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ACOUSTIC PROPAGATION MODELLING

Contents

Part I

Helmholtz Equation

Taken from Computational Ocean Acoustics[1].

$$
\begin{aligned}
O(\omega^2):& \qquad |\nabla\tau|^2 = \frac{1}{c^2(\vec{x})} \\
O(\omega^1):& \qquad 2\nabla\tau\cdot\nabla A_0 + \left(\nabla^2\tau\right)A_0 = 0 \\
O(\omega^{1-j}):& \qquad 2\nabla\tau\cdot\nabla A_j + \left(\nabla^2\tau\right)A_j = -\nabla^2 A_{j-1}
\end{aligned}
$$

[1] Jensen, F. B., Kuperman, W. A., Porter, M. B., and Schmidt, H. (2011). *Computational ocean acoustics*. Springer Science & Business Media

Eikonal Equation

The eikonal equation

$$|\nabla\tau|^2 = \frac{1}{c^2(\vec{x})}$$

is a first-order nonlinear PDE for modelling the path taken by a ray.

First-Order System

In cylindrical coordinates,

$$\frac{dr}{ds} = c\xi(s) \qquad \frac{d\xi}{ds} = \frac{-1}{c(r,z)^2}\frac{\partial c}{\partial r}$$

$$\frac{dz}{ds} = c\zeta(s) \qquad \frac{d\zeta}{ds} = \frac{-1}{c(r,z)^2}\frac{\partial c}{\partial z}$$

$$\frac{d\tau}{ds} = \frac{1}{c(r,z)}$$

with initial conditions

$$r = r_0 \qquad \xi = \frac{\cos(\theta_0)}{c(r_0,z_0)}$$

$$z = z_0 \qquad \zeta = \frac{\sin(\theta_0)}{c(r_0,z_0)}$$

$$\tau = 0$$

and boundary conditions defined as reflection off the bathymetry $z_{\text{bty}}(r)$ and altimetry $z_{\text{ati}}(r)$.

$$\begin{aligned}
\theta_i &= c(r,z)\cos^{-1}(\xi_i) \\
&= c(r,z)\sin^{-1}(\zeta_i) \\
\theta_r &= 2\theta_{\text{bnd}} - \theta_i \\
\xi_r &= \frac{\cos(\theta_r)}{c(r,z)} \\
\zeta_r &= \frac{\sin(\theta_r)}{c(r,z)}
\end{aligned}$$

Boundary Reflection

$$\vec{t}_{\mathrm{rfl}} = \vec{t}_{\mathrm{inc}} - 2\left(\vec{t}_{\mathrm{inc}} \cdot \vec{n}_{\mathrm{bnd}}\right)\vec{n}_{\mathrm{bnd}}$$

Sonar Equations

Detection Threshold

Detection Index

The detection index is expressed as

$$d = \left(\frac{\mu_{\text{spn}} - \mu_{\text{nse}}}{\sigma_{\text{nse}}}\right)^2$$

For a Gaussian noise and signal-plus-noise with a non-fluctuating signal, note that $\sigma_{\text{nse}} = \sigma_{\text{spn}}$ so

$$\begin{aligned} f_{\text{nse}}(x) &= \frac{1}{\sqrt{2\pi\sigma_{\text{nse}}^2}} \exp\left\{\frac{-1}{2}\left(\frac{x - \mu_{\text{nse}}}{\sigma_{\text{nse}}}\right)^2\right\} \\ f_{\text{spn}}(x) &= \frac{1}{\sqrt{2\pi\sigma_{\text{nse}}^2}} \exp\left\{\frac{-1}{2}\left(\frac{x - \mu_{\text{spn}}}{\sigma_{\text{nse}}}\right)^2\right\} \end{aligned}$$

Their respective cumulative density functions are

$$\begin{aligned} F_{\text{nse}}(x) &= \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x - \mu_{\text{nse}}}{\sqrt{2}\sigma_{\text{nse}}}\right)\right] \\ F_{\text{spn}}(x) &= \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x - \mu_{\text{spn}}}{\sqrt{2}\sigma_{\text{nse}}}\right)\right] \end{aligned}$$

The probability of detection and probability of false alarm are defined via the point at which the two density values meet, integrated above as

$$\begin{aligned} p_{\text{dtc}} &= 1 - \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x - \mu_{\text{spn}}}{\sqrt{2}\sigma_{\text{nse}}}\right)\right], \quad & p_{\text{fal}} &= 1 - \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x - \mu_{\text{nse}}}{\sqrt{2}\sigma_{\text{nse}}}\right)\right] \\ \Rightarrow p_{\text{dtc}} &= \frac{1}{2}\operatorname{erfc}\left(\frac{x - \mu_{\text{spn}}}{\sqrt{2}\sigma_{\text{nse}}}\right), & p_{\text{fal}} &= \frac{1}{2}\operatorname{erfc}\left(\frac{x - \mu_{\text{nse}}}{\sqrt{2}\sigma_{\text{nse}}}\right) \\ \Rightarrow x &= \mu_{\text{spn}} + \sqrt{2}\sigma_{\text{nse}}\operatorname{erfc}^{-1}\left(2p_{\text{dtc}}\right), & x &= \mu_{\text{nse}} + \sqrt{2}\sigma_{\text{nse}}\operatorname{erf}^{-1}\left(2p_{\text{fal}}\right) \end{aligned}$$

