Problem 1 [40 points] Lyapunov stability for a non-autonomous nonlinear system

Consider the continuous-time non-autonomous system

$$\dot{x}_1 = -x_1 - g(t)x_2,$$
$$\dot{x}_2 = x_1 - x_2,$$

where the state vector $(x_1, x_2)^\top \in \mathrm{R}^2$. The function $g(t)$ is C^1 in time t, satisfies $0 \le g(t) \le k$ for some constant k, and $\dot{g}(t) \le g(t)$ for all $t \ge 0$.

(a) [10 points] Fixed point

Prove that the origin $(0, 0)^\top$ is a fixed point.

1.

$$\begin{aligned} &\forall t \ge 0, \\ \dot{x}_1 &= -0 - g(t) * 0 &&= 0, \\ \dot{x}_2 &= 0 - 0 &&= 0, \end{aligned}$$

Thus by definition $(0, 0)^\top$ is a fixed point

(b) [30 points] Origin is GES

Use the Lyapunov function $V(t, x_1, x_2) := x_1^2 + (1 + g(t))x_2^2$ to **prove that** the origin is globally exponentially stable (GES).

(**Hint:** Lec. 5, p. 3).

1.

$$\begin{aligned} 0 \le x_2^2 &\le x_2^2 + x_1^2 \\ 0 \le x_2^2 &\le \|x\|^2 \\ 0 \le g(t)x_2^2 &\le k\|x\|^2 \\ \|x\|^2 \le g(t)x_2^2 + \|x\|^2 &\le (1+k)\|x\|^2 \\ \|x\|^2 \le g(t)x_2^2 + x_2^2 + x_1^2 &\le (1+k)\|x\|^2 \\ \|x\|^2 \le (g(t)+1)x_2^2 + x_1^2 &\le (1+k)\|x\|^2 \\ \|x\|^2 \le V(t, x_1, x_2) &\le (1+k)\|x\|^2 \end{aligned}$$

1. We take 1. above $k_1 = 1, k_2 = (1+k), \alpha = 2$:

$$k_1\|x\|^\alpha \le V(t, x_1, x_2) \le k_2\|x\|^\alpha$$

2.

$$\begin{aligned} \frac{\partial V}{\partial t} &= 0 + 0 + \frac{\partial}{\partial t} g(t)x_2^2 \\ &= x_2^2\dot{g}(t) \end{aligned}$$

4.

$$\frac{\partial V}{\partial x} = \begin{bmatrix} \frac{\partial V}{\partial x_1} \\ \frac{\partial V}{\partial x_2} \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 \\ 2x_2(1+g) \end{bmatrix}$$

3. Combining 3. 4. above:

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x_1, x_2) \\ &= x_2^2\dot{g}(t) + 2x_1(-x_1 - gx_2) + (x_1 - x_2)(2x_2)(1+g) \\ &= -2x_1^2 + 2x_1x_2 + x_2^2(\dot{g} - 2 - 2g) \end{aligned}$$

4. Since $\dot{g} \le g$:

$$\dot{g} - 2 - 2g \le g - 2 - 2g = -2 - g$$

and further $-2 - g \le -2$:

$$\begin{aligned} 0 \le g &\le k \\ -k \le -g &\le 0 \\ -2 - k \le -2 - g &\le -2 \end{aligned}$$

therefore $\dot{g} - 2 - 2g \le -2$
5. Combinining 5. and 6. above: $\dot{V} \le 2x_1x_2 - 2x_1^2 - 2x_2^2$
6. $(x_1 - x_2)^2$ is non-negative, so:

$$\begin{aligned} \dot{V} &\le 2x_1x_2 - 2x_1^2 - 2x_2^2 \\ &\le 2x_1x_2 - 2x_1^2 - 2x_2^2 + (x_1 - x_2)^2 \\ &= 2x_1x_2 - 2x_1^2 - 2x_2^2 + x_1^2 - 2x_1x_2 + x_2^2 \\ &= -2x_1^2 - 2x_2^2 + x_1^2 + x_2^2 \\ &= -x_1^2 - x_2^2 \\ &= -1(x_1^2 + x_2^2) \\ &= -1\|x\|^2 \\ &\le k_3\|x\|^\alpha, k_3 = 1, \alpha = 2 \end{aligned}$$

