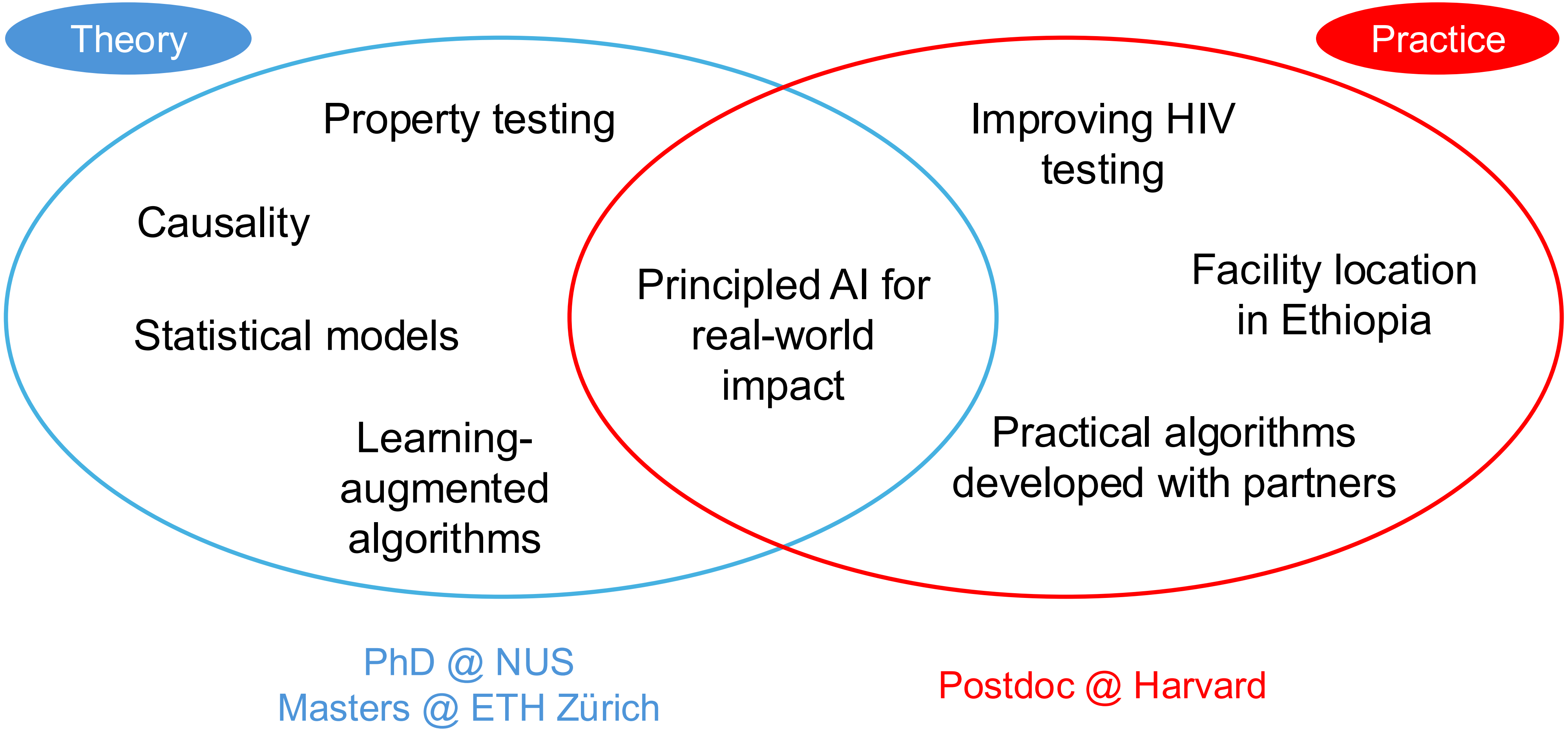
Principled AI for Real-world Impact : Structured Decision-Making under Uncertainty

Davin Choo

Postdoctoral Fellow @ Harvard SEAS

Research vision: Principled AI for real-world impact



Research vision: Principled AI for real-world impact

For this talk, I will share about two projects that I (co-)led at Harvard

Roadmap for this talk

Project 1:
Adaptive disease testing on graphs [1]

Project 2:
Health facility planning in Ethiopia [2]

Lessons learnt and personal takeaways

[1] **Davin Choo**, Yuqi Pan, Tonghan Wang, Milind Tambe, Alastair van Heerden, and Cheryl Johnson. Adaptive Frontier Exploration on Graphs with Applications to Network-Based Disease Testing. Under submission, 2025.

[2] **Davin Choo**, Yohai Trabelsi, Fentabil Getnet, Samson Warkaye Lamma, Wondesen Nigatu, Kasahun Sime, Lisa Matay, Milind Tambe, and Stéphane Verguet. Optimizing Health Coverage in Ethiopia: A Learning-augmented Approach and Persistent Proportionality Under an Online Budget. Under submission, 2025.

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Project 1

Adaptive disease testing on graphs [1]

Yuqi Pan
Harvard University

Tonghan Wang
Harvard University

Milind Tambe
Harvard University

Alastair van Heerden
University of Witwatersrand
Wits Health Consortium

Cheryl Johnson
World Health Organization

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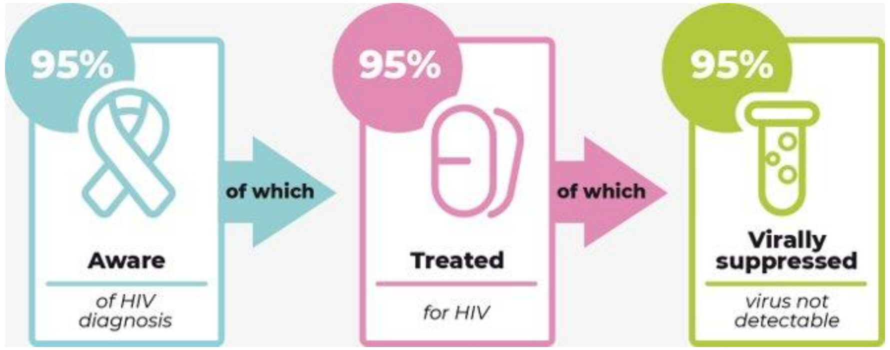
UN 95-95-95 HIV target: what and why



Global initiative by UNAIDS to control HIV epidemic

- 95% of people living with HIV know their status
- 95% of those diagnosed receive sustained treatment
- 95% of those on treatment achieve viral suppression

Undetectable = Untransmittable (U = U)

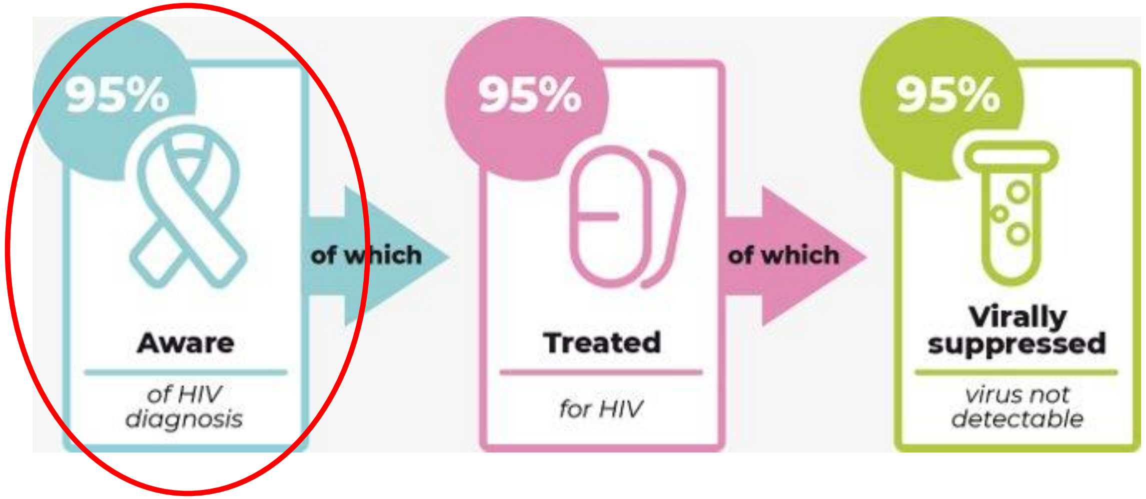
- "A person living with HIV who is on treatment and maintains an undetectable viral load has zero risk of transmitting HIV to their sexual partners."

In line with UN Sustainable Development Goal 3.3

- "By 2030, end the epidemics of AIDS, tuberculosis, malaria and neglected tropical diseases and combat hepatitis, water-borne diseases and other communicable diseases"

https://www.unaids.org/en/resources/documents/2024/global-aids-update-2024
https://www.cdc.gov/global-hiv-tb/php/our-approach/undetectable-untransmittable.html
https://sdgs.un.org/goals/goal3 - targets_and_indicators
Image credit: https://awarehiv.com/en/about-aware-hiv/our-goals

Targeting the first 95% testing goal



Motivation

- The faster we detect positive cases, the faster we can start treatment cascade
- Once individuals learn about their status, they will also typically undergo behavioral changes which help limit spread

Challenges

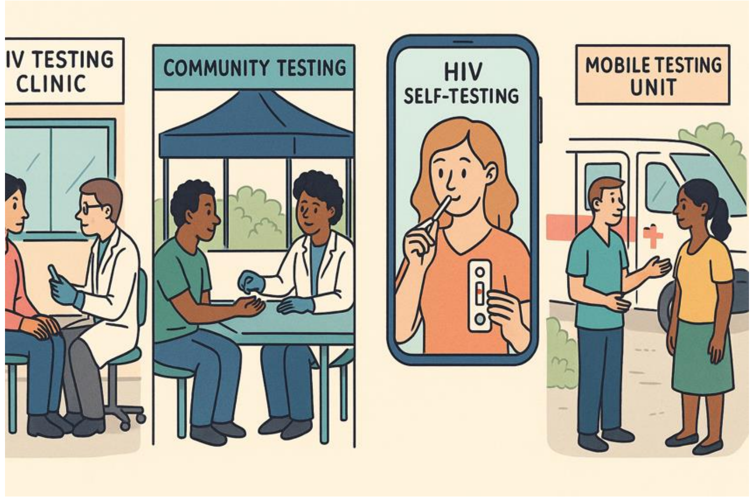
- About 1 in 7 HIV positive individuals do not know they are infected
- Resource limitations due to funding cuts

https://www.unaids.org/en/resources/documents/2024/global-aids-update-2024
https://www.unaids.org/en/impact-US-funding-cuts
Image credit: https://awarehiv.com/en/about-aware-hiv/our-goals

Current testing approaches and challenges

Variety of differentiated testing services address different settings and populations

- Demand creation: Encourage individuals to seek testing via advertisements and outreach
- Self-testing: teach individuals how use and interpret rapid test results
- Facility-based testing: offer tests to individuals visiting clinics
- Community-based testing: going beyond standard health facilities
  - Examples: workplace, parks, bars, clubs, events, schools, places of worship, etc.
- Network-based testing: Offer tests to individuals who interacted with known infected individuals



https://www.who.int/publications/i/item/9789240096394

Image credit: ChatGPT with prompt "Please generate me a cartoonish image of different HIV testing services"

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Reduced effectiveness / Diminishing returns as we approach 95% in the testing goal

- Currently, questionnaires are used to collect data from individuals then derive a “risk score”
    - Each person is associated with an anonymous ID
    - Individual covariates: age, occupation, living region, personal beliefs, alcohol/drug tendencies, etc.
    - Identifiers of sexual and drug-injecting partners
- Current testing prioritization does not effectively adapt to and exploit disease transmission network
- New approaches and ideas are needed for increasing reach and testing effectiveness

https://www.who.int/publications/i/item/9789240096394

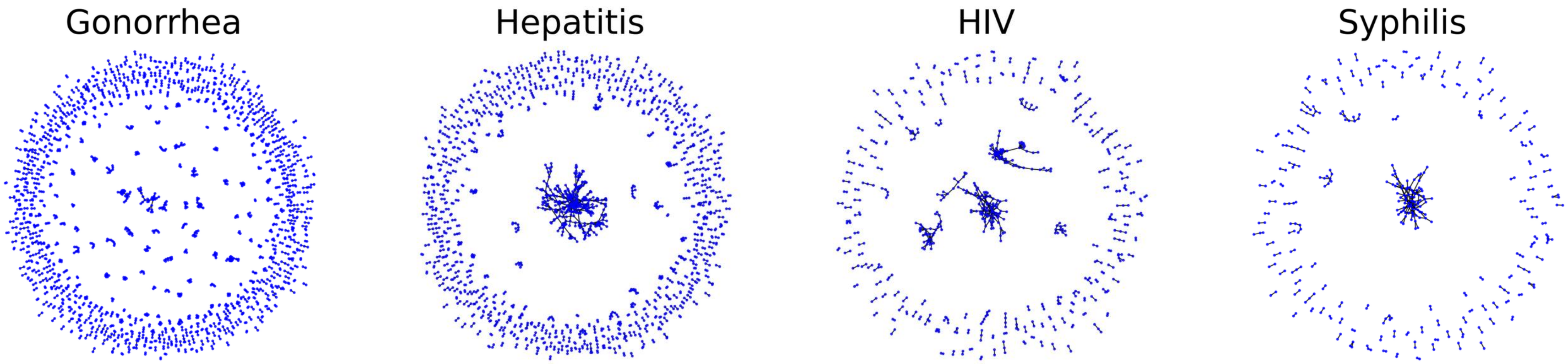
Re-imagining a new testing approach

Objective

- Maximize efficiency of testing resources in detecting HIV+ cases as quickly as possible
  - Resource can be # test kits, or where to focus efforts of human workers in recruiting people for testing

