Problem 1. [2 x 4 = 8 points] Activation Function and Sector Nonlinearity

In Lec. 11, p. 14-16, we introduced sector nonlinearity. In this exercise, we examine some concrete examples/non-examples.

Explain which of the following commonly used neural network activation functions $\sigma : \mathbb{R}^m \mapsto \mathbb{R}^m$, are sector bounded and which are not? If any of these are sector-bounded then **derive** the corresponding sectors $[\alpha, \beta]$ as in Lec. 11, p. 15-16.

(a) ReLU activation $\sigma(x) = \max\{0_{m\times 1}, x\}$ where $x \in \mathbb{R}^m$ and $\max\{\cdot,\cdot\}$ is elementwise.

(b) Leaky ReLU activation $\sigma(x) = \max\{ax, x\}$ where $x \in \mathbb{R}^m$, $a > 0$, and $\max\{\cdot,\cdot\}$ is elementwise.

(c) Sigmoid activation $\sigma(x) = \exp(x) \oslash (1 + \exp(x))$ where $x \in \mathbb{R}^m$, 1 is all-ones column vector, $\oslash$ denotes elementwise division, and $\exp(\cdot)$ is elementwise.

(d) Softmax activation $\sigma(x) = \dfrac{\exp(x)}{1^\top \exp(x)}$ where $x \in \mathbb{R}^m$, 1 is all-ones column vector, and $\exp(\cdot)$ is elementwise.

a. It **is** sector bounded, with sector [0, 1]

1. Consider the function $f(x) = \max\{0_{m\times 1}, x\}(\max\{0_{m\times 1}, x\} - x)$
2. When $x \leq 0, f(x) = 0(0 - x) = 0 \leq 0$
3. When $x > 0, f(x) = x(x - x) = 0 \leq 0$
4. So we see that $\forall x, f(x) = \max\{0_{m\times 1}, x\}(\max\{0_{m\times 1}, x\} - x) \leq 0$
5. Therefore, by Lec 11, pg 16, the sector bound is [0, 1]

b. It **is** sector bounded, with sector [0, ∞]

1. When $x \leq 0, \max\{ax, x\}x = xx = x^2 \geq 0$
2. When $x > 0, \max\{ax, x\}x = axx = ax^2 \geq 0$
3. From above, we see that $\forall x, \max\{ax, x\}x \geq 0$
4. From Lec 11, pg 16, the sector bound is [0, ∞]

c. **Not** sector bounded

d. **Not** sector bounded

Problem 2. [50 points] Feedback Stabilization

Consider the scalar control system $\dot{x} = -x^3 + u$. We want to design (static) state feedback control $u = u(x)$ such that origin of the closed-loop system is GAS. We will design multiple stabilizing controllers for this system, and compare their performance.

(a) **[5 points]** Design a **feedback linearizing controller** $u_{FL}(x)$ by applying "cancel the nonlinearity and get a stable linear closed-loop system" idea.

1. Let $u_{FL}(x) = x^3 - x$, then $\dot{x} = -x$
2. Let $V(x) = \frac{x^2}{2}, \dot{V(x)} = x(-x) = -x^2$
3. So by Lasalle, origin is **A.S.**, and since V is radially unbounded, origin is **G.A.S.**

(b) **[5 points]** Prove that a **linear feedback controller** $u_L(x) = -x$ also makes the origin of the closed-loop system GAS. You will need to use the Barbashin-Krtasovskii theorem.

1. $u_L(x) = -x$, so $\dot{x} = -x^3 - x$
2. Let $V(x) = \frac{x^2}{2}, \quad \dot{V(x)} = -(x^4 + x^2)$, negative semi-definite.
3. Let $S = \{x \quad s.t. \dot{V} = 0\}$
4. Observe

$$
\begin{aligned}
x \quad s.t. \dot{V} &= 0 \qquad (1)\\
\Rightarrow -(x^4 + x^2) &= 0 \qquad (2)\\
\Rightarrow x^4 + x^2 &= 0 \qquad (3)\\
\Rightarrow x^2(x^2 + 1) &= 0 \qquad (4)\\
\Rightarrow x &= 0 \qquad (5)
\end{aligned}
$$

5. From 4., $x \in S \Rightarrow x = 0$, so by Barbashin-Krtasovskii, origin is **A.S.**
6. In addition, since V is radially unbounded, origin is **G.A.S.**

(c) **[5 + 5 = 10 points] Give two reasons** why the controller $u_L(x)$ in part (b) is a better controller than $u_F L(x)$ in part (a). (**Hint:** think rate-of-convergence of the closed-loop system, and magnitude of control signal for large x. For the latter, you may find it insightful to plot $|u|$ as function of x.)

