Linear time series

- AR
  - AR(1) without a drift : $r_t = \phi_1 r_{t-1} + a_t$
    - another formula

$$\left.\begin{aligned} r_t &= \phi_1 r_{t-1} + a_t \\ r_{t-1} &= \phi_1 r_{t-2} + a_{t-1} \\ &\vdots \\ r_1 &= \phi_1 r_0 + a_1 \end{aligned}\right\}$$

$$\begin{aligned} r_t &= \phi_1(\phi_1 r_{t-2} + a_{t-1}) + a_t = \phi_1^2 r_{t-2} + \phi_1 a_{t-1} + a_t \\ &= \phi_1^2(\phi_1 r_{t-3} + a_{t-2}) + \phi_1 a_{t-1} + a_t \\ &= \phi_1^3 r_{t-3} + \phi_1^2 a_{t-2} + \phi_1 a_{t-1} + a_t \\ &\vdots \\ &= \phi_1^t r_0 + \phi_1^{t-1} a_1 + \dots + \phi_1 a_{t-1} + a_t \end{aligned}$$

我们可以一直往下写

We can let $r_0 = 0 \Leftarrow |\phi_1| < 1$. So ϕ_1^t 是很小. r_0 的影响小

$$\Rightarrow r_t = \sum_{i=0}^{\infty} \phi_1^i a_{t-i}$$

- stationary: if $|\phi_1| >= 1 \rightarrow r_t$不收敛
  - $|\phi_1| < 1$ mean reverting behavior：什么意思？当t-1时的收益小于0时，收益率的变化率的期望大于0，也就是说很有可能收益率会上升，也就是回归0均值，反之

$|\phi_1| < 1$, $-1 < \phi_1 < 1$ $\quad \Delta r_t = r_t - r_{t-1}$ $\quad r_t = \phi_1 r_{t-1} + a_t$

$$E[\Delta r_t \mid F_{t-1}] = E[\underbrace{(\phi_1 - 1)}_{<0} r_{t-1} + a_t \mid F_{t-1}] < 0 \quad \text{if } r_{t-1} > 0$$

$$E[\Delta r_t \mid F_{t-1}] > 0 \quad \text{if } r_{t-1} < 0$$

  - $|\phi_1| = 1$ random walk
  - $|\phi_1| > 1$ infinity, explosive process
- $E[r_t] = 0$
- $Var(r_t) = \frac{\sigma_a^2}{1-\phi_1^2}$
- ACF 指数递减

$$E[r_t] = E\left[\sum_{i=0}^{\infty} \phi_1^i a_{t-i}\right] = 0$$

$$Var(r_t) = \sum_{i=0}^{\infty} \phi_1^{2i} Var(a_{t-i}) = \sum_{i=0}^{\infty} \phi_1^{2i} \sigma_t^2 = \sigma_t^2 \frac{1}{1-\phi_1^2} = \frac{\sigma_t^2}{1-\phi_1^2}$$

$$\begin{aligned} \gamma_k = E[r_t r_{t-k}] &= E\left[\left(\sum_{i=0}^{\infty} \phi_1^i a_{t-i}\right)\left(\sum_{j=0}^{\infty} \phi_1^j a_{t-k-j}\right)\right] \\ &= E\left[\begin{aligned} &(a_t + \phi_1 a_{t-1} + \phi_1^2 a_{t-2} + \dots + \phi_1^k a_{t-k} + \phi_1^{k+1} a_{t-k-1} + \dots) \\ &(a_{t-k} + \phi_1 a_{t-k-1} + \dots) \end{aligned}\right] \\ &= (\phi_1^k + \phi_1^{k+2} + \dots)\sigma_a^2 \\ &= \frac{\phi_1^k}{1-\phi_1^2}\sigma_a^2 \end{aligned}$$

$$\Rightarrow \rho_k = \frac{\phi_1^k}{1-\phi_1^2}\sigma_a^2 \Big/ \frac{\sigma_a^2}{1-\phi_1^2} = \phi_1^k \quad \Rightarrow \text{ACF 指数递减}$$

- AR(1) with a drift: $r_t = \phi_1 r_{t-1} + a_t + \phi_0$
  - properties

$$r_t = \phi_0 + \phi_1 r_{t-1} + a_t$$
$$= \phi_0 + \phi_1(\phi_0 + \phi_1 r_{t-2} + a_{t-1}) + a_t$$
$$= \phi_0 + \phi_0\phi_1 + \phi_1^2 r_{t-2} + \phi_1 a_{t-1} + a_t$$
$$= \phi_0 + \phi_0\phi_1 + \phi_1^2(\phi_0 + \phi_1 r_{t-3} + a_{t-2}) + \phi_1 a_{t-1} + a_t$$
$$= \phi_0(1 + \phi_1 + \phi_1^2) + \phi_1^3 r_{t-3} + \phi_1^2 a_{t-2} + \phi_1 a_{t-1} + a_t$$
$$\vdots$$
$$= \phi_0 \frac{1}{1-\phi_1} + \sum_{i=0}^{\infty} \phi_1^i a_{t-i}$$
$$\Rightarrow \mathbb{E}[r_t] = \frac{\phi_0}{1-\phi_1}$$
$$Var(r_t) = \frac{\phi_0}{1-\phi_1} + \sum_{i=0}^{\infty} \phi_1^{2i} \sigma_t^2 = \underbrace{\frac{\phi_0}{1-\phi_1} + \frac{\sigma_t^2}{1-\phi_1^2}} \text{ also not changed}$$
$$r_k = \mathbb{E}\left[\left(r_t - \frac{\phi_0}{1-\phi_1}\right)\left(r_{t-k} - \frac{\phi_0}{1-\phi_1}\right)\right] \Rightarrow \text{not changed}$$

