MVA HW#4

6.7. Using the summary statistics for the electricity-demand data given in Example 6.4, compute T^2 and test the hypothesis $H_0: \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 = \mathbf{0}$, assuming that $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2$. Set $\alpha = .05$. Also, determine the linear combination of mean components most responsible for the rejection of H_0.

$$\bar{\mathbf{x}}_1 = \begin{bmatrix} 204.4 \\ 556.6 \end{bmatrix}, \quad \mathbf{S}_1 = \begin{bmatrix} 13825.3 & 23823.4 \\ 23823.4 & 73107.4 \end{bmatrix}, \quad n_1 = 45$$

$$\bar{\mathbf{x}}_2 = \begin{bmatrix} 130.0 \\ 355.0 \end{bmatrix}, \quad \mathbf{S}_2 = \begin{bmatrix} 8632.0 & 19616.7 \\ 19616.7 & 55964.5 \end{bmatrix}, \quad n_2 = 55$$

$$\mathbf{S}_{\text{pooled}} = \frac{n_1 - 1}{n_1 + n_2 - 2}\mathbf{S}_1 + \frac{n_2 - 1}{n_1 + n_2 - 2}\mathbf{S}_2 = \begin{bmatrix} 10963.7 & 21505.5 \\ 21505.5 & 63661.3 \end{bmatrix}$$

and

$$c^2 = \frac{(n_1 + n_2 - 2)p}{n_1 + n_2 - p - 1} F_{p, n_1+n_2-p-1}(\alpha) = \frac{98(2)}{97} F_{2,97}(.05)$$

$$= (2.02)(3.1) = 6.26$$

6.8. Observations on two responses are collected for three treatments. The observation vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ are

$$\text{Treatment 1:} \quad \begin{bmatrix} 6 \\ 7 \end{bmatrix}, \begin{bmatrix} 5 \\ 9 \end{bmatrix}, \begin{bmatrix} 8 \\ 6 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 7 \\ 9 \end{bmatrix}$$

$$\text{Treatment 2:} \quad \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\text{Treatment 3:} \quad \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

(a) Break up the observations into mean, treatment, and residual components, as in (6-39). Construct the corresponding arrays for each variable. (See Example 6.9.)

(b) Using the information in Part a, construct the one-way MANOVA table.

(c) Evaluate Wilks' lambda, Λ^*, and use Table 6.3 to test for treatment effects. Set $\alpha = .01$. Repeat the test using the chi-square approximation with Bartlett's correction. [See (6-43).] Compare the conclusions.

6.21. Using Moody's bond ratings, samples of 20 Aa (middle-high quality) corporate bonds and 20 Baa (top-medium quality) corporate bonds were selected. For each of the corresponding companies, the ratios

X_1 = current ratio (a measure of short-term liquidity)
X_2 = long-term interest rate (a measure of interest coverage)
X_3 = debt-to-equity ratio (a measure of financial risk or leverage)
X_4 = rate of return on equity (a measure of profitability)

were recorded. The summary statistics are as follows:

Aa bond companies: $n_1 = 20$, $\bar{\mathbf{x}}_1' = [2.287, 12.600, .347, 14.830]$, and

$$\mathbf{S}_1 = \begin{bmatrix} .459 & .254 & -.026 & -.244 \\ .254 & 27.465 & -.589 & -.267 \\ -.026 & -.589 & .030 & .102 \\ -.244 & -.267 & .102 & 6.854 \end{bmatrix}$$

Baa bond companies: $n_2 = 20$, $\bar{\mathbf{x}}_2' = [2.404, 7.155, .524, 12.840]$,

$$\mathbf{S}_2 = \begin{bmatrix} .944 & -.089 & .002 & -.719 \\ -.089 & 16.432 & -.400 & 19.044 \\ .002 & -.400 & .024 & -.094 \\ -.719 & 19.044 & -.094 & 61.854 \end{bmatrix}$$

and

$$\mathbf{S}_{\text{pooled}} = \begin{bmatrix} .701 & .083 & -.012 & -.481 \\ .083 & 21.949 & -.494 & 9.388 \\ -.012 & -.494 & .027 & .004 \\ -.481 & 9.388 & .004 & 34.354 \end{bmatrix}$$

(a) Does pooling appear reasonable here? Comment on the pooling procedure in this case.