1. For $\|x\| \geq 0$, $\|\dot{x_L}\| > \|\dot{x_{FL}}\|$, so $x_L(x)$ converges faster than $x_{FL}(x)$.
2. $u_L(x)$ is less controller signal magnitude, so less risk of saturating / wrap-around issues or numerical issues. You can see the plots below to see this.

Magnitude plot

(d) **[5 points]** The answer in part (c) tells us that it is better not to kill "friendly nonlinearity". Consider another design idea: **doing nothing controller**, i.e., $u_0(x) \equiv 0$ for all $x \in \mathbb{R}$. Prove that $u_0(x)$ also makes the origin GAS.

1. For the **unforced system**, $\dot{x} = -x^3$
2. Let $V(x) = \frac{x^2}{2}, \quad \dot{V(x)} = -x^4$, negative semi-definite.
3. Let $S = \{x \quad s.t. \dot{V} = -x^4 = 0\} = \{x \quad s.t. x = 0\}$
4. From 3. and Barbashin-Krtasovskii, origin is **A.S.** and since V is **radially unbounded**, origin is also **G.A.S.**

(e) **[5 + 5 = 10 points] Give one advantage and one disadvantage** of $u_0(x)$ compared to $u_L(x)$. Again think in terms of the hint in part (c).

1. The advantage of $u_0(x)$ is that the controller output magnitude is always **0**.
2. The disadvantage of $u_0(x)$ is that it converges slower because there is no additional $-x$ input to $\dot{x}$ to aid in convergence.

(f) **[5 points]** Design another globally asymptotically stabilizing controller $u_S(x)$ using **Sontag's formula** in Lec. 13. For this purpose, you need to use a CLF: use what comes to your mind without much thought.

1. Let $V(x) = \frac{x^2}{2}$
2. $\langle \nabla_x V, g \rangle = x$
3. $\langle \nabla_x V, f \rangle = -x^4$

4. $u_s(x) = \psi(x) = 0$ if $\langle \nabla_x V, g \rangle = x = 0$
5. Otherwise, $u_s(x) = \psi(x) = -\frac{-x^4 + \sqrt{(-x^4)^2 + x^4}}{x} = x = x^3 - x\sqrt{x^4 + 1}$

(g) **[5 + 5 = 10 points]** From your answer in part (f), **argue that** near $x = 0$, we have $u_S(x) \approx u_L(x)$; and for $|x| \to \infty$, we have $u_S(x) \approx u_0(x)$, and therefore, $u_S(x)$ outperforms all the previous controllers.

In a single figure, **plot all the four controllers** as functions of x.

1. As $x \to 0$

$$
\begin{aligned}
&x^3 - x\sqrt{x^4 + 1} \qquad (6)\\
&\approx 0 - x\sqrt{0 + 1} \qquad (7)\\
&\approx -x = u_L(x) \qquad (8)
\end{aligned}
$$

2. 1 gives us **good convergence**, as shown above
3. As $x \to \infty$

$$
\begin{aligned}
&x^3 - x\sqrt{x^4 + 1} \qquad (9)\\
&\approx \infty - x\sqrt{\infty} \qquad (10)\\
&\approx \infty - \infty \qquad (11)\\
&\approx 0 = u_0(x) \qquad (12)
\end{aligned}
$$

4. 3 gives us **good controller magnitude**, as shown above
5. By 2. and 4., this controller is better than prior ones.

all controllers

Problem 3. [42 points] Backstepping

Consider the following 3 state control system which is a modification of the worked out example in Lec. 13, p. 14-16, with an additional integrator at the input side:

$$
\begin{aligned}
\dot{x}_1 &= x_1^2 - x_1^3 + x_2,\\
\dot{x}_2 &= x_3,\\
\dot{x}_3 &= u.
\end{aligned}
$$

(a) [20 points] Controller synthesis

Design an integrator backstepping controller to make the origin GAS. In other words, find the feedback $u(x_1, x_2, x_3)$ and the overall Lyapunov certificate $V(x_1, x_2, x_3)$.

