Part 3: Supplementary material
Dynamics

Robots as controlled multi-body dynamics systems

Figure 1.1: A serial manipulator (left), the ABB IRB1400, and a parallel manipulator (right), the ABB IRB940Tricept. Photos courtesy of ABB Robotics.

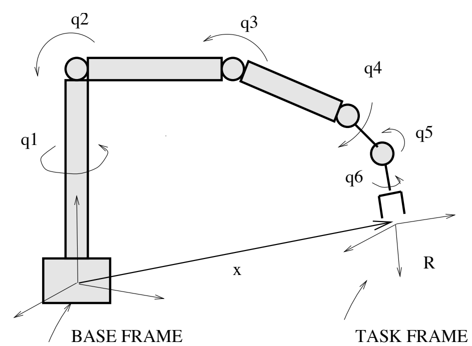

Figure 1.2: A Serial Link Manipulator showing the attached Base Frame, Task Frame, and configuration variables.

Robot Dynamics and Control
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a) b)

Fig. 1. Natural impedance of human legs: a) leg muscles enable running by changing their strength characteristics, b) leg muscles and ligaments as natural body “actuators” and natural “shock-absorbers”.

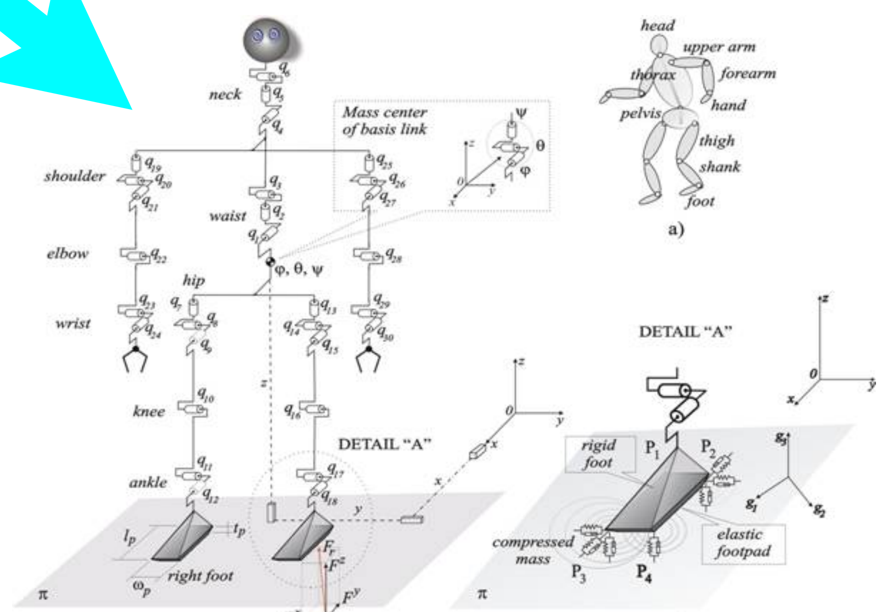

Fig. 3. a) Spinal robot model with two-segments trunk considered in the project, b) Spatial model of a biped robot mechanism interacting with dynamic environement used for verification of the adaptive leg imepdance modulation algorithms, c) compliance model of the ground support used in simulation.

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Newton's Laws

First Law: If there are no forces acting upon a particle, then the particle will move in a straight line with constant velocity.

$$F = 0 \rightarrow v = ct$$

Second Law: A particle acted upon by a force moves so that the force vector is equal to the time rate of change of the linear momentum vector.

$$\underbrace{F = \frac{dp}{dt}, \quad p = mv = m\dot{x}}$$
$$\Downarrow$$
$$F = m\dot{v} = m\ddot{x} = ma$$

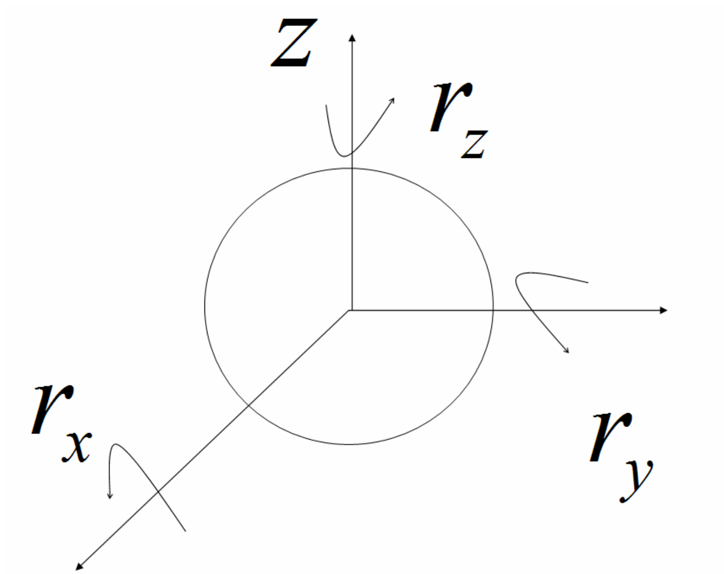
Third Law: When two particles exert forces upon one another, the forces lie along the line joining the particles and the corresponding force vectors are the negative of each other.

Note: 1st and 3rd laws are special cases of 2nd law.

- In the 1st law a=0. In the 3rd one f-R=0 ⇨ a=0
- In the 3nd law F is the balance of forces.

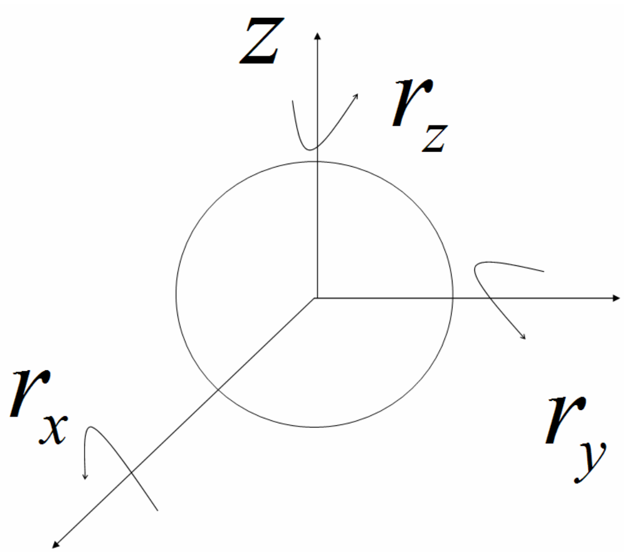
Degrees of Freedom:

1. In physical terms there is nothing to stop the side-to-side motion of the mass.
2. Note that unconstrained matter can undergo three translational motions along x,y and z directions and rotate about each of these degrees of freedom, i.e. rotations rx, ry and rz.
3. Therefore, unconstrained matter can possess 6 degrees of freedom.
4. Such matter do not exist in the universe or in any engineering machine or mechanism.



Degrees of Freedom - Definition

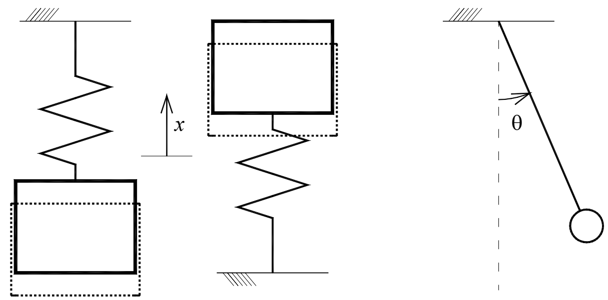
The minimum number of independent coordinates required to determine completely the position of all parts of a system at any instant of time



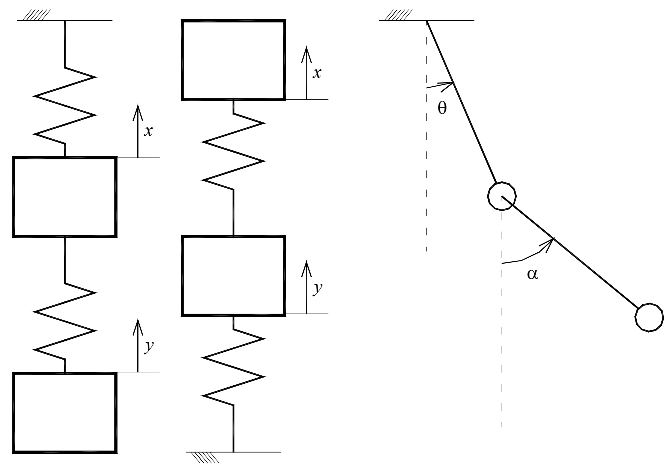
6 degrees of freedom

Degrees of Freedom - Examples”

Single degree of freedom system:

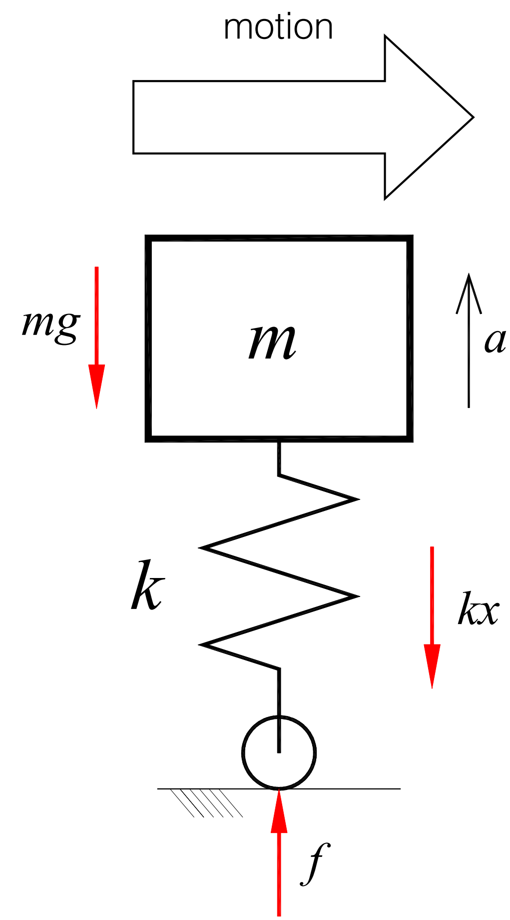


Two degrees of freedom system:



Single Degree of Freedom System

A mass-spring system can represent a suspension system as shown in the figure



Assumptions:

- the spring can act in tension and compression
- *f* is the reaction force
- the load is given the weight *mg* (no other external load is considered)
- the rolling friction is neglected

- Applying Newton's second law of motion:

$$F = ma = f - mg - kx$$

- Thus: $$a = \frac{1}{m}(f - mg - kx)$$

- The second order differential equation of motion is: $$m\ddot{x} = \frac{1}{m}(f - mg - kx)$$

- Note that all accelerated motions are the second order in nature.

