weak stationary: E、Var constant 且 $\neq\infty$ + ACF is constant.

strict stationary: iid

ACF. $|\gamma_k| \le \gamma_0$ $|\rho_k| \le 1$

$\gamma_k = \gamma_{-k}$ $\rho_k = \rho_{-k}$

$\rho_k = 0 \Rightarrow$ white noise $\Rightarrow$ can't predict

$\rho_k \neq 0 \Rightarrow$ $\underbrace{a_t\text{与}a_{t-k}\text{有关}}_{\text{有dependence}}$ $\Rightarrow$ can used to predict

ACF ≠0 不意味着 market inefficiency

conditional part is more important.

a_t. $E[a_t | F_{t-1}] = 0 \rightarrow E[a_t] = 0 \Rightarrow \gamma_k = 0.$

$= E[a_t a_{t-k}]$

$= E[E(a_t a_{t-k} | F_{t-k})]$

$= 0$

$\sigma_t^2 = Var(r_t | F_{t-1})$

$= E[(r_t - \mu_t)^2 | F_{t-1}] = E[a_t^2 | F_{t-1}]$

$E[\varepsilon_t^2] = E\left(\frac{a_t^2}{\sigma_t^2}\right) = E\left[E\left[\frac{a_t^2}{\sigma_t^2} \middle| F_{t-1}\right]\right]$

$= E\left[\frac{1}{\sigma_t^2} E[a_t^2 | F_{t-1}]\right]$ 在已知 F_{t-1} 只情况下. σ_t^2 is a constant

$= E\left[\frac{1}{\sigma_t^2}\sigma_t^2\right] = 1$

AR(1) $r_t = \phi_1 r_{t-1} + a_t = a_t + \sum_{i=1}^{\infty} \phi_1^i a_{t-i}$ a_t iid $\sim (0, \sigma_a^2)$

1.645 90%

1.96 95%

2.54 99%

how to know this model is correct?

$E[r_t] = 0$ $Var(r_t) = E[(r_t - 0)^2]$

$= E\left[\left(\sum_{i=0}^{\infty} \phi_1^i a_{t-i}\right)^2\right]$

$= E\left[\sum_{i=0}^{\infty}(\phi_1^i a_{t-i})^2 + \sum_{i\neq j} \phi_1^i a_{t-i} \phi_1^j a_{t-j}\right]$

$= \sigma_a^2 \sum_{i=0}^{\infty} (1 + \phi_1^2 + \dots)$

$= \sigma_a^2 \frac{1}{1-\phi_1^2} = \frac{\sigma_a^2}{1-\phi^2}$

Stationary: $|\phi_1| < 1$

ACF. $\gamma_0 = E[r_t r_t] = \frac{\sigma_a^2}{1-\phi^2}$ $\rho_0 = 1$

$\gamma_1 = E[r_t r_{t-1}] = E[(a_t + \phi_1 a_{t-1} + \phi_1^2 a_{t-2} + \dots)$

$(a_{t-1} + \phi_1 a_{t-2} + \phi_1^2 a_{t-3} + \dots)]$ $\rho_1 = \phi_1$

$= \sigma_a^2 (\phi_1 + \phi_1^3 + \dots)$ ⋮

$= \frac{\phi_1 \sigma_a^2}{1-\phi_1^2}$ $\rho_k = \phi_1^k$

⋮

$\gamma_k = \frac{\phi_1^k \sigma_a^2}{1-\phi_1^2}$

mean reverting behavior

$r_t = \phi_0 + \phi_1 r_{t-1} + a_t$ $E(r_t) = E[E[r_t | F_{t-1}]]$

$= E[\phi_0 + \phi_1 r_{t-1}]$

$= \phi_0 + \phi_1 E[r_{t-1}]$ $\therefore E = \frac{\phi_0}{1-\phi_1}$

$\Leftarrow E r_t = E r_{t-1} \Leftarrow$ stationary

$Var(r_t) = Var(\phi_1 r_{t-1} + a_t)$ 不变

ACF. 不变

$\Leftarrow$ 要减均值.

