Question 1.

Discussion

- We will show that $\overline{S \cap T} \subset \overline{S} \cup \overline{T}$.
- We will show that $\overline{S} \cup \overline{T} \subset \overline{S \cap T}$.

Proof

Proof. To prove that $\overline{S \cap T} = \overline{S} \cup \overline{T}$, we will prove the following two subset inclusions: $\overline{S \cap T} \subset \overline{S} \cup \overline{T}$ and $\overline{S} \cup \overline{T} \subset \overline{S \cap T}$.

Let us consider the first inclusion. Assume that $x \in \overline{S \cap T}$. Thus, $x \notin S \cap T$. Then both $x \notin S$ and $x \notin T$ must be true. Since $x \notin S$, $x \in \overline{S}$. Likewise, since $x \notin T$, $x \in \overline{T}$. Therefore, $x \in \overline{S} \cup \overline{T}$. Thus, $\overline{S \cap T} \subset \overline{S} \cup \overline{T}$ is true.

Let us now consider the second inclusion. Assume that $x \in \overline{S} \cup \overline{T}$. Then $x \in \overline{S}$ or $x \in \overline{T}$. Since $x \in \overline{S}$ or $x \in \overline{T}$, it is not true that $x \in S$ or $x \in T$. By DeMorgan's Logic Laws, this is equivalent to $x \notin S$ and $x \notin T$ being true. Therefore, $x \in \overline{S \cap T}$ must also be true. Thus, $x \in \overline{S} \cup \overline{T} \subset \overline{S \cap T}$.

Since we have shown both inclusions to be true, then we can conclude that $\overline{S \cap T} = \overline{S} \cup \overline{T}$. □

Question 2.

Discussion

- We will show that $S \cap (T \cup R) \subset (S \cap T) \cup (S \cap R)$.
- We will show that $(S \cap T) \cup (S \cap R) \subset S \cap (T \cup R)$.

Proof

Proof. To prove that $S \cap (T \cup R) = (S \cap T) \cup (S \cap R)$, we will prove the following two subset inclusions: $S \cap (T \cup R) \subset (S \cap T) \cup (S \cap R)$ and $(S \cap T) \cup (S \cap R) \subset S \cap (T \cup R)$.

Let us consider the first inclusion. Assume that $x \in S \cap (T \cup R)$. Then $x \in S$ and $x \in T \cup R$. Since $x \in T \cup R$, $x \in T$ or $x \in R$. If $x \in T$ is true, then

$x \in S \cap T \subset (S \cap T) \cup (S \cap R)$. Similarly, if $x \in R$ is true, then $x \in S \cap R \subset (S \cap T) \cup (S \cap R)$. Thus, in either case $S \cap (T \cup R) \subset (S \cap T) \cup (S \cap R)$.

Next, let us consider the second inclusion. Assume that $x \in (S \cap T) \cup (S \cap R)$. Then $x \in S$ and $x \in T$ or $x \in S$ and $x \in R$. Therefore, $x \in T$ or $x \in R$. If $x \in T$ is true, then $x \in S \cap T \subset S \cap (T \cup R)$. If $x \in R$ is true, then $x \in S \cap R \subset S \cap (T \cup R)$. Thus, in either case $(S \cap T) \cup (S \cap R) \subset S \cap (T \cup R)$.

Since we have shown both inclusions to be true, then we can conclude that $S \cap (T \cup R) = (S \cap T) \cup (S \cap R)$. □

Question 3.

Discussion

- We will show that $A \subset B$ and $C \subset D \Rightarrow A \times C \subset B \times D$.

Proof

Proof. We will show that the statement "If $A \subset B$ and $C \subset D$, then $A \times C \subset B \times D$" is true. Assume that $A \subset B$ and $C \subset D$. Let there be an arbitrary (x, y) such that $(x, y) \in A \times C$. Then $x \in A$ and $x \in B$. Similarly, $y \in C$ and $y \in D$. Thus, $(x, y) \in B \times D$ is also true. Therefore, we can conclude that $A \times C \subset B \times D$. □

Question 4.

Discussion

- We will show that $A \times (B \cap C) \subset (A \times B) \cap (A \times C)$.
- We will show that $(A \times B) \cap (A \times C) \subset A \times (B \cap C)$.

Proof

Proof. To prove that $A \times (B \cap C) = (A \times B) \cap (A \times C)$, we will prove the following two subset inclusions:

$$A \times (B \cap C) \subset (A \times B) \cap (A \times C) \text{ and}$$

$$(A \times B) \cap (A \times C) \subset A \times (B \cap C).$$

Let us consider the first inclusion. Assume that $(x, y) \in A \times (B \cap C)$. Then $x \in A$. Likewise, $y \in B \cap C$, so $y \in B$ and $y \in C$. Since $x \in A$ and $y \in B$, $(x, y) \in A \times B$. Similarly, since $x \in A$ and $y \in C$, $(x, y) \in A \times C$. Because (x, y) is in both sets, we can conclude $(x, y) \in (A \times B) \cap (A \times C)$. Thus, $A \times (B \cap C) \subset (A \times B) \cap (A \times C)$.

Next, let us consider the second inclusion. Assume that $(x, y) \in (A \times B) \cap (A \times C)$. Thus, $(x, y) \in A \times B$ and $(x, y) \in A \times C$. Therefore, $x \in A$, $y \in B$, and $y \in C$. Since $y \in B$ and $y \in C$, $y \in B \cap C$. So, $(x, y) \in A \times (B \cap C)$. Thus, $(A \times B) \cap (A \times C) \subset A \times (B \cap C)$.

Since we have shown both inclusions to be true, we can conclude $A \times (B \cap C) = (A \times B) \cap (A \times C)$. $\square$

Question 5.

Discussion

- We will show that $A \times (B \cup C) \subset (A \times B) \cup (A \times C)$.
- We will show that $(A \times B) \cup (A \times C) \subset A \times (B \cup C)$.

Proof

Proof. To prove that $A \times (B \cup C) = (A \times B) \cup (A \times C)$, we will prove the following two subset inclusions:

$$A \times (B \cup C) \subset (A \times B) \cup (A \times C) \text{ and}$$

$$(A \times B) \cup (A \times C) \subset A \times (B \cup C).$$

Let us consider the first inclusion. Assume that $(x, y) \in A \times (B \cup C)$. Then, $x \in A$ and $y \in B$ or $y \in C$. If $y \in B$, then $(x, y) \in A \times B \subset (A \times B) \cup (A \times C)$. If $y \in C$, then $(x, y) \in A \times C \subset (A \times B) \cup (A \times C)$. Therefore, in either case $(x, y) \in (A \times B) \cup (A \times C)$, and thus $A \times (B \cup C) \subset (A \times B) \cup (A \times C)$.

Next, let us consider the second inclusion. Assume that $(x, y) \in (A \times B) \cup (A \times C)$. Then, $(x, y) \in A \times B$ or $(x, y) \in A \times C$. If $(x, y) \in A \times B$ is true, then $x \in A$ and $y \in B$. Therefore, $(x, y) \in A \times (B \cup C)$. If $(x, y) \in A \times C$, then $x \in A$ and $x \in C$. Therefore, $(x, y) \in A \times (B \cup C)$. Since in both cases $(x, y) \in A \times (B \cup C)$, we can conclude that $(A \times B) \cup (A \times C) \subset A \times (B \cup C)$.

Because we have shown both inclusions to be true, we can conclude $A \times (B \cup C) = (A \times B) \cup (A \times C)$. $\square$