Chapter 2

- White noises
  - 不一定相互独立+$Var < +\infty$

    $$E\varepsilon_t = 0 \ \ and \ \ \text{cov}(\varepsilon_t, \varepsilon_s) = \begin{cases} \sigma^2, & t = s, \\ 0, & t \neq s, \end{cases}$$

- measure of dependence
  - Autocovariance

    $$\gamma_{s,t} = Cov(r_s, r_t) = E[(r_s - \mu_s)(r_t - \mu_t)]$$

  - Autocorrelation

    $$\rho_{s,t} = corr(r_s, r_t) = \frac{\gamma_{s,t}}{\sqrt{\gamma_{s,s}\gamma_{t,t}}}.$$

  - weak stationary
    - 前两个moment是constant
    - 只与k有关，与t无关

      . Lag$-k$ autocovariance:

      $$\gamma_k = Cov(r_t, r_{t-k}) = E[(r_t - \mu)(r_{t-k} - \mu)].$$

      与t无关，只与k有关，即时间差

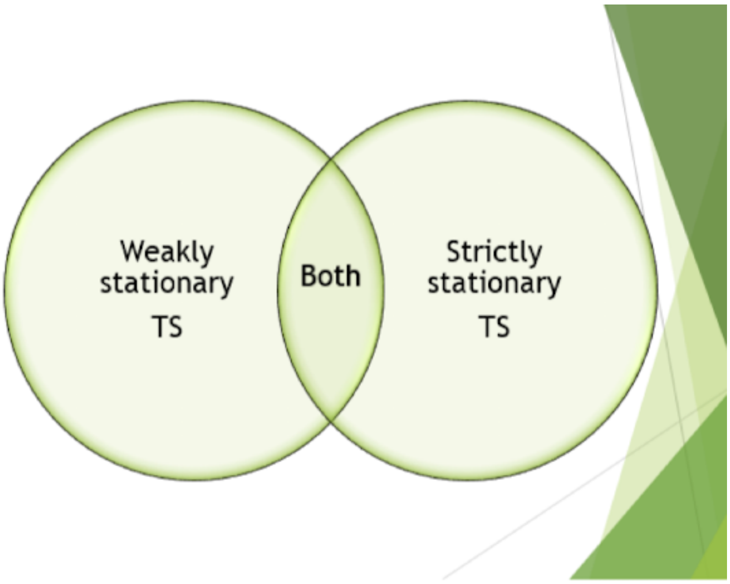
    - ACF不等于0, 意味着cov不等于0，他们是related的，所以我可以根据之前的信息predict, inefficiency
  - strict stationary
    - definition

      Strict: distributions are time-invariant, i.e.,

      $$P(r_{t_1} \le z_1, \cdots, r_{t_n} \le z_n) \quad \text{joint distribution.}$$
      $$= P(r_{t_1+k} \le z_1, \cdots, r_{t_n+k} \le z_n),$$

      for $\forall t_1, \cdots, t_n, k$ and $(z_1, \cdots, z_n)$ and n.

  - 没有办法判断到底是不是station的，只能assump, 然后判断assumption是否正确？close enough
  - relationship: strict + finite variance -->weak



- property of WS
  - $\rho_0 = 1, \gamma_0 = var(r_t)$

    $$\gamma_0 = cov(r_t, r_{t-0}) = cov(r_t, r_t) = Var(r_t)$$

    $$\rho_0 = \frac{Var(r_t)}{\sqrt{Var(r_t)}\sqrt{Var(r_t)}} = 1$$

  - $|\gamma_k| \leq \gamma_0, |\rho_k| \leq 1$

    $$|\gamma_k| = |\mathbb{E}[(r_t-\mu)(r_{t-k}-\mu)]| \leq \sqrt{\underbrace{E(r_t-\mu)^2}\cdot\mathbb{E}[\underbrace{(r_{t-k}-\mu)^2}]} \Leftarrow \overset{\text{Schwarz}}{\mathbb{E}[XY] \leq (\mathbb{E}[X^2]\cdot\mathbb{E}[Y^2])^{\frac{1}{2}}}$$

    $$= \sqrt{(Var(r_t))^2}$$

    $$= Var(r_t) = \gamma_0$$

    $$\rho_k \leq \frac{\gamma_0}{\gamma_0} = 1$$

  - $\gamma_k = \gamma_{-k}, \rho_k = \rho_{-k}$
  - $\rho_k = 0$ for all k不等于0 --> cov =0 --> white noise（应该有WS的condition）
  - estimate:
    - $\mu = 0$

      Test $H_0 : \mu = 0$ vs $H_a : \mu \neq 0$. Compute

      $$t = \frac{\sqrt{T}\bar{r}}{\hat{\sigma}_r}. \qquad \frac{\bar{r}}{\frac{\hat{\sigma}_r}{\sqrt{T}}}$$

      Compare t ratio with $N(0, 1)$ dist.

