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Hough Transform from the Radon Transform

STANLEY R. DEANS

***Abstract*—An appropriate special case of a transform developed by J. Radon in 1917 is shown to have the major properties of the Hough transform which is useful for finding line segments in digital pictures. Such an observation may be useful in further efforts to generalize the Hough transform. Techniques for applying the Radon transform to lines and pixels are developed through examples, and the appropriate generalization to arbitrary curves is discussed.**

***Index Terms*—Curve detection, Hough transform, image processing, line detection, Radon transform.**

I. INTRODUCTION

The transform developed by Hough [1] for detecting straight lines in digital pictures has been discussed recently by several authors with interests in image analysis; for example, see Pratt [2], Duda and Hart [3], and Shapiro and Iannino [4] and references contained therein. The purpose of this communication is to point out that the Hough transform may be deduced as a special case of a much more general transform now known as the Radon transform which has been around for well over half a century [5]. This observation is especially important since efforts to further generalize or extend the Hough transform might profit from readily available results in the mathematics literature; see [6]-[9] and references contained therein. It is also important to emphasize that the theory of the Radon transform and its inversion coupled with the development of efficient methods for approximating the inversion have had a profound impact upon the recent development of computerized tomography. Reviews are available which cover this area [10]-[14].

The two-dimensional form of the Radon transform is given in Section II, followed by some properties and connection with the Hough transform in Section III. Simple examples are reviewed in Sections IV and V. Finally, some concluding remarks are in Section VI.

II. RADON TRANSFORM

The Radon transform may be defined on various spaces of arbitrary dimension [7]; however, to make contact with the Hough transform it is sufficient to consider the special case of two-dimensional Euclidean space and study transforms of characteristic functions [15] of sets of points lying along line segments.

Let $F(x, y)$ be an arbitrary generalized function [16] defined on the xy plane D. The Radon transform of F is given by [5], [7]

$$f(\theta, p) = R\{F\} = \iint_D F(x, y)\, \delta(p - x\cos\theta - y\sin\theta)\, dx\, dy. \tag{1}$$

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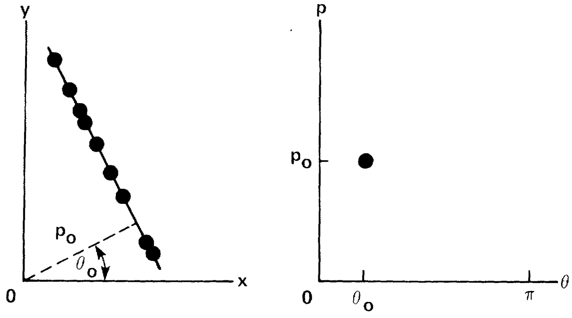

Fig. 1. The xy and θp coordinate systems. F is defined on the xy plane and f is defined on the θp plane.

If θ and/or p remain fixed then one has a sample of the transform. To obtain the full transform it is understood that θ and p vary so that f is determined for arbitrary values of θ and p. Notice that the presence of the Dirac delta function forces an integration of $F(x, y)$ along a line whose normal form is $p = x \cos\theta + y \sin\theta$.

III. Properties

Prior to any detailed calculations it may be useful to consider simple properties of the Radon transform which follow directly from the definition. It will be useful to think of $f(\theta, p)$ defined on a *rectangular* grid in the θp plane in the same fashion as $F(x, y)$ was defined on a rectangular grid in the xy plane. The geometry is illustrated in Fig. 1. Observations which follow immediately include the following.

1) If F is concentrated at a point (x_0, y_0) then f is nonzero along a sinusoidal curve $p = x_0 \cos\theta + y_0 \sin\theta$. This is easy to verify by substituting $F = \delta(x - x_0)\,\delta(y - y_0)$ directly into (1). (Also note that $p = x_0 \cos\theta + y_0 \sin\theta$ is the polar form for a circle with center at $(x_0/2, y_0/2)$ and radius $\frac{1}{2}(x_0^2 + y_0^2)^{1/2}$ where p is the radial coordinate and θ is the angular coordinate.)

2) A given point (θ_0, p_0) in the θp plane corresponds to a line in the xy plane defined by $p_0 = x \cos\theta_0 + y \sin\theta_0$. This follows from (1) with θ and p fixed at $\theta = \theta_0$ and $p = p_0$.

From 1) and 2) along with (1) we also have the following.

3) Collinear points in the xy plane along the line parameterized by θ_0 and p_0 map to sinusoidal curves in the θp plane, but these curves all intersect at the point (θ_0, p_0).

4) Points lying along the curve $p = x_0 \cos\theta + y_0 \sin\theta$ in the θp plane correspond to lines in the xy plane all of which pass through the point (x_0, y_0).