$r_t = \phi_0 + \phi(\phi_0 + \phi_1 r_{t-2} + a_{t-1}) + a_t$

$= \phi_0(1+\phi_1) + \phi_1^2 r_{t-2} + a_t + \phi_1 a_{t-1}$

$= \phi_0(1+\phi_1) + \phi_1^2(\phi_0 + \phi_1 r_{t-3} + a_{t-2}) + a_t + \phi_1 a_{t-1}$

$= \phi_0(1+\phi_1+\phi_1^2) + \phi_1^3 r_{t-3} + a_t + \phi_1 a_{t-1} + \phi_1^2 a_{t-2}$

⋮

$= \phi_0 \frac{1}{1-\phi_1} + \sum_{i=0}^{\infty} \phi_1^i a_{t-i}$

$r_{n+1} = \hat{r}_n(1) = \mathbb{E}[r_{n+1} | F_n] = \phi_0 + \phi_1 r_n$

$e_n(1) = r_{n+1} - \hat{r}_n(1) = \phi_0 + \phi_1 r_n + a_{n+1} - \phi_0 - \phi_1 r_n = a_{n+1}$

$\therefore Var(e_n(1)) = \sigma_a^2$

$\hat{r}_n(2) = \mathbb{E}[r_{n+2} | r_n] = \mathbb{E}[\phi_0 + \phi_1 r_{n+1} + a_{n+2} | r_n]$

$= \phi_0 + \phi_1 \mathbb{E}[r_{n+1} | r_n]$

$= \phi_0 + \phi_1(\phi_0 + \phi_1 r_n) = \phi_0 + \phi_0\phi_1 + \phi_1^2 r_n$

$e_n(2) = \phi_0 + \phi_1 r_{n+1} + a_{n+2} - \hat{r}_{n+2}$

$= \underline{\phi_0} + \underline{\phi_1(\phi_0} + \phi_1 r_n + a_{n+1}) + a_{n+2} - \underline{\phi_0} - \underline{\phi_0\phi_1} - \underline{\phi_1^2 r_n}$

$= \phi_1 a_{n+1} + a_{n+2}$

$Var(e_n(2)) = (\phi_1^2 + 1)\sigma_a^2$

$\hat{r}_n(l) = \mathbb{E}[r_{n+l} | r_n] = \mathbb{E}[\frac{\phi_0}{1-\phi_1} + a_{n+l} + \phi_1 a_{n+l-1} + \phi_1^2 a_{n+l-2} + \dots | F_n]$

$= \frac{\phi_0}{1-\phi_1} + \phi_1^l a_n + \phi_1^{l+1} a_{n-1} + \dots$

$\therefore e_n(l) = r_{n+l} - \hat{r}_n(l) = a_{n+l} + \phi_1 a_{n+l-1} + \dots + \phi_1^{l-1} a_{n+l-(l-1)}$

$= \sum_{j=0}^{l-1} \phi_1^j a_{n+l-j}$

$\therefore Var(e_n(l)) = [1 + (\phi_1)^2 + (\phi_1^2)^2 \dots + (\phi_1^{l-1})^2]\sigma_a^2 = \sigma_a^2 \sum_{j=0}^{l-1} \phi_1^{2j}$

forcast Interval. $[r_n(l) - N_{\frac{\alpha}{2}} \sigma_a \sqrt{\sum_{j=0}^{l-1} \phi_1^{2j}},\ r_n(l) + N_{\frac{\alpha}{2}} \sigma_a \sqrt{\sum_{j=0}^{l-1} \phi_1^{2j}}]$

AR(2) : $r_t = \phi_1 r_{t-1} + \phi_2 r_{t-2} + a_t$

Stationary:

$(1-\alpha_1 B)(1-\alpha_2 B) r_t = a_t$

$w_t = (1-\alpha_2 B) r_t = a_t + \sum_{i=1}^{\infty} \alpha_1^i a_{t-i}$

$r_t = \alpha_2 r_{t-1} + w_t = w_t + \sum_{i=1}^{\infty} \alpha_2^i w_{t-i}$

保证 $|\alpha_1| < 1$. $|\alpha_2| < 1$

$|z_1| = |\frac{1}{\alpha_1}| > 1$ $|z_2| = |\frac{1}{\alpha_2}| > 1$ $\Rightarrow z_1, z_2$ 解 outside circle

