Enumerating Visible Tilings

Joint work with Michael Doerfler and Benton Tyler

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History of Tilings

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Girih tiles in modern day Iran (1453 C.E.)

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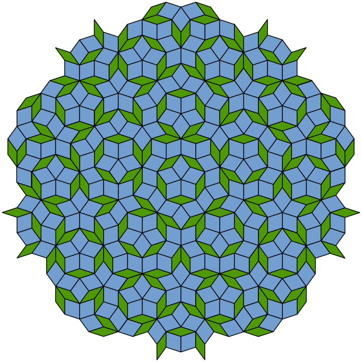

Penrose Tiling

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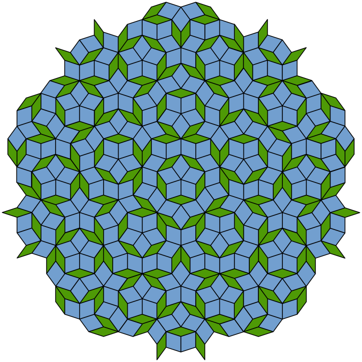
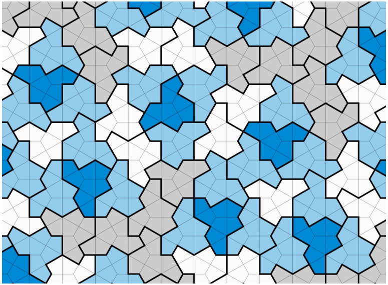


Penrose Tiling



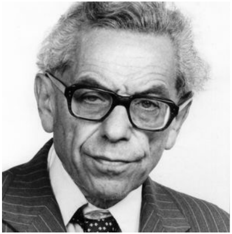
Einstein Tiling

Paul Erdős Revitalizes Interest

Erdős famously offered monetary awards for those who could prove his conjectures.

Erdős offered several conjectures in problems dealing with tilings.

His questions lead to subsequent interest in tiling problems.



Paul Erdős

What is a square tiling?

Definition

To **tile** a square is to completely fill it with square(s), in such a way that none of those squares have any interior points in common.

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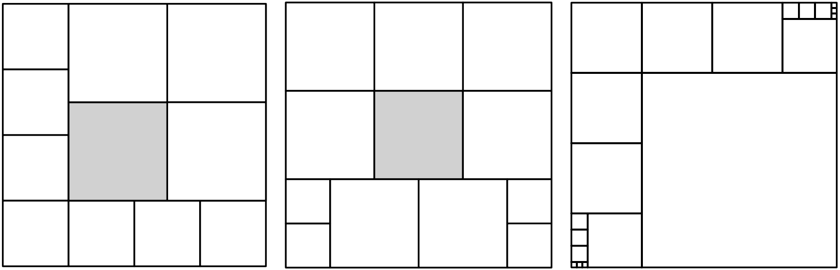

Tilings with $11, 12$, and 20 squares

When can we tilea square?

Theorem

A square can be tiled with k squares for any $k \in \mathbb{N}$ except when $k = 2, 3,$ or 5.

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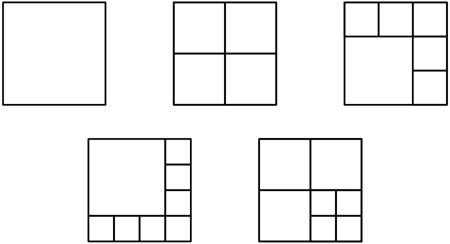

Tilings with $1, 4, 6, 8$, and 7 squares

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Why can we not tile a square with 2 or 3 squares?

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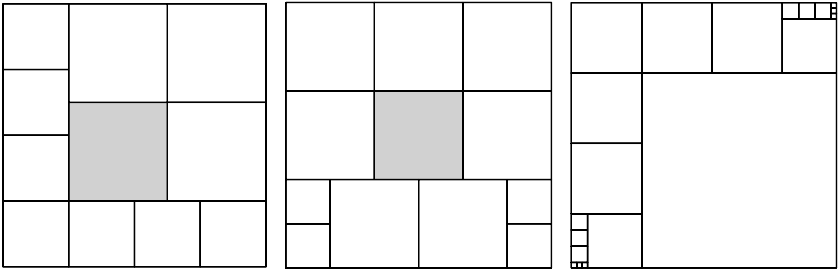

Three tilings, the first two of which are not visible. (Gray tiles are not visible)

Visible Tiling Theorem

Theorem

For any natural number $k \geq 6$, there exists a visible tiling with k squares.