- forecast
  - 1-2 step

$$\hat{r}_n(1) = \mathbb{E}[r_{n+1} | r_n] = \mathbb{E}[\phi_0 + \phi_1 r_n + a_{n+1} | r_n]$$
$$= \phi_0 + \phi_1 r_n$$
$$e_n(1) = r_{n+1} - \hat{r}_n(1) = \phi_0 + \phi_1 r_n + a_{n+1} - \phi_0 - \phi_1 r_n = a_{n+1}$$
$$Var(e_n(1)) = Var(a_{n+1}) = \sigma_a^2$$
$$\hat{r}_n(2) = \mathbb{E}[r_{n+2} | r_n] = \mathbb{E}[\phi_0 + \phi_1(\phi_0 + \phi_1 r_n + a_{n+1}) + a_{n+2} | r_n]$$
$$= \phi_0 + \phi_1\phi_0 + \phi_1^2 r_n$$
$$e_n(2) = r_{n+2} - \hat{r}_n(2) = \phi_1 a_{n+1} + a_{n+2}$$
$$Var(e_n(2)) = (\phi_1^2 + 1)\sigma_a^2 > Var(e_n(1)) \Rightarrow \text{step} \uparrow \text{ uncertainty} \uparrow$$

  - l step

$$
\begin{aligned}
\hat{r}_n(l) &= \mathbb{E}[r_{n+l} \mid r_n] = \mathbb{E}[\phi_0 + \phi_1 r_{n+l-1} + a_{n+l} \mid r_n] \\
&= \phi_0 + \phi_1 \hat{r}_n(l-1) \\
&= \phi_0 + \phi_1(\phi_0 + \phi_1 \hat{r}_n(l-2)) \\
&= \phi_0 + \phi_0\phi_1 + \phi_1^2(\phi_0 + \phi_1 \hat{r}_n(l-3)) \\
&= \phi_0(1+\phi_1+\phi_1^2) + \phi_1^3 \hat{r}_n(l-3) \\
&\vdots \\
&= \phi_0(1+\phi_1+\phi_1^2+\dots+\phi_1^{l-2}) + \phi_1^{l-1}\underbrace{\hat{r}_n(1)}_{\phi_0+\phi_1 r_n} \\
&= \phi_0(1+\phi_1+\dots+\phi_1^{l-1}) + \phi_1^l r_n
\end{aligned}
$$

$$
\begin{aligned}
r_{n+l} &= \phi_0 + \phi_1 r_{n+l-1} + a_{n+l} \\
&= \phi_0 + \phi_1(\phi_0 + \phi_1 r_{n+l-2} + a_{n+l-1}) + a_{n+l} \\
&= \phi_0(1+\phi_1) + \phi_1^2 r_{n+l-2} + \phi_1 a_{n+l-1} + a_{n+l} \\
&= \phi_0(1+\phi_1+\phi_1^2) + \phi_1^3 r_{n+l-3} + \phi_1^2 a_{n+l-2} + \phi_1 a_{n+l-1} + a_{n+l} \\
&\vdots \\
&= \phi_0(1+\phi_1+\dots+\phi_1^{l-1}) + \phi_1^l r_n + \phi_1^{l-1} a_{n+1} + \dots + a_{n+l}
\end{aligned}
$$

$$\Rightarrow \hat{e}_n(l) = \phi_1^{l-1} a_{n+1} + \dots + a_{n+l} = \sum_{i=0}^{l-1} \phi_1^i a_{n+l-i}$$

$$Var(\hat{e}_n(l)) = (\phi_1^{2(l-1)} + \dots + 1)\sigma_a^2 = \frac{1-\phi_1^{2l}}{1-\phi_1^2}\sigma_a^2 = \sigma_a^2 \sum_{i=0}^{l-1} \phi_1^{2i}$$

- Forcast interval

$$r_{n+l} = \underbrace{\hat{r}_n(l)}_{\mu} + \underbrace{\hat{e}_n(l)}_{Var} \leftarrow \phi_1^{l-1} a_{n+1} + \dots + a_{n+l}$$

也不是iid呀?

WN 但不是 normal? 难道是因为 $n \to \infty$, $S_0 \to N(0, \sigma_a^2)$

$$\Rightarrow r_{n+1} \mid F_n \sim N\left(\hat{r}_n(1), \sqrt{\sigma_a^2 \sum_{j=0}^{l-1} \phi_1^{2j}}\right)$$ r_n此时是 constant

$$\Rightarrow \text{F.I.}\left(\hat{r}_n(1) - N_{\frac{\alpha}{2}} \sigma_a \sqrt{\sum_{j=0}^{l-1} \phi_1^{2j}},\ \hat{r}_n(1) + N_{\frac{\alpha}{2}} \sigma_a \sqrt{\sum_{j=0}^{l-1} \phi_1^{2j}}\right)$$

- compact form

$$
\begin{aligned}
r_t &= \phi_0 + \phi_1 r_{t-1} + a_t \\
&= \phi_0 + \phi_1 B r_t + a_t \\
(1-\phi_1 B) r_t &= \phi_0 + a_t
\end{aligned}
$$

- AR(2) without a drift: $r_t = \phi_1 r_{t-1} + \phi_2 r_{t-2} + a_t$
  - stationary condition

$r_t = \phi_0 + \phi_1 r_{t-1} + a_t$

$= \phi_0 + \phi_1 B r_t + a_t$

$(1-\phi_1 B) r_t = \phi_0 + a_t$

$r_t = \phi_1 r_{t-1} + \phi_2 r_{t-2} + a_t$

$= \phi_1 B r_t + \phi_2 B^2 r_t + a_t$

$(1-\phi_1 B-\phi_2 B^2) r_t = a_t$

=0 的解在单位圆之外 $\Uparrow$

$= (1-\alpha_1 z)(1-\alpha_2 z) \quad z_1 = \frac{1}{\alpha_1} \underset{>1}{} \ z_2 = \frac{1}{\alpha_2} > 1 \Leftarrow (|\alpha_1|<1 \quad |\alpha_2|<1)$

$\Rightarrow (1-\alpha_1 B)(1-\alpha_2 B) r_t = a_t$

We let $u_t = (1-\alpha_2 B) r_t$

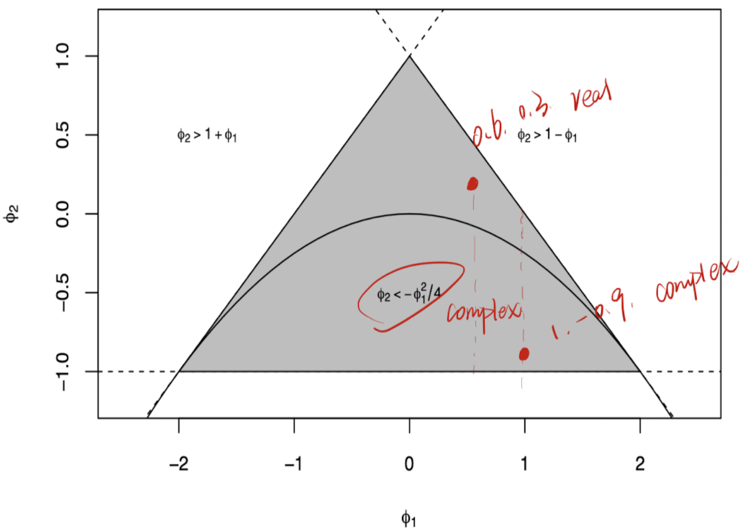
$(1-\alpha_1 B) u_t = a_t \quad u_t = \alpha_1 u_{t-1} + a_t = \sum_{i=0}^{\infty} \alpha_1^i a_{t-i}$ $\left| \begin{array}{l} |\alpha_1|<1 \\ |\alpha_2|<1 \end{array} \right. \Rightarrow$ stationary

