Chapter 1

Mathematical Logic

In its most basic form, Mathematics is the practice of assigning truth to well-defined statements. In this course, we will develop the skills to use known true statements to create newer, more complicated true statements. Thus, we begin our course with how to use logic to connect what we know to what we wish to know.

1.1 Logical Statements

The kinds of statements studied by mathematicians are called **logical statements**, which are defined as meaningful sentences that can be either true or false. It important to note that we need not know if a statement is actually true or false to be a statement; we need to only know that it *can be* either true (T) or false (F). In fact, the job of Mathematicians is to decide which mathematical statements are true or false using a proof.

Below are examples and non-examples of mathematical statements:

- "31 is a prime number" is a mathematical statement (which happens to be true).
- "$x \geq 1$" is a mathematical statement, which is either true or false, depending on the particular x we have in mind.
- "$x + 1$" is not a mathematical statement because it cannot be given a truth value. Notice that $x + 1$ is not even a complete sentence (as it lacks a verb).
- "Factor the polynomial $x^2 + 2x + 1$" is not a mathematical statement as it cannot be assigned a truth value.
- "There are infinitely many prime numbers" is a mathematical statement (which can be proven to be true)
- "There are infinitely many prime numbers p such that $p + 2$ is also prime" is also a mathematical statement, but its truth value is not yet know. Such statements, which are believed to be true, are called *conjectures*. This particular example is known as the "Twin Primes Conjecture" and has eluded mathematicians for almost two hundred years.

In what follows, we will begin to manipulate logical statements and their truth values to construct more complicated statements from simpler pieces.

1.1.1 Negations

If p is a mathematical statement, then its **negation**, written $\neg p$ is the exact opposite statement, which is usually obtained by placing a "not" in the grammatically appropriate place. For example, if p is the statement "20 is a prime number", then its negation $\neg p$ is the statement "20 is not a prime number." Notice that our original statement p has a truth value of False (F) and that $\neg p$ has a truth value of true (T) since 20 is not prime.

In general, negating a statement will always switch its truth value. Thus, if p is T, then $\neg p$ is F; if p is F, then $\neg p$ is T. Below are the negations of the previous mathematical statement examples:

- If p is "31 is a prime number", then $\neg p$ is "31 is not a prime number" or, equivalently, that "31 is composite". Notice that p has a truth value of T and $\neg p$ has a truth value of F.

- If p is "$x \geq 1$" then $\neg p$ is that "$x \ngeq 1$" or, equivalently, "$x < 1$". Again, if p was T for the x we had in mind, then $\neg p$ will be F for that same x.

- If p is "There are infinitely many primes," then $\neg p$ is "There are finitely many primes," a false statement.

- If p is the Twin Primes Conjecture "There are infinitely many prime numbers p such that $p + 2$ is also prime", then $\neg p$ is the statement "There are only finitely many primes p such that $p + 2$ is also prime.

1.1.2 Compound Statements

If we are given two statements p and q, then we can build a more complex statement called a **compound statement** by joining p and q together with the conjunctions "and" or "or". The truth value of these compound statements will depend, in a natural way, on the individual truth values of p and q.

The **conjunction** of the statements p and q is the compound statement read "p and q" and denoted by $p \wedge q$. As in usual English, the statement $p \wedge q$ will be true only when both p and q are true. In other words, if even one of them is false, then $p \wedge q$ is false as well. Below are some examples of conjunctions:

- If p is "31 is a prime number" and q is "31 is positive," then the conjunction $p \wedge q$ is "31 is a prime number and positive". Since both p and q are T, the conjunction $p \wedge q$ is also T.

- If p is "There are infinitely many primes" and q is "The sun is a planet", the $p \wedge q$ is "There are infinitely many primes, and the sun is a planet," which is F because q is F.