So equating these expressions yields

$$\begin{aligned} \mu_{\text{spn}} + \sqrt{2}\sigma_{\text{nse}}\operatorname{erfc}^{-1}\left(2p_{\text{dtc}}\right) &= \mu_{\text{nse}} + \sqrt{2}\sigma_{\text{nse}}\operatorname{erfc}^{-1}\left(2p_{\text{fal}}\right) \\ \Rightarrow \left(\frac{\mu_{\text{spn}} - \mu_{\text{nse}}}{\sigma_{\text{nse}}}\right)^2 &= 2\left[\operatorname{erfc}^{-1}\left(2p_{\text{fal}}\right) - \operatorname{erfc}^{-1}\left(2p_{\text{dtc}}\right)\right]^2 \end{aligned}$$

where the left hand side is the definition of the detection index. Thus,

$$d = 2\left[\operatorname{erfc}^{-1}\left(2p_{\text{fal}}\right) - \operatorname{erfc}^{-1}\left(2p_{\text{dtc}}\right)\right]^2$$

Rearranged,

$$p_{\text{dtc}} = \frac{1}{2}\operatorname{erfc}\left(\operatorname{erfc}^{-1}\left(2p_{\text{fal}}\right) - \sqrt{\frac{d}{2}}\right)$$

Bibliography

Jensen, F. B., Kuperman, W. A., Porter, M. B., and Schmidt, H. (2011). *Computational ocean acoustics*. Springer Science & Business Media.

Part II

Appendix

Concocting Equations

Celerity

$$
\begin{aligned}
c_{\text{max}} &= 1600 \\
c_{\text{min}} &= 1500 \\
c(r,z) &= c_0 + c_1 z + c_2 z^2 \\
c_0 + c_1 z_{\text{ati}} + c_2 z_{\text{ati}}^2 &= c(r, z_{\text{ati}}) = c_{\text{max}} \\
c_0 + c_1 \frac{z_{\text{ati}} + z_{\text{bty}}}{2} + c_2 \left(\frac{z_{\text{ati}} + z_{\text{bty}}}{2} \right)^2 &= c(r, \frac{z_{\text{ati}} + z_{\text{bty}}}{2}) = c_{\text{min}} \\
c_0 + c_1 z_{\text{bty}} + c_2 z_{\text{bty}}^2 &= c(r, z_{\text{bty}}) = c_{\text{max}} \\
\begin{pmatrix} 1 & z_{\text{ati}} & z_{\text{ati}}^2 \\ 1 & \frac{z_{\text{ati}} + z_{\text{bty}}}{2} & \left(\frac{z_{\text{ati}} + z_{\text{bty}}}{2} \right)^2 \\ 1 & z_{\text{bty}} & z_{\text{bty}}^2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} &= \begin{pmatrix} c_{\text{max}} \\ c_{\text{min}} \\ c_{\text{max}} \end{pmatrix}
\end{aligned}
$$

Bathymmetry

$$\begin{aligned}
z_{\text{min}} &= 700 \\
z_{\text{max}} &= 1000 \\
r_0 &= 300 \\
z_{\text{bty}}(r) &= z_{\text{max}} - (z_{\text{max}} - z_{\text{min}}) \exp\left(-\frac{(r-r_0)^2}{A_r}\right) \\
z_{\text{bty}}\left(\frac{r_0}{3}\right) &= z_{\text{min}} + \frac{z_{\text{min}} + z_{\text{max}}}{10} = z_{\text{max}} - (z_{\text{max}} - z_{\text{min}}) \exp\left(-\frac{(r-r_0)^2}{A_r}\right) \\
z_{\text{min}} + \frac{z_{\text{min}} + z_{\text{max}}}{10} &= z_{\text{max}} - (z_{\text{max}} - z_{\text{min}}) \exp\left(-\frac{(r-r_0)^2}{A_r}\right) \\
(z_{\text{max}} - z_{\text{min}}) \exp\left(-\frac{(r-r_0)^2}{A_r}\right) &= z_{\text{max}} - z_{\text{min}} - \frac{z_{\text{min}} + z_{\text{max}}}{10} \\
\exp\left(-\frac{(r-r_0)^2}{A_r}\right) &= \frac{\frac{9}{10} z_{\text{max}} - \frac{11}{10} z_{\text{min}}}{z_{\text{max}} - z_{\text{min}}} \\
-\frac{(r-r_0)^2}{A_r} &= \ln\left(\frac{9 z_{\text{max}} - 11 z_{\text{min}}}{10\left(z_{\text{max}} - z_{\text{min}}\right)}\right) \\
A_r &= \frac{4r_0^2/9}{\ln\left(\frac{9 z_{\text{max}} - 11 z_{\text{min}}}{10\left(z_{\text{max}} - z_{\text{min}}\right)}\right)}
\end{aligned}$$

Sonar Equation Manipulation

Calculating Probability of Detection

$$
\begin{aligned}
d &= Bt\left(\frac{\mathrm{SL}-\mathrm{TL}}{B\mathrm{NL}}\right)^2 \\
p_{\mathrm{dtc}} &= \frac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{d}{2}}+\mathrm{erfc}^{-1}\left(2p_{\mathrm{fal}}\right)\right)
\end{aligned}
$$