Question 1.

Let $\alpha, \beta \in \mathbb{R}$. Notice that $e^{i(\alpha+\beta)} = e^{ia} \cdot e^{ib}$. Therefore, we can obtain the following:

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = (\cos\alpha + i\sin\alpha)\cdot(\cos\beta + i\sin\beta) \Rightarrow$$

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = \cos\alpha\cos\beta + i\cos\alpha\sin\beta + i\sin\alpha\cos\beta + i^2\sin\alpha\sin\beta \Rightarrow$$

$$\cos(\alpha+\beta) + i\sin(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta)$$

For two complex numbers to be equal to each other, their real and imaginary components must be equal. Thus, we obtain the following two angle-sum formulae:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha.$$

Question 2a.

Proof. To prove the biconditional statement, we will prove two conditional statements.

Let us first consider the statement "If $|z| = \mathrm{Re}(z)$, then z is a non-negative real number." Let $z = a + bi$, for $a, b \in \mathbb{R}$. Assume that $|z| = \mathrm{Re}(z)$. Then

$$\sqrt{a^2+b^2} = a \Rightarrow a^2 + b^2 = a^2 \Rightarrow b^2 = 0 \Rightarrow b = 0.$$

Therefore, $z \in \mathbb{R}$. Note that since $|z| \geq 0$, z must also be non-negative.

Let us next consider "If z is a non-negative real number, then $|z| = \mathrm{Re}(z)$." Assume that $z \in \mathbb{R}$ and non-negative. Then z can be written as $z = a + 0i$, for $a \in \mathbb{R}$ and $a \geq 0$. Notice that

$$|z| = \sqrt{a^2+b^2} = \sqrt{a^2+0^2} = a = \mathrm{Re}(z).$$

Thus, since we have proven both conditional statements, we can conclude that the biconditional statement must be true. □

Question 2b.

Proof. To prove this biconditional statement, we will prove two conditional statements.

Let us first show that "If $(\overline{z})^2 = z^2$, then z is purely real or purely imaginary." Let $z = a + bi$ for $a, b \in \mathbb{R}$. Assume that $(\overline{z})^2 = z^2$. Then $(a - bi)^2 = (a + bi)^2$. Expanding this, we obtain

$$a^2 - 2abi - b^2 = a^2 + 2abi - b^2 \Rightarrow -2abi = 2abi.$$

Since a or b must be 0 for $-2abi = 2abi$ to hold, there are three cases to consider:

Case 1. $a = 0$ **and** $b \neq 0$ **:** If $a = 0$ and $b \neq 0$, the condition $-2abi = 2abi$ is satisfied. Then $z = 0 + bi = bi$ and must be purely imaginary.

Case 2. $a \neq 0$ **and** $b = 0$**:** If $a \neq 0$ and $b = 0$, the condition $-2abi = 2abi$ is satisfied. Then $z = a + 0i = a$ and must be purely real.

Case 3. $a = 0$ **and** $b = 0$**:** If $a = 0$ and $b = 0$, the condition $-2abi = 2abi$ is satisfied. Then $z = 0 + 0i = 0$ and must be purely real.

Therefore, we have proven the statement "If $(\overline{z})^2 = z^2$, then z is purely real or purely imaginary."

Next, we will prove the statement "If z is purely real or purely imaginary, then $(\overline{z})^2 = z^2$". Assume that z is purely real or purely imaginary. Then there are two cases to consider:

Case 1. z **is purely real:** If z is purely real, then it can be written as $z = a+bi$ for $a, b \in \mathbb{R}$ and $b = 0$. Notice that $z = a + 0i = a$. Then, we can obtain

$$(\overline{z})^2 = (a - 0i)^2 = a^2 = (a + 0i)^2 = z^2.$$

Case 2. z **is purely imaginary:** If z is purely imaginary, then z can be written as $z = a + bi$ for $a, b \in \mathbb{R}$ and $a = 0, b \neq 0$. Notice that $z = 0 + bi = bi$. Then, we can obtain

$$(\overline{z})^2 = (0 - bi)^2 = -b^2 = (0 + bi)^2 = z^2.$$

Thus, in both cases, $(\overline{z})^2 = z^2$.

Since we have proven both conditional statements, we can conclude that the biconditional statement must be true. $\square$

Question 3a.

Let $z = a + bi$ and $w = c + di$ for $a, b, c, d \in \mathbb{R}$. Then,

$$|z \cdot w| = |(a + bi)(c + di)| = |ac + adi + bci + bdi^2| = |ac - bd + (ad + bc)i|$$

$$= \sqrt{(ac-bd)^2 + (ad+bc)^2} = \sqrt{a^2c^2 - 2abcd + b^2d^2 + a^2d^2 + 2abcd + b^2c^2}$$
$$= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2} = \sqrt{(a^2+b^2)(c^2+d^2)}.$$

Notice that

$$|z| \cdot |w| = \sqrt{a^2+b^2}\sqrt{c^2+d^2} = \sqrt{(a^2+b^2)(c^2+d^2)}.$$

Therefore, $|z \cdot w| = |z| \cdot |w|$.