If we use "or" to join our two statement p and q, then we get the **disjunction**, read "p or q" and written $p \vee q$. It is a little easier for $p \vee q$ to be true: at least one of p or q (or both) need to be true. Thus, the only way that $p \vee q$ will be F is if both p and q are F. Notice that since $p \vee q$ is T when at least one of p or q is T, disjunction is more akin to what computer scientists call *inclusive or*. Below are some examples of disjunctions.

- If p is "25 is a perfect square," and q is "25 is divisible by 3", then $p \vee q$ is "25 is a perfect square or is divisible by 3", which is true since p is true (even though q is not).

- If p is "25 is negative" and q is "25 is prime", then $p \vee q$ is the disjunction "p is negative or prime," which is false since both p and q are false.

We can further construct complicated logical statements by including negations in our compound statements. For example, if we have a statement p and form its conjunction with $\neg p$, then we obtain $p \wedge \neg p$, which is always false, no matter the truth value of p. To see this, note that p and $\neg p$ always have opposite truth values. Thus, $p \wedge \neg p$ will never have both terms being the same truth value and thus will always be F. Such statements, which always have a truth value of F are called *contradictions*. If instead we join p and $\neg p$ via a disjunction, we see that the opposite happens: $p \vee \neg p$ is always T, no matter what the value of p is. This occurs, of course, because p and $\neg p$ have opposite truth values and thus one of them is always T. Such statement that are always true are called *tautologies*.

1.1.3 DeMorgan's Logic Laws

Above, we saw that we obtained interesting truth values when we included negations inside of a conjunction or a disjunction. If we instead take negations of compound statements, we can ask if they are, in some way, equal to some other logical statement that includes negations, conjunctions, and disjunctions.

Specifically, if p and q are logical statements, then consider the negation of their conjunction:

$$\neg(p \wedge q).$$

In other words, we wish to better understand what is meant by "not p and q". Notice, that

- for $\neg(p \wedge q)$ to be T, $p \wedge q$ must be F, and thus at least one of p or q must be F.
- for $\neg(p \wedge q)$ to be F, $p \wedge q$ must be T, and thus both p and q must be T.

Notice that, if we compare $\neg(p \wedge q)$ to $(\neg p) \vee (\neg q)$, then they both have the same truth values given an initial choice of T/F for both p and q. A pair of statements that carry the same truth values given identical inputs are called **logically equivalent** and are considered interchangeable in the field of logic. Thus, we have that $\neg(p \wedge q)$ is logically equivalent to $(\neg p) \vee (\neg q)$ and can state one of **DeMorgan's Logic Laws** symbolically as

$$\neg(p \wedge q) \equiv (\neg p) \vee (\neg q).$$

Thus, negating a conjunction produces the disjunction of two negations. In English, we say "Not p and q" is logically the same as "Not p or not q". For example,

- The negation of the conjunction "31 is prime and positive" is equivalent to the statement "31 is composite or 31 is non-positive".

If we instead choose to negate a disjunction, then another one of DeMorgan's Logic Laws tell us that the negation of a disjunction will give the conjunction of two negations. Symbolically, we have the second of **DeMorgan's Logic Laws**:

$$\neg(p \vee q) \equiv (\neg p) \wedge (\neg q).$$

Again, we see that negation swaps an 'or' for an 'and'. For example,

- The statement "21 is not a multiple of 4 or 5" is logically equivalent to "21 is not a multiple of 4, and 21 is not a multiple of 5."

1.1.4 "If-Then" Statements

Most of the statements that we will encounter in this proof course will come as "If-then" statements, called **conditional statements**. If p and q are statements, then the statement "If p, then q" is denoted by $p \Rightarrow q$. If-then statements work in the following way: for the entire statement "If p, then q" to be true, then whenever p is T, then q should be T; however, if p is F, then q can be either T or F and $p \Rightarrow q$ will still be T. For example,

· If p is "25 is a perfect square", and q is "25 is composite," then $p \Rightarrow q$ is T since p is T and also q is T.

· If p is "25 is a perfect square," and q is "25 is prime", then $p \Rightarrow q$ is F since p is T but q is F.