ACF: $r_t = \phi_1 r_{t-1} + \phi_2 r_{t-2} + a_t$

$\gamma_1 = E\, r_t r_{t-1}$

$= \mathbb{E}[(\phi_1 r_{t-1} + \phi_2 r_{t-2} + a_t) r_{t-1}]$

$\gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_1$ $\quad \gamma_1 = \frac{\phi_1}{1-\phi_2} \gamma_0$

$\gamma_2 = E\, r_t r_{t-2}$

$= \mathbb{E}[(\phi_1 r_{t-1} + \phi_2 r_{t-2} + a_t) r_{t-2}]$

$= \phi_1 \gamma_1 + \phi_2 \gamma_0$ $\quad \gamma_2 = \phi_1 \gamma_1 + \phi_2 \gamma_0$

$\gamma_3 = E\, r_t r_{t-3}$

$= \mathbb{E}[(\phi_1 r_{t-1} + \phi_2 r_{t-2} + a_t) r_{t-3}]$

$= \phi_1 \gamma_2 + \phi_2 \gamma_1$ $\quad \gamma_3 = \phi_1 \gamma_2 + \phi_2 \gamma_1$ $\Rightarrow \gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2}$

$\rho_0 = 1$

$\rho_1 = \frac{\phi_1}{1-\phi_2}$

$\rho_2 = \phi_1 \frac{\gamma_1}{\gamma_0} + \phi_2 = \phi_1 \rho_1 + \phi_2 \rho_0$

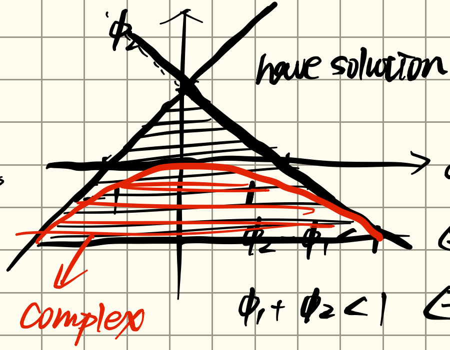
$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}$

$w_{1,2}$ $\quad 1 - \phi_1 z - \phi_2 z^2 = 0$

$\rho_l = \frac{c_1}{w_1^l} + \frac{c_2}{w_2^l}$

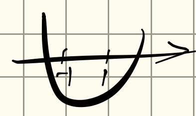
if $w_{1,2}$ all real.

if $w_{1,2}$ complex



$1-\phi_1 x-\phi_2 x^2=0$

out



$\Leftarrow 1+\phi_1-\phi_2>0$

$\phi_1+\phi_2<1 \Leftarrow 1-\phi_1-\phi_2>0$

$-1<\phi_2<1$

$\Delta<0$

$\phi_1^2+4\phi_2<0$

$\phi_2<-\frac{\phi_1^2}{4}$

Stochastic business cycle: 看2个虚根

$k=\frac{2\pi}{\cos^{-1}(\phi_1/2\sqrt{-\phi_2})}$ $\quad k=\frac{2\pi}{\cos^{-1}(a/\sqrt{a^2+b^2})}$

$x_1+x_2=-\frac{\phi_1}{1}=-\phi_1$

$x_1x_2=\frac{-\phi_2}{1}=-\phi_2$.

$x_1, x_2=a\pm bi$

$\frac{\phi_1}{2\sqrt{-\phi_2}}=\frac{-(x_1+x_2)}{2\sqrt{x_1x_2}}$

$=\frac{2a}{2\sqrt{a^2+b^2}}=\frac{a}{\sqrt{a^2+b^2}}$

forcast : $r_t=\phi_0+\phi_1 r_{t-1}+\phi_2 r_{t-2}+a_t$.

$\hat{r}_n(1)=E[r_{n+1}|r_n]=E[\phi_0+\phi_1 r_n+\phi_2 r_{n-1}+a_{t+1}|r_n]$

$=\phi_0+\phi_1 r_n+\phi_2 r_{n+1}$

$\hat{e}_n(1)=a_{n+1}$

$Var(\hat{e}_n(1))=\sigma_a^2$.