7. So from 8. and 2. since they hold true $\forall x \in R^n$ and by Lec 5 pg. 3, the origin **is GES**.

Problem 2 [60 points] Lyapunov stability for a rotating rigid spacecraft

The controlled dynamics for a rotating rigid spacecraft is given by the Euler equation

$$J_1\dot{\omega}_1 = \left(J_2 - J_3\right)\omega_2\omega_3 + \tau_1,$$
$$J_2\dot{\omega}_2 = \left(J_3 - J_1\right)\omega_3\omega_1 + \tau_2,$$
$$J_3\dot{\omega}_3 = \left(J_1 - J_2\right)\omega_1\omega_2 + \tau_3,$$

where the parameters $J_1, J_2, J_3 > 0$ denote the principal moments of inertia; the state vector $(\omega_1, \omega_2, \omega_3)^\top \in \mathrm{R}^3$ denotes the spacecraft's angular velocity (in rad/s) along its principal axes; and the control vector $(\tau_1, \tau_2, \tau_3)^\top$ denotes the torque input applied about the principal axes.

(a) [2 + (2 + 6) = 10 points] Fixed points in the absence of control

Suppose that the controls $\tau_1 = \tau_2 = \tau_3 \equiv 0$.

(i) **Argue that** origin is a fixed point.

1. when $(\omega_1, \omega_2, \omega_3)^T = 0$

$$J_1\dot{\omega}_1 = \left(J_2 - J_3\right)0 + 0 \rightarrow J_1\dot{\omega}_1 = 0 \rightarrow \dot{\omega}_1 = 0,$$
$$J_2\dot{\omega}_2 = \left(J_3 - J_1\right)0 + 0 \rightarrow J_2\dot{\omega}_2 = 0 \rightarrow \dot{\omega}_2 = 0,$$
$$J_3\dot{\omega}_3 = \left(J_1 - J_2\right)0 + 0 \rightarrow J_3\dot{\omega}_3 = 0 \rightarrow \dot{\omega}_3 = 0,$$

(ii) **How many fixed points other than origin** are there? **Explain what physical motions** do these non-origin fixed points correspnd to?

1. All 3 equations above include 2 / 3 state terms. So for any states where 2 of the 3 dimensions of $\omega_i = 0$, all 3 equations will yield 0. i.e. $(\omega_1, 0, 0)^T, (0, \omega_2, 0)^T, (0, 0, \omega_3)^T$

2. For the nonzero elements of those 3 state vectors, it doest *not* matter what the nonzero element is, so there is an *infinite* # of points other than the origin.

3. The physical meaning of those 3 state vectors is when the spacecraft is rotating about *1* of the principal axes while stationary about the other axes. In these states, the spacecraft dynamics do *not* change, the spacecraft will continue rotating about that axis at the same angular velocity indefinitely.

(b) [(10 + 5) + 10 = 25] S/AS in the absence of control

As in part (a), assume that the controls $\tau_1 = \tau_2 = \tau_3 \equiv 0$.

(i) By constructing a suitable Lyapunov function, **prove that** the origin is stable (S). From this analysis, **what can you conclude** about the asymptotic stability (AS) of the origin?

1. Let $V = \frac{1}{2}\begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix}\begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \frac{1}{2}(J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2)$ where $P = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix}$
2. Knowing $J_1, J_2, J_3 > 0$, $V = \frac{1}{2}(J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2) > 0 \; \forall \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix} \rightarrow$ P is positive definite matrix.
3. We can show that $\dot{V}$ is negative **semi-definite**:

$$\begin{aligned} \dot{V} &= \omega_1(J_1 * \dot{\omega}_1) + \omega_2(J_2 * \dot{\omega}_2) + \omega_3(J_3 * \dot{\omega}_3) \\ &= \omega_1(J_2 - J_3)\omega_2\omega_3 + \omega_2(J_3 - J_1)\omega_1\omega_3 + \omega_3(J_1 - J_2)\omega_1\omega_1 \\ &= \omega_1\omega_2\omega_3(J_2 - J_3 + J_3 - J_1 + J_1 - J_2) \\ &= \omega_1\omega_2\omega_3(J_2 - J_3 + J_3 - J_1 + J_1 - J_2) \\ &= 0 \end{aligned}$$

