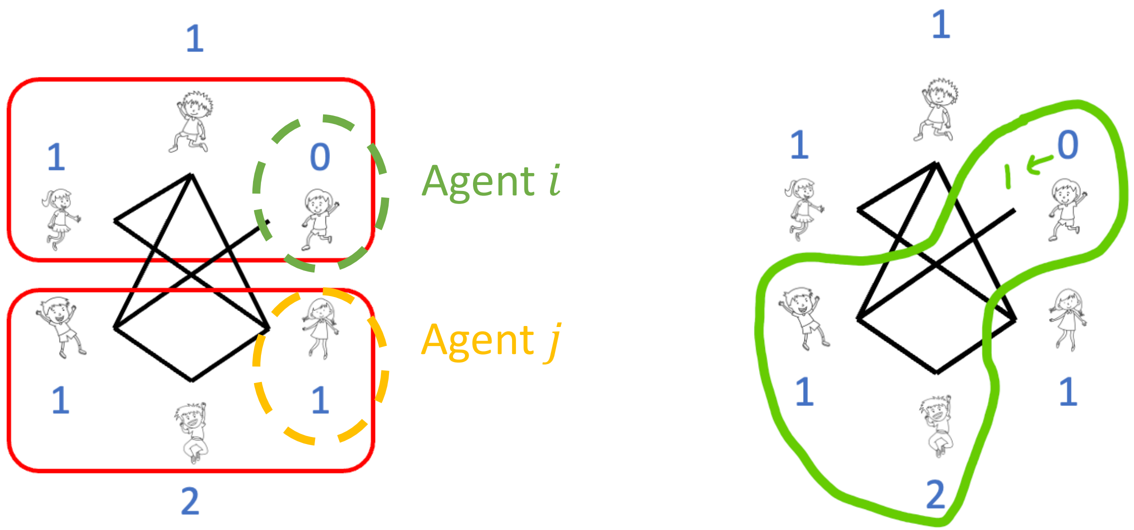
- Relaxation: (α, β)-core
  - A coalition $\mathrm{S} \subseteq V$ is (α, β)-blocking for k-partition X if
  $$u_i(S) > \alpha \cdot u_i\big(X(i)\big) + \beta$$

Fairness notion 2: Envy-free

- The (perceived) own utility is at least any other agent's (perceived) utility. *Note: This is subjective.*

$$\forall j \in [n], \qquad u_i\big(X(i)\big) \geq u_i(X(j) \cup \{i\} \setminus \{j\})$$



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- Relaxation: EF-r

$$\forall j \in [n], \exists \overbrace{g_1, \ldots, g_r}^{\text{Remove as many of agent } i\text{'s friends in } X(j)} \in X(j)$$

$$u_i\big(X(i)\big) \geq u_i\underbrace{(X(j) \cup \{i\} \setminus \{j, g_1, \ldots, g_r\})}_{\text{After removing } r \text{ people from } X(j)\text{, agent } i \text{ no longer envy swapping places with agent } j}$$

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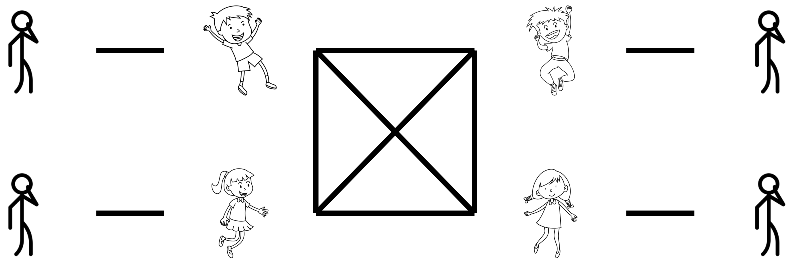
$$\forall j \in [n], \qquad u_i\big(X(i)\big) \geq u_i(X(j) \cup \{i\} \setminus \{j\}) \underbrace{- r}_{\text{Envy-free when } r = 0}$$

Core versus envy-free

$n = 8, \mathrm{k} = 2$

Clique on 4 friends + 4 dangling



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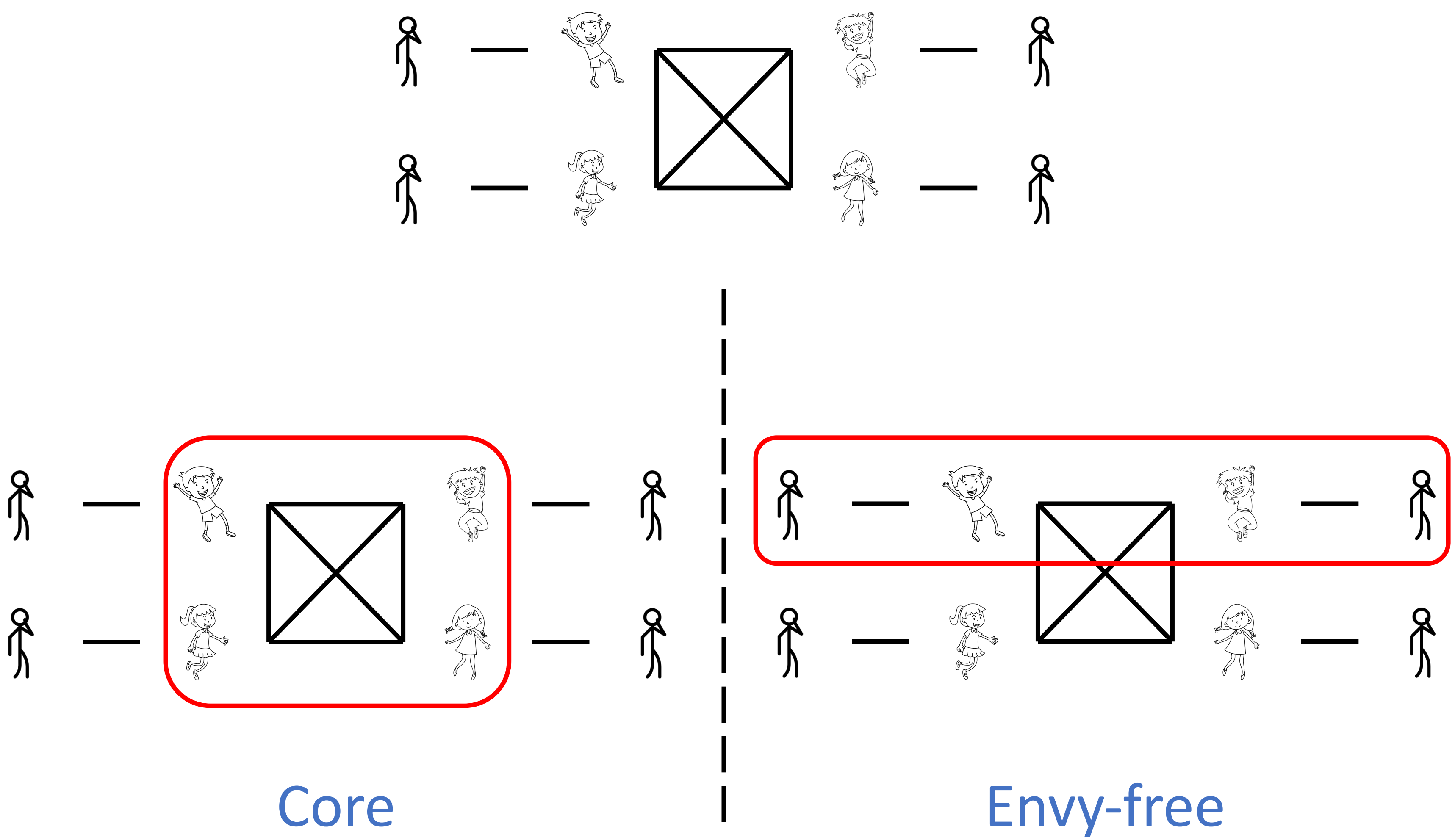

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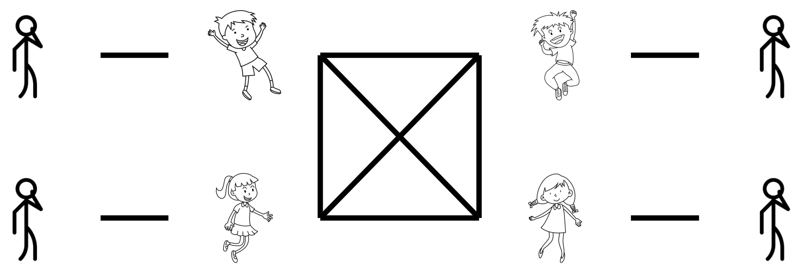
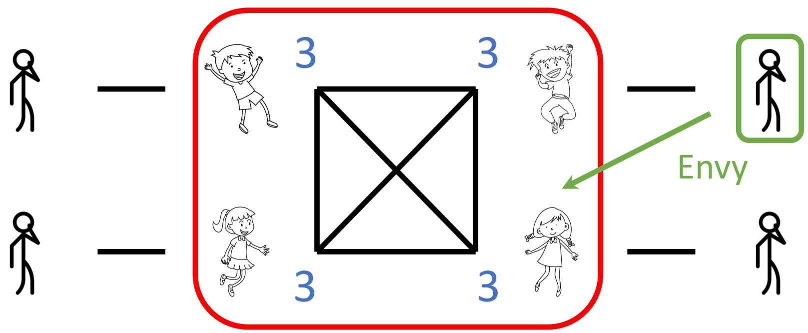


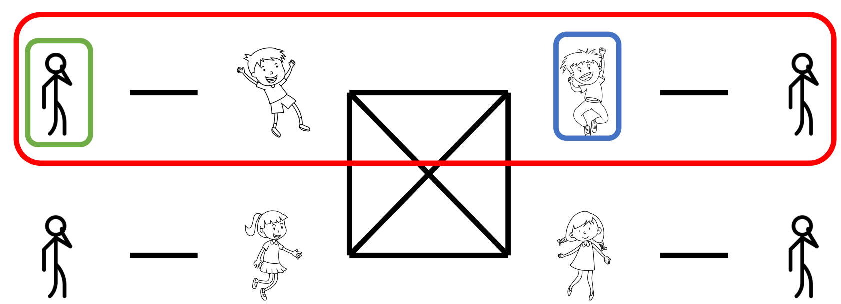
Maximum utility in any group of size 4 is at most 3



Core

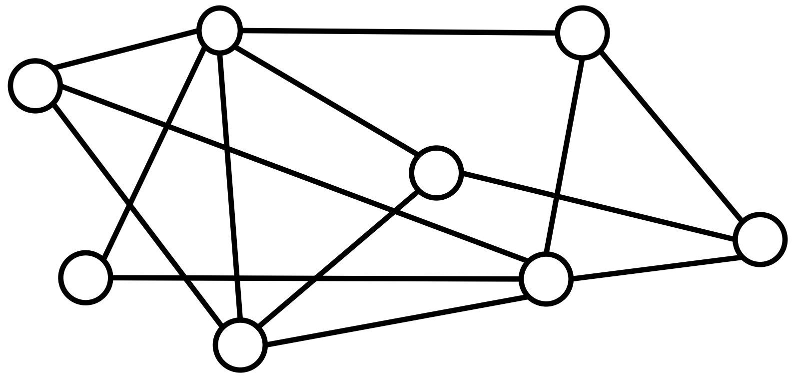
Dangling agents' only friend is in already same group

Clique agents gain ≤ 2 but lose 2 if swap



Envy-free

Min k-cut and $E(A,B)$



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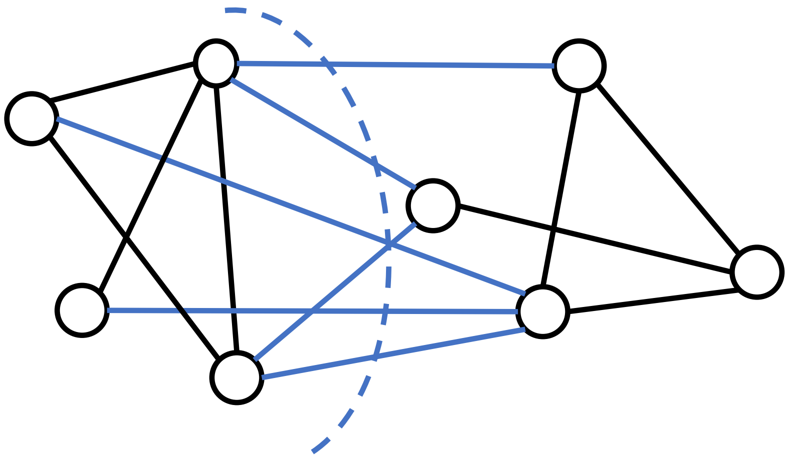
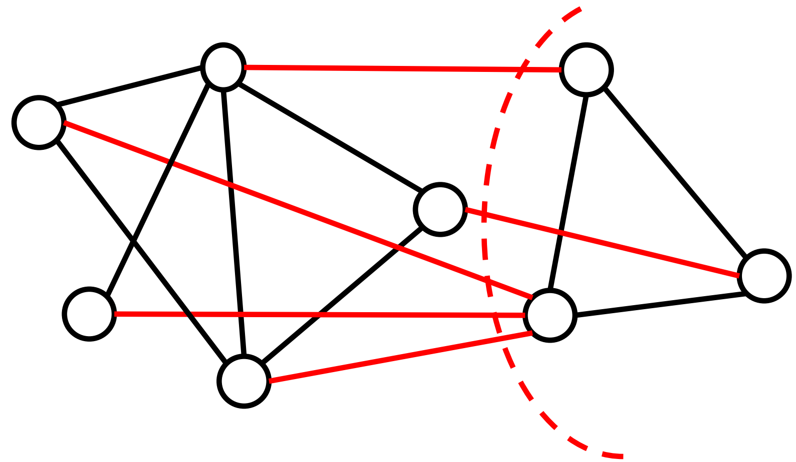
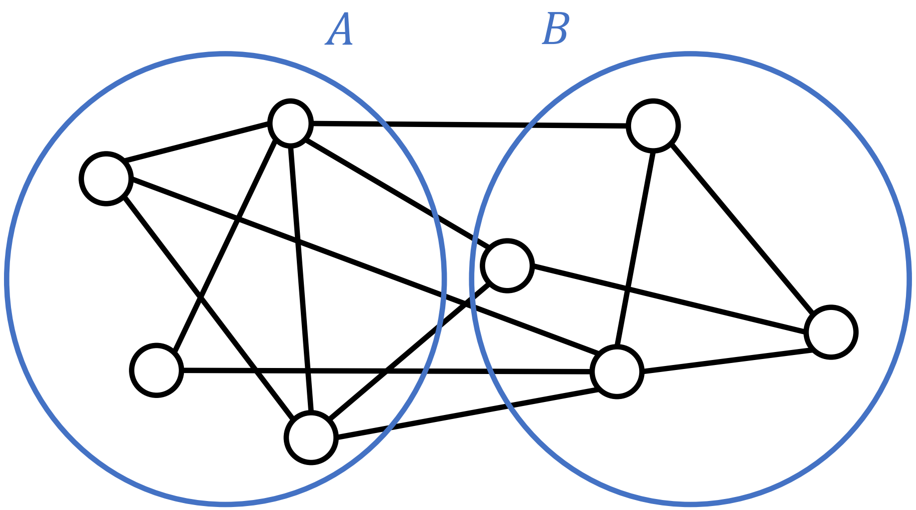


Cut size = 6
balanced

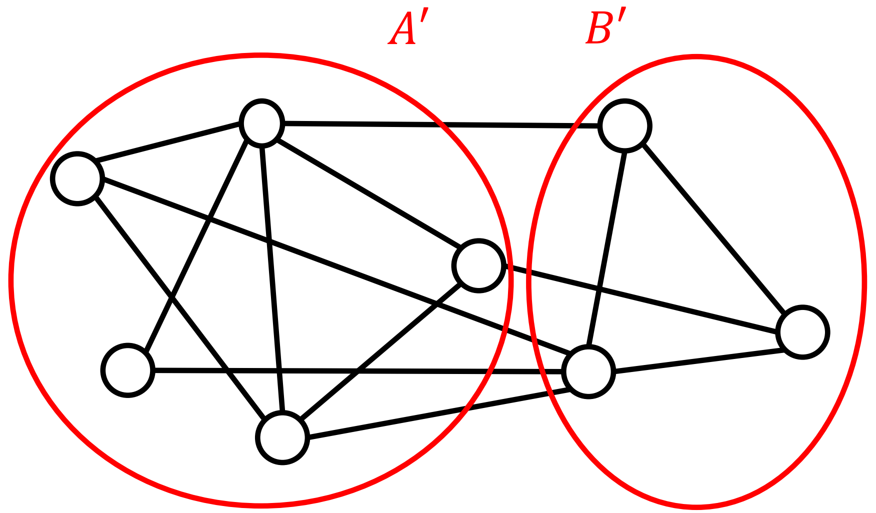


Cut size = 5
imbalanced

Min k-cut and $E(A,B)$

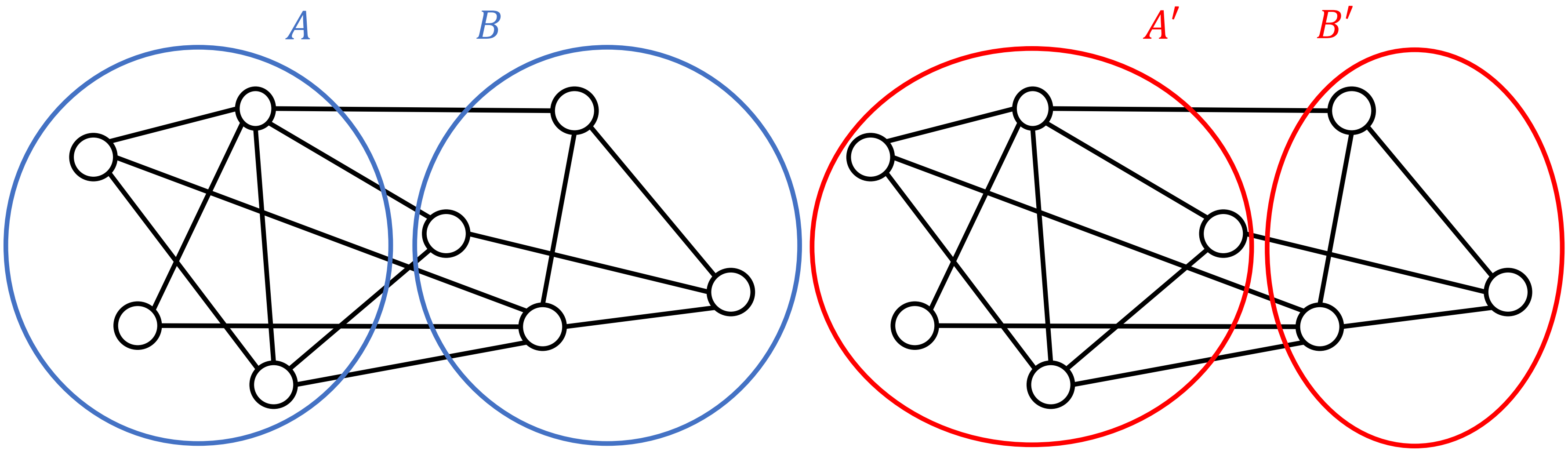


$cut(A,B) = E(A,B) = 6$
balanced



$cut(A',B') = E(A',B') = 5$
imbalanced

Min k-cut and $E(A, B)$



$cut(A, B) = E(A, B) = 6$
balanced

$cut(A', B') = E(A', B') = 5$
imbalanced

- When $k = 2$, can efficiently solve **imbalanced** min 2-cut in poly time
  - Run max flow algorithm for different source and sink nodes
- When $k = 2$ and n is even, **balanced** min 2-cut is the $\underbrace{\text{min-bisection problem}}_{\text{NP-hard}}$
- When $k \geq 3$, NP-hard if k is part of input
  - Polynomial time $2 - \frac{2}{\mathrm{k}}$ approximations exists
  - Under some hardness conjecture, NP-hard to approximate within $2 - \epsilon$

Some background about min cuts… The key point is that balanced min 2-cut is NP-hard.

(Many interesting results. Will only discuss the ones in red)

Results: Core ($k = 2$)

- Open question 1:

  Is there a balanced 2-partitioning in the core?

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- Result 1:

  Min 2-cut is in the $(2,0)$-core

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Can we compute something from $(2,0)$-core in poly time?

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Can we compute something from $(2,0)$-core in poly time?

