Lec. 13 (11/03/2022)

More examples and tricks for converting problems to standard forms :

Example : Minimize ratio of affine functions over polyhedron

$$\left.\begin{array}{ll} \min\limits_{\underline{x} \in \mathbb{R}^n} & f_0(\underline{x}) \\ \text{s.t.} & G\underline{x} \preceq \underline{h} \\ & A\underline{x} = \underline{b} \end{array}\right\} \quad f_0(\underline{x}) := \frac{\langle \underline{c}, \underline{x} \rangle + d}{\langle \underline{e}, \underline{x} \rangle + f}$$

where

$$\underbrace{\operatorname{dom}(f_0) := \left\{ \underline{x} \in \mathbb{R}^n \;\middle|\; \langle \underline{e}, \underline{x} \rangle + f > 0 \right\}}_{\text{convex set}}$$

$$\underbrace{S_\alpha}_{\substack{\alpha \text{ sublevel} \\ \text{set of } f_0(\cdot)}} = \left\{ \underline{x} \in \mathbb{R}^n \;\middle|\; \langle \underline{e}, \underline{x} \rangle + f > 0 \text{ and } \underbrace{\boxed{\frac{\langle \underline{c}, \underline{x} \rangle + d}{\langle \underline{e}, \underline{x} \rangle + f}}}_{f_0(\underline{x})} \leq \alpha \right\}$$

$\underbrace{\hphantom{xxxxxxxxxxxxxxxxxxxx}}$

convex set

because it is an intersection of an open halfspace & a closed halfspace

$\therefore$ $f_0(\underline{x})$ is a quasi-convex function (Lec. 10, p. 10-11)

Similarly, $f_0(\underline{x})$ " " quasi-concave function

$\therefore$ Our $f_0(\underline{x}) = \dfrac{\langle \underline{c}, \underline{x} \rangle + d}{\langle \underline{e}, \underline{x} \rangle + f}$

is quasi-affine

$$f_0(\underline{x}) = \frac{\underline{c}^T \underline{x} + d}{\underline{e}^T \underline{x} + f}$$

$$= \underline{c}^T \underbrace{\left(\frac{x}{\underline{e}^T \underline{x} + f} \right)}_{=: \underline{y}} + d \underbrace{\left(\frac{1}{\underline{e}^T \underline{x} + f} \right)}_{z}$$

$$\Leftrightarrow \tilde{f}_0(\underline{y}, z) = \left\langle \begin{pmatrix} \underline{c} \\ d \end{pmatrix}, \begin{pmatrix} \underline{y} \\ z \end{pmatrix} \right\rangle$$

So if $\underline{x}$ is a feasible solution (i.e., satisfies original constraints in $\underline{x}$) then new variables $(\underline{y}, z)$ satisfy:

$$G\underline{x} \preceq \underline{h} \iff G\underline{y} - \underline{h} z \preceq 0$$

$$A\underline{x} = \underline{b} \iff A\underline{y} - \underline{b} z = \underline{0}$$

$$\frac{1}{\langle \underline{e}, \underline{x} \rangle + f} = z \iff \underline{e}^T \underline{y} + f z = 1$$

$$\langle \underline{e}, \underline{x} \rangle + f > 0 \iff z > 0$$

Recall that

$$\frac{\underline{x}}{\underline{e}^T \underline{x} + f} = \underline{y}$$

$$\Rightarrow \underline{x} z = \underline{y}$$

$$\Rightarrow \underline{x} = \underline{y}/z$$

$\therefore$ Transformed problem in new decision variables $(\underline{y}, z)$ becomes:

$$\left.\begin{aligned}
\min_{(\underline{y},\, z)} \quad & \underline{c}^T \underline{y} + d z \\
\text{s.t.} \quad & G\underline{y} - \underline{h} z \preccurlyeq \underline{0} \\
& A\underline{y} - \underline{b} z = \underline{0} \\
& \underline{e}^T \underline{y} + f z = 1 \\
& z > 0
\end{aligned}\right\} \quad \text{LP!!}$$

Geometric Programming (GP)

Given $\underline{x} \in \mathbb{R}^n$ we say $\underbrace{g(\underline{x}) = c\, x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}}_{\text{a monomial}}$

where $\operatorname{dom}(g) = \mathbb{R}^n_{>0}$

and $c > 0$, $\boxed{a_i \in \mathbb{R}}$

$\boxed{\text{e.g., } 5.1\, x_1^{3/2}\, x_2^{-0.9}\, x_3^{-1.57}}$

We say

$$f(\underline{x}) = \underbrace{\sum_{k=1}^{K} c_k\, x_1^{a_{1k}}\, x_2^{a_{2k}} \cdots x_n^{a_{nk}}}_{\text{linear combination of monomials}} \text{ is called } \boxed{\text{posynominals}}$$