Constraints and structural properties of our problem

- Sexually transmitted diseases do not spread like flu
  - The transmission graph $\mathcal{G}$ is sparse and tree-like
  - Note: We only ever observe a subset of the true graph



https://www.icpsr.umich.edu/web/ICPSR/studies/22140/summary

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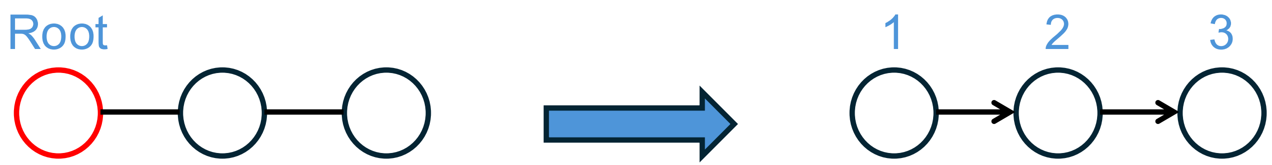
- Policy-wise, it is preferable to test individuals whose neighbors (in $\mathcal{G}$) have been tested
  - This is because it is more informative than randomly picking individuals to test next
  - This maps to a kind of “frontier exploration” constraint on the graph
  - First tested person (the root) of each component is chosen via some domain policy consideration



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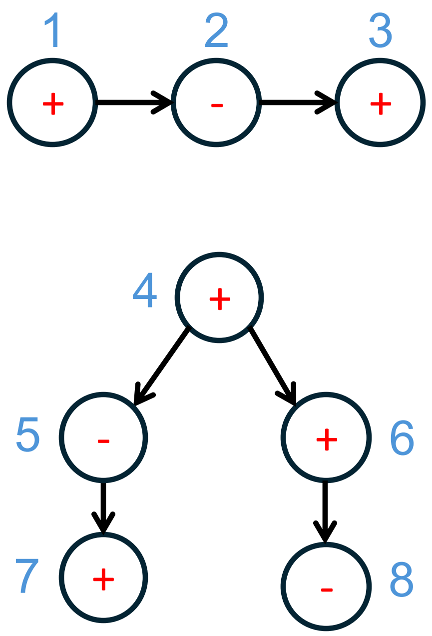
    - First tested person (the root) of each component is chosen via some domain policy consideration
- There is some underlying transmission probability $\mathcal{P}$ that is Markov with respect to $\mathcal{G}$
    - Every individual has an unobserved discrete label of + or –, that is revealed upon testing
    - Markov: $\mathcal{P}$(person = + | revealed status) = $\mathcal{P}$(person = + | revealed statuses of neighbors)

Designing a reward metric for optimization

Illustrative example

- Suppose we know the underlying labels (disease status)
- How good is this sequence of testing?
  - Early detection → early intervention; Also, we may have sudden budget cuts

Testing budget	1	2	3	4	5	6	7	8
# positive detected	1	1	2	3	3	4	5	5



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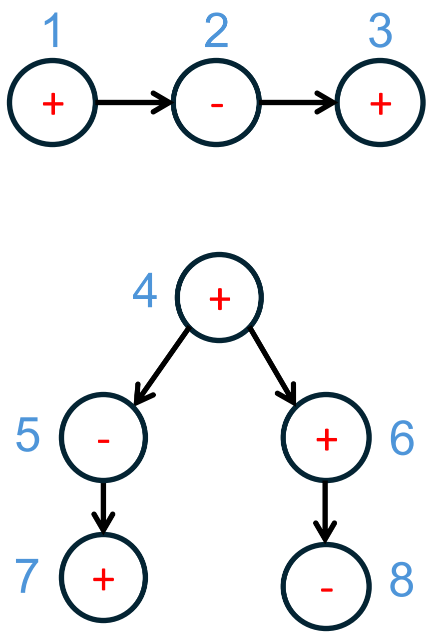

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Reward metric

- Use discount factor $\beta \in (0, 1)$ to favor early discovery of positive cases
  - This sequence : $\beta^0 * 1 + \beta^1 * 0 + \beta^2 * 1 + \beta^3 * 1 + \beta^4 * 0 + \beta^5 * 1 + \beta^6 * 1 + \beta^7 * 0$

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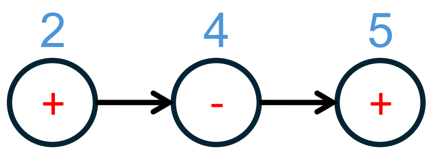

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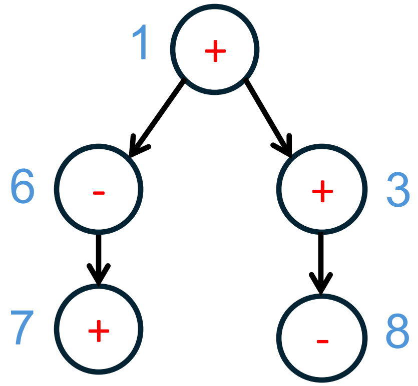

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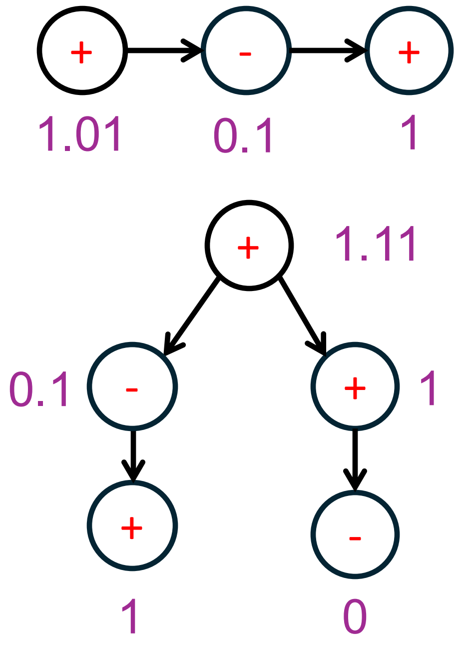
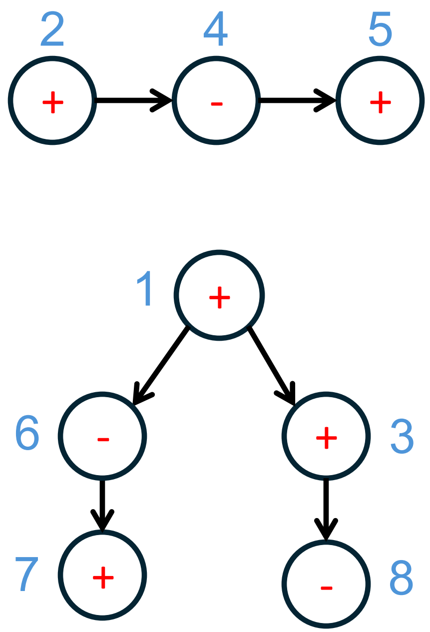
  - This sequence : $\beta^0 * 1 + \beta^1 * 0 + \beta^2 * 1 + \beta^3 * 1 + \beta^4 * 0 + \beta^5 * 1 + \beta^6 * 1 + \beta^7 * 0$
  - A better sequence: $\beta^0 * 1 + \beta^1 * 1 + \beta^2 * 1 + \beta^3 * 0 + \beta^4 * 1 + \beta^5 * 0 + \beta^6 * 1 + \beta^7 * 0$
- Finding a sequence to optimize this reward metric helps detect positive cases faster, for any fixed amount of testing budget

Gittins indices are optimal on rooted forests

Using the idea of Gittins indices for our example

- Compute a score* for each node, then greedily select from the nodes in the frontier
- Known to produce an optimal sequence on our reward metric when the input graph is a forest





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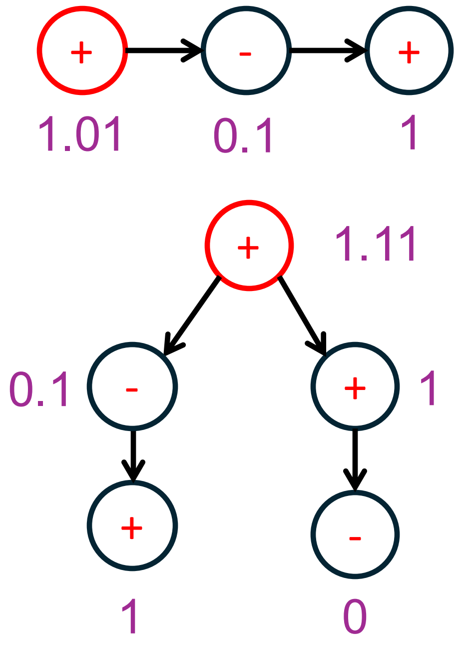

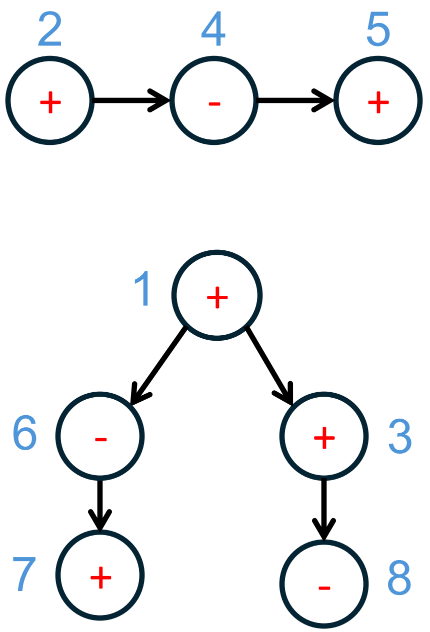

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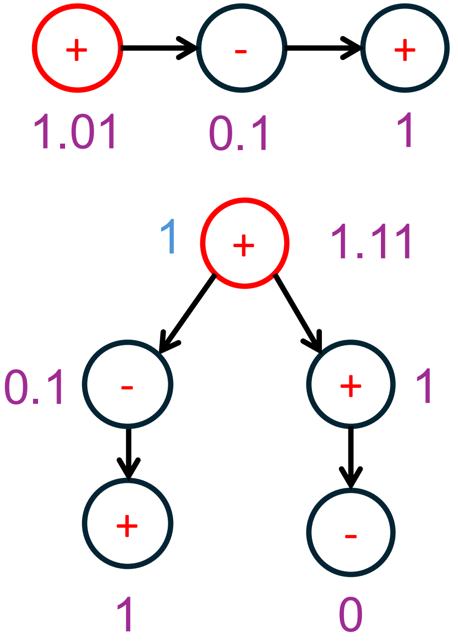

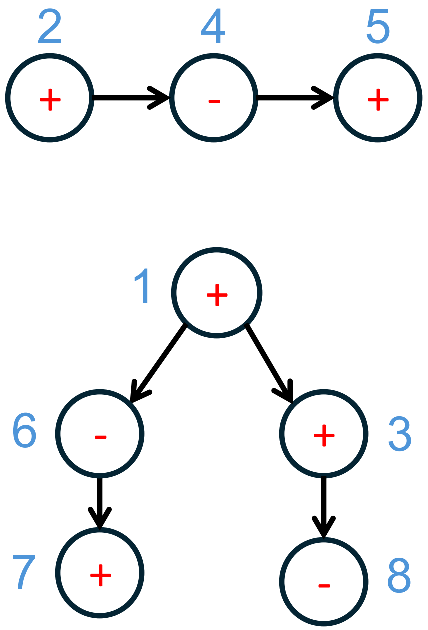

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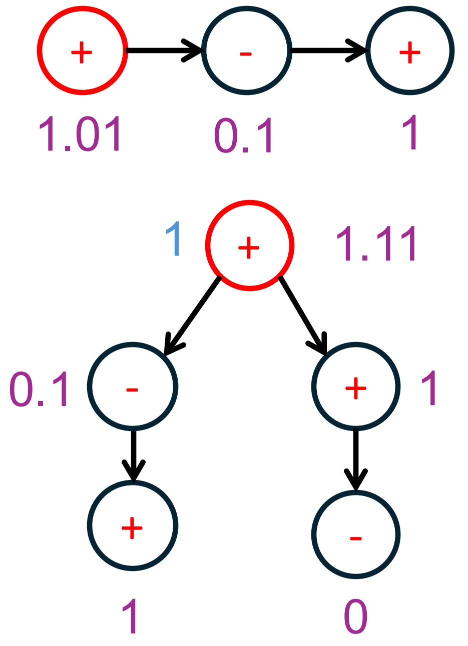