- "Almost" (2,0)-core can be efficiently computed:
  - Partition in the (2,1)-core
  - Partition in the (3,0)-core, when $n \geq k^2 + k$

Corollary of next slide

(Many interesting results. Will only discuss the ones in red)

Results: Core ($k \geq 3$)

- Result 2: There exists instances without balanced k-partition
  - (i) In the $(\alpha, 0)$-core, when $\alpha \geq 1$
  - (ii) In the $(1, \beta)$-core, when $\beta < \frac{k}{2} - 2 = \frac{k-4}{2}$

  Depends on n not dividing nicely by k
- Open question 3: If k divides n, is the core empty?

(Many interesting results. Will only discuss the ones in red)

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- Open question 3: If k divides n, is the core empty?
- Result 3
  1. Every min k-cut is in the $(\mathrm{k}, \mathrm{k}-1)$-core
  2. There is a polynomial time algorithm ALG that returns a k-partition in the $(\mathrm{k}, \mathrm{k}-1)$-core
  3. When $n \geq k^2 + k$, min k-cut is in the $(2\mathrm{k}-1, 0)$-core
  4. When $\mathrm{n} \geq \mathrm{k}^2 + \mathrm{k}$, ALG returns a k-partition in the $(2\mathrm{k}-1, 0)$-core
  5. When $\mathrm{n} < \mathrm{k}^2 + \mathrm{k}$, every balanced k-partition is in the $(1, \mathrm{k})$-core

(Many interesting results. Will only discuss the ones in red)

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Set $k = 2$

“Almost” (2,0)-core can be efficiently computed:
- Partition in the (2,1)-core
- Partition in the (3,0)-core, when $n \geq k^2 + k$

(Many interesting results. Will only discuss the ones in red)

Results: Core ($k \geq 3$)

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  5. When $\mathrm{n} < \mathrm{k}^2 + \mathrm{k}$, every balanced k-partition is in the $(1, \mathrm{k})$-core
- Result 4

  There exists an instance with $\mathrm{n} \geq k^2 + k$ where min k-cut is not in the $(\alpha, 0)$-core, for $\alpha < 2k - 2$

(Many interesting results. Will only discuss the ones in red)

Results: Envy-freeness

- Result 5

EF-1 may not exist even for $k = 2$.

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- Open question 4

  For $k \geq 2$, does EF-2 always exist?

(Many interesting results. Will only discuss the ones in red)

Results: Envy-freeness

- Result 5

  EF-1 may not exist even for $k = 2$.

- Open question 4

  For $k \geq 2$, does EF-2 always exist?

- Result 6

  For $k \geq 2$ and $r \in \mathcal{O}\left(\sqrt{\frac{n}{k} \cdot \ln k}\right)$, EF-$r$ always exists and can be computed in polynomial time.

Relies on known results in discrepancy theory

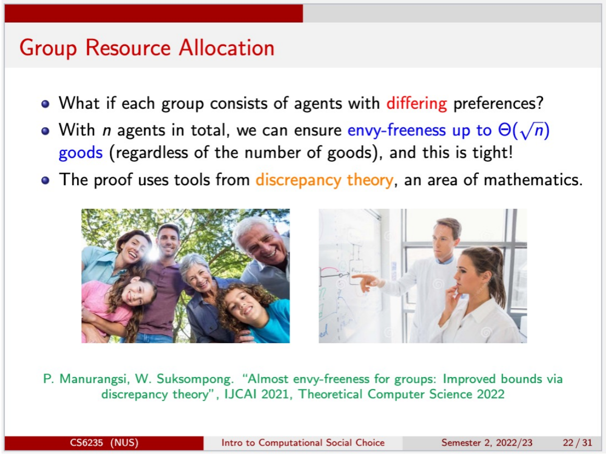
(Many interesting results. Will only discuss the ones in red)

Results: Imbalanced partitioning

- Result 7
  - When $k \geq 2$, can find imbalanced k-partition in the $(1, k-2)$-core in polynomial time
  - When $k \geq 3$, exists instance where no imbalanced k-partition exists in the $(1, \beta)$-core for $\beta < k-2$
- Result 8
  - EF-2 imbalanced 2-partition always exists and can be computed in polynomial time.
- Construction of result 5 can also be used to show that EF-1 may not exist

Future directions

- The many open questions mentioned earlier
- Model extensions
  - Beyond symmetric and binary preferences
  - Assigning items to groups of agents
    - Partition agents in groups, then assign groups to items



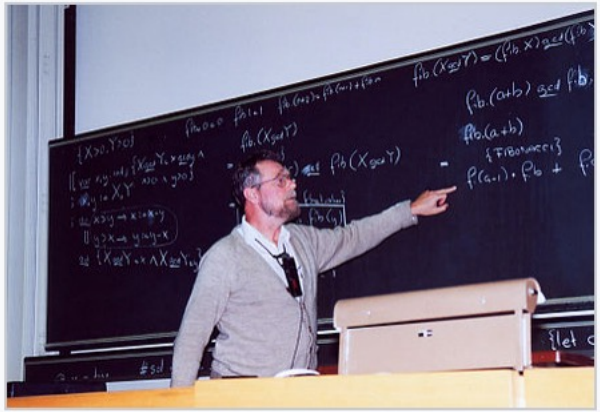
  - What if agents have attributes / types?

The “main part” of the talk is now over.

Since this is a technical class presentation, let’s go into some details.

In the rest of the talk, let’s go through the key ideas behind $1\sim2$ (or more) results.

Some proof ideas and sketches



Edsger Dijkstra at the blackboard during a conference at ETH Zurich in 1994. In Dijkstra's own words, "A picture may be worth a thousand words, a formula is worth a thousand pictures."[12]

"An animated proof is even better!" - Davin

I will animate pictures and equations will be animated to make the key ideas easy to grasp and arguments easy to follow ☺

https://en.wikipedia.org/wiki/A_picture_is_worth_a_thousand_words

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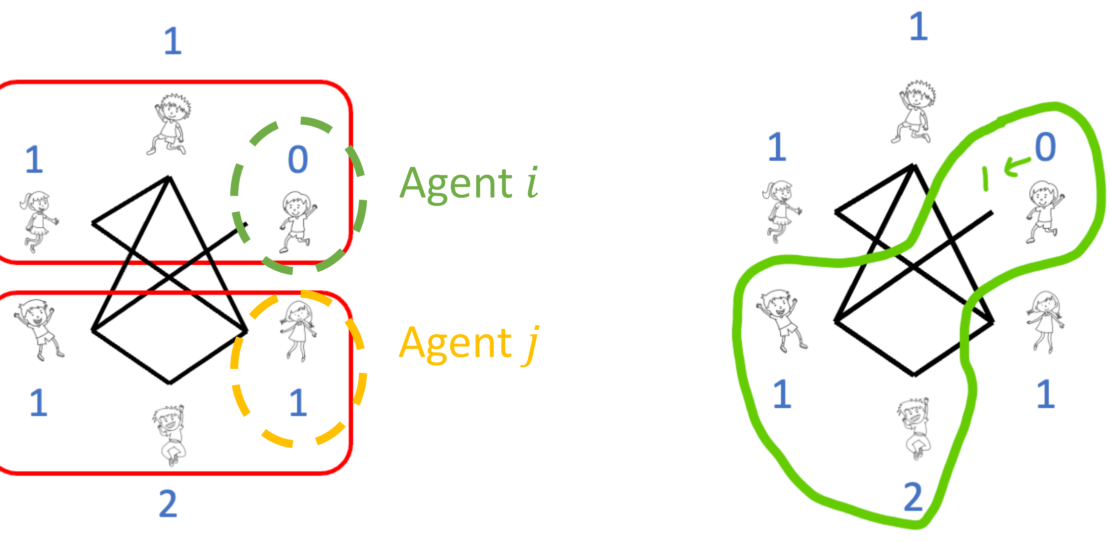
I will share them in descending order of what I think is interesting (and in an ordering that I feel facilitates understanding). **Feel free to ask questions**, it's okay to not complete all the material (I expect not to). Slides are available for your leisure reading.

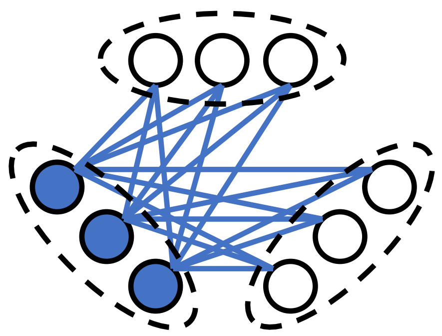
Let's first familiarize ourselves with the notion of **Envy-free** with some lower bound examples



Result 5

Result 5

EF-1 may not exist even for $k = 2$



Graph is complete tri-partite $K_{3,3,3}$ on $n = 9$ agents

Result 5

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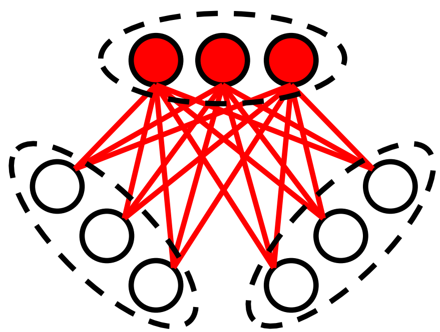

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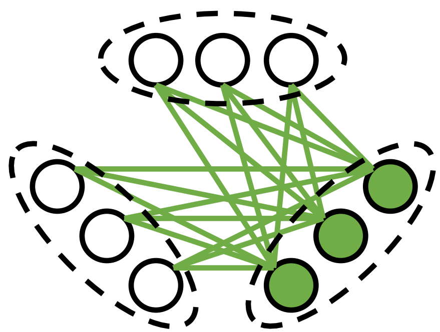

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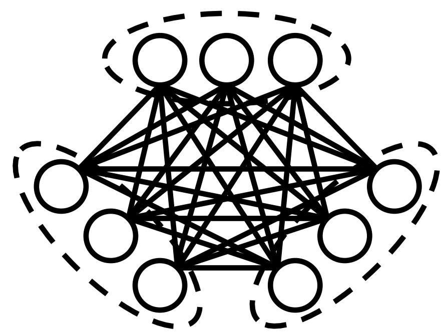

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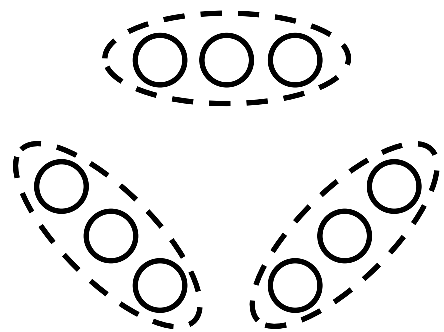

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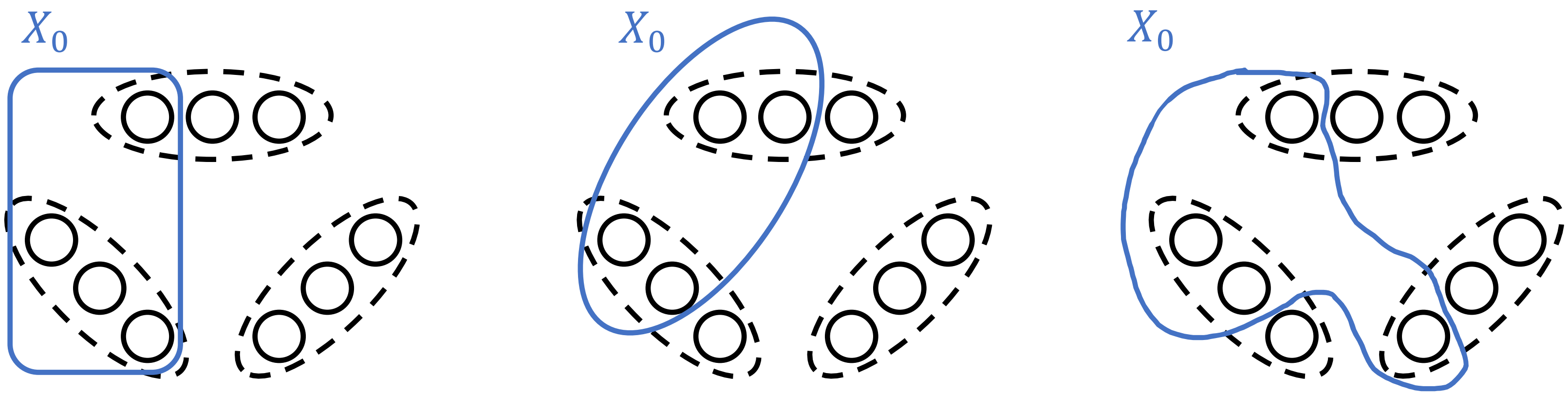


Graph is complete tri-partite $K_{3,3,3}$ on $n = 9$ agents

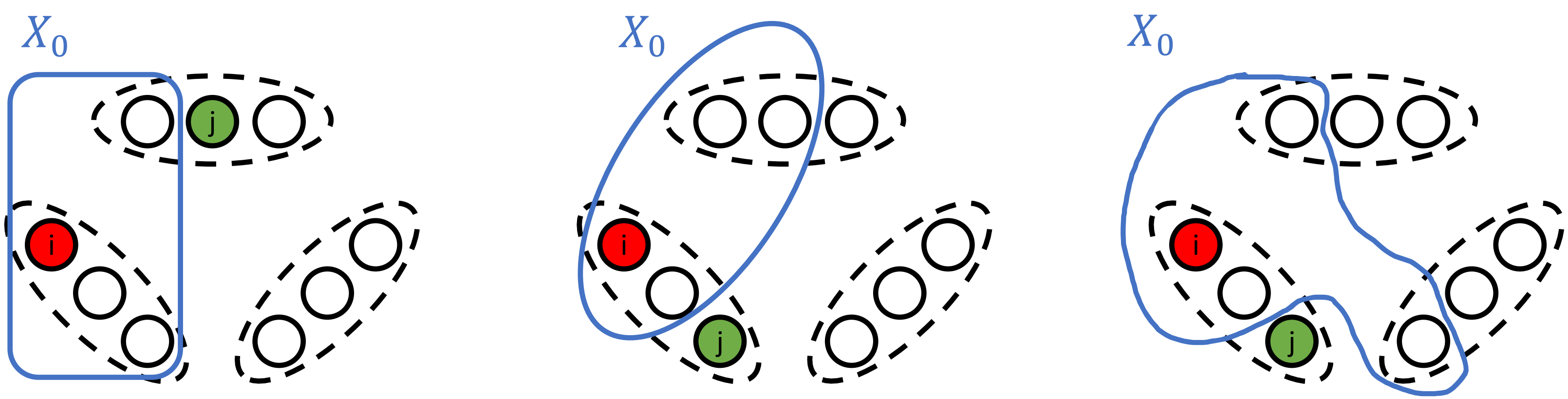
Let (X_0, X_1) be any balanced 2-partition $\Rightarrow 4 = \left\lfloor \frac{9}{2} \right\rfloor = \left\lfloor \frac{n}{k} \right\rfloor \leq |X_0|, |X_1| \leq \left\lceil \frac{n}{k} \right\rceil = \left\lceil \frac{9}{2} \right\rceil = 5$

Without loss of generality, suppose $|X_0| = 4$ and $|X_1| = 5$

Recall definition of **EF-r**: $\forall j \in [n], u_i\big(X(i)\big) \geq u_i(X(j) \cup \{i\} \setminus \{j\}) - r$

Result 5

EF-1 may not exist even for $k = 2$



$u_i(X_0) = 1$
$u_i(X_1 \cup \{i\} \setminus \{j\}) = 3$

$u_i(X_0) = 2$
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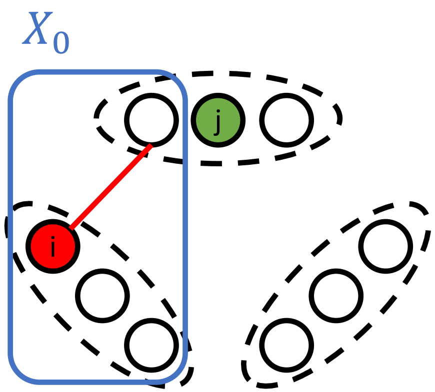

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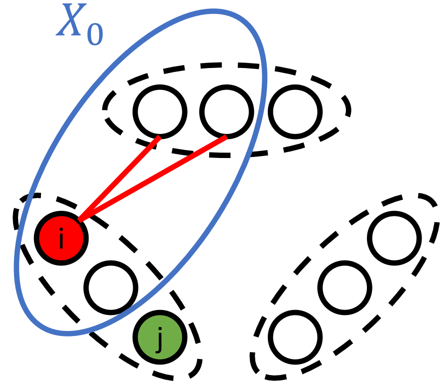

$\boldsymbol{u_i(X_0) = 2}$

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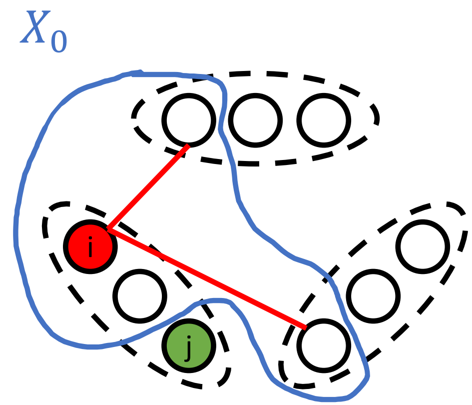

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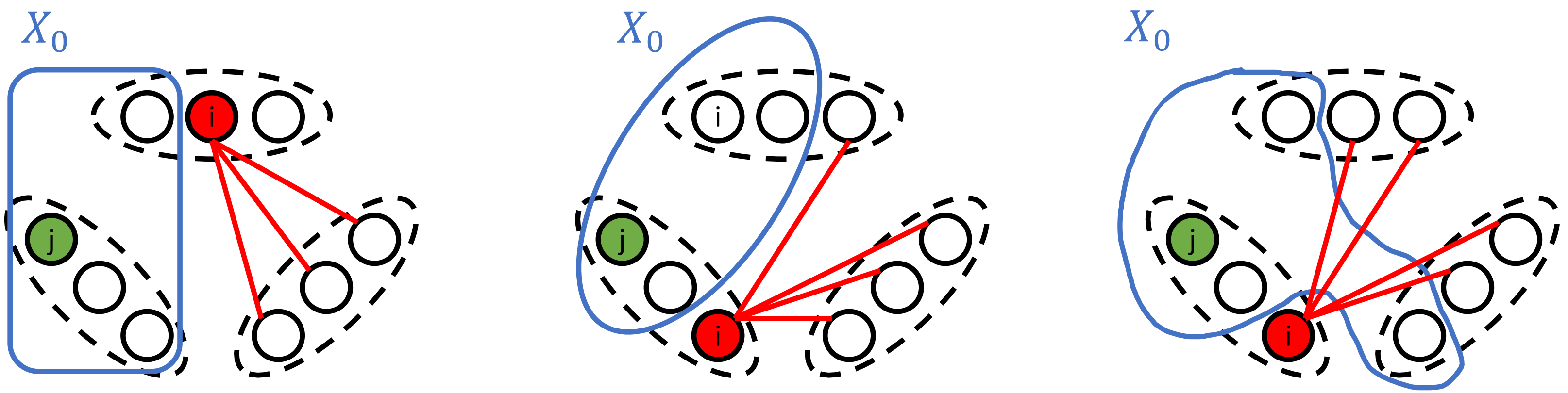

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Discrepancy theory

- Informal: Given subset of elements, assign colors to elements such that each subset has roughly same number of colors of each type

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- Universe $\Omega = [n]$ ⟵ Given
- Set system $\mathcal{S} = \{S_1, \ldots, S_m\}$, where each $S_i \subseteq [n]$ ⟵ Given
- Coloring $\chi: \Omega \to [k]$ ⟵ Find / Compute

Parameters n and m are fixed when Ω and $\mathcal{S}$ are given

Given fixed k, output a coloring χ

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- Discrepancy of $\mathcal{S}$ with respect to coloring χ

$$disc_k(\mathcal{S}, \chi) = \max_{j \in [k],\, i \in [m]} \left| \underbrace{|\chi^{-1}(j) \cap S_i|}_{\substack{\text{All elements in} \\ \text{universe that are} \\ \text{assigned color j}}} - \underbrace{\frac{|S_i|}{k}}_{\substack{\text{If all colors} \\ \text{are balanced} \\ \text{within } S_i}} \right|$$

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- Discrepancy of $\mathcal{S}$ (pick best coloring χ)

$$disc_k(\mathcal{S}) = \min_{\chi: \Omega \to [k]} disc_k(\mathcal{S}, \chi)$$

Discrepancy: What is known?

