Topic 1

- σ-algebra:
  - $\Omega \in F, \emptyset \in F$
  - $\cup E_i \in F$
  - $E^c \in F$
  - $\cap E_i \in F$
- independent
  - $P(A|B) = \frac{P(AB)}{P(B)}$
  - if independent--> $P(AB) = P(A)P(B)$
- Bayes Rule
  - 交换条件和事件的位置

    $$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)} = \frac{P(B|A_i) \times P(A_i)}{\sum_{n=1}^{\infty} P(B|A_n) P(A_n)},$$
- Expectation and variance
  - expectation
    - definition

      $$\mu = E[X] = \int_{-\infty}^{\infty} x dF_X(x) = \begin{cases} \sum_k x_k P(X = x_k) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x f_X(x) dx & \text{if } X \text{ is continuous} \end{cases}$$
    - property

      We let X and Y be two random variables with $\mathrm{E}[X] < \infty$ and $\mathrm{E}[Y] < \infty$. Then
      - $\mathrm{E}[X + Y] = \mathrm{E}[X] + \mathrm{E}[Y]$
      - For any real numbers a and b, we have $\mathrm{E}[aX + b] = a\mathrm{E}[X] + b$
      - If $X \geq Y$ for all ω (If $P(X \geq Y) = 1$), we have $\mathrm{E}[X] \geq \mathrm{E}[Y]$
      - If X and Y are independent, then $\mathrm{E}[XY] = \mathrm{E}[X]\,\mathrm{E}[Y]$
      - If $a \leq X \leq b$, then $a \leq \mathrm{E}[X] \leq b$
  - Variance
    - definition

      $$Var(X) = E\left[(X - E[X])^2\right] = E\left[(X - \mu)^2\right]$$
      $$= \begin{cases} \sum_k (x_k - \mu)^2 P(X = x_k) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} (x - \mu)^2 f_X(x) dx & \text{if } X \text{ is continuous} \end{cases}$$
- Discrete distribution

- Binomial Distribution
  - n次实验，每次实验成功的概率为p

    $$P(N = k) = C_k^n p^k (1-p)^{n-k}, k = 0, 1, 2, \dots, n.$$
  - property

    $$E[N] = np, Var(N) = np(1-p)$$
- Geometric Distribution
  - 第一次成功的实验是哪次实验

    $$P(M = k) = (1-p)^{k-1} p,$$
  - property

    $$E[M] = \frac{1}{p}, Var(M) = (1-p)/p^2.$$
- Multinomial Distribution
  - K 面骰子每一面的成功的概率

    $$P(X_1 = n_1, X_2 = n_2, \dots, X_m = n_m) = \frac{n!}{n_1! n_2! \cdots n_m!} {p_1}^{n_1} {p_2}^{n_2} \cdots {p_m}^{n_m}$$
  - property

    $$E[X_j] = np_j, Var(X_j) = np_j(1-p_j)$$
- Poisson Distribution
  - 一段时间内事件发生次数的概率

    $$P(N = k) = \frac{\lambda^k e^{-\lambda}}{k!}, k = 0, 1, 2, \dots$$
  - property
    - 1.

      $\lambda = E[N]$). Moreover, $Var(N) = \lambda$.
    - 2.

      If $N_1, N_2, \dots N_k$ are k independent Poisson random variable with parameters $\lambda_1, \lambda_2, \dots, \lambda_k$ respectively, then the sum $N_1 + N_2 + \cdots + N_k$ has Poisson distribution with parameter $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_k$.
- Continuous Distribution
  - Uniform Distribution
    - definition

$$f(x)=\begin{cases}\frac{1}{b-a} & if\ x\in[a,b]\\ 0 & if\ otherwise\end{cases}$$

- property

$$E[X]=\frac{a+b}{2} \text{ and } Var[X]=\frac{(b-a)^2}{12}.$$

- Expontential Distribution
  - definition: 平均违约时间，λ是1时间内的违约次数，$\frac{1}{\lambda}$是平均发生的时间

$$f(x)=\lambda e^{-\lambda x},$$

  - property

$$E[X]=\frac{1}{\lambda},\ and\ Var(X)=\frac{1}{\lambda^2}.$$

- Normal distribution
  - definition

$$f(x)=\frac{1}{\sqrt{2\pi}\,\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