AR model $\quad r_t=\phi_0+\phi_1 r_{t-1}+\cdots+\phi_p r_{t-p}+a_t$. $\quad$ AR(p).

PACF. $=corr(r_t, r_{t+k}|r_{t+1}, r_{t+2}\dots r_{t+k-1})$

$=\frac{cov((r_t-\hat{r}_t)(r_{t+k}-\hat{r}_{t+k}))}{\sqrt{Var(r_t-\hat{r}_t)Var(r_{t+k}-\hat{r}_{t+k})}}=\phi_{kk}$.

有$\hat{r}_t$. $\hat{r}_{t+k}$ ⇒ can't calculate.

$\phi_{11}=\rho_1$

$$\phi_{kk}=\frac{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_1 \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_2 \\ \vdots & & \vdots & & \vdots & \vdots \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & \rho_k \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_{k-2} \\ \rho_2 & \rho_1 & 1 & \cdots & \rho_{k-4} & \rho_{k-3} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \rho_{k-2} & \rho_{k-3} & \rho_{k-4} & \cdots & 1 & \rho_1 \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & 1 \end{vmatrix}}$$

feature AR(p). $\quad \phi_{11}, \phi_{22}\dots\phi_{pp}\neq 0$. $\quad$ else $=0$ $\quad \phi_{kk}\sim N(0,\frac{1}{n})$

⇒可用 box test 检验.

estimate parameters: least square or maximum likelihood.

$\sqrt{n}(\hat{\phi}-\phi_0) \sim N(0, \hat{\Omega})$ $\quad \hat{\Omega} = \left(\sum_{t=1}^{n} z_{t-1}\tilde{z}_{t-1}'\right)^{-1}\hat{\sigma}_a^2$.

$\sqrt{n}(\hat{\phi}_i-\phi_{0i}) \sim N(0, \hat{\sigma}_{ii})$

$\hat{\sigma}_a^2 = \frac{1}{n}\sum_{t=1}^{n} a_t(\hat{\phi})^2$

检查 residual 是否是 white noise ⇒ acf of residual =0 ⇒ Box test

是否存在 intercept. $\frac{\bar{r}}{\sqrt{Var(r_t)}} \sim N(0,1)$. 对 $Er_t=0$

intercept 是 $E[r_t]$ 不是 ϕ_0. 但 对 $E[r_t]=0 \Leftrightarrow \phi_0=0$.

MA(1) $\quad r_t=\mu+a_t-\theta a_{t-1}$

$Er_t=\mu$

$Var(r_t)=(1+\theta^2)\sigma_a^2$.

$\gamma_0 = Var(r_t)$ $\quad \rho_0=1$

$\gamma_1 = Er_tr_{t-1} = E[(\cancel{\mu}+a_t-\theta a_{t-1})(\cancel{\mu}+a_{t-1}-\theta a_{t-2})]$ $\quad \rho_1 = -\frac{\theta}{1+\theta^2}$

$= -\theta\sigma_a^2$

$\gamma_2 = Er_tr_{t-2} = E[(a_t-\theta a_{t-1})(a_{t-2}-\theta a_{t-3})]=0$ $\quad \rho_k=0,\ k>1$

forcast: $\hat{r}_n(1) = E[r_{n+1}|r_n] = E[\mu+a_{t+1}-\theta a_t|r_n]$

$= \mu-\theta a_t$

$\hat{e}_n(1) = a_{t+1}$.

$Var=\sigma_a^2$.

$\hat{r}_n(2) = E[r_{n+2}|r_n] = E[\mu+a_{t+2}-\theta a_{t+1}|r_n]=\mu$

$\hat{e}_n(2) = a_{t+2}-\theta a_{t+1}$

$Var=(1+\theta^2)\sigma_a^2$.

$\hat{r}_n(l)=\mu$.