$r_t = \alpha_2 r_{t-1} + u_t = \sum_{i=0}^{\infty} \alpha_2^i u_{t-i}$

Stationarity condition is equivalent to

$$\begin{cases} \phi_2 + \phi_1 < 1, \\ \phi_2 - \phi_1 < 1, \\ -1 < \phi_2 < 1. \end{cases}$$

- but some special cases

  - if in the triangle but not in the semi-circle: two real roots
  - if in the semi-circle but not in the triangle: two complex roots
  - 没有可能一个是实根一个虚数根，边界值not station, outside the triangle--> explosive

- ACF

$$\Rightarrow r_t = \sum_{i=0}^{\infty} \alpha_2^i u_{t-i} = \sum_{i=0}^{\infty} \alpha_2^i \left(\sum_{j=0}^{\infty} \alpha_1^j a_{t-i-j}\right)$$

$$\mathbb{E}[r_t] = 0$$

$$\gamma_1 = \mathbb{E}[r_t r_{t-1}] = \mathbb{E}[(\phi_1 r_{t-1} + \phi_2 r_{t-2} + a_t) r_{t-1}]$$

$$= \mathbb{E}[\phi_1 r_{t-1}^2 + \phi_2 \underbrace{r_{t-2} r_{t-1}} + a_t r_{t-1}]$$

$$= \phi_1 \gamma_0 + \phi_2 \gamma_1$$

$$\gamma_1 = \frac{\phi_1 \gamma_0}{1-\phi_2} \quad \rho_1 = \frac{\phi_1}{1-\phi_2}$$

$$\gamma_k = \mathbb{E}[r_t r_{t-k}] = \mathbb{E}[(\phi_1 r_{t-1} + \phi_2 r_{t-2} + a_t) r_{t-k}]$$

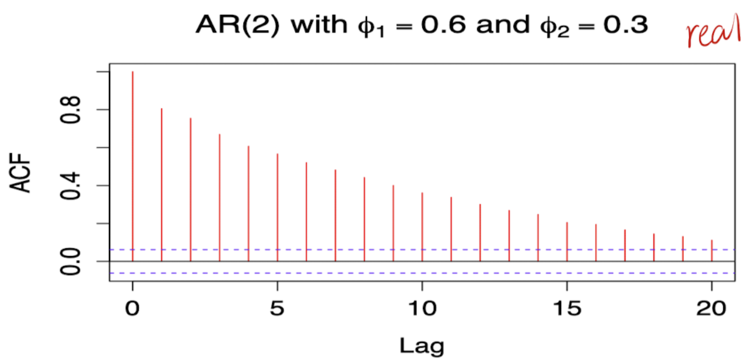
$$= \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2}$$

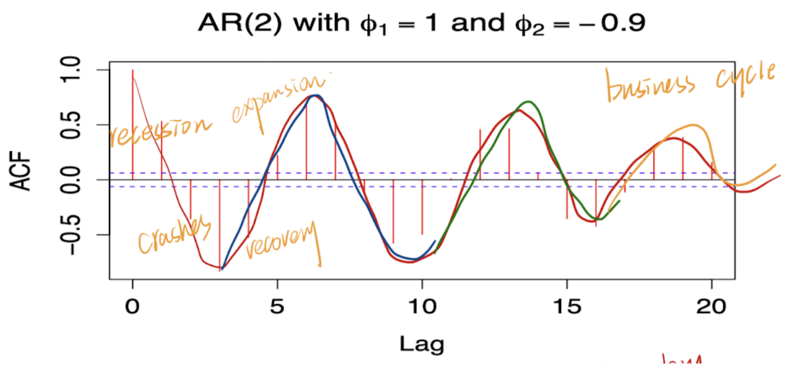
$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}$$

- ω_1, ω_2是系数方程式的、两个解
    - if two real solutions

$$\rho_l = \frac{c_1}{\omega_1^l} + \frac{c_2}{\omega_2^l}, \quad \omega_1 > 1 \quad \omega_2 > 1$$

    - if two complex solutions,damoing sine and cosine waves





- 根据上述的图-->business cycle <-two complex solutions
    - average length

$$k = \frac{2\pi}{cos^{-1}[\phi_1/(2\sqrt{-\phi_2})]}$$

- forcast
  - E

$$E[r_t] = \phi_0 + \phi_1 E[r_{t-1}] + \phi_2 E[r_{t-2}]$$

$$E = \frac{\phi_0}{1-\phi_1-\phi_2}$$

- AR(p)

$$r_t = \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + \cdots + \phi_p r_{t-p} + a_t$$

  - stationary : roots all are outside the unit circle
  - order specification:
    - ACF 递减
    - PACF找不等于0的点，ϕ_{kk}

$$Corr(r_t, r_{t+k}|r_{t+1}, \cdots, r_{t+k-1}) = \frac{Cov[(r_t - \hat{r}_t)(r_{t+k} - \hat{r}_{t+k})]}{\sqrt{Var(r_t - \hat{r}_t)Var(r_{t+k} - \hat{r}_{t+k})}}$$

Formula: $\phi_{11} = \rho_1$,

$$\phi_{kk} = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_1 \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_2 \\ & & \cdots & & & \\ & & \cdots & & & \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & \rho_k \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{k-2} & \rho_{k-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{k-3} & \rho_{k-2} \\ & & \cdots & & & \\ & & \cdots & & & \\ \rho_{k-1} & \rho_{k-2} & \rho_{k-3} & \cdots & \rho_1 & 1 \end{vmatrix}}$$