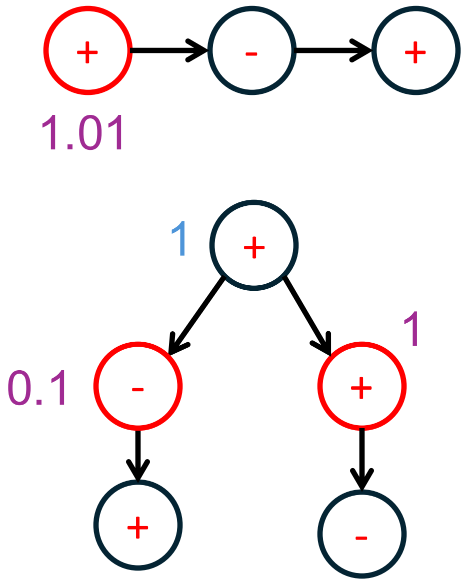

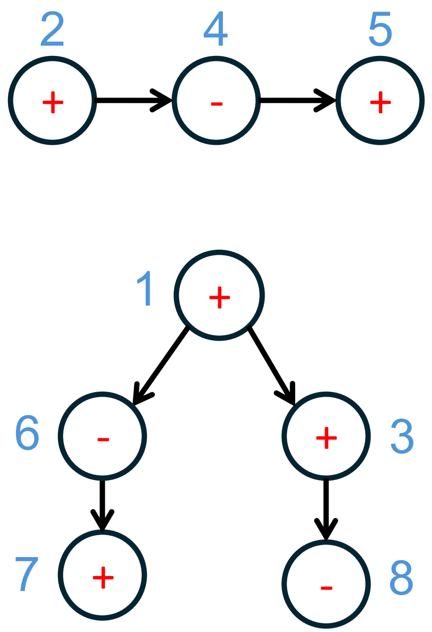

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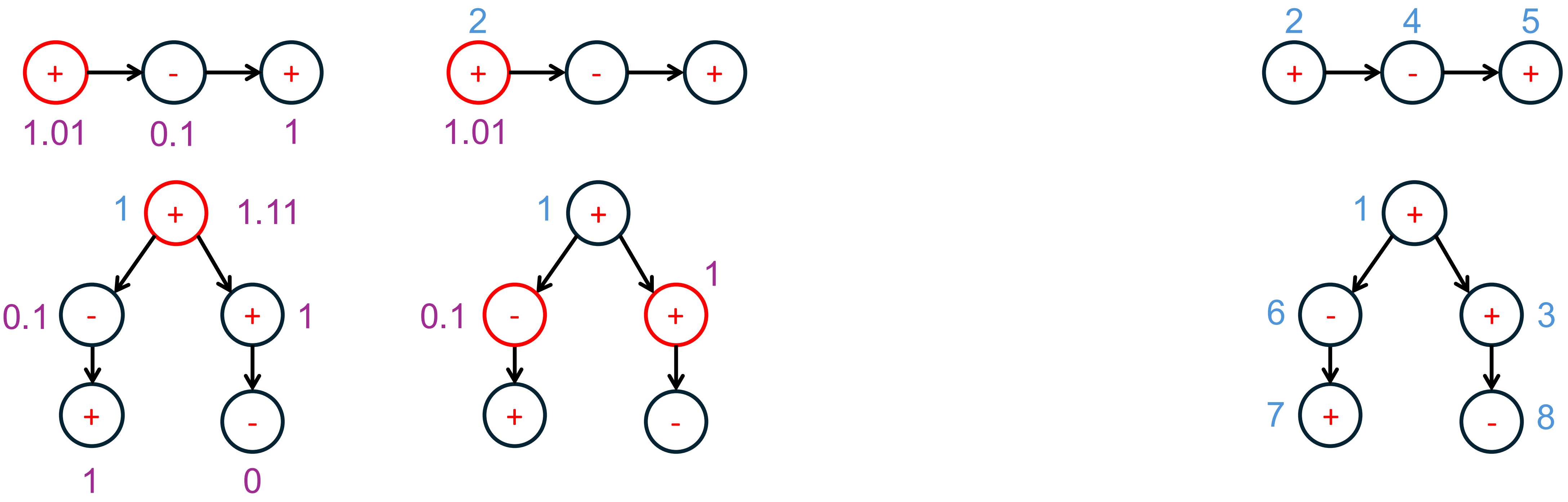

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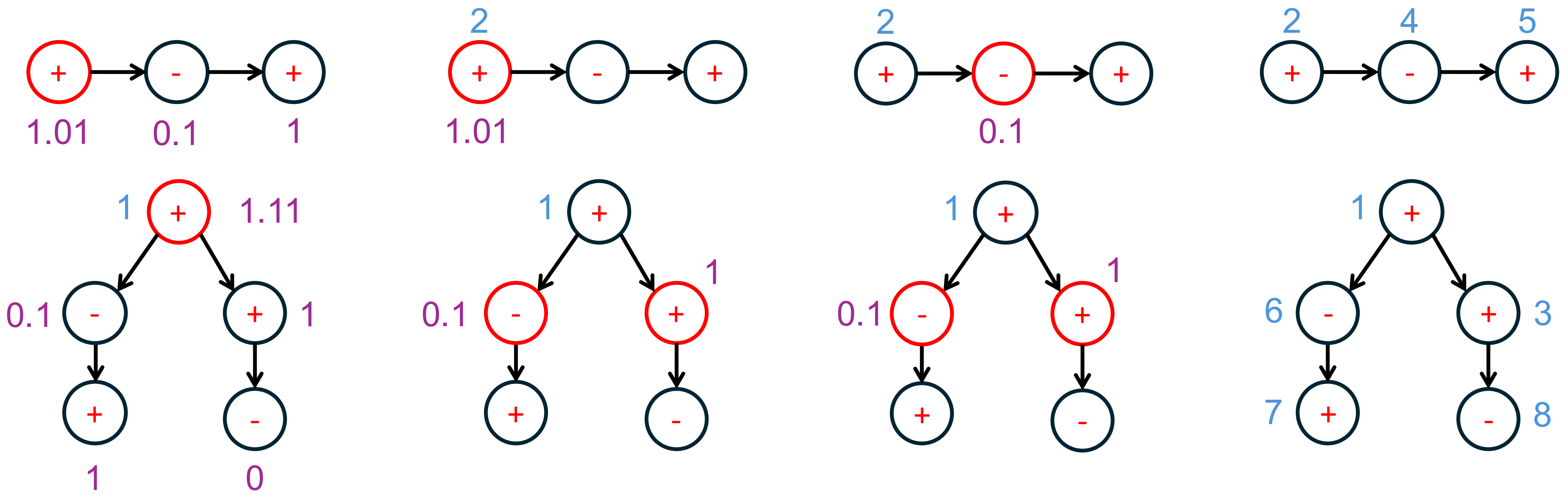

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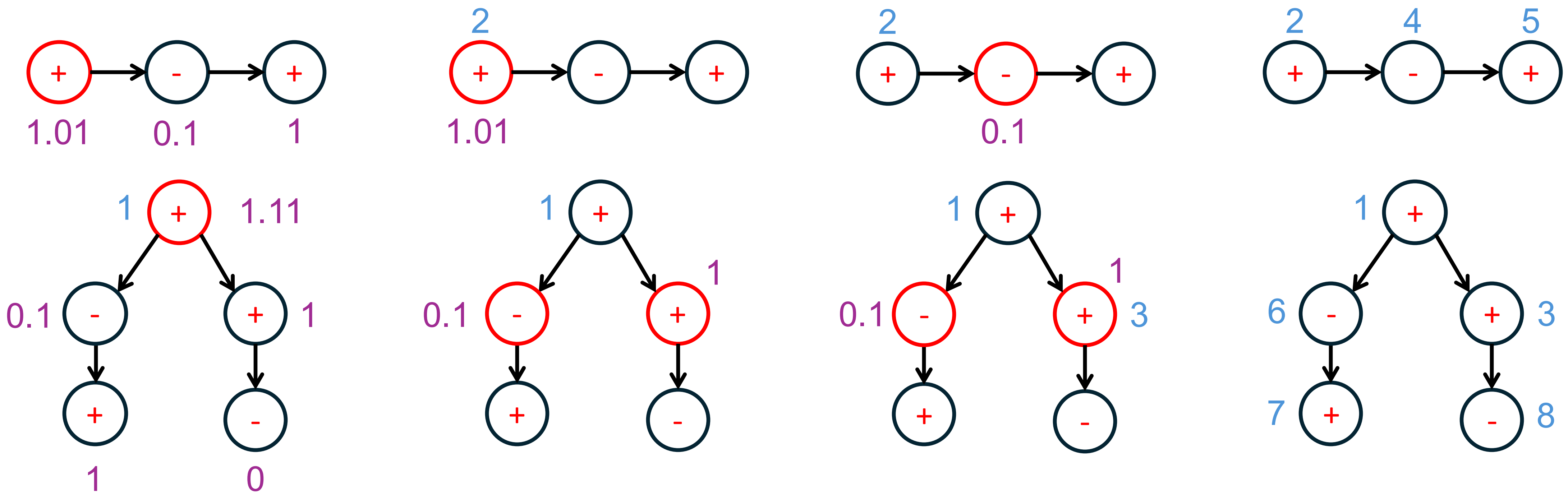

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Gittins index from earlier generalizes to rooted forests and probabilistic labels

- Statuses can be a function of the revealed parent's status, based on a known distribution $\mathcal{P}$
- Now, we pre-compute a score for node X based on every possible parent status, then use the score that corresponds to revealed parent status when ranking node X in the frontier

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Gittins indices are optimal on rooted forests

To define the Gittins index, let us first define two recursive functions ϕ and Φ, as per [KO03]. For any non-root node $X \in \mathbf{X}$, label $b \in \mathbf{\Sigma}$, and value $0 \leq m \leq \frac{\bar{r}}{1-\beta}$,

$$\phi_{X,b}(m) = \max \left\{ m, \sum_{v \in \mathbf{\Sigma}} \mathcal{P}(X = v \mid \mathrm{Pa}(X) = b) \cdot \left[r(X,v) + \beta \cdot \Phi_{\mathrm{Ch}(X),v}(m) \right] \right\} \quad (1)$$

If X is the root, we define $\phi_{X,\emptyset}(m) = \max \left\{ m, \sum_{v \in \mathbf{\Sigma}} \mathcal{P}(X = v) \cdot \left[r(X,v) + \beta \cdot \Phi_{\mathrm{Ch}(X),v}(m) \right] \right\}$.

For any subset of nodes $\mathbf{S} \in \mathbf{X}$, label $b \in \mathbf{\Sigma}$, and value $0 \leq m \leq \frac{\bar{r}}{1-\beta}$,

$$\Phi_{\mathbf{S},b}(m) = \begin{cases} \frac{\bar{r}}{1-\beta} - \int_m^{\frac{\bar{r}}{1-\beta}} \prod_{Y \in \mathbf{S}} \frac{\partial \phi_{Y,b}(k)}{\partial k} \, dk & \text{if } \mathbf{S} \neq \emptyset \\ m & \text{if } \mathbf{S} = \emptyset \end{cases} \quad (2)$$

Gittins index for node X when parent's realization is b

$$g(X,b) = \min \left\{ m \in \left[0, \frac{\bar{r}}{1-\beta}\right] : \phi_{X,b}(m) \geq m \right\}$$

This "min ≥" captures the intuition of "fair value"

We show that our adaptive testing problem can be reduced to "branching bandits"

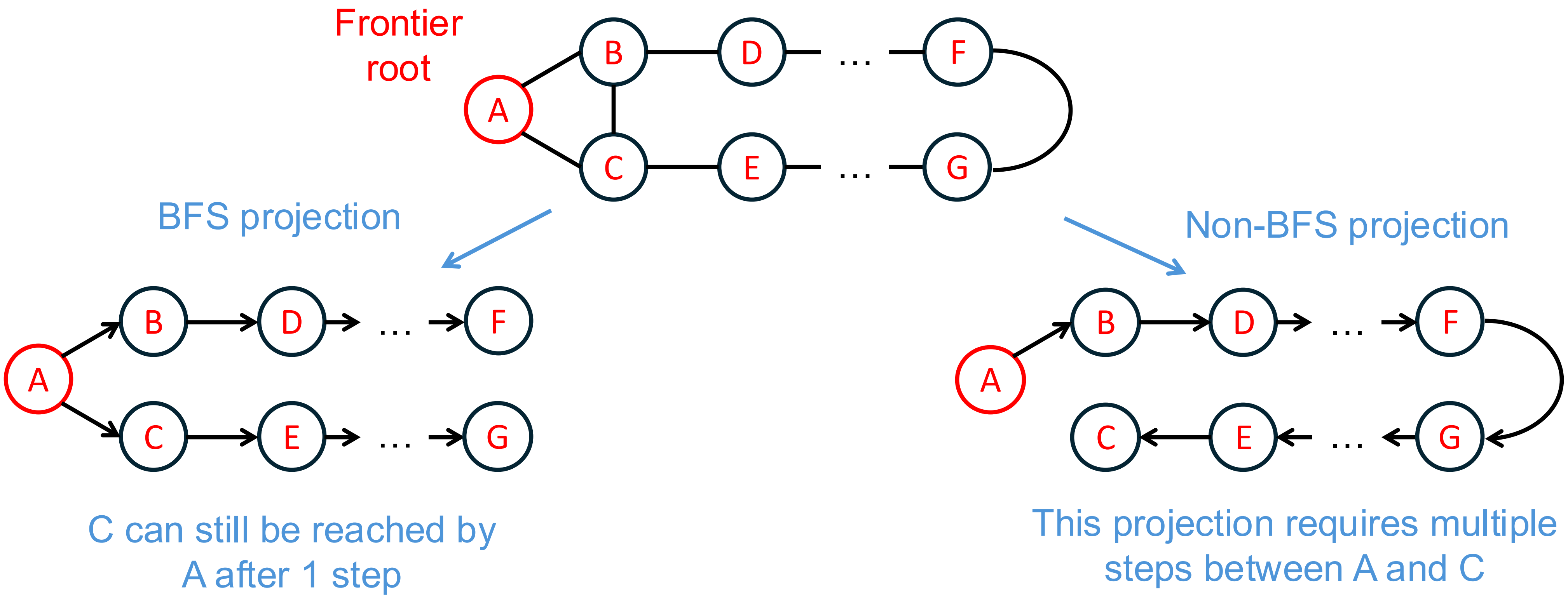
- Optimality of Gittins indices for rooted forests proven in [KO03]
- We provide the first efficient polynomial time dynamic programming (DP) method (with working Python code) in the 20 years since [KO03] for computing ϕ and Φ for *discrete labels*
  - Prove and exploit piecewise linearity of ϕ and Φ, enabling efficient representation of ϕ and Φ in the DP
  - Number of pieces scale with the number of nodes and number of labels

[KO03] Godfrey Keller and Alison Oldale. Branching bandits: a sequential search process with correlated pay-offs. Journal of Economic Theory, 113(2):302–315, 2003.

