Introduction to Algorithmic Trading Strategies
Lecture 8

Performance Measures

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Outline

▶ Sharpe Ratio

▶ Problems with Sharpe Ratio

▶ Omega

▶ Properties of Omega

▶ Portfolio Optimization

References

▸ Connor Keating, William Shadwick. A universal performance measure. Finance and Investment Conference 2002. 26 June 2002.

▸ Connor Keating, William Shadwick. An introduction to Omega. 2002.

▸ Kazemi, Scheeweis and Gupta. Omega as a performance measure. 2003.

▸ S. Avouyi-Dovi, A. Morin, and D. Neto. Optimal asset allocation with Omega function. Tech. report, Banque de France, 2004. Research Paper.

Notations

- $r = (r_1, \ldots, r_n)'$: a *random* vector of returns, either for a single asset over n periods, or a basket of n assets

- Q : the covariance matrix of the returns

- $x = (x_1, \ldots, x_n)'$: the weightings given to each holding period, or to each asset in the basket

Portfolio Statistics

▸ Mean of portfolio

  ▸ $\mu(x) = x'E(r)$

▸ Variance of portfolio

  ▸ $\sigma^2(x) = x'Qx$

Sharpe Ratio

▸ $\text{sr}(x) = \dfrac{\mu(x) - r_f}{\sigma^2(x)} = \dfrac{x'E(r) - r_f}{x'Qx}$

▸ r_f: a benchmark return, e.g., risk-free rate

▸ In general, we prefer

  ▸ a bigger excess return

  ▸ a smaller risk (uncertainty)

▸

Sharpe Ratio Limitations

▸ Sharpe ratio does not differentiate between winning and losing trades, essentially ignoring their likelihoods (odds).

▸ Sharpe ratio does not consider, essentially ignoring, all higher moments of a return distribution except the first two, the mean and variance.

Sharpe's Choice

▸ Both A and B have the same mean.

▸ A has a smaller variance.

▸ Sharpe will always chooses a portfolio of the smallest variance among all those having the same mean.

  ▸ Hence A is preferred to B by Sharpe.

Avoid Downsides and Upsides

▶ Sharpe chooses the smallest variance portfolio to reduce the chance of having extreme losses.

▶ Yet, for a Normally distributed return, the extreme gains are as likely as the extreme losses.

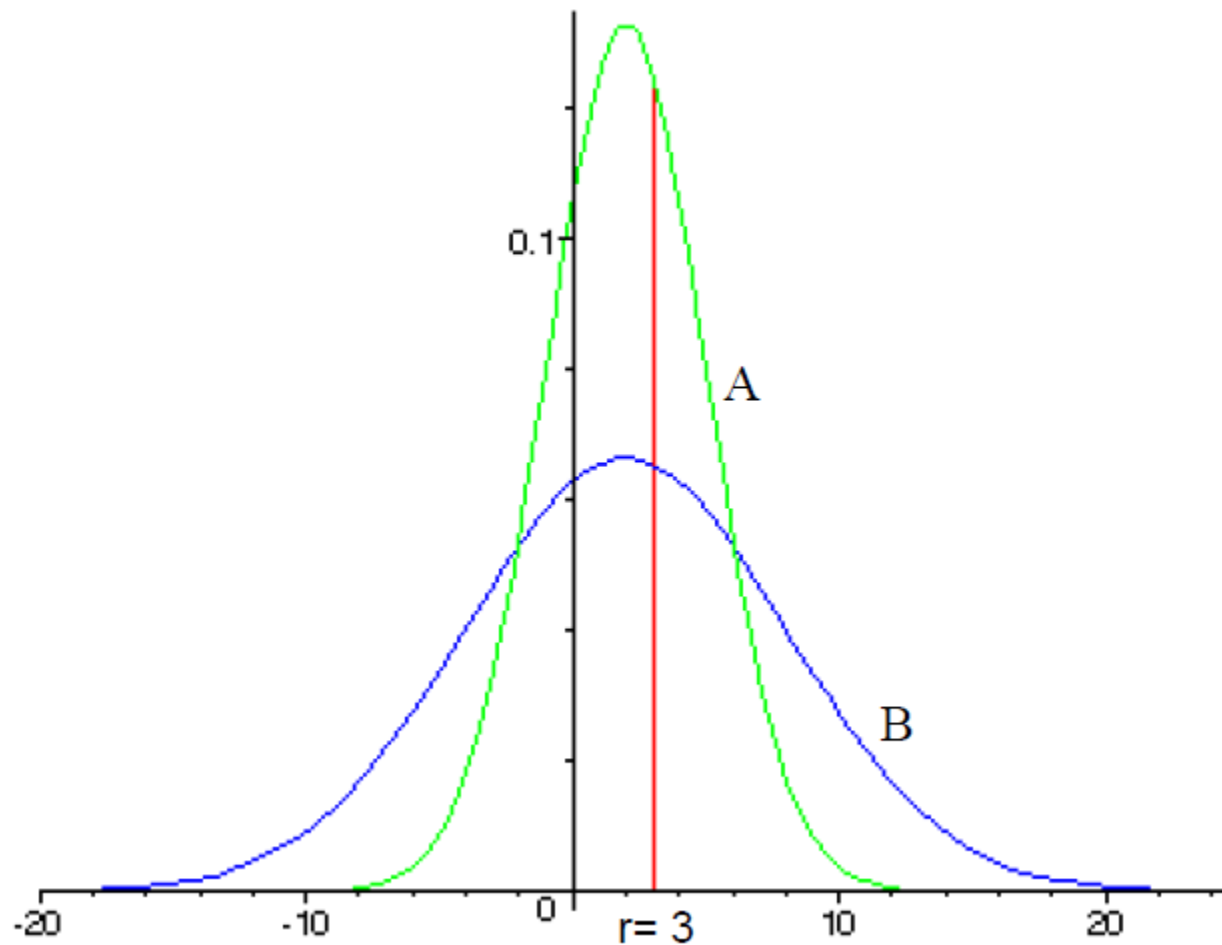▶ Ignoring the downsides will inevitably ignore the potential for upsides as well.

