

Module 3, Part 1: Generalized Linear Models

BIOS 526

Concepts

- Link function.
- Logistic regression and odds ratio.
- Probit regression.
- Poisson regression.

Readings

- Chapter 3, Wood, S. *Generalized Additive Models*, 2017. Has a nice, self-contained introduction to generalized linear models.

Linear Regression Model

Consider the following multiple linear regression model. For $i = 1, \dots, n$,

$$y_i = \beta_0 + \sum_{k=1}^p \beta_k x_{ik} + \epsilon_i \quad \epsilon_i \stackrel{iid}{\sim} N(0, \sigma^2),$$

where x_{ik} is the k th linear predictor for observation i .

The above model assumes

- $\beta_0, \beta_1, \dots, \beta_p$ are fixed unknown constants;
- only the residual error ϵ_i is **random**.

Therefore,

The **linear regression** part is used to model only the **mean function** of y_i .

Generalized Linear Regression Model

A generalized linear model (GLM) extends linear regression to other distributions, where the response variable is generated from a distribution in the **exponential family**.

A GLM involves three ingredients:

1. An exponential family of probability distributions.
2. A linear model $\mathbf{x}_i' \boldsymbol{\beta}$.
3. A **link function** $g()$ and its inverse $g^{-1}()$ relates the linear model to its expectation:

$$E[y_i | \mathbf{x}_i] = \mu_i = g^{-1}(\mathbf{x}_i' \boldsymbol{\beta})$$
$$Var[y_i | \mathbf{x}_i] = V(\mu_i) = V(g^{-1}(\mathbf{x}_i' \boldsymbol{\beta}))$$

Note: unlike ordinary least squares, the basic form of the GLM does not involve a noise variance (no σ^2).

Exponential Family

The basic form for an exponential family density is

$$f_{\theta}(y) = \exp [\{y\theta - b(\theta)\} / a(\phi) + c(y, \phi)],$$

where b , a , and c are known functions, and ϕ is a known scale parameter. There is only **one** unknown parameter: θ .

In the GLM, θ will be a function of $x'_i\beta$.

Examples of distributions in the exponential family include: normal distribution with **known** variance (link = identity), Bernoulli (logit or probit link), binomial (with fixed number of trials), gamma, exponential (link: negative inverse), and others.

https://en.wikipedia.org/wiki/Generalized_linear_model

Rather than derive expressions for the general case, we will focus on the two most popular models:

1. Binary outcome: $y_i \overset{ind}{\sim}$ Bernoulli (p_i), where p_i is the probability of success.
2. Poisson outcome: $y_i \overset{ind}{\sim}$ Poisson (λ_i), λ_i is the rate parameter, equal to expected number of events.

GLMs

The mean and variance function of y_i can be expressed as a function of the distribution parameters (i.e. p_i for Bernoulli and λ_i for Poisson).

1. Binary outcome:

$$E[y_i] = \mu_i = p_i,$$

$$Var[y_i] = V[\mu_i] = V[p_i] = p_i(1 - p_i).$$

2. Poisson outcome:

$$E[y_i] = \mu_i = \lambda_i,$$

$$Var[y_i] = V[\mu_i] = V[\lambda_i] = \lambda_i.$$

A natural approach is to model the mean as a function of linear predictors. A difficulty in modeling non-normal data is that the distributional parameters often have constraints.

1. Binary outcome has expected value $p_i \in (0, 1)$
2. Poisson outcome has expected value $\lambda_i > 0$.

Workspace

Generalized Linear Regression Model

Our solution is to model the **transformed** mean function:

The function $g(\cdot)$ is known as the **link function**.

The link function should have some desirable properties:

GLM for Binary Outcome

For $i = 1, \dots, n$, assume

$$y_i \stackrel{ind}{\sim} \text{Bernoulli}(p_i),$$

where $y_i \in \{0, 1\}$, $y_i|p_i$ for $i = 1, \dots, n$ are independent, and p_i is the probability $y_i = 1$. The probability mass function is given by

We know that

$$\mu_i = p_i = P(Y_i = 1).$$

So we wish to model

The two most commonly used link functions are the [logistic function](#) and the [probit function](#).

Logistic Regression

The logistic regression is formulated as follows. For $i = 1, \dots, n$,

- $g(\mu_i)$ is the **log odds** of success probability.
- $\log\left(\frac{p_i}{1-p_i}\right) \rightarrow -\infty$ when $p_i \rightarrow 0$; and $\log\left(\frac{p_i}{1-p_i}\right) \rightarrow \infty$ when $p_i \rightarrow 1$.
- $\log\left(\frac{p_i}{1-p_i}\right)$ is strictly increasing.