GP standard form:

$$\min_{\underline{x} \in \mathbb{R}^n_{++}} f_0(\underline{x})$$

$$\text{s.t.} \quad f_i(\underline{x}) \leq 1 \quad \forall\, i = 1, \dots, m$$

$$h_j(\underline{x}) = 1 \quad \forall\, j = 1, \dots, p$$

where $f_0, \dots, f_m$ are posynomials

and $h_1, \dots, h_p$ " monomials

e.g., $\max \; x/y$

s.t. $2 \leq x \leq 3$, $\boxed{x^2 + \frac{3y}{z} \leq \sqrt{y}}$, $x, y, z > 0$

Motivational example:

Holiday travel planning:

Want to buy $\boxed{\text{suitcase}}$ of maximum volume

3D rectangle of height h, width w, depth d

Numbers A_{wall}, A_{floor}, $\alpha, \beta, \gamma, \delta$ are given in airlines website

$$\underset{(h,w,d)\in\mathbb{R}^3_{++}}{\text{maximize}} \quad \underbrace{hwd}_{\text{volume}}$$

$$\text{s.t.} \quad 2(hw+hd) \le A_{wall}$$

$$wd \le A_{floor}$$

$$\alpha \le h/w \le \beta, \quad \gamma \le w/d \le \delta$$

$$\iff \quad \min_{(h,w,d)\in\mathbb{R}^3_{++}} \; h^{-1} w^{-1} d^{-1}$$

s.t.

$$\boxed{\frac{2}{A_{wall}} hw + \frac{2}{A_{wall}} hd \le 1}$$

$$\boxed{\frac{1}{A_{floor}} wd \le 1}$$

$$\boxed{\alpha h^{-1} w \le 1}$$

$$\boxed{\frac{1}{\beta} h w^{-1} \le 1}$$

$$\boxed{\gamma d w^{-1} \le 1} \quad \text{and} \quad \boxed{\frac{1}{\delta} w d^{-1} \le 1}$$

in GP standard form

GP standard form is nonconvex problem

Do change of variable:

$$y_i = \log x_i \iff x_i = \exp(y_i)$$

Then the GP standard form becomes:

$$\begin{cases} \min\limits_{\underline{y} \in \mathbb{R}^n} & \log\left(f_0\left(\exp\left(\underline{y}\right)\right)\right) \\ \text{s.t.} & \log\left(f_i\left(\exp\left(\underline{y}\right)\right)\right) \leq 0 \quad \forall i = 1, \dots, m \\ & \log\left(h_j\left(\exp\left(\underline{y}\right)\right)\right) = 0 \quad \forall j = 1, \dots, p \end{cases}$$

where $\exp(\cdot)$ is elementwise.

To understand why the transformed problem is convex, consider a special case: $m = p = 1$,

and $$f_0(\underline{x}) = \sum_{k=1}^{K_0} \alpha_k \, x_1^{\beta_{1,k}} \cdots x_n^{\beta_{n,k}}$$

$$f_1(\underline{x}) = \sum_{k=1}^{K_1} a_k \, x_1^{b_{1,k}} \cdots x_n^{b_{n,k}}$$

$$h_1(\underline{x}) = c \, x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n},$$ where $\alpha_k, a_k, c > 0$.

Let $\underline{p}_k := \begin{pmatrix} \beta_{1,k} \\ \vdots \\ \beta_{n,k} \end{pmatrix}$, $a_k = \log \alpha_k$,

$\underline{r}_k := \begin{pmatrix} b_{1,k} \\ \vdots \\ b_{n,k} \end{pmatrix}$, $s_k = \log a_k$,

$\underline{u} = \begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix}$, $t = -\log c$

Then our problem becomes:

$$\min_{\underline{y} \in \mathbb{R}^n} \log\left(\sum_{k=1}^{K_0} \exp\left(\langle \underline{p}_k, \underline{y} \rangle + q_k\right)\right)$$

$$\text{s.t.} \quad \log\left(\sum_{k=1}^{K_1} \exp\left(\langle \underline{r}_k, \underline{y} \rangle + s_k\right)\right) \leq 0$$

$$\langle \underline{u}, \underline{y} \rangle = t$$

This problem is a convex optimization problem: why? log-sum-exp is convex function

convex composed with affine is convex.

