MVA HW#1

1. Reading assignments: Chapters 1 to 3 of the textbook, including Supplement 2A

2. Exercise 1.4 of the textbook, p. 38.
   Note: no need to plot the marginal dot diagram.
   Perform (b) by using both calculator as if it is given in an exam and computer as if you are doing a project with many numbers; no need to submit the code.

1.4. The world's 10 largest companies yield the following data:

The World's 10 Largest Companies[1]

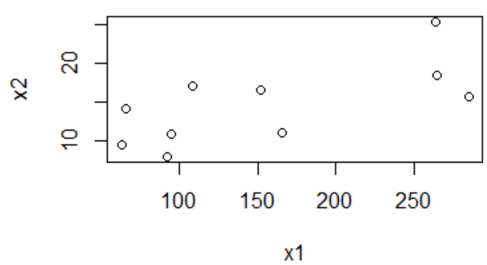
Company	x_1 = sales (billions)	x_2 = profits (billions)	x_3 = assets (billions)
Citigroup	108.28	17.05	1,484.10
General Electric	152.36	16.59	750.33
American Intl Group	95.04	10.91	766.42
Bank of America	65.45	14.14	1,110.46
HSBC Group	62.97	9.52	1,031.29
ExxonMobil	263.99	25.33	195.26
Royal Dutch/Shell	265.19	18.54	193.83
BP	285.06	15.73	191.11
ING Group	92.01	8.10	1,175.16
Toyota Motor	165.68	11.13	211.15

[1]From www.Forbes.com partially based on *Forbes* The Forbes Global 2000, April 18, 2005.

(a) Plot the scatter diagram and marginal dot diagrams for variables x_1 and x_2. Comment on the appearance of the diagrams.

(b) Compute $\bar{x}_1$, $\bar{x}_2$, s_{11}, s_{22}, s_{12}, and r_{12}. Interpret r_{12}.

```
> x1 = c(108.28,152.36,95.04,65.45,62.97,263.99,265.19,285.06,92.01,165.68)
> x2 = c(17.05,16.59,10.91,14.14,9.52,25.33,18.54,15.73,8.10,11.13)
> x3 = c(1484.10,750.33,766.42,1110.46,1031.29,195.26,193.83,191.11,1175.16,2
11.15)
> plot(x1,x2)
```




```
> mean(x1)
[1] 155.603
> mean(x2)
[1] 14.704
```

```
> var(x1)
[1] 7476.453
> var(x2)
[1] 26.19032
```

```
> cov(x1,x2)
[1] 303.6186
> cor(x1,x2)
[1] 0.686136
```

3. 2.6–2.9, p. 104

2.6. Let

$$\mathbf{A} = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$$

(a) Is $\mathbf{A}$ symmetric?

(b) Show that $\mathbf{A}$ is positive definite.

---

a) Yes

b) Show eigen values > 0

```
> A = matrix(c(9,-2,-2,6),2,2)
> eigen(A)
eigen() decomposition
$values
[1] 10  5
```

```
$vectors
           [,1]       [,2]
[1,] -0.8944272 -0.4472136
[2,]  0.4472136 -0.8944272

> ei$values>0
[1] TRUE TRUE
```

2.7. Let $\mathbf{A}$ be as given in Exercise 2.6.

(a) Determine the eigenvalues and eigenvectors of $\mathbf{A}$.

(b) Write the spectral decomposition of $\mathbf{A}$.

(c) Find $\mathbf{A}^{-1}$.

(d) Find the eigenvalues and eigenvectors of $\mathbf{A}^{-1}$.

```
> ei$vectors %*% diag(ei$values) %*% t(ei$vectors)
     [,1] [,2]
[1,]    9   -2
[2,]   -2    6

> eigen(ei$vectors %*% diag(ei$values) %*% t(ei$vectors))
eigen() decomposition
$values
[1] 10  5

$vectors
           [,1]       [,2]
[1,] -0.8944272 -0.4472136
[2,]  0.4472136 -0.8944272
```

2.8. Given the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

find the eigenvalues λ_1 and λ_2 and the associated normalized eigenvectors $\mathbf{e}_1$ and $\mathbf{e}_2$. Determine the spectral decomposition (2-16) of $\mathbf{A}$.

```
> A = matrix(c(1,2,2,-2),2,2)
> eigen(A)
eigen() decomposition
$values
[1]  2 -3

$vectors
           [,1]       [,2]
[1,] -0.8944272 -0.4472136
[2,] -0.4472136  0.8944272
```

2.9. Let $\mathbf{A}$ be as in Exercise 2.8.

(a) Find $\mathbf{A}^{-1}$.

