Topic 3 Quantitive Modeling of Derivative Securities

- In topic 1 & 2, we learn some pricing measures, but for options, we just give some boundaries and don't have the precise price.
- In this topic ,we will use some mathematical method to price the options
- Review of stochastic and Ito calculus
  - Markov process: depend only on the state value in time t, and not depends on the path that arrives on time t
    - weak form of market efficiency: had included all the information in history, and each increments are independent
  - Brownain Motion
    - continuous, stationary and independent increment, B(0)=0
      - 注意！！！⚠️我们在这里定义了continuous，但为什么BM在每一点都non-differentiable呢？可以想象狄拉克函数，他确实连续，但是因为它的变化非常剧烈，接近直角，所以是non differentiable的
    - standard BM: $\mu = 0, \sigma^2 = 1$
      - non overlap:

$$\begin{aligned} P[Z(t) \le z | Z(t_0) = z_0] &= P[Z(t) - Z(t_0) \le z - z_0] \\ \text{Known information} &= \frac{1}{\sqrt{2\pi(t-t_0)}} \int_{-\infty}^{z-z_0} \exp\left(-\frac{x^2}{2(t-t_0)}\right) dx \\ &= N\left(\frac{z-z_0}{\sqrt{t-t_0}}\right). \end{aligned}$$

      - overlap:ρ公式是建立在$t > s$的情况下的

(a) $E[Z(t)^2] = \text{var}(Z(t)) + E[Z(t)]^2 = t.$

(b) $E[Z(t)Z(s)] = \min(t, s).$ Cov = min{t, s}

$$\rho = \frac{E[Z(t)Z(s)]}{\sqrt{\text{var}(Z(t))}\sqrt{\text{var}(Z(s))}} = \frac{s}{\sqrt{st}} = \sqrt{\frac{s}{t}} < 1$$

      - joint distribution

If we define $X_1 = Z(t)/\sqrt{t}$ and $X_2 = Z(s)/\sqrt{s}$, then X_1 and X_2 become standard normal random variables. We then have

$$\begin{aligned} P[Z(t) \le z_t, Z(s) \le z_s] &= P[X_1 \le z_t/\sqrt{t}, X_2 \le z_s/\sqrt{s}] \\ &= N_2(z_t/\sqrt{t}, z_s/\sqrt{s}; \sqrt{s/t}) \end{aligned}$$

where the bivariate normal distribution function is given by

$$\begin{aligned} N_2(x_1, x_2; \rho) = &\int_{-\infty}^{x_2} \int_{-\infty}^{x_1} \frac{1}{2\pi\sqrt{1-\rho^2}} \\ &\exp\left(-\frac{\xi_1^2 - 2\rho\xi_1\xi_2 + \xi_2^2}{2(1-\rho^2)}\right) d\xi_1 d\xi_2. \end{aligned}$$

- Geometric BM
  - $Y(t)=e^{X(t)}$
  - $\ln Y(t)=X(t)$服从于BM，所以$\ln Y(t)-\ln Y(0)$服从于正态分布，也就是$\ln\frac{Y(t)}{Y(0)}$服从于恒泰分布，$\frac{Y(t)}{Y(0)}$服从于lognormal distribution
    - 怎么计算pdf？在这里，$X(t)\sim N(\mu,\sigma^2)$, 但如果是BM，则需要加上t

$$f_Y(x)=\frac{d}{dx}F_Y(x)=\frac{d}{dx}\cdot P(Y\le x)=\frac{d}{dx}P(\ln Y\le \ln x).$$
$$=\frac{d}{dx}F_X(\ln x)$$
$$=f_X(\ln x)\cdot\frac{1}{x}$$
$$=\frac{1}{x\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(\ln x-\mu)^2}{2\sigma^2}\right\}$$

- Quadratic variation of BM
  - non- differentiable --> quadratic variation = T $\neq$ 0 <--$E[Q_\pi]=T, \lim var(Q_\pi - T)=0$ (对切割做极限)
- In general
  - BM(variation rate σ^2) $Q_{[t_1,t_2]}=\sigma^2(t_2-t_1)$
  - $dZ(t)=Z(t)-Z(t-dt), \mu=0, \sigma=1$

$$\mathbb{E}[dZ(t)^2]=\mathbb{E}[Z(t+dt)^2-Z(t)^2]$$
$$=\mathbb{E}[Q_{[t,t+dt]}]$$
$$=t+dt-t=dt$$

$$=dZ(t).$$
$$Z(dt)\sim(0,dt)$$
$$=\sqrt{dt}\cdot G.\quad G\sim N(0,1)$$

$$Var[dZ(t)^2]=\mathbb{E}[dZ(t)^4]-(\mathbb{E}[dZ(t)^2])^2$$
$$=dt^2\cdot\mathbb{E}[G^4]-dt^2$$
$$=2dt^2$$

  -

$$\int_0^T (\mathrm{d}Z(t))^2=\int_0^T \mathrm{d}t=T.$$

- Definition of a stochastic integration
  - 先定义一个函数：non-anticipative function: 非预见函数，不像Markov, 他下一时刻的状态不仅和当前时刻的状态有关，还和到达当前时刻的路径相关，现在我们已知的几乎所有的金融领域的函数都是这样的
  - 因为non-differentiable，只能从最基础的定义Riemann sum出发

$$\int_0^T f(Z,t)\ dZ(t) = \lim_{n\to\infty} \sum_{k=1}^{n} f(Z(\xi_k),\xi_k)[Z(t_k) - Z(t_{k-1})]$$

- 不幸的是，这个值和ξ_k的选取是直接相关的，不像之前的Riemann integration 和中间点的选取无关
- Ito integral
  - choose $\xi_k = t_{k-1}$

    $$\int_0^T f(Z,t)\ dZ(t) = \lim_{n\to\infty} \sum_{k=1}^{n} f(Z(t_{k-1}),t_{k-1})[Z(t_k) - Z(t_{k-1})],$$

  - 如果选取t_{k-1}，那么就是爱t_{k-1}时刻，我们就能已知需要的信息，从而$Z(t_{k-1})$和$Z(t_k) - Z(t_{k-1})$独立，从而期望为0
  - 也就是说，任意一个non anticipative function, 他的Ito Integral 的值都等于0
- Ito process
  - definition, 这是我们的governing equation<--随机变量一定需要满足的条件

    Let $\mathcal{F}_t$ be the natural filtration generated by the standard Brownian motion $Z(t)$ through the observation of the trajectory of $Z(t)$. Let $\mu(t)$ and $\sigma(t)$ be non-anticipative with respect to $Z(t)$ with $\int_0^T |\mu(t)|\ dt < \infty$ and $\int_0^T \sigma^2(t)\ dt < \infty$ (almost surely) for all T. The process $X(t)$ defined by

    $$X(t) = X(0) + \int_0^t \mu(s)\ ds + \int_0^t \sigma(s)\ dZ(s),$$

    is called an *Ito process*. The integral form is more formal since stochastic integrals are well defined. The differential form of the above equation is given as

    $$dX(t) = \mu(t)\ dt + \sigma(t)\ dZ(t).$$

  - Ito's Lemma
    - $Y = f(x,t)$是一个二阶连续可微函数

      $$dY = \left[\frac{\partial f}{\partial t}(X,t) + \mu(t)\frac{\partial f}{\partial x}(X,t) + \frac{\sigma^2(t)}{2}\frac{\partial^2 f}{\partial x^2}(X,t)\right] dt + \sigma(t)\frac{\partial f}{\partial x}(X,t)\ dZ.$$

      $$\partial Y = \frac{\partial Y}{\partial X}dx + \frac{\partial Y}{\partial t}dt + \frac{1}{2}\frac{\partial Y^2}{\partial X^2}\underbrace{dx\,dx}_{\sigma^2(t)dt} + \cancel{\frac{1}{2}\frac{\partial Y^2}{\partial t^2}}\underbrace{dt\,dt}_{=0} + \frac{1}{2}\frac{\partial Y^2}{\partial x\partial t}\underbrace{dx\,dt}_{=0}$$

      当 $\Delta X \to 0$. $\Delta t \to 0$.