The proof of this theorem involves 2 cases. When k is even, and when k is odd.

Case 1: k is even

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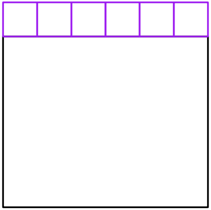
For any natural number $k \geq 6$, there exists a visible tiling with k squares.



Arrange a horizontal series of $\frac{k}{2}$, 1×1 squares.

Case 1: k is even

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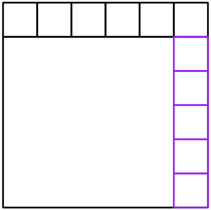

Then, vertically arrange a series of $\frac{k}{2} - 1, 1 \times 1$ squares under the rightmost square.

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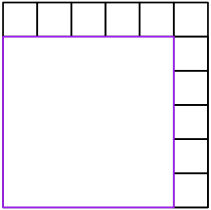
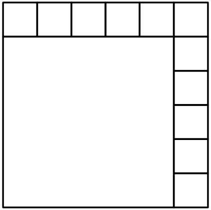


Fill in the empty space with one big square.

Case 2: k is odd

Theorem

For any natural number $k \geq 6$, there exists a visible tiling with k squares.



If k is odd, create a construction for $k - 3$.

Case 2: k is odd

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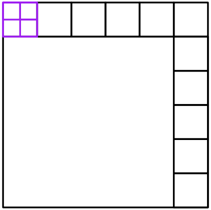

Then cut the top left square into 4 smaller squares.

Case 2: k is odd

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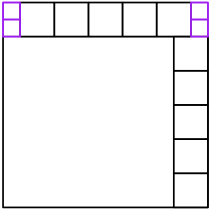

Now shift the two leftmost squares to the other side of the row.

Distinct Tilings

Definition

Tilings will be considered **non-distinct** if they are constructed of the same numbers of squares, and the quantity of all similarly sized squares is the same. Otherwise, the constructions will be **distinct**.

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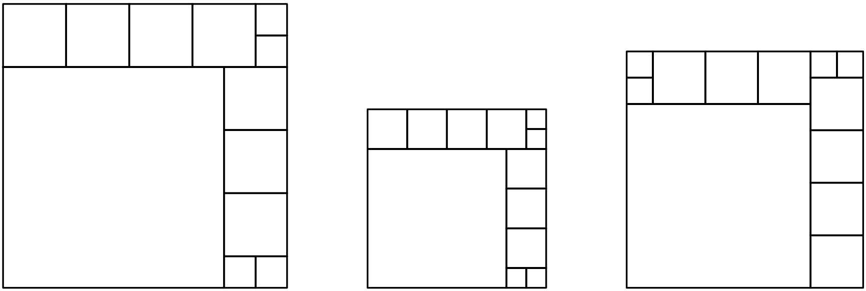
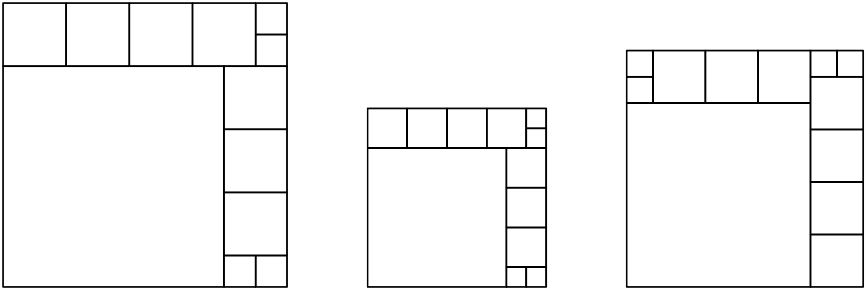


An example of **non-distinct** visible tilings

Enumerating Visible Tilings

While it is an interesting question to ask how many different ways visible tilings may be arranged (as in the figure below), this is not our primary goal here.