- sample PACF --> ρ不是真实的，而是估计出来的

$$\hat{\phi}_{kk} = \frac{\begin{vmatrix} 1 & \hat{\rho}_1 & \hat{\rho}_2 & \cdots & \hat{\rho}_{k-2} & \hat{\rho}_1 \\ \hat{\rho}_1 & 1 & \hat{\rho}_1 & \cdots & \hat{\rho}_{k-3} & \hat{\rho}_2 \\ & & \cdots & & & \\ & & \cdots & & & \\ \hat{\rho}_{k-1} & \hat{\rho}_{k-2} & \hat{\rho}_{k-3} & \cdots & \hat{\rho}_1 & \hat{\rho}_k \end{vmatrix}}{\begin{vmatrix} 1 & \hat{\rho}_1 & \hat{\rho}_2 & \cdots & \hat{\rho}_{k-2} & \hat{\rho}_{k-1} \\ \hat{\rho}_1 & 1 & \hat{\rho}_1 & \cdots & \hat{\rho}_{k-3} & \hat{\rho}_{k-2} \\ & & \cdots & & & \\ & & \cdots & & & \\ \hat{\rho}_{k-1} & \hat{\rho}_{k-2} & \hat{\rho}_{k-3} & \cdots & \hat{\rho}_1 & 1 \end{vmatrix}}$$

- when k>p 时，$\phi_{kk} = 0$

  $j>p,\ \rho_j = \frac{cov(r_t, r_{t-j})}{\gamma_0} = \frac{E[(r_t-\mu)(r_{t-j}-\mu)]}{\gamma_0}$

  r_t 和 r_{t-j} 中没有相同项 $\Rightarrow \rho_j = 0 \Rightarrow \phi_{jj} \approx 0$

  $\hat{\phi}_{jj}$ 是 estimate $\rightarrow$ 不一定是 0，但 ≈ 0

  $\hat{\phi}_{kk} \sim N(0, \frac{1}{n})$ n is number of data

- estimation
  - residuals

    true: $r_t = \phi_{00} + \phi_{01} r_{t-1} + \phi_{02} r_{t-2} + \dots + \phi_{0p} r_{t-p} + a_t$ $\quad \tilde{\phi}_0 = (\phi_{00}, \phi_{01} \dots \phi_{0p})'$

    estimate: $r_t = \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + \dots + \phi_p r_{t-p} + a_t$ $\quad \phi = (\phi_0, \phi_1 \dots \phi_p)'$

    $a_t(\phi) = r_t - \phi_0 - \phi_1 r_{t-1} - \dots - \phi_p r_{t-p}$

    $a_t(\phi) = a_t \Leftrightarrow \phi = \tilde{\phi}_0$

    $a_t(\phi)$ is residuals

  - how to get a best estimator? 判断ϕ是否等于ϕ_0

    min Σ residual^2 $\quad S_*(\phi) = \sum_{t=1}^{n} a_t^2(\phi)$

    derivative

    $S_*'(\phi) = \sum_{t=1}^{n} 2a_t(\phi) \frac{\partial a_t(\phi)}{\partial \phi}$

    $= \sum_{t=1}^{n} 2(r_t - \tilde{z}_{t-1}'\phi)(-\tilde{z}_{t-1}')$

    $= -2\sum_{t=1}^{n} r_t \tilde{z}_{t-1} + 2\sum_{t=1}^{n} \tilde{z}_{t-1}'^2 \phi = 0$

    derivative

    $\phi = (\sum_{t=1}^{n} r_t \tilde{z}_{t-1}) \cdot (\sum_{t=1}^{n} \tilde{z}_{t-1}'^2)^{-1}$ best ϕ

    $S_*''(\phi) = 2\sum_{t=1}^{n} \tilde{z}_{t-1}'^2 \geq 0$

    $\Rightarrow \hat{\sigma}_a^2 = \frac{1}{n} \sum_{t=1}^{n} a_t(\hat{\phi})^2$ 是否等于 σ_a^2 ? $\Rightarrow$ hypothesis

    how to prove ϕ & $\tilde{\phi}_0$ is close? $\Rightarrow$ hypothesis.

    $\sqrt{n}(\hat{\phi} - \phi_0) \sim N(0, \hat{\Omega})$ $\quad \sqrt{n}(\hat{\phi}_i - \phi_{0i}) \sim N(0, \hat{\sigma}_{ii})$ $\quad \hat{\Omega} = (\sum_{t=1}^{n} z_{t-1}^2)^{-1} \hat{\sigma}_a^2$

- model checking 判断residuals 是否约等于0
    - acf of residuals =0
    - Box test: 和前面其实异曲同工，这不是是判断所有的residuals是否都等于0，这里要p足够大，才能认为H_0 正确，也就是residuals 都等于0， g是number of parameters

      $$Q(m) = T(T+2)\sum_{k=1}^{m}\frac{\hat{\rho}_k^2}{T-k} \sim \chi^2_{m-g}$$
- model selection: 很多model其实应该都正确，怎么比较模型之间的好坏呢?
    - AIC

      $$AIC(l) = \ln(\hat{\sigma}_l^2) + \frac{2l}{T},$$
    - BIC

      $$BIC(l) = \ln(\hat{\sigma}_l^2) + \frac{l\ln(T)}{T}.$$

- MA
  - MA(1)
    - $r_t = \mu + a_t - \theta a_{t-1}$
    - always stationary
    - properties

      $E[r_t] = \mu$. $Var(r_t) = \mu + (1+\theta^2)\sigma_a^2$.

      $\gamma_1 = cov(r_t, r_{t-1}) = E[(r_t-\mu)(r_{t-1}-\mu)]$

      $= E[(a_t - \theta a_{t-1})(a_{t-1} - \theta a_{t-2})]$ $\quad r_{t-2}-\mu$

      $= -\theta\sigma_a^2$

      $\gamma_2 = E[(a_t-\theta a_{t-1})(a_{t-2} - \theta a_{t-3})] = 0$.

