Lec. 9 (10/20/2022)

Example: Prove that the function $f : \mathbb{R}^n \times S^n_{++} \mapsto \mathbb{R}$ given by :

$f(\underline{x}, Y) := \underline{x}^T Y^{-1} \underline{x}$ is convex over $\mathrm{dom}(f)$.

$\mathrm{dom}(f) = \mathbb{R}^n \times S^n_{++}$ is a convex set.

For $n = 1$, reduces to last lecture's example: quadratic-over-linear. (x^2/y over $\mathrm{dom}(f) = \mathbb{R} \times \mathbb{R}_{>0}$)

Let's use set convexity to prove function convexity:

$$\mathrm{epi}(f) := \left\{ (\underline{x}, Y, t) \,\middle|\, Y \succ 0,\ \underbrace{\underline{x}^T Y^{-1} \underline{x} \le t} \right\}$$

$$= \left\{ (\underline{x}, Y, t) \,\middle|\, Y \succ 0,\ \underbrace{\begin{bmatrix} Y & \underline{x} \\ \underline{x}^T & t \end{bmatrix} \succeq 0} \right\}$$

Schur complement Lemma (Appendix A.5.5 in textbook)

Intersection of LMIs (convex sets)

$\therefore$ epi(f) is convex set $\Longleftrightarrow$ f is a convex fn.

Schur complement Lemma:

Let $X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \in S^m$ and $\det(A) \neq 0$.

Define $\underbrace{S := C - B^T A^{-1} B}_{\text{Schur complement of } A \text{ in matrix } X}$

The following statements are equivalent:

- $X \succ 0 \iff A \succ 0$ and $S \succ 0$.
- If $A \succ 0$ then $X \succeq 0 \iff S \succeq 0$.

One application of Schur complement:

Computing block determinants:

$$X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix},\quad \det(A) \neq 0, \quad \text{compute } \det(X).$$

$$X = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \rightsquigarrow \det(X) = ac - b^2$$

$$= a\left(c - b^2/a\right)$$

$$\det(X) = \det(A)\,\det(C - B^T A^{-1} B)$$

$$= \det(A)\,\det(S)$$

Revisiting zeroth order characterization/condition of convex functions:

$$\underbrace{f(\theta \underline{x} + (1-\theta)\underline{y}) \leq \theta f(\underline{x}) + (1-\theta) f(\underline{y})}_{\text{Jensen's inequality}}$$

$\forall \; 0 \leq \theta \leq 1$

$\forall \; \underline{x}, \underline{y} \in \mathrm{dom}(f)$.

Jensen's inequality

(Multi-point version):

$$f\left(\sum_{i=1}^{k} \theta_i \underline{x}_i\right) \leq \sum_{i=1}^{k} \theta_i f(\underline{x}_i) \quad \forall \; \theta_i \geq 0$$

and $\sum_{i=1}^{k} \theta_i = 1$

We can interpret this inequality in a probabilistic manner:

$$\underline{x} \in \{ \underline{x}_1, \underline{x}_2, \ldots, \underline{x}_k \}$$

random vector

$$\underbrace{\mathbb{P}}_{\text{Prob}}(\underline{x} = \underline{x}_i) = \theta_i \quad \forall\, i = 1, \ldots, k$$

$$f(\mathbb{E}[\underline{x}]) \leq \mathbb{E}[f(\underline{x})] \quad \text{for } f(\cdot) \text{ convex}$$

$$\forall\, \underline{x} \in \mathrm{dom}(f).$$

Also holds for continuous spaces:

$p : \mathbb{R}^n \mapsto \mathbb{R}$, $p(\underline{x}) \geq 0$, $\int p(\underline{x}) d\underline{x} = 1$

Then Jensen's inequality:

$$f\left(\int \underline{x}\, p(\underline{x})\, d\underline{x} \right) \leq \int f(\underline{x})\, p(\underline{x})\, d\underline{x}$$

for f convex function.

Operations preserving function convexity:

- **Non-neg. weighted sum:**

$f_i : \mathbb{R}^n \mapsto \overline{\mathbb{R}}$, where $f_1, f_2, \ldots, f_m$ are all convex

Then $g(\underline{x}) = \sum_{i=1}^{m} w_i f_i(\underline{x})$, $w_i \geq 0$

is also a convex function.

Also works for continuum:

If $f(\underline{x}, \underline{y})$ is convex in $\underline{x}$, and $w(\underline{y}) \geq 0$ $\forall \underline{y} \in \mathcal{Y}$

Then $\int_{\mathcal{Y}} w(\underline{y}) f(\underline{x}, \underline{y}) \, d\underline{y}$ is convex in $\underline{x}$.

- Composition with affine map:

Let $f : \mathbb{R}^n \mapsto \overline{\mathbb{R}}$ be a convex function.

Let $A \in \mathbb{R}^{m \times n}$, $\underline{b} \in \mathbb{R}^n$

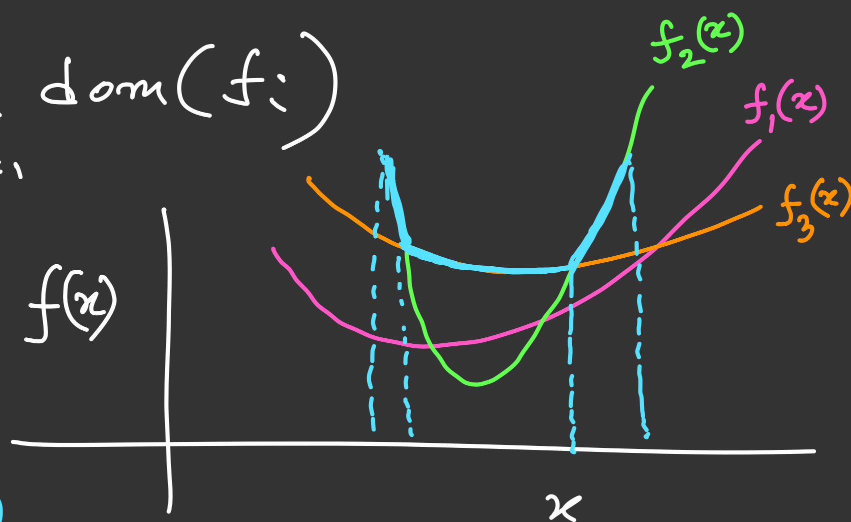
Then $g(\underline{x}) := f(A\underline{x} + \underline{b})$ is a convex function in $\underline{x}$.