(b) Are the financial characteristics of firms with Aa bonds different from those with Baa bonds? Using the pooled covariance matrix, test for the equality of mean vectors. Set $\alpha = .05$.

(c) Calculate the linear combinations of mean components most responsible for rejecting $H_0: \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2 = \mathbf{0}$ in Part b.

(d) Bond rating companies are interested in a company's ability to satisfy its outstanding debt obligations as they mature. Does it appear as if one or more of the foregoing financial ratios might be useful in helping to classify a bond as "high" or "medium" quality? Explain.

(e) Repeat part (b) assuming normal populations with unequal covariance matices (see (6-27), (6-28) and (6-29)). Does your conclusion change?

6.24. Researchers have suggested that a change in skull size over time is evidence of the interbreeding of a resident population with immigrant populations. Four measurements were made of male Egyptian skulls for three different time periods: period 1 is 4000 B.C., period 2 is 3300 B.C., and period 3 is 1850 B.C. The data are shown in Table 6.13 on page 349 (see the skull data on the website www.prenhall.com/statistics). The measured variables are

$$X_1 = \text{maximum breadth of skull (mm)}$$
$$X_2 = \text{basibregmatic height of skull (mm)}$$
$$X_3 = \text{basialveolar length of skull (mm)}$$
$$X_4 = \text{nasal height of skull (mm)}$$

Construct a one-way MANOVA of the Egyptian skull data. Use $\alpha = .05$. Construct 95% simultaneous confidence intervals to determine which mean components differ among the populations represented by the three time periods. Are the usual MANOVA assumptions realistic for these data? Explain.

Table 6.13 Egyptian Skull Data

MaxBreath (x_1)	BasHeight (x_2)	BasLength (x_3)	NasHeight (x_4)	Time Period
131	138	89	49	1
125	131	92	48	1
131	132	99	50	1
119	132	96	44	1
136	143	100	54	1
138	137	89	56	1
139	130	108	48	1
125	136	93	48	1
131	134	102	51	1
134	134	99	51	1
⋮	⋮	⋮	⋮	⋮
124	138	101	48	2
133	134	97	48	2
138	134	98	45	2
148	129	104	51	2
126	124	95	45	2
135	136	98	52	2
132	145	100	54	2
133	130	102	48	2
131	134	96	50	2
133	125	94	46	2
⋮	⋮	⋮	⋮	⋮
132	130	91	52	3
133	131	100	50	3
138	137	94	51	3
130	127	99	45	3
136	133	91	49	3
134	123	95	52	3
136	137	101	54	3
133	131	96	49	3
138	133	100	55	3
138	133	91	46	3

Source: Data courtesy of J. Jackson.

7.3. (*Weighted least squares estimators.*) Let

$$\underset{(n\times 1)}{\mathbf{Y}} = \underset{(n\times(r+1))}{\mathbf{Z}}\ \underset{((r+1)\times 1)}{\boldsymbol{\beta}} + \underset{(n\times 1)}{\boldsymbol{\varepsilon}}$$

where $E(\boldsymbol{\varepsilon}) = \mathbf{0}$ but $E(\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}') = \sigma^2\mathbf{V}$, with $\mathbf{V}(n \times n)$ known and positive definite. For $\mathbf{V}$ of full rank, show that the *weighted least squares* estimator is

$$\hat{\boldsymbol{\beta}}_W = (\mathbf{Z}'\mathbf{V}^{-1}\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{V}^{-1}\mathbf{Y}$$

If σ^2 is unknown, it may be estimated, unbiasedly, by

$$(n - r - 1)^{-1} \times (\mathbf{Y} - \mathbf{Z}\hat{\boldsymbol{\beta}}_W)'\mathbf{V}^{-1}(\mathbf{Y} - \mathbf{Z}\hat{\boldsymbol{\beta}}_W).$$