      $\gamma_k = 0$. $k>1$

      $\Rightarrow$ ACF是什么dependence 现在ACF=0

      $\Rightarrow$ r_t与r_{t-1}, r_{t-2}... 无关.
    - 有讲过PACF的公式吗
    - forecast

$$r_t = \mu + a_t - \theta a_{t-1}$$

$$\hat{r}_n(1) = E[r_{n+1} \mid F_n] = E[\mu + a_{n+1} - \theta a_n \mid F_n]$$
$$= \mu - \theta a_n$$

$$e_n(1) = r_{n+1} - \hat{r}_n(1) = \mu + a_{n+1} - \theta a_n - \mu + \theta a_n = a_{n+1}$$

$$\hat{r}_n(l) = E[r_{n+l} \mid F_n] = E[\mu + a_{n+l} - \theta a_{n+l-1} \mid F_n]$$
$$= \mu$$

MA 只有前一个时间的记忆

$$\hat{e}_n(l) = r_{n+l} - \mu = a_{n+l} - \theta a_{n+l-1}$$

$$Var(\hat{e}_n(l)) = \begin{cases} \sigma_a^2 & l=1 \\ (1+\theta^2)\sigma_a^2 & l>1 \end{cases}$$

- invertibility: the dependence of r_t on r_{t-l}--> 0 when l increases, $|\theta| < 1$
  - get a_t easily
  - dual property

$$r_n = \mu + a_n - \theta a_{n-1} \qquad a_n = r_n + \theta a_{n-1} - \mu$$
$$a_{n-1} = r_{n-1} + \theta a_{n-2} - \mu$$
$$\vdots$$
$$a_2 = r_2 + \theta a_1 - \mu$$

$$MA(1) \Rightarrow a_n = r_n - \mu + \theta(r_{n-1} - \mu + \theta a_{n-2})$$
$$= (r_n - \mu) + \theta(r_{n-1} - \mu) + \theta^2(r_{n-2} - \mu + \theta a_{n-3})$$
$$= \cdots$$
$$= \sum_{i=0}^{n} (r_{n-i} - \mu)\theta^i + \theta^n a_0 \quad \text{when } |\theta|<1 \;\; a_0 \text{ can neglect}$$
$$= \sum_{i=1}^{n} (r_{n-i} - \mu)\theta^i + r_t - \mu \Rightarrow \sum_{i=1}^{\infty} (r_{n-i} - \mu)\theta^i + r_t - \mu$$

$$AR(\infty) \Rightarrow r_n = a_n + \mu - \sum_{i=1}^{\infty} (r_{n-i} - \mu)\theta^i$$
$$= \sum_{i=1}^{\infty} r_{n-i} + a_n + \frac{\mu}{1-\theta} \Rightarrow \text{dual property}$$

- MA(2)
  - $r_t = \mu + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2}$
  - properties

2. $\mathbb{E}[r_t]=\mu$ $\quad Var(r_t)=(1+\theta_1^2+\theta_2^2)\sigma_a^2$

3. Invertibility. $r_t=\mu+(1-\theta_1 B-\theta_2 B^2)a_t$ $\Rightarrow$ solutions outside the unit circle

$$= \mu+(1-\alpha_1 B)\underbrace{(1-\alpha_2 B)a_t}_{u_t}$$

$u_t=\alpha_1 u_{t-1}+r_t-\mu$ $\quad AR(1)\Rightarrow |\alpha_1|<1$

$a_t=\alpha_2 a_{t-2}+u_t$ $\quad AR(1)\Rightarrow |\alpha_2|<1$

4. $\gamma_1=\mathbb{E}[(r_t-\mu)(r_{t-1}-\mu)]=\mathbb{E}[(a_t-\theta_1 a_{t-1}-\theta_2 a_{t-2})(a_{t-1}-\theta_1 a_{t-2}-\theta_2 a_{t-3})]$

$$=(-\theta_1+\theta_1\theta_2)\sigma_a^2=-\sigma_a^2\theta_1(1-\theta_2)$$

$$\rho_1=\frac{-\sigma_a^2\theta_1(1-\theta_2)}{\sigma_a^2(1+\theta_1^2+\theta_2^2)}=\frac{-\theta_1(1-\theta_2)}{1+\theta_1^2+\theta_2^2}$$

$\gamma_2=\mathbb{E}[(r_t-\mu)(r_{t-2}-\mu)]=\mathbb{E}[(a_t-\theta_1 a_{t-1}-\theta_2 a_{t-2})(a_{t-2}-\theta a_{t-3}-\theta_2 a_{t-4})]$

$$=-\theta_2\sigma_a^2$$

$$\rho_2=\frac{-\theta_2\sigma_a^2}{\sigma_a^2(1+\theta_1^2+\theta_2^2)}=\frac{-\theta_2}{1+\theta_1^2+\theta_2^2}$$

$\gamma_3=0$ $\quad \rho_3=0$

$\Rightarrow$ always stationary

5. forcast

$\hat{r}_n(1)=\mathbb{E}[r_{n+1}\mid F_n]=\mathbb{E}[\mu+a_{n+1}-\theta_1 a_n-\theta_2 a_{n-1}\mid F_n]$

$$=\mu-\theta_1 a_n-\theta_2 a_{n-1}$$

$\hat{e}_n(1)=\cancel{\mu}+a_{n+1}-\cancel{\theta_1 a_n}-\cancel{\theta_2 a_{n-1}}-\cancel{\mu}+\cancel{\theta_1 a_n}+\cancel{\theta_2 a_{n-1}}=a_{n+1}$

$\hat{r}_n(2)=\mathbb{E}[r_{n+2}\mid F_n]=\mathbb{E}[\mu+a_{n+2}-\theta_1 a_{n+1}-\theta_2 a_n\mid F_n]=\mu-\theta_2 a_n$

$\hat{e}_n(2)=a_{n+2}-\theta_1 a_{n+1}$

$\hat{r}_n(3)=E[r_{n+3}\mid F_n]=\mathbb{E}[\mu+a_{n+3}-\theta_1 a_{n+2}-\theta_2 a_{n+1}\mid F_n]=\mu$

$\hat{e}_n(3)=a_{n+3}-\theta_1 a_{n+2}-\theta_2 a_{n+1}$

$\Rightarrow$ go to mean after 2 periods.

- MA(q)
  - definition

$$r_t=\mu+a_t-\theta_1 a_{t-1}-\theta_2 a_{t-2}-\cdots-\theta_q a_{t-q},$$

  - invertibility

all the roots of $\theta(z)=1-\theta_1 z-\theta_2 z^2-\cdots-\theta_q z^q=0$
lie outside the unit circle.