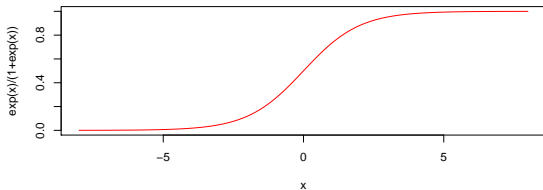
Likelihood function: logistic regression

$$\ell(\boldsymbol{\beta}; \mathbf{y}, \mathbf{x}) =$$

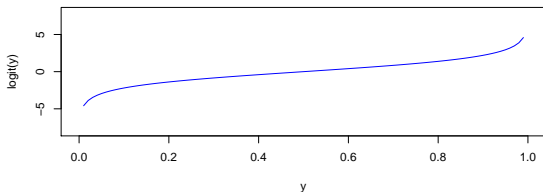
In practice, GLMs are estimated using iteratively reweighted least squares. We won't go into details, but see p. 107 in Wood for more info.

The **logistic** function:

$$p(g) = \exp(g)/(1 + \exp(g)) = 1/(1 + \exp(-g)), \quad (-\infty, \infty) \rightarrow (0, 1).$$



The **logit** function: $g(p) = \log(p/(1 - p))$, $(0, 1) \rightarrow (-\infty, \infty)$.



Logistic Regression: Interpretation

Consider a simple logistic model with only one predictor:

$$y_i \stackrel{\text{ind}}{\sim} \text{Bernoulli}(p_i)$$
$$\log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \beta_1 x_i.$$

When $x_i = 0$:

- $\log\left(\frac{p_i}{1-p_i}\right) = \beta_0.$
- β_0 is interpreted as the **baseline log odds**.

Function of the probability of success at baseline:

$$p_i = \frac{e^{\beta_0}}{1 + e^{\beta_0}}.$$

- Note that the above function satisfies
 - $p_i \in (0, 1)$ for $\beta_0 \in \mathbb{R}.$
 - p_i a strictly increasing function of β_0 .

Logistic Regression: Log odds and log odds ratio

Now we consider the effect of a unit change in x_i :

$$\log\left(\frac{p_i}{1-p_i}\right) = \beta_0 + \beta_1 x_i \quad \text{versus} \quad \log\left(\frac{p_i^*}{1-p_i^*}\right) = \beta_0 + \beta_1 (x_i + 1).$$

Then,

$$\begin{aligned} \log\left(\frac{p_i^*}{1-p_i^*}\right) - \log\left(\frac{p_i}{1-p_i}\right) &= \beta_0 + \beta_1 (x_i + 1) - (\beta_0 + \beta_1 x_i) \\ &= \beta_1. \end{aligned}$$

In words: β_1 is the change in log odds per unit change in x_i .

Equivalently, it is the log odds ratio per unit change in x_i :

Logistic Regression: Odds ratio

e^{β_1} is the **odds ratio**:

$$e^{\beta_1} = \frac{p_i^*/(1 - p_i^*)}{p_i/(1 - p_i)}.$$

Odds ratio interpretation helpful for indicator variables. Let $x_i = 1$ in exposed group, $x_i = 0$ in unexposed group. Then:

$$\begin{aligned} e^{\beta_1} &= \frac{\left\{ \frac{P(y_i=1|x_i=1)}{P(y_i=0|x_i=1)} \right\}}{\left\{ \frac{P(y_i=1|x_i=0)}{P(y_i=0|x_i=0)} \right\}} \\ &= \text{odds(Exposed)}/\text{odds(Unexposed)}. \end{aligned}$$

Logistic Regression: Background

- Logistic regression is commonly used because the slope coefficient corresponds to the log odds ratio (OR), a commonly used measure in epidemiology.
- OR is different from relative risk.
- Risk ratio (RR) is $P(y_i = 1|x_i = 1)/P(y_i = 1|x_i = 0)$
- OR is close to RR when the event $y_i = 1$ is rare, but in general, you need a different model to estimate RR.
- OR can be used in retrospective and observational studies.

Logistic Regression: Probability

The effect of a unit change in x_i depends on x_i on the probability scale.

E.g., $\beta_0 = 0$ and $\beta_1 = 2$:

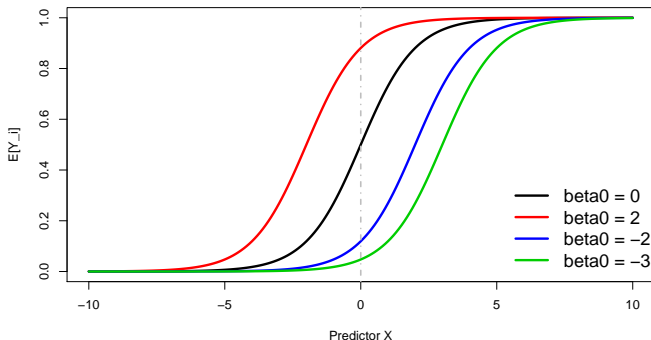
$$\frac{e^{2(0+1)}}{1 + e^{2(0+1)}} - \frac{e^{2(0)}}{1 + e^{2(0)}} \neq \frac{e^{2(1+1)}}{1 + e^{2(1+1)}} - \frac{e^{2(1)}}{1 + e^{2(1)}}$$

A change in x_i from 0 to 1 increases the probability by 0.38, but a change in x_i from 1 to 2 increases the probability by 0.10.

