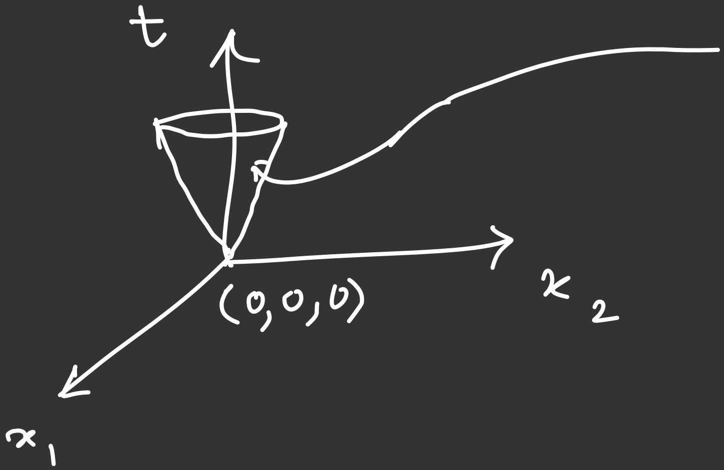
Lec. 6 (10/11/2022)

Euclidean or 2-norm cone for $n = 2$:

$$\mathcal{S} := \{ (x_1, x_2, t) \in \mathbb{R}^3 \mid \sqrt{x_1^2 + x_2^2} \leq t \}$$



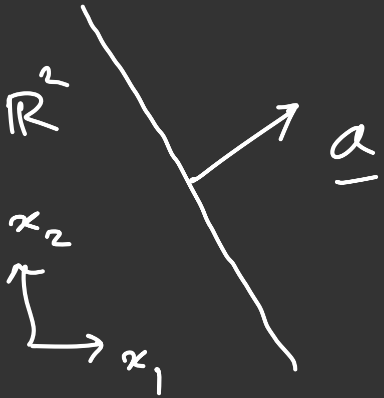
surface for this set $\mathcal{S}$

This cone goes by many names in different disciplines:

- 2-norm cone
- second order cone
- Lorentz cone
- Ice cream cone

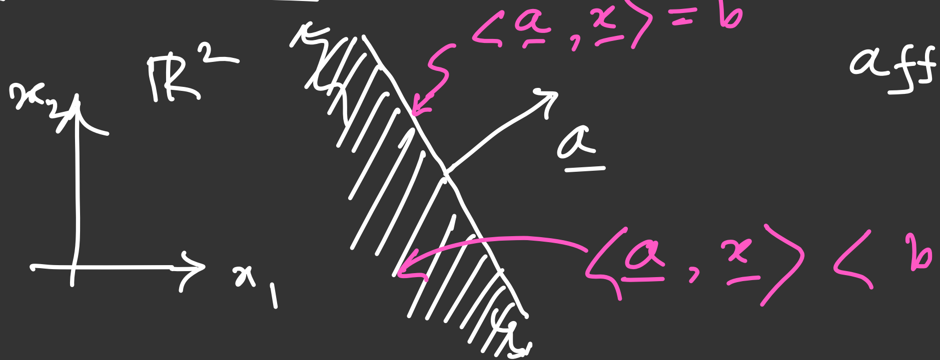
Hyperplane :

$$\{\underline{x} \in \mathbb{R}^n \mid \underbrace{\langle \underline{a}, \underline{x} \rangle = b}_{\text{affine equality}}, \ \underline{a} \in \mathbb{R}^n \setminus \{\underline{0}\}, \ b \in \mathbb{R}\}$$



Halfspace :

$$\{\underline{x} \in \mathbb{R}^n \mid \underbrace{\langle \underline{a}, \underline{x} \rangle \leq b}_{\text{affine inequality}}, \ \underline{a} \in \mathbb{R}^n \setminus \{\underline{0}\}, \ b \in \mathbb{R}\}$$



- Hyperplanes → affine sets
  → all affine sets are convex, so are the hyperplanes.
- Halfspaces → also convex sets (same argument)
  (NOT affine)

- Euclidean ball:

$$B(\underline{x}_c, r) := \{\underline{x} \in \mathbb{R}^n \mid \|\underline{x} - \underline{x}_c\|_2 \le r\}$$

radius $\in \mathbb{R}_+$

center $\in \mathbb{R}^n$ vector

$$= \{\underline{x}_c + r\underline{u} \mid \|\underline{u}\|_2 \le 1\}$$

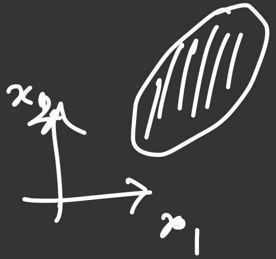
convex set

But sphere (boundary) is nonconvex.