1. Consider

$$
\begin{aligned}
V_1 &= \frac{x_1^2}{2} \qquad (13)\\
\dot{V_1} &= x_1(x_1^2 - x_1^3 + x_2) \qquad (14)
\end{aligned}
$$

2. We want for a positive definite $W_1 = x_1^2$

$$\dot{V_1} = x_1(x_1^2 - x_1^3 + x_2) \leq -W_1(x_1) \qquad (15)$$

3. To the goal in 2, we assert a pseudocontrol $\phi_1(x_1) = -x_1^2 - x_1$ so that when $x_2 = \phi_1(x_1)$:

$$
\begin{aligned}
\dot{V_1} &= x_1(x_1^2 - x_1^3 + \phi_1) \qquad (16)\\
\dot{V_1} &= x_1(x_1^2 - x_1^3 - x_1^2 - x1) \qquad (17)\\
\dot{V_1} &= x_1(-x_1^3 - x1) \qquad (18)\\
\dot{V_1} &= -x_1^4 - x_1^2 \qquad (19)\\
\dot{V_1} &\leq -W_1(x_1) \qquad (20)
\end{aligned}
$$

4. Let $z_2 = x_2 - \phi_1(x_1)$, then with this change of variable the system becomes:

$$
\begin{aligned}
x_2 &= z_2 + \phi_1(x_1) \qquad (21)\\
\dot{x_1} &= x_1^2 - x_1^3 + z_2 + \phi_1(x_1) = -x_1^3 - x_1 + z_2 \qquad (22)\\
\dot{z_2} &= \dot{x_2} - \dot{\phi}_1(x_1) = x_3 - \dot{\phi}_1(x_1) \qquad (23)\\
\dot{x_3} &= u \qquad (24)
\end{aligned}
$$

   Where $\dot{\phi}_1(x_1) = (-2x_1 - 1)(x_1^2 - x_1^3 + x_2)$
5. Now consider $V_2(x_1, z_2) = V_1 + \frac{z_2^2}{2} = \frac{x_1^2}{2} + \frac{z_2^2}{2}$, again we want $\dot{V_2} \leq -W_2(x_1, z_2)$ for some positive definite function $W_2(x_1, z_2) = x_1^2$
6. We expand $\dot{V_2}(x_1, z_2)$:

$$
\begin{aligned}
\dot{V_2} &= \langle \nabla_{x_1} V_2, \dot{x_1} \rangle + \langle \nabla_{z_2} V_2, \dot{z_2} \rangle \qquad (25)\\
&= x_1(-x_1^3 - x_1 + z_2) + z_2(\dot{x_2} - \dot{\phi}_1(x_1)) \qquad (26)\\
&= x_1(-x_1^3 - x_1) + z_2(x_1 + x_3 - \dot{\phi}_1(x_1)) \qquad (27)
\end{aligned}
$$

7. From 6 we see that if we want $\dot{V_2} \leq -W_2(x_1, z_2)$ for a positive definite $W_2(x_1, z_2)$, we can assert a pseudocontrol $\phi_2(x_1, z_2)$ so that when $x_3 = \phi_2(x_1, z_2)$, $\dot{V_2} = x_1(-x_1^3 - x_1) + z_2(-z_2)$
8. For 7 to be true:

$$
\begin{aligned}
x_1 + \phi_2(x_1, z_2) - \dot{\phi}_1(x_1) &= -z_2 \qquad (28)\\
\phi_2(x_1, z_2) &= -z_2 - x_1 + \dot{\phi}_1(x_1) \qquad (29)
\end{aligned}
$$

9. Now let's define another change of variable let $z_3 = x_3 - \phi_2(x_1, z_2)$, then:

$$
\begin{aligned}
z_3 &= x_3 - \phi_2(x_1, z_2) \qquad (30)\\
\dot{z_3} &= \dot{x_3} - \dot{\phi}_2(x_1, z_2) \qquad (31)\\
\dot{z_3} &= u - \dot{\phi}_2(x_1, z_2) \qquad (32)
\end{aligned}
$$

   Furthermore:

$$
\begin{aligned}
z_3 &= x_3 - \phi_2(x_1, z_2) \qquad (33)\\
x_3 &= z_3 + \phi_2(x_1, z_2) \qquad (34)
\end{aligned}
$$