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- Lower bound

$$disc_k(\mathcal{S}) \in \Omega\left(\sqrt{\frac{n}{k}}\right)$$

- Achievable in polynomial time

$$disc_k(\mathcal{S}) \in \mathcal{O}\left(\sqrt{\frac{n}{k} \cdot \ln\left(\frac{km}{n}\right)}\right)$$

Result 6

When $k \geq 2$, EF-r k-partition can be computed in polynomial time, where $r \in \mathcal{O}\left(\sqrt{\frac{n}{k} \cdot \ln k}\right)$

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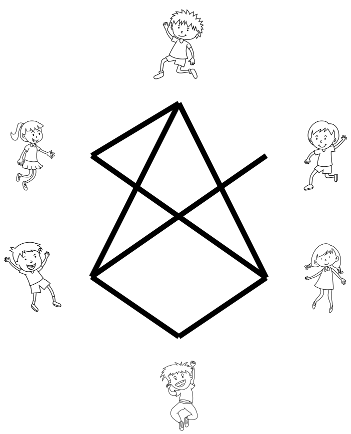

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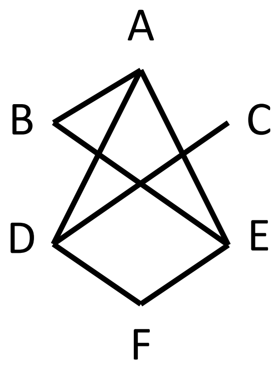

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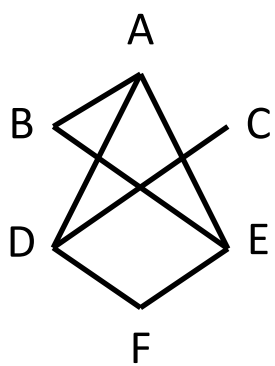

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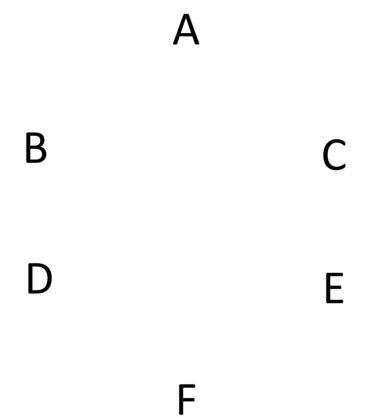

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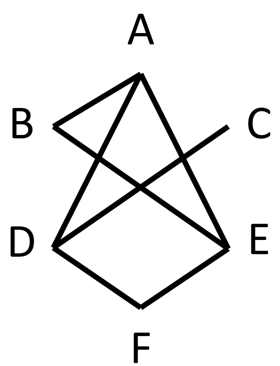

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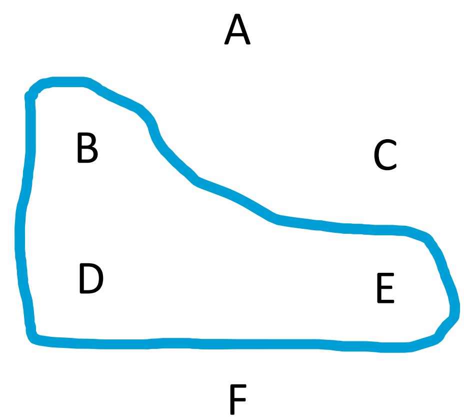

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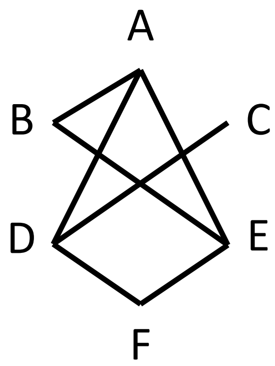

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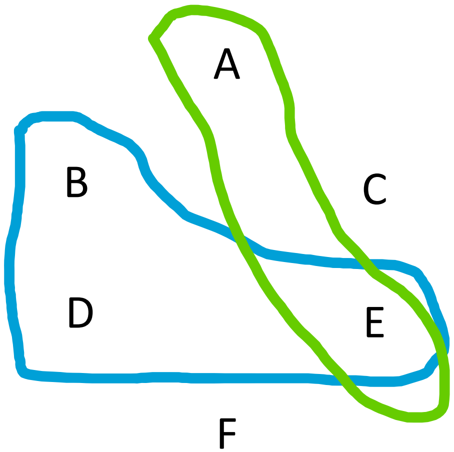

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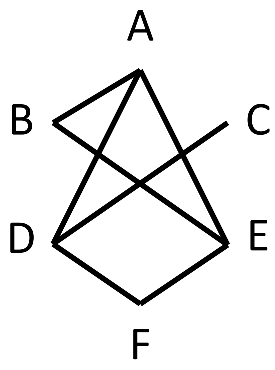

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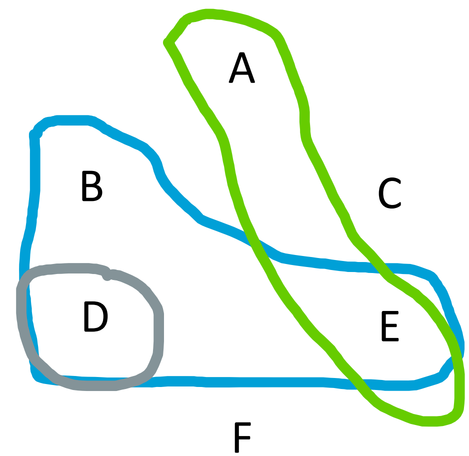

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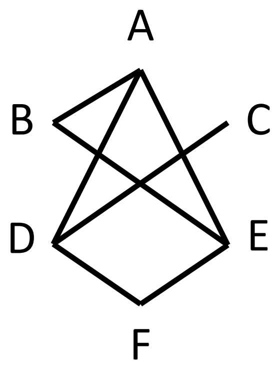

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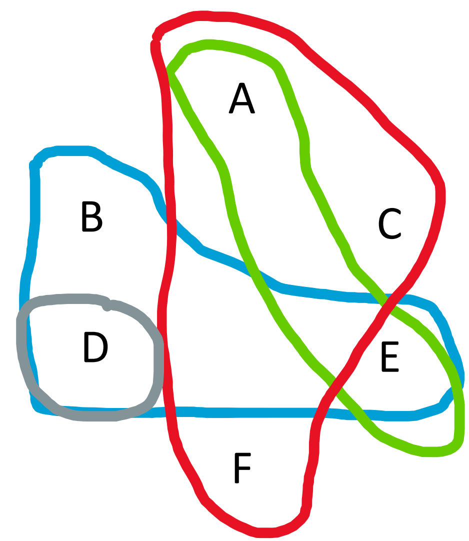

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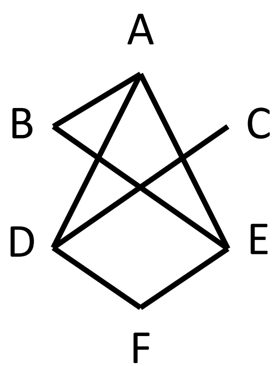

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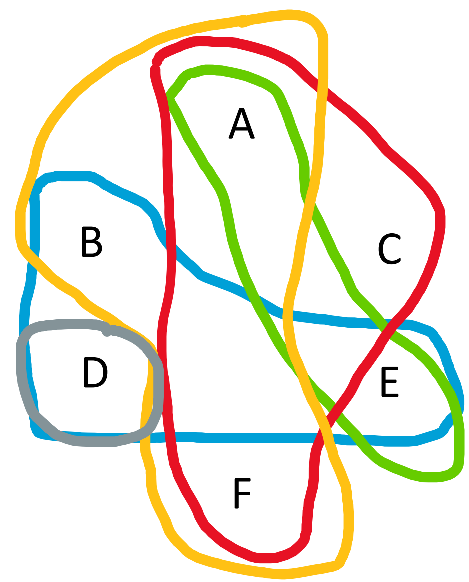

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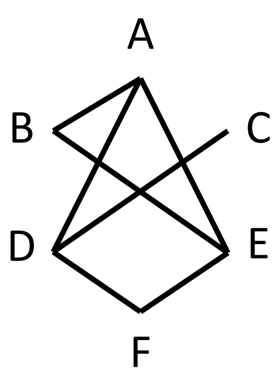

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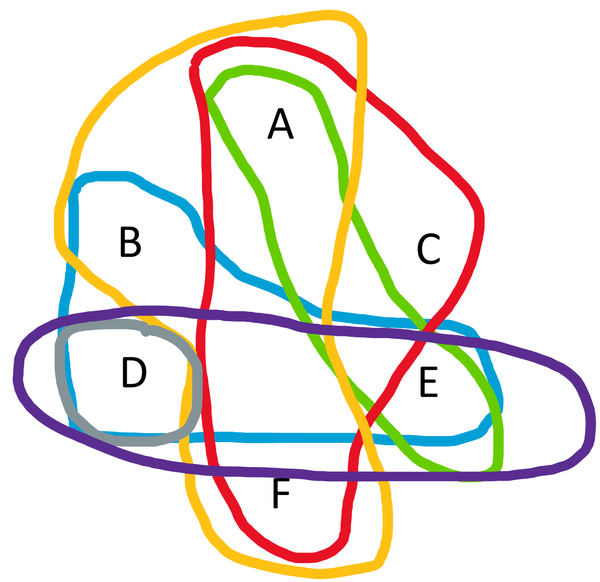

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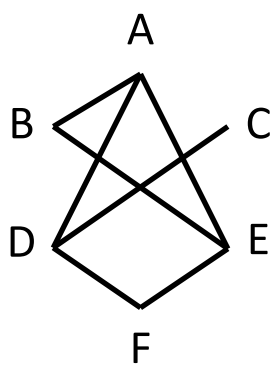

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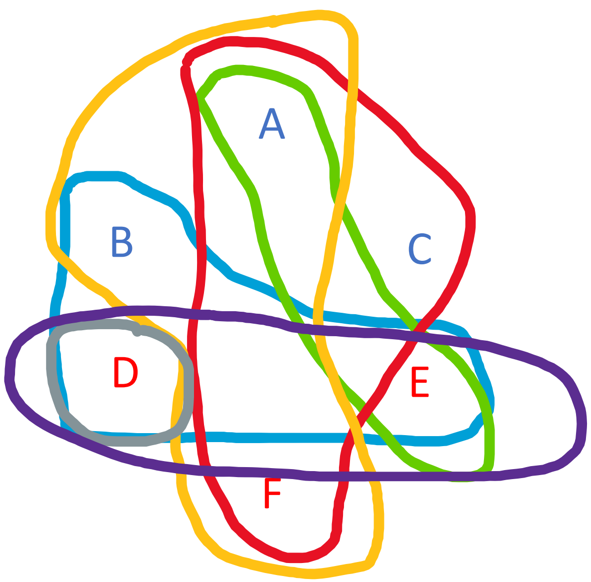

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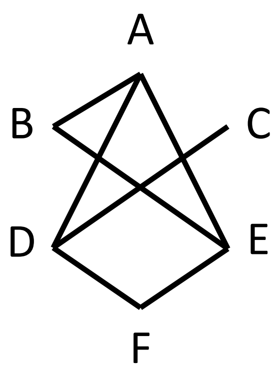

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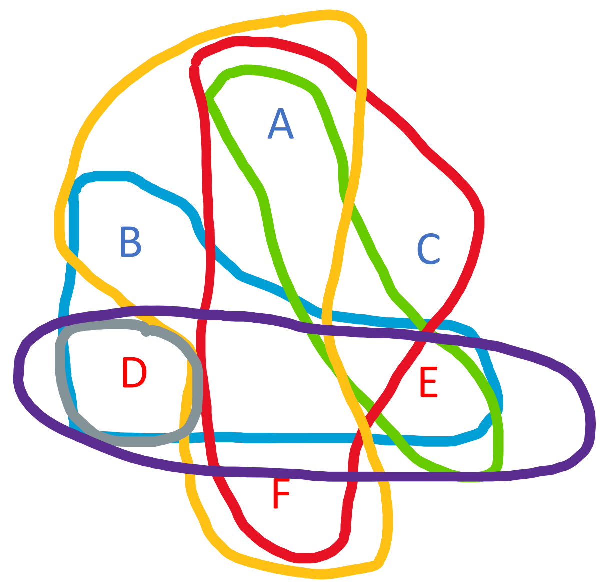

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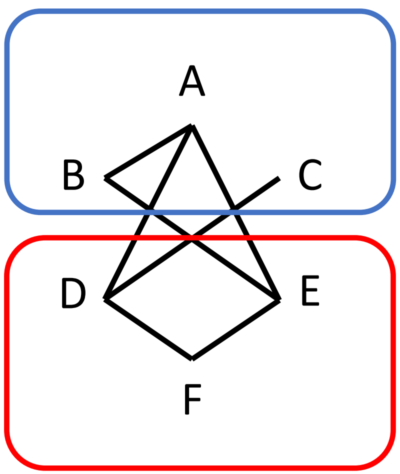

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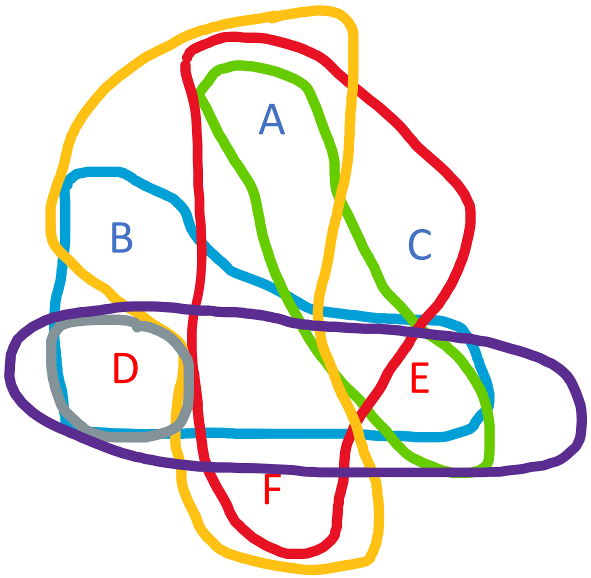

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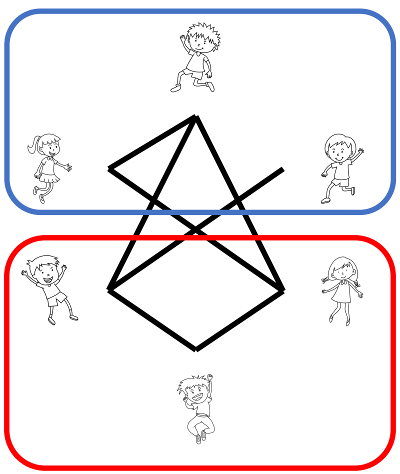

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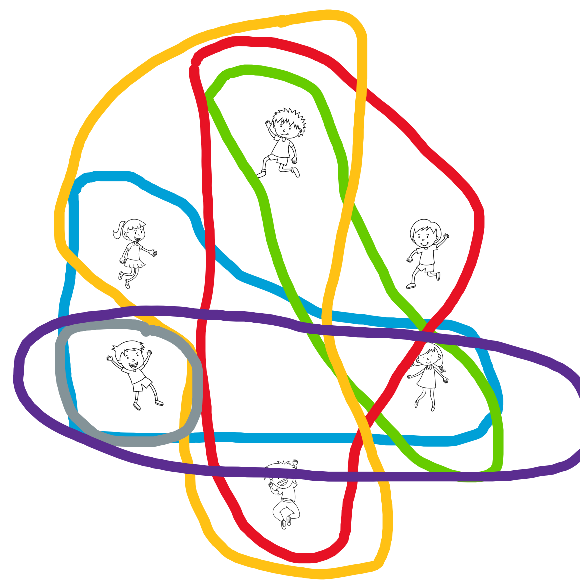

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$N(i)$

$$disc_k(\mathcal{S}, \chi) = \max_{j \in [k],\, i \in [m]} \left| |\chi^{-1}(j) \cap S_i| - \frac{|S_i|}{k} \right|$$

$m = n$ $\quad X_j$ $\quad N(i)$

Result 6

When $k \geq 2$, EF-r k-partition can be computed in polynomial time, where $r \in \mathcal{O}\left(\sqrt{\frac{n}{k} \cdot \ln k}\right)$

Agents $\Omega = [n]$; $\mathrm{S_i} = \mathrm{N(i)}$; $\chi: \Omega \to [k]$

$$disc_k(\mathcal{S}, \chi) = \max_{j \in [k],\, i \in [n]} \left| \left|X_j \cap N(i)\right| - \frac{|N(i)|}{k} \right|$$

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Let $X_j \neq X(i)$ be a partition that i is not in. Then,

$u_i\big(X(i)\big) - u_i\big(X_j\big)$

Result 6

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$$u_i\big(X(i)\big) - u_i\big(X_j\big) = u_i\big(X(i)\big) - \frac{|N(i)|}{k} + \frac{|N(i)|}{k} - u_i\big(X_j\big)$$

Result 6

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$$\leq \left| u_i\big(X(i)\big) - \frac{|N(i)|}{k} \right| + \left| \frac{|N(i)|}{k} - u_i\big(X_j\big) \right|$$

Result 6

When $k \geq 2$, EF-r k-partition can be computed in polynomial time, where $r \in \mathcal{O}\left(\sqrt{\frac{n}{k} \cdot \ln k}\right)$

Agents $\Omega = [n]$; $\mathrm{S_i} = \mathrm{N(i)}$; $\chi: \Omega \to [k]$

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$$\leq \left| u_i\big(X(i)\big) - \frac{|N(i)|}{k} \right| + \left| \frac{|N(i)|}{k} - u_i\big(X_j\big) \right|$$

$$\leq 2 \cdot disc_k(\mathcal{S}, \chi)$$

Result 6

When $k \geq 2$, EF-r k-partition can be computed in polynomial time, where $r \in \mathcal{O}\left(\sqrt{\frac{n}{k} \cdot \ln k}\right)$

- Problem: Partitions may not be balanced

Result 6

When $k \geq 2$, EF-r k-partition can be computed in polynomial time, where $r \in \mathcal{O}\left(\sqrt{\frac{n}{k} \cdot \ln k}\right)$

- Problem: Partitions may not be balanced
- Fix: Add another set $S_{n+1} = V$ (So, $m = n + 1$)

Result 6

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- Fix: Add another set $S_{n+1} = V$ (So, $m = n + 1$)

$$\left| |X_i| - |X_j| \right| = \left| |X_i \cap S_{n+1}| + |X_i \setminus S_{n+1}| - |X_j \cap S_{n+1}| - |X_j \setminus S_{n+1}| \right|$$

Result 6

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$$\begin{aligned}\left||X_i| - |X_j|\right| &= \left||X_i \cap S_{n+1}| + |X_i \setminus S_{n+1}| - |X_j \cap S_{n+1}| - |X_j \setminus S_{n+1}|\right| \\ &= \left||X_i \cap S_{n+1}| + 0 - |X_j \cap S_{n+1}| - 0\right|\end{aligned}$$

Result 6

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Result 6

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- Problem: Partitions may not be balanced
- Fix: Add another set $S_{n+1} = V$ (So, $m = n + 1$)

$$
\begin{aligned}
\left||X_i| - |X_j|\right| &= \left||X_i \cap S_{n+1}| + |X_i \backslash S_{n+1}| - |X_j \cap S_{n+1}| - |X_j \backslash S_{n+1}|\right| \\
&= \left||X_i \cap S_{n+1}| + 0 - |X_j \cap S_{n+1}| - 0\right| \\
&= \left||X_i \cap S_{n+1}| - \frac{|S_{n+1}|}{k} + \frac{|S_{n+1}|}{k} - |X_j \cap S_{n+1}|\right| \\
&\leq \left||X_i \cap S_{n+1}| - \frac{|S_{n+1}|}{k}\right| + \left|\frac{|S_{n+1}|}{k} - |X_j \cap S_{n+1}|\right|
\end{aligned}
$$

Result 6

When $k \geq 2$, EF-r k-partition can be computed in polynomial time, where $r \in \mathcal{O}\left(\sqrt{\frac{n}{k} \cdot \ln k}\right)$

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&= \left||X_i \cap S_{n+1}| + 0 - |X_j \cap S_{n+1}| - 0\right| \\
&= \left||X_i \cap S_{n+1}| - \frac{|S_{n+1}|}{k} + \frac{|S_{n+1}|}{k} - |X_j \cap S_{n+1}|\right| \\
&\leq \left||X_i \cap S_{n+1}| - \frac{|S_{n+1}|}{k}\right| + \left|\frac{|S_{n+1}|}{k} - |X_j \cap S_{n+1}|\right| \\
&\leq 2 \cdot disc_k(\mathcal{S}, \chi)
\end{aligned}$$