Potential for Gains

▸ Suppose we rank A and B by their potential for gains, we would choose B over A.

▸ Shall we choose the portfolio with the biggest variance then?

    ▸ It is very counter intuitive.

Example 1: A or B?

Example 1: L = 3

▸ Suppose the loss threshold is 3.

▸ Pictorially, we see that B has more mass to the right of 3 than that of A.

  ▸ B: 43% of mass; A: 37%.

▸ We compare the likelihood of winning to losing.
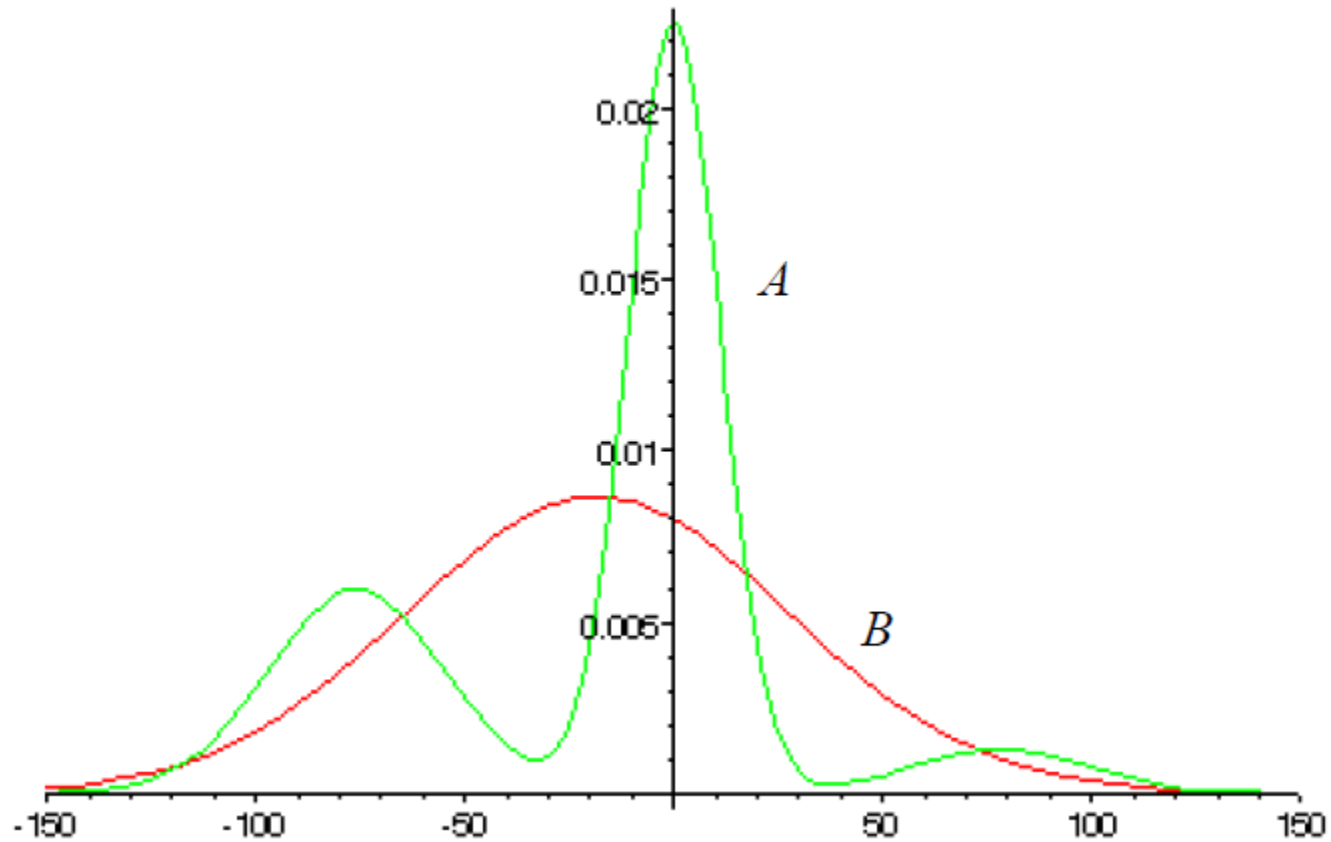
  ▸ B: 0.77; A: 0.59.

