Question 1.

Proof. Since $\mathbb{N}$ is closed under addition, $a - a = 0 \in \mathbb{N}$. Thus,

$$a + b = a + c \Rightarrow a - a + b = a - a + c \Rightarrow 0 + b = 0 + c \Rightarrow b = c.$$

$\square$

Question 2a.

Proof. Let us consider the definition of axiomatic multiplication:

$$a \cdot 0 = 0$$
$$a \cdot S(b) = a + (a \cdot b).$$

If we wish to perform $a \cdot 2$, then we must rewrite it as follows:

$$a \cdot S(1) = a + (a \cdot 1).$$

To obtain $a \cdot 1$, we must rewrite it in a similar fashion:

$$a \cdot S(0) = a + (a \cdot 0) = a.$$

Therefore, $a \cdot 2 = a + a$. $\square$

Question 2b.

Proof. Let us consider the statement $A(n)$ given by

$$\sum_{j=1}^{n} a = n \cdot a.$$

Since the theorem of associativity holds over $\mathbb{N}$, we can equivalently write $A(n)$ as follows:

$$\sum_{j=1}^{n} a = a \cdot n.$$

We will use mathematical induction to show that $A(n)$ is true for all $n \geq 1$. First, we will confirm that the base case $A(1)$ is true. Since

$$\sum_{j=1}^{1} a = a + (a \cdot 0) = a \cdot S(0) = a \cdot 1,$$

$A(1)$ is indeed true.

Next, we will perform the inductive step. Assume that $A(k)$ is true for some $k \geq 1$. Thus, we assume that

$$\sum_{j=1}^{k} a = a \cdot k.$$

We will use this to prove that $A(k+1)$ is true. Notice that

$$\sum_{j=1}^{k+1} a = \left(\sum_{j=1}^{k} a \right) + a = a \cdot k + a = a \cdot (k+1).$$

Thus, using our inductive assumption, we have proven that $A(k+1)$ is true. By induction, we know that the statement $A(n)$ is indeed true for all $n \geq 1$. $\square$

Question 3a.

Proof. We will show that $b \leq a$. Notice that $a = b \cdot c$ can be written as $a = b \cdot S(c-1) = b + (b \cdot (c-1))$. Since $\mathbb{N}$ is closed under multiplication, $b \cdot (c-1) \in \mathbb{N}$. Therefore, $b + (b \cdot (c-1)) \in \mathbb{N}$. Thus, because there exists some $c \in N$ such that $b + c = a$, we can conclude $b \leq a$. $\square$

Question 3b.

Proof. To show that $a \leq a$, we will prove two conditional statements.

First, let us consider the statement "If $a \leq a$, then there exists some $c \in \mathbb{N}$ such that $a + c = a$." Assume that $a \leq a$. Notice that if $c = 0$, then the equation $a + c = a + 0 = a$ is satisfied.

Next, let us consider the statement "If there exists some $c \in \mathbb{N}$ such that $a + c = a$, then $a \leq a$." Assume that there exists some $c \in \mathbb{N}$ satisfying $a + c = a$. Since c can only be 0 to satisfy the equation, $a = a$. Thus, we can conclude that $a \leq a$.

Since we have shown both statements to be true, the biconditional statement is proven. $\square$

Question 3c.

Proof. Assume that $a \leq b$ and $b \leq c$. Then there exists some $c_1, c_2 \in \mathbb{N}$ such that $a + c_1 = b$ and $b + c_2 = c$. Thus, by substituting $a + c_1$ for b, we can obtain $a + c_1 + c_2 = c$. Since $c_1 + c_2 \in \mathbb{N}$, there exists a $c_3 = c_1 + c_2$ such that $a + c_3 = c$. Because we have that $a + c_3 = c$, we can conclude that $a \leq c$. $\square$

Question 3d.

Proof. Assume that $a \leq b$ and $b \leq a$. Then there exists some $c_1, c_2 \in \mathbb{N}$ such that $a + c_1 = b$ and $b + c_2 = a$. Notice that if we substitute $a + c_1$ for b, we obtain $a + c_1 + c_2 = a$. For this equation to be satisfied, both c_1 and c_2 must be 0. Then, $a + 0 = b$ and $b + 0 = a$. Thus, we can conclude that $a = b$. $\square$