Result 6

When $k \geq 2$, EF-r k-partition can be computed in polynomial time, where $r \in \mathcal{O}\left(\sqrt{\frac{n}{k} \cdot \ln k}\right)$

- Problem: Partitions may not be balanced
- Fix: Add another set $S_{n+1} = V$ (So, $m = n + 1$)

$$\left| |X_i| - |X_j| \right| \leq 2 \cdot disc_k(\mathcal{S}, \chi)$$

Recall definition of **EF-r**: $\forall j \in [n], u_i\big(X(i)\big) \geq u_i(X(j) \cup \{i\} \setminus \{j\}) - r$

Result 6

When $k \geq 2$, EF-r k-partition can be computed in polynomial time, where $r \in \mathcal{O}\left(\sqrt{\frac{n}{k} \cdot \ln k}\right)$

- Problem: Partitions may not be balanced
- Fix: Add another set $S_{n+1} = V$ (So, $m = n + 1$)

$$\big||X_i| - |X_j|\big| \leq 2 \cdot disc_k(\mathcal{S}, \chi)$$

- Moving $disc_k(\mathcal{S}, \chi)$ agents between partitions will not affect EF-r when $r \in \mathcal{O}\big(disc_k(\mathcal{S}, \chi)\big)$

Recall definition of **EF-r**: $\forall j \in [n], u_i(X(i)) \geq u_i(X(j) \cup \{i\} \setminus \{j\}) - r$

Result 6

When $k \geq 2$, EF-r k-partition can be computed in polynomial time, where $r \in \mathcal{O}\left(\sqrt{\frac{n}{k} \cdot \ln k}\right)$

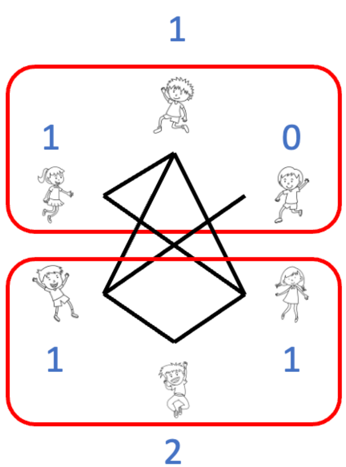
- Problem: Partitions may not be balanced
- Fix: Add another set $S_{n+1} = V$ (So, $m = n + 1$)

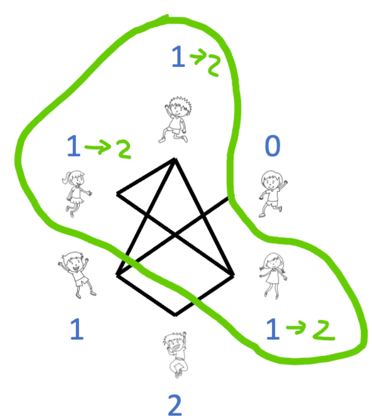
$$\left| |X_i| - |X_j| \right| \leq 2 \cdot disc_k(\mathcal{S}, \chi)$$

- Moving $disc_k(\mathcal{S}, \chi)$ agents between partitions will not affect EF-r when $r \in \mathcal{O}\big(disc_k(\mathcal{S}, \chi)\big)$
- Apply known result (Note: $m = n + 1$)

$$disc_k(\mathcal{S}) \in \mathcal{O}\left(\sqrt{\frac{n}{k} \cdot \ln\left(\frac{km}{n}\right)}\right) \subseteq \mathcal{O}\left(\sqrt{\frac{n}{k} \cdot \ln k}\right)$$

Let's first familiarize ourselves with the notion of **core** and **blocking coalitions** with some lower bound examples





Blocking coalition S

Result 2

Recall definition of (α, β)-blocking coalition S for k-partition X: $u_i(S) > \alpha \cdot u_i\big(X(i)\big) + \beta$

Result 2

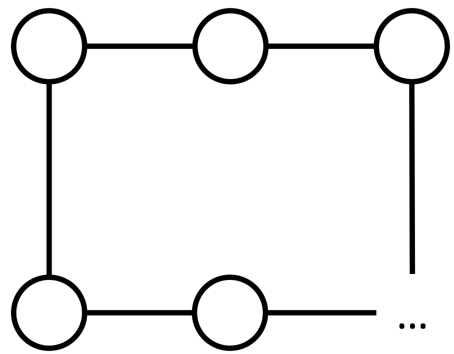
For k ≥ 3, there exists instances where

1. No balanced k-partition in the $(\alpha, 0)$-core
   - For any $\alpha \geq 1$
   - In this instance, there are $n = k + 1$ agents
2. No balanced k-partition in the $(1, \beta)$-core
   - For any $\beta < \frac{k}{2} - 2 = \frac{k-4}{2}$
   - In this instance, there are $n = k^2 - 1$ agents

Depends on n not dividing nicely by k

Result 2(i)

$\mathrm{k} \geq 3$, no $(\alpha, 0)$-core, $\forall \alpha \geq 1$



Graph is cycle on $n = k + 1$ agents

Result 2(i)

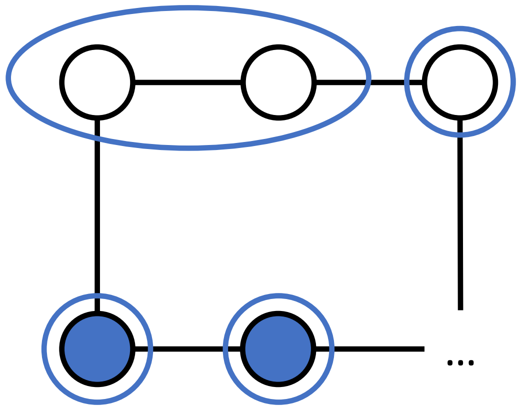
$\mathrm{k} \geq 3$, no $(\alpha, 0)$-core, $\forall \alpha \geq 1$

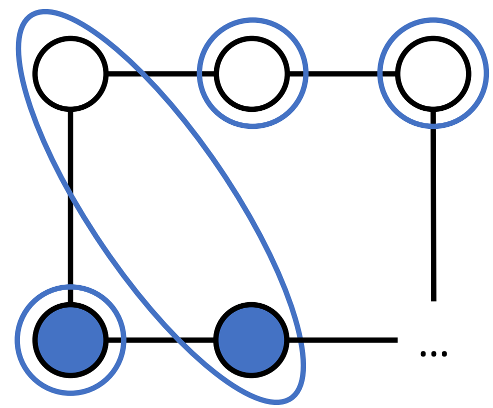
Graph is cycle on $n = k + 1$ agents

In *any* k-partition, we have 1 pair and $k - 1$ singletons

Result 2(i)

$\mathrm{k} \geq 3$, no $(\alpha, 0)$-core, $\forall \alpha \geq 1$





Graph is cycle on $n = k + 1$ agents

In *any* k-partition, we have 1 pair and $k - 1$ singletons

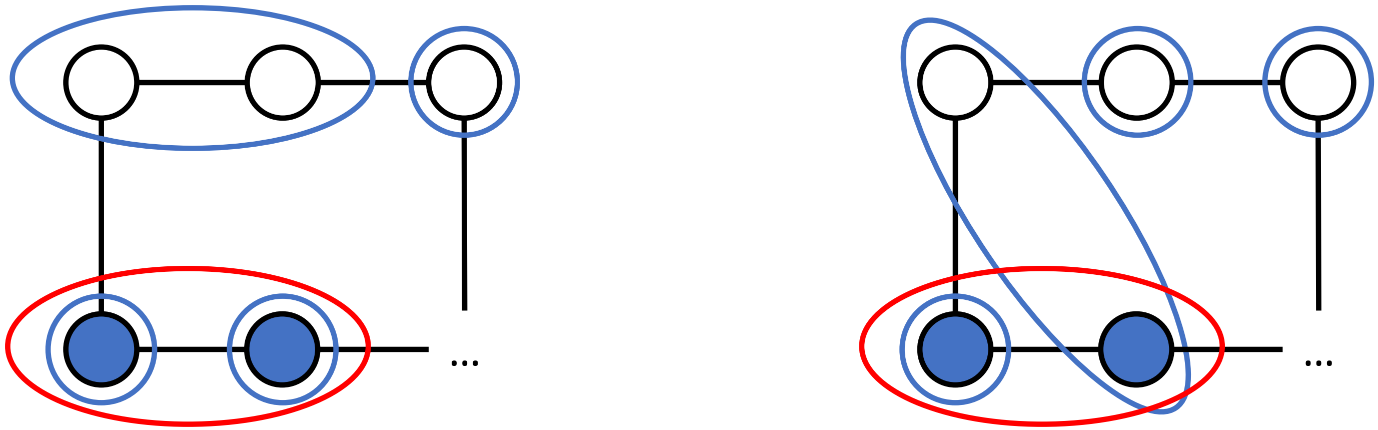
Since $\mathrm{n} = k + 1 \geq 4$, maximal matching size is ≥ 2

There exists two agents (in different groups) who are friends

Recall definition of (α, β)-blocking coalition S for k-partition X: $u_i(S) > \alpha \cdot u_i(X(i)) + \beta$

Result 2(i)

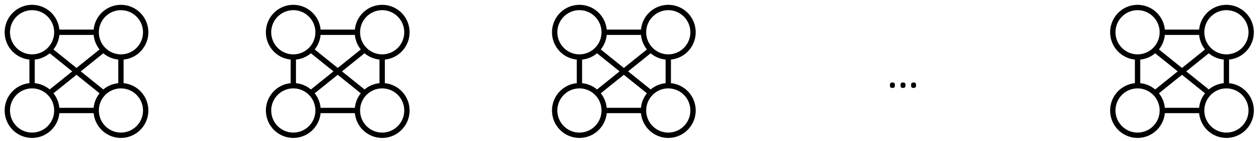
$k \geq 3$, no $(\alpha, 0)$-core, $\forall \alpha \geq 1$



Graph is cycle on $n = k + 1$ agents
In *any* k-partition, we have 1 pair and $k - 1$ singletons
Since $n = k + 1 \geq 4$, maximal matching size is ≥ 2
There exists two agents (in different groups) who are friends
They can increase utility from 0 to 1 → $(\alpha, 0)$-blocking coalition

Result 2(ii)

$k \geq 3$, no $(1, \beta)$-core, $\forall \beta < \frac{k-4}{2}$

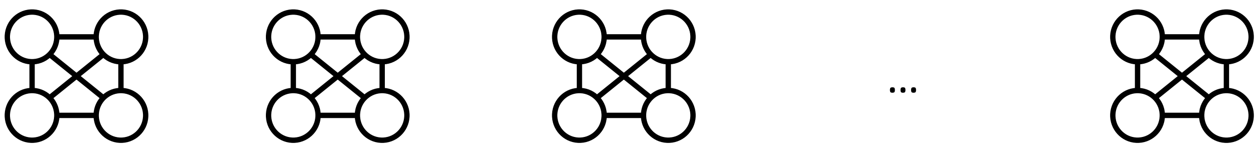


Graph is $k+1$ disjoint cliques $C_0, \ldots, C_k$ each of size $k-1 \Rightarrow n = k^2 - 1$ agents

Under this construction, play around with the inequalities.
The other stuff are more interesting, so we will skip the rest of the details.
You can read the slides at your own leisure.

Result 2(ii)

k ≥ 3, no $(1, \beta)$-core, $\forall \beta < \frac{k-4}{2}$



Graph is $k+1$ disjoint cliques $C_0, \ldots, C_k$ each of size $k-1 \Rightarrow n = k^2 - 1$ agents

There *exists* some clique C_{ℓ^*} such that $\left|C_{\ell^*} \cap X_j\right| \leq \frac{k+1}{2}$ for *any* partition index $j \in [k]$

Suppose not.

For *any* clique index $\ell \in [k+1]$, we have $\left|C_\ell \cap X_{j_\ell}\right| > \frac{k+1}{2}$ for *some* partition index $j_\ell \in [k]$

Observation 1: For partition index $j \in [k]$, we have $\left|X_j\right| \leq \left\lceil \frac{n}{k} \right\rceil = \left\lceil k - \frac{1}{k} \right\rceil = k \leq k+1$

Observation 2: For clique index $\ell \in [k+1]$, index j_ℓ is unique

Otherwise: $|C_\ell| > 2 \cdot \frac{k+1}{2} = k+1$

Observation 3: For different clique indices $\ell \neq \ell'$, we must have $j_\ell \neq j_{\ell'}$

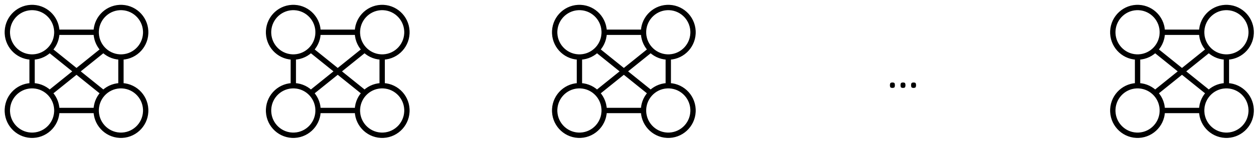
Otherwise: $|X_\ell| = |X_{\ell'}| > 2 \cdot \frac{k+1}{2} = k+1$ since $C_\ell \cap C_{\ell'} = \emptyset$

Contradiction since k+1 cliques but only k partites (cannot have $j_\ell \neq j_{\ell'}$ for all clique indices)

Recall definition of (α, β)-blocking coalition S for k-partition X: $u_i(S) > \alpha \cdot u_i(X(i)) + \beta$

Result 2(ii)

$k \geq 3$, no $(1, \beta)$-core, $\forall \beta < \frac{k-4}{2}$



Graph is $k+1$ disjoint cliques $C_0, \ldots, C_k$ each of size $k-1 \Rightarrow n = k^2 - 1$ agents

There *exists* some clique C_{ℓ^*} such that $\left|C_{\ell^*} \cap X_j\right| \leq \frac{k+1}{2}$ for *any* partition index $j \in [k]$

So, for any agent $i \in C_{\ell^*}$, we have $\mathrm{u_i}\big(\mathrm{X(i)}\big) = |N(i) \cap X(i)| \leq |C_{\ell^*} \cap X(i)| - 1 \leq \frac{k-1}{2}$

Observation 1: $|C_{\ell^*}| = k - 1 = \left\lfloor \frac{n}{k} \right\rfloor$

Observation 2: $\mathrm{u_i}(C_{\ell^*}) = k - 2 \geq u_i\big(X(i)\big) + \frac{k-3}{2} > u_i\big(X(i)\big) + \frac{k-4}{2}$

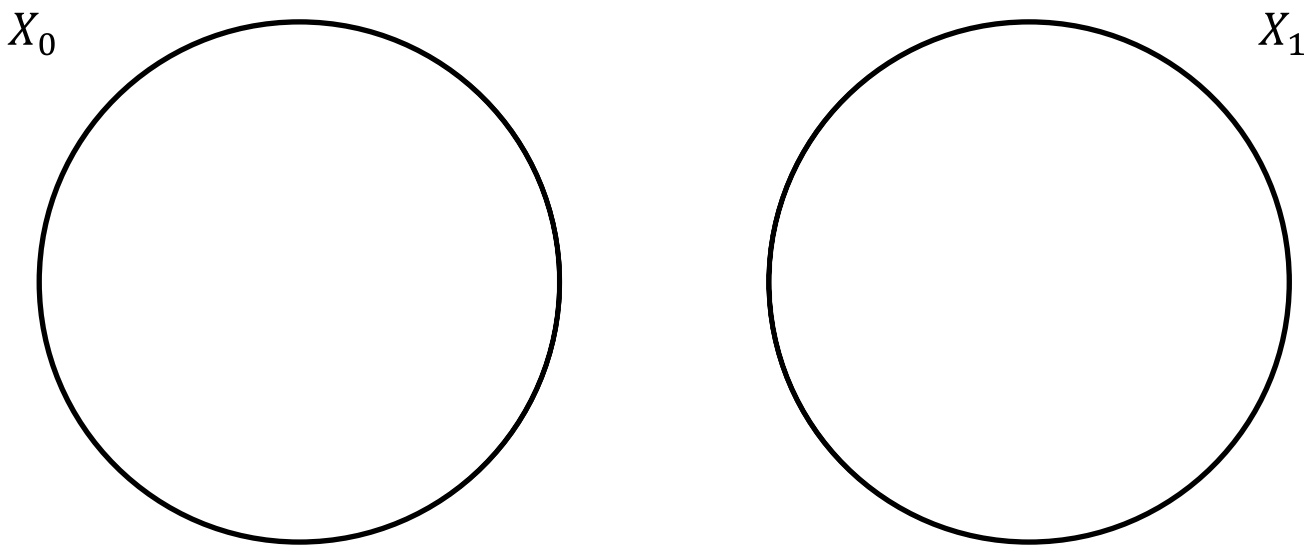
In other words, C_{ℓ^*} is a $(1, \beta)$-blocking coalition

Result 1

Result 1

Min 2-cut is in the (2,0)-core

Let (X_0, X_1) be an arbitrary min 2-cut



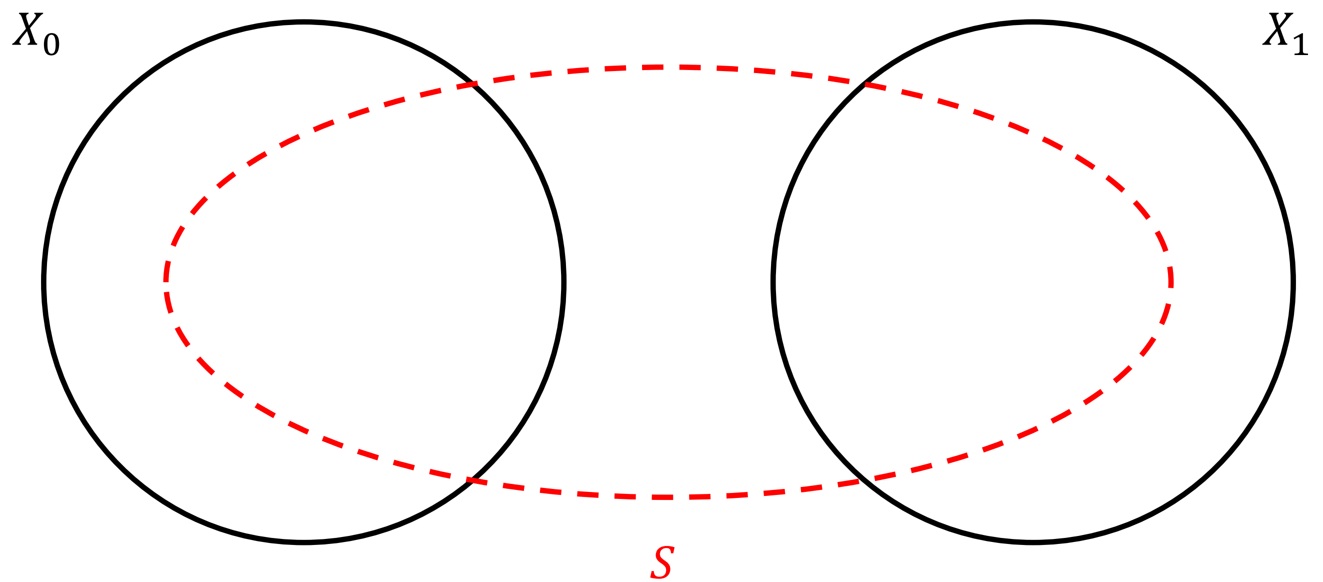
Recall definition of (α, β)-blocking coalition S for k-partition X: $u_i(S) > \alpha \cdot u_i(X(i)) + \beta$