(b) Compute the eigenvalues and eigenvectors of $\mathbf{A}^{-1}$.

(c) Write the spectral decomposition of $\mathbf{A}^{-1}$, and compare it with that of $\mathbf{A}$ from Exercise 2.8.

```
> solve(A)
          [,1]       [,2]
[1,] 0.3333333  0.3333333
[2,] 0.3333333 -0.1666667

> eigen(solve(A))
eigen() decomposition
$values
[1]  0.5000000 -0.3333333

$vectors
           [,1]       [,2]
[1,] -0.8944272  0.4472136
[2,] -0.4472136 -0.8944272
```

Same eigenvectors but the eigen values are the inverse of the previous

4. 2.32, p. 108

2.32. You are given the random vector $\mathbf{X}' = [X_1, X_2, \ldots, X_5]$ with mean vector $\boldsymbol{\mu}'_{\mathbf{X}} = [2, 4, -1, 3, 0]$ and variance–covariance matrix

$$\boldsymbol{\Sigma}_{\mathbf{X}} = \begin{bmatrix} 4 & -1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ -1 & 3 & 1 & -1 & 0 \\ \frac{1}{2} & 1 & 6 & 1 & -1 \\ -\frac{1}{2} & -1 & 1 & 4 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}$$

Partition $\mathbf{X}$ as

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \hdashline X_3 \\ X_4 \\ X_5 \end{bmatrix} = \begin{bmatrix} \mathbf{X}^{(1)} \\ \hdashline \mathbf{X}^{(2)} \end{bmatrix}$$

Let

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

and consider the linear combinations $\mathbf{A}\mathbf{X}^{(1)}$ and $\mathbf{B}\mathbf{X}^{(2)}$. Find

(a) $E(\mathbf{X}^{(1)})$
(b) $E(\mathbf{A}\mathbf{X}^{(1)})$
(c) $\text{Cov}(\mathbf{X}^{(1)})$
(d) $\text{Cov}(\mathbf{A}\mathbf{X}^{(1)})$
(e) $E(\mathbf{X}^{(2)})$
(f) $E(\mathbf{B}\mathbf{X}^{(2)})$
(g) $\text{Cov}(\mathbf{X}^{(2)})$
(h) $\text{Cov}(\mathbf{B}\mathbf{X}^{(2)})$
(i) $\text{Cov}(\mathbf{X}^{(1)}, \mathbf{X}^{(2)})$
(j) $\text{Cov}(\mathbf{A}\mathbf{X}^{(1)}, \mathbf{B}\mathbf{X}^{(2)})$

5. 3.12, p. 146

3.12. Show that $|\mathbf{S}| = (s_{11}s_{22}\cdots s_{pp})|\mathbf{R}|$.

Hint: From Equation (3-30), $\mathbf{S} = \mathbf{D}^{1/2}\mathbf{R}\mathbf{D}^{1/2}$. Taking determinants gives $|\mathbf{S}| = |\mathbf{D}^{1/2}||\mathbf{R}||\mathbf{D}^{1/2}|$. (See Result 2A.11.) Now examine $|\mathbf{D}^{1/2}|$.

6. 3.16, p. 147

3.16. Let $\mathbf{V}$ be a vector random variable with mean vector $E(\mathbf{V}) = \boldsymbol{\mu}_{\mathbf{V}}$ and covariance matrix $E(\mathbf{V} - \boldsymbol{\mu}_{\mathbf{V}})(\mathbf{V} - \boldsymbol{\mu}_{\mathbf{V}})' = \boldsymbol{\Sigma}_{\mathbf{V}}$. Show that $E(\mathbf{V}\mathbf{V}') = \boldsymbol{\Sigma}_{\mathbf{V}} + \boldsymbol{\mu}_{\mathbf{V}}\boldsymbol{\mu}'_{\mathbf{V}}$.