Intuitively, this has to be the case in order for the probability to max out at 1: $\frac{e^{2(100+1)}}{1+e^{2(100+1)}} - \frac{e^{2(100)}}{1+e^{2(100)}} \approx 1 - 1$.

Logistic Regression: Effects of Baseline Odds

$$\text{logit}(p_i) = \beta_0 + x_i.$$
$$\mu_i = P(Y_i = 1) = \frac{e^{\beta_0 + x_i}}{1 + e^{\beta_0 + x_i}}.$$



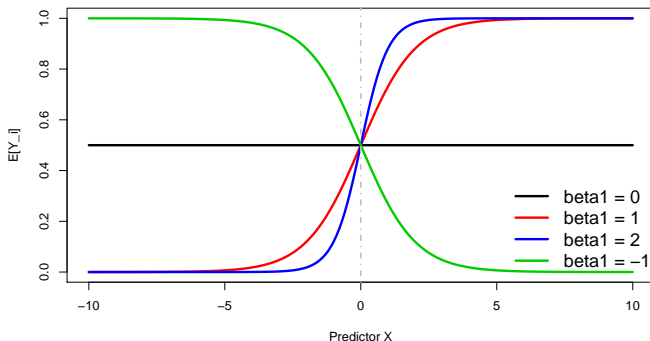
Note:

- The **shape** is maintained.
- The baseline (intercept) probability changes.

Logistic Regression: Effects of Slope

$$\text{logit}(\mu_i) = \beta_1 x_i.$$

$$\mu_i = P(Y_i = 1) = \frac{e^{\beta_1 x_i}}{1 + e^{\beta_1 x_i}}.$$



Note:

- How the steepness changes.
- How the direction of effect changes.

Logistic Regression: Interpretation

In multiple logistic regression, for $i = 1, \dots, n$,

- β_0 is the log odds when all covariate values equal zero.
- β_k is the log odds ratio associated with covariate k while controlling for other covariates.

Logistic Regression: Interpretation

The estimated (predicted) value is given by

$$\mu_i = p_i = \frac{e^{\beta_0 + \sum_{k=1}^p \beta_k x_{ik}}}{1 + e^{\beta_0 + \sum_{k=1}^p \beta_k x_{ik}}}$$

Again, the above function is non-linear in x_k , in contrast with normal regression

Inference: Likelihood Ratio Tests

To conduct inference, we appeal to **asymptotic results** that hold for large-ish n . There are two approaches:

1) Likelihood Ratio Tests (Difference in Deviance)

Let β_F be a vector of coefficients of interest. Then to test $H_0 : \beta_F = 0$, we create a full and reduced model. Let $\ell(\hat{\beta}_{\text{Full}})$ be the log-likelihood of the full model, and $\ell(\hat{\beta}_{\text{Reduced}})$ be the LL for the reduced model. Then we reject H_0 if

$$-2 \left\{ \ell(\hat{\beta}_{\text{Reduced}}) - \ell(\hat{\beta}_{\text{Full}}) \right\} > \chi^2_{\nu, 1-\alpha}$$

where ν is the difference in the number of parameters between the full and reduced, and $\chi^2_{\nu, 1-\alpha}$ is the critical value from a chi-squared distribution with ν degrees of freedom.

Inference: Wald Tests

2) Wald Tests. Under regularity conditions, asymptotically,

$$\begin{aligned}\hat{\beta} &\sim N(\beta, I(\beta)^{-1}), \\ I(\beta) &= E \left\{ \left(\frac{\partial \ell}{\partial \beta} \right) \left(\frac{\partial \ell}{\partial \beta} \right)' \right\} \\ &= -E \frac{\partial^2 \ell}{\partial \beta \partial \beta'},\end{aligned}$$

where the Hessian $I(\beta)$ is called the Fisher information matrix. See Wood p. 106 for details of the expected Hessian which is calculated during iteratively re-weighted least squares.

We can write $\hat{\beta} \sim N(\beta, (\mathbf{X}'\mathbf{W}\mathbf{X})^{-1}\phi)$ where \mathbf{W} contains the “Fisher weights” and $\phi = 1$ in the usual (not overdispersed) GLM.

In R, the default `summary(glmmodel)` is a Wald-type test: $\hat{\beta}_j/se(\hat{\beta}_j)$, where $se(\hat{\beta}_j)$ is extracted from the square root of the j th diagonal of the above covariance.