$$\operatorname{dom}(g) = \{ \underline{x} \in \mathbb{R}^n \mid A\underline{x} + \underline{b} \in \operatorname{dom}(f) \}$$

- Pointwise max or sup :

If $f_1(\underline{x}), \ldots, f_m(\underline{x})$ are convex functions, then so is $f(\underline{x}) := \max\{f_1(\underline{x}), \ldots, f_m(\underline{x})\}$

$$\mathrm{dom}(f) = \bigcap_{i=1}^{m} \mathrm{dom}(f_i)$$



$$\mathrm{epi}(f) = \bigcap_{i=1}^{m} \mathrm{epi}(f_i)$$

Pointwise sup. over uncountable set of convex functions is also convex :

If $f(\underline{x}, \underline{y})$ is convex in $\underline{x}$

then

$$g(\underline{x}) := \sup_{\underline{y} \in \mathcal{Y}} f(\underline{x}, \underline{y})$$ is also convex in $\underline{x}$.

pointwise sup

Similarly,

pointwise inf over concave functions is concave

Example:

Consider the function $f(X) = \lambda_{max}(X)$

where $X \in S^n$.

Is $f(X)$ convex over $dom(f) = S^n$?

For $X \in S^n$, we have:

$$\lambda_{min}(X) \leq \frac{\underline{x}^T X \underline{x}}{\underline{x}^T \underline{x}} \leq \lambda_{max}(X) \quad \forall\, \underline{x} \in \mathbb{R}^n.$$

From $X = VDV^T$ (eig. value decomposition)

$$\because \sup_{\|\underline{x}\|_2=1} \frac{\underline{x}^T X \underline{x}}{\cancel{\underline{x}^T \underline{x}}\ 1} = \lambda_{max}(X)$$

$$\inf_{\|\underline{x}\|_2=1} \frac{\underline{x}^T X \underline{x}}{\cancel{\underline{x}^T \underline{x}}\ 1} = \lambda_{min}(X)$$

$$\lambda_{max}(X) = \underbrace{\sup_{\|\underline{x}\|_2=1} \underline{x}^T X \underline{x}}_{\text{Pointwise sup of linear (convex) functions}}$$

$$\boxed{\therefore \lambda_{max}(X) \text{ is convex function in } X \in S^n}$$

- **Composition:** $f(\underline{x}) = h \circ g(\underline{x})$

$$\left.\begin{aligned} h &: \mathbb{R}^k \mapsto \mathbb{R} \\ g &: \mathbb{R}^n \mapsto \mathbb{R}^k \end{aligned}\right\} \quad f = h \circ g : \mathbb{R}^n \mapsto \mathbb{R}$$

When is the composite nonlinear f^n: $h \circ g$ convex?

$\boxed{\text{p. 84-87 in textbook.}}$

- **Minimization:** If f is convex in $(\underline{x}, \underline{y})$ and $\boxed{\mathcal{S} \text{ is a convex set}}$

Then

$$g(\underline{x}) = \inf_{\underline{y} \in \mathcal{S}} f(\underline{x}, \underline{y}) \quad \text{is convex in } \underline{x}.$$

Example: Distance between a point ($\underline{x}$) and a set S

Let $g(\underline{x}) = \text{dist}(\underline{x}, S)$

$$:= \inf_{\underline{y} \in S} \underbrace{\| \underline{x} - \underline{y} \|_2}_{\text{Jointly convex in } (\underline{x}, \underline{y})}$$

convex

$\therefore$ $g(\underline{x})$ is a convex function in $\underline{x}$.

Convex conjugates / Legendre-Fenchel transform:

$f(\underline{x}) : \mathbb{R}^n \mapsto \mathbb{R}$

Need not be convex

$f^*(\underline{y}) : \mathbb{R}^n \mapsto \mathbb{R}$

convex conjugate or Legendre-Fenchel conjugate of the function f

<u>Definition</u>:

$$f^*(\underline{y}) := \sup_{\underline{x} \in \mathrm{dom}(f)} \left\{ \underline{y}^T \underline{x} - f(\underline{x}) \right\}$$

$$f(\cdot) \longrightarrow \boxed{\quad\quad} \longrightarrow f^*(\cdot)$$

Example: (please check this)

affine: $f(\underline{x}) = \langle \underline{a}, \underline{x} \rangle + b$, $\underline{a} \in \mathbb{R}^n$, $b \in \mathbb{R}$

$$f^*(\underline{y}) = \sup_{\underline{x} \in \mathbb{R}^n} \left(\underline{y}^T \underline{x} - \underline{a}^T \underline{x} - b \right)$$

$$= \begin{cases} -b & \text{for } \underline{y} = \underline{a} \\ +\infty & \text{otherwise} \end{cases}$$

Quadratic: $f(\underline{x}) = \frac{1}{2} \underline{x}^T Q \underline{x}$, $Q \in S^n_{++}$, $\underline{x} \in \mathbb{R}^n$

then $f^*(\underline{y}) = \frac{1}{2} \underline{y}^T Q^{-1} \underline{y}$.