$\hat{e}_n(l) = a_{n+l}-\theta a_{n+l-1}$

$Var=(1+\theta^2)\sigma_a^2$

invertibility; MA(1): $a_t=(r_t-\mu)+\sum_{i=1}^{\infty}\theta^i(r_{t-i}-\mu)$

$$r_t = \frac{\mu}{1-\theta} + a_t - \sum_{i=1}^{\infty} \theta^i r_{t-i}$$

MA(2) $r_t = \mu + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$

$E r_t = \mu$.

$Var = (1+\theta_1^2+\theta_2^2)\sigma_a^2$.

Invertibility: $(1-\theta_1 z - \theta_2 z^2) = 0$ outside.

$r_t = \mu + (1-\alpha_1 B)(1-\alpha_2 B) a_t$.

$u_t = (1-\alpha_2 B) a_t$.

ACF. $\gamma_0 = Var(r_t) = (1+\theta_1^2+\theta_2^2)\sigma_a^2$

$\gamma_1 = E[(r_t-\mu)(r_{t-1}-\mu)]$

$= E[(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2})(a_{t-1} - \theta_1 a_{t-2} - \theta_2 a_{t-3})]$

$= (-\theta_1 + \theta_1\theta_2)\sigma_a^2$

$\gamma_2 = E[(r_t-\mu)(r_{t-2}-\mu)]$

$= E[(a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2})(a_{t-2} - \theta_1 a_{t-3} - \theta_2 a_{t-4})]$

$= -\theta_2 \sigma_a^2$

MA(q) $r_t = \mu + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} \cdots - \theta_q a_{t-q}$

$E = \mu$

$Var = (1+\theta_1^2+\theta_2^2+\ldots+\theta_q^2)\sigma_a^2$

Invertibility: $(1-\theta_1 z - \theta_2 z^2 - \ldots - \theta_q z^q) = 0$. outside.

always stationary

only q 个 ACF $\neq 0$. 其它均为0 ⇒ 用ACF来定阶.

$\gamma_{q+1} = 0$.

① q : PACF.

② estimate parameter.

③ model checking

$\sqrt{n}(\hat\theta - \theta_0) \sim N(0, \hat\Sigma)$ $\quad \hat\Sigma = \left(\sum_{t=1}^{n} \frac{\partial a_{t-1}(\hat\theta)}{\partial\theta} \frac{\partial a_{t-1}(\hat\theta)}{\partial\theta'}\right)^{-1} \hat\sigma_a^2$

$\sqrt{n}(\hat\theta_i - \theta_{0i}) \sim N(0, \hat\sigma_{ii})$

ARMA(1,1) $\quad (1-\phi_1 B) r_t = \phi_0 + \underbrace{(1-\theta B) a_t}_{w_t}$

$$r_t = \phi_0 + \phi_1 r_{t-1} + w_t$$
$$= \phi_0 + \phi_1(\phi_0 + \phi_1 r_{t-2} + w_{t-1}) + w_t$$
$$= \phi_0 + \phi_0\phi_1 + \phi_1^2 r_{t-2} + \phi_1 w_{t-1} + w_t$$
$$= \phi_0 + w_t + \phi_1(\phi_0 + w_{t-1}) + \phi_1^2 r_{t-2}$$
$$= \phi_0(1+\phi_1) + w_t + \phi_1 w_{t-1} + \phi_1^2 r_{t-2}$$

$E r_t = \phi_0 \frac{1}{1-\phi_1} = \frac{\phi_0}{1-\phi_1}$

$$= \frac{\phi_0}{1-\phi_1} + w_t + \sum_{i=1}^{\infty} \phi_1^i w_{t-i}$$
$$= \frac{\phi_0}{1-\phi_1} + a_t - \theta a_{t-1} + \phi(a_{t-1} - \theta a_{t-2}) + \phi^2(a_{t-2} - \theta a_{t-3}) + \dots$$
$$= \frac{\phi_0}{1-\phi_1} + a_t + (\phi-\theta) a_{t-1} + \phi(\phi-\theta) a_{t-2} + \phi^2(\phi-\theta) a_{t-3} \dots$$
$$= \frac{\phi_0}{1-\phi_1} + a_t + (\phi-\theta)\sum_{i=1}^{\infty} \phi^{i-1} a_{t-i}$$

