Fresnel Equations and Transfer Matrix

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Introduction

- Fresnel's equations describe the reflection and transmission of electromagnetic waves at an interface. That is, they give the reflection and transmission coefficients for waves parallel and perpendicular to the plane of incidence.
- Transfer Matrix: a more general approach to calculate the reflection and transmission of electromagnetic waves at an interface.
- Conventional and general transfer Matrix method: coherent and incoherent conditions

Fresnel Equations

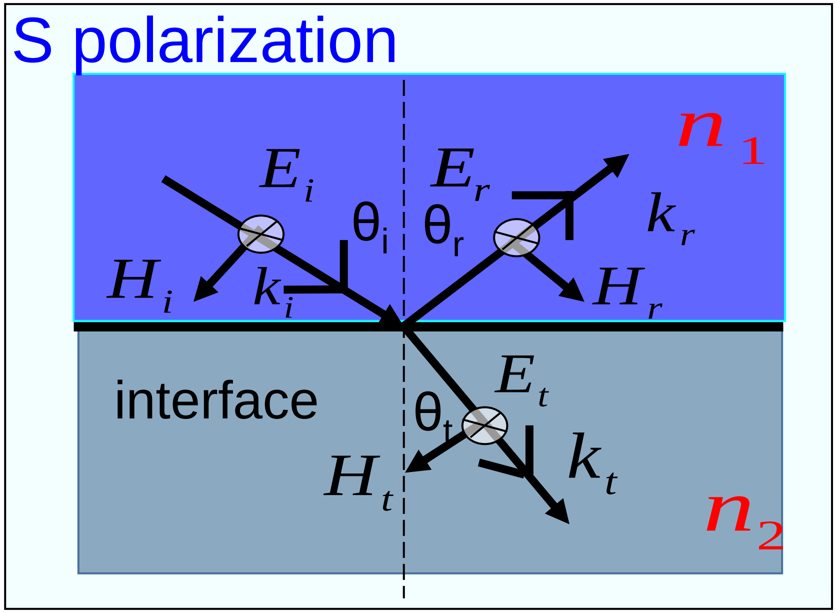
❑Signal interface

--Amplitude reflection coefficient

$$r = \frac{E_r}{E_i}$$

--Amplitude transmission coefficient

$$t = \frac{E_t}{E_i}$$



•For TE waves: S polarization

$$r_s = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t}$$

$$t_s = \frac{2n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t}$$

•For TM waves: P polarization

$$r_p = \frac{n_t \cos\theta_i - n_i \cos\theta_t}{n_t \cos\theta_i + n_i \cos\theta_t}$$

$$t_p = \frac{2n_i \cos\theta_i}{n_t \cos\theta_i + n_t \cos\theta_i}$$

Boundary condition

- Maxwell equations:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = -\frac{\partial \vec{D}}{\partial t} + \vec{J}$$

$$\nabla \cdot \vec{D} = \rho$$

$$\nabla \cdot \vec{B} = 0$$

- Boundary Condition

$$\nabla \times \vec{E} = 0 \qquad E_1^{\parallel} = E_2^{\parallel}$$

$$\nabla \times \vec{H} = \vec{J} \qquad H_1^{\parallel} - H_2^{\parallel} = J_s$$

$$\nabla \cdot \vec{D} = \rho \qquad D_1^{\perp} - D_2^{\perp} = \rho_s$$

$$\nabla \cdot \vec{B} = 0 \qquad B_1^{\perp} = B_2^{\perp}$$

- Boundary Condition at boundary between two general media (finite conductivity):

$$E_1^{\parallel} = E_2^{\parallel}$$

$$H_1^{\parallel} = H_2^{\parallel}$$

Fresnel Equations (S-Polarization)

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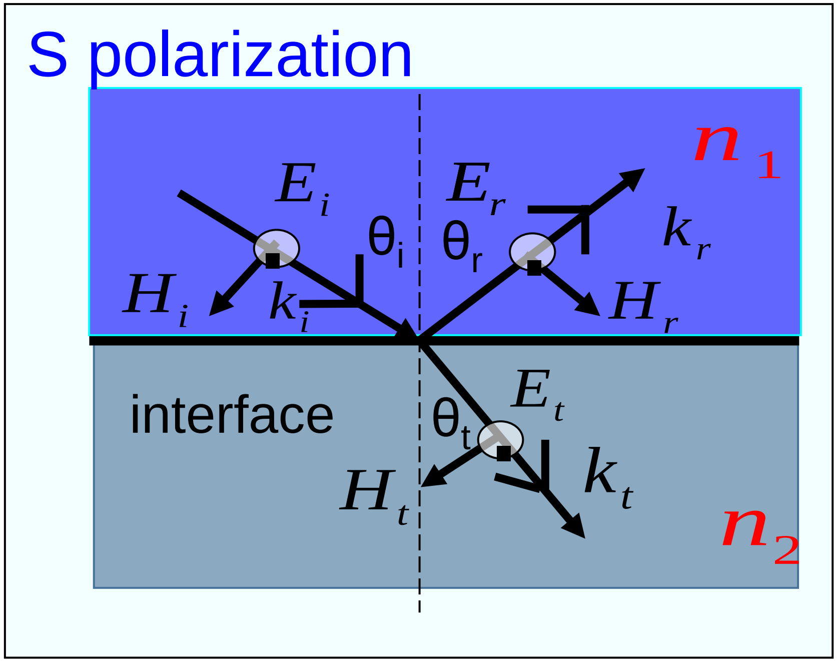
- Maxwell equations:



- Boundary Condition at boundary between two general media (finite conductivity):

$$E_i + E_r = E_t \quad (1)$$

$$-H_i \cos\theta_i + H_r \cos\theta_r = -H_t \cos\theta_t \quad (2)$$

- The amplitude of electric field and magnetic field satisfy

$$H = \sqrt{\frac{\varepsilon}{\mu_o}}E = \sqrt{\frac{\varepsilon_o \varepsilon_r}{\mu_o}}E = \frac{nE}{c}$$

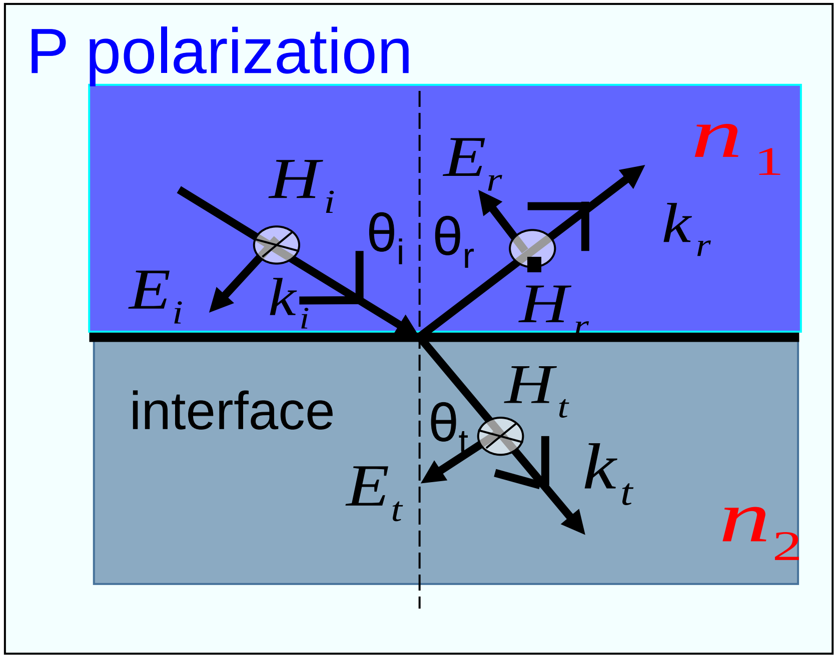
- The Fresnel equation for S- polarization

$$r_s = \frac{E_r}{E_i} = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t}$$

$$t_s = \frac{E_t}{E_i} = \frac{2n_1 \cos\theta_i}{n_1 \cos\theta_i + n_2 \cos\theta_t}$$

Fresnel Equations (P-Polarization)

- Maxwell equations:



- Boundary Condition at boundary between two general media (finite conductivity):

$$E_i \cos\theta_i + E_r \cos\theta_i = E_t \cos\theta_t \qquad (1)$$

$$H_i - H_r = H_t \qquad (2)$$

- The amplitude of electric field and magnetic field satisfy

$$H = \sqrt{\frac{\varepsilon}{\mu_o}} E = \sqrt{\frac{\varepsilon_o \varepsilon_r}{\mu_o}} E = \frac{nE}{c}$$

- The Fresnel equation for P- polarization

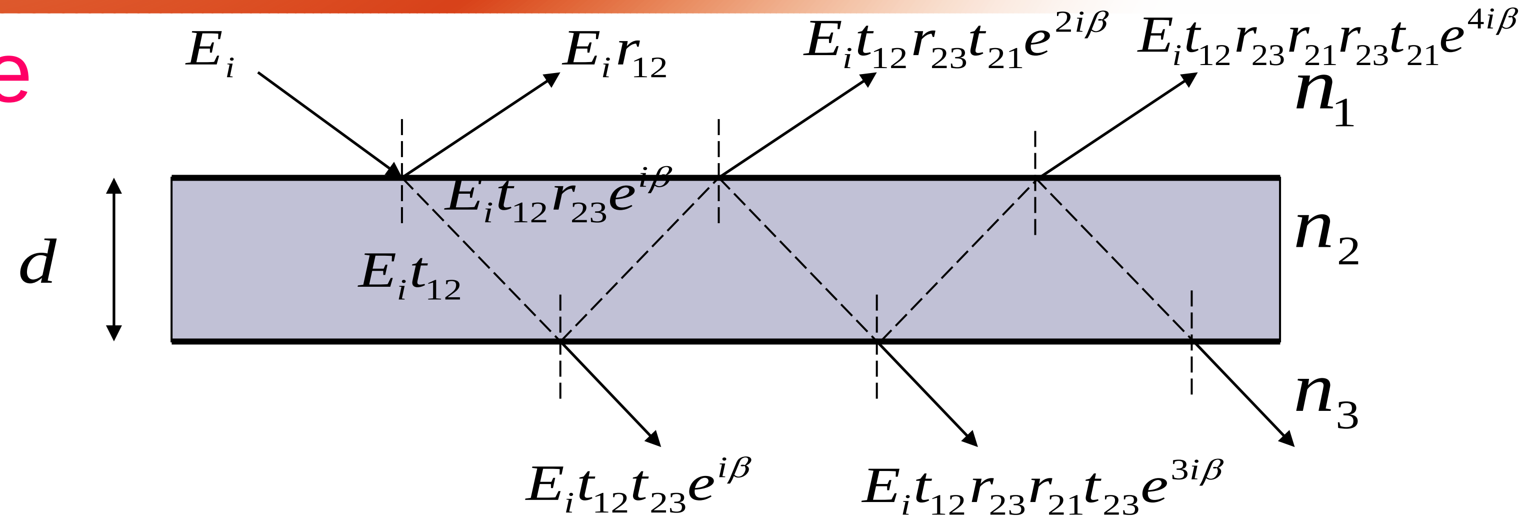
$$r_p = \frac{E_r}{E_i} = \frac{n_1 \cos\theta_t - n_2 \cos\theta_i}{n_1 \cos\theta_t + n_2 \cos\theta_i} \qquad t_p = \frac{E_t}{E_i} = \frac{2n_1 \cos\theta_i}{n_1 \cos\theta_t + n_2 \cos\theta_i}$$