- <u>Ellipsoid</u>:

$$\mathcal{E}(\underset{\substack{\uparrow \\ \text{center vector} \\ \in \mathbb{R}^n}}{\underline{x}_c}, \underset{\substack{\uparrow \\ \text{shape matrix} \\ \in \mathbb{S}^n_{++}}}{P}) := \left\{ \underline{x} \in \mathbb{R}^n \;\middle|\; (\underline{x}-\underline{x}_c)^T P^{-1} (\underline{x}-\underline{x}_c) \leq 1 \right\}$$

Clearly, if $P = r^2 I$ then:



$$= \left\{ \underline{x}_c + M\underline{u} \;\middle|\; \|\underline{u}\|_2 \leq 1 \right\}$$

where $\underset{\substack{\uparrow \\ \text{square root} \\ \text{of matrix } P}}{M} = P^{1/2} \Leftrightarrow MM = P$

e.g., $P = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \in \mathbb{S}^2_{++}$

$a, b \neq 0$

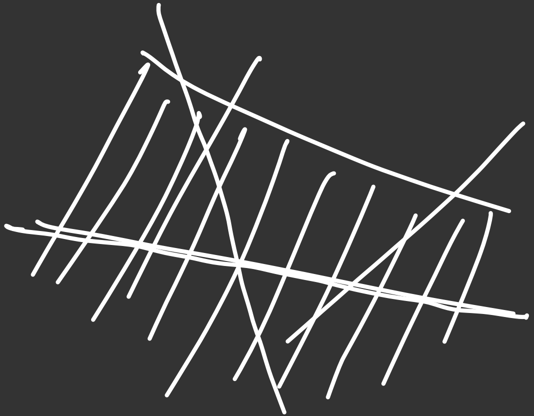
- Polyhedron : Solution set of finite number of affine equations and affine inequalities :

$$P := \left\{ \underline{x} \in \mathbb{R}^n \;\middle|\; \overbrace{\boxed{\underline{a}_i^T \underline{x} \leq b_i, \forall i = 1, \dots, m,}}^{\text{halfspaces}} \; \underbrace{\boxed{\underline{c}_j^T \underline{x} = d_j, \forall j = 1, \dots, p}}_{\text{hyperplanes}} \right\}$$

$$= \left\{ \underline{x} \in \mathbb{R}^n \;\middle|\; A \underline{x} \overset{\text{elementwise}}{\preceq} \underline{b}, \; C \underline{x} = \underline{d} \right\}$$

where $\underbrace{A}_{m \times n} = \begin{pmatrix} \underline{a}_1^T \\ \vdots \\ \underline{a}_m^T \end{pmatrix}$, $\underbrace{C}_{p \times n} = \begin{pmatrix} \underline{c}_1^T \\ \vdots \\ \underline{c}_p^T \end{pmatrix}$

= Finite intersections of halfspaces and hyperplanes



Example:

$$\underbrace{\mathbb{R}^n_+}_{\text{convex cone}} = \{ \underline{x} \in \mathbb{R}^n \mid x_i \geq 0, \; i = 1, \dots, n \}$$

$$= \underbrace{\{ \underline{x} \in \mathbb{R}^n \mid \underline{x} \geq \underline{0} \}}$$

intersection of n halfspaces

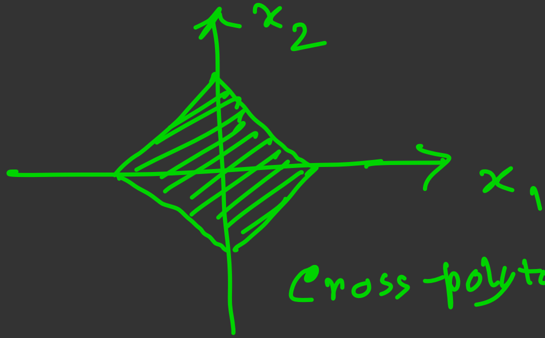
$\Rightarrow$ a polyhedron

Example:

$\overset{\text{unit}}{p}$-norm balls, $p \geq 1$.