Already it is clear that the major properties of the Hough transform may be deduced from the appropriate special case of the Radon transform since observations 1)–4) are just the properties listed by Duda and Hart [3] in their discussion of the Hough transform. We now consider some pedagogical examples to illustrate the use of the Radon transform when $F(x, y)$ is concentrated along a line segment.

IV. Transform Over a Line Segment

Suppose $F(x, y)$ is concentrated along some line segment l. We may write

$$F(x, y) = a(x, y)\,\chi(l) \tag{2}$$

where $a(x, y)$ is a (nonnegative) density along l and $\chi(l)$ is the characteristic function [15] of l. ($\chi(l)$ is unity along l and zero elsewhere.) It is important to keep in mind that now there are two lines, or more precisely one line and a line segment, one defined by $p = x \cos\theta + y \sin\theta$ and a line segment l along which F may assume nonzero values.

To provide a better understanding of calculational techniques we consider a special case. Suppose the line segment l is vertical at a distance x_0 from the y axis as illustrated in Fig. 2. Actually this is not such a special case since by a rotation of coordinates a line with arbitrary slope can be transfored to the situation considered here. Thus, F may be written as

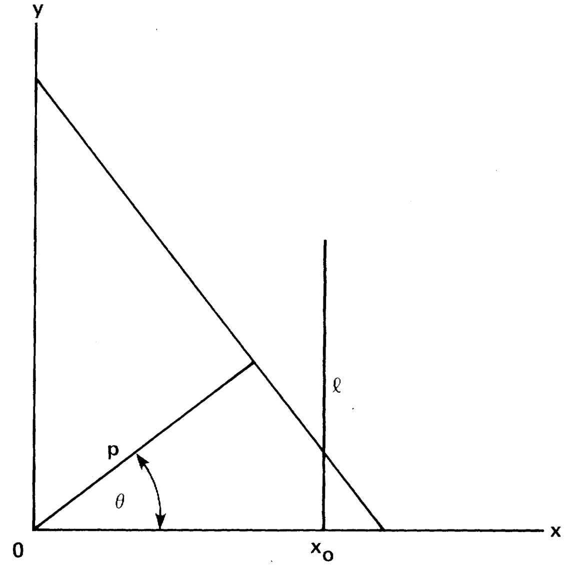

Fig. 2. Transform over a line segment.

$$F(x, y) = a(y)\,\delta(x - x_0), \tag{3}$$

and we assume the density $a(y)$ is such that $\int_{-\infty}^{+\infty} a(y)\,dy$ is finite.

The Radon transform of F defined in (3) is given by

$$\begin{aligned} f(\theta, p) &= \iint_D a(y)\,\delta(x - x_0)\,\delta(p - x\cos\theta - y\sin\theta)\,dx\,dy \\ &= |\csc\theta| \int_{-\infty}^{+\infty} a(y)\,\delta(p\csc\theta - x_0\cot\theta - y)\,dy \\ &= a(p\csc\theta - x_0\cot\theta)/|\sin\theta|. \end{aligned} \tag{4}$$

The $a(p\csc\theta - x_0\cot\theta)$ term has a simple interpretation. It is just the density at the value of y along the line $x = x_0$ where the lines $p = x\cos\theta + y\sin\theta$ and $x = x_0$ intersect. (Just solve $p = x_0\cos\theta + y\sin\theta$ for y.) Of course, since $x = x_0$ is really a segment there may not be a point of intersection in which case f vanishes. The presence of the $|\sin\theta|$ term in the denominator of (4) serves to automatically yield the appropriate limit as the integration is directly along l. To see this go back to (4) and perform the transform for θ fixed at $\theta = 0$,

$$\begin{aligned} f(0, p) &= \iint_D a(y)\,\delta(x - x_0)\,\delta(p - x)\,dx\,dy \\ &= \delta(p - x_0)\int_{-\infty}^{+\infty} a(y)\,dy. \end{aligned} \tag{5}$$

Thus, $f(0, p)$ vanishes except at $p = x_0$ where it is a spike. By selecting an appropriate sequence of values for θ such that the sequence $\{\theta_1, \theta_2, \cdots\}$ converges to $\theta = 0$ it is easy to verify that in fact the corresponding sequence $\{f(\theta_1, p), f(\theta_2, p), \cdots\}$ forms a δ-convergent sequence [16]. The situation is shown in Fig. 3 for three values of θ where, for purposes of illustration, $a(y) = 1$ for $0 < y < L$ and $a(y) = 0$ otherwise. Note