Result 1

Min 2-cut is in the (2,0)-core

Let (X_0, X_1) be an arbitrary min 2-cut

For contradiction, let S be a $(2,0)$-blocking coalition

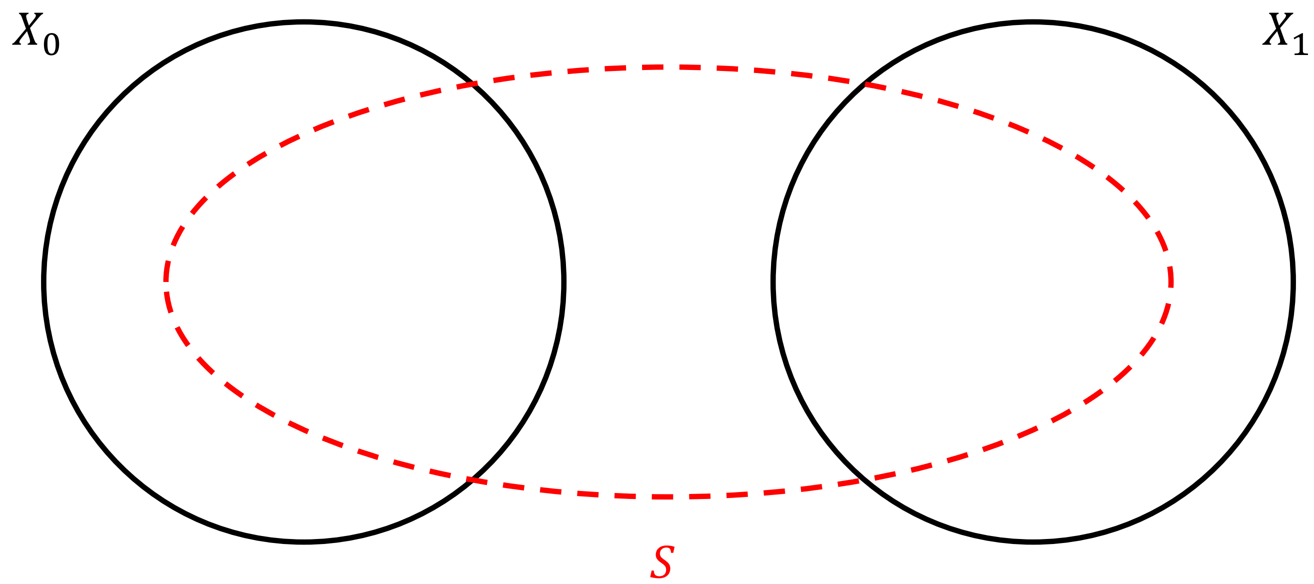


Recall definition of (α, β)-blocking coalition S for k-partition X: $u_i(S) > \alpha \cdot u_i(X(i)) + \beta$

Result 1

Min 2-cut is in the (2,0)-core

For any agent $i \in S$, we have $u_i(S) > 2 \cdot u_i(X(i))$

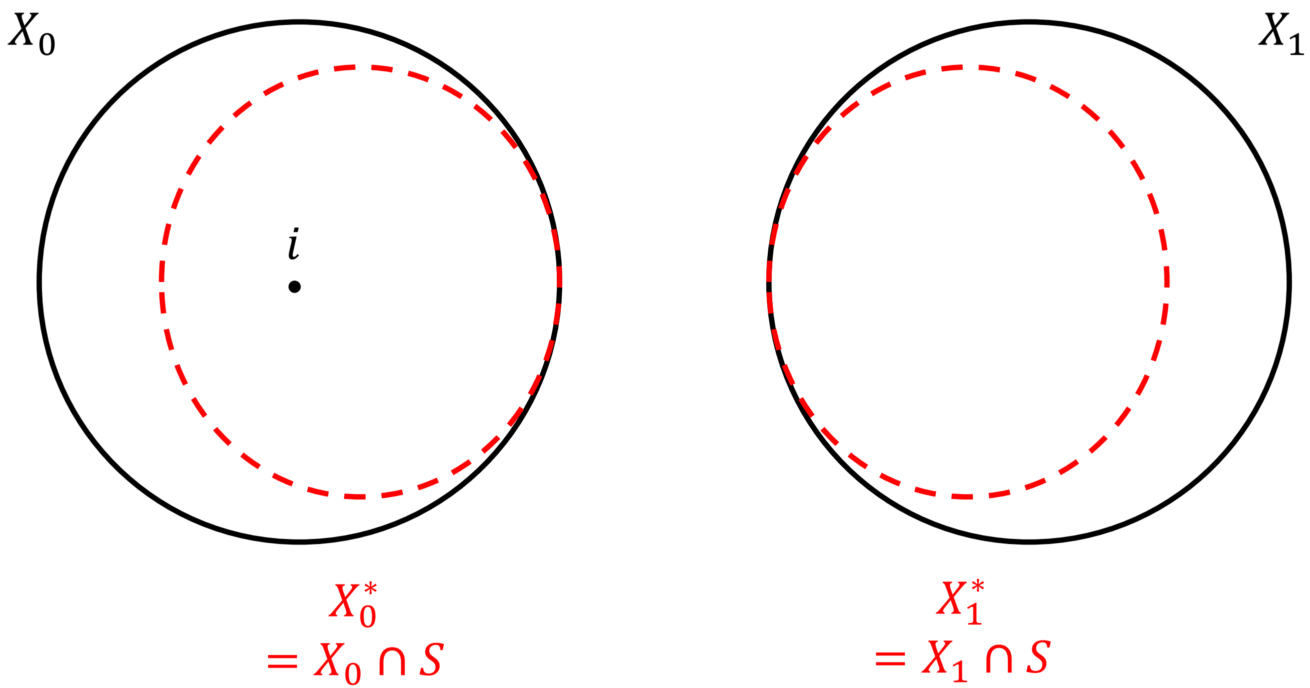


Recall definition of (α, β)-blocking coalition S for k-partition X: $u_i(S) > \alpha \cdot u_i(X(i)) + \beta$

Result 1

Min 2-cut is in the (2,0)-core

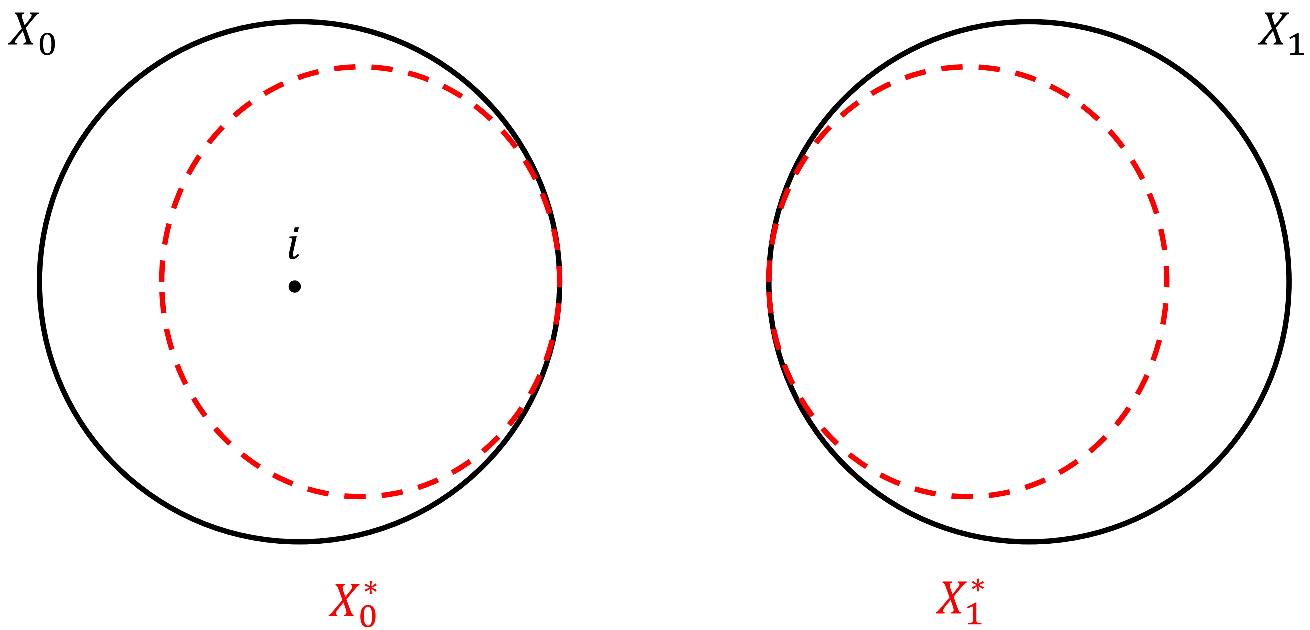
For any agent $i \in S$, we have $u_i(S) > 2 \cdot u_i(X(i))$



Result 1

Min 2-cut is in the (2,0)-core

$$u_i(S) > 2 \cdot u_i\big(X(i)\big)$$

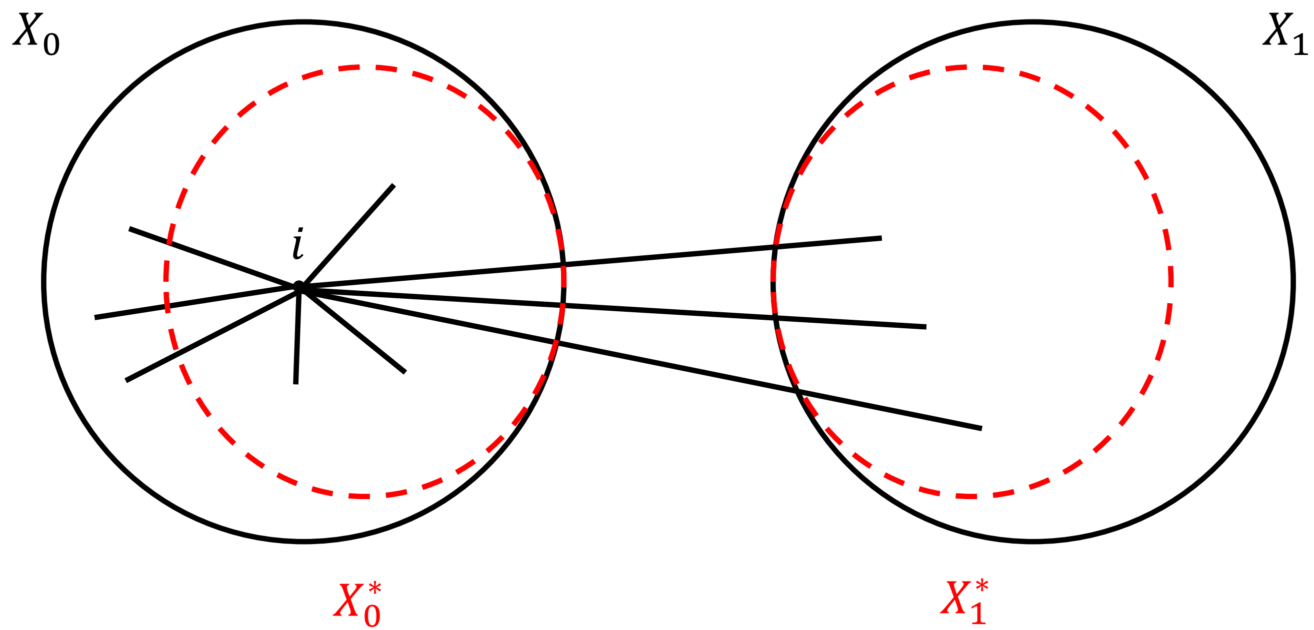


Recall definition of $u_i(S)$: $u_i(S) = |S \cap N(i)|$

Result 1

Min 2-cut is in the (2,0)-core

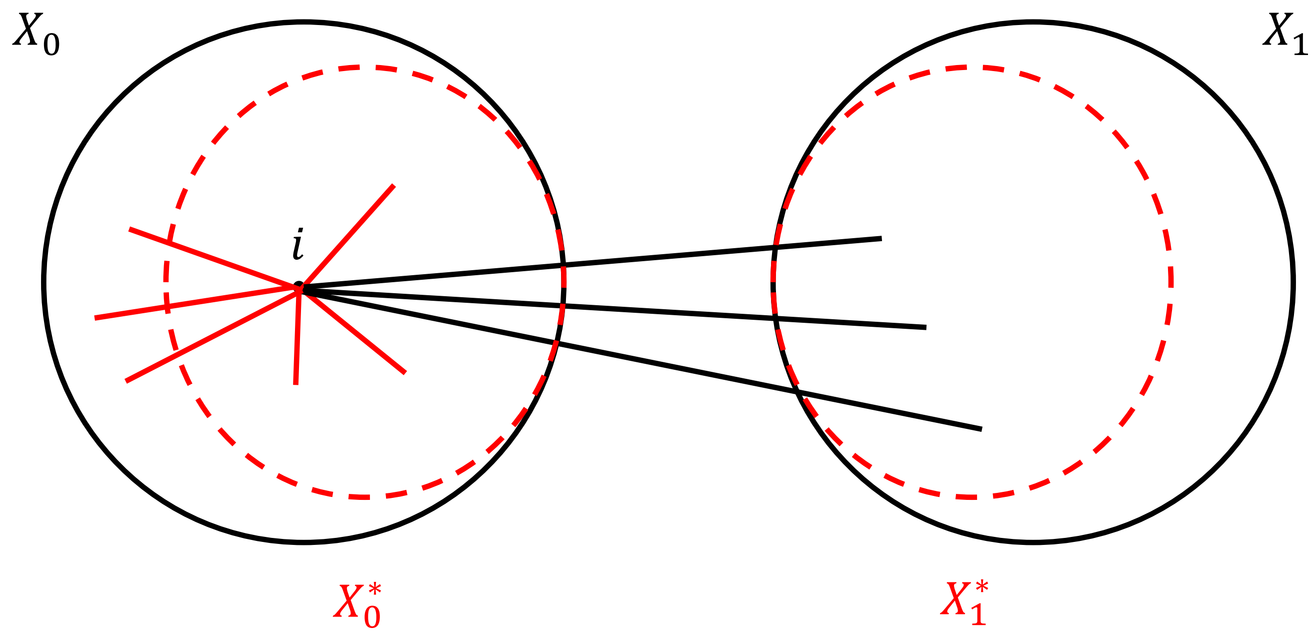
$$\underbrace{|N(i) \cap X_0^*| + |N(i) \cap X_1^*|}_{u_i(S)} > 2 \cdot \underbrace{|N(i) \cap X_0|}_{u_i(X(i))}$$



Result 1

Min 2-cut is in the (2,0)-core

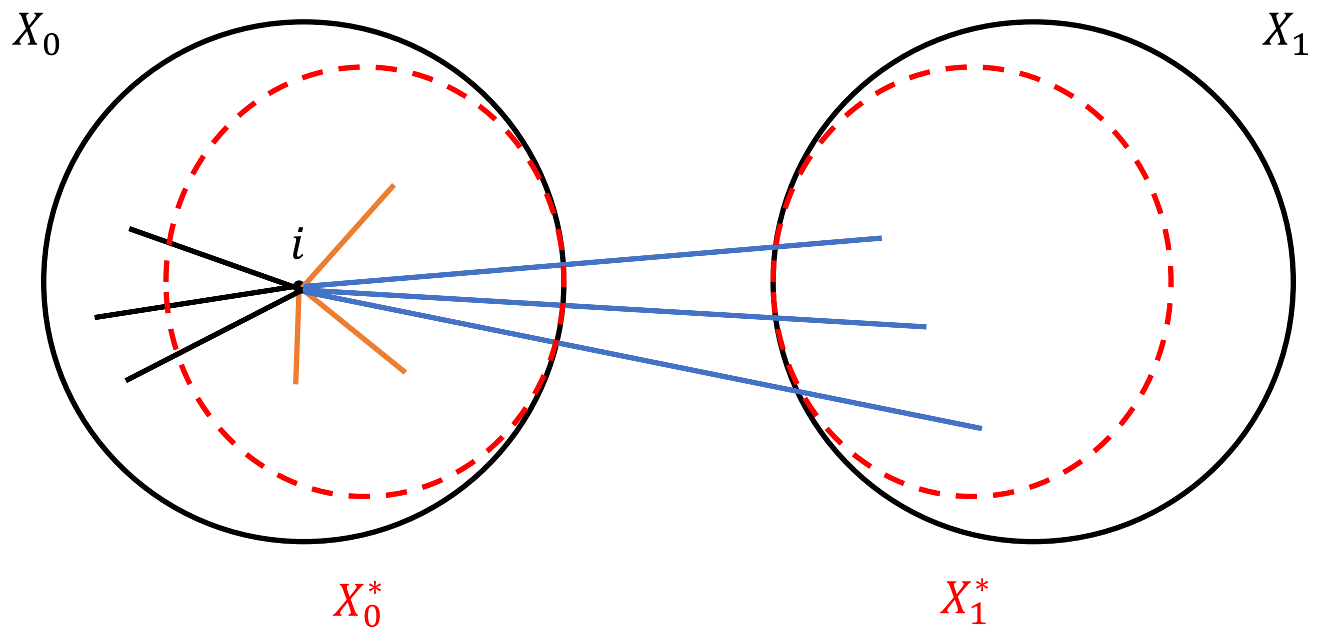
$$|N(i) \cap X_0^*| + |N(i) \cap X_1^*| > 2 \cdot |N(i) \cap X_0|$$



Result 1

Min 2-cut is in the (2,0)-core

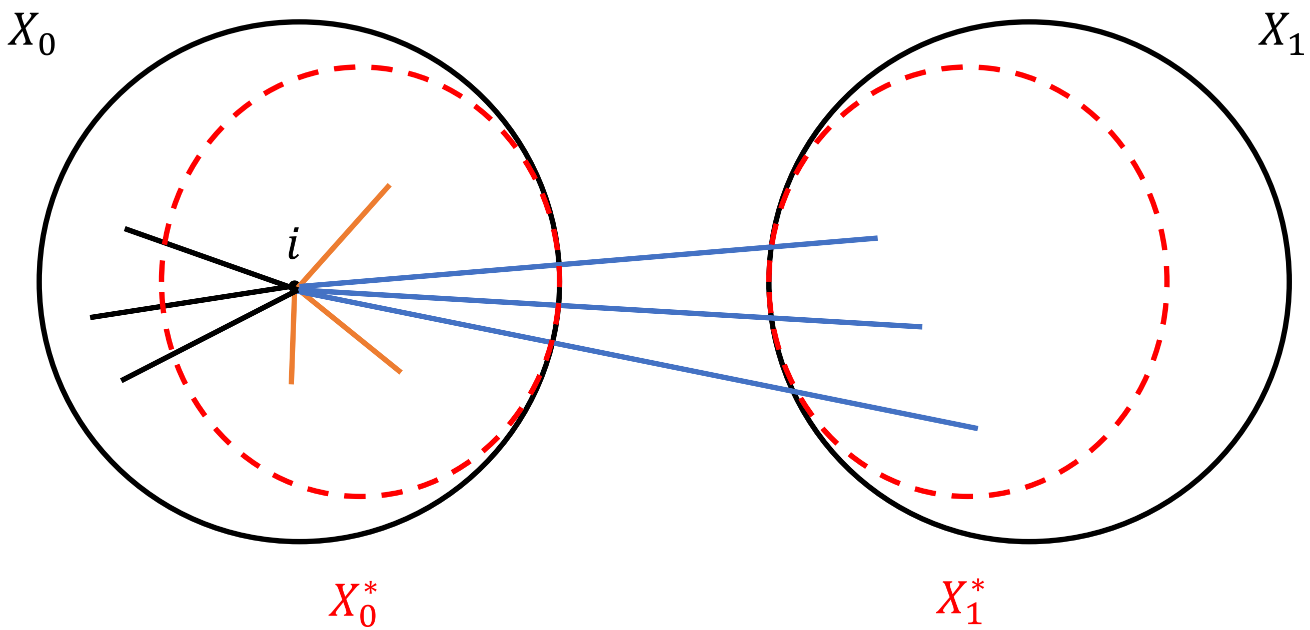
$$|N(i) \cap X_0^*| + |N(i) \cap X_1^*| > 2 \cdot |N(i) \cap X_0|$$