- Log-normal Distribution
  - definition

$$f_X(x)=\frac{d}{dx}F_X(x)=\frac{d}{dx}\Phi\left(\frac{\ln x-\mu}{\sigma}\right)=\varphi\left(\frac{\ln x-\mu}{\sigma}\right)\left(\frac{d}{dx}\left(\frac{\ln x-\mu}{\sigma}\right)\right)$$
$$=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2}\left(\frac{1}{\sigma x}\right)=\frac{1}{x\sqrt{2\pi}\,\sigma}e^{-\frac{1}{2}\left(\frac{\ln x-\mu}{\sigma}\right)^2}$$

  - property

$$E[X]=e^{\mu+\frac{\sigma^2}{2}} \text{ and } Var(X)=\left(e^{\sigma^2}-1\right)e^{2\mu+\sigma^2}.$$

- Gamma Distribution
  - definition

$$f(x;\ \alpha,\beta) = \frac{\beta^{\alpha}\, x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}$$

- property

$$E[X] = \frac{\alpha}{\beta},\ Var(X) = \frac{\alpha}{\beta^2}$$

- gamma function

$$\Gamma(n) = \begin{cases} (n-1)! & n\text{是正整数} \\ \int_0^{\infty} t^{n-1} e^{-t}\, dt & n\text{是有正实部的复数} \end{cases}$$

- Beta Distribution
  - definition

$$f(x;\ \alpha,\beta) = \frac{\Gamma(\alpha+\beta)\, x^{\alpha-1}\, (1-x)^{\beta-1}}{\Gamma(\alpha)\, \Gamma(\beta)}$$

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  - property

Beta distribution has $E[X] = \frac{\alpha}{\alpha+\beta}$,

$$Var(X) = \frac{\alpha\beta}{(\alpha+\beta)^2\, (\alpha+\beta+1)}$$

- Moment
  - definition

$$E[X^k] = \int_{-\infty}^{\infty} x^k dF_X(x) = \begin{cases} \sum_x x^k P(X=x) & if\ X\ is\ discrete \\ \int_{-\infty}^{\infty} x^k f_X(x) dx & if\ X\ is\ continuous \end{cases}$$

  - 注意并不是所有的分布都有所有的k阶moment，如果对于非负rv，某些k是不存在k阶矩的，我们就称这个随机变量have heavy tail
- Conditional expectation
  - property

$$E[X] = E[E[X|Y]]$$
$$Var(X) = E[Var(X|Y)] + Var(E[X|Y])$$

  - random sum expected value and variance

a) $E[S] = E[N]E[X_i]$

b) $Var(S) = Var(X_i)\, E[N] + (E[X_i])^2\, Var(N).$

- MGF

$$M_X(t) = E[e^{tX}],$$

- property

$$E[X^k] = M_X^{(k)}(0)$$

- 其实这里有有迹可循，因为我们知道并不是所有的随机变量都有moment，在t较小的时候更有可能存在，所以我们都是考虑t在0附近的MGF，所以为x=0

- table

Distribution	Probability density function	Moment generating function
Poisson λ	$P(X=k)=\frac{\lambda^k e^{-\lambda}}{k!}$	$e^{\lambda e^t-\lambda}$
Binomial (n,p)	$P(X=k)=C_k^n p^k(1-p)^k$	$(pe^t+(1-p))^n$
Geometric (p)	$P(X=k)=p(1-p)^{k-1}$	$\frac{pe^t}{1-(1-p)e^t},\ t<\ln\frac{1}{1-p}$
Uniform (a,b)	$f(x)=\frac{1}{b-a},\ x\in[a,b]$	$\frac{e^{tb}-e^{ta}}{t(b-a)}$
Exponential (λ)	$f(x)=\lambda e^{-\lambda x},\ x\geq 0$	$\frac{\lambda}{\lambda-t}\ \ for\ t<\lambda$
Normal (μ,σ^2)	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$e^{\mu t+\frac{1}{2}\sigma^2t^2}$
Gamma (α,θ)	$f(x)=\frac{\left(\frac{x}{\theta}\right)^\alpha e^{-\frac{x}{\theta}}}{x\Gamma(\alpha)},\ x>0$	$(1-\theta t)^{-\alpha},\quad t<\frac{1}{\theta}$