  - building an MA model
    - order<- ACF

$E=\mu$

$\gamma_1=\mathbb{E}[(r_t-\mu)(r_{t-1}-\mu)]$

$$=\mathbb{E}[(a_t-\theta_1 a_{t-1}-\theta_2 a_{t-2}-\dots\theta_q a_{t-q})(a_{t-1}-\theta_1 a_{t-2}-\dots-\theta_q a_{t-q-1})]$$

$$=-\theta_1+\theta_1\theta_2+\dots+\theta_q\theta_{q+1}$$

when $k=q+1$

$\gamma_{q+1}=\mathbb{E}[(r_t-\mu)(r_{t-q-1}-\mu)]$

$$=\mathbb{E}[(a_t-\theta_1 a_{t-1}-\theta_2 a_{t-2}-\dots-\theta_q a_{t-q})(a_{t-q-1}\dots)]=0$$

$\Rightarrow$ ACF 在 $q+1$ 之后 $=0$

    - estimation
      - MA(1) with $|\theta_0<1|$
      - 我们之前这里是利用求导，令一阶导等于0，求最大值，但这里很困难

$\Rightarrow$ ACF 在 q+1之后=0

$r_t = a_t - \theta_0 a_{t-1}$ $\hat{r}_t = -\theta a_{t-1}$ $r_t = a_t(\theta) - \theta a_{t-1}$

$S_n(\theta) = \sum_{i=1}^{n} [r_t - (-\theta a_{t-1})]^2$ 对θ求导 $= \sum_{i=1}^{n} a_t(\theta)^2$

我们不知道a_{t-1} $a_1(\theta) = r_1 + \theta \times 0$

$a_2(\theta) = r_2 + \theta a_1(\theta)$

$\vdots$

$a_t(\theta) = r_t + \theta a_{t-1}(\theta)$

$\frac{\partial S_n(\theta)}{\partial \theta} = \sum_{i=1}^{n} 2 \cdot a_t(\theta) \frac{\partial a_t(\theta)}{\partial \theta}$

$\frac{\partial^2 S_n(\theta)}{\partial \theta^2} = \sum_{i=1}^{n} 2\left(\frac{\partial a_t(\theta)}{\partial \theta}\frac{\partial a_t(\theta)}{\partial \theta} + a_t(\theta)\frac{\partial^2 a_t(\theta)}{\partial \theta^2}\right)$ too complicated

$\Rightarrow$ iteration we want $\frac{\partial S_n(\theta)}{\partial \theta} = 0$ can't find θ

$\Rightarrow$ find $\xi_0 \in (-1,1)$ $\underbrace{\frac{\partial S_n(\theta)}{\partial \theta}}_{=0} - \frac{\partial S_n(\xi_0)}{\partial \theta} = \frac{\partial^2 S_n(\xi_1^*)}{\partial \theta^2}(\hat{\theta} - \xi_0)$ $f(x) = f(a) + f'(a)(x-a) + f''(a)(x-a)^2 + \dots$

$\hat{\theta} = \xi_0 - \left[\frac{\partial^2 S_n(\xi_1^*)}{\partial \theta^2}\right]^{-1} \frac{\partial S_n(\xi_0)}{\partial \theta}$ 有余项 最后一项 can't find ξ_1^* $a \to \xi$

$\Rightarrow \xi_0 \to \xi_1^*$ $\xi_1 = \xi_0 - \left[\frac{\partial^2 S_n(\xi_0)}{\partial \theta^2}\right]^{-1} \frac{\partial S_n(\xi_0)}{\partial \theta}$ 令$a = \xi_0$ $f(x) = f(a) + f'(\xi)(x-a)$

$\Rightarrow \underbrace{\frac{\partial S(\theta)}{\partial \theta}}_{=0} - \frac{\partial S_n(\xi_1)}{\partial \theta} = \frac{\partial^2 S_n(\xi_2^*)}{\partial \theta^2}(\hat{\theta} - \xi_1)$ 在ξ_0处展开

$\hat{\theta} = \xi_1 - \left[\frac{\partial^2 S_n(\xi_2^*)}{\partial \theta^2}\right]^{-1} \frac{\partial S_n(\xi_1)}{\partial \theta}$ can't find ξ_2^*

$\Rightarrow \xi_1 \to \xi_2^*$ $\xi_2 = \xi_1 - \left[\frac{\partial^2 S_n(\xi_1)}{\partial \theta^2}\right]^{-1} \frac{\partial S_n(\xi_1)}{\partial \theta}$

$\vdots$

$\xi_m = \xi_{m-1} - \left[\frac{\partial^2 S_n(\xi_{m-1})}{\partial \theta^2}\right]^{-1} \frac{\partial S_n(\xi_{m-1})}{\partial \theta}$

$\xi_{m-1} \to \hat{\theta}_n$

- ϕ 是否close toϕ_0

Theory: $\hat{\theta} \approx \theta_0$ MA(1) $\frac{\partial a_t}{\partial \theta}$

true

$$\sqrt{n}(\hat{\theta} - \theta_0) \sim N(0, \hat{\Omega}).$$

$\frac{\partial a_t}{\partial \theta}$

$$\sqrt{n}(\hat{\theta}_i - \theta_{0i}) \sim N(0, \hat{\sigma}_{ii}).$$

some variable may be useless

where

test

$$\hat{\Omega} = \left(\sum_{t=1}^{n} \frac{\partial a_{t-1}(\hat{\theta})}{\partial \theta} \frac{\partial a_{t-1}(\hat{\theta})}{\partial \theta'}\right)^{-1} \hat{\sigma}_a^2$$

- Model check
  - box test of residuals <-- residuals are white noise <-- ACF =0 (independent)<-- p足够大

- ARMA
  - ARMA(1,1):

$$(1-\phi_1 B)r_t = \phi_0 + (1-\theta B)a_t,$$

$$r_t = \phi_1 r_{t-1} + \phi_0 + a_t - \theta_1 a_{t-1}.$$

- stationary: same as AR
- invertibilty: same as MA

invertible. $a_t = \underbrace{r_t - \phi_0 - \phi_1 r_{t-1}}_{W_t} + \theta_1 a_{t-1}$

$a_t = \theta_1 a_{t-1} + W_t$. AR(1) $\Rightarrow |\theta_1| < 1$.

$r_t = \phi_1 r_{t-1} + W_t + \phi_0$ AR(1) $\Rightarrow |\phi_1| < 1$.

