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ACOUSTIC PROPAGATION MODELLING

Contents

Part I

Helmholtz Equation

Taken from Computational Ocean Acoustics[1].

[1] Jensen, F. B., Kuperman, W. A., Porter, M. B., and Schmidt, H. (2011). *Computational ocean acoustics*. Springer Science & Business Media

$$
\begin{aligned}
O(\omega^2): &\qquad |\nabla\tau|^2 = \frac{1}{c^2(\vec{x})} \\
O(\omega^1): &\qquad 2\nabla\tau\cdot\nabla A_0 + \left(\nabla^2\tau\right) A_0 = 0 \\
O(\omega^{1-j}): &\qquad 2\nabla\tau\cdot\nabla A_j + \left(\nabla^2\tau\right) A_j = -\nabla^2 A_{j-1}
\end{aligned}
$$

Eikonal Equation

The eikonal equation

$$|\nabla \tau|^2 = \frac{1}{c^2(\vec{x})}$$

is a first-order nonlinear PDE for modelling the path taken by a ray.

First-Order System

In cylindrical coordinates,

$$\frac{dr}{ds} = c\xi(s) \qquad \frac{d\xi}{ds} = \frac{-1}{c(r,z)^2}\frac{\partial c}{\partial r}$$

$$\frac{dz}{ds} = c\zeta(s) \qquad \frac{d\zeta}{ds} = \frac{-1}{c(r,z)^2}\frac{\partial c}{\partial z}$$

$$\frac{d\tau}{ds} = \frac{1}{c(r,z)}$$

with initial conditions

$$r = r_0 \qquad \xi = \frac{\cos(\theta_0)}{c(r_0, z_0)}$$

$$z = z_0 \qquad \zeta = \frac{\sin(\theta_0)}{c(r_0, z_0)}$$

$$\tau = 0$$

and boundary conditions defined as reflection off the bathymetry $z_{\text{bty}}(r)$ and altimetry $z_{\text{ati}}(r)$.

$$\begin{aligned}
\theta_i &= c(r,z)\cos^{-1}(\xi_i) \\
&= c(r,z)\sin^{-1}(\zeta_i) \\
\theta_r &= 2\theta_{\text{bnd}} - \theta_i \\
\xi_r &= \frac{\cos(\theta_r)}{c(r,z)} \\
\zeta_r &= \frac{\sin(\theta_r)}{c(r,z)}
\end{aligned}$$

Boundary Reflection

$$\vec{t}_{\mathrm{rfl}} = \vec{t}_{\mathrm{inc}} - 2\left(\vec{t}_{\mathrm{inc}} \cdot \vec{n}_{\mathrm{bnd}}\right)\vec{n}_{\mathrm{bnd}}$$

Bibliography

Jensen, F. B., Kuperman, W. A., Porter, M. B., and Schmidt, H. (2011). *Computational ocean acoustics*. Springer Science & Business Media.

Part II

Appendix

Concocting Equations

Celerity

$$
\begin{aligned}
c_{\mathrm{max}} &= 1600 \\
c_{\mathrm{min}} &= 1500 \\
c(r,z) &= c_0 + c_1 z + c_2 z^2 \\
c_0 + c_1 z_{\mathrm{ati}} + c_2 z_{\mathrm{ati}}^2 &= c(r, z_{\mathrm{ati}}) = c_{\mathrm{max}} \\
c_0 + c_1 \frac{z_{\mathrm{ati}} + z_{\mathrm{bty}}}{2} + c_2 \left(\frac{z_{\mathrm{ati}} + z_{\mathrm{bty}}}{2} \right)^2 &= c(r, \frac{z_{\mathrm{ati}} + z_{\mathrm{bty}}}{2}) = c_{\mathrm{min}} \\
c_0 + c_1 z_{\mathrm{bty}} + c_2 z_{\mathrm{bty}}^2 &= c(r, z_{\mathrm{bty}}) = c_{\mathrm{max}} \\
\begin{pmatrix} 1 & z_{\mathrm{ati}} & z_{\mathrm{ati}}^2 \\ 1 & \frac{z_{\mathrm{ati}} + z_{\mathrm{bty}}}{2} & \left(\frac{z_{\mathrm{ati}} + z_{\mathrm{bty}}}{2} \right)^2 \\ 1 & z_{\mathrm{bty}} & z_{\mathrm{bty}}^2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} &= \begin{pmatrix} c_{\mathrm{max}} \\ c_{\mathrm{min}} \\ c_{\mathrm{max}} \end{pmatrix}
\end{aligned}
$$

Bathymmetry

$$
\begin{aligned}
z_{\text{min}} &= 700 \\
z_{\text{max}} &= 1000 \\
r_0 &= 300 \\
z_{\text{bty}}(r) &= z_{\text{max}} - (z_{\text{max}} - z_{\text{min}}) \exp\left(-\frac{(r-r_0)^2}{A_r}\right) \\
z_{\text{bty}}\left(\frac{r_0}{3}\right) &= z_{\text{min}} + \frac{z_{\text{min}} + z_{\text{max}}}{10} = z_{\text{max}} - (z_{\text{max}} - z_{\text{min}}) \exp\left(-\frac{(r-r_0)^2}{A_r}\right) \\
z_{\text{min}} + \frac{z_{\text{min}} + z_{\text{max}}}{10} &= z_{\text{max}} - (z_{\text{max}} - z_{\text{min}}) \exp\left(-\frac{(r-r_0)^2}{A_r}\right) \\
(z_{\text{max}} - z_{\text{min}}) \exp\left(-\frac{(r-r_0)^2}{A_r}\right) &= z_{\text{max}} - z_{\text{min}} - \frac{z_{\text{min}} + z_{\text{max}}}{10} \\
\exp\left(-\frac{(r-r_0)^2}{A_r}\right) &= \frac{\frac{9}{10} z_{\text{max}} - \frac{11}{10} z_{\text{min}}}{z_{\text{max}} - z_{\text{min}}} \\
-\frac{(r-r_0)^2}{A_r} &= \ln\left(\frac{9 z_{\text{max}} - 11 z_{\text{min}}}{10\,(z_{\text{max}} - z_{\text{min}})}\right) \\
A_r &= \frac{4 r_0^2 / 9}{\ln\left(\frac{9 z_{\text{max}} - 11 z_{\text{min}}}{10\,(z_{\text{max}} - z_{\text{min}})}\right)}
\end{aligned}
$$