· If p is "25 is prime" and q is "25 is even", then $p \Rightarrow q$ is T since p is F. In fact, q can be either T or F. Since p is F, $p \Rightarrow q$ is always true.

1.1.5 Direct Proofs of If-Then Statements

The more interesting If-then statements usually include a variable. For example, consider a statement like "If $x > 1$, then $x^2 > 1$." It is precisely these kinds of statements that we will focus on in this course. For this statement, depending on the value of x, the p (hypothesis) statement "$x > 1$" will be either T of F. Similarly, the q (conclusion) statement "$x^2 > 1$" will either be T of F. For our $p \Rightarrow q$ statement to be T, we must show that whenever x satisfies $x > 1$, then it must also satisfy $x^2 > 1$. If $x > 1$ is not true, then the entire If-then statement is automatically true. Thus, we are only interested in situations where $x > 1$ is satisfied. So, to verify that the conditional statement "If $x > 1$, then $x^2 > 1$" is true, we must begin our mathematical argument (known as a **proof**) by assuming that $x > 1$. Then, we will use known algebraic laws to conclude that $x^2 > 1$.

Soon, we will see our first proof of the truth of a mathematical statement. Before we begin, since this course focuses on proof-writing, we will not just give our statement and prove it; instead, we will offer a *Discussion* section, where the proof-methodologies and thought processes are discussed. Then, after appropriately outlining and planning our arguments, we will implement the proof in the *Proof* section. You will notice that our proofs are written *in full, English sentences with proper syntax and grammar.*

Proposition. If $x > 1$, then $x^2 > 1$.

Discussion.

`We will assume:` $x > 1$

`We will show:` $x^2 > 1$.

`What we will do:` We will begin with the fact that $x > 1$ and use the well-known algebraic rule that if $a > b$ and $c > d$ and all numbers are positive, then $ac > bd$.

Proof. Assume $x > 1$. Since $x > 1 > 0$, then both x and 1 are positive. Thus, multiplying the inequality $x > 1$ with itself yields $x^2 > 1 \cdot 1 = 1$, which is equivalent to $x^2 > 1$. Thus, $x^2 > 1$.

□

Of course, our mathematical statements will be significantly more complicated than this example. The above is an illustration of what is expected of a proof and how we use the logic behind mathematical statements to develop proof techniques.

Now that our proof is complete, we can confidently (mathematically) say that anytime we have a number x such that $x > 1$, then we can automatically conclude that $x^2 > 1$. Notice that we indicate that our proof is complete by placing a $\square$ to the bottom right of the end of the proof.

1.2 Manipulating Conditional Statements

Given the importance of If-then statements, we wish to understand which other conditional statements are logically equivalent to $p \Rightarrow q$. To this end, we will investigate three different possibilities to the conditional statement $p \Rightarrow q$:

- **The Converse:** $q \Rightarrow p$
- **The Inverse:** $\neg p \Rightarrow \neg q$
- **The Contrapositive:** $\neg q \Rightarrow \neg p$

To investigate these different possibilities of conditional statements that might be logically equivalent to the original statement $p \Rightarrow q$, we will use the above example of "If $x > 1$, then $x^2 > 1$". In this example, p is "$x > 1$" and q is "$x^2 > 1$." Since we proved above that conditional statement is true, then it is true for any x. Thus, if inverse, converse, or contrapositive is not always true, then it cannot be logically equivalent to $p \Rightarrow q$.

1.2.1 The Converse and Inverse

For our conditional statement, the converse is $q \Rightarrow p$, which is read as "If $x^2 > 1$, then $x > 1$". However, we note that for $x = -3$, the hypothesis $x^2 > 1$ is T; however, $x > 1$ is F. Thus, the converse is not true and thus cannot be logically equivalent to the original $p \Rightarrow q$.

Next, we investigate the inverse statement. Before we begin, note that $\neg p$ is the statement "$x \leq 1$" and $\neg q$ is the statement "$x^2 \leq 1$." So, the inverse statement is $\neg p \Rightarrow \neg q$ and is read "If $x \leq 1$, then $x^2 \leq 1$." Again, if we choose $x = -3$ we see that the hypothesis is T, yet the conclusion is F. Thus, the converse cannot be logically equivalent to the original conditional statement.