      $dx^2 = \sigma^2(t)dt$

      $dx = \sigma(t)dt^{\frac{1}{2}}$

      $dx\,dt = \sigma(t)dt^{\frac{3}{2}}$ the linear function

      $\therefore dx\,dt = 0$

      $$dX = \mu(t)dt + \sigma(t)dz$$

      $$\partial Y = \frac{\partial Y}{\partial X}\mu(t)dt + \frac{\partial Y}{\partial X}\sigma(t)dz + \frac{\partial Y}{\partial t}dt + \frac{1}{2}\frac{\partial Y^2}{\partial X^2}\sigma^2(t)dt$$

      $$= \left(\frac{\partial Y}{\partial X}\mu(t) + \frac{\partial Y}{\partial t} + \frac{1}{2}\sigma^2(t)\frac{\partial Y^2}{\partial X^2}\right)dt + \frac{\partial Y}{\partial X}\sigma(t)dz$$

  - Martingale property of a zero-drift process
    - 注意⚠️：必须是没有漂移项！

$B_{t+s}-B_s \sim N(0,\sigma^2 t)$

$E[B_t|F_{t-1}]=E[B_{t-1}+B_t-B_{t-1}|F_{t-1}]=B_{t-1}$　B_t(无漂移项)是martingale

如果B_t存在漂移项

$B_{t+s}-B_s \sim N(\mu t,\sigma^2 t)$　$B_t-B_{t-1}\sim N(\mu,\sigma^2)$.

$E[B_t|F_{t-1}]=E[B_{t-1}+B_t-B_{t-1}|F_{t-1}]=B_{t-1}+\mu \neq B_{t-1}$

∴不是martingale

$X(t)=X(0)+\int_0^t \mu(s)ds+\int_0^t \sigma(s)dz(s)$

若$\mu(s)$有漂移项　$X(t)-X(0)=\int_0^t \sigma(s)dz(s)$

∴ $X(T)-X(t)=\int_t^T \sigma(s)dz(s)$

$E[X(T)|F_t]=E[X(t)|F_t]+\underbrace{E\left[\int_t^T \sigma(s)dz(s)|F_t\right]}_{\text{Ito Integral}=0}$

$=X(t)$. ⇒ martingale.

若有漂移项　$X(T)-X(t)=\int_t^T \mu(s)ds+\int_t^T \sigma(s)ds$,

$E[X(T)]=E[X(t)]+\underbrace{E[\int_t^T \mu(s)ds]}_{\neq 0} \neq E[X(t)]$

⇒不是martingale

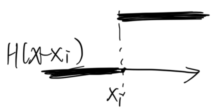
- 这其实给我们的计算带来了很大的困难，如果不满足martingale, 我们很难计算期望
- 那怎么办呢？我们想到了一种方法，本质就是坐标转化，假设我们考虑一个最简单的，他的漂移项固定为a，我们将横坐标向右移动a，那现在的漂移项就变成了0，诶，满足martingale， 有很多好的性质可以用

- Change of measure
  - 为什么有一个transition density function?
    - 我觉得这里是在推导我们有了X_t的governing function，但是X_t的概率密度函数到底是多少呢？
    - 我们suppose $P(X_t \in (x-\frac{dx}{2},x+\frac{dx}{2}))=u(x,t)$

(这只是一个对H和S函数的介绍

$F_X(x)=P(X\le x)=\sum p(X=x_i)\cdot 1_{\{x_i\le x\}}$

$=\sum p(X=x_i)H(x-x_i)$



如果$X_t=\xi$，为一个constant.

∴ $F_X(x)=H(x-\xi)$

$f_X(x)=\frac{d}{dx}H(x-x_i)=\delta(x-x_i)$

$H(x-x_i)$在x_i处的导数为∞，在其它处的导数为0

∴ $\frac{d}{dx}H(x-x_i)=\delta(x-x_i)$

- 而$\frac{\partial u}{\partial t}=\frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}$ (为什么满足这个条件？）
  - 在SC中讲过4.6，在已知B_t=X时，B_T的density function是什么，其中就满足这个式子，这里只是把B_T换成了X_T,X_T在这里是没有漂移项的，满足这个式子，可以求

出X_t的density function--> normal distribution

$$u(x,t)=\frac{1}{\sigma\sqrt{2\pi t}}\exp\left(-\frac{(x-\xi)^2}{2\sigma^2 t}\right).$$

- 那有漂移项咋办呢？change of measure
  - process:
    - equation:

$$u(x,t)=\frac{1}{\sigma\sqrt{2\pi t}}\exp\left(-\frac{(x-\mu t-\xi)^2}{2\sigma^2 t}\right)$$

    - 这时，governing equation 也改变了,称$\frac{d\hat{P}}{dP}$ 为Radon-Nikodym derivative
      - example

不再是0-drift的 $\frac{\partial u}{\partial t}=\frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}$.

而是 $\frac{\partial u}{\partial t}=-\mu\frac{\partial u}{\partial x}+\frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}$. $u(x,0)=\delta(x-\xi)$

我们想要 $u(y,t)=\frac{1}{\sigma\sqrt{2\pi t}}\exp\left(-\frac{(y-\xi)^2}{2\sigma^2 t}\right)$ $(y-\xi)^2\to y$

$u(x,t)=\frac{1}{\sigma\sqrt{2\pi t}}\exp\left(-\frac{(x-\mu t-\xi)^2}{2\sigma^2 t}\right)$

→这两个式子是一个 ∴ $x-\mu t=y$

$\boxed{x=y+\mu t}$

$x-\mu t=y$

为了简便、我们令$\sigma^2=1$ $\xi=0$

likelihood ratio $=\frac{x\text{-frame}}{y\text{-frame}}=\frac{\exp\left(-\frac{(y+\mu t)^2}{2t}\right)}{\exp\left(-\frac{y^2}{2t}\right)}$ ← 在x上 0 drift / ← 在y上 0 drift

$=\exp\left(-\frac{1}{2t}[2\mu t y+\mu^2t^2]\right)$

$=\exp\left(-\mu y-\frac{1}{2}\mu^2 t\right)$

推广 generalize

推广 generalize

现在我们有个P下的BM. 0 drift, 1 variance $Z_P(t)$

add the drift term to $Z_P(t)$. $\Rightarrow Z_P^{\mu}(t)=\mu t+Z_P(t)$

现在想将 $Z_P^{\mu}(t)$ change measure to 0-drift $\tilde{Z}$.

根据上面的例子. $\frac{x\text{-frame}}{y\text{-frame}}=\frac{Z_P(t)}{\tilde{Z}_P(t)}$ 这就是. likelihood ratio.

如何通过 likelihood ratio 修改 E

$dP_X(x)=P[X\in(x-\frac{dx}{2},x+\frac{dx}{2})]=f_X^P(x)\,dx$.