- The relations of Fresnel equations

$$r_{12} = -r_{21} \qquad t_{12}t_{21} - r_{12}r_{21} = 1$$

Transmittance of a three-layerd Structure

❑Two interface



- The total amplitude reflection coefficient

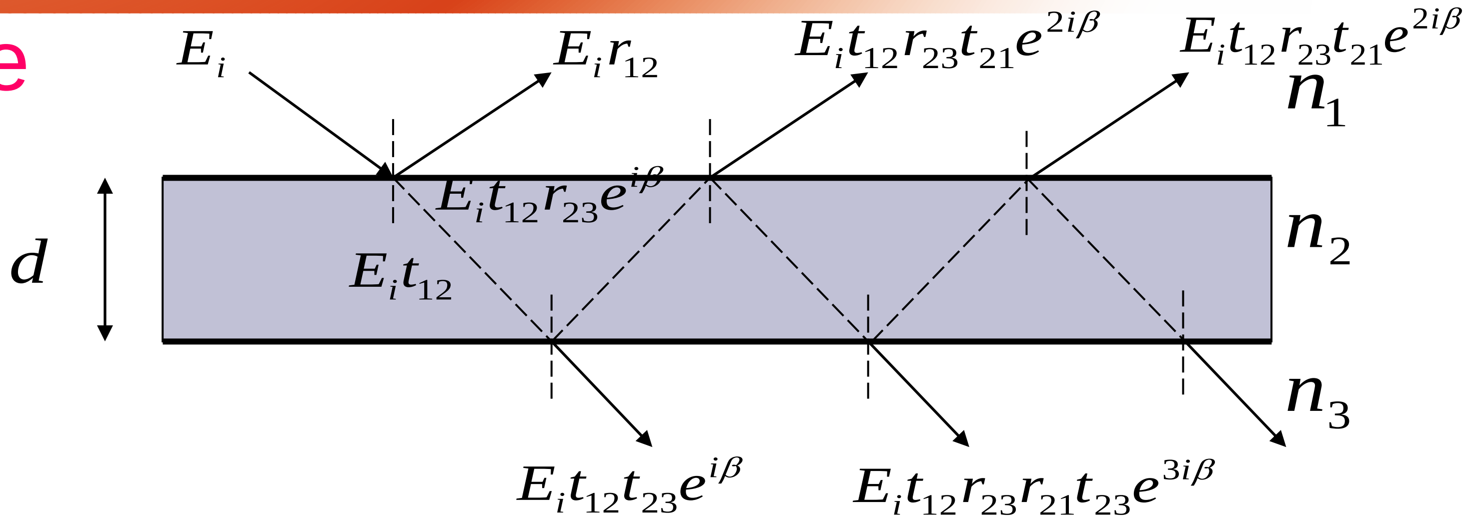
$$r = \frac{E_r}{E_i} = r_{12} + t_{12} r_{23} t_{21} e^{i2\beta} + t_{12} r_{23} t_{21} (r_{23} r_{21}) e^{i4\beta} + ...) = \frac{r_{12} + r_{23} e^{i2\beta}}{1 - r_{21} r_{23} e^{i2\beta}}$$

$$= \frac{r_{12} + r_{23} e^{i2\beta}}{1 + r_{12} r_{23} e^{i2\beta}} \quad \text{where } \beta = (2\pi / \lambda) n_2 d_2$$

- The reflection R

$$R = |r|^2$$

❑Two interface



- The total amplitude transmission coefficient

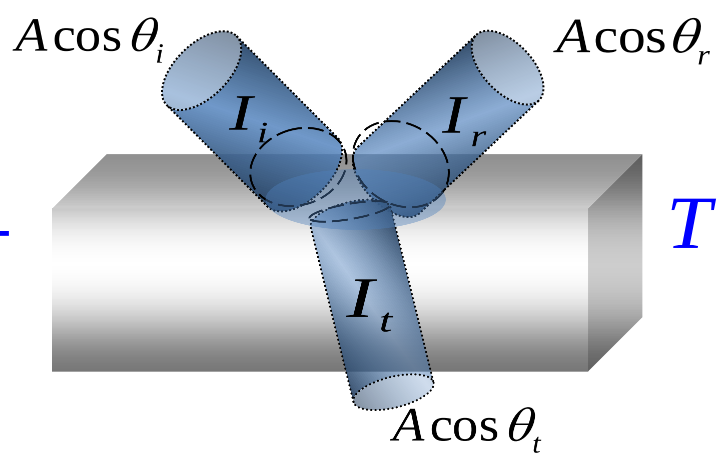
$$t = \frac{E_t}{E_i} = t_{12} t_{23} e^{i\beta}\left(1 + r_{23} r_{21} e^{i2\beta} + (r_{23} r_{21})^2 e^{i4\beta} + \ldots\right) = \frac{t_{12} t_{23} e^{i\beta}}{1 + r_{12} r_{23} e^{i2\beta}}$$