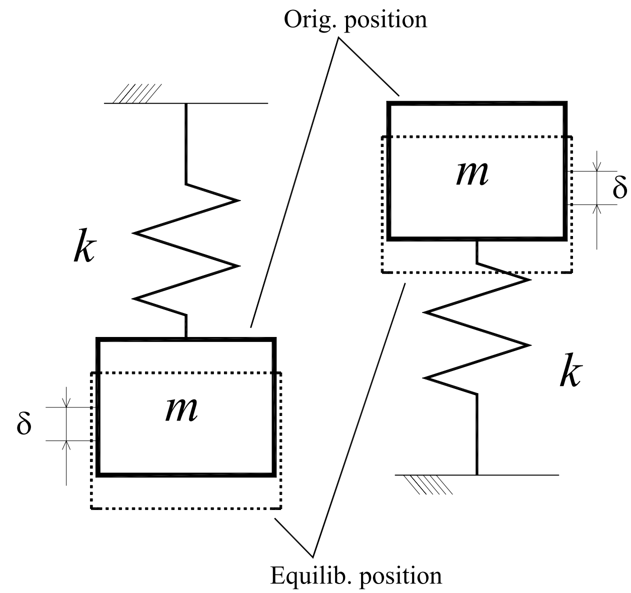
Static Deflection and Natural Frequency:

The equilibrium position is obtained, when:

$$\dot{x} = \ddot{x} = 0$$

Also assume $f=0$.

Then: $mg = k\delta$ or: $\delta = \dfrac{mg}{k}$



Where:

δ: static deflection (i.e. movement of body under its own weight, due to gravity)

We can write the equation of motion for small amplitude oscillations about the equilibrium position as:

$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \omega_n^2 x = 0$$

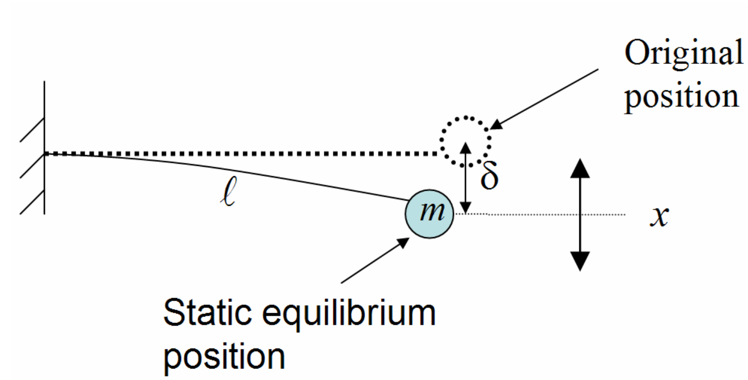
where: $\omega_n = \sqrt{\dfrac{k}{m}} \rightarrow \text{natural frequency}$

A vibrating beam as a Single Degree of Freedom system

Assumption: small vibrations!

(If the vibrations are large, “Euler’s Elastica” model must be used)

- A vibrating beam can also act as a single degree of freedom system. For example, a cantilever beam subjected to an impulse.
- An impulse is a force applied for a very short period

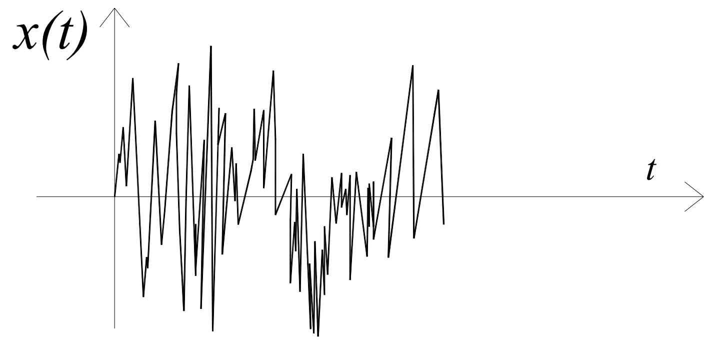


- Remember: $$m\ddot{x} + kx = 0$$
- we need to determine *k*, which is a function of beam static deflection δ.
- The static deflection is movement of the beam due to the attached mass under equilibrium condition.
- When the beam is struck at this location it will oscillate as as shown by x.

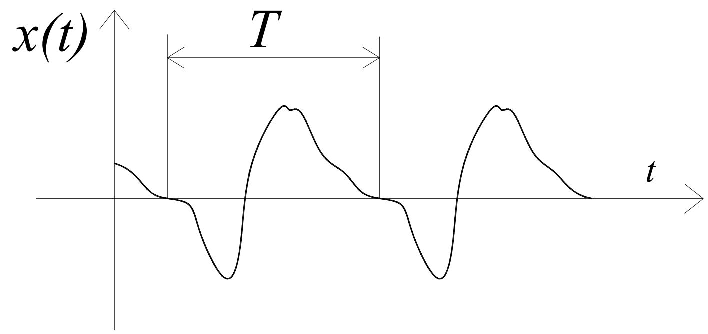
Random, Periodic and Harmonic Motions:

Possible time histories:

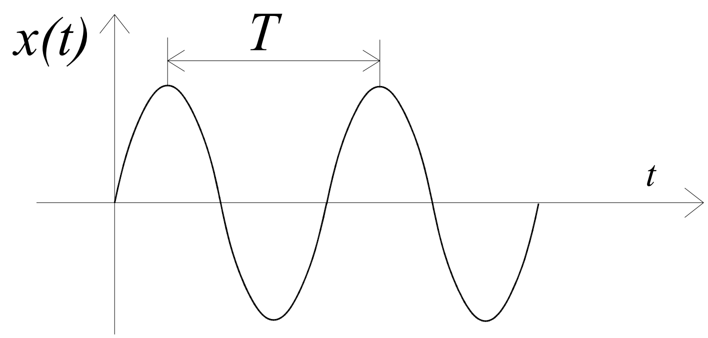
1) Random Motion



2) Periodic Motion

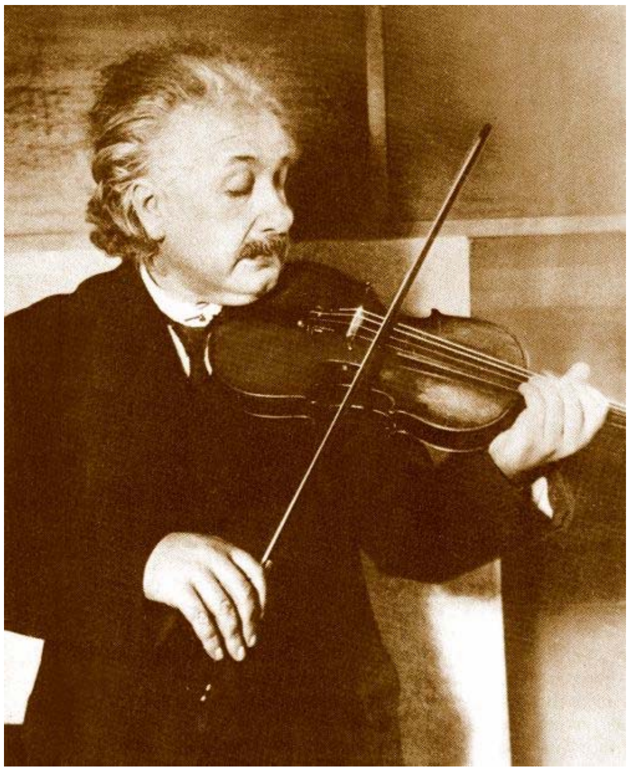


3) Harmonic Motion

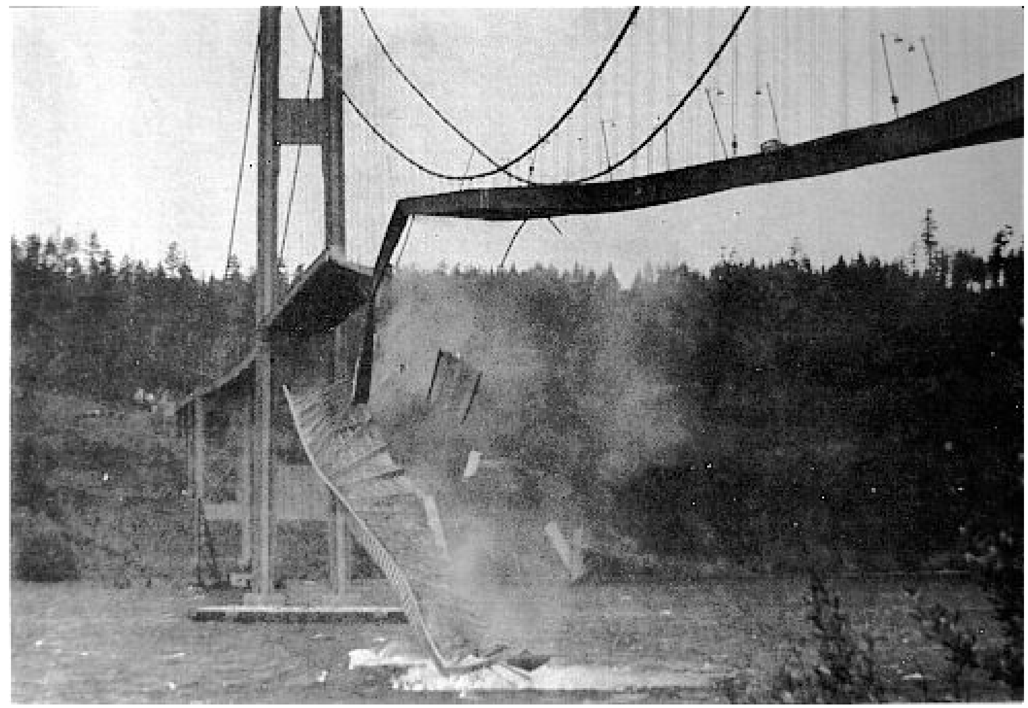


Practical examples:

Good Vibrations



Bad Vibrations
(Tacoma Narrows Bridge)



Two approaches:

There are two approaches, which predict the behavior of a dynamic system:

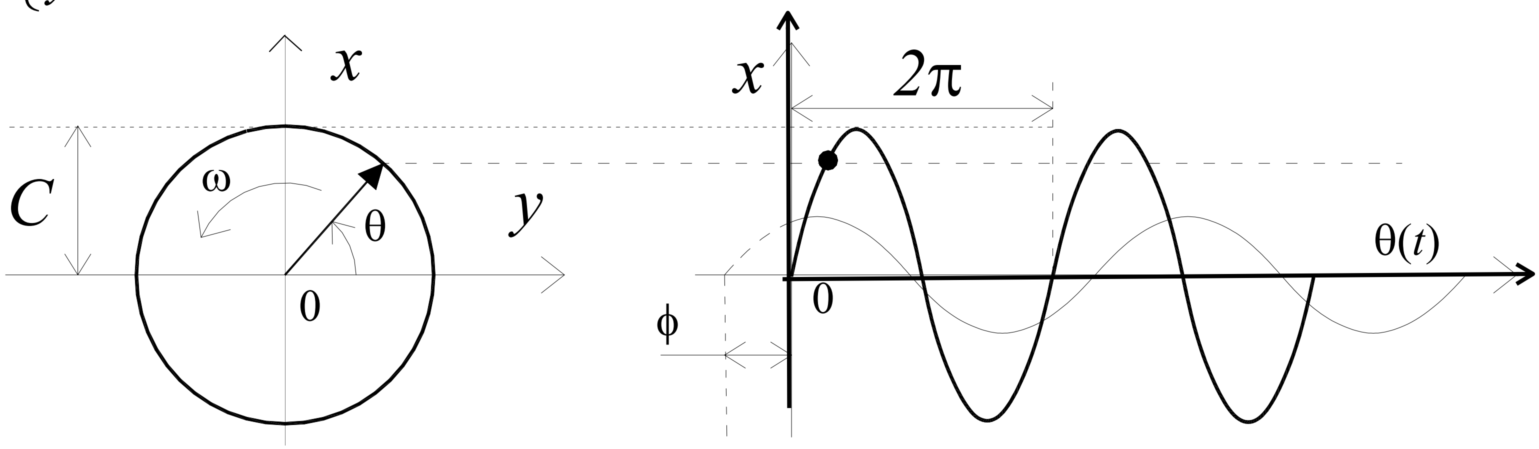
1. Analytic:- fast, but not always applicable

2. Numerical: - slower, but significantly more versatile.

Revision - Vectorial Representation of Harmonic Motion

$$\begin{cases} x = C \sin \omega t \\ y = C \cos \omega t \end{cases}$$

Either projection can be taken to represent a reciprocating motion



Considering the vertical projection:

$$\begin{cases} x = C \sin \omega t \\ v = \dfrac{dx}{dt} = \omega C \cos \omega t \\ a = \dfrac{d^2x}{dt^2} = -\omega^2 C \sin \omega t = -\omega^2 x \end{cases}$$

Definitions:

$T = 2\pi/\omega$	$\rightarrow$ Period
$f = 1/T = \omega/2\pi$	$\rightarrow$ Frequency
$\theta = \omega t$	$\rightarrow$ Angular Displacement
C	$\rightarrow$ Amplitude
ϕ	$\rightarrow$ Phase Angle

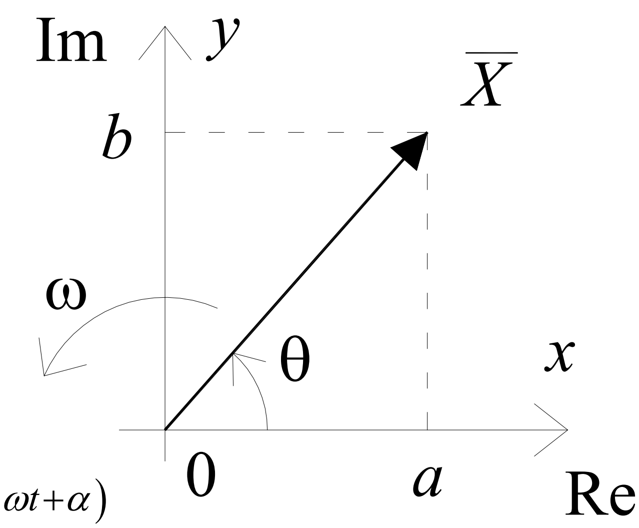
Revision - Complex Representation of Harmonic Motion

Vectorial representation for one single vector (Developed by Leonhard Euler)

$$\bar{X} = a + ib$$

$$\bar{X} = A\cos\theta + iA\sin\theta = Ae^{i\theta}$$

$$A = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \tan^{-1}\frac{a}{b}$$

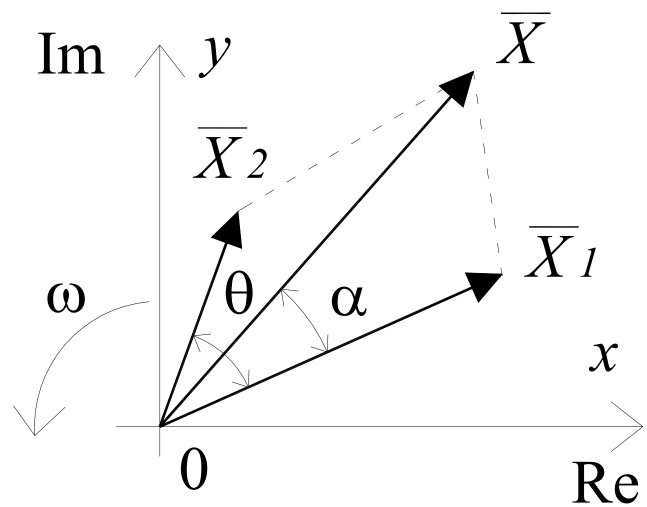


Vectorial addition

$$\bar{X} = \bar{X}_1 + \bar{X}_2 = A_1 e^{i\omega t} + A_2 e^{i(\omega t + \theta)} = Ae^{i(\omega t + \alpha)}$$

$$A = \sqrt{\left(A_1 + A_2\cos\theta\right)^2 + \left(A_2\sin\theta\right)^2}$$

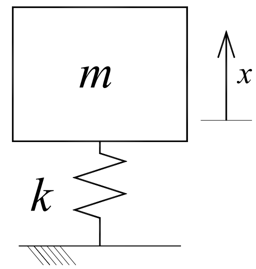
$$\alpha = \tan^{-1}\left(\frac{A_2\sin\theta}{A_1 + A_2\cos\theta}\right)$$



Where: $\theta = \omega t$ and $i = \sqrt{-1}$

Undamped Single Degree of Freedom System [1]

Recall that for the system schematically represented in the figure:



$$m\ddot{x}(t)+kx(t)=0 \rightarrow \boxed{\ddot{x}(t)+\omega_n^2 x(t)=0} \text{ where: } \omega_n=\sqrt{k/m}$$

The solution for this equation is subjected to two initial conditions:

$$x(0)=x_0, \quad \dot{x}(0)=v_0$$

The solutions for the differential equation have the following exponential form:

$$x(t)=Ae^{st}$$

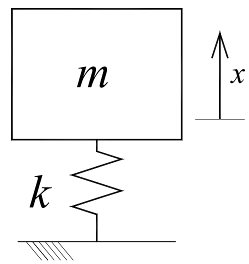
The differential equation becomes the <u>*characteristic equation*</u> which has two pure imaginary complex conjugate roots:

$$\boxed{s^2+\omega_n^2=0} \longrightarrow \left\{\begin{matrix} s_1 \\ s_2 \end{matrix}\right. = \pm i\omega_n \qquad i=\sqrt{-1}$$

[1] Leonard Meirovitch: <u>*Fundamental of Vibrations*</u>

Therefore, the general solution for the differential equation can be written as:

$$x(t) = A_1 e^{i\omega_n t} + A_2 e^{-i\omega_n t}$$



- A_1 and A_2 are constant of integration, both complex quantities
- $x(t)$ must be real

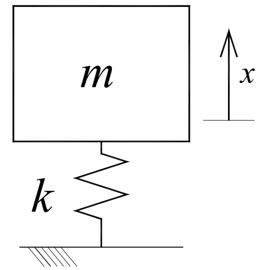
↓

Therefore: $e^{i\omega_n t}$ and $e^{-i\omega_n t}$ are complex conjugate

↓

A_2 is the complex conjugate of A_1

Any complex number can be expressed as the product of its magnitude multiplied by an exponential with pure imaginary exponent.



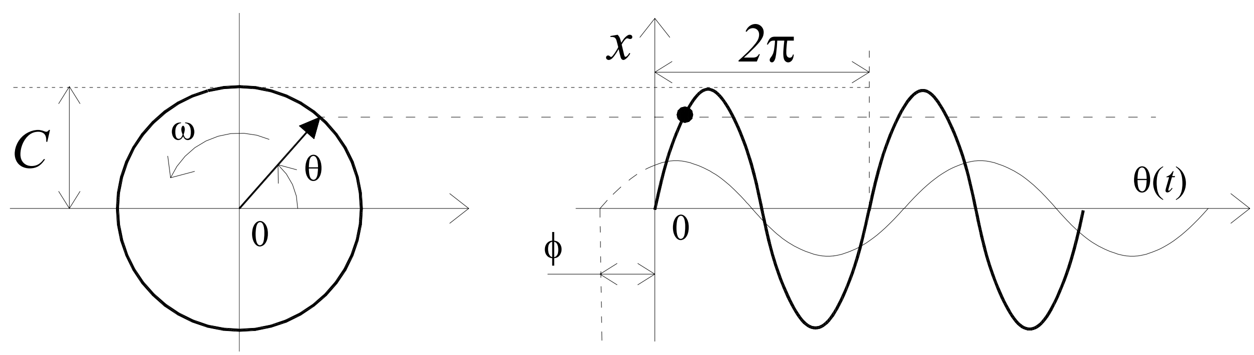
For convenience we use:

$$A_1 = \frac{C}{2} e^{-i\phi}, A_2 = \bar{A}_1 = \frac{C}{2} e^{i\phi}$$

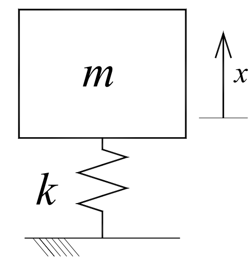
Where C and ϕ are real constants.