1.2.2 The Contrapositive

We now investigate the *contrapositive* $\neg q \Rightarrow \neg p$ to our original conditional statement $p \Rightarrow q$. In our example, it would read "If $x^2 \leq 1$, then $x \leq 1$." Now, our previous counterexample of $x = -3$ no longer works since it does not make our hypothesis $x^2 \leq 1$ true. In fact, if we choose a variety of x values satisfying the hypothesis $x^2 \leq 1$, it is always true that $x \leq 1$. This leads us to believe that the original conditional statement is logically equivalent to the contrapositive. In fact, we have that

$$p \Rightarrow q \equiv \neg q \Rightarrow \neg p.$$

Thus, if we wish to prove the statement $p \Rightarrow q$, then we could instead prove $\neg q \Rightarrow \neg p$. Proving the contrapositive to be true would mean that the original conditional statement is true as well. The technique of proving the contrapositive is an extremely helpful one. Below are some examples of conditional statements and their (logically equivalent) contrapositives.

- $p \Rightarrow q$: "If a polygon is a square, it is also a rectangle"; its contrapositive $\neg p \Rightarrow \neg q$: "If a polygon is not a rectangle, then it is not a square". The two statements are true.

- $p \Rightarrow q$: "If n is a multiple of 6, then n is even." Its contrapositive $\neg q \Rightarrow \neg p$: "If n is odd, then it is not a multiple of 6.

- $p \Rightarrow q$: "If n is even, then $n + 3$ is odd." Its contrapositive $\neg q \Rightarrow \neg p$: "If $n + 3$ is even, then n is odd."

We investigate this next example of finding a contrapositive in a bit more depth. Consider the statement $p \Rightarrow q$ given by "If a is a rational number and b is an irrational number, then $a + b$ is an irrational number." In this case, p is the statement "a is a rational number and b is an irrational number", which is a conjunction; the statement q is "$a + b$ is an irrational number." To find the contrapositive, we must form $\neg q \Rightarrow \neg p$. We can easily see that $\neg q$ is given by "$a + b$ is a rational number." Computing $\neg p$ is a little more subtle. Since p is a conjunction, to negate it, we must use DeMorgan's Logic Laws:

$$\begin{gathered}\neg(\text{ } a \text{ is a rational number } \wedge \text{ } b \text{ is an irrational number }) \equiv \\ \neg(a \text{ is a rational number}) \vee \neg(\text{ } b \text{ is an irrational number }) \equiv \\ a \text{ is an irrational number or } b \text{ is a rational number.}\end{gathered}$$

Thus, the contrapositive to

$$\begin{gathered}\text{If } a \text{ is a rational number and } b \text{ is an irrational number,} \\ \text{then } a + b \text{ is an irrational number.}\end{gathered}$$

is logically equivalent to the contrapositive

$$\begin{gathered}\text{If } a + b \text{ is a rational number,} \\ \text{then } a \text{ is an irrational number or } b \text{ is a rational number.}\end{gathered}$$

1.2.3 Proof by Contrapositive

We will now prove a conditional statement $p \Rightarrow q$ by instead proving the logically equivalent contrapositive $\neg q \Rightarrow \neg p$.

Proposition. If $x^3 < 0$, then $x < 0$.

Discussion.
The original conditional statement is $p \Rightarrow q$, with p being "$x^3 < 0$" and q being "$x < 0$." This statement is rather difficult to work with because, if we were to prove it directly, we would begin with knowing a fact about x^3, not about x. In the contrapositive, however, this will change. So, the contrapositive can be constructed by first noting that $\neg p$ is "$x^3 \geq 0$" and that $\neg q$ is "$x \geq 0$". Thus, the contrapositive is the statement $\neg q \Rightarrow \neg p$ given by

$$\text{If } x \geq 0, \text{ then } x^3 \geq 0.$$

Thus, we will prove this easier-to-handle contrapositive; proving this will prove the original (and more complicated) conditional statement.