4. Therefore, from 2. and 3. above, and by Lec 3. pg 2 pg 3, we prove that the origin **is S, not AS**

(ii) Given initial conditions $(\omega_{10}, \omega_{20}, \omega_{30})^\top$ and parameters $J_1 \ne J_2 \ne J_3$, it is possible to explicitly solve for the flow (still assuming $\tau_1 = \tau_2 = \tau_3 = 0$) as

$$\omega_1(t) = \bar{\omega}_{10}\mathrm{cn}\left(\omega_p t + \varepsilon, m\right),$$
$$\omega_2(t) = \bar{\omega}_{20}\mathrm{sn}\left(\omega_p t + \varepsilon, m\right),$$
$$\omega_3(t) = \bar{\omega}_{30}\mathrm{dn}\left(\omega_p t + \varepsilon, m\right),$$

where $\bar{\omega}_{10}, \bar{\omega}_{20}, \bar{\omega}_{30}, \omega_p, \varepsilon, m$ depend only on $(\omega_{10}, \omega_{20}, \omega_{30})^\top$ and J_1, J_2, J_3. The functions $\mathrm{cn}, \mathrm{sn}, \mathrm{dn}$ are the so-called Jacobi elliptic functions and are periodic in time t. **Using this information alone, and ignoring the previous Lyapunov analysis, explain what can you conclude** about the asymptotic stability of the origin?

1. If the $\omega_i(t)$ terms are **periodic** then by definition they by definition they do **not** converge to any one value i.e. at some starting time and period $t_1, p_1, \omega_i(t_1) = \omega_i(t_1 + k_1p_1) = \alpha$ for $k_1 = 1, 2, 3, \ldots\infty$ and for some other $t_2, p_2, \omega_i(t_2) = \omega_i(t_2 + k_2p_2) = \beta$ for $k_2 = 1, 2, 3, \ldots\infty$ s.t. $\alpha \ne \beta$
2. So if $\omega_i(t)$ converges, then that means $\lim_{t \rightarrow +\infty} w_i(t) = w_i(t_1 + k_1p_1) = w_i(t_2 + k_2p_2) = \gamma$ but this contradicts the statement in 1. So $\omega_i(t)$ **does not converge**
3. Thus by definition, given $\omega_i(t)$ the origin is **not AS**

(c) [25 points] Global asymptotic stabilization using feedback control

For $i = 1, 2, 3$, consider the state feedback control law $\tau_i = -k_i\omega_i$, where $k_i > 0$ are constants. **Prove that** origin of the closed-loop system is globally asymptotically stable (GAS).

1. Let $V = \frac{1}{2}\begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix}\begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \frac{1}{2}(J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2)$ where $P = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix}$
2. Knowing $J_1, J_2, J_3 > 0$, $V = \frac{1}{2}(J_1\omega_1^2 + J_2\omega_2^2 + J_3\omega_3^2) > 0 \; \forall \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix} \rightarrow$ P is positive definite matrix.
3. We can also see that since all terms in V are non-negative, $\lim_{\|w\| \rightarrow +\infty} V = +\infty$ so by definition V is radially unbounded.
4. We can show that $\dot{V}$ is negative definite:

$$\begin{aligned} \dot{V} &= \omega_1(J_1 * \dot{\omega}_1) + \omega_2(J_2 * \dot{\omega}_2) + \omega_3(J_3 * \dot{\omega}_3) \\ &= \omega_1(J_2 - J_3)\omega_2\omega_3 - k_1\omega_1^2 + \omega_2(J_3 - J_1)\omega_1\omega_3 - k_2\omega_2^2 + \omega_3(J_1 - J_2)\omega_1\omega_1 - k_3\omega_3^2 \\ &= \omega_1\omega_2\omega_3(J_2 - J_3 + J_3 - J_1 + J_1 - J_2) - k_1\omega_1^2 - k_2\omega_2^2 - k_3\omega_3^2 \\ &= \omega_1\omega_2\omega_3(J_2 - J_3 + J_3 - J_1 + J_1 - J_2) - k_1\omega_1^2 - k_2\omega_2^2 - k_3\omega_3^2 \\ &= 0 - k_1\omega_1^2 - k_2\omega_2^2 - k_3\omega_3^2 \\ &= -(k_1\omega_1^2 + k_2\omega_2^2 + k_3\omega_3^2) < 0 \end{aligned}$$

5. From 2., 3. 4., and by Lecture 3 pg 9, we prove that **the origin is GAS**