   And:

$$
\begin{aligned}
\dot{z_2} &= \dot{x_2} - \dot{\phi}_1(x_1) \qquad (35)\\
\dot{z_2} &= x_3 - \dot{\phi}_1(x_1) \qquad (36)\\
\dot{z_2} &= z_3 + \phi_2(x_1, z_2) - \dot{\phi}_1(x_1) \qquad (37)\\
\dot{z_2} &= z_3 + (-z_2 - x_1 + \dot{\phi}_1(x_1)) - \dot{\phi}_1(x_1) \qquad (38)\\
\dot{z_2} &= z_3 - z_2 - x_1 \qquad (39)
\end{aligned}
$$

   Then the new system is:

$$
\begin{aligned}
\dot{x_1} &= -x_1^3 - x_1 + z_2 \qquad (40)\\
\dot{z_2} &= z_3 - z_2 - x_1 \qquad (41)\\
\dot{z_3} &= u - \dot{\phi}_2(x_1, z_2) \qquad (42)
\end{aligned}
$$

10. Now consider $V_3(x_1, z_2, z_3) = V_2 + \frac{z_3^2}{2} = \frac{x_1^2}{2} + \frac{z_2^2}{2} + \frac{z_3^2}{2}$. Then:

$$
\begin{aligned}
\dot{V_3} &= \langle \nabla_{x_1} V_3, \dot{x_1} \rangle + \langle \nabla_{z_2} V_3, \dot{z_2} \rangle + \langle \nabla_{z_3} V_3, \dot{z_3} \rangle \qquad (43)\\
\dot{V_3} &= x_1\dot{x_1} + z_2\dot{z_2} + z_3\dot{z_3} \qquad (44)\\
\dot{V_3} &= x_1(-x_1^3 - x_1 + z_2) + z_2(z_3 - z_2 - x_1) + z_3(u - \dot{\phi}_2(x_1, z_2)) \qquad (45)\\
\dot{V_3} &= x_1(-x_1^3 - x_1) + x_1 z_2 + z_2(-x_1) + z_2(z_3 - z_2) + z_3(u - \dot{\phi}_2(x_1, z_2)) \qquad (46)\\
\dot{V_3} &= x_1(-x_1^3 - x_1) + z_2(-z_2) + z_3(z_2 + u - \dot{\phi}_2(x_1, z_2)) \qquad (47)\\
\dot{V_3} &= x_1(-x_1^3 - x_1) - z_2^2 + z_3(z_2 + u - \dot{\phi}_2(x_1, z_2)) \qquad (48)
\end{aligned}
$$

11. From 10, we see that if we want $\dot{V_3} \leq -W_3(x_1, z_2, z_3)$ for a positive definite $W_3(x_1, z_2, z_3)$, we can **derive u** so that:

$$
\begin{aligned}
z_2 + u - \dot{\phi}_2(x_1, z_2) &= -z_3 \qquad (49)\\
u &= -z_3 - z_2 + \dot{\phi}_2(x_1, z_2) \qquad (50)
\end{aligned}
$$

    Where

$$\dot{\phi}_2(x_1, z_2) = \langle \nabla_{x_1} \phi_2, \dot{x_1} \rangle + \langle \nabla_{z_2} \phi_2, \dot{z_2} \rangle \qquad (51)$$

12. So from 11, if we define $u = -z_3 - z_2 + \dot{\phi}_2(x_1, z_2)$, then:

$$\dot{V_3}(x_1, z_2, z_3) = x_1(-x_1^3 - x_1) - z_2^2 - z_3^2 \qquad (52)$$

    Which we see is a negative of a non-negative function. And our goal in 11:

$$\dot{V_3} \leq -W_3(x_1, z_2, z_3) \qquad (53)$$

    is achieved for a positive definite function $W_3(x_1, z_2, z_3) = x_1^2$

(b) [22 points] Numerical simulation

Use your answer in part (a) to write a MATLAB function `BacksteppingClosedLoop.m` that can be called by the supplied executable `Backstepping.m` in CANVAS Files section. Submit the two plots generated by `Backstepping.m`:

(i) a phase portrait of the closed loop dynamics for 10 randomly generated initial conditions,

(ii) a representative time series plot for a specific controlled trajectory.

The plot commands are already there in `Backstepping.m`. So your job is to correctly implement the function `BacksteppingClosedLoop.m`.

phase portrait

time series plot