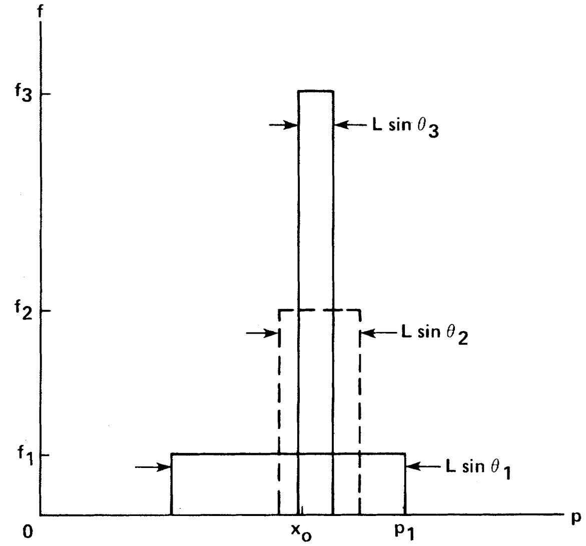

Fig. 3. First three terms of δ-convergent sequence.

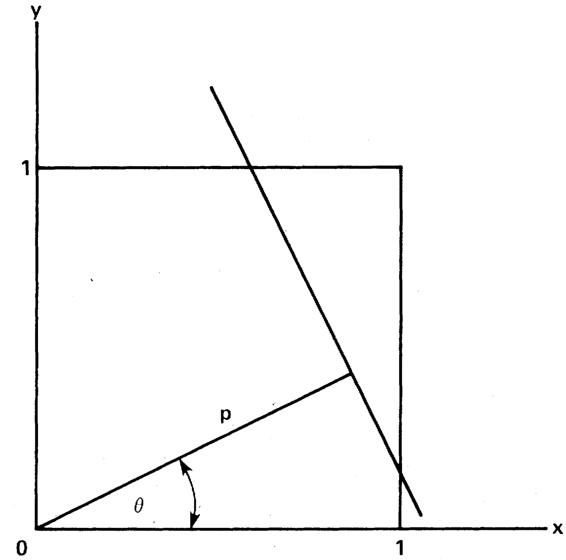

Fig. 4. Radon transform over unit pixel.

that for $\theta = \theta_1$ the function f vanishes for $p > p_i = x_0 \cos \theta_i + L \sin \theta_i$ and for $p < x_0 \cos \theta_i$. Over the intermediate region of length $L \sin \theta$ along the p axis the value of f is constant at $f_i = 1/|\sin \theta_i|$. Consequently, the "area" is always L. Note that from (5) one also gets $\int_{-\infty}^{+\infty} f(0, p)\, dp = \int_{-\infty}^{+\infty} a(y)\, dy = L$.

V. Transform Over a Pixel

In most applications the segment l is not a true line segment but has finite width and is actually a strip. The essential properties associated with the transform over a strip are illustrated by the transform over a single pixel. To form a strip one merely lines up several pixels.

Suppose F is unity on a pixel of unit length and unit width and zero otherwise; that is, F is identified with the characteristic function of the pixel. As we shall see, now the Radon transform of F is just the length of the line contained with the pixel. Since it is always possible to transform coordinates to one corner of the pixel it is sufficient to consider the situation illustrated in Fig. 4. The Radon transform is given by (1) where D is now the unit pixel and it is adequate to consider angles in the range $0 \leqslant \theta \leqslant \pi/4$. The integration must be carried out over three separate regions for p. The calculation for the first two regions is not shown in detail since the technique should be clear from the calculation for the third region.

1) $$f(\theta, p) = \frac{p}{\sin \theta \cos \theta}, \qquad 0 < p \leqslant \sin \theta$$

2) $$f(\theta, p) = \frac{1}{\cos \theta}, \qquad \sin \theta \leqslant p \leqslant \cos \theta$$

3) $$\begin{aligned} f(\theta, p) &= \iint_D F(x, y)\, \delta(p - x \cos \theta - y \sin \theta)\, dx\, dy \\ &= \frac{1}{\cos \theta} \int_{p \csc \theta - \cot \theta}^{1} F(p \sec \theta - y \tan \theta, y)\, dy \\ &= \frac{1}{\cos \theta} (1 - p \csc \theta + \cot \theta) \\ &= \frac{\sin \theta + \cos \theta - p}{\sin \theta \cos \theta}, \qquad \cos \theta \leqslant p \leqslant \sin \theta + \cos \theta. \end{aligned}$$

VI. Transform Over a Curve

The generalization from line detection to curve detection is straightforward, although the computational details are more complicated. Let

$$p = C(x, y; \xi) \tag{6}$$

be the equation for some family of curves in the xy plane parametrized by the scalar p and the components $\xi_1, \xi_2, \cdots, \xi_n$ of the vector ξ. Suppose, for example, we are interested in parabolas of the form

$$p = y - \xi_1 x^2 - \xi_2 x. \tag{7}$$

Then, $C(x, y; \xi) = y - \xi_1 x^2 - \xi_2 x$ and $\xi = (\xi_1, \xi_2)$. Moreover, note that for $\xi = (\cos \theta, \sin \theta)$ and $r = (x, y)$, (6) has the form

$$p = r \cdot \xi = x \cos \theta + y \sin \theta$$

which was used in line detection.

