PDF evolution for the unforced/uncontrolled dynamics: $(u_1 \equiv u_2 \equiv u_3 \equiv 0)$

$\boxed{\text{Axi-symmetric case}}$: $(J_1 = J_2 \neq J_3)$

$$x_3(t) = x_{30} \quad \text{(constant)}$$

$$\dot{x}_1 = \alpha_1 x_{30} x_2$$

$$\Rightarrow \ddot{x}_1 = \alpha_1 x_{30} \dot{x}_2$$

$$= \alpha_1 x_{30} (\alpha_2 x_{30} x_1)$$

$$= \underbrace{\alpha_1}_{-\alpha_2} \alpha_2 x_{30}^2 x_1$$

$$= -(\alpha_2^2 x_{30}^2) x_1 \quad \text{(simple harmonic oscillator with ang. freq. } \omega := \alpha_2 x_{30})$$

$$\Rightarrow x_1(t) = A \cos(\alpha_2 x_{30} t + \phi)$$

Determine A, ϕ from IC : $x_1(t=0) = x_{10}$, $\dot{x}_1(0) = \alpha_1 x_{30} x_{20}$

This yields the flow map $(x_{10}, x_{20}, x_{30}) \mapsto (x_1, x_2, x_3)$ given by :

$$\left.\begin{aligned} x_1 &= A\cos(\alpha_2 x_{30} t + \phi) \\ x_2 &= A\sin(\alpha_2 x_{30} t + \phi) \\ x_3 &= x_{30} \end{aligned}\right\}$$

where

$$A = \sqrt{x_{10}^2 + x_{20}^2}$$

$$\phi = \arctan\left(\frac{x_{20}}{x_{10}}\right)$$

------ $(\cancel{*}**)$

But the sol$^{\underline{n}}$ of the Liouville PDE, in this case, is

$$\rho(t, x_1, x_2, x_3) = \rho_0(\underbrace{x_{10}, x_{20}, x_{30}}_{\substack{\text{need inverse}\\ \text{flow map to}\\ \text{substitute here}}}) \cdot \underbrace{1}_{(\because \text{div.} = 0)}$$

To obtain inverse flow map from :

$$x_1 = A\cos(\omega t + \phi)$$
$$x_2 = A\sin(\omega t + \phi)$$
where ω depends on x_3

we get A, ϕ in terms of x_1, x_2 :

$$\tan(\omega t + \phi) = \frac{x_2}{x_1} \Rightarrow \phi = \arctan\left(\frac{x_2}{x_1}\right) - \omega t$$

and $A = \sqrt{x_1^2 + x_2^2}$

Finally, back out A, ϕ to x_{10}, x_{20} using $\phi = \arctan\left(\frac{x_{20}}{x_{10}}\right)$ and $A = \sqrt{x_{10}^2 + x_{20}^2}$:

$$\omega t = \arctan\left(\frac{x_2}{x_1}\right) - \arctan\left(\frac{x_{20}}{x_{10}}\right)$$

$$\Rightarrow \tan(\omega t) = \frac{x_2/x_1 - \cancel{x_{20}/x_{10}}^{\,c_0}}{1 + \frac{x_2}{x_1}\cancel{\frac{x_{20}}{x_{10}}}^{\,c_0}}$$

c_0 (say,) From here, solve for c_0

$$\Rightarrow \frac{x_{20}}{x_{10}} = \underbrace{\frac{x_2 - x_1 \tan(\omega t)}{x_1 + x_2 \tan(\omega t)}}_{\text{call it } \gamma}$$

$$\Rightarrow \boxed{x_{20} = \gamma x_{10}}$$

OTOH:

$$A = \sqrt{x_{10}^2 + x_{20}^2} = \sqrt{x_1^2 + x_2^2}$$

$$\Rightarrow x_{10}\sqrt{1+\gamma^2} = \sqrt{x_1^2 + x_2^2}$$

$$\Rightarrow x_{10} = \sqrt{\frac{x_1^2 + x_2^2}{1+\gamma^2}}$$

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$\therefore$ The inverse flow map $(x_1, x_2, x_3) \mapsto (x_{10}, x_{20}, x_{30})$ is :

$$x_{10} = \sqrt{\frac{x_1^2 + x_2^2}{1+\gamma^2}}$$

$$x_{20} = \gamma\, x_{10} = \gamma \sqrt{\frac{x_1^2 + x_2^2}{1+\gamma^2}}$$

$$x_{30} = x_3$$

where $\gamma := \dfrac{x_2 - x_1 \tan(\alpha_2 x_3 t)}{x_1 + x_2 \tan(\alpha_2 x_3 t)}$.

Hence the transient joint PDF :

$$\boxed{\rho(t, x_1, x_2, x_3) = \rho_0\left(\sqrt{\frac{x_1^2+x_2^2}{1+\gamma^2}},\ \gamma\sqrt{\frac{x_1^2+x_2^2}{1+\gamma^2}},\ x_3\right)}.$$

Plot $\rho_0 \equiv \mathcal{N}(0, I)$ trivariate std. normal $\to \rho(t, \cdot, \cdot, \cdot)$ (3D contours ?)

$\boxed{\text{Generic case}}$: $J_1 \neq J_2 \neq J_3$

In this case, the flow map (see e.g. [Landau-Lifschitz, vol. 1, eq. (37.10)])

$$\left.\begin{aligned} \omega_1 &= \overline{\omega}_{10}\, cn(\omega_p t + \varepsilon, m) \\ \omega_2 &= \overline{\omega}_{20}\, sn(\omega_p t + \varepsilon, m) \\ \omega_3 &= \overline{\omega}_{30}\, dn(\omega_p t + \varepsilon, m) \end{aligned}\right\}$$

where $cn(\cdot)$, $sn(\cdot)$, $dn(\cdot)$ are Jacobi elliptic f-s

The $\overline{\omega}_{i0}$, ω_p, ε, m only depend on IC

Need to compute inverse flow map to be used in

$$\rho(t, x_1, x_2, x_3) = \rho_0(x_{10}, x_{20}, x_{30}) \cdot \underset{\substack{\uparrow \\ \text{since} \\ \text{div} = 0}}{1}$$