- Intensity

$$I = 2nc\varepsilon_o |E|^2$$

- The transmittance T

$$T = \frac{n_3 \cos\theta_3}{n_1 \cos\theta_1} |t|^2$$



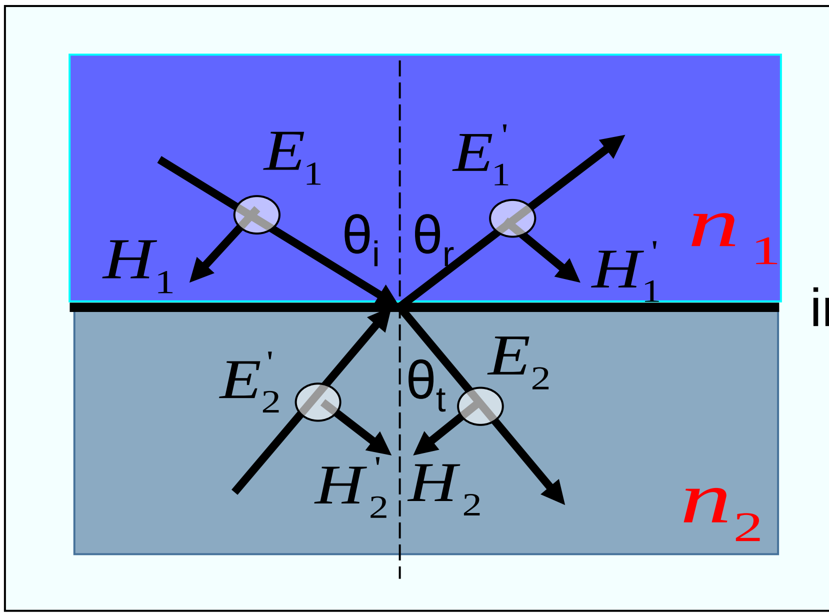
$$T \propto |t|^2$$

- $|t|^2$ is just amplitude transmittance of refractive wave

Transfer Matrix

- Imposing the continuity of tangential component of electric field and magnetic field

S polarization



interface

$$E_1 + E_1' = E_2 + E_2'$$

$$H_1 \cos\theta_1 - H_1' \cos\theta_1 = H_2 \cos\theta_2 - H_2' \cos\theta_2$$

$$H = \frac{nE}{c}$$

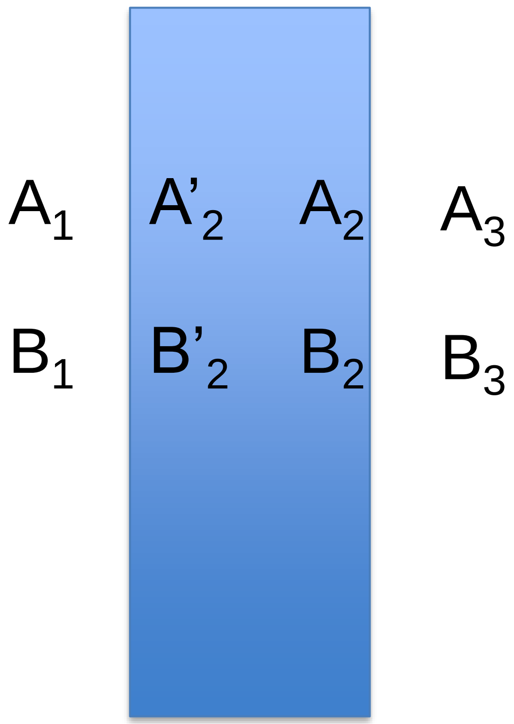
$$D_1 \begin{pmatrix} E_1 \\ E_1' \end{pmatrix} = D_2 \begin{pmatrix} E_2 \\ E_2' \end{pmatrix}$$

$$D_1 = \begin{pmatrix} 1 & 1 \\ n_i \cos\theta_i & -n_i \cos\theta_i \end{pmatrix} \quad i = 1, 2$$

Transfer Matrix

- The waves on the two interfaces of the layer are related to each other

$$E(x) = \mathrm{R}e^{-ik_x x} + L\,e^{+ik_x x} = A(x) + B(x)$$



$$D_1 \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = D_2 \begin{pmatrix} A_2^{'} \\ B_2^{'} \end{pmatrix}$$

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = D_1^{-1} D_2 \begin{pmatrix} A_2^{'} \\ B_2^{'} \end{pmatrix} = D_{12} \begin{pmatrix} A_2^{'} \\ B_2^{'} \end{pmatrix}$$

