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ACOUSTIC PROPAGA-TION MODELLING

Part I

Helmholtz Equation

Taken from Computational Ocean Acoustics[1].

[1] Jensen, F. B., Kuperman, W. A., Porter, M. B., and Schmidt, H. (2011). *Computational ocean acoustics*. Springer Science & Business Media

$$O(\omega^2): \qquad |\nabla\tau|^2 = \frac{1}{c^2(\vec{x})}$$

$$O(\omega^1): \qquad 2\nabla\tau \cdot \nabla A_0 + \left(\nabla^2\tau\right) A_0 = 0$$

$$O(\omega^{1-j}): \qquad 2\nabla\tau \cdot \nabla A_j + \left(\nabla^2\tau\right) A_j = -\nabla^2 A_{j-1}$$

Eikonal Equation

The eikonal equation

$$|\nabla \tau|^2 = \frac{1}{c^2(\vec{x})}$$

is a first-order nonlinear PDE for modelling the path taken by a ray.

First-Order System

In cylindrical coordinates,

$$\frac{dr}{ds} = c\xi(s) \qquad\qquad \frac{d\xi}{ds} = \frac{-1}{c(r,z)^2} \frac{\partial c}{\partial r}$$

$$\frac{dz}{ds} = c\zeta(s) \qquad\qquad \frac{d\zeta}{ds} = \frac{-1}{c(r,z)^2} \frac{\partial c}{\partial z}$$

$$\frac{d\tau}{ds} = \frac{1}{c(r,z)}$$

with initial conditions

$$r = r_0 \qquad\qquad \xi = \frac{\cos(\theta_0)}{c(r_0, z_0)}$$

$$z = z_0 \qquad\qquad \zeta = \frac{\sin(\theta_0)}{c(r_0, z_0)}$$

$$\tau = 0$$

and boundary conditions defined as reflection off the bathymetry $z_{\text{bty}}(r)$ and altimetry $z_{\text{ati}}(r)$.

$$\theta_i = c(r,z) \cos^{-1}(\xi_i)$$
$$= c(r,z) \sin^{-1}(\zeta_i)$$
$$\theta_r = 2\theta_{\text{bnd}} - \theta_i$$
$$\xi_r = \frac{\cos(\theta_r)}{c(r,z)}$$
$$\zeta_r = \frac{\sin(\theta_r)}{c(r,z)}$$

Asymptotic Analysis

Repeated,

$$O(\omega^2): \qquad |\nabla\tau|^2 = \frac{1}{c^2(\vec{x})}$$

$$O(\omega^1): \qquad 2\nabla\tau \cdot \nabla A_0 + \left(\nabla^2\tau\right) A_0 = 0$$

$$O(\omega^{1-j}): \qquad 2\nabla\tau \cdot \nabla A_j + \left(\nabla^2\tau\right) A_j = -\nabla^2 A_{j-1}$$

where the unknowns are $\tau(\vec{x})$ and $A_j(\vec{x})$ for $j = 0, 1, 2, \dots$.

Bibliography

Jensen, F. B., Kuperman, W. A., Porter, M. B., and Schmidt, H. (2011).
   Computational ocean acoustics. Springer Science & Business Media.

Part II

Appendix

Concocting Equations

Celerity

$$c_{max} = 1600$$

$$c_{min} = 1500$$

$$c(r, z) = c_0 + c_1 z + c_2 z^2$$

$$c_0 + c_1 z_{ati} + c_2 z_{ati}^2 = c(r, z_{ati}) = c_{max}$$

$$c_0 + c_1 \frac{z_{ati} + z_{bty}}{2} + c_2 \left(\frac{z_{ati} + z_{bty}}{2} \right)^2 = c(r, \frac{z_{ati} + z_{bty}}{2}) = c_{min}$$

$$c_0 + c_1 z_{bty} + c_2 z_{bty}^2 = c(r, z_{bty}) = c_{max}$$

$$\begin{pmatrix} 1 & z_{ati} & z_{ati}^2 \\ 1 & \frac{z_{ati} + z_{bty}}{2} & \left(\frac{z_{ati} + z_{bty}}{2} \right)^2 \\ 1 & z_{bty} & z_{bty}^2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_{max} \\ c_{min} \\ c_{max} \end{pmatrix}$$

Bathymmetry

$$z_{\min} = 700$$

$$z_{\max} = 1000$$

$$r_0 = 300$$

$$z_{\text{bty}}(r) = z_{\max} - (z_{\max} - z_{\min}) \exp\left(-\frac{(r - r_0)^2}{A_r}\right)$$

$$z_{\text{bty}}\left(\frac{r_0}{3}\right) = z_{\min} + \frac{z_{\min} + z_{\max}}{10} = z_{\max} - (z_{\max} - z_{\min}) \exp\left(-\frac{(r - r_0)^2}{A_r}\right)$$

$$z_{\min} + \frac{z_{\min} + z_{\max}}{10} = z_{\max} - (z_{\max} - z_{\min}) \exp\left(-\frac{(r - r_0)^2}{A_r}\right)$$

$$(z_{\max} - z_{\min}) \exp\left(-\frac{(r - r_0)^2}{A_r}\right) = z_{\max} - z_{\min} - \frac{z_{\min} + z_{\max}}{10}$$

$$\exp\left(-\frac{(r - r_0)^2}{A_r}\right) = \frac{\frac{9}{10}z_{\max} - \frac{11}{10}z_{\min}}{z_{\max} - z_{\min}}$$

$$-\frac{(r - r_0)^2}{A_r} = \ln\left(\frac{9z_{\max} - 11z_{\min}}{10(z_{\max} - z_{\min})}\right)$$

$$A_r = \frac{4r_0^2/9}{\ln\left(\frac{9z_{\max} - 11z_{\min}}{10(z_{\max} - z_{\min})}\right)}$$