Question 1.

Proof. To show that the biconditional statement is true, we will prove two conditional statements.

We will first prove the statement "If $4 \mid a^2 - b^2$, then a and b are of the same parity" by proving its contrapositive "If a and b are not of the same parity, then $4 \nmid a^2 - b^2$." Let us consider two cases:

Case 1. a is even and b is odd: If a is even and b is odd, then they can be written as $a = 2k$ and $b = 2l + 1$ for some $k, l \in \mathbb{Z}$. Notice that $a^2 - b^2$ can be rewritten as

$$(2k)^2 - (2l + 1)^2 = 4k^2 - 4l^2 - 4l - 1 = 4(k^2 - l^2 - l) - 1.$$

Case 2. a is odd and b is even: If a is odd and b is even then they can be written as $a = 2k + 1$ and $b = 2l$ for some $k, l \in \mathbb{Z}$. Observe that $a^2 - b^2$ can be rewritten as

$$(2k + 1)^2 - (2l)^2 = 4k^2 + 4k + 1 - 4l^2 = 4(k^2 + k - l^2) + 1.$$

Thus, in both cases, $4 \nmid a^2 - b^2$.

Next, we will prove the statement "If a and b are of the same parity, then $4 \mid a^2 - b^2$." Let us consider two cases:

Case 1. a and b are both even: If a and b are both even, then they can be written as $a = 2k$ and $b = 2l$ for some $k, l \in \mathbb{Z}$. Note that $a^2 - b^2$ can be rewritten as

$$(2k)^2 - (2l)^2 = 4k^2 - 4l^2 = 4(k^2 - l^2).$$

Case 2. a and b are both odd: If a and b are both odd, then they can be written as $a = 2k + 1$ and $b = 2l + 1$ for some $k, l \in \mathbb{Z}$. Note that $a^2 - b^2$ can be rewritten as

$$(2k+1)^2 - (2l+1)^2 = 4k^2+4k+1-4l^2-4l-1 = 4k^2+4k-4l^2-4l = 4(k^2+k-l^2-l).$$

Thus, in both cases, $4 \mid a^2 - b^2$.

Since we have shown both conditional statements are true, we can conclude that the biconditional statement is true. $\square$

Question 2a.

Proof. To prove the biconditional statement, we must prove two conditional statements.

Let us first show that the statement "If $3 \mid a$, then $3 \mid a^2$" is true. Assume that $3 \mid a$. Then a can be written as $a = 3k$ for some $k \in \mathbb{Z}$. Notice then that a^2 can be written as $a^2 = 9k^2 = 3(3k^2)$. Thus, $3 \mid a^2$.

Next, we will prove the statement "If $3 \mid a^2$, then $3 \mid a$" by proving its contrapositive "If $3 \nmid a$, then $3 \nmid a^2$". Assume that $3 \nmid a$. Then a can be written as either $a = 3k + 1$ or $a = 3k + 2$ for some $k \in \mathbb{Z}$. Therefore, there are two cases to consider:

Case 1. $a = 3k + 1$**:** If $a = 3k + 1$, then $a^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$.

Case 2. $a = 3k + 2$**:** If $a = 3k + 2$, then $a^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k) + 4$.

Thus, in either case, $3 \nmid a^2$.

Since we have proven both conditional statements, the biconditional statement is proven. □

Question 2b.

Proof. To show that $\sqrt{3}$ is irrational, let us first assume the contrary, that $\sqrt{3}$ is rational. If $\sqrt{3}$ is rational, then it can be written as

$$\sqrt{3} = \frac{p}{q}$$

where p and q have no common divisors. Squaring both sides, we obtain

$$3 = \frac{p^2}{q^2}.$$

This is equivalent to $3q^2 = p^2$. Since $3 \mid p^2$, $3 \mid p$. Thus, p can be written as $p = 3k$ for some $k \in \mathbb{Z}$. Substituting, $3q^2 = (3k)^2 = 9k^2 \Rightarrow q^2 = 3k^2$. Notice that $3 \mid q^2$, thus $3 \mid q$. Since both p and q are divisible by 3, this contradicts the original assumption that p and q have no common divisors.

So, our initial assumption that $\sqrt{3}$ is rational must be false, thus we can conclude that $\sqrt{3}$ is irrational. □

Question 3.

Proof. We will show that the statement "If $a + b$ is rational, then a is irrational or b is rational" is true by proving its contrapositive "If a is rational and b is irrational, then $a + b$ is irrational." Let us first assume the contrary, that $a + b$ is rational. Notice that we can subtract a from $a + b$, yielding

$$a - a + b = 0 + b = b.$$

Since a is a rational number, it is closed under additive inverses, thus 0 is also a rational number. However, $0 + b = b$, an irrational number, which contradicts the original assumption. Therefore, if a is rational and b is irrational, then $a + b$ is irrational. $\square$