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = D_{12} P_2 D_{23} \begin{pmatrix} A_3 \\ B_3 \end{pmatrix}$$

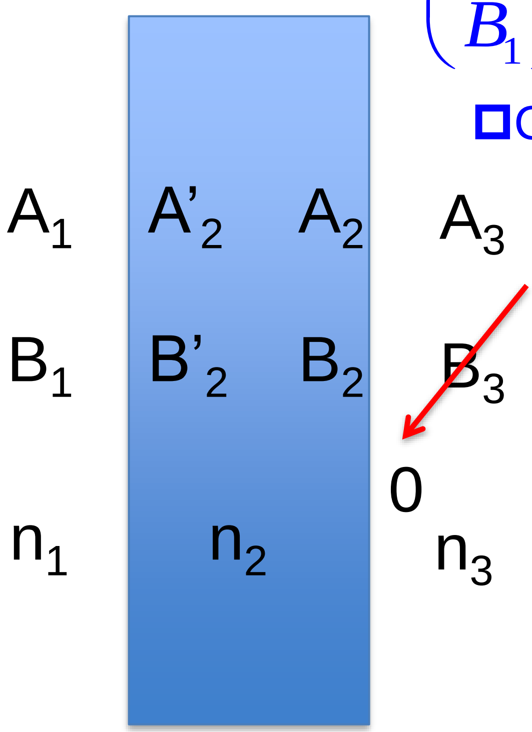
- Propagation Matrix

$$P_2 = \begin{pmatrix} \exp(ik_{2x}d) & 0 \\ 0 & \exp(-ik_{2x}d) \end{pmatrix}$$ $K_{2x}=n_2(\omega/c)\mathrm{Cos}\theta_2$

$$D_{12} = \frac{1}{t_{12}} \begin{pmatrix} 1 & r_{12} \\ r_{12} & 1 \end{pmatrix} \qquad D_{23} = \frac{1}{t_{23}} \begin{pmatrix} 1 & r_{23} \\ r_{23} & 1 \end{pmatrix}$$

Transfer Matrix

- The amplitudes A_1, B_1, and A_3, B_3 are related by the following equation



$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = D_{12} P_2 D_{23} \begin{pmatrix} A_3 \\ B_3 \end{pmatrix} = M \begin{pmatrix} A_3 \\ B_3 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} A_3 \\ B_3 \end{pmatrix}$$

- Consider n_3 layer is semi-infinite, there is no reflection in n_3 layer

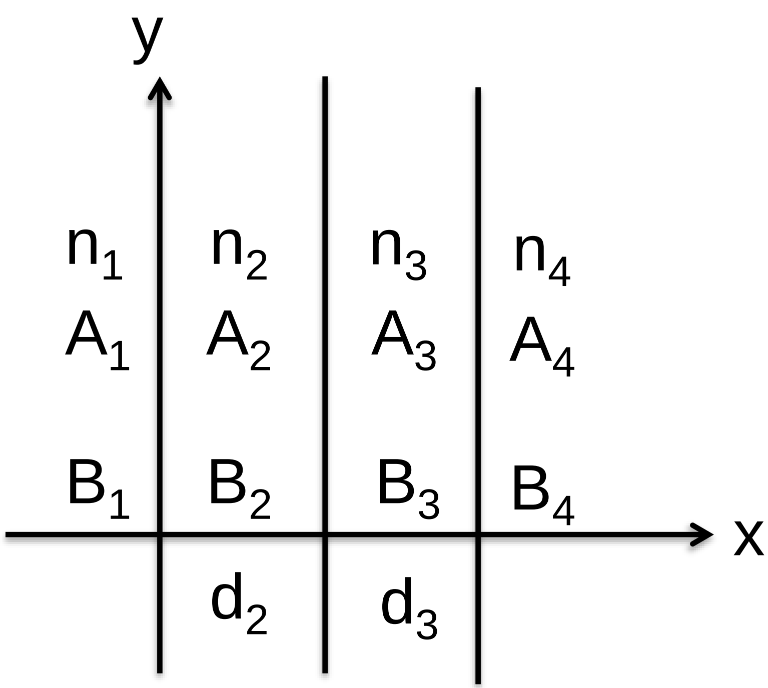
$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} A_3 \\ 0 \end{pmatrix}$$

$$R = |r|^2 ; r = \left(\frac{B_1}{A_1} \right) = \frac{M_{21}}{M_{11}}$$

$$T = \frac{n_3 \cos \theta_3}{n_1 \cos \theta_1} |t|^2 ; t = \left(\frac{A_3}{A_1} \right) = \frac{1}{M_{11}}$$

Transfer Matrix

Extension to 4 layers



$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = D_1^{-1} D_2 P_2 D_2^{-1} D_3 P_3 D_3^{-1} D_4 \begin{pmatrix} A_4 \\ B_4 \end{pmatrix}$$

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = D_{12} P_2 D_{23} P_3 D_{34} \begin{pmatrix} A_4 \\ B_4 \end{pmatrix}$$

$$D_{n,n+1} = \frac{1}{t_{n,n+1}} \begin{pmatrix} 1 & r_{n,n+1} \\ r_{n,n+1} & 1 \end{pmatrix}; n = 1, 2, 3$$