▸ We therefore prefer B to A.

Example 1: L = 1

- Suppose the loss threshold is 1.
- A has more mass to the right of L than that of B.
- We compare the likelihood of winning to losing.
  - A: 1.71; B: 1.31.
- We therefore prefer A to B.

Example 2

Example 2: Winning Ratio

▸ It is evident from the example(s) that, when choosing a portfolio, the likelihoods/odds/chances/potentials for upside and downside are important.

▸ Winning ratio $\frac{W_A}{W_B}$:

  ▸ 2σ gain: 1.8
  ▸ 3σ gain: 0.85
  ▸ 4σ gain: 35

Example 2: Losing Ratio

▸ Losing ratio $\frac{L_A}{L_B}$:

  ▸ 1σ loss: 1.4
  ▸ 2σ loss: 0.7
 ▸ 3σ loss : 80
  ▸ 4σ loss : 100,000!!!

Higher Moments Are Important

▸ Both large gains and losses in example 2 are produced by moments of order 5 and higher.

　▸ They even shadow the effects of skew and kurtosis.

　▸ Example 2 has the same mean and variance for both distributions.

▸ Because Sharpe Ratio ignores all moments from order 3 and bigger, it treats all these very different distributions the same.

▸

How Many Moments Are Needed?
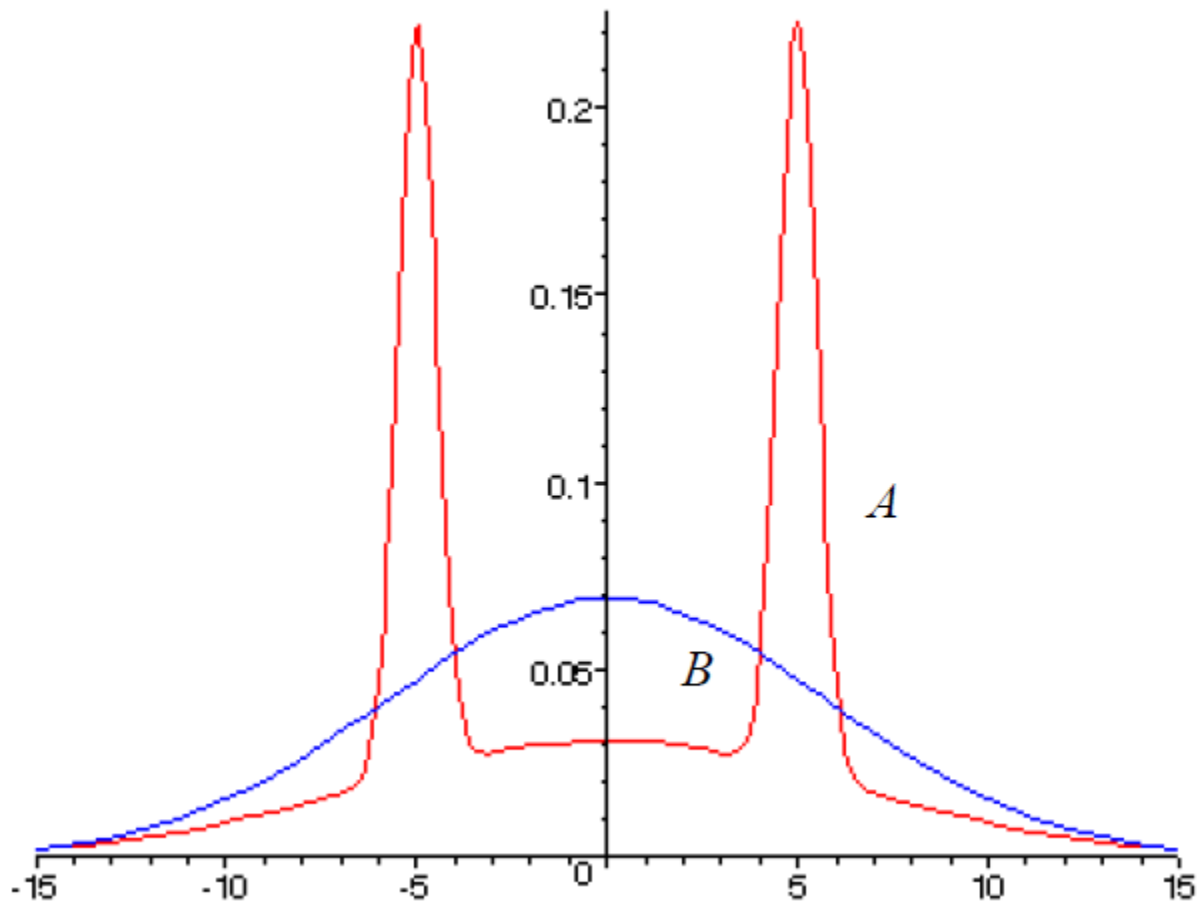
Distribution A

‣ Combining 3 Normal distributions

  ‣ N(-5, 0.5)

  ‣ N(0, 6.5)

  ‣ N(5, 0.5)

‣ Weights:

  ‣ 25%

  ‣ 50%

  ‣ 25%

Moments of A

- Same mean and variance as distribution B.
- Symmetric distribution implies all odd moments (3rd, 5th, etc.) are 0.
- Kurtosis = 2.65 (smaller than the 3 of Normal)
  - Does smaller Kurtosis imply smaller risk?
- 6th moment: 0.2% different from Normal
- *8th moment: 24% different from Normal*
- *10th moment: 55% bigger than Normal*

Performance Measure Requirements

▸ Take into account the odds of winning and losing.