Result 1

Min 2-cut is in the (2,0)-core

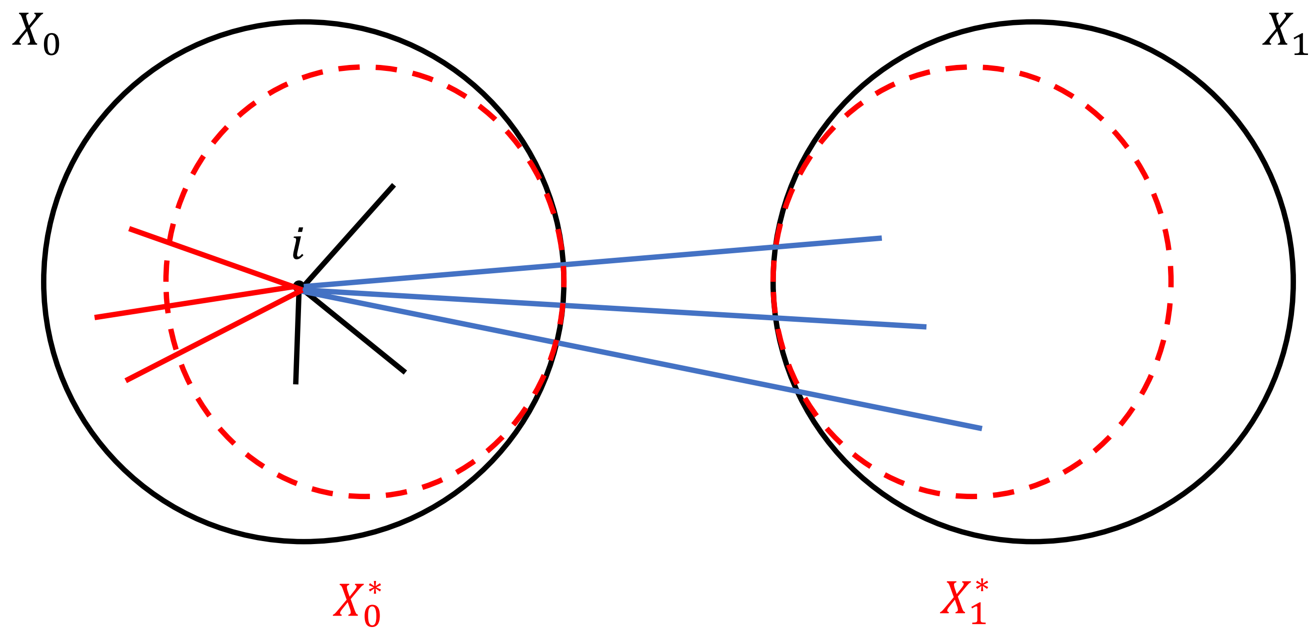
$$|N(i) \cap X_1^*| > 2 \cdot |N(i) \cap X_0| - |N(i) \cap X_0^*|$$



Result 1

Min 2-cut is in the (2,0)-core

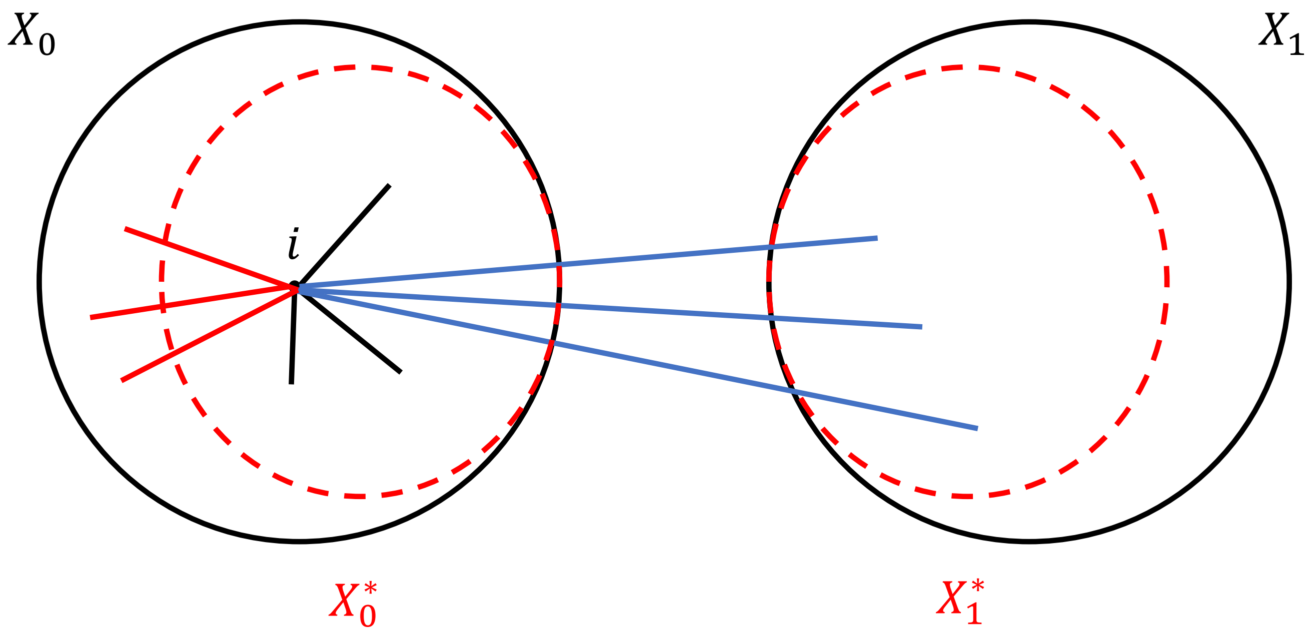
$$|N(i) \cap X_1^*| > 2 \cdot |N(i) \cap X_0 \setminus X_0^*|$$



Result 1

Min 2-cut is in the (2,0)-core

$$\sum_{i \in X_0^*} |N(i) \cap X_1^*| > 2 \cdot \sum_{i \in X_0^*} |N(i) \cap X_0 \setminus X_0^*|$$



Recall definition of $E(A,B)$: Edges between sets A and B

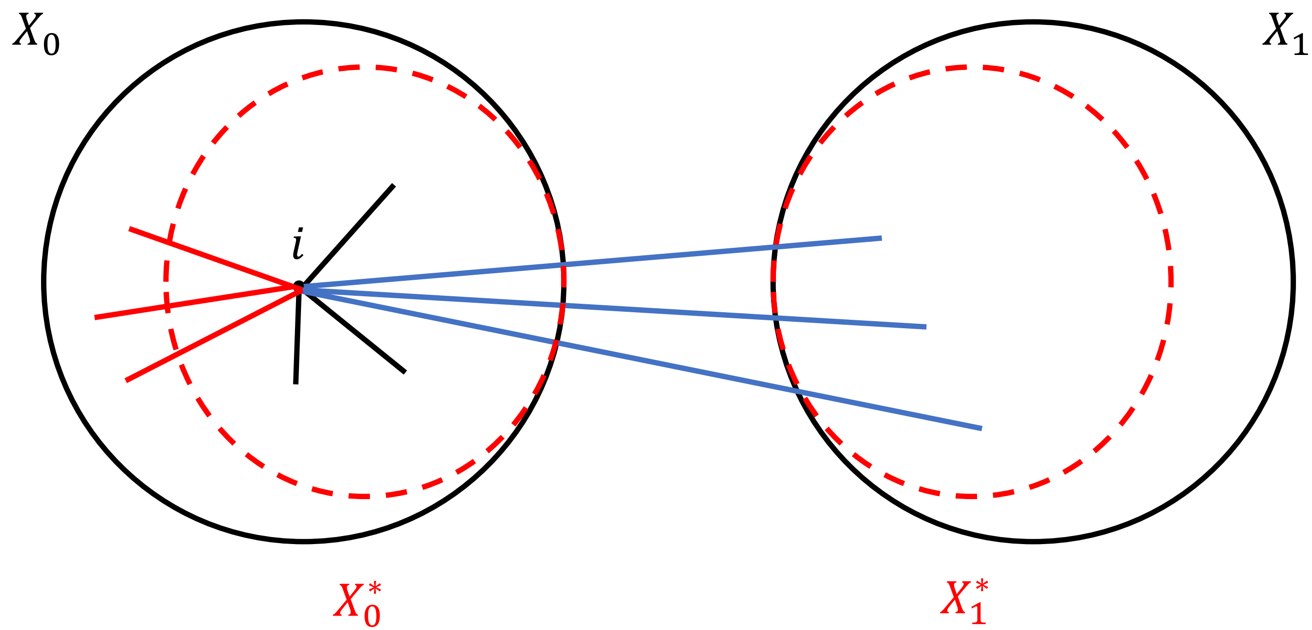
Result 1

Min 2-cut is in the (2,0)-core

$$\sum_{i \in \boldsymbol{X_0^*}} |N(i) \cap \boldsymbol{X_1^*}| > 2 \cdot \sum_{i \in \boldsymbol{X_0^*}} |N(i) \cap \boldsymbol{X_0} \setminus \boldsymbol{X_0^*}|$$

Edges between $\boldsymbol{X_0^*}$ and $\boldsymbol{X_1^*}$

Edges between $\boldsymbol{X_0^*}$ and $\boldsymbol{X_0} \setminus \boldsymbol{X_0^*}$

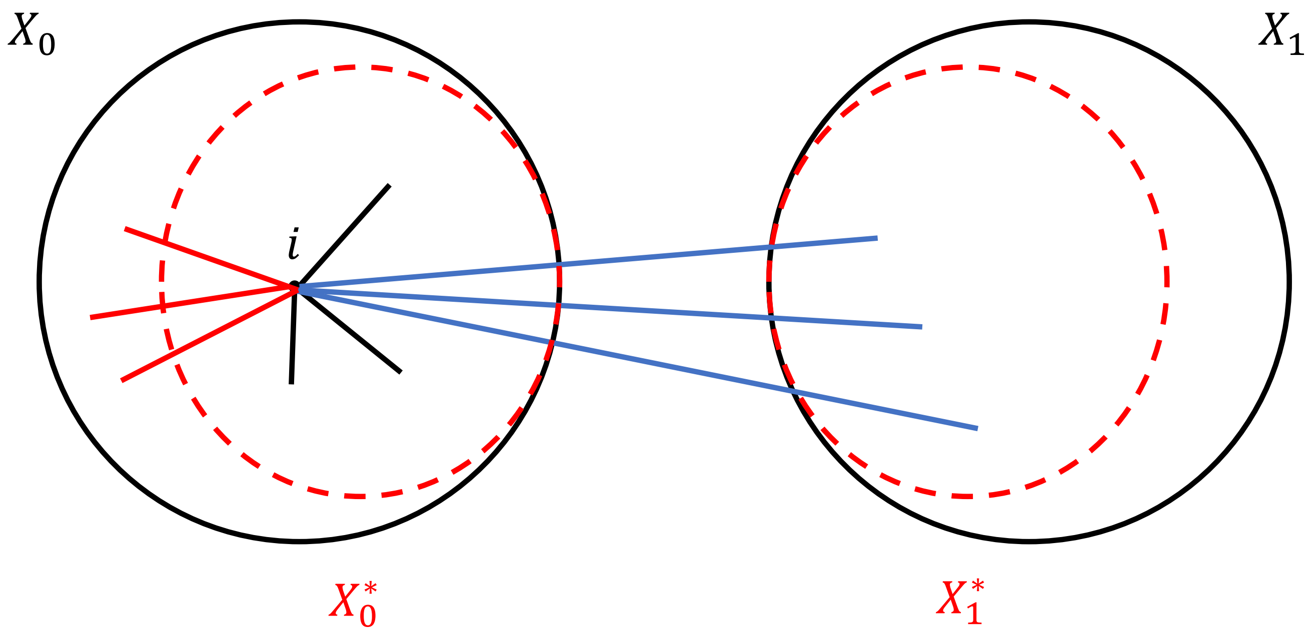


Recall definition of $E(A,B)$: Edges between sets A and B

Result 1

Min 2-cut is in the (2,0)-core

$$E(X_0^*, X_1^*) > 2 \cdot E(X_0^*, X_0 \setminus X_0^*)$$



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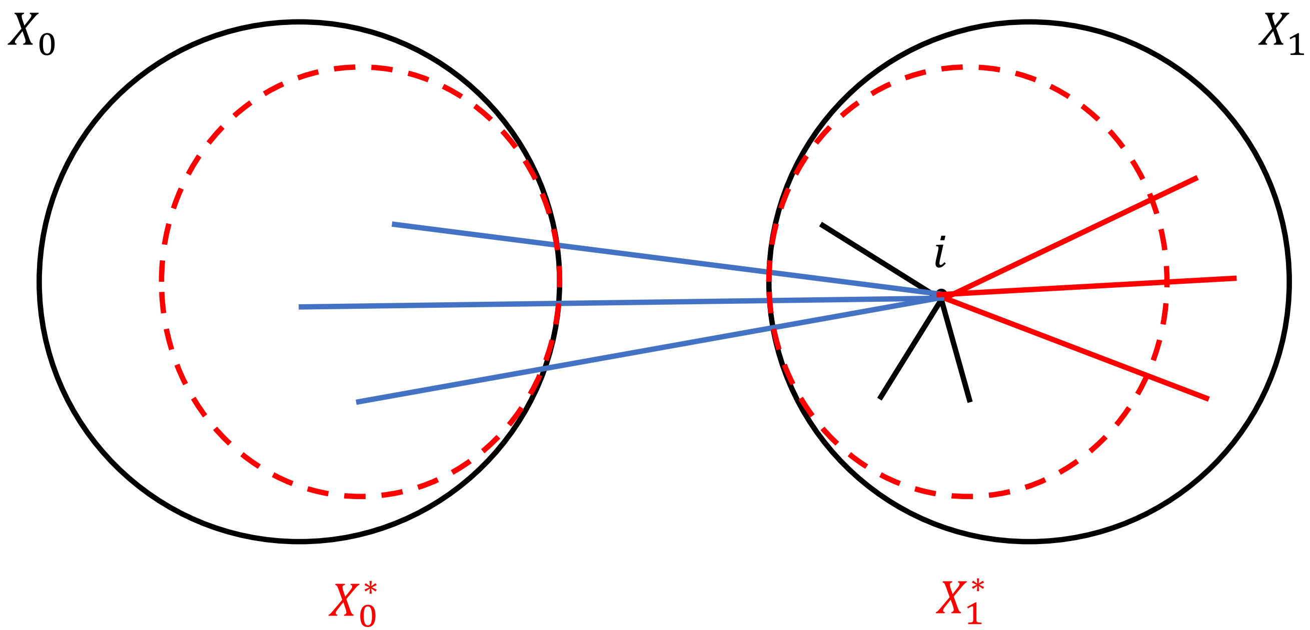
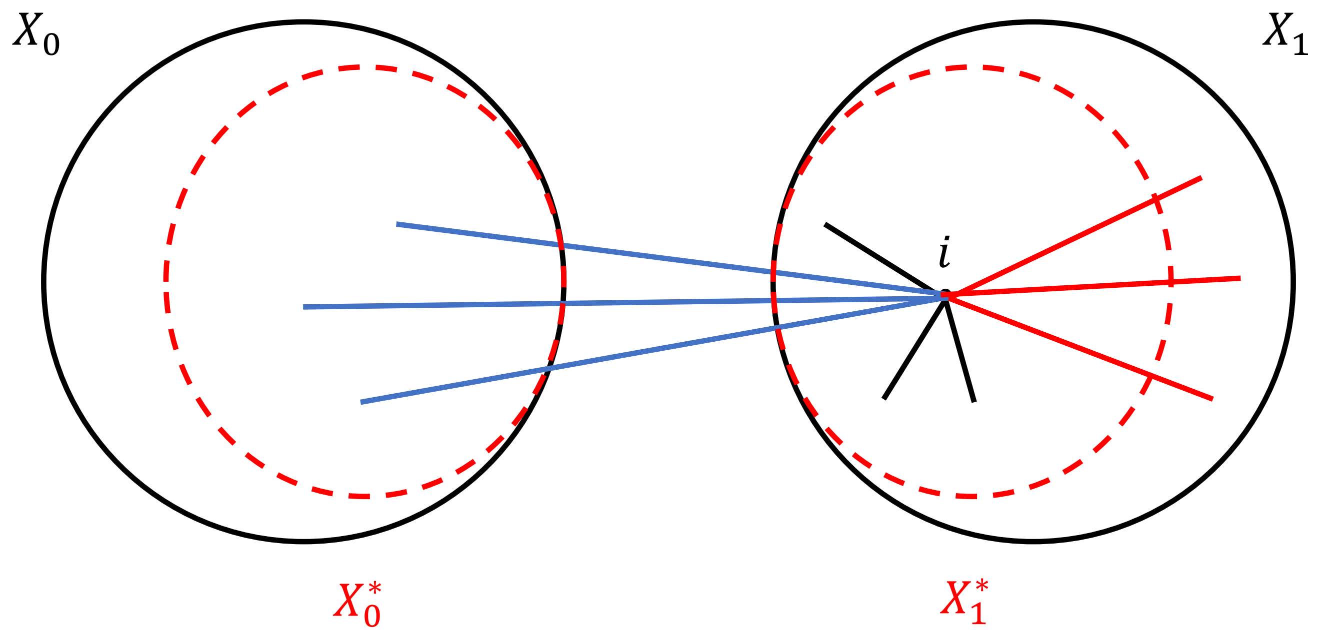


Recall definition of $E(A, B)$: Edges between sets A and B

Result 1

Min 2-cut is in the (2,0)-core

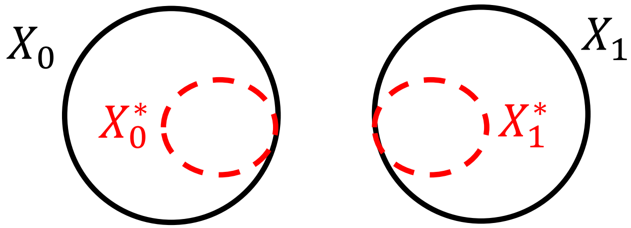
$$E(X_0^*, X_1^*) > 2 \cdot \max\{E(X_0^*, X_0 \setminus X_0^*), E(X_1^*, X_1 \setminus X_1^*)\}$$
$$\geq E(X_0^*, X_0 \setminus X_0^*) + E(X_1^*, X_1 \setminus X_1^*)$$



Result 1

Min 2-cut is in the (2,0)-core

$$E(X_0^*, X_1^*) > E(X_0^*, X_0 \setminus X_0^*) + E(X_1^*, X_1 \setminus X_1^*)$$

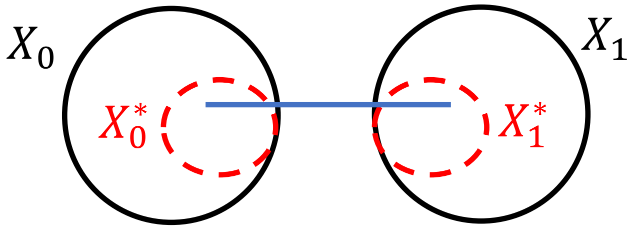


$cut(X_0, X_1) = E(X_0, X_1)$

Result 1

Min 2-cut is in the (2,0)-core

$$E(X_0^*, X_1^*) > E(X_0^*, X_0 \setminus X_0^*) + E(X_1^*, X_1 \setminus X_1^*)$$

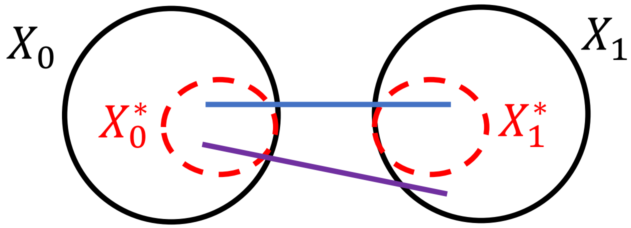


$$cut(X_0, X_1) = E(X_0, X_1)$$
$$= E(X_0^*, X_1^*)$$

Result 1

Min 2-cut is in the (2,0)-core

$$E(X_0^*, X_1^*) > E(X_0^*, X_0 \setminus X_0^*) + E(X_1^*, X_1 \setminus X_1^*)$$

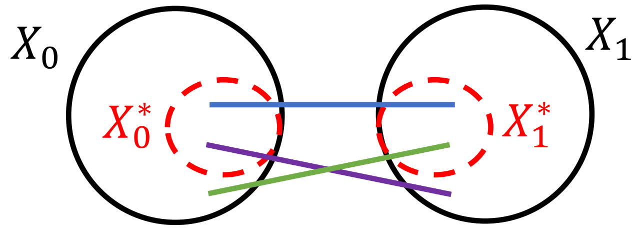


$$\begin{aligned} cut(X_0, X_1) &= E(X_0, X_1) \\ &= E(X_0^*, X_1^*) + E(X_0^*, X_1 \setminus X_1^*) \end{aligned}$$

Result 1

Min 2-cut is in the (2,0)-core

$$E(X_0^*, X_1^*) > E(X_0^*, X_0 \setminus X_0^*) + E(X_1^*, X_1 \setminus X_1^*)$$



$$cut(X_0, X_1) = E(X_0, X_1)$$
$$= E(X_0^*, X_1^*) + E(X_0^*, X_1 \setminus X_1^*) + E(X_1^*, X_0 \setminus X_0^*)$$

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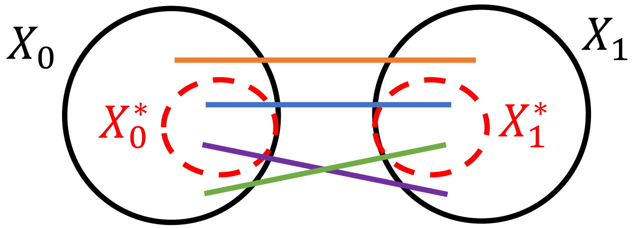

$$cut(X_0, X_1) = E(X_0, X_1)$$
$$= E(X_0^*, X_1^*) + E(X_0^*, X_1 \setminus X_1^*) + E(X_1^*, X_0 \setminus X_0^*) + E(X_0 \setminus X_0^*, X_1 \setminus X_1^*)$$