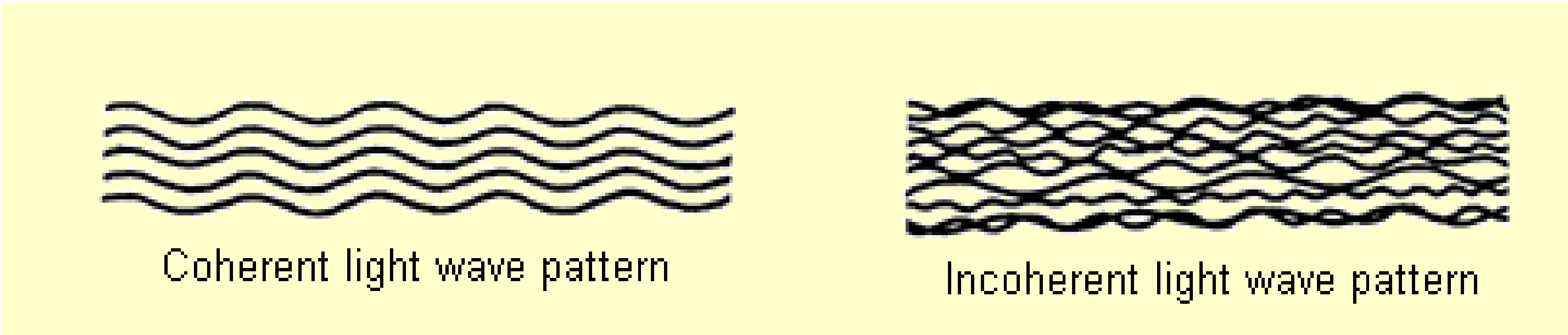
$$P_n = \begin{pmatrix} \exp(ik_{nx} d) & 0 \\ 0 & \exp(-ik_{nx} d) \end{pmatrix}; n = 1, 2$$

$K_{nx} = n_n(\omega/c)\mathrm{Cos}\theta_n = 2\pi n_n \mathrm{Cos}\theta_n/\lambda$

Discussion

- The discussions above are limited to coherent condition.
- If the thin film thickness is larger than coherence length of incident light, the incoherent effect is required to be included.
  - —. Summary
- Coherent interference : thin films
- Incoherent interference : thick films
- Partial interference : thin films with rough surfaces

What is Coherence ?



Fabry-Perot oscillations:

- The thin film thickness close to wavelength range
- Good resolution of spectroscopic measurement
- The coherent length of the light source larger than sample thickness

Conventional Transfer Matrix

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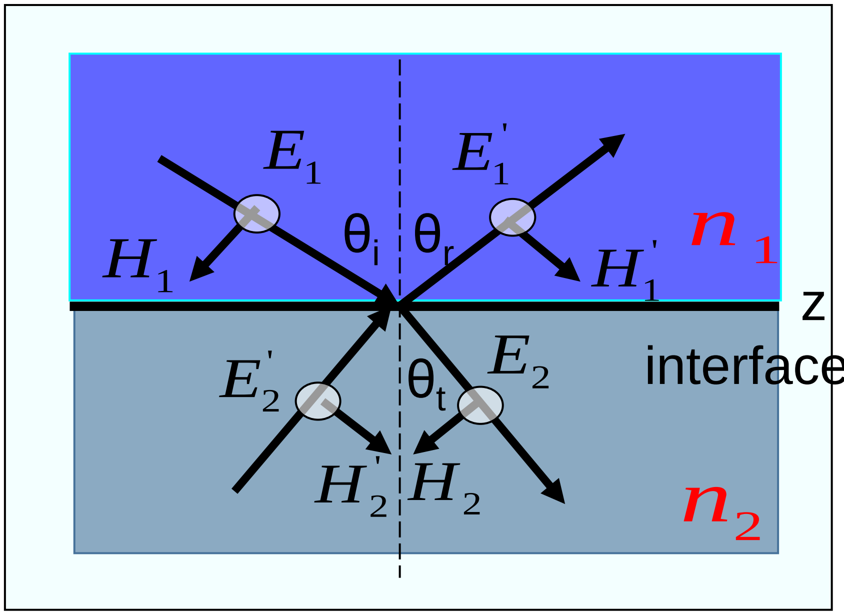
- Imposing the continuity of tangential component of electric field and magnetic field at the perfect interface

S polarization



$$E_1 + E_1' = E_2 + E_2'$$

$$H_1 \cos\theta_1 - H_1' \cos\theta_1 = H_2 \cos\theta_2 - H_2' \cos\theta_2$$

$$H = \frac{nE}{c}$$

$$D_1 \begin{pmatrix} E_1 \\ E_1' \end{pmatrix} = D_2 \begin{pmatrix} E_2 \\ E_2' \end{pmatrix} \qquad D_i = \begin{pmatrix} 1 & 1 \\ n_i \cos\theta_i & -n_i \cos\theta_i \end{pmatrix}$$

$$i = 1, 2$$

$$D_{12} = D_2^{-1} D_1 = \begin{pmatrix} \dfrac{n_2 \cos\theta_2 + n_1 \cos\theta_1}{2n_2 \cos\theta_2} & \dfrac{n_2 \cos\theta_2 - n_1 \cos\theta_1}{2n_2 \cos\theta_2} \\ \dfrac{n_2 \cos\theta_2 - n_1 \cos\theta_1}{2n_2 \cos\theta_2} & \dfrac{n_2 \cos\theta_2 + n_1 \cos\theta_1}{2n_2 \cos\theta_2} \end{pmatrix}$$