$$Var = \left(1 + (\phi-\theta)^2 [1^2 + \phi^2 + \phi^4 + \dots]\right)\sigma_a^2$$
$$= \left(1 + (\phi-\theta)^2 \frac{1}{1-\phi^2}\right)\sigma_a^2$$

$$\gamma_1 = \mathbb{E}[(r_t-\mu)(r_{t-1}-\mu)]$$
$$= \mathbb{E}[(a_t + (\phi-\theta)a_{t-1} + (\phi-\theta)\phi a_{t-2} + (\phi-\theta)\phi^2 a_{t-3} \dots)(a_{t-1} + (\phi-\theta)a_{t-2} + (\phi-\theta)\phi a_{t-3} + \dots)]$$
$$= [(\phi-\theta) + (\phi-\theta)^2\phi + (\phi-\theta)^2\phi^3 + \dots]\sigma_a^2$$
$$= \left[(\phi-\theta) + (\phi-\theta)^2 \frac{\phi}{1-\phi^2}\right]\sigma_a^2$$

$$\therefore \rho_1 = \frac{(1-\phi^2)(\phi-\theta) + (\phi-\theta)^2\phi}{(1-\phi^2) + (\phi-\theta)^2}$$

[假设 $\mu=0$, $\phi_0=0$]

$$\gamma_1 = \mathbb{E}[(\phi_1 r_{t-1} + a_t - \theta a_{t-1}) r_{t-1}]$$
$$= \phi_1\gamma_0 + 0 - \theta\sigma_a^2$$
$$= \underbrace{\phi_1\gamma_0}_{AR} - \theta\sigma_a^2$$

$\rho_2 = \phi_1\rho_1 \quad \dots \quad \rho_k = \phi_1^{k-1}\rho_1$

$\therefore$ ACF 和 PACF 均不等于0

$\downarrow$

similiar to AR

$$r_t = \phi_0 + \phi_1 r_{t-1} + a_t - \theta a_{t-1}$$

forcast $\quad (1-\phi_1 B) r_t = \phi_0 + \underbrace{(1-\theta B) a_t}_{w_t}$

$$\hat{r}_n(1) = \mathbb{E}[r_{n+1} \mid r_n] = \mathbb{E}[\phi_0 + \phi_1 r_n + a_{n+1} - \theta a_n \mid F_n]$$
$$= \phi_0 + \phi_1 r_n - \theta a_n$$

$$e_n(1) = a_{n+1}$$

$\hat{r}_n(2) = E[r_{n+2}|r_n] = E[\phi_0 + \phi_1 r_{n+1} + a_{n+2} - \theta a_{n+1} | F_n]$

$= \phi_0 + \phi_1 \hat{r}_n(1)$

$\hat{r}_n(3) = E[\phi_0 + \phi_1 r_{n+2} + a_{n+3} - \theta a_{n+2} | F_n]$

$= \phi_0 + \phi_1 \hat{r}_n(2) \quad \Rightarrow$ AR model.

ARMA (p,q) $\quad r_t = \phi_0 + \phi_1 r_{t-1} + \dots + \phi_p r_{t-p}$

$+ a_t - \theta_1 a_{t-1} - \dots - \theta_q a_{t-q}$

$1 - \sum \phi_i z^i$ and $1 - \sum \theta_i z^i$ all outside + no common solution.

representation: $\phi_p(z)\theta_q^{-1}(z) = 1 - \sum_{i=1}^{\infty} \pi_i z^i$

$\phi_p(z) r_t = \theta_q(z) a_t + \phi_0$

① AR: $\quad a_t = \phi_p(z)\theta^{-1}(z) r_t - \theta_q^{-1}(z)\phi_0$

$= \left(1 - \sum_{i=1}^{\infty} \pi_i z^i\right) r_t - \theta_q^{-1}(z)\phi_0$

$= r_t - \pi_1 r_{t-1} - \pi_2 r_{t-2} \dots - \theta_q^{-1}(z)\phi_0$

$r_t = \theta_q^{-1}(z)\phi_0 + a_t + \pi_1 r_{t-1} + \dots$

② MA: $\quad \phi_p^{-1}(z)\theta_q(z) = 1 + \sum \psi_i z^i$

$r_t = \phi_p^{-1}(z)\theta_q(z) a_t + \phi_p^{-1}(z)\phi_0$

$= a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \dots + \phi_p^{-1}(z)\phi_0.$

forcast error 看 MA representation.