An example of **non-distinct** visible tilings

Enumerating Visible Tilings

How many distinct visible tilings of a square exist for each natural number k where k is the quantity of tiles?

Enumerating Visible Tilings

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When $k = 26$, for example:

- $\underbrace{12}_{r} = \underbrace{3 + 4 + 1 + 1 + 3}_{n \text{ values}}$

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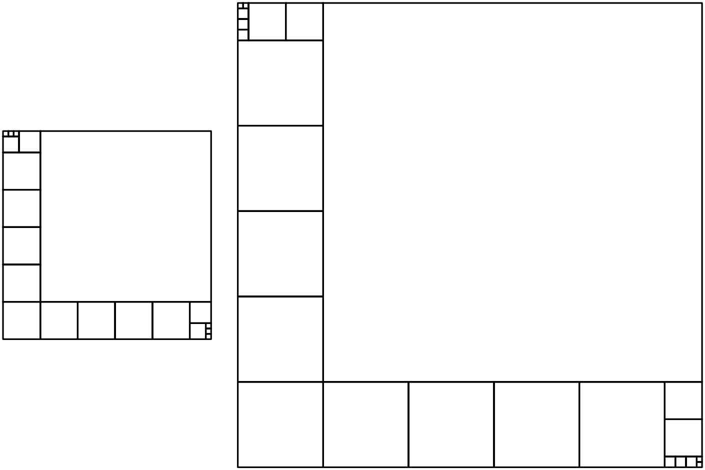
- $\underbrace{12}_{r} = \underbrace{3 + 4 + 1 + 1 + 3}_{n \text{ values}}$
- $12 = 4 + 2 + 2 + 1 + 2 + 1$
- $12 = 2 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$
- ...

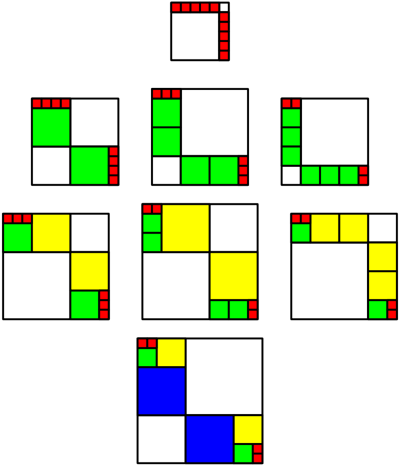
Sample Visible Tilings



$k = 20$, $r = 3 + 1 + 1 + 4$ (1 of 128 distinct tilings)
$k = 24$, $r = 2 + 3 + 2 + 4$ (1 of 512 distinct tilings)

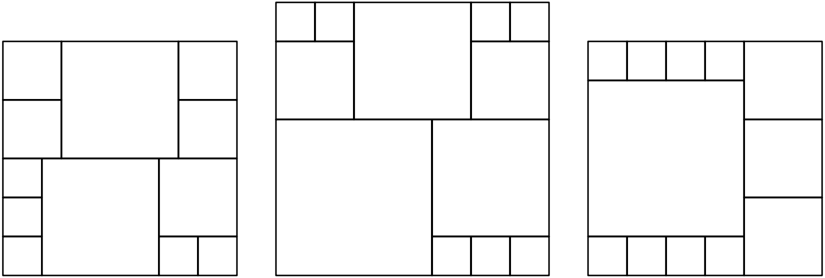
Enumerating Visible Tilings

The 8 visible tilings generated using our technique for $k = 12$



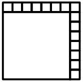
Enumerating Visible Tilings

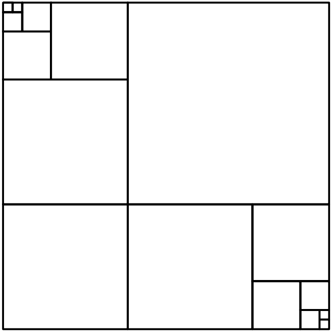
Interestingly, our technique does not generate all of the possible visible tilings for each k value.