Conventional Transfer Matrix

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- Imposing the continuity of tangential component of wavenumber at the interface

$$k \equiv \frac{2\pi}{\lambda} n$$

$$k_z = \frac{2\pi}{\lambda} n \,\mathrm{Sin}\,\theta$$

$$k_x = \sqrt{k^2 - (\frac{2\pi}{\lambda} n \,\mathrm{Sin}\,\theta)^2}$$

$$D_{12} = D_2^{-1} D_1 = \begin{pmatrix} \frac{n_2 \cos\theta_2 + n_1 \cos\theta_1}{2n_2 \cos\theta_2} & \frac{n_2 \cos\theta_2 - n_1 \cos\theta_1}{2n_2 \cos\theta_2} \\ \frac{n_2 \cos\theta_2 - n_1 \cos\theta_1}{2n_2 \cos\theta_2} & \frac{n_2 \cos\theta_2 + n_1 \cos\theta_1}{2n_2 \cos\theta_2} \end{pmatrix}$$

$$D_{12} = D_2^{-1} D_1 = \begin{pmatrix} \frac{k_{2x} + k_{1x}}{2k_{1x}} & \frac{k_{2x} - k_{1x}}{2k_{1x}} \\ \frac{k_{2x} - k_{1x}}{2k_{1x}} & \frac{k_{2x} - k_{1x}}{2k_{1x}} \end{pmatrix} = \frac{1}{t_{12}} \begin{pmatrix} 1 & r_{12} \\ r_{12} & 1 \end{pmatrix}$$

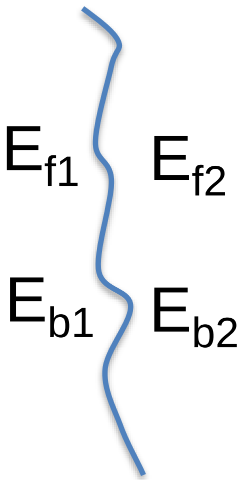
For s polarization

$$D_{12} = D_2^{-1} D_1 = \begin{pmatrix} \frac{n_1^2 k_{2x} + n_2^2 k_{1x}}{2n_1^2 k_{2x}} & \frac{n_1^2 k_{2x} - n_2^2 k_{1x}}{n_1^2 2k_{2x}} \\ \frac{n_1^2 k_{2x} - n_2^2 k_{1x}}{2k_{2x}} & \frac{n_1^2 k_{2x} - n_2^2 k_{1x}}{2k_{2x}} \end{pmatrix} = \frac{1}{t_{12}} \begin{pmatrix} 1 & r_{12} \\ r_{12} & 1 \end{pmatrix}$$

For p polarization

General Transfer Matrix

- A slight extension to have waves coming from both sides of the rough interfaces at once



$$E(x) = E_f\, e^{-ik_z z} + E_b\, e^{+ik_z z}$$

$$E_{b1} = E_{f1} r_{12} + E_{b2} t_{21}$$

$$E_{f2} = E_{f1} t_{12} + E_{b2} r_{21}$$

$$E_{f1} = \frac{E_{f2}}{t_{12}} - \frac{E_{b2} r_{21}}{t_{12}}$$

$$E_{b1} = \frac{E_{f2} - E_{b2} r_{21}}{t_{12}} r_{12} + E_{b2} t_{21} = \frac{r_{12}}{t_{12}} E_{f2} + \left(-\frac{r_{21} r_{12}}{t_{12}} + t_{21}\right) E_{b2}$$

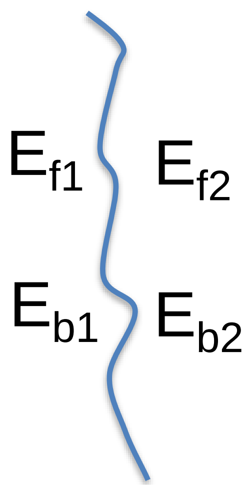
$$\begin{pmatrix} E_{f1} \\ E_{b1} \end{pmatrix} = \frac{1}{t_{12}} \begin{pmatrix} 1 & -r_{21} \\ r_{12} & t_{12} t_{21} - r_{21} r_{12} \end{pmatrix} \begin{pmatrix} E_{f2} \\ E_{b2} \end{pmatrix}$$

General Transfer Matrix

- A slight extension to have waves coming from both sides of the rough interfaces at once

$$\begin{pmatrix} E_{f1} \\ E_{b1} \end{pmatrix} = \frac{1}{t_{12}} \begin{pmatrix} 1 & -r_{21} \\ r_{12} & t_{12}t_{21} - r_{21}r_{12} \end{pmatrix} \begin{pmatrix} E_{f2} \\ E_{b2} \end{pmatrix}$$