- mean: $\frac{\phi_0}{1-\phi_1}$

$\gamma_1 = \mathbb{E}[(r_t-\mu)(r_{t-1}-\mu)] = \mathbb{E}[(\phi_1 r_{t-1} + \phi_0 + a_t - \theta_1 a_{t-1})(r_{t-1})]$

$= \phi_1 \mathbb{E}[r_{t-1}^2] - \theta_1 E[a_{t-1} r_{t-1}]$

$\mathbb{E}[a_{t-1} r_{t-1}] = \mathbb{E}[a_{t-1}(\phi_1 r_{t-2} + a_{t-1} - \theta_1 a_{t-2})]$

$= \sigma_a^2$

$= \phi_1 \gamma_0 - \theta_1 \sigma_a^2$.

$\rho_1 = \phi_1 - \frac{\theta_1 \sigma_a^2}{\gamma_0} \neq \phi_1$

$\gamma_2 = \mathbb{E}[r_t r_{t-2}] = \mathbb{E}[(\phi_1 r_{t-1} + \phi_0 + a_t - \theta_1 a_{t-1}) r_{t-2}]$

$= \phi_1 \gamma_1$

$\Rightarrow \gamma_k = \phi_1 \gamma_{k-1} = \phi_1^{k-1}\gamma_1$. $\rho_k = \phi_1^{k-1}\rho_1$. ACF $\downarrow$

PACF never equal to 0

- consider $\phi_0 = 0$
- forcast

$\hat{r}_n(1) = \mathbb{E}[r_{n+1} | F_n] = \mathbb{E}[\phi_1 r_n + \phi_0 + a_{n+1} - \theta_1 a_n | F_n]$

$= \phi_1 r_n + \phi_0 - \theta_1 a_n$.

$\hat{e}_n(1) = a_{n+1}$

$\hat{r}_n(2) = \mathbb{E}[r_{n+2} | F_n] = \mathbb{E}[\phi_1 r_{n+1} + \phi_0 + a_{n+2} - \theta_1 a_{n+1} | F_n]$

$= \phi_1 \underbrace{\hat{r}_n(1)}_{\text{affect indirectly}} + \phi_0$.

- ARMA(p,q)
  - definition

$$\begin{aligned} r_t \;&=\; \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + \cdots + \phi_p r_{t-p} \\ &\quad + a_t - \theta_1 a_{t-1} - \cdots - \theta_q a_{t-q}, \end{aligned}$$

- condition

  > where all the roots of
  >
  > $$\phi_p(z) = 1 - \sum_{i=1}^{p} \phi_i z^i \; and \; \theta_q(z) = 1 - \sum_{i=1}^{q} \theta_i z^i$$
  >
  > lie outside the unit circle, and they have no common roots.

- representations

  > **Representations**:
  >
  > $$\phi_p(z)\theta_q^{-1}(z) = 1 - \sum_{i=1}^{\infty} \pi_i z^i,$$
  >
  > $\pi_i = O(\rho^i)$ with $\rho \in (0, 1)$. AR representation:
  >
  > $$r_t = \theta_q^{-1}(1)\phi_0 + a_t + \pi_1 r_{t-1} + \pi_2 r_{t-2} + \cdots .$$
  >
  > It tells how r_t depends on its past values. r_t
  >
  > $$\phi_p^{-1}(z)\theta_q(z) = 1 + \sum_{i=1}^{\infty} \psi_i z^i,$$
  >
  > $\psi_i = O(\rho^i)$ with $\rho \in (0, 1)$. MA representation:
  >
  > $$r_t = \phi_0 \phi_p^{-1}(1) + a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots .$$
  >
  > It tells how r_t depends on the past shocks. a_t

- FI
  - 对$e_n(l)$的估计

    > $r_{n+l} = \cancel{\mu +} a_{n+l} + \psi_1 a_{n+l-1} + \psi_2 a_{n+l-2} + \cdots$
    >
    > $E[r_{n+l} | F_n] = \mu + \cancel{\psi_l a_n + \psi_{l+1} a_{n-1} + \cdots}$
    >
    > $\therefore e_n(l) = a_{n+l} + \psi_1 a_{n+l-1} + \psi_2 a_{n+l-2} + \cdots + \psi_{l-1} a_{n+1}$
    >
    > $\therefore Var(e_n(l)) = (1 + \psi_1^2 + \cdots + \psi_{l-1}^2)\sigma_a^2$

  - confidence interval

    > $$\left[\hat{r}_n(l) - N_{\frac{\alpha}{2}}\sigma_a\sqrt{\sum_{j=0}^{l-1}\psi_j^2}, \hat{r}_n(l) + N_{\frac{\alpha}{2}}\sigma_a\sqrt{\sum_{j=0}^{l-1}\psi_j^2}\right]$$

- non-stationary time series
  - 我们考虑random walk: $p_t = p_{t-1} + a_t + \mu$
    - properties:
      - non-stationary:

$$
\begin{aligned}
P_t &= P_{t-1} + a_t + \mu \\
&= a_t + \mu + \mu + a_{t-1} + P_{t-2} + \dots \\
&= t\mu + P_0 + a_t + a_{t-1} + \dots \to \infty
\end{aligned}
$$

$$\mathbb{E}[P_t] = \mu t + P_0. \quad \text{straight line}$$

- strong memory<--ACF $\longrightarrow$ 1

考虑 $\mu = 0$

$$
\begin{aligned}
\gamma_{tk} &= E[(Y_t - \mu)(Y_{t-k} - \mu)] \\
&= E[(Y_t - P_0)(Y_{t-k} - P_0)] \\
&= \mathbb{E}[(a_t + a_{t-1} + \dots + a_1)(a_{t-k} + a_{t-k-1} + \dots + a_1)] \\
&= E[a_{t-k}^2 + a_{t-k-1}^2 + \dots + a_1^2] \\
&= (t-k)\sigma_a^2
\end{aligned}
$$

$k=0$ 时. $Var = t\sigma_a^2$

$$\rho_{tk} = \sqrt{\frac{(t-k)\sigma_a^2}{t\sigma_a^2}} = \sqrt{1 - \frac{k}{t}} \to 1$$