$d\tilde{P}_X(x)=\tilde{P}[X\in(x-\frac{dx}{2},x+\frac{dx}{2})]=f_X^{\tilde{P}}(x)\,dx$

$\mathbb{E}_{\tilde{P}}[X]=\int x\,d\tilde{P}(x)=\int x f_X^{\tilde{P}}(x)\,dx=\int x\frac{f_X^{\tilde{P}}(x)}{f_X^P(x)}\cdot f_X^P(x)\,dx$

$=\int x\frac{f_X^{\tilde{P}}(x)}{f_X^P(x)}\cdot dP(x)$

$=\mathbb{E}_P\left[X\frac{f_X^{\tilde{P}}(x)}{f_X^P(x)}\right]$

$=\mathbb{E}_P\left[X\frac{d\tilde{P}_X}{dP_X}\right]$

$\frac{d\hat{P}_X}{dP_X}$

$=\exp\left(-\mu Z_P(x)-\frac{\mu^2}{2}T\right)$

    - 下面证明为什么现在$\hat{P}$ 下的BM是0 漂移项的了
      - proof

proof. $\mathbb{E}_{\tilde{P}}[\exp(\alpha Z_{\tilde{P}}(T))]$

$= \mathbb{E}_P[\exp(\alpha Z_P(T) + \alpha\mu T)\frac{d\tilde{P}}{dP}]$

$= \mathbb{E}_P[\exp(\alpha Z_P(T) + \alpha\mu T - \mu Z_P(x) - \frac{\mu^2}{2}T)]$

$= e^{\alpha\mu T - \frac{\mu^2}{2}T}\,\mathbb{E}_P[\exp((\alpha-\mu)Z_P(x))]$ $\quad Z_P(x) = \exp(\frac{x^2}{2})$

$= \exp(\frac{(\alpha-\mu)^2}{2} + \alpha\mu T - \frac{\mu^2}{2}T)\,M_P(\alpha-\mu)$

$= \exp(\frac{\alpha^2}{2})$

(正态分布)

$Z_{\tilde{P}}(t) = Z_P(T)$

在$\tilde{P}$ 0 drift

在P下 μ drift.

- 或者这样想吧，如果我们在x处静止看这个BM，他就是有漂移项，但如果我们从x处移动着看BM，速度就等于这个漂移项，那么就是相对静止的，这样就可以认为不存在漂移项了，这时$y = x + \mu t$

- 以上我们都是认为漂移项是μt,但如果这个系数不是常数呢，而是一个变化的数值，我们就需要引入积分，公式就变成了

$$\frac{d\tilde{P}}{dP} = \rho(t) = \exp\left(\int_0^t -\gamma(s)\ dZ_P(s) - \frac{1}{2}\int_0^t \gamma(s)^2\ ds\right)$$

- 接下来我们真正运用这个公式，从P测度到Q测度，但需要注意的是，我们致力于令*具有零漂移项，那么S_t就拥有r漂移项

- process

$\frac{dS_t}{S_t} = \rho\, dt + \sigma\, dZ_t^P \qquad \frac{dS_t^*}{S_t^*} = (\rho - r)dt + \sigma\, dZ_t^P.$

$\quad = r\,dt + \sigma\, dZ_t^Q \qquad\qquad = \sigma\, dZ_t^Q.$

$\therefore dZ_t^Q = \frac{(\rho - r)dt}{\sigma} + dZ_t^P.$

$= \exp(-\mu Z_t^P - \frac{1}{2}\mu^2 t)$

$= \exp(-\frac{\rho - r}{\sigma}\cdot Z_t^P - \frac{t}{2}\frac{(\rho - r)^2}{\sigma^2})$

- Application: Feynaman-Kac representation formula governing equation. $dX(s) = \mu(X(s), s)ds + \sigma(X(s), s)\,dZ(s)$

F. Itô's Lemma

$$dF = [\underbrace{\frac{\partial F}{\partial t} + \mu(X,t)\frac{\partial F}{\partial x} + \frac{\sigma^2(X,t)}{2}\frac{\partial^2 F}{\partial x^2}}_{A}]dt + \sigma\frac{\partial F}{\partial x}dZ$$

如果F满足 $A=0$. $\Rightarrow dF = \sigma\frac{\partial F}{\partial x}dZ \Rightarrow$ martingale

$$\mathbb{E}_{x,t}[F(X,T)] = F(x,t) = h(X(T))$$

如果不是martingale，就利用 change of measure $X \to \tilde{X}$.

也就能得到类似的式子。

- Riskless hedging principle and dynamic replicatig strategy
  - Riskless hedging principle
    - 这一节的逻辑是利用两种replication的方法计算如何dynamic hedging，一种是两个securities： option, asset,一种是三种securities：option, asset, money in bank
    - 整个过程我们建立在risk neutral 情况下，也就是说我们致力于达到$d\Pi(t) = r\Pi(t)dt$
    - 且满足self-financing
    - method 1: 两个securities： option, asset
      - process

$\Pi_t = -C_t + \Delta_t S_t$ $\quad dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{\sigma^2}{2}S^2\frac{\partial^2 C}{\partial S^2}dt$

$d\Pi_t = -dC + \Delta_t dS_t$ $\quad \frac{dS}{S} = \mu dt + \sigma dZ$

$= -\frac{\partial C}{\partial t}dt - \frac{\partial C}{\partial S}dS - \frac{\sigma^2}{2}S^2\frac{\partial^2 C}{\partial S^2}dt + \Delta_t dS_t$

$= -\frac{\partial C}{\partial t}dt + (\Delta_t - \frac{\partial C}{\partial S})(S\mu dt + S\sigma dZ) - \frac{\sigma^2}{2}S^2\frac{\partial^2 C}{\partial S^2}dt$

$= dt(-\frac{\partial C}{\partial t} + \Delta_t S\mu - \frac{\partial C}{\partial S}S\mu - \frac{\sigma^2}{2}S^2\frac{\partial^2 C}{\partial S^2})dt$

$+ (\Delta_t - \frac{\partial C}{\partial S})\sigma S dZ$

因为是risk neutral $\Rightarrow \Delta = \frac{\partial C}{\partial S}$

$\Rightarrow d\Pi_t = (-\frac{\partial C}{\partial t} - \frac{\sigma^2}{2}S^2\frac{\partial^2 C}{\partial S^2})dt$

$d\Pi_t = r\Pi_t dt$

$= (-rC_t + \frac{\partial C}{\partial S}\cdot S r)dt$

$\therefore \frac{\partial C}{\partial S}\cdot Sr + \frac{\partial C}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 C}{\partial S^2} - rC = 0$ BS formula.

      - remark:
        - 1.其实按理说，$d(\triangle_t S_t) = S_t d(\triangle_t) + \triangle_t d(S_t)$,但这里我们忽略了第一项，因为一段时间的利润的结算是按照这段时间开始之前的asset的数量来结算的，我们认为$\triangle_t$在这段时间内是不发生改变的
        - 2.μ不见了！为什么呢，这是我们对asset的预期收益，他和risk neutral是相互独立的！！！因为我们转换到了Q测度，Q测度的drift term 就是r， 与μ无关
        - 3.这里面几个参数：X, T, S, r, σ除了σ其他的我们都是可以直接观察到的
      - Deficiencies：
        - 其实这个模型我们是做了很多假设的，比如r是常数，没有transaction cost等

(i) Trading takes place continuously in time.
(ii) The riskless interest rate r is known and constant over time. 是常数.
(iii) The asset pays no dividend. 无dividend.
(iv) There are no transaction costs in buying or selling the asset or the option, and no taxes.
(v) The assets are perfectly divisible. Hard
(vi) There are no penalties to short selling and the full use of proceeds is permitted.
(vii) There are no arbitrage opportunities.