Incoherent interface



Conditions obtained from boundary conditions at the perfect interface

$$r_{12} = -r_{21}$$

$$t_{12}t_{21} - r_{12}r_{21} = 1$$

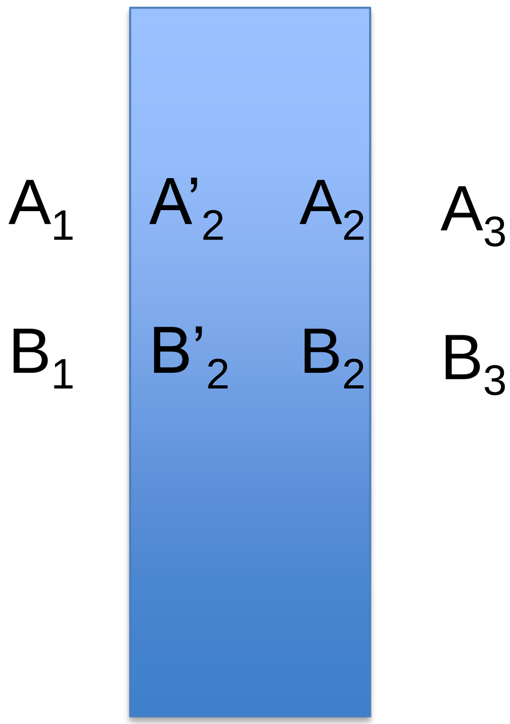
$$\begin{pmatrix} E_{f1} \\ E_{b1} \end{pmatrix} = \frac{1}{t_{12}} \begin{pmatrix} 1 & r_{12} \\ r_{12} & 1 \end{pmatrix} \begin{pmatrix} E_{f2} \\ E_{b2} \end{pmatrix}$$

coherent interface

Propagation Matrix

- The waves on the two interfaces of the layer are related to each other

$$E(x) = \mathrm{R}e^{-ik_x x} + L\,e^{+ik_x x} = A(x) + B(x)$$



$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = D_{12} P_2 D_{23} \begin{pmatrix} A_3 \\ B_3 \end{pmatrix}$$

- Propagation Matrix for coherent propagation

$$P_2 = \begin{pmatrix} \exp(ik_{2x}d) & 0 \\ 0 & \exp(-ik_{2x}d) \end{pmatrix}$$

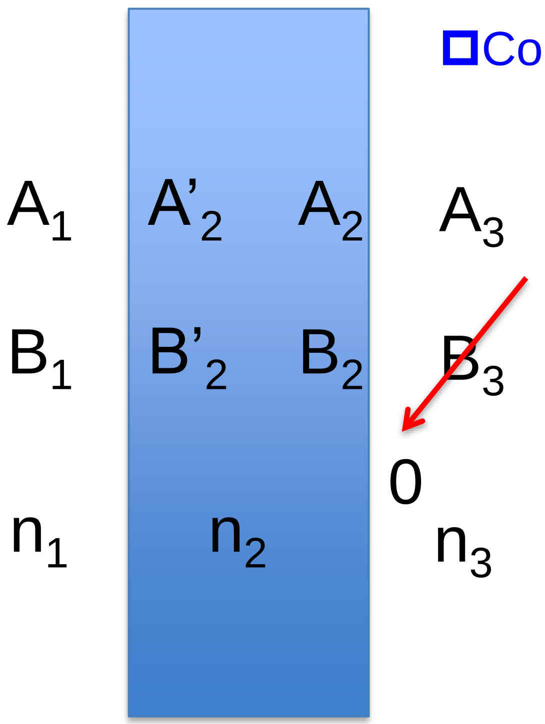
- Propagation Matrix for incoherent propagation

$$P_2 = \begin{pmatrix} 1/\exp(-\alpha_2 d) & 0 \\ 0 & \exp(-\alpha_2 d) \end{pmatrix} \qquad \alpha_2 = \mathrm{Im}[k_{2x}]$$

General Transfer Matrix

- The amplitudes A_1, B_1, and A_3, B_3 are related by the following equation

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = D_{12} P_2 D_{23} \begin{pmatrix} A_3 \\ B_3 \end{pmatrix} = M \begin{pmatrix} A_3 \\ B_3 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} A_3 \\ B_3 \end{pmatrix}$$



- Consider n_3 layer is semi-infinite, there is no reflection in n_3 layer

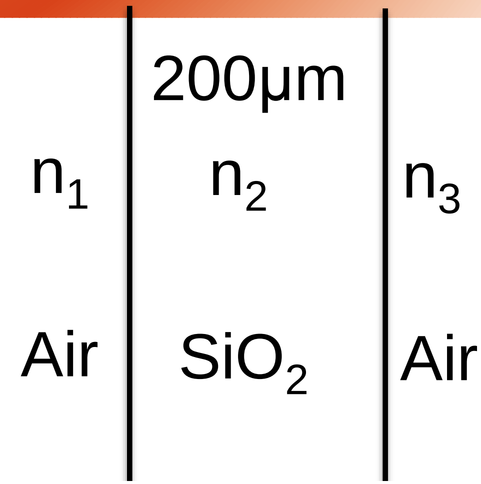
$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} A_3 \\ 0 \end{pmatrix}$$

$$R = |r|^2 ; r = \left(\frac{B_1}{A_1} \right) = \frac{M_{21}}{M_{11}}$$

$$T = \frac{n_3 \cos \theta_3}{n_1 \cos \theta_1} |t|^2 ; t = \left(\frac{A_3}{A_1} \right) = \frac{1}{M_{11}}$$

Example: Thick SiO_2 Layer

Example: Thick SiO_2 Layer has around 0.67 reflectance



Calculation results: