Linear Algebra Review
BIOS 526

Definitions

1. Inner product: $\mathbf{a}'\mathbf{b} = \Sigma_i a_i b_i$.
2. Euclidean norm for vectors (length): $||\mathbf{a}||_2 = \sqrt{\mathbf{a}'\mathbf{a}}$.
3. Vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if $\mathbf{a}'\mathbf{b} = 0$.
4. Linear independence: vectors $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are linearly independent if $\sum_i c_i \mathbf{a}_i \neq 0$ unless $c_i = 0$ for all i.
5. Column space of matrix $\mathbf{A}$ = the linear space spanned by the columns of $\mathbf{A}$.
6. Rank of matrix $\mathbf{A}$ = minimum(n,k) where n is the number of linearly independent columns of $\mathbf{A}$ and k is the number of linearly independent rows of $\mathbf{A}$.
7. Trace: $tr(\mathbf{A}) = \sum_i \mathbf{A}_{ii}$
8. Eigenvalue: If $\mathbf{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ where $\boldsymbol{x} \neq 0$, then λ is an eigenvalue of $\mathbf{A}$ and $\boldsymbol{x}$ is a corresponding eigenvector.
9. A matrix $\mathbf{P}$ is idempotent if $\mathbf{P}^2 = \mathbf{P}$. A symmetric idempotent matrix is called a projection matrix.

Properties of Matrix Operations

1. If $\mathbf{A}$ and $\mathbf{B}$ are invertible, then $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
2. $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$
3. $(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$.

Properties of Covariance Matrix

Let $\mathbf{X}$ be a random vector in $\mathbb{R}^p$.

1. Symmetric: $\text{cov}(\mathbf{X}) = [\text{cov}(\mathbf{X})]'$.

2. $\text{cov}(\mathbf{X})$ is positive semidefinite.

3. $\text{cov}(\mathbf{X}+\mathbf{a}) = \text{cov}(\mathbf{X})$ if $\mathbf{a}$ is a constant vector.

4. $\text{cov}(\mathbf{A}\mathbf{X}) = \mathbf{A}\text{cov}(\mathbf{X})\mathbf{A}'$ if $\mathbf{A}$ is a constant matrix.

5. $\text{cov}(\mathbf{X}) = E[\mathbf{X}\mathbf{X}'] - E[\mathbf{X}](E[\mathbf{X}])'$.

6. $\text{cov}(\mathbf{A}\mathbf{X}, \mathbf{B}\mathbf{Y}) = \mathbf{A}\text{cov}(\mathbf{X}, \mathbf{Y})\mathbf{B}'$ if $\mathbf{A}$ and $\mathbf{B}$ are constant matrices.

Properties of Multivariate Normal Distribution:

1. If $\mathbf{Y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\mathbf{C}_{p\times n}$ is a constant matrix, then $\mathbf{C}\mathbf{Y} \sim N_p(\mathbf{C}\boldsymbol{\mu}, \mathbf{C}\boldsymbol{\Sigma}\mathbf{C}')$.

2. Partition $\mathbf{Y} = [\mathbf{Y}_1, \mathbf{Y}_2]$. The conditional distribution of $\mathbf{Y}_1$ given $\mathbf{Y}_2$ is

$$\mathbf{Y}_1|\mathbf{Y}_2 = \mathbf{y}_2 \sim N_n(\boldsymbol{\mu}_1 + \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}(\mathbf{y}_2 - \boldsymbol{\mu}_2), \boldsymbol{\Sigma}_{11} - \boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\Sigma}_{21}).$$

3. If $\mathbf{Y} \sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then $(\mathbf{Y}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{Y}-\boldsymbol{\mu}) \sim \chi^2_n$.