Question 1a.

Discussion

- We will show that f is injective.
- We will show that f is surjective.

Proof

Proof. To show that f is bijective, we must show that it is both injective and surjective.

To prove that f is injective, let us first assume $f(s_1) = f(s_2)$. Then, the statement can be rewritten as follows: $ms_1 + b = ms_2 + b \Rightarrow ms_1 = ms_2 \Rightarrow s_1 = s_2$. Thus, f is injective.

Next, we will prove that f is surjective. Let t be an element in the co-domain of f. Assume that there exists a pre-image in the domain $s \in \mathbb{R}$. Then s must satisfy the equation $f(s) = ms + b = t$. Notice that if $s = \frac{-b+t}{m}$, the equation holds. Thus, every $t \in \mathbb{R}$ has the pre-image $\frac{-b+t}{m} \in \mathbb{R}$.

Since we have shown f to be both injective and surjective, we can conclude that f is bijective. □

Question 1b.

Since f is a bijection, it is invertible. The inverse $f^{-1}(x) = \frac{1}{m}(x - b)$. We can confirm this by noting that

$$f^{-1}(f(x)) = \frac{1}{m}(mx + b - b) = \frac{1}{m}(mx) = x.$$

Question 2.

Discussion

- We will show that f is injective.
- We will show that f is surjective.

Proof

Proof. To show that f is bijective, we must show that it is both injective and surjective.

First, we will prove that f is injective. Assume that $f(s_1) = f(s_2)$. Notice that the statement can be rewritten as follows:

$$\frac{\gamma s_1 + 1}{s_1 + \rho} = \frac{\gamma s_2 + 1}{s_2 + \rho} \Rightarrow (\gamma s_1 + 1)(s_2 + \rho) = (\gamma s_2 + 1)(s_1 + \rho) \Rightarrow$$

$$\gamma s_1 s_2 + \gamma s_1 p + s_2 + \rho = \gamma s_1 s_2 + \gamma s_2 \rho + s_1 + \rho \Rightarrow \gamma s_1 \rho + s_2 = \gamma s_2 \rho + s_1 \Rightarrow$$

$$s_2 - \gamma s_2 \rho = s_1 - \gamma s_1 \rho \Rightarrow s_2(1 - \gamma\rho) = s_1(1 - \gamma\rho) \Rightarrow s_2 = s_1.$$

Since when $f(s_1) = f(s_2)$, $s_1 = s_2$, we can conclude that f is injective.

Next, we will prove that f is bijective. Let $t \in \mathbb{R} - \{\gamma\}$ be an element in the co-domain of f. Assume that there exists a pre-image in the domain $s \in \mathbb{R} - \{-\rho\}$. Then s must satisfy the equation $f(s) = \frac{\gamma s+1}{s+\rho} = t$. To obtain the solution, we will solve for s:

$$\frac{\gamma s + 1}{s + \rho} = t \Rightarrow \gamma s + 1 = t(s + \rho) \Rightarrow \gamma s + 1 = st + \rho t \Rightarrow \gamma s - st = \rho t - 1 \Rightarrow$$

$$s(\gamma - t) = \rho t - 1 \Rightarrow s = \frac{\rho t - 1}{\gamma - t}.$$

Thus, every $t \in \mathbb{R} - \{\gamma\}$ has the pre-image $\frac{\rho t-1}{\gamma-t} \in \mathbb{R} - \{-\rho\}$.

Since we have proven that f is both injective and surjective, we can conclude that f is a bijection. □

Question 3.

Discussion

- **What we know:** Since $g \circ f$ is injective, when $g(f(s_1)) = g(f(s_2))$, $f(s_1) = f(s_2)$.
- We will show that if $g \circ f$ is injective, then f is injective.

Proof

Proof. Let us consider the statement "If $g \circ f$ is injective, then f is injective." Assume that $g \circ f$ is injective. Then, by the definition of injectivity, when

$g(f(s_1)) = g(f(s_2))$, $f(s_1) = f(s_2)$. Let us assume that f is not injective. Therefore, when $f(s_1) = f(s_2)$, $s_1 \neq s_2$. However, notice that if $s_1 \neq s_2$ and $f(s_1) = f(s_2)$, then $g \circ f$ is not injective — a contradiction. Thus, f must be injective. $\square$

Question 4a.

Discussion

- **What we know:** Let $a \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = a$. Then
$$\int_0^1 f(x)dx = \int_0^1 adx = ax\Big|_0^1 = a - 0 = a.$$
- We will show that φ is surjective.

Proof

Proof. We will show that φ is surjective. Let $a \in \mathbb{R}$ be an element in the co-domain of φ. Assume that there exists a pre-image in the domain $f \in C([0,1])$. Then f must satisfy the equation
$$\varphi(f) = \int_0^1 f(x)dx = a.$$
Notice that if $f(x) = a$, then the equation holds, as
$$\int_0^1 f(x)dx = \int_0^1 adx = ax\Big|_0^1 = a - 0 = a.$$
Also note that since $f(x) = a$ is continuous for all $x \in \mathbb{R}$, $f \in C([0,1])$. Thus, every $a \in \mathbb{R}$ has the pre-image $f(x) = a \in C([0,1])$. $\square$

Question 4b.

Discussion

- **What we know:** Consider the element $\frac{1}{3}$ in the co-domain of the function φ. If $f(x) = \frac{1}{3}$ and $g(x) = x^2$, then
$$\varphi(f) = \int_0^1 f(x)dx = \frac{1}{3}x\Big|_0^1 = \frac{1}{3} = \frac{1}{3}x^3\Big|_0^1 = \int_0^1 x^2dx = \varphi(g).$$
- We will show that the function φ is not injective.

Proof

Proof. To prove that φ is not injective, let us first assume the contrary, that φ is injective. Let $f(x) = \frac{1}{3}$ and $g(x) = x^2$ be functions in the domain of φ. Observe that

$$\varphi(f) = \int_0^1 f(x)dx = \frac{1}{3}x\Big|_0^1 = \frac{1}{3}.$$

Also notice that

$$\varphi(g) = \int_0^1 g(x)dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}.$$

Since $\varphi(f) = \varphi(g)$, but $f \neq g$, φ cannot be injective. $\square$