Binary Outcome Example

Dataset: a cohort of live births from Georgia born in the year 2001 ($N = 77,340$).

Variables:

- *ptb*: indicator for whether the baby from pregnancy i was born preterm (< 37 weeks).
- *age*: the mother's age at delivery (centered at age 25).
- *male*: indicator of the baby's sex (1 = male; 0=female).
- *tobacco*: indicator for mother's tobacco use during pregnancy (1 = yes; 0 = no)

The *glm* () Function

Fitting a GLM model in R is very similar to a linear regression model. We need to specify the distribution (**binomial**) and the link function (**logit**).

```
glm(formula = ptb ~ age + male + tobacco, family = binomial(link = "logit"),  
    data = dat)
```

Deviance Residuals:

| Min | 1Q | Median | 3Q | Max |
|---------|---------|---------|---------|--------|
| -0.5160 | -0.4236 | -0.4103 | -0.4088 | 2.2500 |

Coefficients:

| | Estimate | Std. Error | z value | Pr(> z) |
|-------------|------------|------------|----------|--------------|
| (Intercept) | -2.4370033 | 0.0200791 | -121.370 | < 2e-16 *** |
| age | -0.0006295 | 0.0021596 | -0.291 | 0.77068 |
| maleM | 0.0723659 | 0.0258672 | 2.798 | 0.00515 ** |
| tobacco | 0.4096495 | 0.0534627 | 7.662 | 1.83e-14 *** |

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Birth Outcome Analysis

- Preterm delivery was significantly associated with male babies (p -value= 0.005) when controlling for age and mother's smoking status. The odds ratio of a preterm birth for a male baby versus a female baby was 1.07 (95% CI: 1.02, 1.13).

$$\text{OR} = e^{0.0723} = 1.07$$

$$\text{CI} = e^{(0.0723 \pm 1.96 * 0.0258)} = (1.02, 1.13)$$

- Note: **transform the intervals**. Do NOT transform standard errors (requires delta method).
- Preterm delivery was significantly associated with whether the mother smoked during pregnancy ($p < 0.001$) when controlling for age and the baby's sex. The odds ratio for mother's that smoked versus did not smoke was $e^{0.409} = 1.51$ (95% CI: 1.36, 1.67).

Birth outcome analysis, cont.

- The baseline proportion (female babies born to mother of age 25 who didn't smoke) of preterm delivery was

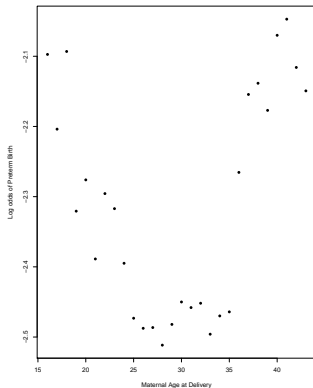
$$\frac{e^{-2.437}}{1 + e^{-2.437}} = 0.080.$$

- We didn't find an effect of mother's age.

Birth Outcome Analysis - Mother's Age

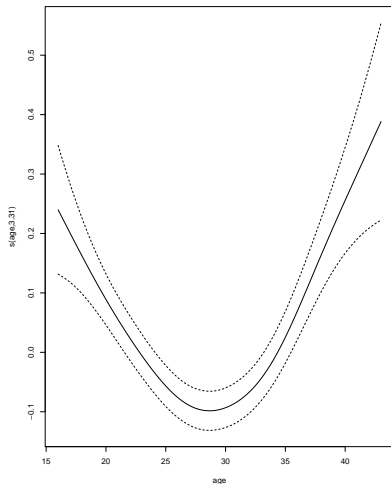
We assumed that the mother's age has a linear effect on log odds of preterm birth. Is this a reasonable assumption?

Explore this by calculating % preterm births for each age group.



Generalized Additive Model

In Module 5, we will model this non-linearly using splines:



P-values using LRTs

```
> library(car)
> # LRTs:
> Anova(fit)
Analysis of Deviance Table (Type II tests)
```

Response: ptb

| | LR | Chisq | Df | Pr(>Chisq) |
|---------|--------|-------|----|---------------|
| age | 0.085 | 1 | | 0.770669 |
| male | 7.834 | 1 | | 0.005126 ** |
| tobacco | 53.648 | 1 | | 2.399e-13 *** |

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Probit Regression

Probit regression is an alternative approach to model binary data. It still assumes the Bernoulli model, but uses a different **link** function. For

$i = 1, \dots, n$,

$$y_i \stackrel{ind}{\sim} \text{Bernoulli}(p_i)$$

where Φ^{-1} is the **inverse cumulative distribution function** of a standard normal distribution. Recall $\Phi^{-1}(x)$ asks what Z-value gives a cumulative probability of x ?