Result 1

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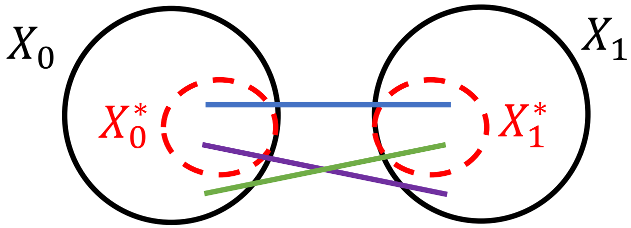
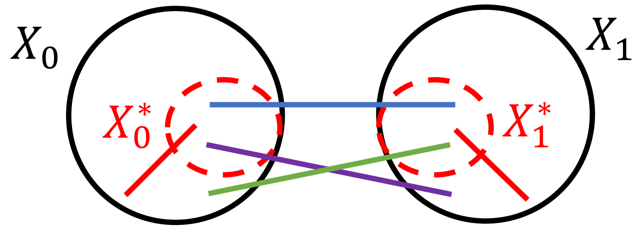


$$\begin{aligned} cut(X_0, X_1) &= E(X_0, X_1) \\ &= E(X_0^*, X_1^*) + E(X_0^*, X_1 \setminus X_1^*) + E(X_1^*, X_0 \setminus X_0^*) + \underbrace{E(X_0 \setminus X_0^*, X_1 \setminus X_1^*)}_{\text{drop this}} \\ &\geq E(X_0^*, X_1^*) + E(X_0^*, X_1 \setminus X_1^*) + E(X_1^*, X_0 \setminus X_0^*) \end{aligned}$$

Result 1

Min 2-cut is in the (2,0)-core

$$E(X_0^*, X_1^*) > E(X_0^*, X_0 \setminus X_0^*) + E(X_1^*, X_1 \setminus X_1^*)$$

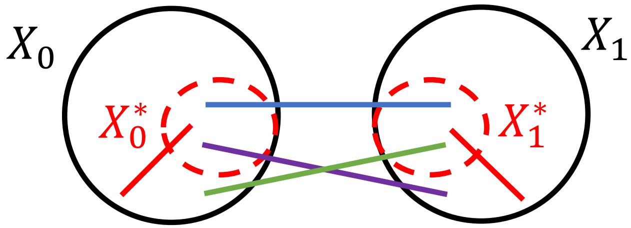


$$\begin{aligned} cut(X_0, X_1) &= E(X_0, X_1) \\ &= E(X_0^*, X_1^*) + E(X_0^*, X_1 \setminus X_1^*) + E(X_1^*, X_0 \setminus X_0^*) + E(X_0 \setminus X_0^*, X_1 \setminus X_1^*) \\ &\geq E(X_0^*, X_1^*) + E(X_0^*, X_1 \setminus X_1^*) + E(X_1^*, X_0 \setminus X_0^*) \end{aligned}$$

Result 1

Min 2-cut is in the (2,0)-core

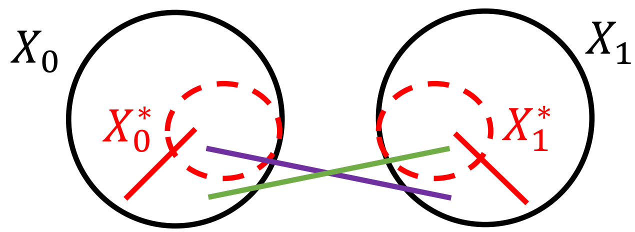
$$E(X_0^*, X_1^*) > E(X_0^*, X_0 \setminus X_0^*) + E(X_1^*, X_1 \setminus X_1^*)$$



$$\begin{aligned}
cut(X_0, X_1) &= E(X_0, X_1) \\
&= E(X_0^*, X_1^*) + E(X_0^*, X_1 \setminus X_1^*) + E(X_1^*, X_0 \setminus X_0^*) + E(X_0 \setminus X_0^*, X_1 \setminus X_1^*) \\
&\geq E(X_0^*, X_1^*) + E(X_0^*, X_1 \setminus X_1^*) + E(X_1^*, X_0 \setminus X_0^*) \\
&> E(X_0^*, X_0 \setminus X_0^*) + E(X_1^*, X_1 \setminus X_1^*) + E(X_0^*, X_1 \setminus X_1^*) + E(X_1^*, X_0 \setminus X_0^*)
\end{aligned}$$

Result 1

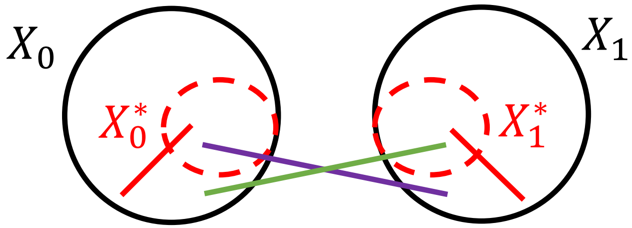
Min 2-cut is in the (2,0)-core



$$
\begin{aligned}
cut(X_0, X_1) &= E(X_0, X_1) \\
&= E(X_0^*, X_1^*) + E(X_0^*, X_1 \setminus X_1^*) + E(X_1^*, X_0 \setminus X_0^*) + E(X_0 \setminus X_0^*, X_1 \setminus X_1^*) \\
&\geq E(X_0^*, X_1^*) + E(X_0^*, X_1 \setminus X_1^*) + E(X_1^*, X_0 \setminus X_0^*) \\
&> {\color{red} E(X_0^*, X_0 \setminus X_0^*) + E(X_1^*, X_1 \setminus X_1^*)} + {\color{purple} E(X_0^*, X_1 \setminus X_1^*)} + {\color{green} E(X_1^*, X_0 \setminus X_0^*)}
\end{aligned}
$$

Result 1

Min 2-cut is in the (2,0)-core



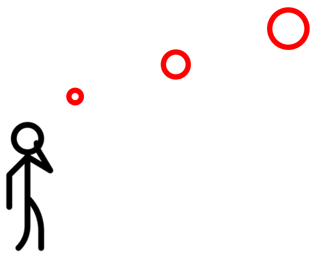
$$
\begin{aligned}
cut(X_0, X_1) &= E(X_0, X_1) \\
&= E(X_0^*, X_1^*) + E(X_0^*, X_1 \setminus X_1^*) + E(X_1^*, X_0 \setminus X_0^*) + E(X_0 \setminus X_0^*, X_1 \setminus X_1^*) \\
&\geq E(X_0^*, X_1^*) + E(X_0^*, X_1 \setminus X_1^*) + E(X_1^*, X_0 \setminus X_0^*) \\
&> E(X_0^*, X_0 \setminus X_0^*) + E(X_1^*, X_1 \setminus X_1^*) + E(X_0^*, X_1 \setminus X_1^*) + E(X_1^*, X_0 \setminus X_0^*) \\
&= cut(S, V \setminus S)
\end{aligned}
$$

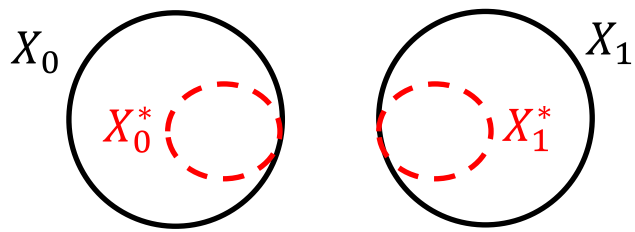
Result 1

Min 2-cut is in the (2,0)-core

Let (X_0, X_1) be an arbitrary min 2-cut

For contradiction, let S be a $(2,0)$-blocking coalition





$$
\begin{aligned}
cut(X_0, X_1) &= E(X_0, X_1) \\
&= E(X_0^*, X_1^*) + E(X_0^*, X_1 \setminus X_1^*) + E(X_1^*, X_0 \setminus X_0^*) + E(X_0 \setminus X_0^*, X_1 \setminus X_1^*) \\
&\geq E(X_0^*, X_1^*) + E(X_0^*, X_1 \setminus X_1^*) + E(X_1^*, X_0 \setminus X_0^*) \\
&> E(X_0^*, X_0 \setminus X_0^*) + E(X_1^*, X_1 \setminus X_1^*) + E(X_0^*, X_1 \setminus X_1^*) + E(X_1^*, X_0 \setminus X_0^*) \\
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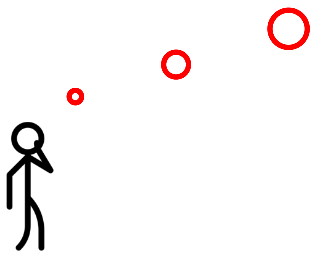
$$

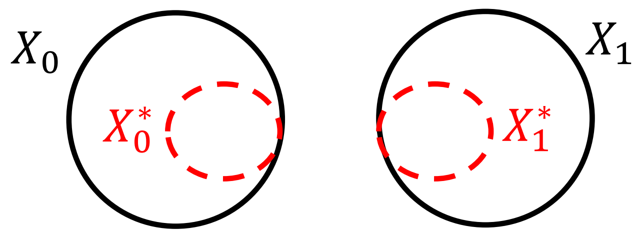
Result 1

Min 2-cut is in the (2,0)-core

Let (X_0, X_1) be an arbitrary min 2-cut

For contradiction, let S be a $(2,0)$-blocking coalition





$$
\begin{aligned}
\boldsymbol{cut(X_0, X_1)} &= E(X_0, X_1) \\
&= E(X_0^*, X_1^*) + E(X_0^*, X_1 \setminus X_1^*) + E(X_1^*, X_0 \setminus X_0^*) + E(X_0 \setminus X_0^*, X_1 \setminus X_1^*) \\
&\geq E(X_0^*, X_1^*) + E(X_0^*, X_1 \setminus X_1^*) + E(X_1^*, X_0 \setminus X_0^*) \\
&> E(X_0^*, X_0 \setminus X_0^*) + E(X_1^*, X_1 \setminus X_1^*) + E(X_0^*, X_1 \setminus X_1^*) + E(X_1^*, X_0 \setminus X_0^*) \\
&= \boldsymbol{cut(S, V \setminus S)}
\end{aligned}
$$

Contradiction since (X_0, X_1) was supposed to be min 2-cut!

Result 3

Result 3

For $\mathrm{k} \geq 3$, the following statements hold:

1. Every min k-cut is in the $(\mathrm{k}, \mathrm{k}-1)$-core
2. There is a polynomial time algorithm ALG that returns a k-partition in the $(\mathrm{k}, \mathrm{k}-1)$-core
3. When $n \geq k^2 + k$, min k-cut is in the $(2\mathrm{k}-1{,}0)$-core
4. When $\mathrm{n} \geq \mathrm{k}^2 + \mathrm{k}$, ALG returns a k-partition in the $(2\mathrm{k}-1{,}0)$-core
5. When $\mathrm{n} < \mathrm{k}^2 + \mathrm{k}$, every balanced k-partition is in the $(1, \mathrm{k})$-core

Result 3

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2. There is a polynomial time algorithm ALG that returns a k-partition in the $(\mathrm{k}, \mathrm{k}-1)$-core
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4. When $\mathrm{n} \geq \mathrm{k}^2 + \mathrm{k}$, ALG returns a k-partition in the $(2\mathrm{k}-1,0)$-core
5. **When $\mathbf{n} < \mathbf{k}^2 + \mathbf{k}$, every balanced k-partition is in the $(\mathbf{1}, \mathbf{k})$-core**

Recall definition of (α, β)-blocking coalition S for k-partition X: $u_i(S) > \alpha \cdot u_i\big(X(i)\big) + \beta$

Theorem 3(iv)

When $\mathrm{n} < \mathrm{k}^2 + \mathrm{k}$, every balanced k-partition is in the $(1, \mathrm{k})$-core

- Largest partition size is $\left\lceil \frac{\mathrm{n}}{\mathrm{k}} \right\rceil < \left\lceil k + \frac{1}{k} \right\rceil = k + 1$
- Extreme: Initially 0, then gain k friends by deviating
- Formally, for *any* agent i in *any* blocking coalition S,

$$u_i(S) \leq k \leq u_i\big(X(i)\big) + k$$

- That is, *no* possible coalition S such that

$$u_i(S) > u_i\big(X(i)\big) + k$$

- So, every balanced k-partition is in the $(1, \mathrm{k})$-core

Result 3

For $\mathrm{k} \geq 3$, the following statements hold:

1. **Every min k-cut is in the $(\mathbf{k}, \mathbf{k}-\mathbf{1})$-core**
2. **There is a polynomial time algorithm ALG that returns a k-partition in the $(\mathbf{k}, \mathbf{k}-\mathbf{1})$-core**
3. When $n \geq k^2 + k$, min k-cut is in the $(2\mathrm{k}-1,0)$-core
4. When $\mathrm{n} \geq \mathrm{k}^2 + \mathrm{k}$, ALG returns a k-partition in the $(2\mathrm{k}-1,0)$-core
5. When $\mathrm{n} < \mathrm{k}^2 + \mathrm{k}$, every balanced k-partition is in the $(1, \mathrm{k})$-core

Lemma

Suppose S is $(\mathrm{k}, \mathrm{k}-1)$-blocking coalition of X.
Then, for all $i \in S$,

$$u_i(S \cap X_j) > u_i(X(i)) + 1\text{, for all } j \in [k]$$

Suppose, for contradiction, that there exists $i \in S$ such that

$$u_i(S \cap X_j) \leq u_i(X(i)) + 1\text{, for all } j \in [k]$$

$$u_i(S) = \sum_{j \in [k]} u_i(S \cap X_j)$$

Lemma

Suppose S is $(\mathrm{k}, \mathrm{k}-1)$-blocking coalition of X.
Then, for all $i \in S$,
$$u_i(S \cap X_j) > u_i(X(i)) + 1, \text{ for all } j \in [k]$$

Suppose, for contradiction, that there exists $i \in S$ such that
$$u_i(S \cap X_j) \leq u_i(X(i)) + 1, \text{ for all } j \in [k]$$

$$\begin{aligned} u_i(S) &= \sum_{j \in [k]} u_i(S \cap X_j) \\ &\leq u_i(S \cap X(i)) + \sum_{\substack{j \in [k] \\ X_j \neq X(i)}} (u_i(X_i) + 1) \end{aligned}$$

Lemma

Suppose S is $(\mathrm{k}, \mathrm{k}-1)$-blocking coalition of X.
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$$u_i(S \cap X_j) \leq u_i(X(i)) + 1, \text{ for all } j \in [k]$$

$$u_i(S) = \sum_{j \in [k]} u_i(S \cap X_j)$$

$$\leq u_i(S \cap X(i)) + \sum_{\substack{j \in [k] \\ X_j \neq X(i)}} (u_i(X_i) + 1)$$

Remove intersection

$$\leq u_i(X(i)) + (k-1) \cdot (u_i(X_i) + 1)$$

Lemma

Suppose S is $(\mathrm{k}, \mathrm{k}-1)$-blocking coalition of X.
Then, for all $i \in S$,

$$u_i(S \cap X_j) > u_i(X(i)) + 1, \text{ for all } j \in [k]$$

Suppose, for contradiction, that there exists $i \in S$ such that

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$$\begin{aligned}
u_i(S) &= \sum_{j \in [k]} u_i(S \cap X_j) \\
&\leq u_i(S \cap X(i)) + \sum_{\substack{j \in [k] \\ X_j \neq X(i)}} (u_i(X_i) + 1) \\
&\leq u_i(X(i)) + (k-1) \cdot (u_i(X_i) + 1) \\
&= k \cdot u_i(X(i)) + (k-1)
\end{aligned}$$

Recall definition of (α, β)-blocking coalition S for k-partition X: $u_i(S) > \alpha \cdot u_i\big(X(i)\big) + \beta$

Lemma

Suppose S is $(\mathrm{k}, \mathrm{k}-1)$-blocking coalition of X.
Then, for all $i \in S$,

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Suppose, for contradiction, that there exists $i \in S$ such that

$$u_i\big(S \cap X_j\big) \leq u_i\big(X(i)\big) + 1, \text{ for all } j \in [k]$$

$$\begin{aligned} u_i(S) &= \sum_{j \in [k]} u_i\big(S \cap X_j\big) \\ &\leq u_i\big(S \cap X(i)\big) + \sum_{\substack{j \in [k] \\ X_j \neq X(i)}} \big(u_i(X_i) + 1\big) \\ &\leq u_i\big(X(i)\big) + (k-1) \cdot \big(u_i(X_i) + 1\big) \\ &= k \cdot u_i\big(X(i)\big) + (k-1) \end{aligned}$$

Contradiction to S being a $(\mathrm{k}, \mathrm{k}-1)$-blocking coalition.

Result 3(i)

For $\mathrm{k} \geq 3$,
every min k-cut is in the $(\mathrm{k}, \mathrm{k}-1)$-core

Let X be an arbitrary min k-cut.

Suppose, for a contradiction, that S is a $(k, k-1)$-blocking coalition.

Result 3(i)

For $\mathrm{k} \geq 3$,
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Let X be an arbitrary min k-cut.

Suppose, for a contradiction, that S is a $(k, k-1)$-blocking coalition.

For all $i_1 \in S, j \in [k]$,

$$u_{i_1}\left(X(i_1)\right) + 1 < u_{i_1}\left(S \cap X_j\right)$$

$\boxed{\text{Suppose } S \text{ is } (\mathrm{k}, \mathrm{k}-1)\text{-blocking coalition of } X. \text{ Then, for all } i \in S,\ u_i\left(S \cap X_j\right) > u_i\left(X(i)\right) + 1, \text{ for all } j \in [k]}$

Lemma

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Lemma

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Let X be an arbitrary min k-cut.

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For all $i_1 \in S, i_2 \in [n]$,

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Result 3(i)

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For $\mathrm{k} \geq 3$,
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Let X be an arbitrary min k-cut.

Suppose, for a contradiction, that S is a $(k, k-1)$-blocking coalition.

For all $i_1 \in S, i_2 \in [n]$,

$$u_{i_1}(X(i_1)) + 1 < u_{i_1}(S \cap X(i_2)) \leq u_{i_1}(X(i_2))$$

Consider the longest possible sequence $i_1 \to i_2 \to \cdots \to i_t$ where an arc $i_j \to i_{j+1}$ means that $u_{i_j}\left(X\left(i_{j+1}\right)\right) > u_{i_j}\left(X\left(i_j\right)\right) + 1$

<u>Sequence forms cycle</u>

<u>Sequence is acyclic</u>

Result 3(i)

For $\mathrm{k} \geq 3$,
every min k-cut is in the $(\mathrm{k}, \mathrm{k}-1)$-core

Let X be an arbitrary min k-cut.

Suppose, for a contradiction, that S is a $(k, k-1)$-blocking coalition.

For all $i_1 \in S, i_2 \in [n]$,

$$u_{i_1}\big(X(i_1)\big) + 1 < u_{i_1}\big(S \cap X(i_2)\big) \leq u_{i_1}\big(X(i_2)\big)$$

Consider the longest possible sequence $i_1 \to i_2 \to \cdots \to i_t$ where an arc $i_j \to i_{j+1}$ means that $u_{i_j}\left(X\left(i_{j+1}\right)\right) > u_{i_j}\left(X\left(i_j\right)\right) + 1$

Sequence forms cycle	Sequence is acyclic
Rotate agents along cycle	Swap agents i_{t-1} and i_t

(Some details...)

Remark: The strictness in the inequality is crucial.

Result 3(i)

For $\mathrm{k} \geq 3$,
every min k-cut is in the $(\mathrm{k}, \mathrm{k}-1)$-core

Let X be an arbitrary min k-cut.

Suppose, for a contradiction, that S is a $(k, k-1)$-blocking coalition.

For all $i_1 \in S, i_2 \in [n]$,

$$u_{i_1}(X(i_1)) + 1 < u_{i_1}(S \cap X(i_2)) \leq u_{i_1}(X(i_2))$$

Consider the longest possible sequence $i_1 \to i_2 \to \cdots \to i_t$ where an arc $i_j \to i_{j+1}$ means that $u_{i_j}\left(X(i_{j+1})\right) > u_{i_j}\left(X(i_j)\right) + 1$

Sequence forms cycle

Rotate agents along cycle

$$u_{i_j}\left(X(i_{j+1})\right) > u_{i_j}\left(X(i_j)\right) + 1$$

So, cut drops by at least 1, even in the worst case where i_{j+1} is a friend of i_j that is leaving $X(i_{j+1})$.

