Question 1.

Proof. Consider the statement $A(n)$ given by

$$\sum_{j=0}^{n} r^j = \frac{1 - r^{n+1}}{1 - r}.$$

We will use mathematical induction to show that $A(n)$ is true for all $n \geq 0$. First, we will confirm that the base case $A(0)$ is true. Since

$$\sum_{j=0}^{0} r^j = r^0 = 1 = \frac{1 - r}{1 - r} = \frac{1 - r^{0+1}}{1 - r},$$

$A(0)$ is indeed true.

Next, we will perform the inductive step. Assume that $A(k)$ is true for some $k \geq 0$. Thus, we assume

$$\sum_{j=0}^{k} r^j = \frac{1 - r^{k+1}}{1 - r}.$$

We will use this to prove that $A(k+1)$ is true. Notice that

$$\sum_{j=0}^{k+1} r^j = \left(\sum_{j=0}^{k} r^j\right) + r^{k+1} = \frac{1 - r^{k+1}}{1 - r} + r^{k+1} = \frac{1 - r^{k+1} + (1 - r)r^{k+1}}{1 - r}$$

$$= \frac{1 - r^{k+1} + r^{k+1} - r^{k+2}}{1 - r} = \frac{1 - r^{k+2}}{1 - r}.$$

Thus, using our inductive assumption, we have proven that $A(k+1)$ is true. By induction, we know that the statement $A(n)$ is indeed true for all $n \geq 0$. □

Question 2a.

$$f'(x) = -\frac{1}{(1 - x)^2} \cdot -1 = \frac{1}{(1 - x)^2}$$

$$f''(x) = -2 \cdot \frac{1}{(1 - x)^3} \cdot -1 = \frac{2}{(1 - x)^3}$$

$$f'''(x) = -3 \cdot \frac{2}{(1 - x)^4} \cdot -1 = \frac{6}{(1 - x)^4}$$

$$f^{(4)}(x) = -4 \cdot \frac{6}{(1 - x)^5} \cdot -1 = \frac{24}{(1 - x)^5}$$

From our observations above, it can be conjectured that

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}.$$

Question 2b.

Proof. Consider the statement $A(n)$ given by

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}.$$

We will use mathematical induction to show that $A(n)$ is true for all $n \geq 0$. First, we will confirm that the base case $A(0)$ is true. Since

$$f^{(0)}(x) = f(x) = \frac{1}{1-x} = \frac{0!}{(1-x)^{0+1}},$$

$A(0)$ is indeed true.

Next, we will perform the inductive step. Assume that $A(k)$ is true for some $k \geq 0$. Thus, we assume

$$f^{(k)}(x) = \frac{k!}{(1-x)^{k+1}}.$$

We will use this to prove that $A(k+1)$ is true. Notice that

$$f^{(k+1)}(x) = \frac{\mathrm{d}}{\mathrm{d}x} f^{(k)}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{k!}{(1-x)^{k+1}}\right) = k!\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{(1-x)^{k+1}}\right)$$
$$= k! \cdot -(k+1) \cdot \frac{1}{(1-x)^{k+2}} \cdot -1 = \frac{(k+1)!}{(1-x)^{k+2}}.$$

Thus, using our inductive assumption, we have proven that $A(k+1)$ is true. By induction, we know that the statement $A(n)$ is indeed true for all $n \geq 0$. $\square$

Question 3.

Proof. Consider the statement $A(n)$ given by

$$(1+x)^n \geq 1 + nx, \text{ for } x > -1.$$

We will use mathematical induction to show that $A(n)$ is true for all $n \geq 1$. First, we will confirm the base case $A(1)$. Since

$$(1+x)^1 = 1 + x \geq 1 + 1 \cdot x \Rightarrow 1 + x \geq 1 + x \Rightarrow 1 \geq 1,$$

$A(1)$ is indeed true.

Next, we will perform the inductive step. Assume that $A(k)$ is true for some $k \geq 1$. Thus, we assume

$$(1+x)^k \geq 1 + kx.$$

We will use this to prove that $A(k+1)$ is true. Notice that

$$(1+x)^{k+1} = (1+x)^k(1+x) = (1+x)^k + (1+x)^k x.$$

Since $(1+x)^k \geq 1 + kx$, we can rewrite the above expression to obtain

$$\begin{aligned}(1+x)^k + (1+x)^k x = 1 + kx + (1+kx)x &= 1 + kx + x + kx^2 \\ &= 1 + (k+1)x + kx^2.\end{aligned}$$

Notice that $1 + (k+1)x + kx^2 \geq 1 + (k+1)x$, since $kx^2 \geq 0$. Thus, using our inductive assumption, we have proven that $A(k+1)$ is true. By induction, we know that the statement $A(n)$ is indeed true for all $n \geq 1$. $\square$