- Example: $\Phi^{-1}(0.5) = 0$ and $\Phi^{-1}(0.975) = 1.96$.
- Note $\Phi^{-1}(p_i)$ is strictly increasing, has range $(-\infty, \infty)$ and domain $(0, 1)$.

Probit Regression

The probit link function results in

$$p_i = \Phi\left(\beta_0 + \sum_{k=1}^p \beta_k x_{ik}\right).$$

Therefore, $P(y_i = 1)$ is viewed as the probability of a standard normal variable being less than $\beta_0 + \sum_{k=1}^p \beta_k x_{ik}$.

Probit regression has a very attractive **latent variable** (i.e., unobserved) interpretation. Let Z_i denote a latent variable associated with each binary outcome.

$$Z_i \stackrel{ind}{\sim} N\left(\beta_0 + \sum_{k=1}^p \beta_k x_{ik}, 1\right).$$

Then

$$\begin{aligned} P(Z_i > 0) &= 1 - P(Z_i < 0) = 1 - P\left(\frac{0 - (\beta_0 + \sum_{k=1}^p \beta_k x_{ik})}{1} < 0\right) \\ &= 1 - \Phi\left(-(\beta_0 + \sum_{k=1}^p \beta_k x_{ik})\right) = \Phi\left(\beta_0 + \sum_{k=1}^p \beta_k x_{ik}\right). \end{aligned}$$

Probit Regression: Latent Variable Representation

We can rewrite

$$y_i \stackrel{ind}{\sim} \text{Bernoulli}(p_i) \quad \Phi^{-1}(p_i) = \beta_0 + \sum_{k=1}^p \beta_k x_{ik}$$

as a hierarchical model:

$$\begin{aligned} 1. \quad & Z_i \stackrel{ind}{\sim} N\left(\beta_0 + \sum_{k=1}^p \beta_k x_{ik}, 1\right) \\ 2. \quad & y_i = \begin{cases} 0 & \text{if } Z_i < 0 \\ 1 & \text{if } Z_i > 0 \end{cases} \end{aligned}$$

Therefore we assume the binary outcome $y_i = 1$ when its latent variable Z_i **passes the threshold 0**.

The probability of this occurring depends on the mean of the latent variable Z_i . Larger mean $(\beta_0 + \sum_{k=1}^p \beta_k x_{ik})$ increases the the probability of $y_i = 1$.

Probit Regression: Interpretation

Consider a simple probit model with only one predictor:

$$y_i \overset{ind}{\sim} \text{Bernoulli}(p_i)$$

$$p_i = \Phi(\beta_0 + \beta_1 x_i).$$

Interpretation of the regression coefficients is arguably more challenging.
Represents change in z-score.

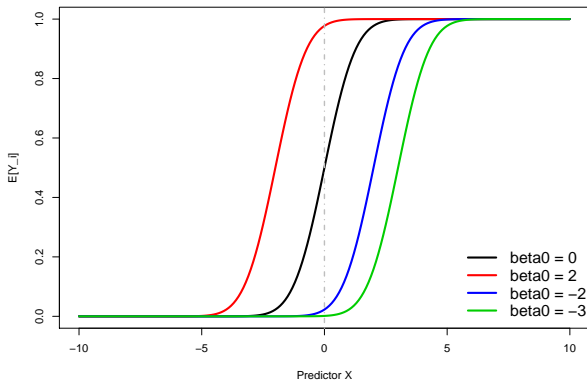
- The baseline probability is $\Phi(\beta_0)$.
- The effect of a unit increase in x_i on $P(y_i = 1)$ is

$$\Phi(\beta_0 + \beta_1 x_i + \beta_1) - \Phi(\beta_0 + \beta_1 x_i),$$

which again depends on the value of x_i .