    - $ACF = 0$

$$\hat{\rho}_k = \frac{\sum_{t=1}^{T-k}(r_t - \bar{r})(r_{t+k} - \bar{r})}{\sum_{t=1}^{T}(r_t - \bar{r})^2},$$

- individual

  $H_0 : \rho_1 = 0$ vs $H_a : \rho_1 \neq 0$

  ⇒ 不能

  $\rho \neq 0$ ⇒ prec

  $$t = \frac{\hat{\rho}_1}{\sqrt{1/T}} = \sqrt{T}\hat{\rho}_1 \sim \boxed{N(0,1).}$$

  can't do one i

- joint: 要拒绝H_0 <-- p足够小

  Joint test (Ljung-$\boxed{\text{Box}}$ statistics): 要拒绝 ρ ⇒ p_value<

  m不能太大 ⇒ it is not

  H_0 : $\rho_1 = \cdots = \rho_m = 0$ vs H_a : $\rho_i \neq 0$ for some i.

  it has d

  $$Q(m) = T(T+2)\sum_{k=1}^{m}\frac{\hat{\rho}_k^2}{T-k} \sim \chi_m^2.$$

  Asym. χ^2 dist with m degrees of freedom.

  - 注意：m不能太大，m太大，ACF用的数据太少

- notation: $Br_t = r_{t-1}, B2 = 2$
- $r_t = \mu_t + a_t$ μ_t : predict part, the best predictor of r_t
  - 如果我们令$g_t \in F_t \to g_t \in f(F_{t-1})$

    $$E(r_t - g_t)^2 > E(r_t - \mu_t)^2 \ if \ g_t \neq \mu_t.$$

    - proof

      $$\mathbb{E}(r_t - g_t)^2 = \mathbb{E}[((r_t-\mu_t)-(g_t-\mu_t))^2] = \mathbb{E}[(r_t-\mu_t)^2] + \underbrace{\mathbb{E}[(\mu_t-g_t)^2]}_{\neq 0} \ > \mathbb{E}[(r_t-\mu_t)^2]$$

      $$- \underbrace{2\mathbb{E}[(r_t-\mu_t)(g_t-\mu_t)]}_{=0}$$

      $$\mathbb{E}[(r_t-\mu_t)(g_t-\mu_t)] = \mathbb{E}[\mathbb{E}[(r_t-\mu_t)(g_t-\mu_t)|F_{t-1}]]$$

      $$= \mathbb{E}[(g_t-\mu_t)\mathbb{E}[(r_t-\mu_t)|F_{t-1}]]$$

      $$= \mathbb{E}[(g_t-\mu_t)(\mu_t-\mu_t)]$$

      $$= 0$$

  - a_t is white noise but no sure it's iid

$$\sigma_t^2 = Var(r_t \mid F_{t-1})$$
$$= \mathbb{E}[(r_t-\mu_t)^2 \mid F_{t-1}] \underbrace{-(\mathbb{E}[r_t-\mu_t \mid F_{t-1}])^2}_{=0} \qquad \mu_t = \mathbb{E}[r_t \mid F_{t-1}]$$
$$= \mathbb{E}[a_t^2 \mid F_{t-1}] = Var(a_t)$$

$$\Rightarrow r_t = \mu_t + \underline{\sigma_t \varepsilon_t} \ a_t$$

① $\mathbb{E}[\varepsilon_t] = \mathbb{E}[\mathbb{E}[\varepsilon_t \mid F_{t-1}]] = \mathbb{E}[\mathbb{E}[\frac{a_t}{\sigma_t} \mid F_{t-1}]] = \mathbb{E}[\frac{1}{\sigma_t}\mathbb{E}[a_t \mid F_{t-1}]] = 0$

② $Var(\varepsilon_t) = \mathbb{E}[\varepsilon_t^2] = \mathbb{E}[\mathbb{E}[\frac{a_t^2}{\sigma_t^2} \mid F_{t-1}]] = \mathbb{E}[\frac{1}{\sigma_t^2}\mathbb{E}[a_t^2 \mid F_{t-1}]]$
$= 1$

③ 对 a_t、 $\gamma_k = \mathbb{E}[a_t \, a_{t-k}] = \mathbb{E}[\mathbb{E}[a_t a_{t-k} \mid F_{t-k}]]$
$= \mathbb{E}[a_{t-k}\mathbb{E}[a_t \mid F_{t-k}]] = 0$

$\Rightarrow a_t$ is unrelated. $\Rightarrow \varepsilon_t$ is unrelated.

$\Rightarrow \varepsilon_t \sim WN(0,1)$