`We will assume:` $x \geq 0$

`We will show:` $x^3 \geq 0$.

`What we will do:` We will state at the beginning that we will prove instead the contrapositive. Directly proving the contrapositive is very straightforward.

Proof. To prove our statement, we will instead prove the contrapositive, which states "If $x \geq 0$, then $x^3 \geq 0$." Thus, we assume that $x \geq 0$. Since all terms are non-negative, we can multiply $x \geq 0$ by itself thrice to obtain $x \cdot x \cdot x \geq 0 \cdot 0 \cdot 0$, which is equivalent to $x^3 \geq 0$. Thus, since the contrapositive is true, the original statement "If $x^3 < 0$, then $x < 0$" is also true, as desired.

$\square$

1.2.4 "If and Only If" Statements

For the conditional statement $p \Rightarrow q$ to be true, we must show that whenever the hypothesis p is true, then the conclusion q must also be true. It is not necessary that whenever q is true that p must be true as well. If we do wish to form a logical statement where the truth value of p and q are identical, then we can construct the **biconditional statement** "p if and only if q", written $p \Leftrightarrow q$.

We should think about the biconditional statement $p \Leftrightarrow q$ as two conditional statements: $p \Rightarrow q$ and $q \Rightarrow p$. Thus, when proving a biconditional statement to be true, we must essentially prove that two conditional statements are true.

Consider the statement $p \Leftrightarrow q$ "n is even if and only if n^2 is even". We will prove that this statement is true by proving two statements:

· $p \Rightarrow q$: "If n is even, then n^2 is even."

· $q \Rightarrow p$: "If n^2 is even, then n is even."

Before we begin with the proof, we will take this time to remember that a number n is even if it can be written as $n = 2k$ for some whole number k. Furthermore, n is odd if it can be written as $2k + 1$ for some whole number k.

Proposition. n is even if and only if n^2 is even.

Discussion. This biconditional statement will be broken up as $p \Rightarrow q$: "If n is even, then n^2 is even"; and $q \Rightarrow p$: "If n^2 is even, then n is even." Notice that the first conditional $p \Rightarrow q$ is a fairly straightforward proof obtained easily by writing $n = 2k$ and showing that $n^2 = 2(\text{whole number})$.

The other conditional statement, $q \Rightarrow p$, is a little bit more complicated: "If n^2 is even, then n is even." Since the hypothesis of this statement gives information about n^2, we should consider the contrapositive, which would place the statement about n in the hypothesis. So, we instead consider the contrapositive of $q \Rightarrow p$, which is $\neg p \Rightarrow \neg q$: "If n is odd, then n^2 is odd." Thus, we will assume that $n = 2k + 1$ and use this to show that $n^2 = 2(\text{whole number}) + 1$.

Proof. To prove this biconditional statement, we will prove two conditional statements.

First, we consider the statement "If n is even, then n^2 is even." Since n is even, we can write it as $n = 2k$, where k is a whole number. Thus, $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. Since k is a whole number, $2k^2$ is also a whole number. Thus, we n^2 is even.

The remaining statement to be proven is "If n^2 is even, then n is even." We will instead prove its contrapositive "If n is odd, then n^2 is odd." Since n is odd, then we can write it as $n = 2k + 1$, where k is a whole number. Squaring, we get that

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Since k is a whole number, $2k^2 + 2k$ is a whole number and thus n^2 is odd. Since we have proven the contrapositive, the original statement "If n^2 is even, then n is even" is true as well.

Since we proved above the two conditional statements, the biconditional statement "n is even if and only if n^2 is even" is proven.

$\square$

1.3 Quantified Statements

For many statements involving a variable or parameter, it is useful to specify for how many such values we mean the statement to be true. We investigate the two major forms of quantified statements: those we wish to be valid for many values of the parameter and those we wish to be valid for at least one value.