Distinct visible tilings for $k = 12$ not generated using our technique

Enumerating Visible Tilings

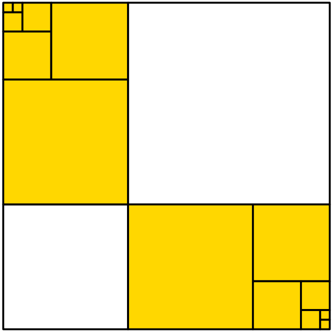




Two extreme examples of tilings when $n = 1$ and $n = r - 1$

Fibonacci Tilings

We call tilings when $n = r - 1$ Fibonacci tilings because of the golden rectangles located on either side of the square. These rectangles are generated by the Fibonacci sequence that appears in a version of our constructions.



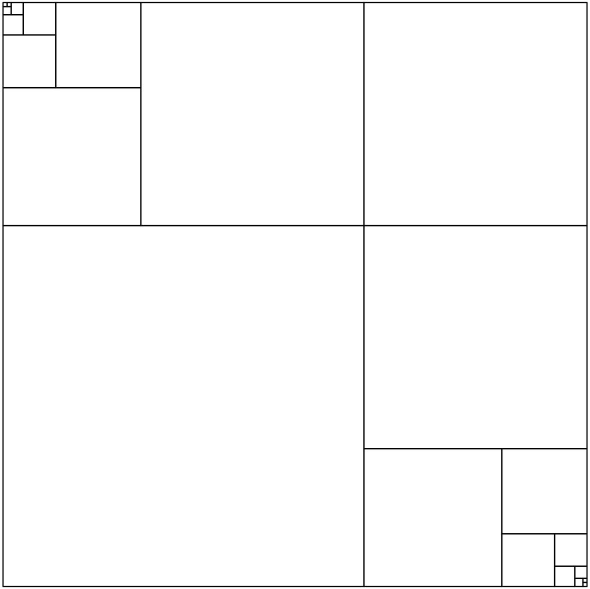
A Fibonacci tiling for which $k = 16$, and $r = 2 + 1 + 1 + 1 + 1 + 1$

Square Side Lengths of Consecutive Fibonacci Tilings

In the table below, k is equal to the number of squares in the visible tiling.

k	6	8	10	12	14	16	18	20	22	24
Tiled square side length	3	5	8	13	21	34	55	89	144	233

Fibonacci Tiling with 22 Squares

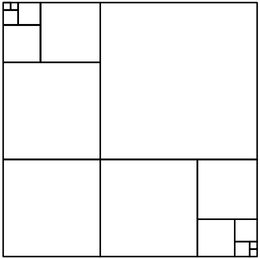


The Area of a Fibonacci Tiled Square

$$\left[\sum_{n=1}^{\frac{k}{2}-1} 2\left[\frac{\Phi^{n}-\phi^{n}}{\sqrt{5}}\right]^{2}\right]+\left[\frac{\Phi^{\frac{k}{2}-1}-\phi^{\frac{k}{2}-1}}{\sqrt{5}}\right]^{2}+\left[\frac{\Phi^{\frac{k}{2}}-\phi^{\frac{k}{2}}}{\sqrt{5}}\right]^{2}=\left[\frac{\Phi^{\frac{k}{2}+1}-\phi^{\frac{k}{2}+1}}{\sqrt{5}}\right]^{2}$$

What's Next?

We believe that Fibonacci tilings hold great promise in answering several other questions that we are pursuing as they are both efficient and visible.



Article References

- Erdős, P., Soifer, A., Squares in Square. Geombinatorics, volume IV, issue 4 (1995), 110-114.
- Erdős, P., Some of my favorite problems in number theory. Comb. Week. Resenhas 2 (1995), no. 2, 165-186.
- Burt, J., Staton, W., Tyler, B., On Visible Tilings. Geombinatorics, volume XXV, issue 3 (2016), 103-112.
- Doerfler, M,. Swain, C., Tyler, B., Enumerating Visible Tilings. Geombinatorics, submitted (2023).

Image References

- Mesopotamia image: https://cargocollective.com/klink/History-Cone-Mosaic-Mesopotamia
- Girih Tiles Image: https://www.sciencenews.org/article/ancient-islamic-penrose-tiles-0
- Penrose Tiling Image: Inductiveload/Wikipedia
- Einstein Tiling Image: David Smith, Joseph Samuel Myers, Craig S. Kaplan, Chaim Goodman-Strauss (CC BY-SA 4.0)
- Erdős Image: https://en.wikipedia.org/wiki/Penrose_tiling

Thank You

Questions?