Therefore: $$\boxed{x(t) = \frac{C}{2}\left[e^{i(\omega_n t-\phi)} + e^{-i(\omega_n t-\phi)}\right] = C\cos(\omega_n t-\phi)}$$

Recall that: →



To determine C and ϕ, we use the initial conditions *(t=0):*



$$x(0) = x_0 = C\cos\phi$$

$$\dot{x}(0) = v_0 = \omega_n C\sin\phi$$

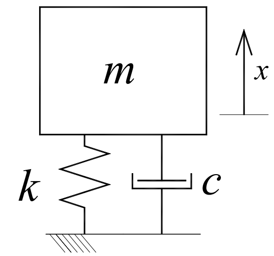
Therefore:

$$C = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}$$

$$\phi = \tan^{-1}\frac{v_0}{x_0\omega_n}$$

Damped Single degree of freedom systems

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=0 \rightarrow \boxed{\ddot{x}(t)+2\zeta\omega_n\dot{x}(t)+\omega_n^2 x(t)=0}$$



$$\begin{cases} \omega_n = \sqrt{k/m} & \rightarrow \text{ natural frequency} \\ \zeta = c/(2m\omega_n) & \rightarrow \text{ viscous damping factor} \end{cases}$$

The solution for this equation is subjected to two initial conditions:

$$x(0)=x_0, \quad \dot{x}(0)=v_0$$

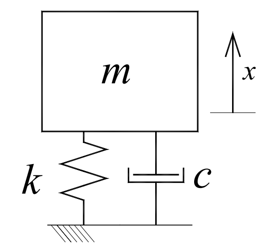
The differential equation solutions have the following exponential form:

$$x(t)=Ae^{st}$$

Using a similar technique the *characteristic equation* can be computed as:

$$\boxed{s^2+2\zeta\omega_n s+\omega_n^2=0} \longrightarrow \begin{cases} s_1 \\ s_2 \end{cases} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

The system is <u>*critically damped*</u> if:



$$\sqrt{\zeta^2 - 1} = 0$$

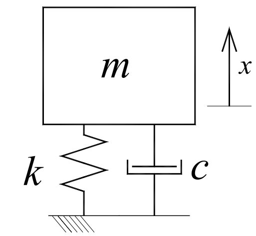
The <u>*critical damping constant*</u> can be determined:

$$\left(\frac{c_c}{2m\omega_n}\right)^2 - 1 = 0 \rightarrow \boxed{c_c = 2m\omega_n}$$

Discussion:

$$\begin{cases} \zeta = 0 & \rightarrow \text{ harmonic oscillator} \\ 0 < \zeta < 1 & \rightarrow \text{ underdamped systems} \\ \zeta = 1 & \rightarrow \text{ critical damped systems} \\ \zeta > 1 & \rightarrow \text{ overdamped systems} \end{cases}$$

Recall that: $$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$



For the initial conditions *(t=0):*

$$x(0) = x_0 = A_1 + A_2$$

$$\dot{x}(0) = v_0 = s_1 A_1 + s_2 A_2$$

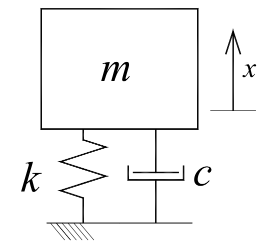
Therefore:

$$A_1 = \frac{-s_2 x_0 + v_0}{s_1 - s_2}, \qquad A_2 = \frac{s_1 x_0 - v_0}{s_1 - s_2}$$

The solution becomes:

$$\boxed{x(t) = \frac{-s_2 x_0 + v_0}{s_1 - s_2} e^{s_1 t} + \frac{s_1 x_0 - v_0}{s_1 - s_2} e^{s_2 t}}$$

<u>Underdamped systems</u>: $0<\zeta<1$



Recall the characteristic equation:

$$s^2+2\zeta\omega_n s+\omega_n^2=0 \longrightarrow \left\{\begin{matrix} s_1 \\ s_2 \end{matrix}\right. =-\zeta\omega_n \pm i\omega_d$$

where: $\omega_d=\sqrt{1-\zeta^2}\,\omega_n$

Inserting s_1 and s_2 in the characteristic equation it can be determined that:

$$\boxed{x(t)=Ce^{-\zeta\omega_n t}\cos(\omega_d t-\phi)}$$

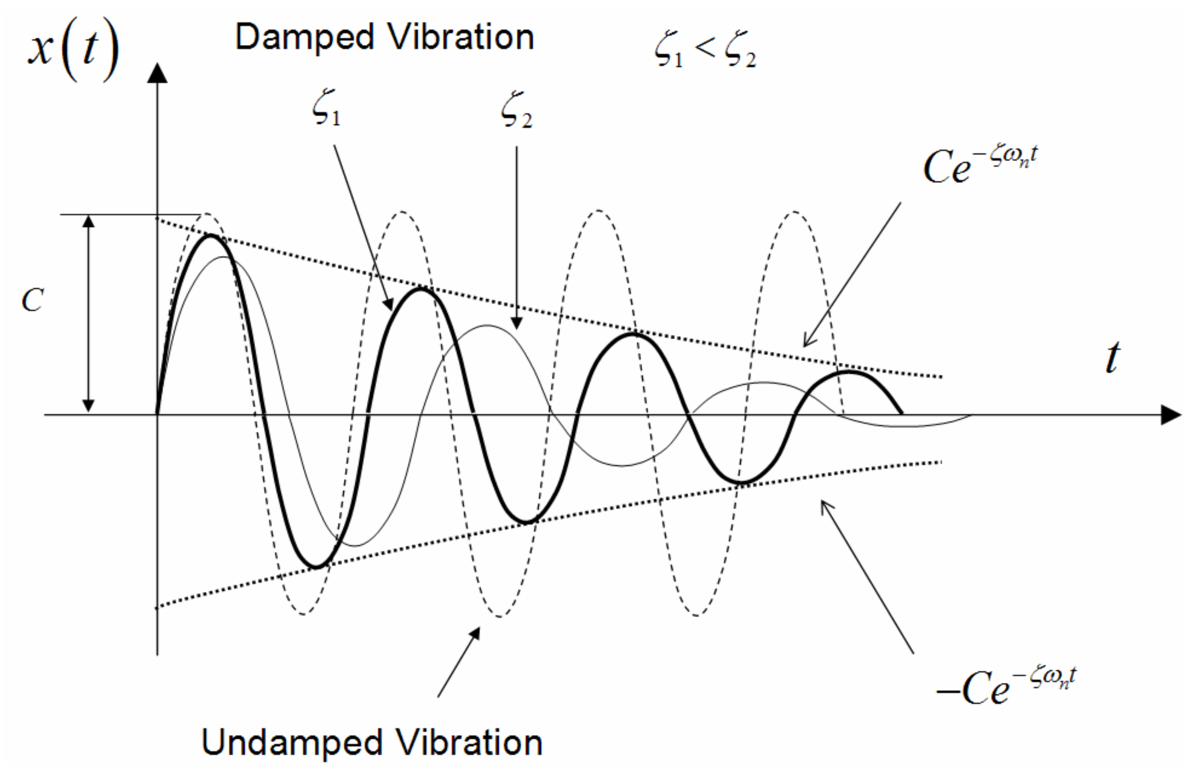
$\omega_n \rightarrow$ natural frequency

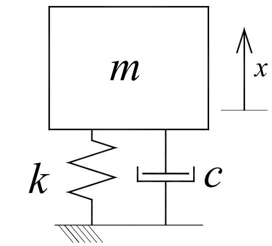
$\omega_d \rightarrow$ frequency of damped vibration

C and ϕ can be determined from the initial conditions as:

$$C=\sqrt{x_0^2+\left(\frac{\zeta\omega_n x_0+v_0}{\omega_d}\right)^2} \quad \text{and} \quad \phi=\tan^{-1}\frac{\zeta\omega_n x_0+v_0}{x_0\omega_d}$$

Underdamped systems: $0 < \zeta < 1$

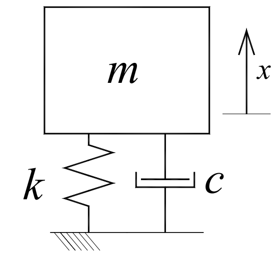




$$\boxed{x(t) = \underbrace{Ce^{-\zeta\omega_n t}}_{1}\underbrace{\cos(\omega_d t - \phi)}_{2}}$$

1. Exponential decay
2. Harmonic function

Overdamped systems: $\zeta > 1$



Recall that:

$$x(t) = \frac{-s_2 x_0 + v_0}{s_1 - s_2} e^{s_1 t} + \frac{s_1 x_0 - v_0}{s_1 - s_2} e^{s_2 t}$$

$$\begin{cases} s_1 \\ s_2 \end{cases} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$$

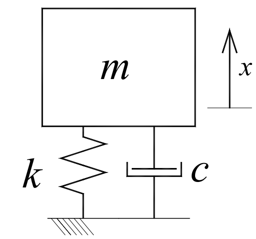
For $\zeta>1$ both roots are real and negative.

Remember that: $e^{\alpha} - e^{-\alpha} = 2\sinh\alpha \quad \text{and} \quad e^{\alpha} + e^{-\alpha} = 2\cosh\alpha$

$$\boxed{x(t) = e^{-\zeta\omega_n t}\left(\frac{\zeta\omega_n x_0 + v_0}{\sqrt{\zeta^2 - 1}\;\omega_n}\sinh\left(\sqrt{\zeta^2 - 1}\;\omega_n t\right) + x_0 \cosh\left(\sqrt{\zeta^2 - 1}\;\omega_n t\right)\right)}$$

This represents an *aperiodic decay*

Critically damped systems: $\zeta = 1$



The system has double roots. Therefore:

$$s_1 = s_2 = -\omega_n$$

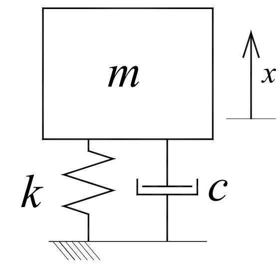
It could be shown that (the prove of which is not within the scope of this module):

$$\lim_{\zeta \to 1} \frac{\sinh \sqrt{\zeta^2 - 1}\ \omega_n t}{\sqrt{\zeta^2 - 1}\ \omega_n t} = t, \qquad \lim_{\zeta \to 1} \cosh \sqrt{\zeta^2 - 1}\ \omega_n t = 1$$

