Lec. # 3 (09/29/2022)

Same Example from last lecture

$$X = \begin{bmatrix} \boxed{2} & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \in S^3$$

$$2 > 0, \quad 3 > 0, \quad 2(4-1) + 1(-2-0) + 0$$

$$= 6 - 2 = 4 > 0$$

$$\boxed{\therefore \; X \in S^3_{++}}$$

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Testing $\mathbb{S}^n_+$ via principal minors (needs to be ≥ 0)

From the square matrix, delete specific rows & columns such that the row and column indices being deleted are the same

↳ principal minors are determinants of the resulting matrices.

Example: Consider $X = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix} \in \mathbb{S}^3$

delete nothing: $\det(X) = 1(4-4) - 2(2-2) - 1(-4+4)$
$= 0$

delete one row-col. pair:

delete $(r1, c1) \rightsquigarrow \det([\;]_{2\times2}) = 4-4=0$

" $(r2, c2) \rightsquigarrow \det([\;]_{2\times2}) = 1-1=0$

" $(r3, c3) \rightsquigarrow \det([\;]_{2\times2}) = 4-4=0$

delete two row-col. pairs:

$$\left.\begin{array}{l} \text{delete } (r1, c1) \;\&\; (r2, c2) \rightsquigarrow \\ \text{" } (r2, c2) \;\&\; (r3, c3) \rightsquigarrow \\ \text{" } (r3, c3) \;\&\; (r1, c1) \rightsquigarrow \end{array}\right\} \begin{array}{l} \text{main diag. entries} \\ 1>0,\; 4>0,\; 1>0. \end{array}$$

$$\boxed{\therefore \quad X \in S^3_+}$$

End of Example.

Application Example:

$X \in S^2_+$, we have $X = \begin{bmatrix} x & z \\ z & y \end{bmatrix}$

Principal minor condition: $x \geq 0$

$y \geq 0$

$xy - z^2 \geq 0$

$\therefore \; S^2_+ = \{ (x, y, z) \in \mathbb{R}^3 \mid x \geq 0, y \geq 0, xy \geq z^2 \}$

$\subset \mathbb{R}^3$

Visualize this set (code in CANVAS)

Boundary of this surface/set: $z^2 = xy$

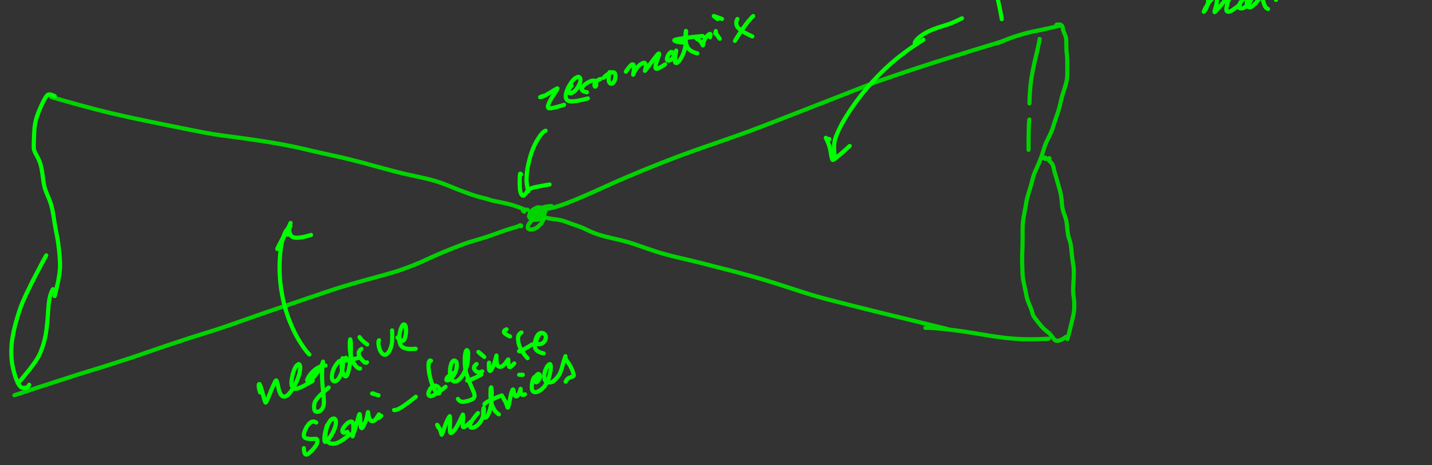
$\Leftrightarrow z = \pm\sqrt{xy}$

$x \geq 0, y \geq 0$.

$\therefore$ Interior of S^2_+ is precisely S^2_{++}

$\Longleftrightarrow$ Boundary of S^2_+ comprises of rank deficient matrices

Interior satisfies the leading principal minor condition: $xy > z^2$ for $x, y > 0$.