- 所以这个模型的缺陷非常明显
    - 1.asset的价格变动是BM的，这个条件非常强，实际上，asset的价格变动非常复杂
    - 2.continuous hedging: 这会导致很大的transaction cost
    - 3.interest rate 不可能是一个常数
    - 但优点也很明显，就是除了σ其他的我们都很容易观察到
- method 2: three securities: option,asset, monry in the bank
    - process

$$\Pi(t) = M_V(t) + M_S(t) + M(t)$$
$$= Q_V(t)V + Q_S(t)S + M(t) = 0$$
$$d\Pi(t) = \underbrace{V\,dQ_V(t) + S_t\,dQ_S(t) + dM(t)}_{=0} \text{ 因为 self-financing}$$
$$Q_V(t)dV_t + Q_S(t)dS_t + rM(t)dt$$
$$dV_t = \Big(\underbrace{\frac{\partial V}{\partial t} + \rho\frac{\partial V}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}}_{V\rho_V}\Big)dt + \underbrace{\sigma S\frac{\partial V}{\partial S}}_{V\sigma_V}dz \qquad \frac{dS_t}{S} = \rho\,dt + \sigma\,dz$$
$$\therefore d\Pi_t = Q_V(t)(V\rho_V dt + V\sigma_V dz) + Q_S(t)(S\rho dt + S\sigma dz) + rM(t)dt$$
$$= dt\,(V\rho_V Q_V(s) + S\rho Q_S(t) + rM(t))$$
$$+ dz\,\underbrace{(VQ_V(t)\sigma_V + SQ_S(t)\sigma)}_{=0}$$
① $VQ_V(t)\sigma_V + SQ_S(t)\sigma = 0$.
$$\cancel{V}Q_V(t)\sigma S\frac{\partial V}{\partial S} = -Q_S(t)\sigma S \qquad \frac{Q_S(t)}{Q_V(t)} = -\frac{\partial V}{\partial S}$$
令 $Q_V(t) = -1$
则 $Q_S(t) = \frac{\partial V}{\partial S}$
② $d\Pi(t) = 0 \Rightarrow V\rho_V Q_V(s) + S\rho Q_S(t) + rM(t) = 0$
$$V\rho_V Q_V(s) - \cancel{S\rho\frac{\partial V}{\partial S}}Q_V(t) + rM(t) = 0$$
$$Q_V(s)\Big(\frac{\partial V}{\partial t} + \cancel{\rho S\frac{\partial V}{\partial S}} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}\Big) + rM(t) = 0$$
$$M(t) = -Q_V(t)V - Q_S(t)S$$
$$\therefore \cancel{Q_V(s)}\Big(\frac{\partial V}{\partial t} + \rho S\frac{\partial V}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}\Big) - r\cancel{Q_V(t)}V + r\cancel{Q_V(t)}S\frac{\partial V}{\partial S} = 0$$
$$\Rightarrow \frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} - rV + rS\frac{\partial V}{\partial S} = 0. \text{ 这也是我们的BS equation}$$

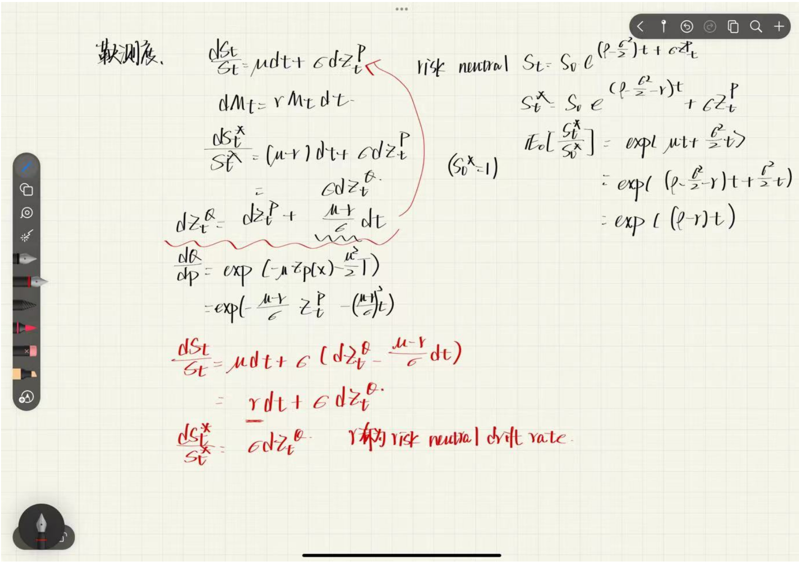
    - market price of risk:
        - 意思是，两个可以hedge的asset本质上应该是market price of risk 相同

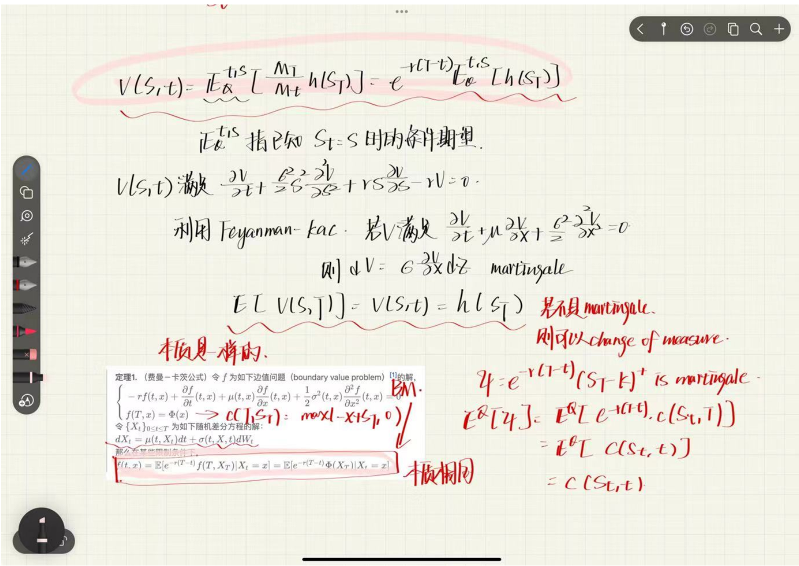
$$\underbrace{\frac{\rho_V - r}{\sigma_V}}_{\lambda_V} = \underbrace{\frac{\rho - r}{\sigma}}_{\lambda_S} \quad \Rightarrow \quad \text{Black-Scholes equation.}$$

        - 如果risk averse, 讨厌risk, 期望在loss的概率小的情况下，起码预期收益率是正的，如果是负的就不投资，所以他的market price of risk 是正的，

- 但如果是风险偏好，哪怕他有很大可能loss，但如果他的收益是非常大的正的，那他也会选择投资，所以预期的收益率并不一定非要大于r，所以market price of risk 是负的
- risk neutral , market price of risk is 0