▸ Take into account the sizes of winning and losing.

▸ Take into account of (all) the moments of a return distribution.

Loss Threshold

▸ Clearly, the definition, hence likelihoods, of winning and losing depends on how we define loss.

▸ Suppose L = Loss Threshold,

  ▸ for return < L, we consider it a loss

  ▸ for return > L, we consider it a gain

An Attempt

- To account for
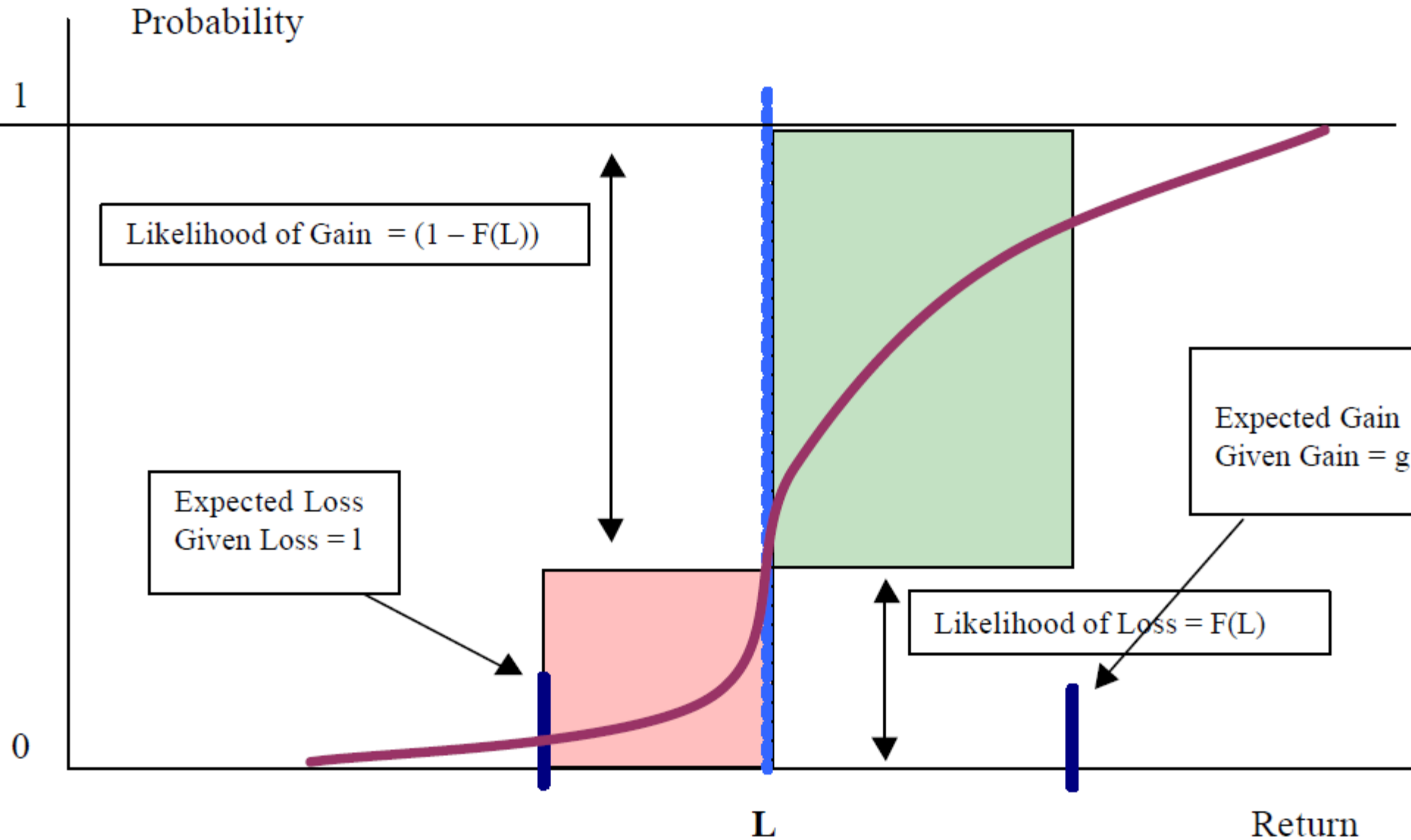 - the odds of wining and losing
  - the sizes of wining and losing
- We consider
  - $\Omega = \dfrac{E(r|r>L) \times P(r>L)}{E(r|r\leq L) \times P(r\leq L)}$
- $\Omega = \dfrac{E(r|r>L)\left(1-F(L)\right)}{E(r|r\leq L)F(L)}$