In practice, use BFS to project non-trees into trees before applying Gittins computation

Transmission graphs $\mathcal{G}$ may not be forests in general

- Run breadth-first search (BFS) on $\mathcal{G}$ from the frontier roots in each component
- This minimizes height to root, reducing artificial frontier constraint due to projection



Some experimental results

Assume that $\mathcal{G}$ and $\mathcal{P}$ are provided as input

- $\mathcal{G}$ is subgraph of disease-specific interaction graph involving ~300 nodes
- $\mathcal{P}$ is an Ising model, i.e., Markov random field (MRF) with unary and pairwise potentials θ
- Train $\mathcal{P}$ from disease-specific past data, i.e., fit θ via pseudo-likelihood
- Measure *expected* performance, with respect to $\mathcal{P}$

Baselines

- Random: Randomly select next frontier node to test
- Greedy: Select frontier node with highest posterior probability of being positive
- DQN: Deep Q-network using message-passing GNN with edge-conditioned weights

Additional remarks

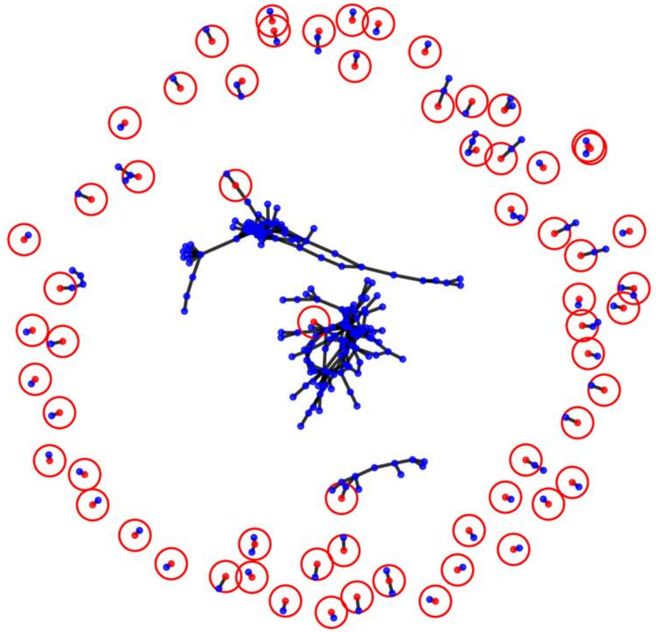
- MRF computations are reasonable on low treewidth graphs via Junction tree algorithm
- For small graphs (~10 nodes) where we can brute-force compute the optimal policy, we verified that Gittins literally traces the optimal curve, unsurprisingly

Empirical evaluation on HIV interaction graph

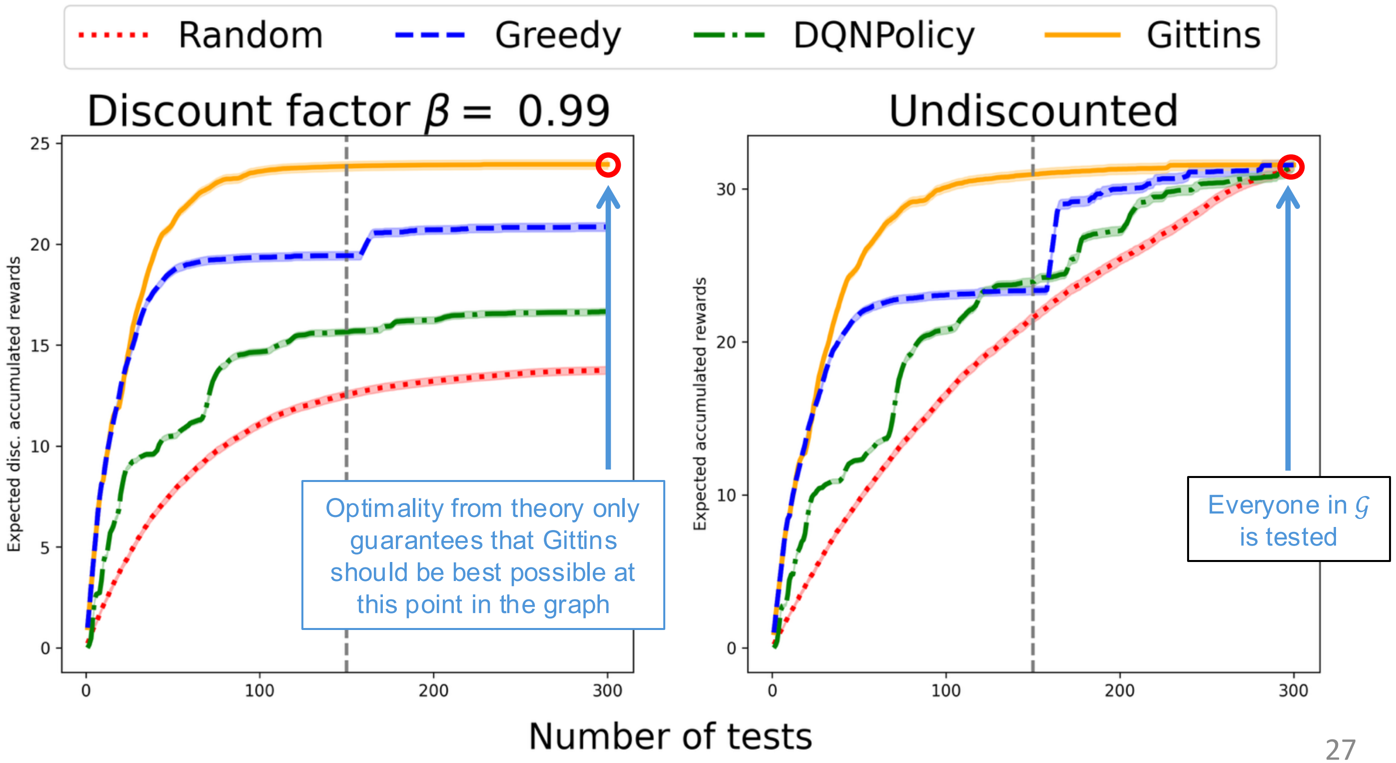
With only budget to test half the population, Gittins detects almost all positive cases in expectation while the other methods still miss about 25% of the positive cases

- Similar trend for other disease graphs; Gittins always “the best” (see backup slides)

HIV sex interaction graph



Frontier roots are circled in red



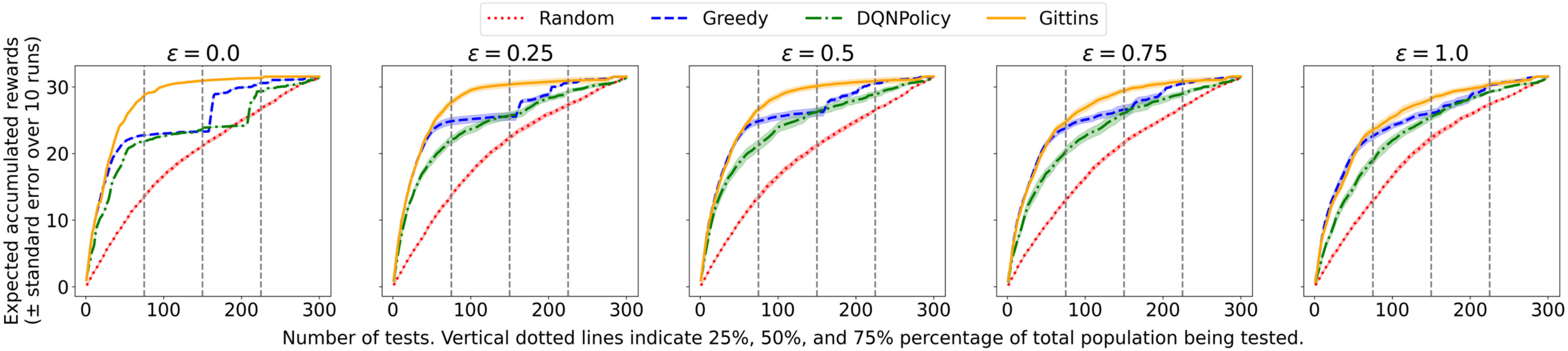
Trend remains empirically robust even when policies only have access to noisy estimate $\mathcal{Q}$ of $\mathcal{P}$

Generate noisy distribution $\mathcal{Q}$ by adding noise to $\mathcal{P}$

- $\mathcal{P}$ is an Ising model, i.e., Markov random field with unary and pairwise potentials θ
- $\mathcal{Q}$ is defined by adding noise to θ
- For different ε, add ε * u * $|\theta_i|$ to i-th coordinate of θ, where u ~ Uniform(-1, 1)

Empirical advantage of Gittins decreases as noise increases

- Below: Undiscounted accumulated reward plot for the same HIV interaction graph



Next steps: Working towards a field trial and a more comprehensive retrospective study

Trial-run deployment

- In collaboration with WHO and the Gates foundation, under the HIV LIFT project
- Deployment area: *KwaZulu-Natal, South Africa*

World Health Organization

Gates Foundation

Comprehensive retrospective study

- In collaboration with FHI 360[1]
- Currently drafting concept note and data agreement to get access anonymized *Tanzanian* program data to perform a more comprehensive retrospective study of our method

fhi360

[1] “FHI 360 is a nonprofit organization that mobilizes research, resources and relationships so that people everywhere can access the opportunities they need to lead full, healthy lives. Our staff of more than 2,000 experts work in over 50 countries around the world.”

Roadmap for this talk

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Adaptive disease testing on graphs [1]

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Health facility planning in Ethiopia [2]

Lessons learnt and personal takeaways

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Project 2

Health facility planning in Ethiopia [2]

Yohai Trabelsi
Harvard University

Fentabil Getnet
National Data Management and Analytics Center for Health, Ethiopian Public Health Institute

Samson Warkaye Lamma
National Data Management and Analytics Center for Health, Ethiopian Public Health Institute

Wondesen Nigatu
Primary Healthcare and Community Engagement Lead Executive Office, Ministry of Health

Kasahun Sime
Primary Healthcare and Community Engagement Lead Executive Office, Ministry of Health

Lisa Matay
Department of Global Health and Population, Harvard T.H. Chan School of Public Health

Stéphane Verguet
Department of Global Health and Population, Harvard T.H. Chan School of Public Health

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The Ethiopia setting: Background

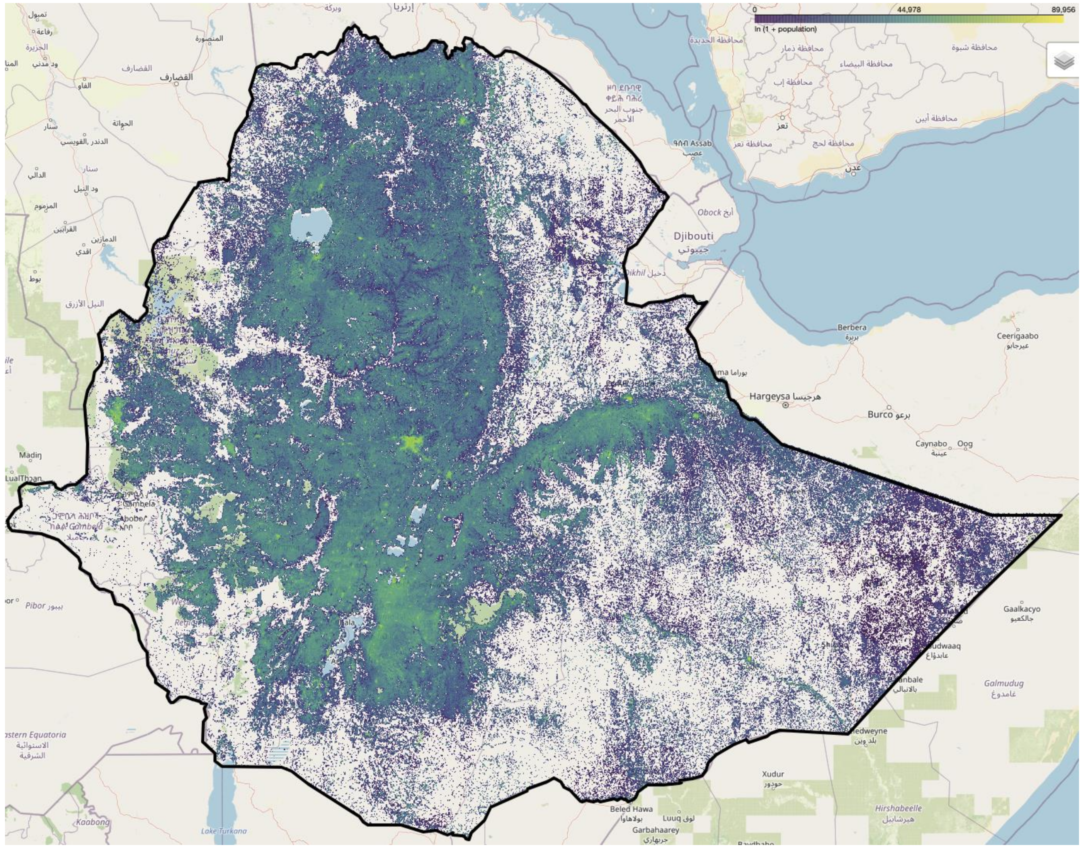
Fact sheet

- 2nd most populous country in Africa, with ~130 million people
- 12 regional states and 2 chartered cities
- Level of administration: Country → Region → Woredas (districts) → Kebeles
- ~76% live in rural areas, where access to healthcare is limited
- Walking is most common form of transport in rural areas
- Different tiers of healthcare provided by different kinds of facilities