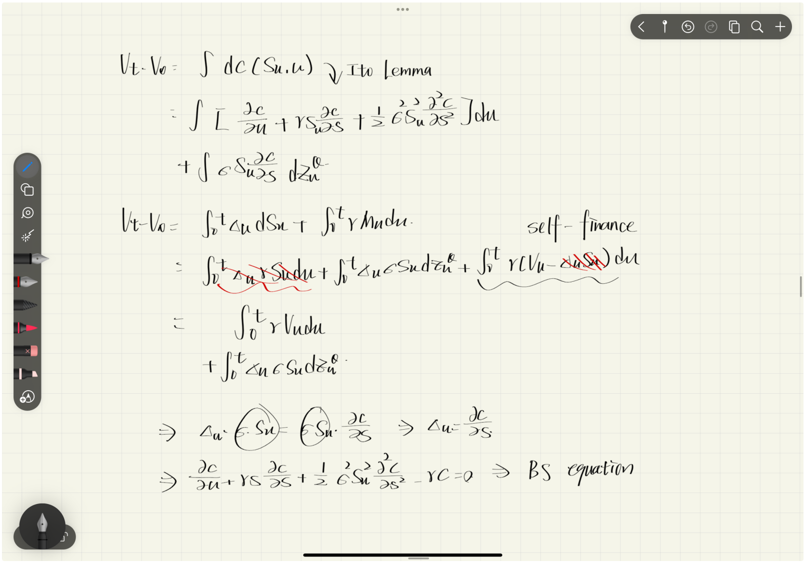
- method 3 : Risk neutral measure
  - 感觉这里像是在考虑风险中性测度和鞅测度两个测度下是相同的
  - 为什么相同



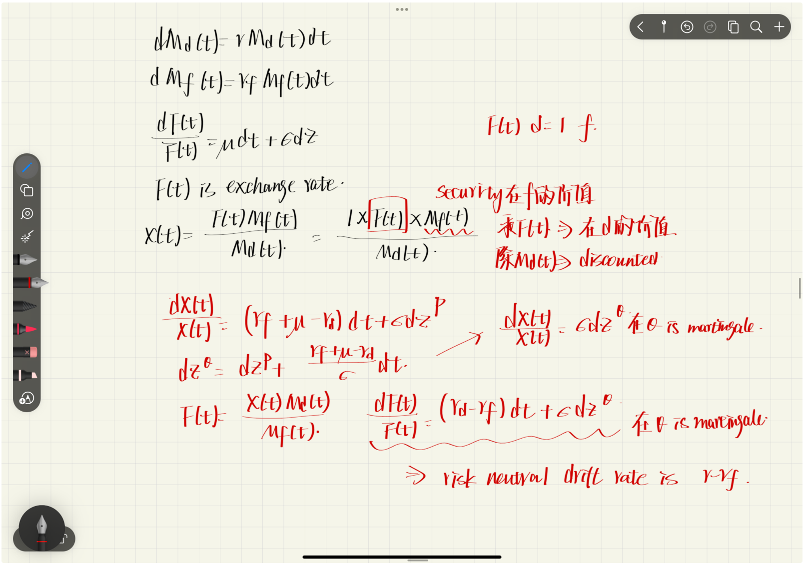


  - 这里可以和上面的method 1和method 2 相同

- method 4: BS equation
  - process



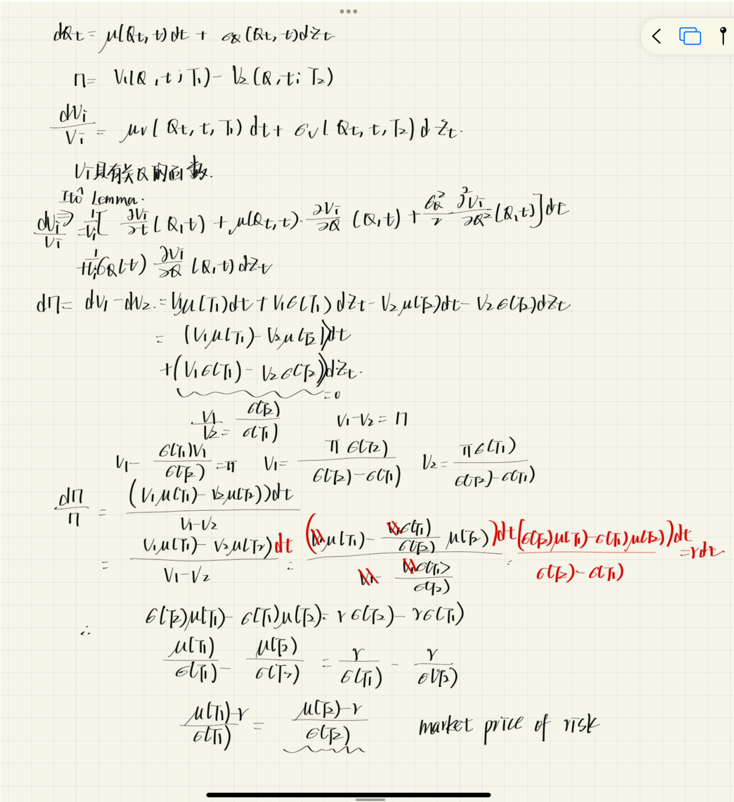
- example: exchange rate



- $\frac{\partial V}{\partial S}$ 有正有负
  - for call option, S越大，现货价值越大，执行的概率越大，c越大： $\max(S_T - X)$, $\frac{\partial V}{\partial S}$ >0
  - for put option, $\max(X - S_T)$, $\frac{\partial V}{\partial S}$ <0
- 总的理解，看一个这整个BS equation的式子，S_T是一个stochastic process, 由Z_t这个standard Brownian Motion组成，我们通过distribution判断这个S_t也是服从于正态分布，而c是由S_t和t共同决定的函数，是根据Ito process来求解的，我们认为若c=f(S_t),认为他和t没有直接关系，求偏导就直接等于0
- Remarks（纯理解）
  - 风险是什么？风险决定了超额收益，如果风险等于0，那么超额收益就等于0，根据上面market price of risk, risk=0,market price of risk =0, $\rho_v = \rho = r$
  - 我们假设的c只和S_t和t相关，S_t和process相关，我们并没有假设S_t和investors（ρ）相关，然后我们得到了一个和ρ相关的BS equation，但实际情况下，S_t肯定是和ρ相关的，所以这其实也是一个缺陷

- 上面推导的BS equation都是针对一个tradable的产品来讲的，如果是一个non-tradable呢？

- 首先注意：perfect hedging is impossible
- 因为不再是tradable security了，我们没办法直接知道他的μ和σ,只能通过用两个derivative（必须都是基于Q的这样就可以保证risk来源相同，但是又要让T不一样，这样才可以对冲，T_1, T_2）来对冲
- process



$$dQ_t = \mu(Q_t, t)dt + \sigma_Q(Q_t, t)dz_t$$

$$\Pi = V_1(Q, t; T_1) - V_2(Q, t; T_2)$$

$$\frac{dV_i}{V_i} = \mu_V(Q_t, t, T_1)dt + \sigma_V(Q_t, t, T_2)dz_t$$

V_i是有关Q的函数.

