Chapter 4

Question 1. Let $a, b \in \mathbb{Z}$. Show that $4 \mid a^2 - b^2$ iff a and b have the same parity.

Proof. Assume that $4 \mid a^2 - b^2$, then $a^2 - b^2 = 4n$ for some $n \in \mathbb{Z}$, so (a+b)(a-b) = 4n. If a and b had opposite parity, then both (a+b) and (a-b) would be odd, and so (a+b)(a-b) would also be odd. However, 4n = 2(2n), and since $2n \in \mathbb{Z}$, 4n is even. We conclude that a and b must have the same parity.

Conversely, assume that a and b have the same parity.

Case 1: *a* and *b* are even. Then a = 2n and b = 2m for some $n, m \in \mathbb{Z}$. So $a^2 - b^2 = 4n^2 - 4m^2 = 4(n^2 - m^2)$. Since $n^2 - m^2 \in \mathbb{Z}$, $4 \mid a^2 - b^2$.

Case 2: Now assume both a and b are odd. Then a=2n+1 and b=2m+1 for some $n,m\in\mathbb{Z}$. So $a^2-b^2=(2n+1)^2-(2m+1)^2=(4n^2+4n+1)-(4m^2+4m+1)=4n^2+4n-4m^2-4m=4(n^2+n-m^2-m)$. Now $\mathbb Z$ is closed under multiplication and addition, so $n^2+n-m^2-m\in\mathbb Z$ if $n,m\in\mathbb Z$. This means we can write a^2-b^2 in the form 4k where $k\in\mathbb Z$, and we conclude that $4\mid a^2-b^2$.

Question 2. Let $a \in \mathbb{Z}$. Show that $3 \mid a$ iff $3 \mid a^2$.

Proof. Assume $3 \mid a$. Then a = 3n for some $n \in \mathbb{Z}$, so $a^2 = (3n)^2 = 9n^2 = 3(3n^2)$. Since $3n^2 \in \mathbb{Z}$, we have $3 \mid a^2$.

Conversely, assume that $3 \mid a^2$ and $3 \nmid a$ by contradiction. Then $a^2 = 3n$ for some $n \in \mathbb{Z}$, and either a = 3k + 1 or a = 3k + 2 for some $k \in \mathbb{Z}$.

Case 1: a = 3k + 1 for some $k \in \mathbb{Z}$. Then $a^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$. Since \mathbb{Z} is closed under multiplication and addition, we have $3k^2 + 2k \in \mathbb{Z}$. So a^2 is of the form 3m+1 for some $m \in \mathbb{Z}$, which is a contradiction since $3 \mid a^2$.

Case 2: a = 3k + 2 for some $k \in \mathbb{Z}$. It follows that $a^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 6k + 1) + 1$. Since $3k^2 + 6k + 1$ is an integer, a^2 is of the form 3m + 1 for some $m \in \mathbb{Z}$, a contradiction. We conclude that $3 \mid a$.

Question 3. Use the result from Question 2 to show that $\sqrt{3}$ is irrational.

Proof. We prove the desired result by contradiction. Assume $\sqrt{3}$ is rational, that is, it can be expressed in the form $\frac{m}{n}$ where $m,n\in\mathbb{Q},\ n\neq 0$, and m and n are in lowest terms. Then $m=\sqrt{3}n$ which implies $m^2=3n^2$, which subsequently implies that $3\mid m^2$. But then $3\mid m$ by our previous result, and so m=3k for some $k\in\mathbb{Z}$. Therefore $3n^2=(3k)^2=9k^2$, which implies that $n^2=3k^2$ which leads to the fact that $3\mid n^2$. However, this implies $3\mid n$ by our previous result, a contradiction since we assumed that m and n had no common divisors. Therefore $\sqrt{3}$ is irrational.

Question 4. Let $a, b \in \mathbb{R}$. Show that if a + b is rational, then a is irrational or b is irrational.

Proof. Assume by the contrapositive statement that a is rational and b is irrational. We want to show that a+b is not rational—we do this by contradiction.

Assume that $a+b\in\mathbb{Q},$ then $a+b=\frac{m}{n}$ for some $m,n\in\mathbb{Z}.$ We have $a=\frac{k}{q}$ by assumption. So

$$b = \frac{m}{n} - a = \frac{m}{n} - \frac{k}{q} = \frac{mq + nk}{nq}$$

which is rational since $\mathbb Q$ is a field. But b is irrational by assumption, a contradiction, and we are done.