1.3.1 "For all" statements

If we consider the statement $x^2 \geq 0$, it's clear that this is true for all real numbers. Thus, we can more precisely say that "For all $x \in \mathbb{R}$, $x^2 \geq 0$". This statement is called a **universally quantified** or a **for all statement**. This statement says that the statement $x^2 \geq 0$ is meant for all real numbers x (this is what $x \in \mathbb{R}$ means). The phrase "for all" is frequently notated by $\forall$.

In general, universally quantified statements are of the form

$$\forall x \, p(x),$$

where $p(x)$ is some statement involving the variable x. Thus, to prove the statement $\forall x \, p(x)$, we must prove that the statement $p(x)$ is true for all values of x.

1.3.2 "There exists" statements

If we instead consider a statement like "$3x - 1 = 0$", it is clear that this is only valid for one value of x. In these situations, where we seek the mere existence of a value for x satisfying some condition, we use an **existentially quantified** or **there exists statement**. Thus, we are more interested in statements like "There exists an $x \in \mathbb{R}$ such that $3x - 1 = 0$." This statement means that there is at least one value of x for which the statement "$3x - 1 = 0$" is true. The phrase "there exists" is frequently notated by $\exists$.

In general, existentially quantified statements are of the form

$$\exists x \, p(x),$$

where, once again, $p(x)$ is a statement involving the variable x. Thus, to prove the statement $\exists x \, p(x)$ is true, we need to find at least one value of x that satisfies $p(x)$. Many times, we go further by asking that there be *exactly* one value of x satisfying $p(x)$. In these kinds of statement, we ask for a *unique* value of x satisfying $p(x)$. These unique existence statement are notated by

$$\exists! x \, p(x).$$

For our particular example, the value $x = \frac{1}{3}$ makes the statement $3x - 1 = 0$ true, then it is true that "There exists an $x \in \mathbb{R}$ such that $3x - 1 = 0$." In fact, since $x = \frac{1}{3}$ is the only answer, then the unique existence statement "There exists a unique $x \in \mathbb{R}$ such that $3x - 1 = 0$" is also true.

1.3.3 Negating Quantified Statements

If we wish to know when a universally quantified statement like $\forall x\, p(x)$ is false, we should understand what it means to negate such statement. Since $\forall x\, p(x)$ is true when $p(x)$ is true for every single value of x, it will be false when there is at least one x so that $p(x)$ is not true. In terms of negations, we have

$$\neg(\forall x\, p(x)) \equiv \exists x\, \neg p(x).$$

Such an x is called a **counterexample** to our mathematical statement.

Similarly, we wish to know when an existentially quantified statement like $\exists x\, p(x)$ is false. For $\exists x\, p(x)$ to be true, there must be at least one value of x so that $p(x)$ is true; thus, to make the entire quantified statement false, we need to show that for all values of x, $p(x)$ is false (in other words, $\neg p(x)$ is true). In terms of negations, we have

$$\neg(\exists x p(x)) \equiv \forall x\, \neg p(x).$$

Below are some examples of negated quantified statements:

- Consider the existential statement "There exists an $x \in \mathbb{R}$ such that $x^2 = -1$". This is known to be a false statement; its negation is the universally quantified statement "For all $x \in \mathbb{R}$, $x^2 \neq -1$," which is a true statement.
- Consider the universally quantified statement "For all $x \in \mathbb{R}$, x is positive or x is negative." This statement is actually false because $x = 0$ is a real number for which "x is positive or x is negative" is false. The negation is the existentially quantified statement "There exists an $x \in \mathbb{R}$ such that x is non-positive and x is non-negative," which is a true statement (with $x = 0$ being the value that exists). Notice that we utilized DeMorgan's Logic Laws to negate the "or" statement.

1.3.4 Proving and Disproving Quantified Statements

The above negation rules give us a guide as to how to go about both proving and disproving both universally and existentially quantified statements.

