# Chapter 2

## measure of dependence

Autocovariance

$$\gamma_{s,t} = Cov(r_s, r_t) = E[(r_s - \mu_s)(r_t - \mu_t)]$$

Autocorrelation

$$\rho_{s,t} = corr(r_s, r_t) = \frac{\gamma_{s,t}}{\sqrt{\gamma_{s,s}\gamma_{t,t}}}.$$

### weak stationary

前两个moment是constant

只与k有关，与t无关

. Lag$-k$ autocovariance:

$$\gamma_k = Cov(r_t, r_{t-k}) = E[(r_t - \mu)(r_{t-k} - \mu)].$$

与t无关 只与k有关 时间间隔

ACF不等于0，意味着cov不等于0，他们是related的，所以我可以根据之前的信息predict, inefficiency

### strict stationary

definition

Strict: distributions are time-invariant, i.e.,

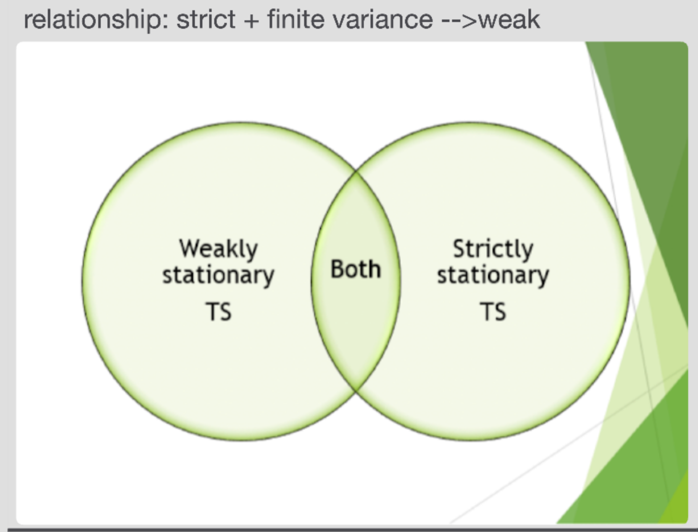
$$P(r_{t_1} \le z_1, \cdots, r_{t_n} \le z_n) = P(r_{t_1+k} \le z_1, \cdots, r_{t_n+k} \le z_n),$$

joint distribution

for $\forall t_1, \cdots, t_n, k$ and $(z_1, \cdots, z_n)$ and $n$.

没有办法判断到底是不是station的，只能assump, 然后判断assumption是否正确？close enough

relationship: strict + finite variance -->weak




### property of WS

$\rho_0 = 1, \gamma_0 = var(r_t)$

$$\gamma_0 = cov(r_t, r_{t-0}) = cov(r_t, r_t) = Var(r_t)$$

$$\rho_0 = \frac{Var(r_t)}{\sqrt{Var(r_t)}\sqrt{Var(r_t)}} = 1$$

$|\gamma_k| \le \gamma_0, |\rho_k| \le 1$

$$|\gamma_k| = |E[(r_t-\mu)(r_{t-k}-\mu)]| \le \sqrt{E(r_t-\mu)^2 \, E[(r_{t-k}-\mu)^2]} \Leftarrow \text{Schwarz } E[XY] \le (E[X^2]E[Y^2])^{\frac{1}{2}}$$

$$= \sqrt{(Var(r_t))^2}$$

$$= Var(r_t) = \gamma_0$$

$$\rho_k \le \frac{\gamma_0}{\gamma_0} = 1$$

$\gamma_k = \gamma_{-k}, \rho_k = \rho_{-k}$

$\rho_k = 0$ for all k不等于0 --> cov =0 --> white noise（应该有WS的condition）

#### estimate:

$\mu = 0$

Test $H_0: \mu = 0$ vs $H_a: \mu \ne 0$. Compute

$$t = \frac{\sqrt{T}\bar{r}}{\hat{\sigma}_r}. \qquad \frac{\bar{r}}{\frac{\hat{\sigma}_r}{\sqrt{T}}}$$

Compare $t$ ratio with $N(0,1)$ dist.