Optimization nomenclature:

Optimization problem looks like:

$$\begin{aligned} \min \quad & f_0(\underline{x}) \\ \text{s.t.} \quad & \underline{x} \in \mathcal{X} \end{aligned} \iff \min_{\underline{x} \in \mathcal{X}} f_0(\underline{x})$$

$\boxed{\text{Decision variable}}$

↳ Returns minimum value $f_0(\underline{x}^*)$

Here, $f_0 : \mathcal{X} \longmapsto \overline{\mathbb{R}}$

$\boxed{\text{Objective function}}$

"argmin" $\iff$ argument of the minimizer: $\underline{x}^*$

Set $\mathcal{X} \to$ feasible set

Similarly, we can do:

$$\boxed{\begin{aligned} & \min f_0(X) \\ & X \in \mathcal{X} \end{aligned}}$$

Examples of scalar functions of scalars:

linear function: ax, $a \neq 0$

affine function: $ax + b$, $a \neq 0, b \neq 0$

quadratic " : $ax^2 + bx + c$, etc.

trig. functions: $\sin(x)$, $\cos(x)$ etc.

Scalar functions of vectors:

$f(\underline{x}) = \langle \underline{a}, \underline{x} \rangle$ (linear function), $\underline{a} \in \mathbb{R}^n \setminus \{\underline{0}\}$

$\equiv \underline{a}^T \underline{x} \equiv \underline{a} \cdot \underline{x} = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$

inner product

$f(\underline{x}) = \langle \underline{a}, \underline{x} \rangle + b$ (affine function)

$f(\underline{x}) = \underline{x}^T A \underline{x} + \langle \underline{b}, \underline{x} \rangle + c,$ $\quad A \in \mathbb{R}^{n \times n}, \underline{b} \in \mathbb{R}^n, c \in \mathbb{R}.$

quadratic function.

vector p norms. $\quad 0 \le p \le +\infty.$

$$f(\underline{x}) = \|\underline{x}\|_p := \begin{cases} \text{cardinality}(\underline{x}) & \text{for } p = 0 \\ \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} & \text{for } 0 < p < +\infty \\ \max_{i=1,\dots,n} |x_i| & \text{for } p = +\infty \end{cases}$$

e.g., $\|\underline{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$, $\quad \|\underline{x}\|_1 = |x_1| + \dots + |x_n|$

$\|\underline{x}\|_3 = \left(|x_1|^3 + \dots + |x_n|^3\right)^{1/3}$

$$f(\underline{x}) = \frac{\underline{x}^T P \underline{x} + \langle \underline{q}, \underline{x} \rangle + r}{\langle \underline{a}, \underline{x} \rangle + b}$$ (quadratic over linear)

Scalar functions of matrix :

$$f(X) = \mathrm{tr}(X), \quad X \in \mathbb{R}^{n \times n}$$

$$f(X) = \det(X), \quad X \in \mathbb{R}^{n \times n}$$

$$f(X) = \mathrm{tr}(X^{-1}), \quad X \in GL(n)$$

$$f(X) = \langle A, X \rangle + b, \quad X \in \mathbb{R}^{m \times n}$$

Affine function

where $\langle A, X \rangle := \mathrm{tr}(A^T X)$

matrix Euclidean/ Frobenius inner product

$$f(X) = \underbrace{\rho(X)}_{\text{spectral radius of matrix } X \in \mathbb{R}^{n\times n}} := \max_{i=1,\dots,n} |\lambda_i(X)|$$

$$f(X) = \lambda_{min}(X), \quad X \in S^n.$$

<u>Matrix norms:</u> $\boxed{X \in \mathbb{R}^{m\times n}}$

Frobenius or Hilbert-Schmidt norm:

$$\|X\|_F = \sqrt{\langle X, X\rangle} = \sqrt{tr(X^T X)} = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}^2}$$

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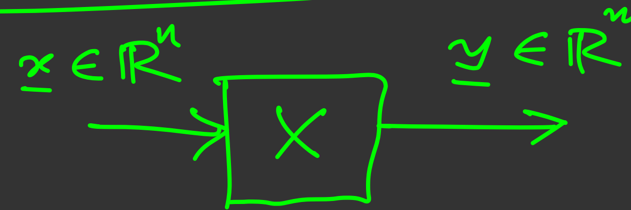
Induced Matrix Norms / Operator matrix norm:

$$\|X\|_p = \max_{\underline{x} \in \mathbb{R}^n \setminus \{0\}} \frac{\|\underline{y}\|_p}{\|\underline{x}\|_p}$$



$$= \max_{\|\underline{x}\|_p = 1} \|X \underline{x}\|_p$$

Special cases:

$$\|X\|_1 = \max_{j=1,\dots,n} \sum_{i=1}^{m} |X_{ij}|$$ → maximum col. sum

$$\|X\|_\infty = \max_{i=1,\dots,m} \sum_{j=1}^{n} |X_{ij}|$$ → maximum row sum

$$\|X\|_2 = \sqrt{\lambda_{max}(X^T X)} = \underbrace{\sigma_{max}(X)}_{\text{maximum singular value}}$$

$\|X\|_2$ ← spectral norm

Singular values of $X \in \mathbb{R}^{m \times n}$:

$$\sigma_i(X) := \sqrt{\lambda_i(X^T X)}$$

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0 \quad \Big\} \text{ order the singular values of } X$$

$\#$ of nonzero singular values $= \text{rank}(X)$

$$\text{rank}(X) \leq \min\{m, n\}$$

if equality then full rank.

Highly popular Misconceptions

#1 Function f is convex (Subtle!)

Convexity is a topological notion/idea, NOT a geometric notion. Convexity depends on BOTH f and $\boxed{\text{domain}(f)}$ (denoted as $\text{dom}(f)$)

By changing $\text{dom}(f)$, the same function f maybe convex or non-convex!!