Hint: $\mathbf{V}^{-1/2}\mathbf{Y} = (\mathbf{V}^{-1/2}\mathbf{Z})\boldsymbol{\beta} + \mathbf{V}^{-1/2}\boldsymbol{\varepsilon}$ is of the classical linear regression form $\mathbf{Y}^* = \mathbf{Z}^*\boldsymbol{\beta} + \boldsymbol{\varepsilon}^*$, with $E(\boldsymbol{\varepsilon}^*) = \mathbf{0}$ and $E(\boldsymbol{\varepsilon}^*\boldsymbol{\varepsilon}^{*\prime}) = \sigma^2\mathbf{I}$. Thus, $\hat{\boldsymbol{\beta}}_W = \hat{\boldsymbol{\beta}}^* = (\mathbf{Z}^*\mathbf{Z}^*)^{-1}\mathbf{Z}^{*\prime}\mathbf{Y}^*$.

7.4. Use the weighted least squares estimator in Exercise 7.3 to derive an expression for the estimate of the slope β in the model $Y_j = \beta z_j + \varepsilon_j, j = 1, 2, \ldots, n$, when (a) $\mathrm{Var}(\varepsilon_j) = \sigma^2$, (b) $\mathrm{Var}(\varepsilon_j) = \sigma^2 z_j$, and (c) $\mathrm{Var}(\varepsilon_j) = \sigma^2 z_j^2$. Comment on the manner in which the unequal variances for the errors influence the optimal choice of $\hat{\beta}_W$.

7.6. *(Generalized inverse of* $\mathbf{Z'Z}$*)* A matrix $(\mathbf{Z'Z})^-$ is called a generalized inverse of $\mathbf{Z'Z}$ if $\mathbf{Z'Z}(\mathbf{Z'Z})^-\mathbf{Z'Z} = \mathbf{Z'Z}$. Let $r_1 + 1 = \text{rank}(\mathbf{Z})$ and suppose $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r_1+1} > 0$ are the nonzero eigenvalues of $\mathbf{Z'Z}$ with corresponding eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{r_1+1}$.

(a) Show that

$$(\mathbf{Z'Z})^- = \sum_{i=1}^{r_1+1} \lambda_i^{-1}\mathbf{e}_i\mathbf{e}_i'$$

is a generalized inverse of $\mathbf{Z'Z}$.

(b) The coefficients $\hat{\boldsymbol{\beta}}$ that minimize the sum of squared errors $(\mathbf{y} - \mathbf{Z}\boldsymbol{\beta})'(\mathbf{y} - \mathbf{Z}\boldsymbol{\beta})$ satisfy the *normal equations* $(\mathbf{Z'Z})\hat{\boldsymbol{\beta}} = \mathbf{Z'y}$. Show that these equations are satisfied for any $\hat{\boldsymbol{\beta}}$ such that $\mathbf{Z}\hat{\boldsymbol{\beta}}$ is the projection of $\mathbf{y}$ on the columns of $\mathbf{Z}$.

(c) Show that $\mathbf{Z}\hat{\boldsymbol{\beta}} = \mathbf{Z}(\mathbf{Z'Z})^-\mathbf{Z'y}$ is the projection of $\mathbf{y}$ on the columns of $\mathbf{Z}$. (See Footnote 2 in this chapter.)

(d) Show directly that $\hat{\boldsymbol{\beta}} = (\mathbf{Z'Z})^-\mathbf{Z'y}$ is a solution to the normal equations $(\mathbf{Z'Z})[(\mathbf{Z'Z})^-\mathbf{Z'y}] = \mathbf{Z'y}$.

Hint: (b) If $\mathbf{Z}\hat{\boldsymbol{\beta}}$ is the projection, then $\mathbf{y} - \mathbf{Z}\hat{\boldsymbol{\beta}}$ is perpendicular to the columns of $\mathbf{Z}$.