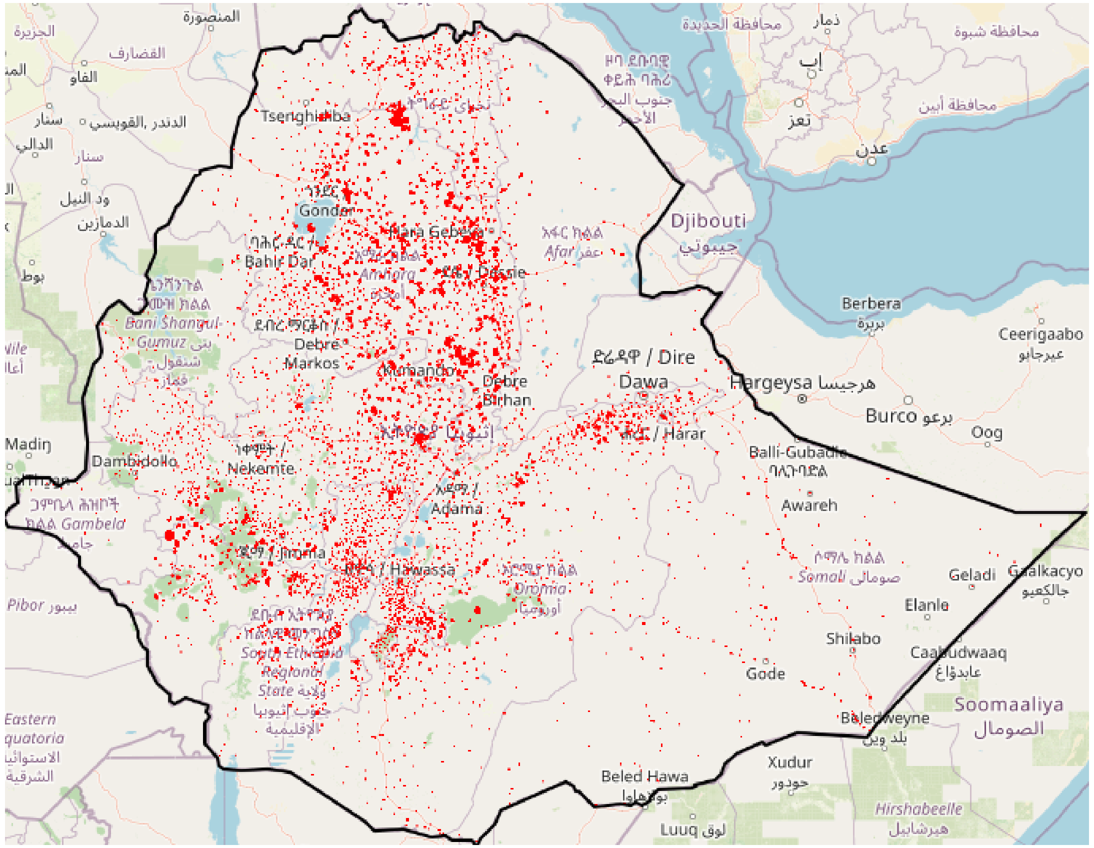
In line with UN Sustainable Development Goal 3.8

- "Achieve universal health coverage, including financial risk protection, access to quality essential health-care services and access to safe, effective, quality and affordable essential medicines and vaccines for all"

72% of the population lack access to “comprehensive health care” within 2 hours of walking



Population of Ethiopia in log scale
(brighter spots are denser areas)



Areas within 2 hours of walking by existing facility capable of “comprehensive health care”

Formalizing the facility location problem

Objective

- Maximize coverage over 5-year planning horizon

Constraints and structural properties of our problem

- Decision making and budget availability mainly at regional level
    - Each region has own budget and independently decides where to build facilities within their region Central government only provide recommendations
- Each region has different “fairness” prioritization over districts
    - Prioritize districts based on income, or number of unassisted home births (proxy for lack of “comprehensive health care”), etc.
- Uncertainty in future budget due to unforeseen budget cuts or donors

Available data and problem modeling

Available data (that is also used by the Ethiopian officials when planning)

- Population forecasts via WorldPop projections in 1km-by-1km grids
- Point-to-point walking distance estimates [WNVR+20]
- List of existing facilities
- Estimated yearly regional budget from expert consultation with Ethiopia officials

Problem modeling: "Some variant of maximum coverage"

- Maximize coverage → Maximize monotone submodular function
- 5-year horizon with budget uncertainty → Online setting with irrevocable decisions
- Budget and proportionality constraints → Matroid constraints

Translates to a complicated formula of "satisfaction ratio" that we need to optimize at any point in time, which will not be the focus for this talk

WorldPop: https://www.worldometers.info/world-population/ethiopia-population/

[WNVR+20] D. J. Weiss, A. Nelson, C. A. Vargas-Ruiz, K. Gligori´c, S. Bavadekar, E. Gabrilovich, A. Bertozzi-Villa, J. Rozier, H. S. Gibson, T. Shekel, and et al. Global maps of travel time to healthcare facilities. Nature Medicine, 26:1835–1838, 2020.

Legend

☐ Grid

Illustrative example

Characteristics in this 2-year 5x5 grid, where coverage is Manhattan distance 1

- Population can change over the years

3	3	5	1	5
3	3	2	5	1
1	5	10	5	1
1	7	5	5	5
3	1	3	5	3

Year 1

4	4	2	1	4
2	4	2	7	2
0	5	11	6	1
1	7	5	7	1
4	2	1	3	3

Year 2

[redacted]

? facilities for Year 1

[redacted]

? facilities for Year 2

<u>Legend</u>

- Grid
- Covered by existing facility
- Facility built in this time step

Illustrative example

Characteristics in this 2-year 5x5 grid, where coverage is Manhattan distance 1

- Population can change over the years
- Future budget is uncertain: must plan based on existing facilities and current budget

3	3	5	1	5
3	3	2	5	1
1	5	10	5	1
1	7	5	5	5
3	1	3	5	3

Year 1

4	4	2	1	4
2	4	2	7	2
0	5	11	6	1
1	7	5	7	1
4	2	1	3	3

Year 2

3	3	5	1	5
3	3	2	5	1
1	5	10	5	1
1	7	5	5	5
3	1	3	5	3

2 facilities for Year 1

[redacted]

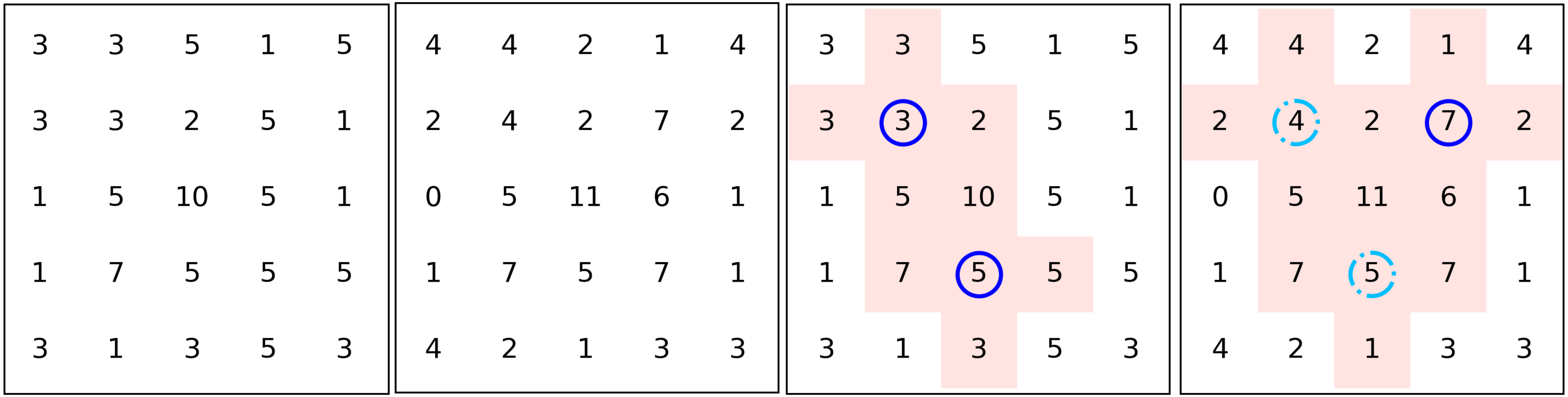
? facilities for Year 2

Legend
- Grid
- Covered by existing facility
- Facility built in this time step
- Facility built in the past

Illustrative example

Characteristics in this 2-year 5x5 grid, where coverage is Manhattan distance 1

- Population can change over the years
- Future budget is uncertain: must plan based on existing facilities and current budget
- Once facility is built, it continues to provide coverage for future time steps



Year 1 | Year 2 | 2 facilities for Year 1 | 1 facility for Year 2

Maximizing non-decreasing submodular functions are well-studied in the literature

Select subset S of elements such that f(S) is maximized

- f is a non-decreasing and submodular function

Classic setting: selection S satisfies cardinality constraint, i.e., $|S| \leq b$

- 1 – 1/e approximation via simple greedy algorithm [NWF78]
  - Until you run out of budget, pick the next item that maximizes marginal gain
- Cannot obtain better approximation ratio unless P = NP [Fei98]

Extension: selection S satisfies k matroid constraints

- 1 / (k+1) approximation achievable and tight for "local greedy" [FNW78]
  - High-level idea: Run greedy for each matroid
- For single matroid (k = 1), 1 – 1/e approx. achievable via complicated rounding [CCPV11]
- Remark: cardinality constraint is a special case of partition matroid

[NWF78] G. L. Nemhauser, L. A. Wolsey, and M. L. Fisher. An analysis of approximations for maximizing submodular set functions – I. Mathematical Programming, 14:265–294, 1978.
[Fei98] Uriel Feige. A threshold of ln n for approximating set cover/ Journal of the ACM (JACM), 45:634-652, 1998.
[FNW78] M. L. Fisher, G. L. Nemhauser, and L. A. Wolsey. An analysis of approximations for maximizing submodular set functions – II, volume 8. Springer, 1978.
[CCPV11] Gruia Calinescu, Chandra Chekuri, Martin Pal, and Jan Vondrak. Maximizing a monotone submodular function subject to a matroid constraint. SIAM Journal on Computing, 40(6):1740–1766, 2011.