- have a time trend
- extension

$$P_t = P_{t-1} - \theta a_{t-1} + a_t$$

Expansion:

$$
\begin{aligned}
a_t &= \theta a_{t-1} + \underset{(AR)}{(P_t - P_{t-1})} = \sum_{i=0}^{\infty} \theta^i (P_{t-i} - P_{t-i-1}) \\
&= P_t + \theta P_{t-1} + \theta^2 P_{t-2} \dots \\
&\quad - P_{t-1} - \theta P_{t-2} \dots \\
&= P_t - (1-\theta) P_{t-1} - \theta(1-\theta) P_{t-2} \dots \\
&= P_t - (1-\theta) \sum_{i=1}^{\infty} \theta^{i-1} P_{t-i}
\end{aligned}
$$

- how to test the unit root?
  - Dickey-Fuller test:

我们知道，如果 $|\hat{\phi}_1|<1 \Rightarrow \sqrt{n}(\hat{\phi}_1-\phi_1) \sim N(0, \hat{\sigma})$ $\quad \hat{\phi}_1 = \frac{\sum_{t=1}^{n} P_t P_{t-1}}{\sum_{t=1}^{n} P_{t-1}^2}$

$H_0: \phi_1 = 1$. $\quad H_a: \phi_1 \neq 1$ $\quad \rho_n = \hat{\phi}_1 - 1 = \frac{\sum_{t=1}^{n} P_{t-1} e_t}{\sum_{t=1}^{n} P_{t-1}^2}$

$\sqrt{n}(\hat{\phi}_1 - 1) \sim N(0, \hat{\sigma})$? 但这是不对的，因为我们事先不知道 $|\hat{\phi}_1|<1$.

因此我们要重新找个方法估计 $\hat{\phi}_1$

if $\phi_1 = 1$. $\Rightarrow P_t = e_t + e_{t-1} + \ldots + e_1 + P_0 = \sum_{i=1}^{t} e_i$

$P_{[n\tau]} = \sum_{i=1}^{[n\tau]} e_i$. random fun on $\tau \in [0,1]$ $\quad T_n = n(\hat{\phi}_1 - 1)\sqrt{\sum_{t=1}^{n} \frac{P_{t-1}^2}{n^2}}$

$\frac{1}{\sqrt{n}} P_{[n\tau]} = \frac{1}{\sqrt{n}} \sum_{i=1}^{[n\tau]} e_i \to \sigma_a B(\tau)$ Brownian Motion.

$\Rightarrow \sum_{t=1}^{n} P_{[n\tau]} \Delta P_{[n\tau]} \to \sigma_a^2 \int_0^1 B(\tau)\, dB(\tau)$

$\sum_{n=1}^{n} [\frac{1}{\sqrt{n}} P_{[n\tau]}]^2 \to \sigma_a^2 \int_0^1 B^2(\tau)\, d\tau$

$$\rho_n \xrightarrow{d} \frac{\int_0^1 B(\tau)\, dB(\tau)}{\int_0^1 B^2(\tau)\, d\tau}$$

$$T_n \xrightarrow{d} \frac{\int_0^1 B(\tau)\, dB(\tau)}{\sqrt{\int_0^1 B^2(\tau)\, d\tau}}$$

T_n 是什么?

- ARIMA:
  - $\phi_p(B)(1-B)^d p_t = \theta_q(B) a_t$ $\phi_p(z) = 0, \theta_q(z) = 0$ 的解在单位圆外
  - Var

$\psi(B) = \phi_p(B)(1-B)^d = 1 - \psi_1 B - \psi_2 B^2 - \ldots \psi_{p+d} B^{p+d}$.

$\psi(B) p_t = \theta_q(B) a_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} \ldots - \theta_q a_{t-q}$.

$p_t = \psi_1 p_{t-1} + \psi_2 p_{t-2} + \ldots + \psi_{p+d} p_{t-p-d}$

$+ a_t - \theta_1 a_{t-1} - \ldots - \theta_q a_{t-q}$.

error must be form by the unpredicted part. 设 $e_n(l) = \sum_{j=0}^{l-1} \psi_j a_{n+l-j}$.

剩下的问题是如何计算 ψ_j.

$p_{t+l} = \sum_{j=1}^{\infty} \pi_j p_{t+l-j} + a_{t+l} = \sum_{j=1}^{\infty} \pi_j B^j p_{t+l} + a_{t+l}$

$= p_{t+l} \sum_{j=1}^{\infty} \pi_j B^j + a_{t+l}$

$p_{t+l}(1 - \sum_{j=1}^{\infty} \pi_j B^j) = a_{t+l}$

$\therefore 1 - \sum_{j=1}^{\infty} \pi_j B^j = \frac{a_{t+l}}{p_{t+l}} = \frac{\phi_p(B)(1-B)^d}{\theta_q(B)}$

$\Rightarrow \psi_j = \sum_{i=0}^{j-1} \pi_{j-i} \psi_i \Leftarrow$ why?

  - example

example 1. $(1-B)p_t = (1-\theta B)a_t$.

$$\frac{a_t}{p_t} = \frac{1-B}{1-\theta B} = (1-B)(1+\theta B+\theta^2B^2 \dots)$$

$$= 1+\theta B+\theta^2B^2+\dots$$

$$-B-\theta B^2-\dots$$

$$= 1-(1-\theta)B-\theta(1-\theta)B^2-\dots$$

$$\therefore \pi_j = (1-\theta)\theta^{j-1}$$

$\psi_0 = 1$. $\quad \psi_1 = \sum_{i=0}^{0} \pi_{1-i}\psi_i = \pi_1\psi_1 = \pi_1 = 1-\theta$

$$\psi_2 = \sum_{i=0}^{1} \pi_{2-i}\psi_i = \pi_2\psi_0 + \pi_1\psi_1 = \theta(1-\theta)+(1-\theta)^2 = 1-\theta$$

$$\psi_3 = \sum_{i=0}^{2} \pi_{3-i}\psi_i = \pi_3\psi_0 + \pi_2\psi_1 + \pi_1\psi_2$$

$$= \theta^2(1-\theta)+\theta(1-\theta)^2+(1-\theta)^2$$

$$= \theta(1-\theta)+(1-\theta)^2 = (1-\theta)$$

$$\vdots$$

So $\psi_j = 1-\theta$.

$e_n(l) = \sum_{j=0}^{l-1}(1-\theta)a_{n+l-j}$ $\quad Var(e_n(l)) = (1-\theta)^2 l$.