Stochastic LP / Chance-constrained LP :

$$\min_{\underline{x} \in \mathbb{R}^n} \langle \underline{c}, \underline{x} \rangle$$

$$\text{s.t.} \quad \underline{a}_i^T \underline{x} \leq b_i \;\; \forall i = 1, \ldots, m$$

$$\underline{a}_i \sim \mathcal{N}(\underline{\mu}_i, \Sigma_i), \qquad \underline{\mu}_i \in \mathbb{R}^n, \quad \Sigma_i \in \mathbb{S}^n_+$$

random vector

(follows the law/distribution of)

Normal/Gaussian probability distribution

mean vector

Covariance matrix

Suppose we want to solve:

$$\min_{\underline{x} \in \mathbb{R}^n} \underline{c}^T \underline{x}$$

$$\text{s.t.} \quad \mathbb{P}(\underline{a}_i^T \underline{x} \leq b) \geq \eta \quad \forall\, i = 1, \dots, m$$

Typically 0.5 or more

$\boxed{\text{text p. 157 bottom - 158}}$

$$\boxed{\Phi(\eta) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\eta} \exp(-t^2/2)\, dt}$$

standard form SOCP

$$\min_{\underline{x} \in \mathbb{R}^n} \underline{c}^T \underline{x}$$

$$\text{s.t.} \quad \underline{\mu}_i^T \underline{x} + \Phi^{-1}(\eta) \| \Sigma_i^{1/2} \underline{x} \|_2 \leq b_i \quad \forall\, i = 1, \dots, m$$

Next example: Minimize maximum eig. value

Let $A_0, A_1, \ldots, A_n \in S^m$

and $A(\underline{x}) := A_0 + A_1 x_1 + A_2 x_2 + \ldots + A_n x_n$

where $\underline{x} \equiv \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

$$A(\underline{x}) \in S^m$$

Consider the problem: $\min_{\underline{x} \in \mathbb{R}^n} \underbrace{\lambda_{max}(A(\underline{x}))}_{\text{convex function}}$

because $\lambda_{max}(\cdot)$ is convex in S^m (Lec. 9, p. 12-13)

convex opt. problem

BUT in what standard form?

epigraph form (Lec. 12, p. 9)

$$\min_{(\underline{x}, t)} \quad t$$
$$\text{s.t.} \quad \lambda_{max}(A(\underline{x})) \leq t$$

SDP standard form

$$\min_{\underline{x} \in \mathbb{R}^n} \quad \underline{c}^T \underline{x}$$
$$\text{s.t.} \quad F(\underline{x}) \succeq 0$$

From linear algebra:

$$\lambda_{max}(A) \leq t$$
$$\Updownarrow$$
$$A - tI \preceq 0$$

$$\min_{(\underline{x}, t)} \quad t$$
$$\text{s.t.} \quad A(\underline{x}) - tI \preceq 0$$
$$\Updownarrow$$
$$tI - A(\underline{x}) \succeq 0$$

Example: Minimize induced 2 norm / maximum singular value of a matrix $A \in \mathbb{R}^{m \times n}$

Recall:

$$\|A\|_2 = \sigma_{max}(A) = \sqrt{\lambda_{max}(A^T A)}$$

Problem:

$$\min_{\underline{x} \in \mathbb{R}^n} \|A(\underline{x})\|_2$$

$$\Updownarrow$$

$$\min_{(\underline{x}, t) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}} t$$

$$\text{s.t.} \quad \|A(\underline{x})\|_2 \leq t$$

$$\Leftrightarrow \quad \min_{(\underline{x},t)\in\mathbb{R}^n\times\mathbb{R}_{>0}} \; t$$

s.t. $\boxed{\lambda_{max}\left(\left(A(\underline{x})\right)^T A(\underline{x})\right) \leq t^2}$

$\Updownarrow$

$\boxed{\left(A(\underline{x})\right)^T A(\underline{x}) \preccurlyeq t^2 I}$

$\Updownarrow$ Schur complement Lemma (Lec. 9, p. 2-3)

$$\Leftrightarrow \quad \min_{(\underline{x},t)\in\mathbb{R}^n\times\mathbb{R}_{>0}} \; t \quad \text{s.t.} \begin{bmatrix} tI & A \\ A^T & tI \end{bmatrix} \succcurlyeq 0$$

and $t > 0$

LMI

SDP standard form

Vector version of Schur complement reverse engineering:

$$\|\underline{u}\|_2 \leq t$$

$$\Leftrightarrow \sqrt{\underline{u}^T \underline{u}} \leq t$$

$$\Leftrightarrow \underline{u}^T \underline{u} \leq t^2$$

$$\boxed{\Leftrightarrow \begin{bmatrix} tI & \underline{u} \\ \underline{u}^T & \underset{1\times 1}{t} \end{bmatrix} \succeq 0 \quad \text{and} \quad t > 0}$$