Itô Lemma.

$$\frac{dV_i}{V_i} = \frac{1}{V_i}\left[\frac{\partial V_i}{\partial t}(Q, t) + \mu(Q_t, t)\frac{\partial V_i}{\partial Q}(Q, t) + \frac{\sigma_Q^2}{2}\frac{\partial^2 V_i}{\partial Q^2}(Q, t)\right]dt + \frac{1}{V_i}\sigma_Q(t)\frac{\partial V_i}{\partial Q}(Q, t)dz_t$$

$$d\Pi = dV_1 - dV_2 = V_1\mu(T_1)dt + V_1\sigma(T_1)dz_t - V_2\mu(T_2)dt - V_2\sigma(T_2)dz_t$$
$$= (V_1\mu(T_1) - V_2\mu(T_2))dt + \underbrace{(V_1\sigma(T_1) - V_2\sigma(T_2))}_{=0}dz_t$$

$$\frac{V_1}{V_2} = \frac{\sigma(T_2)}{\sigma(T_1)} \qquad V_1 - V_2 = \Pi$$

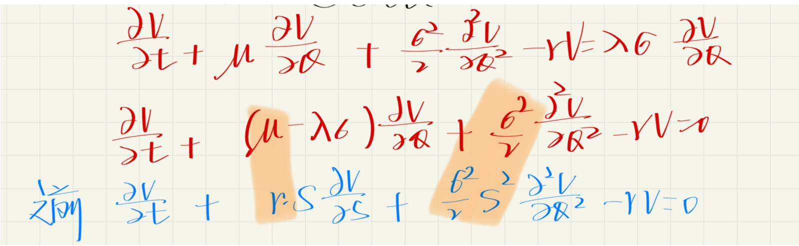
$$V_1 - \frac{\sigma(T_1)V_1}{\sigma(T_2)} = \Pi \qquad V_1 = \frac{\Pi\sigma(T_2)}{\sigma(T_2) - \sigma(T_1)} \qquad V_2 = \frac{\Pi\sigma(T_1)}{\sigma(T_2) - \sigma(T_1)}$$

$$\frac{d\Pi}{\Pi} = \frac{(V_1\mu(T_1) - V_2\mu(T_2))dt}{V_1 - V_2} = \frac{V_1\mu(T_1) - V_2\mu(T_2)dt}{V_1 - V_2} = \frac{\left(\cancel{V_1}\mu(T_1) - \frac{\cancel{V_1}\sigma(T_1)}{\sigma(T_2)}\mu(T_2)\right)dt}{\cancel{V_1} - \frac{\cancel{V_1}\sigma(T_1)}{\sigma(T_2)}} = \frac{(\sigma(T_2)\mu(T_1) - \sigma(T_1)\mu(T_2))dt}{\sigma(T_2) - \sigma(T_1)} = rdt$$

$$\therefore \sigma(T_2)\mu(T_1) - \sigma(T_1)\mu(T_2) = r\sigma(T_2) - r\sigma(T_1)$$

$$\frac{\mu(T_1)}{\sigma(T_1)} - \frac{\mu(T_2)}{\sigma(T_2)} = \frac{r}{\sigma(T_1)} - \frac{r}{\sigma(T_2)}$$

$$\frac{\mu(T_1) - r}{\sigma(T_1)} = \underbrace{\frac{\mu(T_2) - r}{\sigma(T_2)}}$$ market price of risk

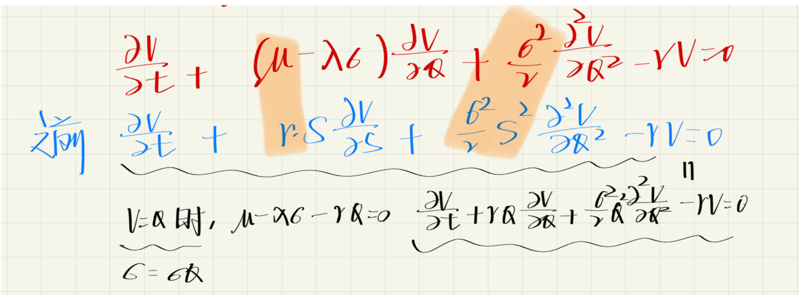


$$\frac{\partial V}{\partial t} + \mu\frac{\partial V}{\partial Q} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial Q^2} - rV = \lambda\sigma\frac{\partial V}{\partial Q}$$

$$\frac{\partial V}{\partial t} + (\mu - \lambda\sigma)\frac{\partial V}{\partial Q} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial Q^2} - rV = 0$$

对比 $\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} - rV = 0$

- 上述过程中，我们是利用两个衍生品来定价Q，但如果Q是tradable的，就直接是V=Q，
  - 如果加上$\sigma_Q = \sigma \cdot Q$，可以重新得到BS equation

  $$\frac{\partial V}{\partial t} + (\mu - \lambda\sigma)\frac{\partial V}{\partial Q} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial Q^2} - rV = 0$$

  对比 $\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} - rV = 0$

  $V = Q$时，$\mu - \lambda\sigma - rQ = 0$ $\quad \frac{\partial V}{\partial t} + rQ\frac{\partial V}{\partial Q} + \frac{\sigma^2}{2}Q^2\frac{\partial^2 V}{\partial Q^2} - rV = 0$

  $\sigma = \sigma Q$

  - 实际上，non-tradable 是没办法让V=Q的
- Dynamic hedging of a call option
  - 这里就是将存在银行的钱的时间价值和银行给的利息分开了，详细看/Users/xiaohehe/Desktop/hkust/Financial Derivatives/dynamic hedging.xlsx

- 这里的delta怎么算？根据市场上可以直接观察到的$\frac{\triangle V}{\triangle S}$
- option vale: 在dynamic replicate过程中，总共花费了5，263，300元，我们的X=50，所以在T时获得了5，000，000，多花了263，000元，折现到0时刻就是这个option的fair value

0.458×100,000×48.12

Week	Stock Price	Delta	Shares Purchased	Cost of Shares Purchased ($000)	Cumulative Cash Outflow ($000)	Interest Cost ($000)
0	49.00	0.522	52,200	2,557.8	2,557.8	2.5
1	48.12	0.458	(6,400)	(308.0)	2,252.3	2.2
2	47.37	0.400	(5,800)	(274.7)	1,979.8	1.9
3	50.25	0.596	19,600	984.9	2,966.6	2.9
4	51.75	0.693	9,700	502.0	3,471.5	3.3
5	53.12	0.774	8,100	430.3	3,905.1	3.8
6	53.00	0.771	(300)	(15.9)	3,893.0	3.7
7	51.87	0.706	(6,500)	(337.2)	3,559.5	3.4
8	51.38	0.674	(3,200)	(164.4)	3,398.5	3.3
9	53.00	0.787	11,300	598.9	4,000.7	3.8
10	49.88	0.550	(23,700)	(1,182.2)	2,822.3	2.7
11	48.50	0.413	(13,700)	(664.4)	2,160.6	2.1
12	49.88	0.542	12,900	643.5	2,806.2	2.7
13	50.37	0.591	4,900	246.8	3,055.7	2.9
14	52.13	0.768	17,700	922.7	3,981.3	3.8
15	51.88	0.759	(900)	(46.7)	3,938.4	3.8
16	52.87	0.865	10,600	560.4	4,502.6	4.3
17	54.87	0.978	11,300	620.0	5,126.9	4.9
18	54.62	0.990	1,200	65.5	5,197.3	5.0
19	55.87	1.000	1,000	55.9	5,258.2	5.1
20	57.25	1.000	0	0.0	5,263.3	

- 如果没有hedging,我必须在T时刻花5，725，000元买这么多股票去exercise，就亏损了725，000元，所以hedging了risk
- 注意，这里是没有transcation cost的，实际生活里还要考虑transcation cost, 所以不存在perfect hedging

- 如果exercise，$\triangle \to 1$, 反之趋于0
- 有没有发现，价格升高时，是购买stock，可以理解为，价格升高，exercise的几率变大所以需要屯股票