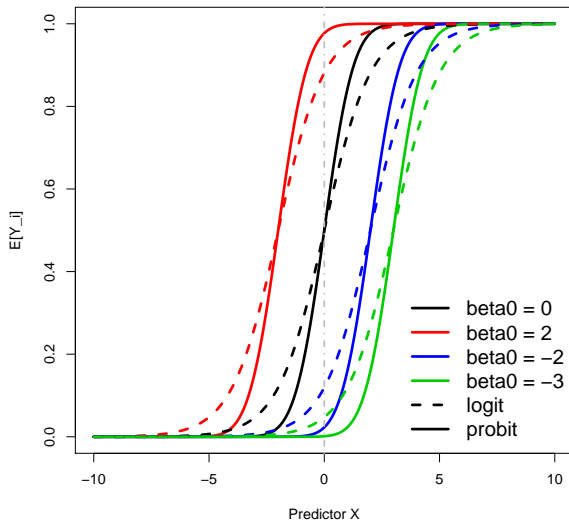
Probit Regression: Effects of Intercept

$$p_i = \mu_i = \Phi(\beta_0 + \beta_1 x_i).$$



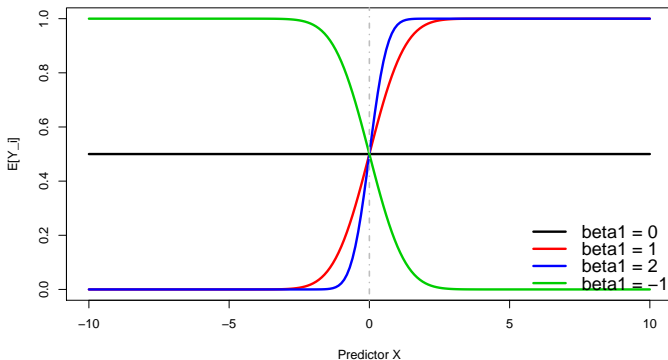
Very similar behaviors as logistic regression. Slightly different tail behaviors compared to a logit link function.

Logit and Probit Regression: Effects of Intercept



Probit Regression: Effects of Slope

$$p_i = \mu_i = \Phi(\beta_1 x_i).$$



The *glm* () Function with probit

```
glm(formula = ptb ~ age + male + tobacco, family = binomial(link = "probit"),  
    data = dat)
```

Deviance Residuals:

| Min | 1Q | Median | 3Q | Max |
|---------|---------|---------|---------|--------|
| -0.5158 | -0.4237 | -0.4104 | -0.4087 | 2.2509 |

Coefficients:

| | Estimate | Std. Error | z value | Pr(> z) | |
|-------------|------------|------------|----------|----------|-----|
| (Intercept) | -1.4024285 | 0.0099738 | -140.612 | < 2e-16 | *** |
| age | -0.0003746 | 0.0010793 | -0.347 | 0.7285 | |
| maleM | 0.0363566 | 0.0129215 | 2.814 | 0.0049 | ** |
| tobacco | 0.2102264 | 0.0281156 | 7.477 | 7.59e-14 | *** |

Comparing the logistic and probit regression model, we note the regression coefficients are **qualitatively** similar but the magnitude differs.

Poisson Regression

A Poisson regression is specified as follows. For $i = 1, \dots, n$,

For a Poisson distributed random variable,

Poisson regression is often used to model **count data**. Examples include daily mortality in a city, number of HIV infected individuals in a neighborhood, and number of medical errors at a hospital.

- Here the link function is $\log(\cdot)$.
- $\log(\cdot)$ has domain $(0, \infty)$ and range $(-\infty, \infty)$, and is strictly increasing.

Poisson Regression Interpretation

Consider a simple Poisson regression model with only one covariate:

$$y_i \overset{ind}{\sim} \text{Poisson}(\lambda_i)$$

$$\log(\lambda_i) = \beta_0 + \beta_1 x_i.$$

When $x_i = 0$, $\log(\lambda_i) = \beta_0$.

- $e^{\beta_0} = \lambda_i$ is the **baseline expected counts**.

Poisson Regression Interpretation

Now consider a unit change in x

$$\log(\lambda_i) = \beta_0 + \beta_1 x_i \quad \log(\lambda_i^*) = \beta_0 + \beta_1 (x_i + 1).$$

Note that

$$\beta_1 = \log(\lambda_i^*) - \log(\lambda_i)$$

- $e^{\beta_1} = \lambda_i^* / \lambda_i$ is the **relative change (rate)** in count per unit change in x .
- For continuous variables with $\beta_1 > 0$, the rate increases by $100 * (e^{\beta_1} - 1)\%$ for every unit increase in x .
- For factors with $\beta_1 > 0$, the rate increases by $100 * (e^{\beta_j} - 1)\%$ for level j relative to baseline.

Covariate impacts are **multiplicative** rather than additive (applies to log models in general) on the count scale:

Example: bacteria counts

Dataset: antibiotic resistance in a mutation of *E. coli*.

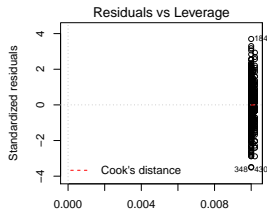
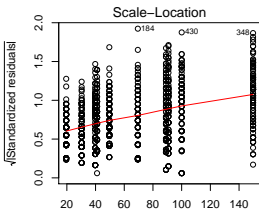
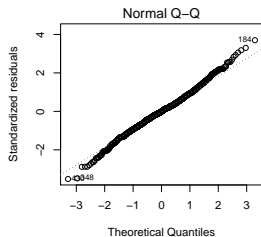
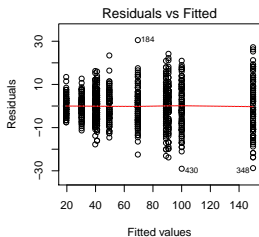
Variables:

- *Colony*: the number of ampicillin-resistant mutant colonies
- *Conc*: the concentration of novobiocin ($\mu\text{g}/\text{ml}$)
- *Media*: the type of media used for bacterial growth.