First Attempt

Probability

Likelihood of Gain $= (1 - F(L))$

Expected Loss
Given Loss $= 1$

Expected Gain
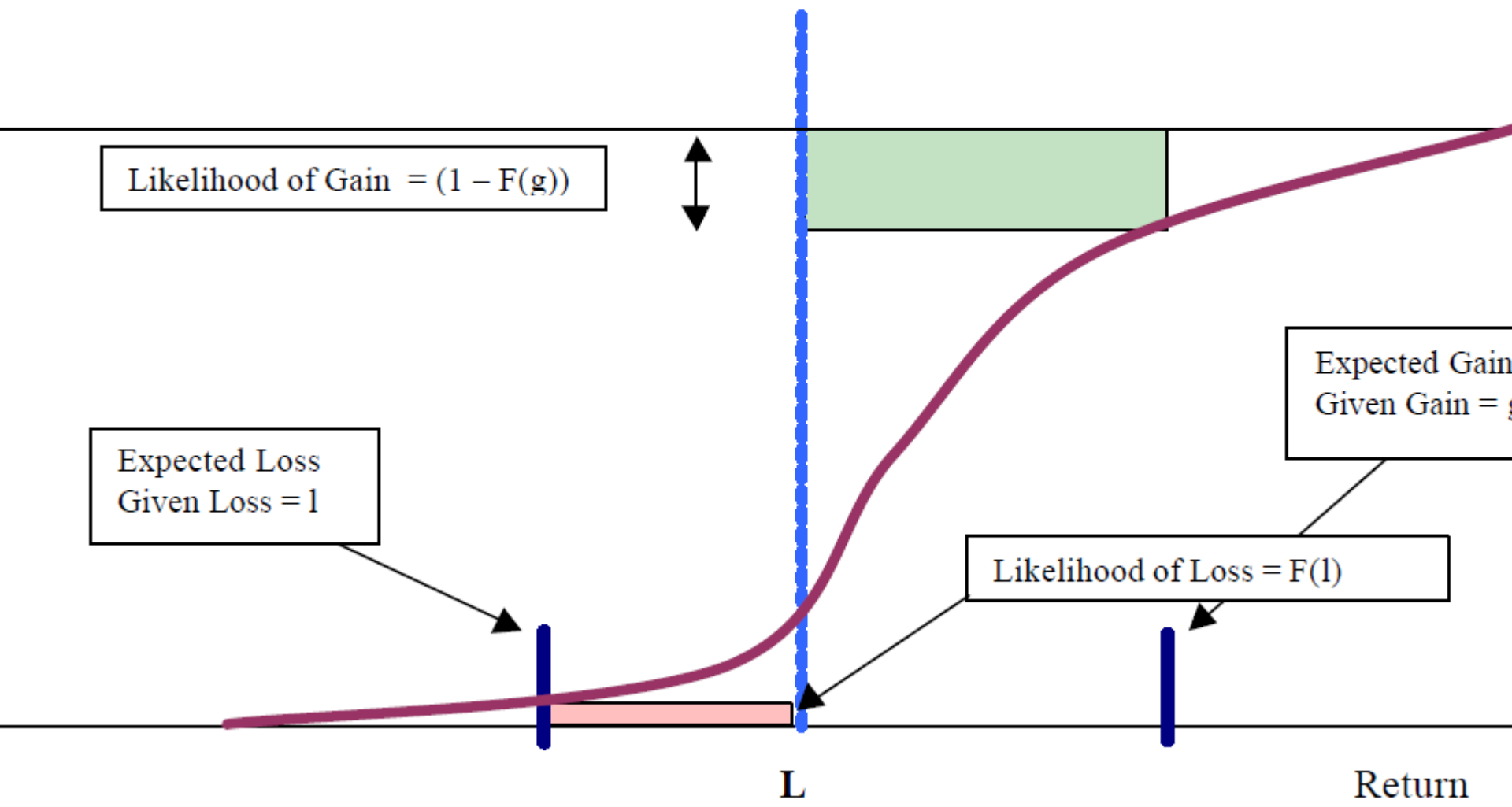Given Gain $= g$

Likelihood of Loss $= F(L)$

L

Return

First Attempt Inadequacy

‣ Why F(L)?

‣ Not using the information from the entire distribution.