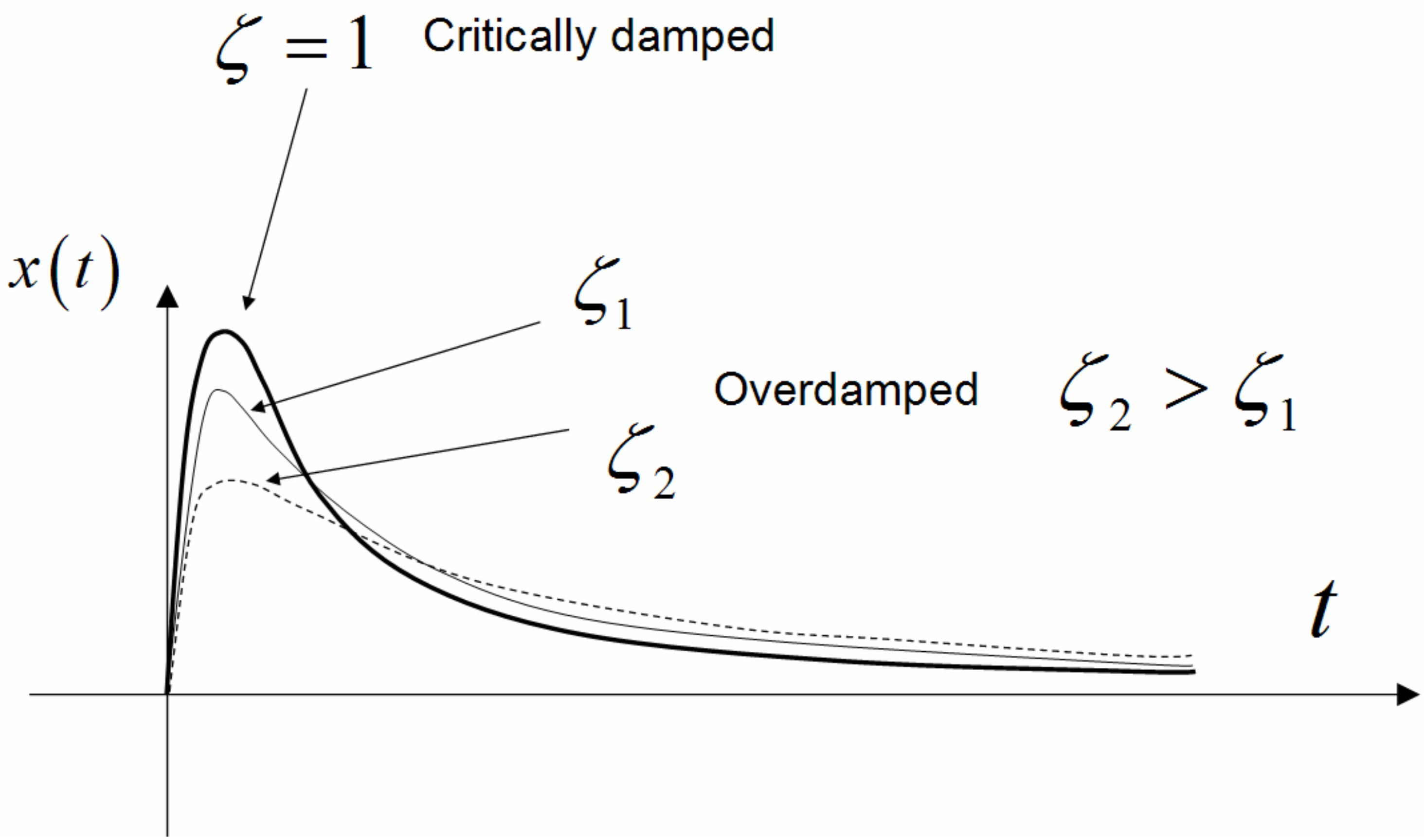
Therefore:

$$\boxed{x(t) = \left[x_0 + \left(\omega_n x_0 + v_0 \right) t \right] e^{-\omega_n t}}$$

This as is an *aperiodic decay* as in the case of overdamping

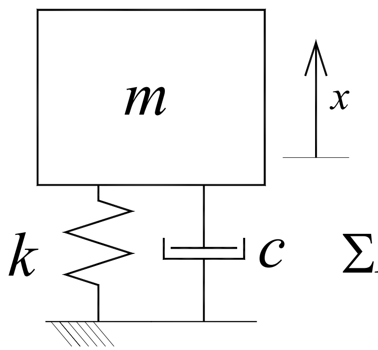
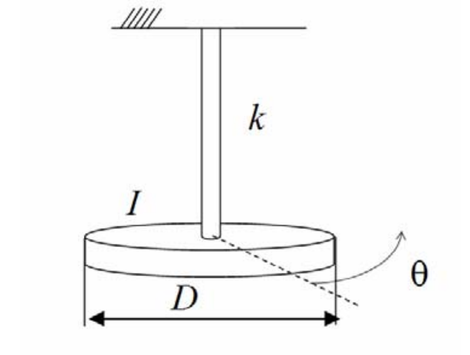


m
x
k
c



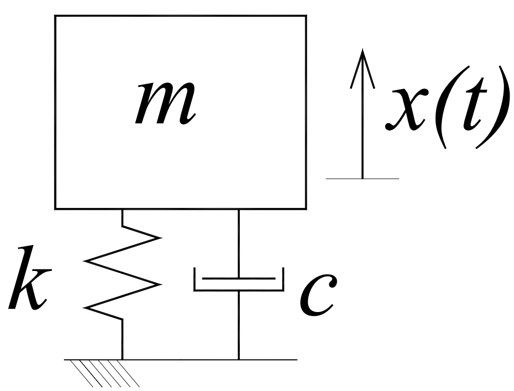
$\zeta = 1$ Critically damped
$x(t)$
ζ_1
Overdamped $\zeta_2 > \zeta_1$
ζ_2
t

Equivalence between linear and torsional vibration

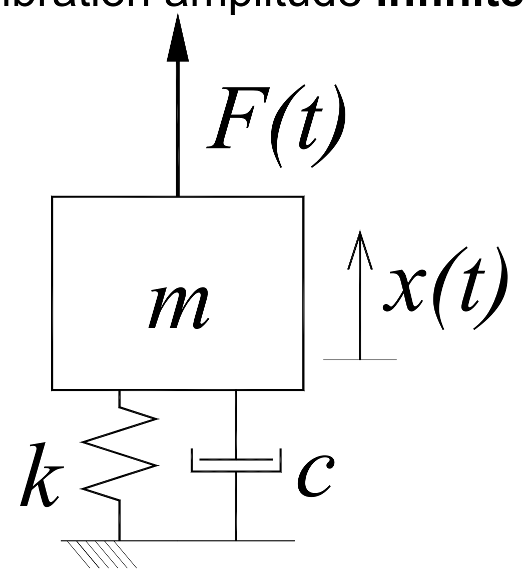
Linear Vibration	**Torsional Vibration**
 $\Sigma F = 0$	 $\Sigma M = 0$
m [kg]	$I\ [kg \cdot m^2] = \dfrac{m \times D^2}{8}$
$k\ \left[\dfrac{\text{N}}{\text{m}}\right] = \dfrac{F}{x}$	$k\ \left[\dfrac{\text{Nm}}{\text{rad}}\right] = \dfrac{T}{\theta} = \dfrac{G \times J}{L}$ where: $J = \dfrac{\pi D^4}{32}$
$\omega_n\ [\text{rad/s}] = \sqrt{\dfrac{k}{m}}$	$\omega_n\ [\text{rad/s}] = \sqrt{\dfrac{k}{I}}$
$f_n\ [Hz] = \dfrac{\omega_n}{2\pi}$	$f_n\ [Hz] = \dfrac{\omega_n}{2\pi}$
$\ddot{x}(t) + 2\zeta\omega_n \dot{x}(t) + \omega_n^2 x(t) = 0$	$\ddot{\theta}(t) + 2\zeta\omega_n \dot{\theta}(t) + \omega_n^2 \theta(t) = 0$

Single Degree of Freedom - Periodic Excitation

1. Our previous analyses might be regarded as special cases of the forced, damped “complete” case. For free, damped vibrations the input force is zero (during the period under consideration) and in free, undamped vibrations both the input force and the system damping are zero.
2. In forced vibrations we rarely consider undamped problems because this would make the resonant vibration amplitude **infinite**.



Damped System



Damped System with excitation

Free Body Diagrams:

To solve vibration problems you should always draw a free body diagram. Consider the earlier systems. In each case imagine that the mass or inertia is displaced in the $+x$ direction.

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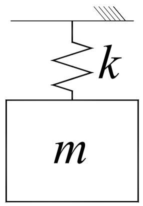
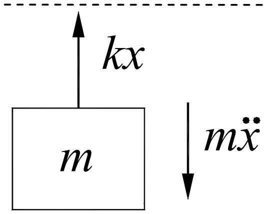
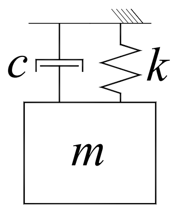
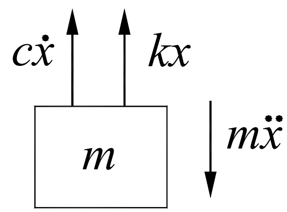
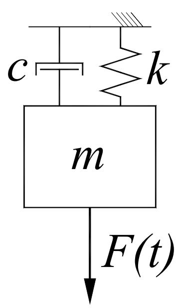
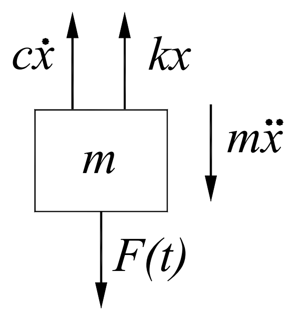
<tr><th></th><th>System</th><th>Free Body Diagram</th><th>Equation of motion</th></tr>
<tr><td>Undamped</td><td>
</td><td>
</td><td>$m\ddot{x}(t)+kx(t)=0$</td></tr>
<tr><td>Damped</td><td>
</td><td>
</td><td>$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=0$</td></tr>
<tr><td>Forced and Damped</td><td>
</td><td>
</td><td>$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=F(t)$</td></tr>
</table>

Harmonic excitation:

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=F(t)$$

The solution for this equation is subjected to two initial conditions:

$$x(0)=x_0,\quad \dot{x}(0)=v_0$$

For convenience the harmonic excitation can be expressed as:

$$F(t)=F_0\sin\omega t=kA\sin\omega t$$

Harmonic and periodic forces belong to a very important class of excitations, namely, the **steady-state** excitations

Considering the last formulation for the applied force the equation of motion is:

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=kA\sin\omega t$$

Dividing the equation of motion by m, we obtain the *characteristic equation* :

$$\boxed{\ddot{x}(t)+2\zeta\omega_n\dot{x}(t)+\omega_n^2x(t)=\omega_n^2A\cos\omega t}$$

$\omega \rightarrow$ excitation frequency

$\omega_n=\sqrt{k/m} \rightarrow$ natural frequency

$\zeta=c/(2\omega_n m) \rightarrow$ viscous damping factor

Consider the equation of motion:

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=F_0 \sin \omega t$$

The full solution of this equation of motion consists of **two** parts.