The experiment involved two media preparations (LB and M9), 5 concentrations of novobiocin, and 100 replicates for each media-concentration combination. TNTC (too numerous to count) were recorded when the number of colonies exceeded 300.

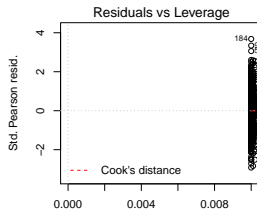
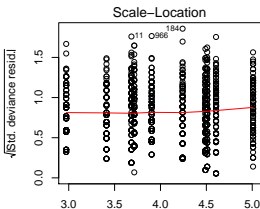
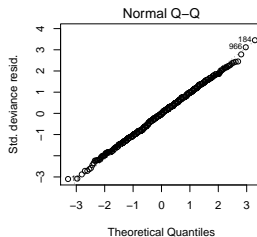
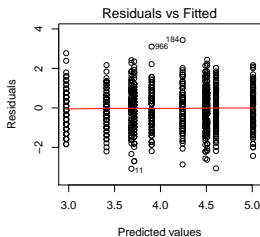
Modeling count data

```
lm_colony = lm(Colony_numeric~factor(Conc)*Media,data=colonydata)
```



Modeling count data: Poisson

```
glm_colony = glm(Colony_numeric~factor(Conc)*Media,data=colonydata,family = "poisson")
```



Residuals in glms

The usual plot of residual versus fitted is not useful in GLMs because of the relationship between the mean and variance.

The **deviance residuals** have an approximate normal distribution.

The deviance residual is

$$\hat{\epsilon}_i^d = \text{sign}(y_i - \hat{\mu}_i) \sqrt{d_i}.$$

where d_i is the i th term in the calculation of the deviance. For details, see p.113 in Wood 2017. (This plot is not useful for 0/1 data as common in logistic regression.)

Goodness of fit tests, quasipoisson

In glms, the **deviance** performs a role similar to the sum of squared errors in OLS:

where $\ell(\hat{\beta}_{\max}; \mathbf{y})$ is the “saturated model,” equal to likelihood evaluated at $\hat{\mu}_i = y_i$.

Asymptotically, $D(\hat{\beta}) \sim \chi^2_{n-p}$

One can perform a deviance test to examine goodness of fit. The null hypothesis is that the model fits the data. $p < 0.05$ indicates a problem (i.e., lack of fit).

```
with(glm_colony, cbind(res.deviance = deviance, df = df.residual,  
p = pchisq(deviance, df.residual, lower.tail=FALSE)))
```

```
res.deviance  df      p  
[1,]      973.8841 987 0.6108368
```

Here, $p > 0.05$, from which we conclude that the model provides an adequate fit.

Overdispersion

In Poisson, the assumption that $Ey_i = Vy_i$ is often violated.

One can add an additional **overdispersion parameter**, also called a **scale parameter**.

One can adjust the parameter variances by the scale parameter:

$$\hat{\beta} \sim N(0, I(\beta)^{-1} \phi)$$

We will see this again in GEEs.

Section 3.1.5 in Wood describes three different estimators of ϕ .

Quasipoisson in GLM

```
> glm_colony_quasi = glm(Colony_numeric~factor(Conc)*Media,data=colonydata,family = "quasi")
> summary(glm_colony_quasi)
```

Call:

```
glm(formula = Colony_numeric ~ factor(Conc) * Media, family = "quasipoisson",
    data = colonydata)
```

Deviance Residuals:

| Min | 1Q | Median | 3Q | Max |
|---------|---------|---------|--------|--------|
| -3.0815 | -0.7615 | -0.0047 | 0.6573 | 3.4344 |

Coefficients:

| | Estimate | Std. Error | t value | Pr(> t) |
|-------------------------|-----------|------------|---------|--------------|
| (Intercept) | 3.683308 | 0.015807 | 233.023 | < 2e-16 *** |
| factor(Conc)100 | 0.557587 | 0.019786 | 28.181 | < 2e-16 *** |
| factor(Conc)200 | 0.806339 | 0.018981 | 42.481 | < 2e-16 *** |
| factor(Conc)250 | 1.325726 | 0.017764 | 74.631 | < 2e-16 *** |
| factor(Conc)300 | 0.922162 | 0.018660 | 49.418 | < 2e-16 *** |
| MediaM9 | -0.710845 | 0.027448 | -25.898 | < 2e-16 *** |
| factor(Conc)100:MediaM9 | -0.118573 | 0.034923 | -3.395 | 0.000713 *** |
| factor(Conc)200:MediaM9 | -0.064499 | 0.033221 | -1.942 | 0.052480 . |
| factor(Conc)250:MediaM9 | 0.210650 | 0.030490 | 6.909 | 8.75e-12 *** |
| factor(Conc)300:MediaM9 | 0.009164 | 0.032406 | 0.283 | 0.777404 |