$\hat{r}_n(l) = E[a_{t+l} + \psi_1 a_{t+l-1} + \dots + \psi_l a_t + \psi_{l+1} a_{t-1} + \dots | F_n]$

$= \psi_l a_t + \psi_{l+1} a_{t-1} \dots$

$e_n = a_{n+l} + \psi_1 a_{n+l-1} + \dots + \psi_{l-1} a_{n+1}$

Var $= 1 + \psi_1^2 + \dots + \psi_{l-1}^2 \qquad (1-\phi_1 B) r_t = \phi_0 + \underbrace{(1-\theta B) a_t}_{w_t}$

$|\phi| = 1 \Rightarrow$ non-stationary

strong memory

unit-root test

Seasonal: $\quad w_t = (1 - \theta B^s) a_t \quad$ only $r_s \neq 0$.

$w_t = (1-\theta B)(1 - \Theta B^s) a_t \quad$ 对称.

how to make sure if σ_t is a constant

if constant, ACF $=0$

ARCH:

GARCH. $\sigma_t^2 = \alpha_0 + \sum^m \alpha_i a_{t-i}^2 + \sum^s \beta_i \sigma_{t-i}^2$

$\eta_t = a_t^2 - \sigma_t^2$.

$$\underbrace{a_t^2}_{r_t} = \alpha_0 + \sum^{\max(m,s)} (\alpha_i + \beta_i) \underbrace{a_{t-i}^2}_{r_{t-i}} - \sum^s \beta_i \underbrace{\eta_{t-i}}_{a_{t-i}} + \underbrace{\eta_t}_{a_t}$$

GARCH (1,1) $\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2$.

① $E(a_t)=0$ σ_t和ε_t is independent.

② $$\begin{aligned}\sigma_t^2 &= \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 \sigma_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \\ &= \alpha_0 + (\alpha_1 \varepsilon_{t-1}^2 + \beta_1)\sigma_{t-1}^2 \\ &= \alpha_0 + (\alpha_1 \varepsilon_{t-1}^2 + \beta_1)[\alpha_0 + (\alpha_1 \varepsilon_{t-2}^2 + \beta_1)\sigma_{t-2}^2] \\ &= \alpha_0 + \alpha_0(\alpha_1 \varepsilon_{t-1}^2 + \beta_1) + (\alpha_1 \varepsilon_{t-1}^2 + \beta_1)(\alpha_1 \varepsilon_{t-2}^2 + \beta_1)\sigma_{t-2}^2 \ldots \\ &= \alpha_0 \left[1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} (\alpha_1 \varepsilon_{t-j}^2 + \beta_1)\right]\end{aligned}$$

$\Leftarrow$ if $E\ln(\alpha_1 \varepsilon_t^2 + \beta_1) < 0 \Rightarrow$ converge.

Cauchy $S_{n+m} - S_n \to 0$

③ $$\begin{aligned}E[a_t^2] &= E[\varepsilon_t^2]E[\sigma_t^2] = E[\sigma_t^2] \\ &= \alpha_0\left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} E[\alpha_1 \varepsilon_{t-j}^2 + \beta_1]\right) \\ &= \alpha_0\left(1 + \sum_{i=1}^{\infty} \prod_{j=1}^{i} \alpha_1 + \beta_1\right) \\ &= \alpha_0\left(1 + \sum_{i=1}^{\infty} (\alpha_1 + \beta_1)^i\right) \\ &= \alpha_0\left(1 + \frac{\alpha_1 + \beta_1}{1 - (\alpha_1 + \beta_1)}\right) \\ &= \alpha_0 \frac{1}{1 - (\alpha_1 + \beta_1)} \quad \text{if } 0 < \alpha_1 + \beta_1 < 1\end{aligned}$$

for Garch $(m,s) \Rightarrow \sum^{\max(m,s)} (\alpha_i + \beta_i) < 1$

④ if $1 - 2\alpha_1^2 - (\alpha_1 + \beta_1)^2 > 0$. $\dfrac{E[a_t^4]}{[E(a_t^2)]^2} > 3$.