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Hard problem, even to approximate

Greedy (based on marginal gain) is more or less the best we can hope for in terms of approximation guarantee, when restricting to polynomial time algorithms

[NWF78] G. L. Nemhauser, L. A. Wolsey, and M. L. Fisher. An analysis of approximations for maximizing submodular set functions – I. Mathematical Programming, 14:265–294, 1978.
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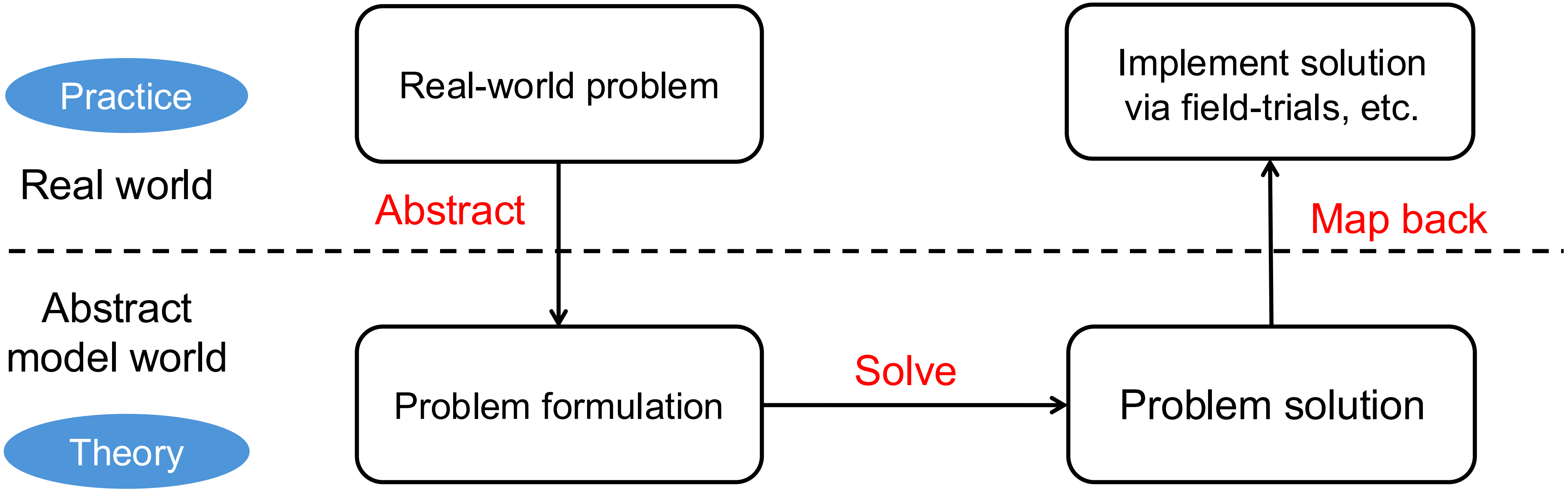
A general problem-solving framework



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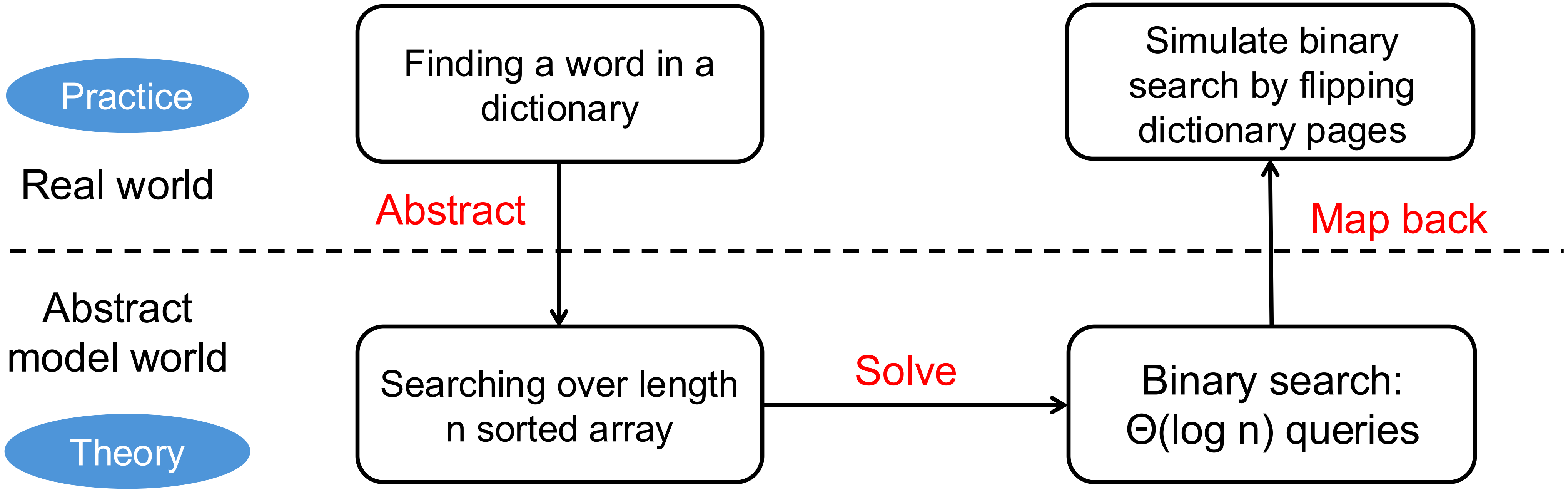
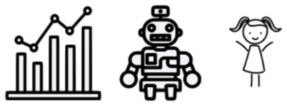


Learning-augmented algorithms: Exploiting instance-specific imperfect advice



The word you are looking for is on page x

Practice

Real world

Finding a word in a dictionary

Abstract

Simulate binary search by flipping dictionary pages

Map back

Abstract model world

Theory

Searching over length n sorted array

Solve

Binary search: $\Theta(\log n)$ queries

https://thenounproject.com/icon/statistics-7090732/
https://thenounproject.com/icon/robot-7098785/
https://thenounproject.com/icon/girl-1257314/

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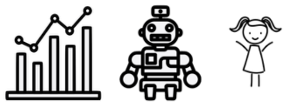

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Searching over length n sorted array

Solve

Algorithm with imperfect advice: $\Theta(\log |x - x^*|)$ queries

https://thenounproject.com/icon/statistics-7090732/
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Leveraging on domain expertise for health facility location in Ethiopia

Currently: Given budget of *b* facilities, regional experts propose *b* locations

- Consult and interview citizens, conduct field measurements, etc.
- Then, produce a selection using domain expertise accumulated over years of fieldwork, local knowledge, and institutional memory

Leveraging on domain expertise for health facility location in Ethiopia

Currently: Given budget of *b* facilities, regional experts propose *b* locations

- Consult and interview citizens, conduct field measurements, etc.
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Learning-augmented approach

- Take the proposed set of locations **A** as additional advice input to the problem instance
- Produce a new "unified" selection **U** such that
  - Selection **U** respects same constraints as **A**

If advice is somehow "perfect",
then f(**U**) = f(**A**) = f(**OPT**)

Theoretical guarantee we can show
f(**U**) ≥ max { f(**A**), f(**G**) }

Never worse than blindly trusting advice

Never worse than advice-free baseline

  - Here, **G** is selection produced by the best advice-free algorithm
    (Note: For the problem of submodular maximization, greedy achieves best poly-time approximation)
  - Remark: Can show a more complicated guarantee of f(**U**) that varies according to "advice quality"

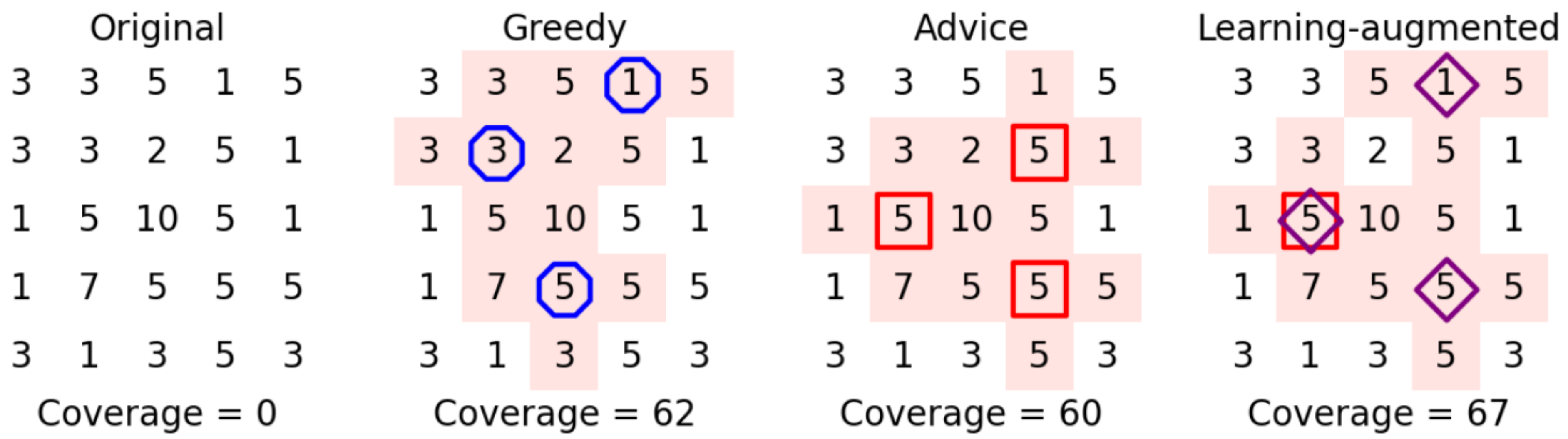
Leveraging on domain expertise for health facility location in Ethiopia

Can consider polynomially many subsets

High-level idea of our learning-augmented algorithm

- Run the greedy algorithm with different subsets $\mathbf{A}_i$ of advice selection $\mathbf{A}$ as an initial selection
- That is, maximize marginal gain when selecting next element, starting from $\mathbf{A}_i$ instead of $\emptyset$
- When $\mathbf{A}_i = \emptyset$, we recover $\mathbf{U} = \mathbf{G}$
- When $\mathbf{A}_i = \mathbf{A}$, we recover $\mathbf{U} = \mathbf{A}$

$f(\mathbf{U}) \geq \max \{ f(\mathbf{A}), f(\mathbf{G}) \}$



Next steps: A tool and a public health study

With Harvard T.H. Chan School of Public Health and officials from Ethiopia

- Build an actual tool and perform user studies with Ethiopian regional planners
- Study the usefulness and impact of our proposed method from a public health perspective

Image credit: ChatGPT with prompt “Can you generate a cartoonish planning tool (with a map) being shown in a big screen and two people chatting and pointing to it?”

Roadmap for this talk

Project 1:
Adaptive disease testing on graphs [1]

Project 2:
Health facility planning in Ethiopia [2]

Lessons learnt and personal takeaways

[1] **Davin Choo**, Yuqi Pan, Tonghan Wang, Milind Tambe, Alastair van Heerden, and Cheryl Johnson. Adaptive Frontier Exploration on Graphs with Applications to Network-Based Disease Testing. Under submission, 2025.
[2] **Davin Choo**, Yohai Trabelsi, Fentabil Getnet, Samson Warkaye Lamma, Wondesen Nigatu, Kasahun Sime, Lisa Matay, Milind Tambe, and Stéphane Verguet. Optimizing Health Coverage in Ethiopia: A Learning-augmented Approach and Persistent Proportionality Under an Online Budget. Under submission, 2025.

Lessons learnt and personal takeaways

1. **A useful problem formulation is a non-trivial contribution**
   - A good problem formulation is half the battle won
   - Real-world are high-stakes yet often vaguely defined
     - "I want my process to be more efficient" → In what aspect?
     - "I want the allocation to be fair" → How to quantify?
   - Problem formulation isn't about over-simplification, but about abstracting the key essence of the problem: "All models are wrong, but some are useful" – George Box [Box79]
   - Careful modeling requires *close collaboration with domain experts* who share a mutual understanding of the problem and mutual trust, and appreciation of the work involved
2. **Real-world settings often have structures to exploit**
   - Principally designed methods can beat simple heuristics, even if assumptions are imperfect
3. **Learning-augmented algorithms are a powerful and practical tool**
   - Improve processes while respecting domain expertise in the age of AI and getting buy-in

[Box79] George E. P. Box. Robustness in the Strategy of Scientific Model Building. In Robustness in Statistics, pages 201–236. Elsevier, 1979.

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3. **Learning-au ... werful and pra ...**
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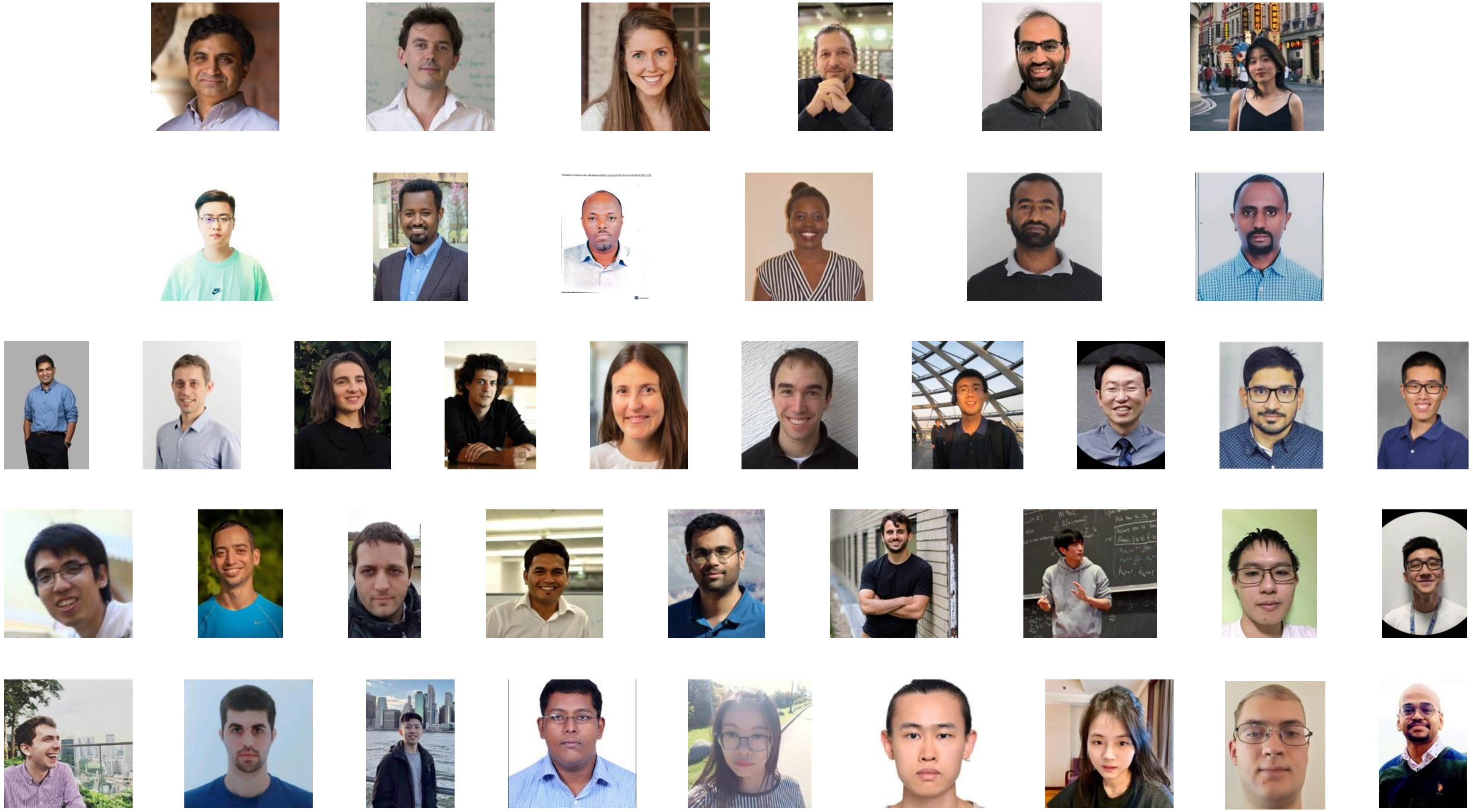
"Often, the most important step is making the right notion, defining the right notion. Once you have the right notion, you know, the rest of the theory, theorems, proofs, [and] constructions follow."