 ‣ hence ignoring higher moments

Another Attempt
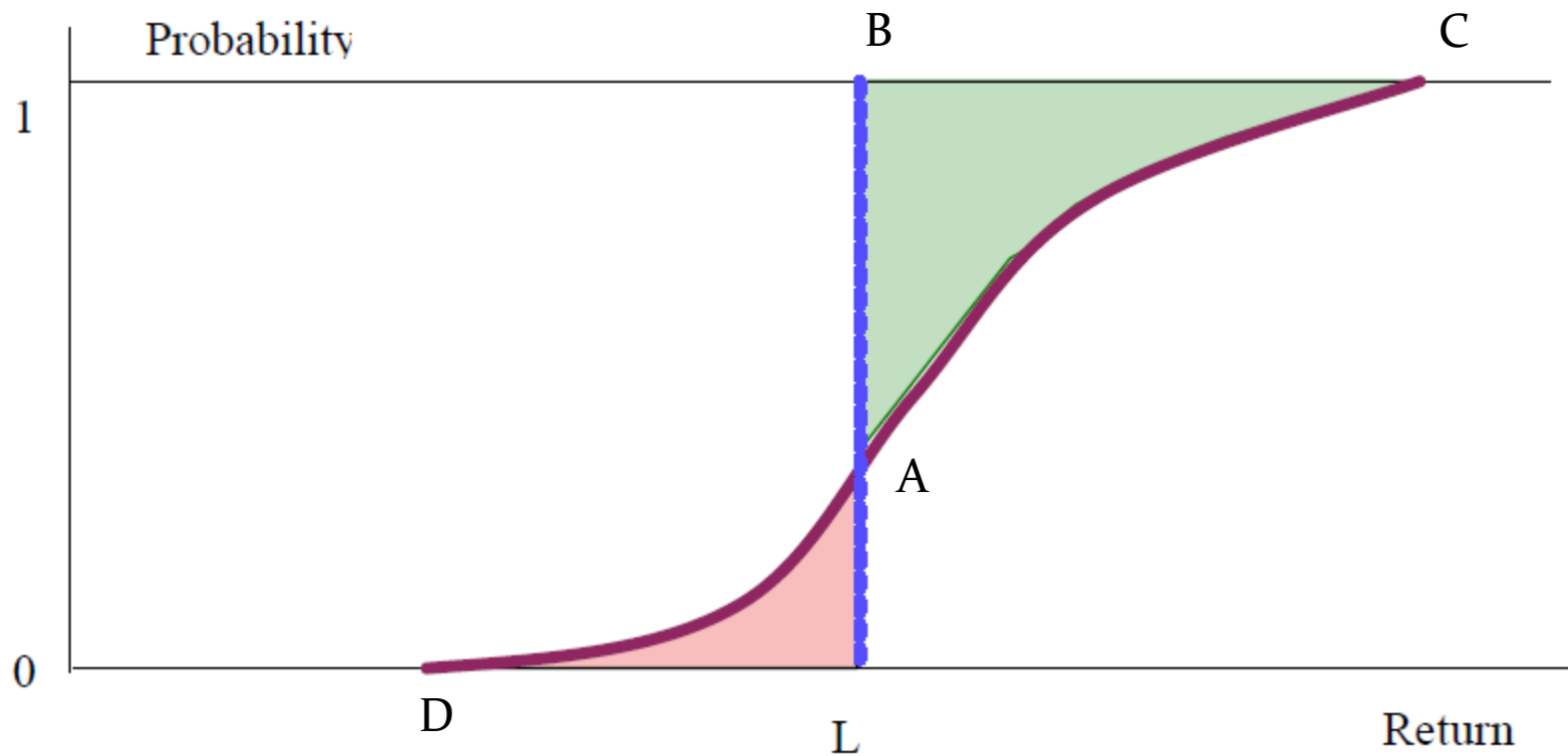
Probability

Likelihood of Gain $= (1 - F(g))$

Expected Gain
Given Gain $= g$

Expected Loss
Given Loss $= 1$

Likelihood of Loss $= F(l)$

L

Return

Yet Another Attempt

Omega Definition

- Ω takes the concept to the limit.

- Ω uses the whole distribution.

- Ω definition:

  - $\Omega = \dfrac{ABC}{ALD}$

- $\Omega = \dfrac{\int_{L}^{b=\max\{r\}}[1-F(r)]dr}{\int_{a=\min\{r\}}^{L}F(r)dr}$

Intuitions

▸ Omega is a ratio of winning size weighted by probabilities to losing size weighted by probabilities.

▸ Omega considers size and odds of winning and losing trades.

▸ Omega considers all moments because the definition incorporates the whole distribution.

Omega Advantages

‣ There is no parameter (estimation).

‣ There is no need to estimate (higher) moments.

‣ Work with all kinds of distributions.

‣ Use a function (of Loss Threshold) to measure performance rather than a single number (as in Sharpe Ratio).

‣ It is as smooth as the return distribution.

‣ It is monotonic decreasing.