$\Rightarrow$ can catch fat tail feature

σ_t is strict stationary

forcast: $\alpha_1+\beta_1$ is important.

IGARCH. $\alpha_1+\beta_1=1$

GARCH-M : $r_t = \mu + c\sigma_t^2 + a_t$

$a_t = \sigma_t \varepsilon_t$.

$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2$ c is risk premium.

EGARCH: $g(\varepsilon_t) = \theta\varepsilon_t + |\varepsilon_t| - \mathbb{E}|\varepsilon_t|$ asymmetry.

$a_t = \sigma_t\varepsilon_t$. $\ln\sigma_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i g(\varepsilon_{t-i}) + \sum_{i=1}^{s} \beta_i \ln(\sigma_{t-i}^2)$

θ_1/θ is leverage. $\theta_1 = \theta\alpha$. (EGARCH(1,1))

CHARMA : $a_t = \delta_{1t}a_{t-1} + \delta_{2t}a_{t-2} + \cdots + \delta_{mt}a_{t-m} + \eta_t$.

RCA : $r_t = \phi_0 + \sum_{i=1}^{p}(\phi_i + \delta_{it}) r_{t-i} + a_t$

when 市场 stable: garch interval narrow

市场 not stable. garch interval. wide

Vector: mean: $\begin{bmatrix} \mu_{1t} \\ \mu_{2t} \\ \vdots \\ \mu_{kt} \end{bmatrix} = \mu$.

γ: variance $= \Gamma(0) = \begin{bmatrix} \gamma_{11}(0) & \gamma_{12}(0) & \cdots & \gamma_{1k}(0) \\ \vdots & \ddots & & \\ \vdots & & \ddots & \\ \gamma_{k1}(0) & \cdots & & \gamma_{kk}(0) \end{bmatrix}$

$\gamma_{ii}(0) = E[(z_{it}-\mu_i)^2]$ $\gamma_{ij}(0) = \mathbb{E}[(z_{it}-\mu_i)(z_{jt}-\mu_j)]$

ρ: D $\rho_{ij}(0) = \dfrac{\gamma_{ij}(0)}{\sqrt{\gamma_{ii}(0)\gamma_{jj}(0)}}$ $D = \begin{pmatrix} \sqrt{\gamma_{11}(0)} & & & \\ & \sqrt{\gamma_{22}(0)} & & \\ & & \ddots & \\ & & & \sqrt{\gamma_{kk}(0)} \end{pmatrix}$

$\rho(0) = D^{-1}\Gamma(0)D^{-1}$

$\Gamma(l) = \mathbb{E}[(z_t-\mu)(z_{t-l}-\mu)]$

$= \begin{pmatrix} \gamma_{11}(l) & \gamma_{12}(l) & \cdots \\ \vdots & & \\ \vdots & & \end{pmatrix}$ $\gamma_{ij}(l) = E[(z_{it}-\mu_i)(z_{j,t-l}-\mu_j)]$

cointegration: $<k$. 个 independent β.

partial stationary: some root $=1$. others outside the unit circle.

$$\Phi(B) = \Phi^*(B)(1-B) + \Phi(1)B,$$

$$\Phi^*(B)(1-B)\mathbf{y}_t = \alpha\beta'\mathbf{y}_{t-1} + \varepsilon_t$$

$$\phi(B)\, y_t = \varepsilon_t$$

$$(\phi^*(B)(1-B) + \phi(1)B)\, y_t = \varepsilon_t$$

$$\phi^*(B)(1-B)\, y_t + \phi(1)\, y_{t-1} = \varepsilon_t$$

$$\phi^*(B)(1-B)\, y_t = -\phi(1)\, y_{t-1} + \varepsilon_t$$

$$= \alpha\underbrace{\beta' y_{t-1}}_{\text{stationary}} + \varepsilon_t$$

VECM $\rightarrow \varepsilon_t$

β. long-run parameter

$r = k-d$. d 是 unit root #