Sequence is acyclic

Swap agents i_{t-1} and i_t

$$u_{i_{t-1}}(X(i_t)) > u_{i_{t-1}}(X(i_{t-1})) + 1$$

So, cut drops by at least 2. Meanwhile,

$$u_{i_t}(X(i_t)) \leq u_j(X(j)) + 1, \text{ for any j} \in [n]$$

Plug $\mathrm{j} = t-1$:

$$u_{i_t}(X(i_t)) \leq u_{i_{t-1}}(X(i_{t-1})) + 1$$

So, cut increases by at most 1.

Result 3(i)

For $\mathrm{k} \geq 3$,
every min k-cut is in the $(\mathrm{k}, \mathrm{k}-1)$-core

Let X be an arbitrary min k-cut.

Suppose, for a contradiction, that S is a $(k, k-1)$-blocking coalition.

For all $i_1 \in S, i_2 \in [n]$,

$$u_{i_1}\big(X(i_1)\big) + 1 < u_{i_1}\big(S \cap X(i_2)\big) \leq u_{i_1}\big(X(i_2)\big)$$

Consider the longest possible sequence $i_1 \to i_2 \to \cdots \to i_t$ where an arc $i_j \to i_{j+1}$ means that $u_{i_j}\left(X\left(i_{j+1}\right)\right) > u_{i_j}\left(X\left(i_j\right)\right) + 1$

<u>Sequence forms cycle</u>	<u>Sequence is acyclic</u>
Rotate agents along cycle	Swap agents i_{t-1} and i_t

In either cases, cut size drops.
Contradiction to the assumption that X was a min k-cut

Algorithm ALG: Local search

1. Let X be an arbitrary balanced k-partition
2. Repeat until fixed point:
   1. Build a directed graph G' using current partitioning X
   2. If there is an "envy cycle" in G', rotate to eliminate
   3. Else if ∃"swappable pair", swap one such pair
   4. Else, break
3. Return X

Algorithm ALG: Local search

Repeat until fixed point:

1. Build a directed graph G' using current partitioning X
   - $G' = (V', E')$
   - $V' = V$
   - $E' = \{(i, j) \colon u_i(X(j)) > u_i\big(X(i)\big) + 1\}$
2. If there is an "envy cycle" in G', rotate to eliminate
3. Else if $\exists$"swappable pair", swap one such pair
4. Else, break

Algorithm ALG: Local search

Repeat until fixed point:

1. Build a directed graph G' using current partitioning X
   - $G' = (V', E')$
   - $V' = V$
   - $E' = \{(i,j)\colon u_i(X(j)) > u_i\big(X(i)\big) + 1\}$

   The exact condition from the proof earlier
2. If there is an "envy cycle" in G', rotate to eliminate
3. Else if ∃"swappable pair", swap one such pair
4. Else, break

Algorithm ALG: Local search

Repeat until fixed point:

1. Build a directed graph G' using current partitioning X
   - $E' = \{(i,j)\colon u_i(X(j)) > u_i\big(X(i)\big) + 1\}$
2. If there is an "envy cycle" in G', rotate to eliminate
   - Envy cycle: $i_0 \to i_1 \to \cdots \to i_{s-1} \to i_0$ in E'
   - Shift agent i_j into partition $X\big(i_{j+1 \, mod \, s}\big)$
3. Else if ∃"swappable pair", swap one such pair
4. Else, break

Just like proof earlier

Observe that cut(X) always decreases if step 2 triggers.
Shifting can be done in polynomial time.

Algorithm ALG: Local search

Repeat until fixed point:

1. Build a directed graph G' using current partitioning X
   - $E' = \{(i,j)\colon u_i(X(j)) > u_i\big(X(i)\big) + 1\}$
2. If there is an “envy cycle” in G', rotate to eliminate
3. Else if ∃“swappable pair”, swap one such pair
   - $\{i,j\}$ are swappable if **all** 3 following conditions hold:
     1. $u_j\big(X(j)\big) = 0$
     2. $u_i\big(X(j)\big) > u_i\big(X(i)\big)$
     3. i and j are **not** friends or $u_i\big(X(j)\big) > u_i\big(X(i)\big) + 1$
4. Else, break

Jointly guarantee that cut drops if swapped

Observe that cut(X) always decreases if step 3 triggers.
Swapping can be done in polynomial time.

Algorithm ALG: Local search

Repeat until fixed point:

1. Build a directed graph G' using current partitioning X
   - $E' = \{(i,j): u_i(X(j)) > u_i(X(i)) + 1\}$
2. If there is an "envy cycle" in G', rotate to eliminate
3. Else if ∃"swappable pair", swap one such pair
   - $\{i,j\}$ are swappable if **all** 3 following conditions hold:
     1. $u_j(X(j)) = 0$
     2. $u_i(X(j)) > u_i(X(i))$
     3. i and j are **not** friends or $u_i(X(j)) > u_i(X(i)) + 1$
4. Else, break

Same condition

Used in the $(2k-1,0)$-core proof

If not friends, enough to have condition 2 to swap

Observe that cut(X) always decreases if step 3 triggers.
Swapping can be done in polynomial time.

Algorithm ALG: Local search

Repeat until fixed point:

1. Build a directed graph G' using current partitioning X
   - $E' = \{(i,j): u_i(X(j)) > u_i(X(i)) + 1\}$
2. If there is an "envy cycle" in G', rotate to eliminate
3. Else if $\exists$"swappable pair", swap one such pair
   - $\{i,j\}$ are swappable if **all** 3 following conditions hold:
     1. $u_j(X(j)) = 0$
     2. $u_i(X(j)) > u_i(X(i))$
     3. i and j are **not** friends or $i \rightarrow j$ in E'
4. Else, break

Same condition

Used in the $(2k-1,0)$-core proof

If not friends, enough to have condition 2 to swap

Observe that cut(X) always decreases if step 3 triggers.
Swapping can be done in polynomial time.

Algorithm ALG: Local search

1. Let X be an arbitrary balanced k-partition
2. Repeat until fixed point:
    1. Build a directed graph G' using current partitioning X
    2. If there is an "envy cycle" in G', rotate to eliminate
    3. Else if ∃"swappable pair", swap one such pair
    4. Else, break
3. Return X

Since cut(X) is initially at most n^2 and cut(X) always decreases if step 2 or 3 triggers, while loop terminates in polynomial number of steps. Furthermore, each iteration runs in polynomial time.

Result 3(iii)

The algorithm ALG returns a k-partition in the $(\mathrm{k}, \mathrm{k}-1)$-core

Let X be output of ALG.

Suppose, for a contradiction, that S is a $(\mathrm{k}, \mathrm{k}-1)$-blocking coalition.

Result 3(iii)

The algorithm ALG returns a k-partition in the $(\mathrm{k}, \mathrm{k}-1)$-core

Let X be output of ALG.

Suppose, for a contradiction, that S is a $(\mathrm{k}, \mathrm{k}-1)$-blocking coalition.

For all $i_1 \in S, j \in [k]$,

$$u_{i_1}(X(i_1)) + 1 < u_{i_1}(S \cap X_j)$$

> Suppose S is $(\mathrm{k}, \mathrm{k}-1)$-blocking coalition of X.
> Then, for all $\mathrm{i} \in S$,
> $$\mathrm{u_i}(\mathrm{S} \cap X_j) > u_i(X(i)) + 1, \text{ for all j} \in [k]$$

Lemma

Result 3(iii)

The algorithm ALG returns a k-partition in the $(\mathrm{k}, \mathrm{k}-1)$-core

Let X be output of ALG.

Suppose, for a contradiction, that S is a $(\mathrm{k}, \mathrm{k}-1)$-blocking coalition.

For all $i_1 \in S, i_2 \in [n]$,

$$u_{i_1}\big(X(i_1)\big) + 1 < u_{i_1}\big(S \cap X(i_2)\big)$$

Result 3(iii)

The algorithm ALG returns a k-partition in the $(\mathrm{k}, \mathrm{k}-1)$-core

Let X be output of ALG.

Suppose, for a contradiction, that S is a $(\mathrm{k}, \mathrm{k}-1)$-blocking coalition.

For all $i_1 \in S, i_2 \in [n]$,

$$u_{i_1}\big(X(i_1)\big) + 1 < u_{i_1}\big(S \cap X(i_2)\big) \leq u_{i_1}\big(X(i_2)\big)$$

So, $i_1 \to i_2 \in E'$.

Result 3(iii)

The algorithm ALG returns a k-partition in the $(\mathrm{k}, \mathrm{k}-1)$-core

Let X be output of ALG.

Suppose, for a contradiction, that S is a $(\mathrm{k}, \mathrm{k}-1)$-blocking coalition.

For all $i_1 \in S, i_2 \in [n]$,

$$u_{i_1}\big(X(i_1)\big) + 1 < u_{i_1}\big(S \cap X(i_2)\big) \leq u_{i_1}\big(X(i_2)\big)$$

So, $i_1 \to i_2 \in E'$.

Consider the longest possible sequence $i_1 \to i_2 \to \cdots \to i_t$ in E'.

<u>Sequence forms cycle</u> | <u>Sequence is acyclic</u>

Result 3(iii)

The algorithm ALG returns a k-partition in the $(\mathrm{k}, \mathrm{k}-1)$-core

Let X be output of ALG.

Suppose, for a contradiction, that S is a $(\mathrm{k}, \mathrm{k}-1)$-blocking coalition.

For all $i_1 \in S, i_2 \in [n]$,

$$u_{i_1}\big(X(i_1)\big) + 1 < u_{i_1}\big(S \cap X(i_2)\big) \leq u_{i_1}\big(X(i_2)\big)$$

So, $i_1 \to i_2 \in E'$.

Consider the longest possible sequence $i_1 \to i_2 \to \cdots \to i_t$ in E'.

<u>Sequence forms cycle</u>	<u>Sequence is acyclic</u>
ALG would have rotated the cycle	$\{i_{t-1}, i_t\}$ is “swappable pair”

In either cases, ALG would not have terminated.
Contradiction to the assumption that X was output of ALG

Result 3

Will not present. Idea: use a variant of Lemma.

For $\mathrm{k} \geq 3$, the following statements hold:

1. Every min k-cut is in the $(\mathrm{k}, \mathrm{k}-1)$-core
2. There is a polynomial time algorithm ALG that returns a k-partition in the $(\mathrm{k}, \mathrm{k}-1)$-core
3. **When $\boldsymbol{n} \geq \boldsymbol{k}^{2}+\boldsymbol{k}$, min k-cut is in the $(\mathbf{2k}-\mathbf{1}, \mathbf{0})$-core**
4. **When $\mathbf{n} \geq \mathbf{k}^{2}+\mathbf{k}$, ALG returns a k-partition in the $(\mathbf{2k}-\mathbf{1}, \mathbf{0})$-core**
5. When $\mathrm{n}<\mathrm{k}^{2}+\mathrm{k}$, every balanced k-partition is in the $(1, \mathrm{k})$-core

Lemma'

Suppose S is $(2\mathrm{k}-1,0)$-blocking coalition of X.
Then, for all $\mathrm{i} \in S$,
If $u_i(S \cap X_j) \leq u_i(X(i)) + 1$ for all $j \in [k]$,
then $u_i(X(i)) = 0$.

Suppose, for contradiction, that there exists $i \in S$ such that

$$\mathrm{u_i}(\mathrm{S} \cap X_j) \leq u_i(X(i)) + 1, \text{ for } \textit{some } j \in [k]$$

and $u_i(X(i)) \geq 1$

$$u_i(S) = \sum_{j \in [k]} u_i(S \cap X_j)$$

$$\leq u_i(S \cap X(i)) + \sum_{\substack{j \in [k] \\ X_j \neq X(i)}} (u_i(X_i) + 1)$$

$$\leq u_i(X(i)) + (k-1) \cdot (u_i(X_i) + 1)$$

$$= k \cdot u_i(X(i)) + (k-1)$$

$$\leq (2k-1) \cdot u_i(X(i))$$

The only changes to Lemma.

Result 3(iv)

When $n \geq k^2 + k$, ALG returns a k-partition in the $(2\mathrm{k} - 1{,}0)$-core

Let X be output of ALG.

Suppose, for a contradiction, that S is a $(2\mathrm{k} - 1{,}0)$-blocking coalition.

Result 3(iv)

When $n \geq k^2 + k$, ALG returns a k-partition in the $(2\mathrm{k}-1{,}0)$-core

Let X be output of ALG.

Suppose, for a contradiction, that S is a $(2\mathrm{k}-1{,}0)$-blocking coalition.

From earlier, "$u_i(S \cap X_j) > u_i(X(i)) + 1$" leads to contradiction.

Suppose now that "$u_i(S \cap X_j) \leq u_i(X(i)) + 1$".

Hiding the "for all $j \in [k]$"

Result 3(iv)

When $n \geq k^2 + k$, ALG returns a k-partition in the $(2\mathrm{k}-1{,}0)$-core

Let X be output of ALG.

Suppose, for a contradiction, that S is a $(2\mathrm{k}-1{,}0)$-blocking coalition.

From earlier, "$u_i(S \cap X_j) > u_i(X(i)) + 1$" leads to contradiction.

Suppose now that "$u_i(S \cap X_j) \leq u_i(X(i)) + 1$".

Suppose S is $(2\mathrm{k}-1{,}0)$-blocking coalition of X. Then, for all $\mathrm{i} \in S$,
If $u_i(S \cap X_j) \leq u_i(X(i)) + 1$ for all $j \in [k]$, then $u_i(X(i)) = 0$.

Lemma'

Result 3(iv)

When $n \geq k^2 + k$, ALG returns a k-partition in the $(2\mathrm{k}-1{,}0)$-core

Let X be output of ALG.

Suppose, for a contradiction, that S is a $(2\mathrm{k}-1{,}0)$-blocking coalition.

From earlier, "$u_i\left(S \cap X_j\right) > u_i\left(X(i)\right) + 1$" leads to contradiction.

Suppose now that "$u_i\left(S \cap X_j\right) \leq u_i\left(X(i)\right) + 1$".

So, $u_i\left(X(i)\right)$ = 0 for all i ∈ S.

> Suppose S is $(2\mathrm{k}-1{,}0)$-blocking coalition of X. Then, for all $\mathrm{i} \in S$,
> If $u_i\left(S \cap X_j\right) \leq u_i\left(X(i)\right) + 1$ for all $j \in [k]$, then $u_i\left(X(i)\right) = 0$.

Lemma'

Result 3(iv)

When $n \geq k^2 + k$, ALG returns a k-partition in the $(2\mathrm{k}-1,0)$-core

Let X be output of ALG.

Suppose, for a contradiction, that S is a $(2\mathrm{k}-1,0)$-blocking coalition.

Suppose now that $u_i\big(X(i)\big) = 0$ for all $i \in S$.

Result 3(iv)

When $n \geq k^2 + k$, ALG returns a k-partition in the $(2\mathrm{k}-1,0)$-core

Let X be output of ALG.

Suppose, for a contradiction, that S is a $(2\mathrm{k}-1,0)$-blocking coalition.

Suppose now that $u_i\big(X(i)\big) = 0$ for all $i \in S$.

Since $\mathrm{n} \geq k^2 + k$, $|S| \geq \left\lfloor \frac{k^2+1}{k} \right\rfloor = \left\lfloor k + \frac{1}{k} \right\rfloor = k + 1$.

By pigeonhole principle, $\exists i_1, i_2 \in S$ such that $X(i_1) = X(i_2)$.

<u>i_1 and i_2 are friends</u>

Then, $u_i\big(X(i)\big) \geq 1$
since $X(i_1) = X(i_2)$

Contradiction to $u_i\big(X(i)\big)$ = 0

<u>i_1 and i_2 are not friends</u>

$\exists i_3 \in S$ such that
$\{i_2, i_3\}$ is "swappable pair"

(Some details…)

ALG would not have terminated.
Contradiction to the assumption that
X was output of ALG

Result 3(iv)

When $n \geq k^2 + k$, ALG returns a k-partition in the $(2\mathrm{k}-1,0)$-core

Let X be output of ALG.

Suppose, for a contradiction, that S is a $(2\mathrm{k}-1,0)$-blocking coalition.

Suppose now that $u_i\big(X(i)\big) = 0$ for all $i \in S$.

Since $\mathrm{n} \geq k^2 + k$, $|S| \geq \left\lfloor \frac{k^2+1}{k} \right\rfloor = \left\lfloor k + \frac{1}{k} \right\rfloor = k + 1$.

By pigeonhole principle, $\exists i_1, i_2 \in S$ such that $X(i_1) = X(i_2)$.

<u>i_1 and i_2 are friends</u>

Then, $u_i\big(X(i)\big) \geq 1$ since $X(i_1) = X(i_2)$

Contradiction to $u_i\big(X(i)\big)$ = 0

<u>i_1 and i_2 are not friends</u>

- Since $k \geq 2$, $|S| \geq 3$.
- By definition of blocking coalition, utility of i_1 strictly increases, so i_1 has a friend in S. Let i_3 be this friend.
- Note that $u_{i_3}\big(X(i_3)\big) = 0$ since $i_3 \in S$.
- Suppose i_2 and i_3 are not friends. Then, $u_{i_3}\big(X(i_3)\big) = 0 < 1 = u_{i_3}\big(X(i_2)\big)$ since as i_1 is friend of i_3.
- Suppose i_2 and i_3 are friends. Then, $1 + u_{i_3}\big(X(i_3)\big) = 1 < 2 = u_{i_3}\big(X(i_2)\big)$ since both are friends of i_3.
- In either case, (i_2, i_3) is a “swappable pair”.

Repeat exact same argument as 3(iv)

Result 3(iii)

When $n \geq k^2 + k$, min k-cut is in the $(2\mathrm{k} - 1{,}0)$-core

Let X be an arbitrary min k-cut.

Suppose, for a contradiction, that S is a $(2k - 1{,}0)$-blocking coalition.

Repeat exact same argument as 3(iv)

Result 3(iii)

When $n \geq k^2 + k$, min k-cut is in the $(2\mathrm{k} - 1{,}0)$-core

Let X be an arbitrary min k-cut.

Suppose, for a contradiction, that S is a $(2k - 1{,}0)$-blocking coalition.

From earlier, "$u_i(S \cap X_j) > u_i(X(i)) + 1$" leads to contradiction.

Suppose now that "$u_i(S \cap X_j) \leq u_i(X(i)) + 1$".

So, $u_i(X(i))$ = 0 for all $i \in S$.

Suppose S is $(2\mathrm{k} - 1{,}0)$-blocking coalition of X. Then, for all $\mathrm{i} \in S$,
If $u_i(S \cap X_j) \leq u_i(X(i)) + 1$ for all $j \in [k]$, then $u_i(X(i)) = 0$.

Lemma'

Repeat exact same argument as 3(iv)

Result 3(iii)

When $n \geq k^2 + k$, min k-cut is in the $(2\mathrm{k}-1{,}0)$-core

Let X be an arbitrary min k-cut.

Suppose, for a contradiction, that S is a $(2k-1{,}0)$-blocking coalition.

Suppose now that $u_i\big(X(i)\big) = 0$ for all $i \in S$.

Since $\mathrm{n} \geq k^2 + k$, $|S| \geq \left\lfloor \frac{k^2+1}{k} \right\rfloor = \left\lfloor k + \frac{1}{k} \right\rfloor = k + 1.$

By pigeonhole principle, $\exists i_1, i_2 \in S$ such that $X(i_1) = X(i_2)$.

<u>i_1 and i_2 are friends</u>

Then, $u_i\big(X(i)\big) \geq 1$
since $X(i_1) = X(i_2)$

Contradiction to $u_i\big(X(i)\big)$ = 0

<u>i_1 and i_2 are not friends</u>

$\exists i_3 \in S$ such that
$\{i_2, i_3\}$ is “swappable pair”

ALG would not have terminated.
Contradiction to the assumption that
X was output of ALG

Result 3(iii)

When $n \geq k^2 + k$, min k-cut is in the $(2\mathrm{k}-1{,}0)$-core

Let X be an arbitrary min k-cut.

- Recall that cut size drops in each iteration of ALG.
- If we pass X to ALG, it will not terminate.
- So, X cannot be min k-cut!

<u>i_1 and i_2 are friends</u>

Then, $u_i\big(X(i)\big) \geq 1$
since $X(i_1) = X(i_2)$

Contradiction to $u_i\big(X(i)\big)$ = 0

<u>i_1 and i_2 are not friends</u>

$\exists i_3 \in S$ such that
$\{i_2, i_3\}$ is “swappable pair”

ALG would not have terminated.
Contradiction to the assumption that
X was output of ALG