- 3.5 European option pricing formulars and their greeks
  - 这一节主要集中在求解上述的BS equation, 通过两种方法
  - Green function
    - perspective

Consider the initial value problem

$$LV(y,\tau) = 0, \quad V(y,0) = V_0(y)$$

where L is a linear differential operator and $V_0(y)$ is the initial condition.

unit impulse (source)

The first step is to solve for the Green function (fundamental solution) $\phi(y,\tau)$, where

$\tau=0$ Initial condition

$$L\phi(y,\tau) = 0, \quad \phi(y,0) = \delta(y).$$

Recall that the initial value function can be expressed as

$(y-\frac{\xi}{2}, y+\frac{\xi}{2})$

$$V_0(y) = \int_{-\infty}^{\infty} V_0(\xi)\delta(y-\xi)\, d\xi,$$

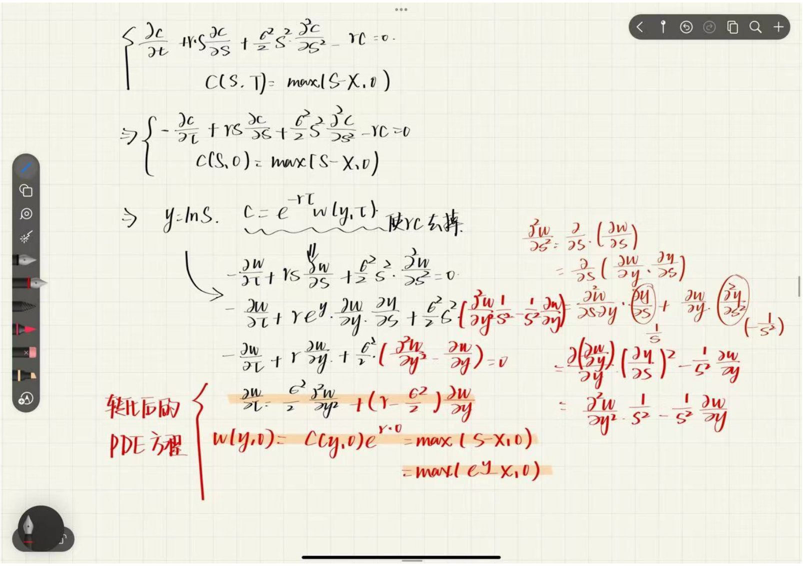
$\xi \to 0^+$

take average of $V_0(\xi)$ over $(y-\frac{\xi}{2}, y+\frac{\xi}{2})$

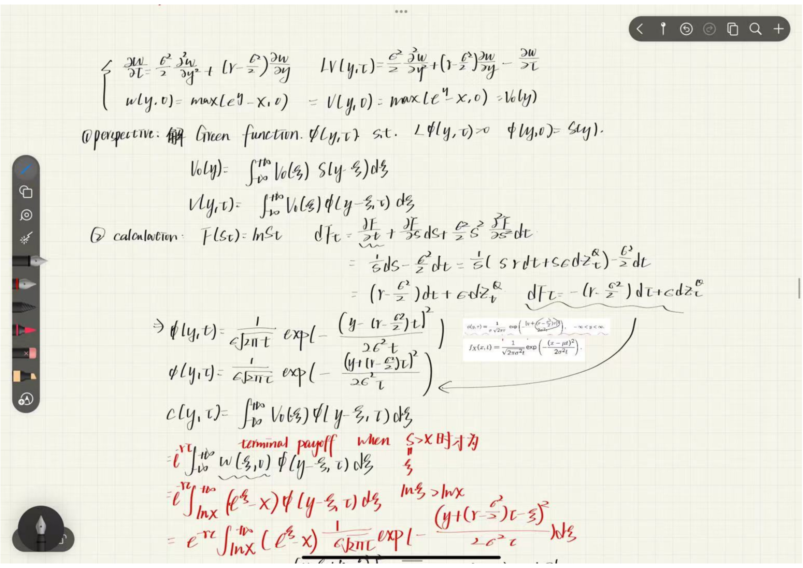
which is visualized as an **infinite sum of impulses**. By the superposition principle of linear differential equation, we obtain

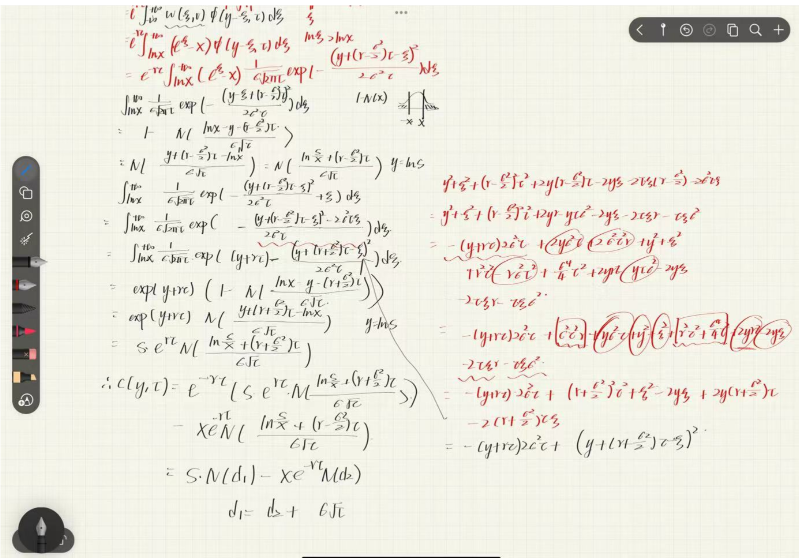
$$V(y,\tau) = \int_{-\infty}^{\infty} V_0(\xi)\phi(y-\xi,\tau)\, d\xi.$$

- 首先我们需要转化一下自变量，因为我们已知的价格是T时的价格，$c_T = \max\{S - X, 0\} = c(S, 0)$，即$\tau = T - t$时，这个option的价格是我们之前已知的max
- 这时这个pde转化成了



