review

- 积分$\int B_s ds$ 的distribution 是Gaussian，把它写成求和之后极限的形式，每一个分量都服从Gaussian，加上独立，求和也就服从Gaussian，均值为0，方差为$\frac{1}{3}T^3$
- quadrtaic variation:$\lim \sum (B_{t_{i+1}} - B_{t_i})^2 = T$，这个是啥啊，这不就是$\int B_s^2 ds$!!!不是！！！这个需要再乘$(t_{i+1} - t_i)$，这个是$\int (dB_t)^2$
- total variation is ∞
- $\int H_u dB_u$
  - H_u 可以是任何函数哦，也可以包含B_u的
  - H_u is simple process --> continuous trajectories, martingale , isometry($E[(\int_0^t H_u dB_u)^2] = E[\int_0^t H_u^2 du]$),相乘($E[(\int_s^{t_1} H_u dB_u) \int_s^{t_2} K_u dB_u)] = E[\int_s^{min(t_1,t_2)} H_u K_u du]$),另一个martingale($(\int_0^t H_u dB_u)^2 - \int_0^t H_u^2 du$)，均值为0
    - quadratic variation: $\int_0^t H_u^2 du$
    - quadratic correlation: $\int_0^t H_u K_u du$
  - $H_u \in H^2, E[\int_0^t H_u^2 du] < +\infty$,可以用simple process 逼近$H^2$, --> 也满足martingale和isometry

    Proposition 5.3.5 (Wiener integral) Let $f: [0,T] \to \mathbb{R}$ be a square-integrable deterministic function, then the process $\left(\int_0^t f(s)\, dB_s\right)_{0 \le t \le T}$ is called Wiener integral and verifies

    $$\int_0^t f(s)\, dB_s \sim \mathcal{N}\left(0, \int_0^t f^2(s)\, ds\right)$$

  - $H_u \in H^1, \int_0^t H_u^2 du < +\infty$ 这时，这个积分不再是martingale，而是local martingale
  - 注意哈，所有的讨论都是针对这一个积分，只不过是H_u不同满足的性质也就不一样
- local martingale + bounded from below --> supermartingale
- martingale representaion

  Thm 7.1.2 Let $(M_t)_{0 \le t \le T}$ be a square-integrable martingale. Then there exists a unique process $H \in \mathbb{H}^2$ s.t. $M_t = M_0 + \int_0^t H_s dW_s$ $0 \le t \le T$. p.a.s
  In particular M has continuous path. (most Important)

  - change of measure : $z = \frac{dQ}{dP}$必须是一个martingale， 且在p的概念下期望为1，在也被称为Radon-Nikodym derivative
    - deterministic :

$$\frac{dQ}{dP} = e^{\int_0^T h(t)\,dW_t - \frac{1}{2}\int_0^T (h(t))^2 dt} \quad (\star)$$ 此时 $h(t)$ is a deterministic function

- BM:

  > $B_t := W_t - \int_0^t h(u)\,du \quad t \in [0,T]$ is a Brownian Motion under Q

- stochastic :
  - ϕ is H^2_{loc}

    > $$Z_t := \exp\left(\int_0^t \phi_s \cdot dW_s - \frac{1}{2}\int_0^t |\phi_s|^2 ds \right) \quad 0 \le t \le T.$$

  - BM:

    > Let Z be given by $(\star)$ and suppose that $\mathbb{E}[Z_T] = 1$. Then the process $\tilde{W}_t := W_t - \int_0^t \phi_s ds \quad t \in T$ is a Brownian Motion under Q defined by $\frac{dQ}{dP} = Z_T$