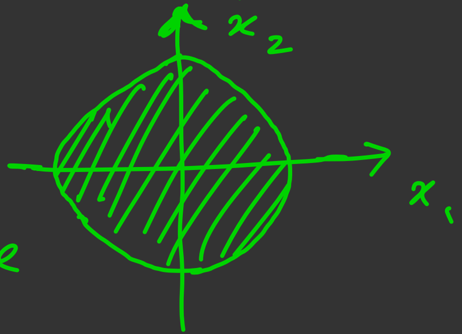
$$\left.\begin{array}{c}\text{unit } p\text{-norm balls, } p \geq 1. \\ \underbrace{\{\underline{x} \in \mathbb{R}^n \mid \|\underline{x}\|_p \leq 1\}}_{\text{unit } p\text{-norm ball}}\end{array}\right\} \text{convex sets}$$

Fix $n = 2$

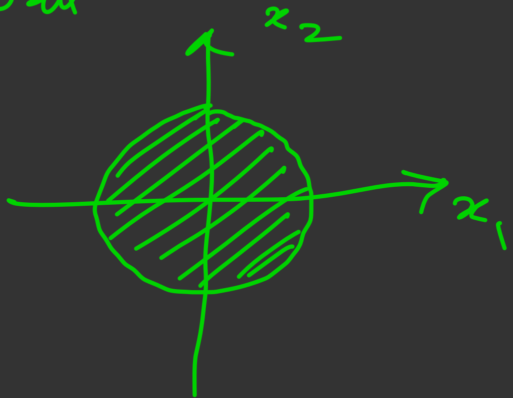


Cross-polytope

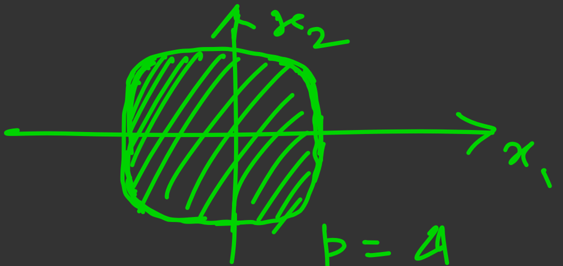
$p = 1$



$p = 1.5$



$p = 2$



$p = 4$

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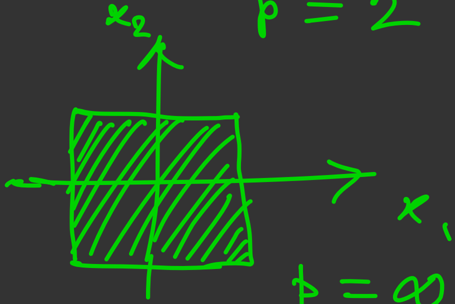

$p = \infty$

Hint for proof of convexity:

use Minkowski inequality: $\|\underline{u}+\underline{v}\|_p \le \|\underline{u}\|_p + \|\underline{v}\|_p$

$\forall\, \underline{u}, \underline{v} \in \mathbb{R}^n,\ p \geq 1.$

& then the def^n. of convex set:

$\theta \underline{x}_1 + (1-\theta)\underline{x}_2$ etc.

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1 norm ball & ∞-norm balls are polyhedra

$\{\underline{x}\in\mathbb{R}^n \mid |x_1|+\dots+|x_n| \le 1\}$

$\{\underline{x}\in\mathbb{R}^n \mid \max_{i=1,\dots,n} |x_i| \le 1\}$

$$= \{(x_i, y_i)_{i=1}^n \mid -y_i \le x_i \le +y_i,\ \forall i=1,\dots,n$$

$$y_1+\dots+y_n \le 1\}$$

← $(2n+1)$ linear inequalities in $2n$ variables

$\Leftrightarrow$ $\therefore$ 1-norm ball is the intersection of $(2n+1)$ halfspaces in $\mathbb{R}^{2n}$

Why $|x_1| + |x_2| + \dots + |x_n| \leq 1$

$\Leftrightarrow$ $-y_i \leq x_i \leq +y_i$, $\sum_{i=1}^{n} y_i \leq 1$

Because: introduce new variables y_i such that:

$|x_i| \leq y_i$ $\qquad \sum_{i=1}^{n} |x_i| \leq 1$

$\Leftrightarrow$ $-y_i \leq x_i \leq +y_i$ $\Leftrightarrow$ $\sum_{i=1}^{n} y_i \leq 1$

- ∞ - norm ball:

Use the fact that

$$\max_{i=1,\dots,n} |x_i| \le 1 \iff \underbrace{|x_i| \le 1}\ \forall i=1,\dots,n$$

$$\iff -1 \le x_i \le +1$$

$$\forall i=1,\dots,n$$

$\therefore$ ∞-norm ball is the intersection of $2n$ halfspaces in $\mathbb{R}^n$.

$\therefore$ Polyhedron.