(d) The eigenvalue–eigenvector requirement implies that $(\mathbf{Z'Z})(\lambda_i^{-1}\mathbf{e}_i) = \mathbf{e}_i$ for $i \leq r_1 + 1$ and $0 = \mathbf{e}_i'(\mathbf{Z'Z})\mathbf{e}_i$ for $i > r_1 + 1$. Therefore, $(\mathbf{Z'Z})(\lambda_i^{-1}\mathbf{e}_i)\mathbf{e}_i'\mathbf{Z}' = \mathbf{e}_i\mathbf{e}_i'\mathbf{Z}'$. Summing over i gives

$$(\mathbf{Z'Z})(\mathbf{Z'Z})^-\mathbf{Z}' = \mathbf{Z'Z}\left(\sum_{i=1}^{r_1+1} \lambda_i^{-1}\mathbf{e}_i\mathbf{e}_i'\right)\mathbf{Z}'$$
$$= \left(\sum_{i=1}^{r_1+1} \mathbf{e}_i\mathbf{e}_i'\right)\mathbf{Z}' = \left(\sum_{i=1}^{r+1} \mathbf{e}_i\mathbf{e}_i'\right)\mathbf{Z}' = \mathbf{IZ}' = \mathbf{Z}'$$

since $\mathbf{e}_i'\mathbf{Z}' = \mathbf{0}$ for $i > r_1 + 1$.

7.7. Suppose the classical regression model is, with rank $(\mathbf{Z}) = r + 1$, written as

$$\underset{(n\times 1)}{\mathbf{Y}} = \underset{(n\times(q+1))}{\mathbf{Z}_1} \underset{((q+1)\times 1)}{\boldsymbol{\beta}_{(1)}} + \underset{(n\times(r-q))}{\mathbf{Z}_2} \underset{((r-q)\times 1)}{\boldsymbol{\beta}_{(2)}} + \underset{(n\times 1)}{\boldsymbol{\varepsilon}}$$

where $\text{rank}(\mathbf{Z}_1) = q + 1$ and $\text{rank}(\mathbf{Z}_2) = r - q$. If the parameters $\boldsymbol{\beta}_{(2)}$ are identified beforehand as being of primary interest, show that a $100(1 - \alpha)\%$ confidence region for $\boldsymbol{\beta}_{(2)}$ is given by

$$(\hat{\boldsymbol{\beta}}_{(2)} - \boldsymbol{\beta}_{(2)})' [\mathbf{Z}_2'\mathbf{Z}_2 - \mathbf{Z}_2'\mathbf{Z}_1(\mathbf{Z}_1'\mathbf{Z}_1)^{-1}\mathbf{Z}_1'\mathbf{Z}_2] (\hat{\boldsymbol{\beta}}_{(2)} - \boldsymbol{\beta}_{(2)}) \leq s^2(r - q)F_{r-q,n-r-1}(\alpha)$$

Hint: By Exercise 4.12, with 1's and 2's interchanged,

$$\mathbf{C}^{22} = [\mathbf{Z}_2'\mathbf{Z}_2 - \mathbf{Z}_2'\mathbf{Z}_1(\mathbf{Z}_1'\mathbf{Z}_1)^{-1}\mathbf{Z}_1'\mathbf{Z}_2]^{-1}, \qquad \text{where } (\mathbf{Z}'\mathbf{Z})^{-1} = \begin{bmatrix} \mathbf{C}^{11} & \mathbf{C}^{12} \\ \mathbf{C}^{21} & \mathbf{C}^{22} \end{bmatrix}$$

Multiply by the square-root matrix $(\mathbf{C}^{22})^{-1/2}$, and conclude that $(\mathbf{C}^{22})^{-1/2}(\hat{\boldsymbol{\beta}}_{(2)} - \boldsymbol{\beta}_{(2)})/\sigma^2$ is $N(\mathbf{0}, \mathbf{I})$, so that

$$(\hat{\boldsymbol{\beta}}_{(2)} - \boldsymbol{\beta}_{(2)})'(\mathbf{C}^{22})^{-1}(\hat{\boldsymbol{\beta}}_{(2)} - \boldsymbol{\beta}_{(2)}) \text{ is } \sigma^2\chi^2_{r-q}.$$