Avi Wigderson,
2023 Turing award winner

Avi Wigderson, Turing Award Lecture: "Alan Turing: A TCS Role Model", 2024. Available at https://www.youtube.com/live/f2NiGO8zC1c?t=3211s

Special thanks to all my amazing collaborators on my research journey!



Back up slides

Sample of questionnaire questions

Each person is associated with an anonymous ID

Individual characteristics

The first thing I would like to do is ask you some general background questions. (race, education, and living situation)

A. DEMOGRAPHICS

1. Record sex as observed: Male 1
Female 2

2a. What is your date of birth, month, day, and year?
____ ____ / ____ ____ / ____ ____
month day year

b. How old are you? ________ years **(Check age with date of birth)**

3. Record racial or ethnic background:

African American, Black Non-Hispanic 1
Black, Hispanic 2
White, Non-Hispanic 3
White, Hispanic 4
Asian/Asian American 5
Other (describe): 6
____

4. What is the highest grade of school you've completed?

Grade (1-11) .... __ __
High School Diploma 12
GED 13
Some College 14
College 15
Graduate School 16
Technical/vocational 17

5a. What is your marital status?

Married 1
Separated 2
Divorced 3
Widowed 4
Single **[Ask 5b]** 5

5b. **[If "Single"]**, Have you ever been married?

No 0
Yes 1

Personal beliefs

H. AIDS INFORMATION

Thank you for all of your answers about HIV and your health status.
Now I'm going to ask you some questions about opinions about HIV/AIDS.

*** If HIV Positive, skip question(s)**

		Strongly Agree	**Kind of Agree**	**Kind of Disagree**	**Strongly Disagree**
1.	Injection drug users are at risk for getting AIDS.	1	2	3	4
2.	Cleaning works with soap and water kills the AIDS virus.	1	2	3	4
3.	Cooking the drugs will kill the HIV virus.	1	2	3	4
4.	Natural skin condoms are protective against HIV.	1	2	3	4
5.	People are likely to get AIDS if they bleach their works before sharing them.	1	2	3	4
6.	The AIDS virus was started by an experiment that went wrong.	1	2	3	4
7.	If it's meant to be I will get the AIDS virus.	1	2	3	4
8.	You've got to die of something someday - it might as well be AIDS!	1	2	3	4
9.	AIDS was created to kill blacks and poor folks.	1	2	3	4
10.	Most research projects don't help the community.	1	2	3	4

Sexual and drug-injecting partners

94 **Please tell me the FIRST NAME and the FIRST INITIAL OF THE LAST NAME of all the different people with whom you have had close personal contact, had sex, taken drugs, or shared needles with during the past <u>6 months</u> (give respondent a specific date [a benchmark] to go back to). Start with current people and work your way back.**

ENTER ALL NAMES ONTO THE FIRST, DOUBLE-SIDED MATRIX . AFTER THE NAMES ARE WRITTEN DOWN:

Now I would like to ask you a few questions about these people.

95 **For each person, please tell me how long you have known them.**

96 **Which of these people are related to you, by either blood or marriage?**

97 **Which of these people are neighbors (live within easy walking distance)?**

98 **Which of these people are coworkers (people you work with)?**

99 **How would you describe the people you've named who are neither relations, neighbors, or coworkers?**

100 **How often have you seen each of these people in person in the last 3 months?**
Codes for Frequency of Contact:
0 = not at all
1 = once or twice
2 = three to six times
3 = at least a couple of times a month
4 = weekly
5 = daily
97 = not asked
98 = refused
99 = not known

101 **Please tell me the age of each person you have listed.**

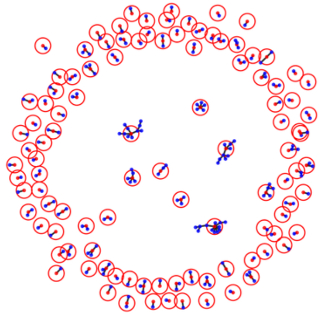
102 **To which ethnic group or race does each of these individuals belong?**

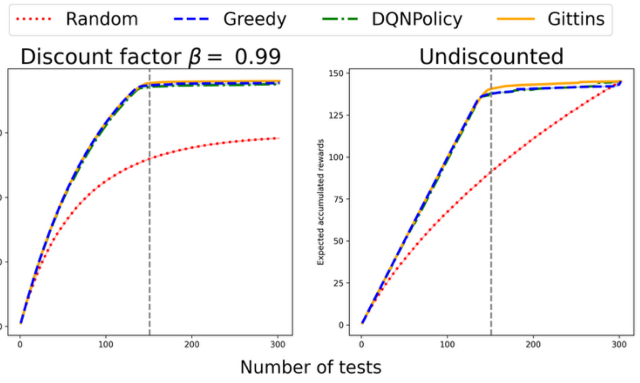
103 **What is the gender of each person you have listed.**

https://www.icpsr.umich.edu/web/ICPSR/studies/22140/summary

Empirical evaluation on other disease graphs

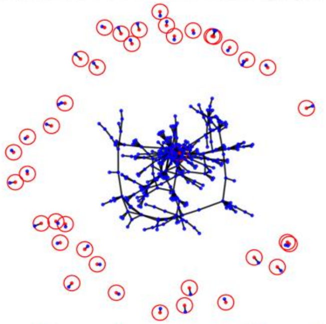
Chlamydia sex interaction graph

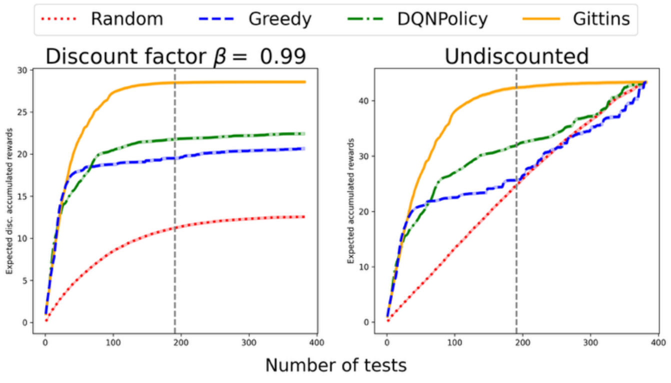




Frontier roots are circled in red

Hepatitis sex interaction graph





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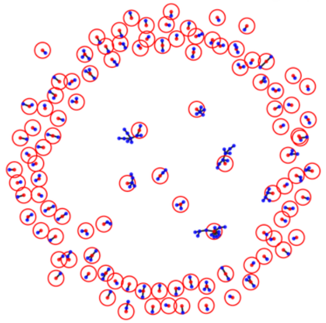
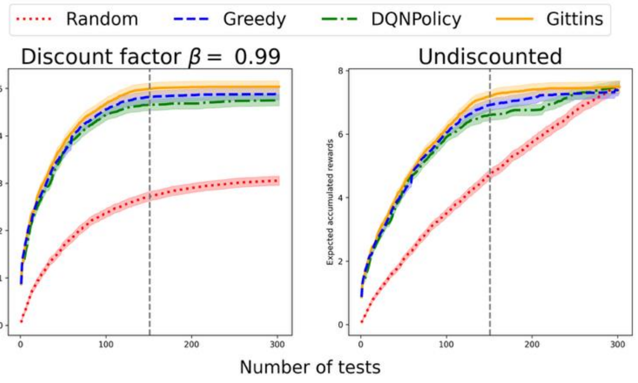
Gonorrhea sex interaction graph





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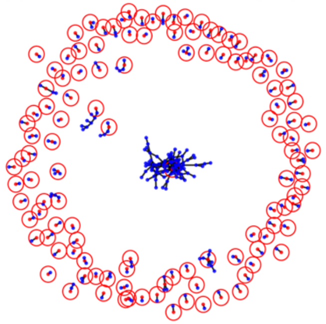
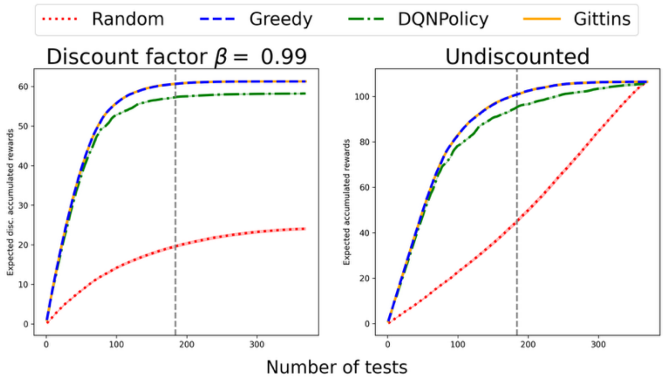
Syphilis sex interaction graph





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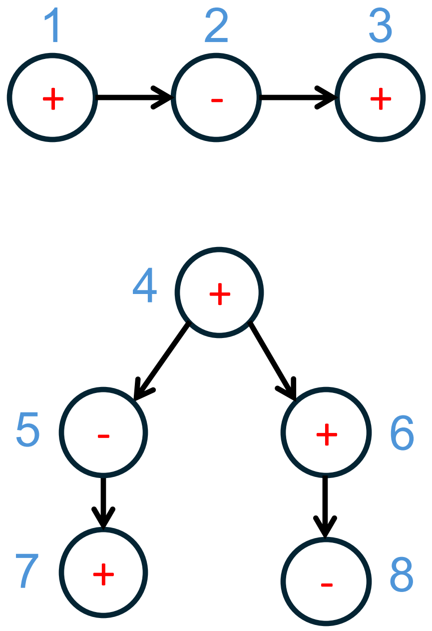
Use discount factor β to favor early discovery of positive cases

Reward metric

- Use discount factor $\beta \in (0, 1)$ to favor early discovery of positive cases
  - Early detection → early intervention; Also, we may have sudden budget cuts
- Example reward on this sequence and realizations
  - $\beta^0 * 1 + \beta^1 * 0 + \beta^2 * 1 + \beta^3 * 1 + \beta^4 * 0 + \beta^5 * 1 + \beta^6 * 1 + \beta^7 * 0$
- Optimal testing sequence may involve switching between trees



Ultimately, what we care about is maximizing the number of positive cases discovered (i.e., undiscounted reward) for any fixed limited number of tests

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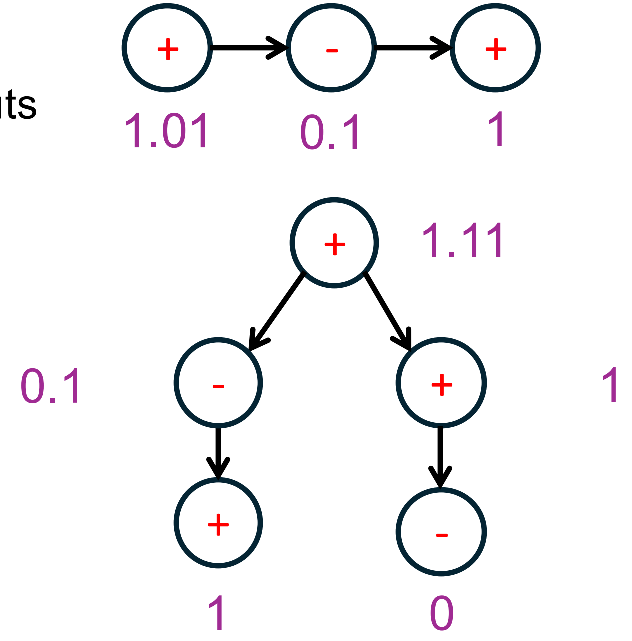

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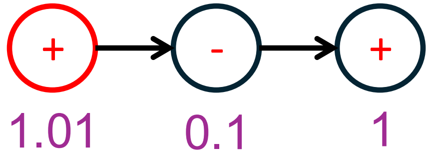

