

# TRANSITION TO MATHEMATICAL PROOFS

## CHAPTER 5 - COMPLEX NUMBERS ASSIGNMENT

INSTRUCTIONS: For the below questions, show all of your work. For the proofs, be sure that you

- (i) write a complete proof in full English sentences;
- (ii) if hand-writing, write legibly and clearly.

NOTE: Discussion sections are no longer required. You may, of course, include them in your assignments, as they may help the grader give more helpful feedback.

**Question 1.** Similar to how we obtained the double-angle formulae in the notes, use the Euler equation to show the two angle-sum formulae hold:

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha;$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta.$$

**Question 2.**

- (a) Show that  $|z| = \operatorname{Re}(z)$  if and only if  $z$  is a non-negative real number.
- (b) Show that  $(\bar{z})^2 = z^2$  if and only if  $z$  is purely real or purely imaginary (i.e., its real part is 0).

**Question 3.** The modulus of a complex number is, in many ways, a generalization of the absolute value of a real number. Here, we give another property of the modulus that the absolute value of a real number already enjoys.

If  $z, w \in \mathbb{C}$ , show that

$$|z \cdot w| = |z| \cdot |w|$$

in the following two ways:

- (a) By using the Cartesian form  $z = a + bi$  and  $w = c + di$  for the complex numbers  $z$  and  $w$ .
- (b) By using the polar form  $z = r_1 e^{i\theta_1}$  and  $w = r_2 e^{i\theta_2}$  for the complex numbers  $z$  and  $w$ .

**Question 4.** Below, we will prove a remarkable fact about real polynomials using complex numbers. For the parts below, let  $z = a + bi$  and  $w = c + di$  be complex numbers.

- (a) Show that  $\overline{z + w} = \bar{z} + \bar{w}$ .
- (b) Show that  $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$ .
- (c) Use (b) to show that  $\overline{z^n} = (\bar{z})^n$  for any natural number  $n \in \mathbb{N}$ .
- (d) Consider the following polynomial  $p(z)$  with *real coefficients*:

$$p(z) = \alpha_n z^n + \alpha_{n-1} z^{n-1} + \cdots + \alpha_1 z + \alpha_0,$$

where each  $\alpha_i$  is a real number. Show that if a complex number  $w$  is a root to the above polynomial with real coefficients, then its conjugate  $\bar{w}$  is also a root to the same polynomial. That is, use (a) - (c) to show that if  $p(w) = 0$ , then  $p(\bar{w}) = 0$ .