- PGF
  - 只有某些discrete random variabke才有PGF

$$P_N(t) = E[t^N] = \sum_{k=0} P(N=k)\, t^k$$

- table

Distribution	Probability density function	Probability generating function
Poisson λ	$P(N=k)=\frac{\lambda^k e^{-\lambda}}{k!}$ For $k=0,1,2,\ldots.$	$e^{\lambda(t-1)}$
Binomial (n,p)	$P(N=k)=C_k^n p^k(1-p)^k$ for $k=0,1,\ldots,n$	$(1+p(t-1))^n$
Geometric (p)	$P(N=k)=p(1-p)^{k-1}$ for $k=1,2,\ldots.$	$\frac{tp}{1-t(1-p)}$ $for\ -\frac{1}{1-p}<t<\frac{1}{1-p}$

- property

$$p_k = \frac{1}{k!} P_N^{(k)}(0)$$

- CF
  - definition

    $$\varphi(t) = \mathrm{E}\left[e^{itX}\right] = \mathrm{E}[\cos tX] + i\,\mathrm{E}[\sin tX]$$

  - table

    Summary of characteristic function of some well-known distributions

Distribution	Probability density function	Characteristic function
Poisson λ	$P(X=k) = \frac{\lambda^k e^{-\lambda}}{k!}$	$e^{\lambda e^{it} - \lambda}$
Binomial (n,p)	$P(X=k) = C_k^n p^k (1-p)^k$	$\left(pe^{it} + (1-p)\right)^n$
Uniform (a,b)	$f(x) = \frac{1}{b-a},\ x \in [a,b]$	$\frac{e^{itb} - e^{ita}}{it(b-a)}$
Exponential (λ)	$f(x) = \lambda e^{-\lambda x},\ x \geq 0$	$\frac{\lambda}{\lambda - it}$
Normal (μ,σ^2)	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	$e^{i\mu t - \frac{1}{2}\sigma^2 t^2}$
Cauchy*	$f(x) = \frac{1}{\pi\sqrt{1+x^2}},\ x \in \mathbb{R}$	$e^{-\lvert t\rvert}$

    (*Note: The MGF of Cauchy distribution does not exist)

  - 虽然CF是每个分布都存在，但是还是不能直接从CF得到moment，因为moment不一定存在，只有在已知moment存在的情况下，才可以使用CF获得moment，同样还是在0附近的值

    $$E[X^k] = (-i)^{-k} \frac{d^k}{dt^k} \varphi(t) \Big|_{t=0}$$

- Basic inequalities
  - Jensen

    1. **Jensen inequality** (powerful and commonly used)
    Suppose $\psi(.)$ is a convex function and X and $\psi(X)$ have finite expectation.
    Then $\psi(E[X]) \leq E[\psi(X)]$. $\psi(\theta x + (1-\theta)y) \leq \theta\psi(x) + (1-\theta)\psi(y)$ 令$\theta=1$

  - Markov

    For any $a > 0$, $P(|X| \geq a) \leq \frac{1}{a} E(|X|)$

  - Chebyshev

$$\text{For any } a > 0,\ P(|X - E(X)| \geq a) \leq \frac{Var(X)}{a^2}$$

- Holder

$$\text{For } \frac{1}{p} + \frac{1}{q} = 1 \text{ with } p > 0 \text{ and } q > 0,\ E|XY| \leq \|X\|_p \|Y\|_q$$

- Schwarz

$$E(XY) \leq \left[E\left(X^2\right)E\left(Y^2\right)\right]^{1/2}$$

- Minkowski

$$\text{For } p \geq 1,\ \|X + Y\|_p \leq \|X\|_p + \|Y\|_p$$