1. **COMPLEMENTARY FUNCTION**, which is the solution of the damped, free vibration equation examined previously. This component of the solution tells us the transient motion that dies away as steady state vibration is achieved.

2. **PARTICULAR INTEGRAL**, which tells us the steady state vibration at the excitation frequency. The particular integral can be found by assuming a solution of the form:

$$x = X \sin(\omega t - \phi)$$

ω can take any value and **is not to be confused with** ω_n (the natural frequency).

$$X \rightarrow \text{vibration amplitude}$$

$$\phi \rightarrow \text{phase angle}$$

$$\omega \rightarrow \text{excitation frequency}$$

• If we wish to examine the motion of the vibrating system right from the introduction of the excitation force, we need **both** functions (the complementary function and the particular integral).

• Often, however, we will only wish to examine the steady state behaviour of the system and we need only the particular integral.

$$x = X \sin(\omega t - \phi)$$

$$\dot{x} = \omega X \cos(\omega t - \phi)$$

$$\ddot{x} = -\omega^2 X \sin(\omega t - \phi)$$

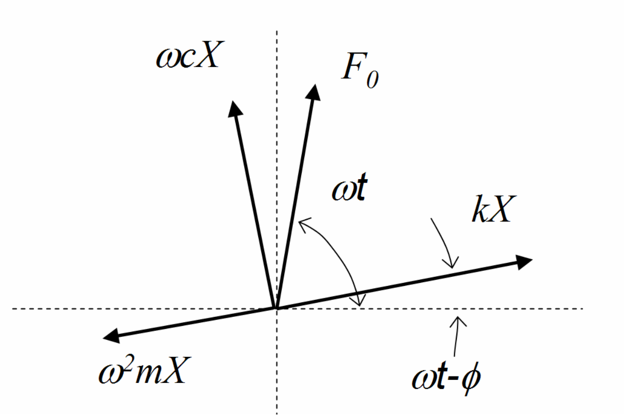
Substituting in the equation of motion we obtain:

$$-\omega^2 mX \sin(\omega t - \phi) + \omega cX \cos(\omega t - \phi) + kX \sin(\omega t - \phi) = F_0 \sin \omega t$$

Phasor Diagram:

This equation could be solved as a trigonometric problem but it is more convenient to consider each of the four terms in the equation as a **phasor**. The length of the phasor is the amplitude of the component, their phase relative to each other is given by the arguments of the sin and cos terms and the phasors rotate at frequency ω.

$$-\omega^2 mX \sin(\omega t - \phi) + \omega cX \cos(\omega t - \phi) + kX \sin(\omega t - \phi) = F_0 \sin \omega t$$



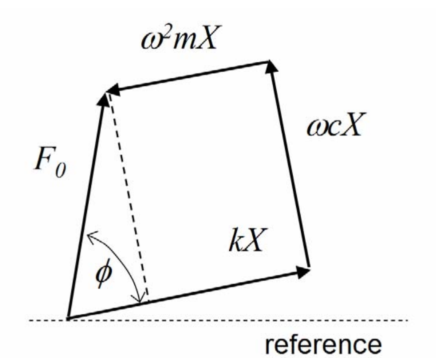
- Phasors rotate at same ω, maintaining relative speed

- The actual values (in the equation of motion) are the projections of the phasors onto the vertical axis.

The phasors are treated just like vectors and a vector polygon can now be drawn representing the equation of motion from which X and ϕ follow easily.

To find the amplitude and phase we first use Pythagoras' Theorem:

$$F_0^2 = \left[kX - \omega^2 mX\right]^2 + \left[\omega cX\right]^2 \quad \text{and} \quad \tan\phi = \frac{\omega c}{k - \omega^2 m}$$



Simple re-arrangement of the equation for amplitude gives an equation relating the vibration displacement amplitude to the force amplitude and frequency and the various properties of the vibrating system.

$$X = \frac{F_0}{\sqrt{\left[k - \omega^2 m\right]^2 + \left[\omega c\right]^2}}$$

Resonance and Dynamic Magnifier

Considering the following notation:

$$\begin{cases} \omega & \rightarrow \text{ excitation frequency} \\ \omega_n = \sqrt{k/m} & \rightarrow \text{ natural frequency} \\ \zeta = c/(2\omega_n m) & \rightarrow \text{ viscous damping factor} \end{cases}$$

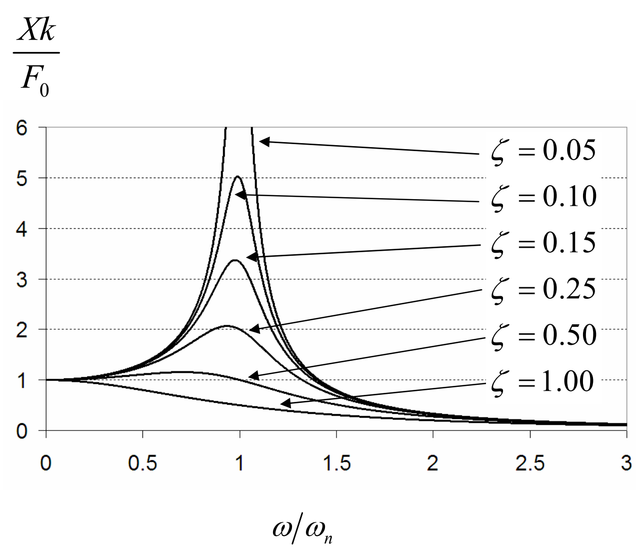
It is very common to express X and ϕ in non-dimensional terms as:

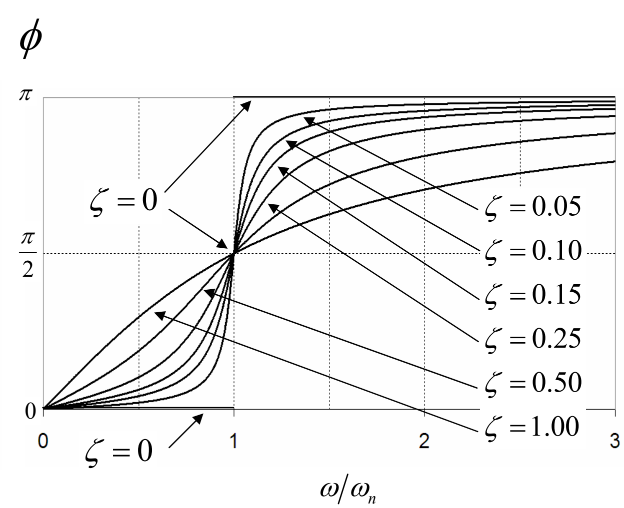
$$\frac{Xk}{F_0} = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\left(\frac{\omega}{\omega_n}\right)\right]^2}} \quad \text{and} \quad \tan\phi = \frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$

These forms of the equation are so convenient that we often consider resonance to be when $(\omega/\omega_n)=1$, i.e. at the undamped natural frequency. This is **not** entirely true.

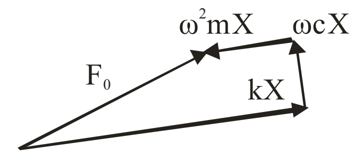
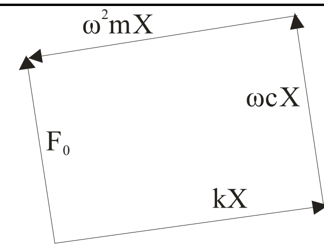
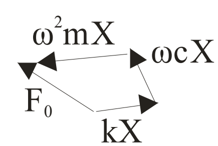
Resonance actually occurs at $\omega_n\sqrt{1-2\zeta^2}$, i.e. a little bit lower than the undamped natural frequency.

$(Xk)/F_0$ is known as the **MAGNIFICATION FACTOR** or the **DYNAMIC MAGNIFIER**. The following non-dimensional graphs show how the magnification factor and the phase angle are functions only of the frequency ratio and the damping factor:





Three frequency regions are often considered as shown in the table below:

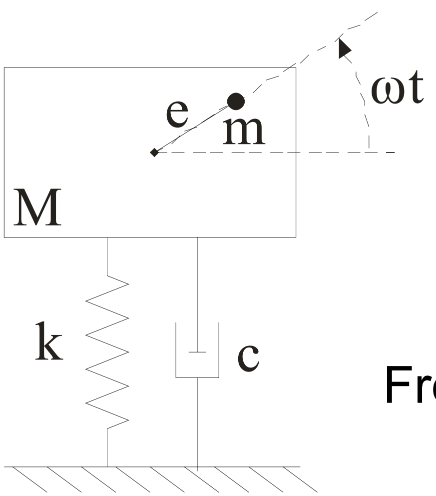
		Vector Polygon
Low frequency: $\omega/\omega_n << 1$	Inertia and damping terms small. Force balances stiffness term. ϕ is very small.	
"Resonance": $\omega/\omega_n = 1$	Inertia and stiffness terms balance. Force overcomes damping term and damping controls amplitude. ϕ =90°	
High frequency: $\omega/\omega_n >> 1$	Stiffness and damping terms small. Force overcomes inertia term. $\phi \cong 180°$	

The total solution, if required in the problem to be solved, is then the **sum** of the complementary function and the particular integral:

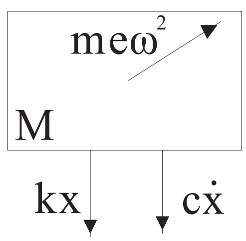
$$x(t) = \overbrace{Ce^{-\zeta\omega_n t}\cos(\omega_d t - \phi)} + \overbrace{\frac{F_0}{k}\frac{1}{\sqrt{\left(1-(\omega/\omega_n)^2\right)^2 + \left(2\zeta(\omega/\omega_n)\right)^2}}\sin(\omega t - \phi)}$$

Example - Excitation by Rotating Unbalance

- One of the most common forms of excitation is through the rotation of an unbalanced rotor. In this case the force is generated internally rather than appearing as an externally applied force.
- Consider a machine vibrating due to a rotating unbalance. The unbalance is usually visualised as a rotating eccentric mass within a non-rotating machine.
- The easiest way to write the equation of motion is to consider the external forces to be equal not to a single inertia term but to the sum of the non-rotating mass (*M*-*m*) inertia term and the rotating mass (*m*) inertia term:



System



Free Body Diagram

$$(M-m)\ddot{x} + m\frac{d^2}{dt^2}\left(x + e\sin\omega t\right) = \sum F$$

$$(M-m)\ddot{x} + m\frac{d^2}{dt^2}\left(x + e\sin\omega t\right) = -c\dot{x} - kx$$

therefore

$$M\ddot{x} + c\dot{x} + kx = me\omega^2 \sin\omega t$$

From the equation of motion the solution proceeds as for the first mass-spring-damper translational system to show:

$$X = \frac{me\omega^2}{\sqrt{\left(k-\omega^2 M\right)^2 + \left(\omega c\right)^2}} \quad \text{and} \quad \tan\phi = \frac{\omega c}{k-\omega^2 M}$$

The exact form of the magnification factor always depends on the type of excitation. For this case the magnification factor and the phase angle are given by:

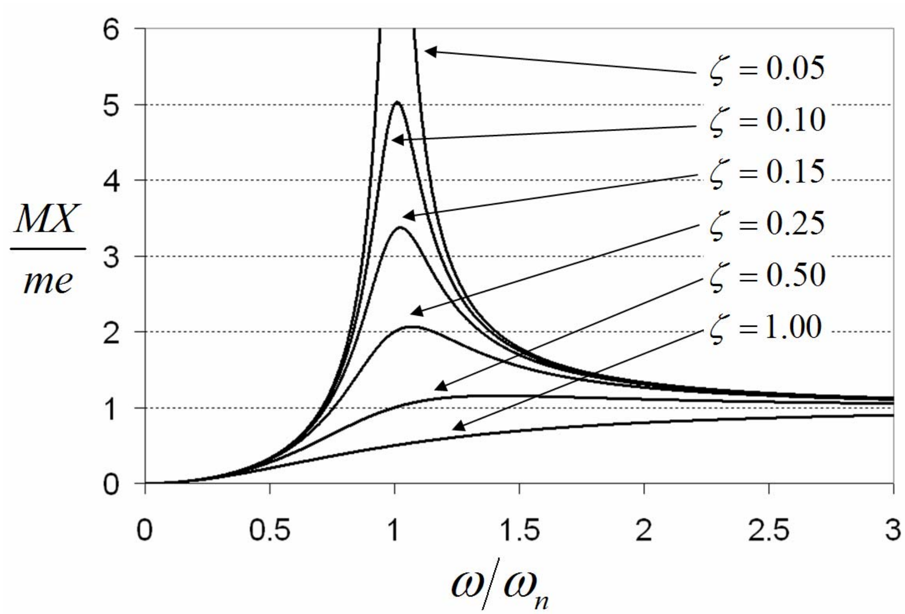
$$\frac{MX}{me} = \frac{\left(\omega/\omega_n\right)^2}{\sqrt{\left(1-\left(\omega/\omega_n\right)^2\right)^2 + \left(2\zeta\left(\omega/\omega_n\right)\right)^2}} \quad \text{and} \quad \tan\phi = \frac{2\zeta\left(\omega/\omega_n\right)}{1-\left(\omega/\omega_n\right)^2}$$

The changing form of the magnification factor illustrates the importance of being able to derive these quantities rather than committing them to memory.

To help your understanding of how to solve these problems you should make sure that you can derive the magnification factor from the original equation of motion.

You should also derive the resonant frequency again by differentiating with respect to frequency and setting the differential to zero to find the maximum. This time it is easiest to differentiate $(me/MX)^2$ with respect to frequency and find the minimum point.

You should find that the resonance occurs at $\omega_n \big/ \sqrt{1-2\zeta^2}$, i.e. a little bit higher than the undamped natural frequency. (Compare this to the previous case where the resonant frequency was a little bit <u>lower</u> than the undamped natural frequency). This is the result of the force amplitude increasing proportionally to the square of frequency.

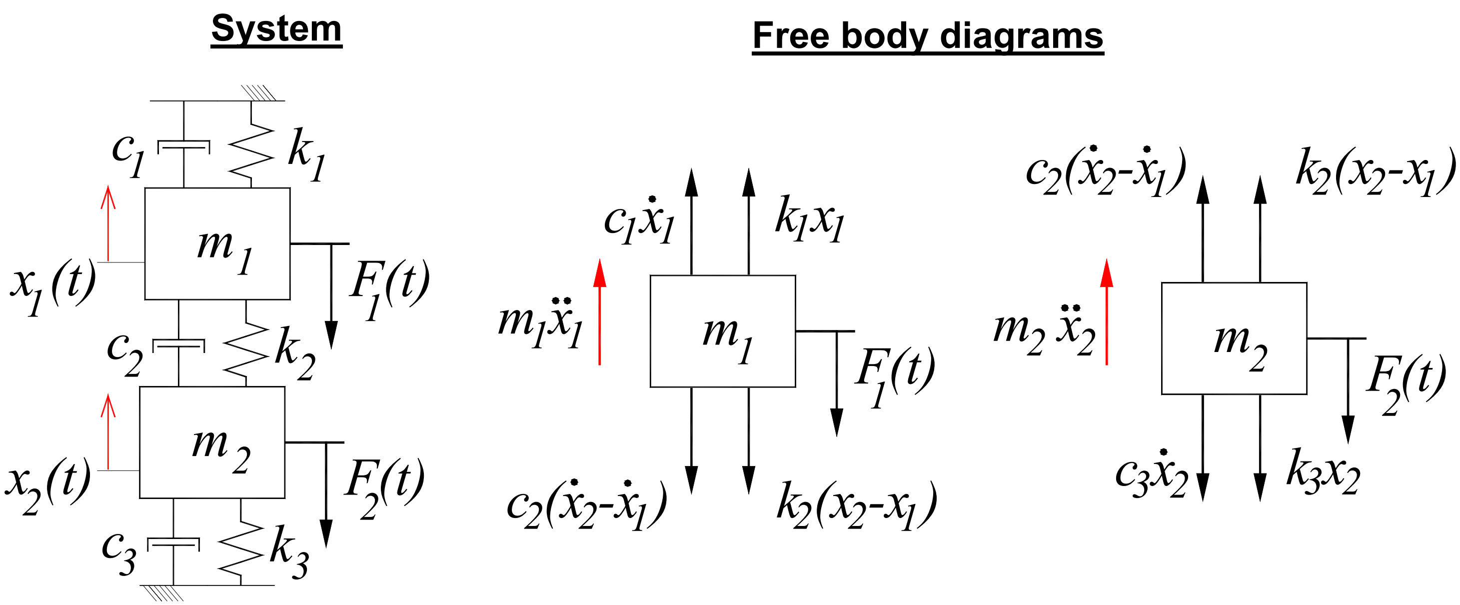


The graph on the left shows the variation of the magnification factor as a function of frequency ratio for a forced vibration due to a rotating imbalance:

Two degrees of freedom - Deriving the equations of motion

If two co-ordinates are required to define fully the motion of a vibrating system, then the system has 2 Degrees of Freedom (DOF) and is described by *two second order differential equations.*

The need for two degrees of freedom may arise because the system consists of two independent rigid bodies or because the system consists of a single rigid body whose motion requires two co-ordinates for a full description.



Using Newton's second law for each mass we obtain the equations of motion:

$$\begin{cases} m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 - c_2\left(\dot{x}_2 - \dot{x}_1\right) - k_2\left(x_2 - x_1\right) = F_1\left(t\right) \\ m_2\ddot{x}_2 + c_2\left(\dot{x}_2 - \dot{x}_1\right) + k_2\left(x_2 - x_1\right) - c_3\dot{x}_2 - k_3x_2 = F_2\left(t\right) \end{cases}$$

which can be rearranged as follow:

$$\begin{cases} m_1\ddot{x}_1 + \left(c_1 + c_2\right)\dot{x}_1 - c_2\dot{x}_2 + \left(k_1 + k_2\right)x_1 - k_2x_2 = F_1\left(t\right) \\ m_2\ddot{x}_2 - c_2\dot{x}_1 + \left(c_2 - c_3\right)\dot{x}_2 - k_2x_1 + \left(k_2 - k_3\right)x_2 = F_2\left(t\right) \end{cases}$$

This can also be written in matrix form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}$$

or more simply:

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F(t)\}$$

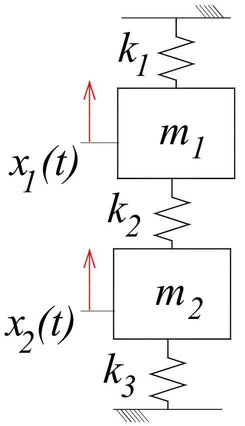
Where:

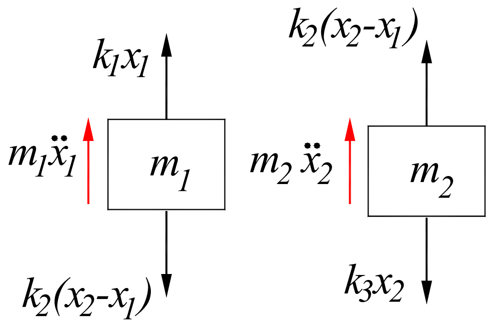
$$\begin{cases} [M] \rightarrow \text{mass matrix} \\ [C] \rightarrow \text{damping matrix} \\ [K] \rightarrow \text{stiffness matrix} \end{cases}$$

Natural Frequencies and Modes of Vibrations

Consider an <u>undamped</u> system oscillating freely:

System | Free-body diagram | Equations of motion





$$\begin{cases} m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0 \\ m_2\ddot{x}_2 + (k_2 + k_3)x_2 - k_2 x_1 = 0 \end{cases}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

Using substitutions of the form:

$$x_1(t) = A_1 e^{st} \quad \text{and} \quad x_2(t) = A_2 e^{st} \quad \text{where } s = i\omega \text{ ; } \left(i = \sqrt{-1}\right)$$

we can solve the original individual equations of motion as simultaneous equations or we can substitute into the matrix form of the equation:

$$\begin{bmatrix} k_1+k_2-m_1\omega^2 & -k_2 \\ -k_2 & k_2+k_3-m_2\omega^2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow \left([K]-[M]\omega^2\right) \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Set the determinant to zero:

$$Det\left([K]-\omega^2[M]\right) = \begin{vmatrix} k_1+k_2-m_1\omega^2 & -k_2 \\ -k_2 & k_2+k_3-m_2\omega^2 \end{vmatrix} = 0$$

The resulting equation is the **CHARACTERISTIC EQUATION**:

$$\left(k_1+k_2-m_1\omega^2\right)\ \left(k_2+k_3-m_2\omega^2\right) - k_2^2 = 0$$

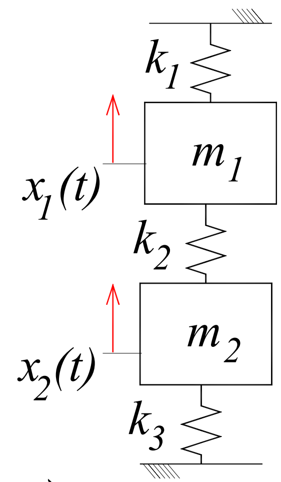
which is a quadratic equation in ω^2. The two roots, ω_1^2 and, ω_2^2 are known as the **EIGENVALUES** or **CHARACTERISTIC VALUES**.