$ACF = 0$

$$\hat{\rho}_k = \frac{\sum_{t=1}^{T-k}(r_t - \bar{r})(r_{t+k} - \bar{r})}{\sum_{t=1}^{T}(r_t - \bar{r})^2},$$

individual

$H_0: \rho_1 = 0$ vs $H_a: \rho_1 \ne 0$

$$t = \frac{\hat{\rho}_1}{\sqrt{1/T}} = \sqrt{T}\hat{\rho}_1 \sim \boxed{N(0,1)}.$$

⇒ 不能

$\rho \ne 0$ ⇒ pred

can't do one

joint: 要拒绝$H_0$ <-- p足够小

Joint test (Ljung-$\boxed{\text{Box}}$ statistics):

要拒绝 $\rho$ ⇒ p_value<

m不能太大 ⇒ it is not

$H_0: \rho_1 = \cdots = \rho_m = 0 \ vs \ H_a: \rho_i \ne 0$ for some $i$.

it has d

$$Q(m) = T(T+2)\sum_{k=1}^{m}\frac{\hat{\rho}_k^2}{T-k} \sim \chi_m^2.$$

Asym. $\chi^2$ dist with $m$ degrees of freedom.

注意：m不能太大，m太大，ACF用的数据太少

#### notation: $Br_t = r_{t-1}, B2 = 2$

#### $r_t = \mu_t + a_t$ $\mu_t$： predict part, the best predictor of $r_t$

如果我们令$g_t \in F_t \to g_t \in f(F_{t-1})$

$$E(r_t - g_t)^2 > E(r_t - \mu_t)^2 \ if \ g_t \ne \mu_t.$$

proof

$$E(r_t - g_t)^2 = E[((r_t-\mu_t)-(g_t-\mu_t))^2] = E[(r_t-\mu_t)^2] + \underbrace{E[(\mu_t-g_t)^2]}_{\ne 0} \underbrace{- 2E[(r_t-\mu_t)(g_t-\mu_t)]}_{=0} > E[(r_t-\mu_t)^2]$$

$$E[(r_t-\mu_t)(g_t-\mu_t)] = E[E[(r_t-\mu_t)(g_t-\mu_t)|F_{t-1}]]$$

$$= E[(g_t-\mu_t)E[(r_t-\mu_t)|F_{t-1}]]$$

$$= E[(g_t-\mu_t)(\mu_t-\mu_t)]$$

$$= 0$$

$a_t$ is white noise but no sure it's iid

$$\sigma_t^2 = Var[r_t|F_{t-1}]$$

$$= E[(r_t-\mu_t)^2|F_{t-1}] \underbrace{-(E[r_t-\mu_t|F_{t-1}])^2}_{=0} \qquad \mu_t = E[r_t|F_{t-1}]$$

$$= E[a_t^2|F_{t-1}] = Var(a_t)$$

$$\Rightarrow r_t = \mu_t + \underbrace{\sigma_t \varepsilon_t}_{} \ a_t$$

① $E[\varepsilon_t] = E[E[\varepsilon_t|F_{t-1}]] = E[E[\frac{a_t}{\sigma_t}|F_{t-1}]] = E[\frac{1}{\sigma_t}E[a_t|F_{t-1}]] = 0$

② $Var(\varepsilon_t) = E[\varepsilon_t^2] = E[E[\frac{a_t^2}{\sigma_t^2}|F_{t-1}]] = E[\frac{1}{\sigma_t^2}E[a_t^2|F_{t-1}]] = 1$

③ 对 $a_t$. $\gamma_k = E[a_t a_{t-k}] = E[E[a_t a_{t-k}|F_{t-k}]]$

$$= E[a_{t-k}E[a_t|F_{t-k}]] = 0$$

⇒ $a_t$ is unrelated ⇒ $\varepsilon_t$ is unrelated

⇒ $\varepsilon_t \sim WN(0,1)$

## White noises

不一定相互独立$+Var < +\infty$

$$E\varepsilon_t = 0 \ \ and \ \ \text{cov}(\varepsilon_t, \varepsilon_s) = \begin{cases} \sigma^2, & t = s, \\ 0, & t \ne s, \end{cases}$$