7. Find the maximum and minimum of $\frac{\mathbf{x}'\mathbf{A}\mathbf{x}}{\mathbf{x}'\mathbf{x}}$, where $\mathbf{x} \neq \mathbf{0}$ is a trivariate real-valued vector and

$$\mathbf{A} = \begin{bmatrix} 1 & .2 & .3 \\ .2 & 1 & .2 \\ .3 & .2 & 1 \end{bmatrix}$$

```
> A = matrix(c(1,0.2,0.3,0.2,1,0.2,0.3,0.2,1),3)
> A
     [,1] [,2] [,3]
[1,]  1.0  0.2  0.3
[2,]  0.2  1.0  0.2
[3,]  0.3  0.2  1.0
> eigen(A)
eigen() decomposition
$values
[1] 1.4701562 0.8298438 0.7000000

$vectors
          [,1]       [,2]          [,3]
[1,] 0.6059128  0.3645129  7.071068e-01
[2,] 0.5154991 -0.8568901  5.551115e-16
[3,] 0.6059128  0.3645129 -7.071068e-01
```

8. Given a random vector $\mathbf{X} = (X_1\ X_2\ X_3)'$ with mean $\boldsymbol{\mu} = (3, 2, 1)'$ and covariance matrix $\boldsymbol{\Sigma}$. The eigenvalues of $\boldsymbol{\Sigma}$ are $\lambda_1 = 12, \lambda_2 = 6, \lambda_3 = 2$ and the corresponding eigenvectors are

$$\mathbf{e}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 2/\sqrt{6} \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

Find the following quantities

(a) $\det(\boldsymbol{\Sigma}) = |\boldsymbol{\Sigma}|$

(b) $\mathrm{tr}(\boldsymbol{\Sigma})$

(c) $\mathrm{var}(\mathbf{e}_1'\mathbf{X})$

(d) $\boldsymbol{\Sigma}$

(e) $\boldsymbol{\Sigma}^{-1}$

(f) $\boldsymbol{\Sigma}^{-1/2}$

(g) Define $Y_1 = \mathbf{e}_1'\mathbf{X}, Y_2 = \mathbf{e}_2'\mathbf{X}, Y_3 = \mathbf{e}_3'\mathbf{X}$ and $\mathbf{Y} = (Y_1, Y_2, Y_3)'$. Compute $E(\mathbf{Y}), \mathrm{Cov}(\mathbf{Y})$

(h) Prove $\mathbf{Y}'\mathbf{Y} = \mathbf{X}'\mathbf{X}$

(i) Are the following true?

i. $|\boldsymbol{\Sigma}_X| = |\boldsymbol{\Sigma}_Y|$

ii. $\mathrm{tr}(\boldsymbol{\Sigma}_X) = \mathrm{tr}(\boldsymbol{\Sigma}_Y)$

iii. $\boldsymbol{\Sigma}_X = \boldsymbol{\Sigma}_Y$

iv. $(\mathbf{X} - \boldsymbol{\mu}_X)'\boldsymbol{\Sigma}_X^{-1}(\mathbf{X} - \boldsymbol{\mu}_X) = (\mathbf{Y} - \boldsymbol{\mu}_Y)'\boldsymbol{\Sigma}_Y^{-1}(\mathbf{Y} - \boldsymbol{\mu}_Y)$

v. $(\mathbf{X} - \boldsymbol{\mu}_X)'(\mathbf{X} - \boldsymbol{\mu}_X) = (\mathbf{Y} - \boldsymbol{\mu}_Y)'(\mathbf{Y} - \boldsymbol{\mu}_Y)$

```
> E = matrix(c(1/sqrt(3),1/sqrt(3),
1/sqrt(3),2/sqrt(6),-1/sqrt(6),-1/s
qrt(6),0,1/sqrt(2),-1/sqrt(2)),3)
> E
          [,1]       [,2]       [,
3]
[1,] 0.5773503  0.8164966  0.000000
0
[2,] 0.5773503 -0.4082483  0.707106
8
[3,] 0.5773503 -0.4082483 -0.707106
8
> lamba = diag(c(12,6,2))
> lamba
     [,1] [,2] [,3]
[1,]   12    0    0
[2,]    0    6    0
[3,]    0    0    2
> S = E %*% lamba %*% t(E)
> S
     [,1] [,2] [,3]
[1,]    8    2    2
[2,]    2    6    4
[3,]    2    4    6
> determinant(S,FALSE)
$modulus
[1] 144
attr(,"logarithm")
[1] FALSE

$sign
[1] 1

attr(,"class")
[1] "det"
> solve(S)
            [,1]        [,2]
 [,3]
[1,]  0.13888889 -0.02777778 -0.027
77778
[2,] -0.02777778  0.30555556 -0.194
44444
[3,] -0.02777778 -0.19444444  0.305
55556
> E %*% sqrt(lamba) %*% t(E)
         [,1]     [,2]     [,3]
[1,] 2.787694 0.338204 0.338204
[2,] 0.338204 2.270056 0.855842
[3,] 0.338204 0.855842 2.270056
> t(E) %*% c(3,2,1)
          [,1]
[1,] 3.4641016
[2,] 1.2247449
[3,] 0.7071068
```