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for quasipoisson family taken to be 0.983908)

Back to original model

When the data are slightly underdispersed, i.e., dispersion parameter < 1 , and there is no evidence of lack of fit, I suggest using the original model:

```
> summary(glm_colony)
```

Call:

```
glm(formula = Colony_numeric ~ factor(Conc) * Media, family = "poisson",  
    data = colonydata)
```

Deviance Residuals:

| Min | 1Q | Median | 3Q | Max |
|---------|---------|---------|--------|--------|
| -3.0815 | -0.7615 | -0.0047 | 0.6573 | 3.4344 |

Coefficients:

| | Estimate | Std. Error | z value | Pr(> z) |
|-------------------------|-----------|------------|---------|--------------|
| (Intercept) | 3.683308 | 0.015935 | 231.140 | < 2e-16 *** |
| factor(Conc)100 | 0.557587 | 0.019947 | 27.953 | < 2e-16 *** |
| factor(Conc)200 | 0.806339 | 0.019136 | 42.138 | < 2e-16 *** |
| factor(Conc)250 | 1.325726 | 0.017908 | 74.028 | < 2e-16 *** |
| factor(Conc)300 | 0.922162 | 0.018812 | 49.019 | < 2e-16 *** |
| MediaM9 | -0.710845 | 0.027671 | -25.689 | < 2e-16 *** |
| factor(Conc)100:MediaM9 | -0.118573 | 0.035208 | -3.368 | 0.000758 *** |
| factor(Conc)200:MediaM9 | -0.064499 | 0.033491 | -1.926 | 0.054126 . |
| factor(Conc)250:MediaM9 | 0.210650 | 0.030738 | 6.853 | 7.23e-12 *** |
| factor(Conc)300:MediaM9 | 0.009164 | 0.032670 | 0.280 | 0.779097 |

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Interpretation

- The rate here is colonies per petri dish.
- Intercept: the log of the expected number of colonies is 3.68 in LB media with no novobiocin. Equivalently, the log rate is 3.68 in LB media with no novobiocin.
- The estimated number of colonies and 95% CI for this baseline is $e^{3.68}$, $e^{3.68-1.96*0.016} - e^{3.68+1.96(0.016)} = 39.6$ (38.4 - 41).
- The relative rate in Media M9 with concentration of novobiocin equal to 300 is $e^{0.009}$, i.e., the rate increases by 0.9%, which is not significant ($p > 0.05$).

Example with overdispersion

For educational purposes, consider this poor fitting model:

```
> glm_nomedia_quasi = glm(Colony_numeric~factor(Conc),data=colonydata,  
family = "quasipoisson")  
> summary(glm_nomedia_quasi)
```

Call:

```
glm(formula = Colony_numeric ~ factor(Conc), family = "quasipoisson",  
data = colonydata)
```

Deviance Residuals:

| | Min | 1Q | Median | 3Q | Max |
|--|---------|---------|---------|--------|--------|
| | -5.6832 | -2.8772 | -0.2983 | 2.5557 | 6.2330 |

Coefficients:

| | Estimate | Std. Error | t value | Pr(> t) |
|-----------------|----------|------------|---------|------------|
| (Intercept) | 3.38805 | 0.03727 | 90.90 | <2e-16 *** |
| factor(Conc)100 | 0.52177 | 0.04701 | 11.10 | <2e-16 *** |
| factor(Conc)200 | 0.78726 | 0.04493 | 17.52 | <2e-16 *** |
| factor(Conc)250 | 1.40432 | 0.04162 | 33.74 | <2e-16 *** |
| factor(Conc)300 | 0.92691 | 0.04400 | 21.06 | <2e-16 *** |

Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for quasipoisson family taken to be 8.185913)

Null deviance: 21484.0 on 996 degrees of freedom
Residual deviance: 8249.4 on 992 degrees of freedom
AIC: NA

GOF test

```
> glm_nomedia = glm(Colony_numeric~factor(Conc),data=colonydata,family = "poisson")
> summary(glm_nomedia)
```

Call:

```
glm(formula = Colony_numeric ~ factor(Conc), family = "poisson",
     data = colonydata)
```

...

(Dispersion parameter for poisson family taken to be 1)

```
Null deviance: 21484.0 on 996 degrees of freedom
Residual deviance: 8249.4 on 992 degrees of freedom
AIC: 14127
```

```
> # versus:
```

```
> with(glm_nomedia, cbind(res.deviance = deviance, df = df.residual, p = pchisq(deviance, df.residual)
  res.deviance df p
[1,]      8249.368 992 0
```