Omega Example
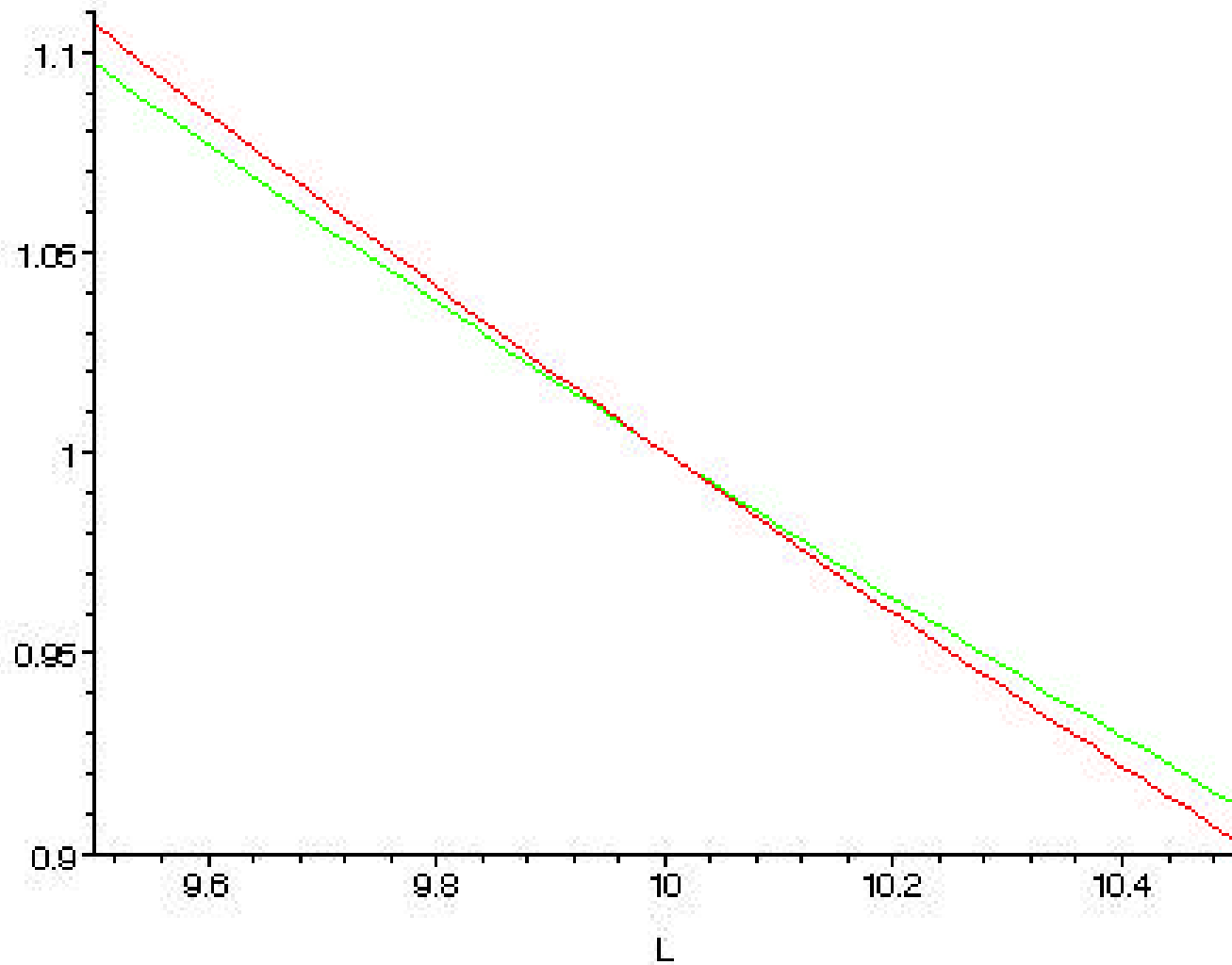
Affine Invariant

- $\varphi: r \rightarrow Ar + B$, iff $\widehat{\Omega}\big(\varphi(L)\big) = \Omega(L)$
- $L \rightarrow AL + B$
- We may transform the returns distribution using any invertible transformation before calculating the Gamma measure.
- The transformation can be thought of as some sort of utility function, modifying the mean, variance, higher moments, and the distribution in general.

Numerator Integral (1)

▸ $\int_L^b d\left[x\left(1 - F(x)\right)\right]$

▸ $= \left[x\left(1 - F(x)\right)\right]_L^b$

▸ $= b\left(1 - F(b)\right) - L\left(1 - F(L)\right)$

▸ $= -L\left(1 - F(L)\right)$

▸

Numerator Integral (2)

- $\int_L^b d\left[x(1 - F(x))\right]$

- $= \int_L^b (1 - F(x)) dx + \int_L^b x d(1 - F(x))$

- $= \int_L^b (1 - F(x)) dx - \int_L^b x dF(x)$

Numerator Integral (3)

- $-L\big(1 - F(L)\big) = \int_L^b \big(1 - F(x)\big)dx - \int_L^b x\,dF(x)$

- $\int_L^b \big(1 - F(x)\big)dx = -L\big(1 - F(L)\big) + \int_L^b x\,dF(x)$

- $= \int_L^b (x - L)f(x)dx$

- $= \int_a^b \max(x - L, 0)f(x)dx$

- $= E\big[\max(x - L, 0)\big]$

undiscounted call option price

Denominator Integral (1)

- $\int_a^L d[xF(x)]$
- $= [xF(x)]^L{}_a$
- $= LF(L) - a(F(a))$
- $= LF(L)$

Denominator Integral (2)

- $\int_a^L d[xF(x)]$
- $= \int_a^L F(x)dx + \int_a^L x\,dF(x)$

Denominator Integral (3)