- To prove a universally quantified statement $\forall x\, p(x)$, we simply take an arbitrary x and directly show that $p(x)$ is true. For example, if we wish to prove "For all whole numbers $n \geq 3$, $n^2 - 1$ is composite." we take an arbitrary n satisfying $n \geq 3$ and show that $n^2 - 1$ is not prime. The proof would go as follows:

  Proposition. For all whole numbers $n \geq 3$, $n^2 - 1$ is composite.

  Proof. Let $n \geq 3$ be any whole number. Notice that $n^2 - 1 = (n+1)(n-1)$. Thus, we have written $n^2 - 1$ as a product of two whole numbers $n + 1$ and $n - 1$. Since $n \geq 3$, then $n - 1 \geq 2$ and thus we have written $n^2 - 1$ as the product of two whole numbers greater or equal to 2. Thus, $n^2 - 1$ is composite.

  □

- To disprove a universally quantified statement $\forall x\, p(x)$, we must prove the negation

  $$\neg(\forall x\, p(x)) \equiv \exists x\, \neg p(x)$$

  to be true. In other words, we must find at least one x such that $p(x)$ is false. For example, if we wish to disprove the universally quantified statement "For all prime numbers n, n is odd," we need to only give one n that is prime and that is even.

Disprove. For all prime numbers n, n is odd.

Disproof. Notice that $n = 2$ is a prime number, but 2 is even. Thus, it is not true that all prime numbers are odd.

$\square$

- To prove an existentially quantified statement $\exists x\, p(x)$, we need only provide an x that makes the statement $p(x)$ true and, depending on how obvious it is, show that $p(x)$ is true. For example, if we wish to prove that "There exists a unique $x \in \mathbb{R}$ such that $3x - 1 = 0$," then our job is two-fold: find the x that works and show that it is the only one. To do this latter step, we assume that both x and y make $3x - 1 = 0$ true. Then, we use this assumption to conclude that $x = y$.

  Proposition. There exists a unique real number x such that $3x - 1 = 0$.

  Proof. Consider the real number $x = \frac{1}{3}$. Notice that $3\left(\frac{1}{3}\right) - 1 = 1 - 1 = 0$. Thus, there exists at least one real number x such that $3x - 1 = 0$. To show that there exists a unique such x, we will assume that x and y both satisfy $3x - 1 = 0$ and show that $x = y$. So, if $3x - 1 = 0$ and $3y - 1 = 0$, then $3x - 1 = 0 = 3y - 1$. So, $3x - 1 = 3y - 1$ and $3x = 3y$. Dividing by 3, we obtain that $x = y$. Thus, there exists a unique x such that $3x - 1 = 0$.

- To disprove an existentially quantified statement, $\exists x\, p(x)$, we need to show that its negation

  $$\neg(\exists x\, p(x)) \equiv \forall x\, \neg p(x)$$

  to be true. Thus, we must show that for every x, $\neg p(x)$ is true. For example, if we wish to disprove the existentially quantified statement "There exists a real number x such that $x^2 < 0$," we will show that for all x, the negation $x^2 \geq 0$ is true.

  Disprove. There exists a real number x such that $x^2 < 0$.

  Disproof. To disprove this statement, we will show that for all real numbers x, $x^2 \geq 0$. Notice that any real number falls into one of the following two cases: $x \geq 0$ or $x < 0$. We will prove $x^2 \geq 0$ for both these cases. In the first case, if $x \geq 0$, then multiplying the inequality $x \geq 0$ with itself will not change the sign, and we have that $x \cdot x \geq 0 \cdot 0$, which is equivalent to $x^2 \geq 0$. In the second case, if $x < 0$, then we can multiply through by -1 and switch the inequality to obtain $-x > 0$. Multiplying $-x > 0$ by itself will not change the sign, and we yield $(-x) \cdot (-x) > 0 \cdot 0$, which is equivalent to $x^2 > 0$ and thus $x^2 \geq 0$. Thus, the statement "there exists a real number x such that $x^2 < 0$" is false.