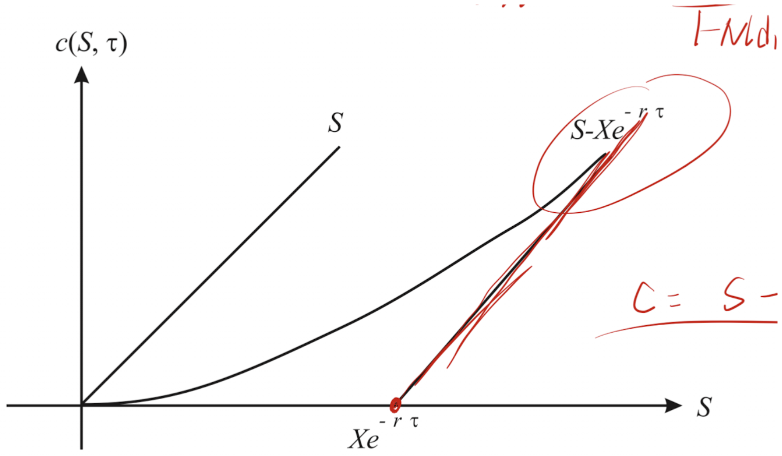
- 如何解这个pde呢？就是根据我们的perspective



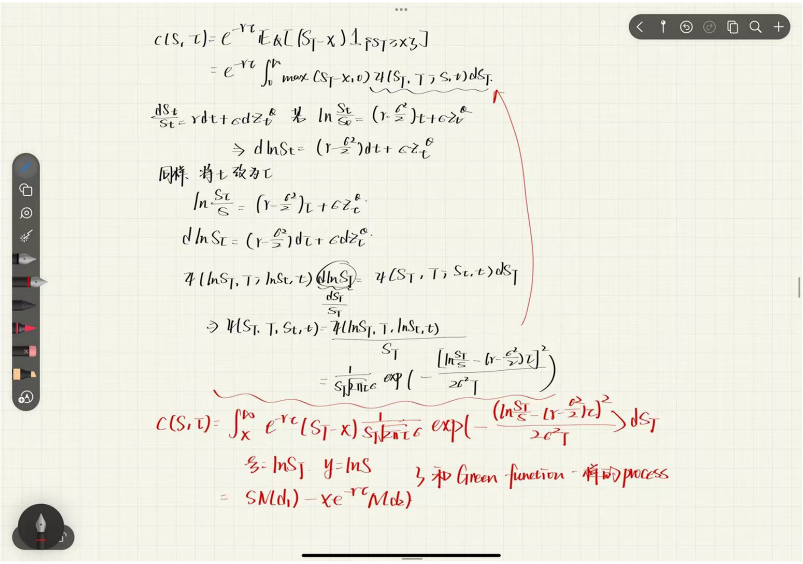


- 得到了解之后要验证这个解的边值条件：

- $\tau = 0, S = X, N(d_1) = N(d_2) = \frac{1}{2}, c = \frac{1}{2}(S - Xe^{-r\tau}) = 0$
- $\lim_{S \to 0} c(S, \tau) = 0$ deep out of money
- $\lim_{S \to \infty} c(S, \tau) = S - Xe^{-r\tau}$ deep in the money
- $\max(S - Xe^{-r\tau}, 0) \le c(S, \tau) \le S$



- Risk neutral transitio density function
  - process



$c(S,\tau) = e^{-r\tau} E_Q[(S_T - X) 1_{\{S_T \ge X\}}]$

$= e^{-r\tau} \int_0^{\infty} \max(S_T - X, 0)\, \psi(S_T, T; S, t)\, dS_T$

$\frac{dS_t}{S_t} = r\,dt + \sigma\, dZ_t^Q$ 若 $\ln \frac{S_t}{S_0} = (r - \frac{\sigma^2}{2})t + \sigma Z_t^Q$

$\Rightarrow d\ln S_t = (r - \frac{\sigma^2}{2})dt + \sigma Z_t^Q$

同样，将 t 改为 τ

$\ln \frac{S_\tau}{S} = (r - \frac{\sigma^2}{2})\tau + \sigma Z_\tau^Q$

$d\ln S_\tau = (r - \frac{\sigma^2}{2})d\tau + \sigma dZ_\tau^Q$

$\psi(\ln S_T, T; \ln S_t, t)\, \underbrace{d\ln S_T}_{\frac{dS_T}{S_T}} = \psi(S_T, T; S_t, t)\, dS_T$

$\Rightarrow \psi(S_T, T, S_t, t) = \frac{\psi(\ln S_T, T, \ln S_t, t)}{S_T}$

$= \frac{1}{S_T \sqrt{2\pi\tau}\,\sigma} \exp\left(-\frac{[\ln\frac{S_T}{S} - (r - \frac{\sigma^2}{2})\tau]^2}{2\sigma^2 \tau}\right)$

$c(S,\tau) = \int_X^{\infty} e^{-r\tau}(S_T - X) \frac{1}{S_T\sqrt{2\pi\tau}\,\sigma} \exp\left(-\frac{(\ln\frac{S_T}{S} - (r - \frac{\sigma^2}{2})\tau)^2}{2\sigma^2 \tau}\right) dS_T$

$\xi = \ln S_T \quad y = \ln S$ } 和 Green function 一样的 process

$= SN(d_1) - Xe^{-r\tau}N(d_2)$

- 通过对比，我们在这里赋予了很多之前未考虑的公式的经济意义：c表示t=0时，如果已知exercise, S的价格减去花的钱的折现，也就是未来如果已知exercise，赚的钱减去花的钱在当下的折现，确实是option的定义，于是在这里我们终于给出了option value的确切的数学公式

$$c(S,\tau) = e^{-r\tau}\mathbb{E}_Q[S_T \mathbb{1}_{\{S_T \geq X\}}] - e^{-r\tau}\mathbb{E}[X \mathbb{1}_{\{S_T \geq X\}}]$$

$$= SN(d_1) - Xe^{-r\tau}N(d_2)$$

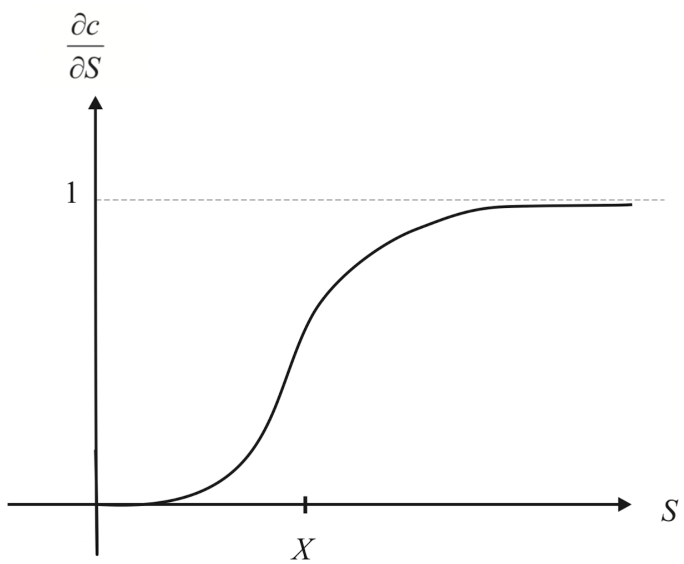
if exercise, S的现值 = $SN(d_1)$ (有折现) (conditioning on exercising S_T 在 $t=0$ 时的价格)

if exercise, $\mathbb{E}[\mathbb{1}_{\{S_T \geq X\}}] = Q(S_T \geq X) = N(d_2)$

$N(d_2)$ 表示 exercise 的概率

$Xe^{-r\tau}N(d_2)$ T时花的钱的期望折现到 $t=0$ 时刻)

- 且通过这个公式，我们还验证之前的一些性质
  - $\triangle_c = \frac{\partial c}{\partial S} = N(d_1) > 0$,也就是option value 随着S的增长而增长（验证完毕）
  - $\tau \to +\infty, \triangle_c = 1, \tau \to 0, \triangle_c \to$注意这里要分情况讨论:
    - 如果$S = X, \frac{1}{2}; S > X, 1; S < X, 0$



    - 凹凸性的转折点，小于是convex，大于是concave

The curve of $\triangle_c$ against S changes concavity at

$$S_c = X \exp\left(-\left(r + \frac{3\sigma^2}{2}\right)\tau\right)$$

so that the curve is concave upward for $0 \leq S < S_c$ and concave downward for $S_c < S < \infty$.

    - 当$\tau \to \infty, d_1 \to \infty, \triangle \to 1$,股价有足够多的时间高于X，所以执行的概率趋于1，$\triangle \to 1$
    - $\triangle_c$有不同的渐进极限，尤其是当S和X很相近时，下一刻不知道到底是会趋于exercise或者不exercise，这时风险很大，但幸运的是，之前已经积累了一部分股票，所以已经降低了风险，达到了hedging的效果

In the middle. depends on $\boxed{\Delta}$