- $LF(L) = \int_a^L F(x)dx + \int_a^L xdF(x)$

- $\int_a^L F(x)dx = LF(L) - \int_a^L xdF(x)$

- $= \int_a^L (L - x)f(x)dx$

- $= \int_a^b \max(L - x, 0)f(x)dx$

- $= E[\max(L - x, 0)]$

undiscounted put option price

Another Look at Omega

▸ $\Omega = \dfrac{\int_{L}^{b=\max\{r\}}[1-F(r)]dr}{\int_{a=\min\{r\}}^{L}F(r)dr}$

▸ $= \dfrac{E[\max(x-L,0)]}{E[\max(L-x,0)]}$

▸ $= \dfrac{e^{-r_f}E[\max(x-L,0)]}{e^{-r_f}E[\max(L-x,0)]}$

▸ $= \dfrac{C(L)}{P(L)}$

▸

Options Intuition

▸ Numerator: the cost of acquiring the return above *L*

▸ Denominator: the cost of protecting the return below *L*

▸ Risk measure: the put option price as the cost of protection is a much more general measure than variance

Can We Do Better?

▸ Excess return in Sharpe Ratio is more intuitive than $C(L)$ in Omega.

▸ Put options price as a risk measure in Omega is better than variance in Sharpe Ratio.

Sharpe-Omega

- $\Omega_S = \dfrac{\bar{r} - L}{P(L)}$

▶ In this definition, we combine the advantages in both Sharpe Ratio and Omega.

  ▶ meaning of excess return is clear

  ▶ risk is bettered measured

▶ Sharpe-Omega is more intuitive.

▶ *Ω_S ranks the portfolios in exactly the same way as Ω.*

Sharpe-Omega and Moments

▸ It is important to note that the numerator relates only to the first moment (the mean) of the returns distribution.

▸ It is the denominator that take into account the variance and all the higher moments, hence the whole distribution.

Sharpe-Omega and Variance

▸ Suppose $\bar{r} > L$. $\Omega_S > 0$.

  ▸ The bigger the volatility, the higher the put price, the bigger the risk, the smaller the Ω_S, the less attractive the investment.

  ▸ We want smaller volatility to be more certain about the gains.

▸ Suppose $\bar{r} < L$. $\Omega_S < 0$.

  ▸ The bigger the volatility, the higher the put price, the bigger the Ω_S, the more attractive the investment.

  ▸ Bigger volatility increases the odd of earning a return above L.

▸

Portfolio Optimization

▸ In general, a Sharpe optimized portfolio is different from an Omega optimized portfolio.

Optimizing for Omega

$$
\begin{cases}
\max_{x} \Omega_S(x) \\
\sum_i^n x_i E(r_i) \geq \rho \\
\sum_i^n x_i = 1 \\
x_i{}^l \leq x_i \leq 1
\end{cases}
$$

- Minimum holding: $x^l = \left(x_1{}^l, \ldots, x_n{}^l\right)'$

Optimization Methods

▸ Nonlinear Programming
  ▸ Penalty Method

▸ Global Optimization
  ▸ Tabu search (Glover 2005)
  ▸ Threshold Accepting algorithm (Avouyi-Dovi et al.)
  ▸ MCS algorithm (Huyer and Neumaier 1999)
  ▸ Simulated Annealing
  ▸ Genetic Algorithm

▸ Integer Programming (Mausser et al.)

3 Assets Example

- $x_1 + x_2 + x_3 = 1$
- $R_i = x_1 r_{1i} + x_2 r_{2i} + x_3 r_{3i}$
- $= x_1 r_{1i} + x_2 r_{2i} + (1 - x_1 - x_2) r_{3i}$

Penalty Method

- $F(x_1, x_2) =$
  $-\Omega(R_i) +$
  $\rho\{[\min(0, x_1)]^2 + [\min(0, x_2)]^2 + [\min(0, 1 - x_1 - x_2)]^2\}$

- Can apply Nelder-Mead, a Simplex algorithm that takes initial guesses.

- F needs not be differentiable.

- Can do random-restart to search for global optimum.

Threshold Accepting Algorithm

▸ It is a local search algorithm.

   ▸ It explores the potential candidates around the current best solution.

▸ It "escapes" the local minimum by allowing choosing a lower than current best solution.

   ▸ This is in very sharp contrast to a hilling climbing algorithm.

Objective

- Objective function
  - $h: X \rightarrow R, X \in R^n$
- Optimum
  - $h_{\text{opt}} = \max\limits_{x \in X} h(x)$

Initialization

▸ Initialize n (number of iterations) and $step$.

▸ Initialize sequence of thresholds th_k, $k = 1, \ldots, step$

▸ Starting point: $x_0 \in X$

Thresholds

▸ Simulate a set of portfolios.

▸ Compute the distances between the portfolios.

▸ Order the distances from smallest to biggest.

▸ Choose the first *step* number of them as thresholds.

Search

- $x_{i+1} \in N_{x_i}$ (neighbour of x_i)
- Threshold: $\Delta h = h(x_{i+1}) - h(x_i)$
- Accepting: If $\Delta h > -th_k$ set $x_{i+1} = x_i$
- Continue until we finish the last (smallest) threshold.
  - $h(x_i) \approx h_{opt}$
- Evaluating h by Monte Carlo simulation.