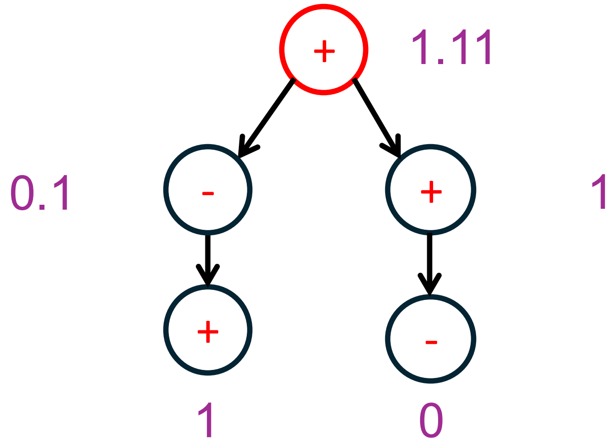

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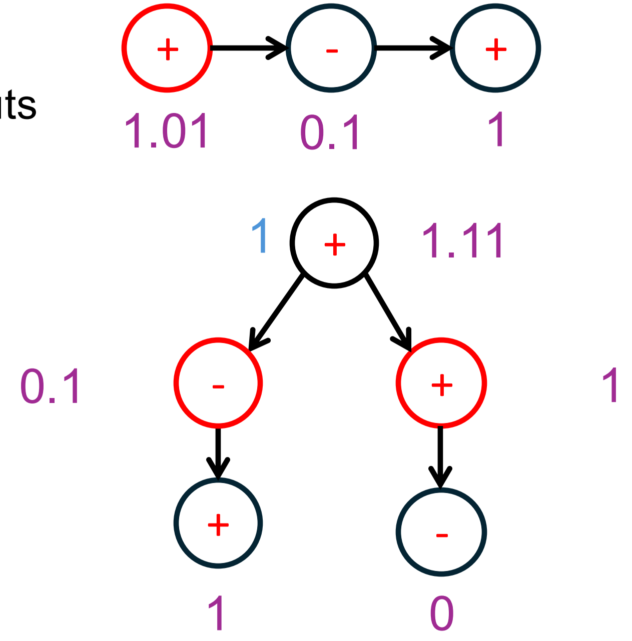

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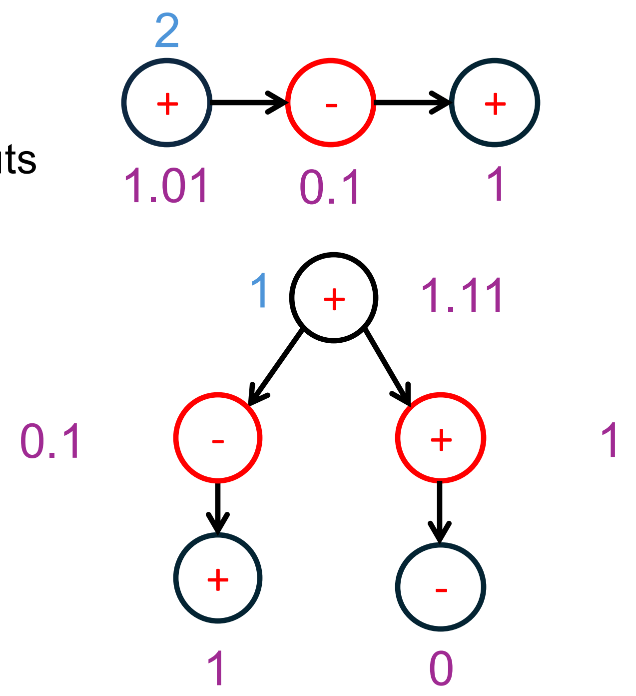
- Example reward on this sequence and realizations
  - $\beta^0 * 1 + \beta^1 * 0 + \beta^2 * 1 + \beta^3 * 1 + \beta^4 * 0 + \beta^5 * 1 + \beta^6 * 1 + \beta^7 * 0$
- Optimal testing sequence may involve switching between trees
- Another sequence for this example
  - Let's assign some scores as a proxy for the *relative* value of the subtree
  - Greedily select individual with highest score in frontier
  - $\beta^0 * 1 + \beta^1 * 1 + \beta^2 * \square + \beta^3 * \square + \beta^4 * \square + \beta^5 * \square + \beta^6 * \square + \beta^7 * \square$



Ultimately, what we care about is maximizing the number of positive cases discovered (i.e., undiscounted reward) for any fixed limited number of tests

Use discount factor β to favor early discovery of positive cases

Reward metric

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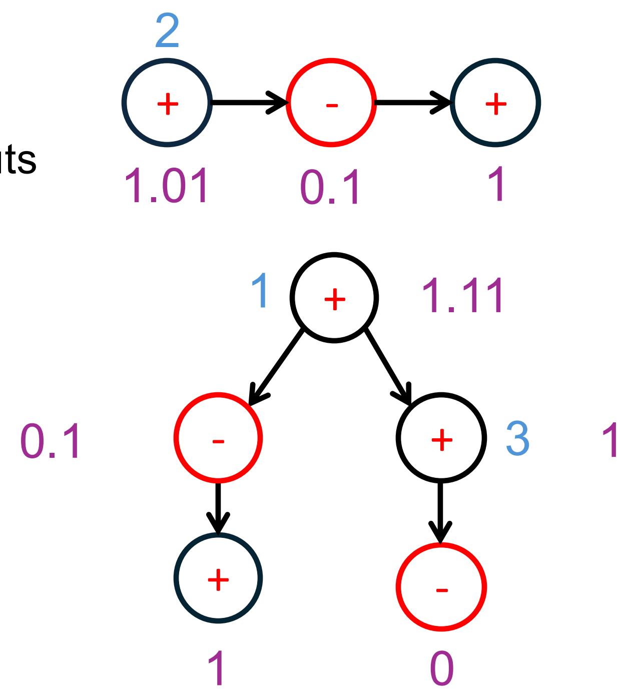

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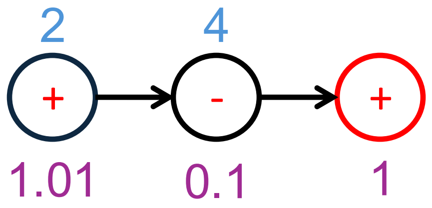

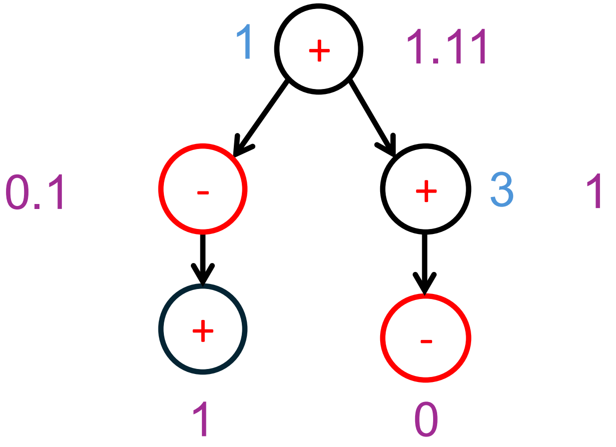

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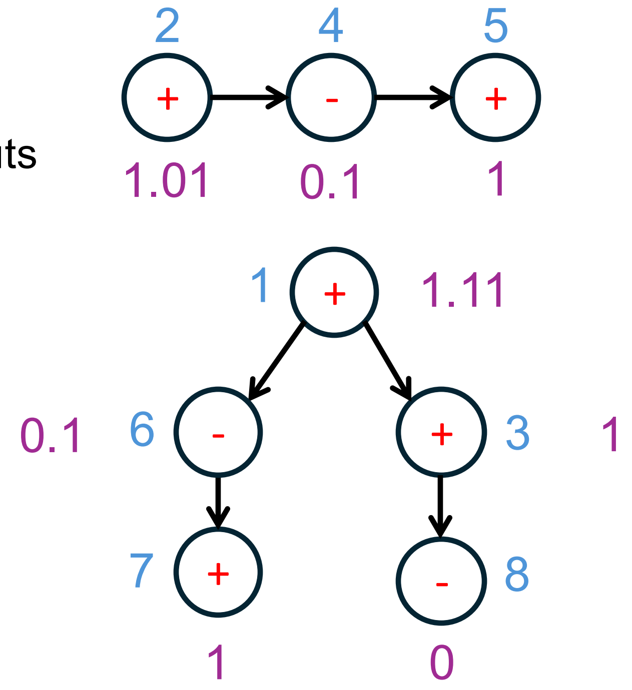

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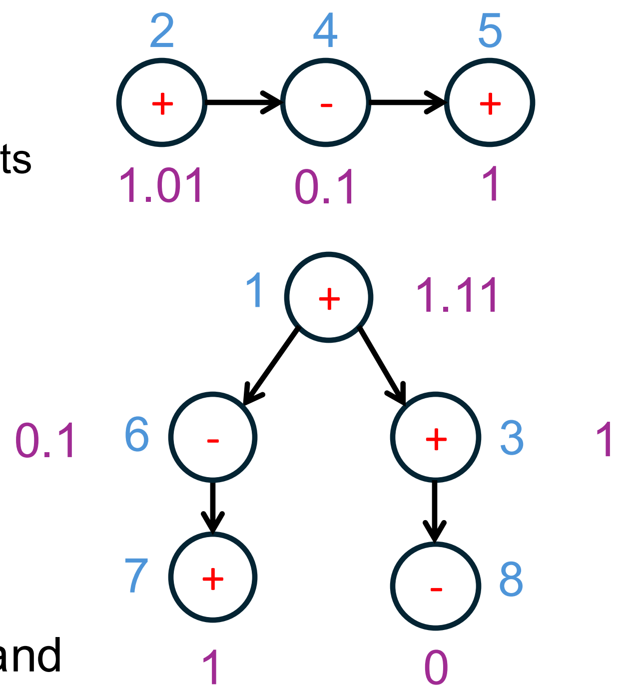
- These scores are also known as Gittin indices* in the bandit literature and such a procedure is known to produce the optimal sequence



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* The actual Gittins indices are not the same exact score numbers shown in this example

Gittins indices are optimal on rooted forests

Gittins index from earlier generalizes to rooted forests and probabilistic labels

- Statuses can be a function of the revealed parent's status, based on a known distribution $\mathcal{P}$
- Now, we pre-compute a score for node X based on every possible parent status, then use the score that corresponds to revealed parent status when ranking node X in the frontier

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Rooted forests are essentially branching bandits; optimality is proven in [KO03]

To define the Gittins index, let us first define $\boxed{\text{two recursive functions } \phi \text{ and } \Phi}$ as per [KO03]. For any non-root node $X \in \mathbf{X}$, label $b \in \mathbf{\Sigma}$, and value $0 \leq m \leq \frac{\bar{r}}{1-\beta}$,

$$\phi_{X,b}(m) = \max\left\{m, \sum_{v \in \mathbf{\Sigma}} \mathcal{P}(X = v \mid \mathrm{Pa}(X) = b) \cdot \left[r(X,v) + \beta \cdot \Phi_{\mathrm{Ch}(X),v}(m)\right]\right\} \tag{1}$$

If X is the root, we define $\phi_{X,\emptyset}(m) = \max\left\{m, \sum_{v\in\mathbf{\Sigma}} \mathcal{P}(X = v) \cdot \left[r(X,v) + \beta \cdot \Phi_{\mathrm{Ch}(X),v}(m)\right]\right\}$.

For any subset of nodes $\mathbf{S} \in \mathbf{X}$, label $b \in \mathbf{\Sigma}$, and value $0 \leq m \leq \frac{\bar{r}}{1-\beta}$,

$$\Phi_{\mathbf{S},b}(m) = \begin{cases} \frac{\bar{r}}{1-\beta} - \int_m^{\frac{\bar{r}}{1-\beta}} \prod_{Y \in \mathbf{S}} \frac{\partial \phi_{Y,b}(k)}{\partial k}\, dk & \text{if } \mathbf{S} \neq \emptyset \\ m & \text{if } \mathbf{S} = \emptyset \end{cases} \tag{2}$$

Gittins index for node X when parent's realization is b $\longrightarrow$

$$g(X,b) = \min\left\{m \in \left[0, \frac{\bar{r}}{1-\beta}\right] : \phi_{X,b}(m) \geq m\right\}$$

$\longleftarrow$ This "min ≥" captures the intuition of "fair value"

[KO03] Godfrey Keller and Alison Oldale. Branching bandits: a sequential search process with correlated pay-offs. Journal of Economic Theory, 113(2):302–315, 2003.

Gittins indices are optimal on rooted forests

$\bar{r}$ is maximum instantaneous reward.
In earlier example, $\bar{r} = 1$

Ch(X) is set of children of node X

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In earlier example, the subscripts b don't matter because parent's realizations don't affect X's value

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In earlier example, the only non-zero term here is when v is exactly the realized +/- label

If X is the root, we define $\phi_{X,\emptyset}(m) = \max\left\{m, \sum_{v \in \boldsymbol{\Sigma}} \mathcal{P}(X = v) \cdot \left[r(X, v) + \beta \cdot \Phi_{\mathrm{Ch}(X),v}(m)\right]\right\}$.

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$$g(X, b) = \min\left\{m \in \left[0, \frac{\bar{r}}{1-\beta}\right] : \phi_{X,b}(m) \geq m\right\}$$

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- We provide the first efficient polynomial time dynamic programming (DP) method (with working Python code) in the 20 years since [KO03] for computing ϕ and Φ for *discrete labels*
  - Prove and then exploit the piecewise linearity of functions ϕ and Φ
  - Piecewise linearity enable efficient representation of these functions in the DP from the leaves
  - Number of pieces scale with the number of nodes and number of labels

$$\phi_{X,b}(m) = \max\left\{m, \sum_{v\in\mathbf{\Sigma}} \mathcal{P}(X = v \mid \mathrm{Pa}(X) = b) \cdot \left[r(X,v) + \beta \cdot \Phi_{\mathrm{Ch}(X),v}(m)\right]\right\}$$

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