In complete analogy with Section IV, suppose $F(x, y)$ is concentrated along some curve c,

$$F(x, y) = a(x, y)\, \chi(c), \tag{8}$$

where $a(x, y)$ is a nonnegative density along c and $\chi(c)$ is the characteristic function of c. The generalization of the transform in (1) is

$$f(\xi, p) = \iint_D a(x, y)\, \chi(c)\, \delta[p - C(x, y; \xi)]\, dx\, dy. \tag{9}$$

Clearly, $f(\xi, p)$ will be concentrated at precisely the values of p and ξ which yield the curve c out of the family designated by C. If c is the curve $y = x^2$, for example, then f will be concentrated at $p = 0$, $\xi_1 = 1$, and $\xi_2 = 0$ provided C is given by (7). Finally, if $F(x, y)$ is concentrated along some "curve of finite width" or curved strip s then (8) is modified to read

$$F(x, y) = a(x, y)\, \chi(s)$$

where $\chi(s)$ is the characteristic function of the region of the strip. Those feature points which lie inside the strip are thus mapped to a single point or dense region of parameter space. The resulting clusters correspond to curves in the original picture.

Similar types of mappings of feature points, which are possible points of curves, into clusters in parameter space have been achieved by use of Hough-type transforms and the results have been analyzed for computer detection of curves in noisy pictures [17], [18]. Further discussion of line and curve detection by use of the Radon transform may be found in [19].

VII. SUMMARY

In the appropriate special case the Radon transform reduces to the Hough transform thus providing a natural formalism for generalizing the Hough transform. The Radon transform has been studied extensively and references to the mathematics literature were given. Due primarily to the advent of computerized tomography there has been great interest in Radon transform inversion in a digital setting. Review articles in this area were cited. Finally, simple examples were presented to illustrate properties and use of the Radon transform for lines and pixels, and generalizations to more complicated curves and regions were given.

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Pixel Labeling by Supervised Probabilistic Relaxation

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***Abstract*—A simple modification to existing probabilistic relaxation procedures is suggested which allows the information contained in initial labels to exert an influence on the direction of relaxation throughout the process. In this manner, the initial labels assume more importance than with conventional algorithms and are used in combination with the outcome of relaxation at each iteration to produce a cooperative estimate of the correct label for a particular object. Pixel labeling examples are presented which show the performance that can be obtained with the modified algorithm. The procedure is readily generalized to allow other data to influence the process.**

Index Terms—**Context, initial-labeling, labeling, pixel, relaxation.**

I. INTRODUCTION

Probabilistic relaxation procedures iteratively modify an initial estimate of the labeling of a scene element by reference to spatial context. Several algorithms have been proposed for this purpose; Rosenfeld *et al.* [1] have devised a technique that introduces context by means of correlations of labels between objects and their neighbors. Zucker and Mohammed [2] have suggested schemes that depend instead upon the conditional probability of occurrence of a particular label on an object in view of the labeling on neighbors. More recently an algorithm, also based upon a probabilistic interpretation of context, and which has a probabilistic rather than heuristic basis, has been proposed [3] as have variations on Rosenfeld's algorithm [4], [5] and an algorithm derived from a constrained optimization of a mixed consistency and ambiguity criterion [6].

It is an essential ingredient of the above schemes that the initial scene labeling is used only once, viz., when the algorithm is initialized, and thereafter the success of the final labeling is dependent upon both the attributes of the algorithm and the accuracy of the contextual data, both of which can be envisaged as assuming more significance relative to the initial labeling as relaxation proceeds. This may not be a difficulty in picture-labeling problems such as the "toy triangle" example often used [1], [2] since the initial labeling is seen mainly as an initialization procedure and the context information is often known with certainty. The situation can be quite different, however, in pixel labeling exercises such as those undertaken in the interpretation of Landsat images. For example, when it is desired to determine a label for every pixel in an image, the contextual information would generally not be known exactly and indeed may only be an estimate based upon "typical" image data of a similar type. Furthermore, the initial labeling, by and large, would represent "the best one could do" based

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