Question 3b.

Let $z = r_1 e^{i\theta_1}$ and $w = r_2 e^{i\theta_2}$ for $r_1, r_2, \theta_1, \theta_2 \in \mathbb{R}$. Then,

$$|z \cdot w| = |r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2}| = |r_1 r_2 e^{i\theta_1 + i\theta_2}| = |r_1 r_2 e^{i(\theta_1+\theta_2)}|$$
$$= \sqrt{r_1^2 r_2^2 \cos^2(\theta_1+\theta_2) + r_1^2 r_2^2 \sin^2(\theta_1+\theta_2)}$$
$$= \sqrt{r_1^2 r_2^2 (\cos^2(\theta_1+\theta_2) + \sin^2(\theta_1+\theta_2))} = r_1 r_2.$$

Notice that

$$|z| \cdot |w| = |r_1 e^{i\theta_1}| \cdot |r_2 e^{i\theta_2}| = \sqrt{r_1^2 \cos^2\theta_1 + r_1^2 \sin^2\theta_1}\sqrt{r_2^2 \cos^2\theta_2 + r_2^2 \sin^2\theta_2}$$
$$\sqrt{r_1^2(\cos^2\theta_1 + \sin^2\theta_1)}\sqrt{r_2^2(\cos^2\theta_2 + \sin^2\theta_2)} = r_1 r_2.$$

Therefore, $|z \cdot w| = |z| \cdot |w|$.

Question 4a.

Proof. We will show that $\overline{z+w} = \overline{z} + \overline{w}$. Let $z = a + bi$ and $w = c + di$ for $a, b, c, d \in \mathbb{R}$. Then, $\overline{z+w}$ can be written as follows:

$$\overline{z+w} = \overline{a+bi+c+di} = \overline{a+c+(b+d)i} = a+c-(b+d)i.$$

Notice that

$$\overline{z} + \overline{w} = \overline{a+bi} + \overline{c+di} = a - bi + c - di = a + c - (b+d)i.$$

Thus, $\overline{z+w} = \overline{z} + \overline{w}$. $\square$

Question 4b.

Proof. We will show that $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$. Let $z = a + bi$ and $w = c + di$ for $a, b, c, d \in \mathbb{R}$. Then $\overline{z \cdot w}$ can be written as follows:

$$\overline{z \cdot w} = \overline{(a+bi)(c+di)} = \overline{ac + adi + bci + bdi^2} = \overline{ac - bd + (ad+bc)i}$$

$$= ac - bd - (ad+bc)i.$$

Notice that

$$\overline{z} \cdot \overline{w} = \overline{a+bi} \cdot \overline{c+di} = (a-bi)(c-di) = ac - adi - bci + bdi^2$$

$$= ac - bd - (ad+bc)i.$$

Thus, $\overline{z \cdot w} = \overline{z} \cdot \overline{w}$. □

Question 4c.

Proof. We will show that $\overline{z^n} = (\overline{z})^n$ for $n \in \mathbb{N}$. Notice that $\overline{z^n}$ can be written as $\overline{z^n} = \overline{z \cdot z^{n-1}}$. Similarly, $(\overline{z})^n$ can be written as $(\overline{z})^n = \overline{z} \cdot \overline{z^{n-1}}$. Since $\overline{a \cdot b} = \overline{a} \cdot \overline{b}$, $\overline{z \cdot z^{n-1}} = \overline{z} \cdot \overline{z^{n-1}} \Rightarrow \overline{z^n} = (\overline{z})^n$. □

Question 4d.

Proof. Given the polynomial $p(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ with real coefficients, we will prove the statement "If $p(w) = 0$, then $p(\overline{w}) = 0$ for $w \in \mathbb{C}$." Assume that $p(w) = 0$. Notice that

$$p(\overline{w}) = a_n (\overline{w})^n + a_{n-1} (\overline{w})^{n-1} + ... + a_1 \overline{w} + a_0$$

$$= a_n \overline{w^n} + a_{n-1} \overline{w^{n-1}} + ... + a_1 \overline{w} + a_0$$
$$= \overline{a_n w^n} + \overline{a_{n-1} w^{n-1}} + ... + \overline{a_1 w} + \overline{a_0}$$
$$= \overline{a_n w^n + a_{n-1} w^{n-1} + ... + a_1 w + a_0}.$$

Since $p(w) = a_n w^n + a_{n-1} w^{n-1} + ... + a_1 w + a_0 = 0$, $p(\overline{w})$ can be written as

$$p(\overline{w}) = \overline{a_n w^n + a_{n-1} w^{n-1} + ... + a_1 w + a_0} = \overline{0 + 0i} = 0 - 0i = 0.$$

Thus, if $p(w) = 0$, then $p(\overline{w}) = 0$ for $w \in \mathbb{C}$. □