The square roots ω_1 and ω_2 of the eigenvalues represent the **NATURAL FREQUENCIES** of the system

From the original equations of motion we can write the ratio of amplitudes:

$$\frac{A_1}{A_2} = \frac{k_2}{k_1 + k_2 - m_1\omega^2} = \frac{k_2 + k_3 - m_2\omega^2}{k_2}$$

If one amplitude is chosen as 1 we say the ratio is normalised and this is then called the **NORMAL MODE**. The two resulting normal modes are referred to as the **EIGENVECTORS**.



Example:

Simplifying the above problem by letting $\boldsymbol{k=k_1=k_2=k_3}$ and $\boldsymbol{m=m_1=m_2}$

Therefore the characteristic equation is:

$$\left(k_1 + k_2 - m_1\omega^2\right)\left(k_2 + k_3 - m_2\omega^2\right) - k_2^2 = \left(2k - m\omega^2\right)^2 - k^2 = 0$$

which has solutions: $\omega_1 = \sqrt{\dfrac{k}{m}}$ and $\omega_2 = \sqrt{\dfrac{3k}{m}}$

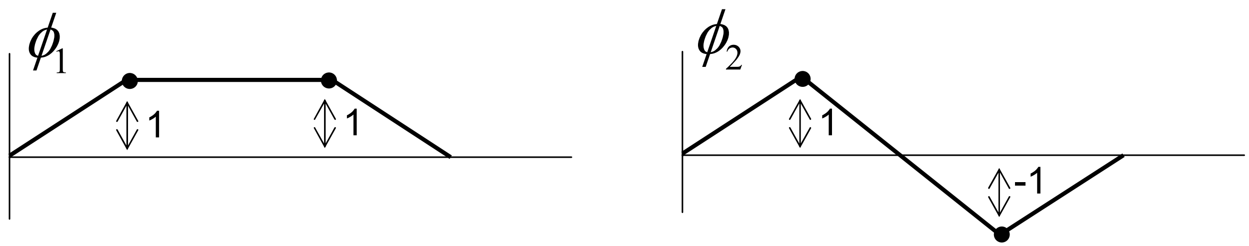
At the first natural frequency the amplitude ratio is : $$\left(\frac{A_1}{A_2}\right)^{(1)} = 1$$

while at the second natural frequency: $$\left(\frac{A_1}{A_2}\right)^{(2)} = -1$$

What do these values mean about the motions of the masses?

The corresponding normal modes are written: $$\phi_1(x) = \left\{\begin{matrix} 1 \\ 1 \end{matrix}\right\}\begin{matrix} \rightarrow A_1 \\ \rightarrow A_2 \end{matrix};\ \phi_2(x) = \left\{\begin{matrix} 1 \\ -1 \end{matrix}\right\}$$

In a system like this one, with rigid masses joined by light springs, the motion is along the axis of the system. However, for all system types (translational, torsional, shear) it is conventional to represent the mode shapes by drawing displacements perpendicular to the axis of the system. It is also common to draw the axis of the system horizontally. For example:



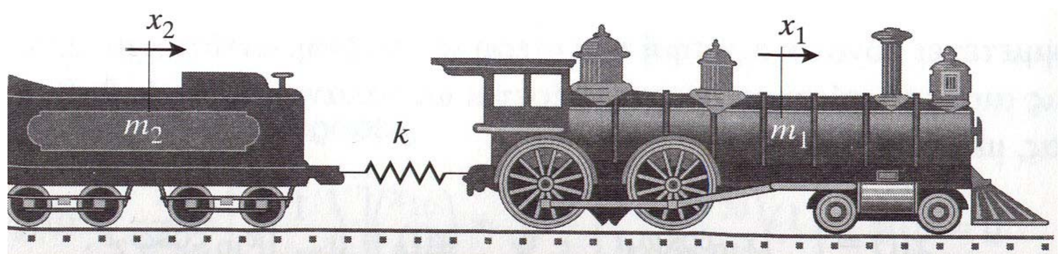
Modal representation for two degrees of freedom

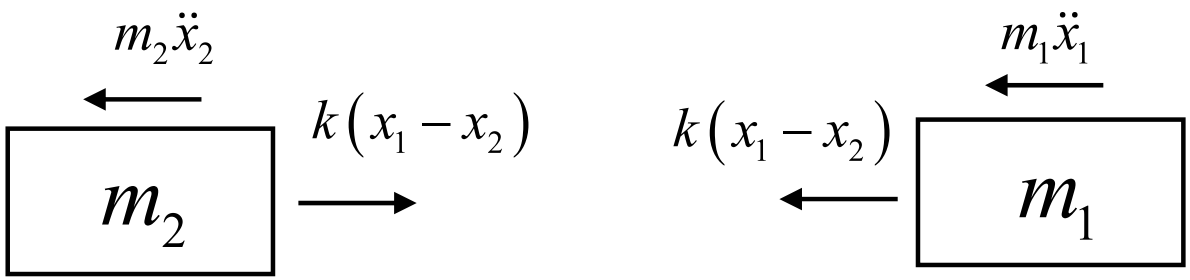
Known displacements, i.e. those calculated at the mass or inertia points, are joined by straight lines. The shape of the normal mode depends on the ratio of the displacements, say X_1/X_2, only and not on the absolute values of the displacements.

The second normal mode shows a point, other than the fixed boundaries, at rest. This point is called a **NODE**.

The normal modes are the shapes in which the system prefers to vibrate just as the natural frequencies are the frequencies at which the system prefers to vibrate.

Example:





Free body diagram

By applying Newton's second law and writing the equations of motion into the matrix form:

$$\begin{cases} m_1\ddot{x}_1 + kx_1 - kx_2 = 0 \\ m_2\ddot{x}_2 - kx_1 + kx_2 = 0 \end{cases} \Rightarrow [M]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

Where: $$[M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad [K] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}, \quad \{x\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

The characteristic equation is obtained from:

$$Det\left([K] - \omega^2 [M]\right) = 0$$

which has solutions:

$$\omega_1 = 0; \quad \omega_2 = \sqrt{\frac{k\left(m_1 + m_2\right)}{m_1 + m_2}}$$

The normal modes are found by using: $$\left([K]-[M]\omega^2\right)\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

At the first natural frequency: $$\left(\frac{A_1}{A_2}\right)^{(1)} = 1$$

while at the second natural frequency: $$\left(\frac{A_1}{A_2}\right)^{(2)} = -\frac{m_2}{m_1}$$

The corresponding normal modes are written:

$$\phi_1(x) = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}, \quad \phi_2(x) = \begin{Bmatrix} 1 \\ -\dfrac{m_1}{m_2} \end{Bmatrix}$$

Draw the modal representation for two degrees of freedom!

System Response - Laplace Transformation Method

The solution of many vibrations problems by direct means can cause serious difficulties and it may not even be possible, unless some type of *transformation* is used.

There is a large variety of transformations, but the general idea behing all of them is the same:

1. **Transform the difficult problem into a simple one**
2. **Solve the simple problem**
3. **Inverse transform the solution of the simple problem to obtain the solution of the original difficult problem.**

Laplace transform:

• Can be used to find the response of a system under any type of excitation, including periodic and harmonic types.

• Provides an efficient method for solving linear ordinary differential equations with constant coefficient

•Takes into account initial conditions automatically

The Laplace transform of a function x(t), denoted symbolically as $\overline{x}(s) = \mathscr{L}x(t)$ is **defined** as:

(1) $$\boxed{\overline{x}(s) = \mathscr{L}x(t) = \int_0^{\infty} e^{-st} x(t)\,dt} \qquad \begin{cases} e^{-st} & \rightarrow \text{kernel of the transformation} \\ s & \rightarrow \text{subsidiary variable} \end{cases}$$

- Since the integration is with respect to t, the transform gives a function of s.

•Eq. (1) is defined in terms of an integral, → is called ***integral transformation***.

Necessary Steps: (to solve a problem using Laplace transform method)

1) *Write the equation of motion of the system.*
2) *Transform each term of the equation, using known initial conditions.*
3) *Solve the transformed response of the system.*
4) *Obtain the desired solution by using inverse Laplace transformation.*

Recall the differential equation of motion for 1DOF:

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t) \qquad (2)$$

To solve the problem by Laplace transform it is necessary to find the Laplace transformation of: $\dot{x}(t)$, $\ddot{x}(t)$ and $F(t)$.

1) <u>The transform of $\dot{x}(t)$ can be integrated by parts:</u>

$$\mathcal{L}\frac{dx(t)}{dt} = \int_0^\infty e^{-st}\frac{dx(t)}{dt}dt = e^{-st}x(t)\Big|_0^\infty + s\int_0^\infty e^{-st}x(t)dt = -x(0) + s\overline{x}(s)$$

$x(0) \rightarrow$ the value of $x(t)$ at $t = 0$ (initial displacement)

2) <u>The transform of $\ddot{x}(t)$ can be integrated by parts:</u>

$$\mathcal{L}\frac{d^2x(t)}{dt^2} = \int_0^\infty e^{-st}\frac{d^2x(t)}{dt^2}dt = -\dot{x}(0) - sx(0) + s^2\overline{x}(s)$$

$\dot{x}(0) \rightarrow$ the value of $\dot{x}(t)$ at $t = 0$ (initial velocity)

3) <u>The transform of the force *F(t)*:</u> $\overline{F}(s) = \mathcal{L}F(t) = \int_0^\infty e^{-st}F(t)dt$

We can transform both sides of equation (2) and obtain:

$$m\mathcal{L}\ddot{x}(t) + c\mathcal{L}\dot{x}(t) + k\mathcal{L}x(t) = \mathcal{L}F(t)$$

or

$$\boxed{\left(ms^2 + cs + k\right)\overline{x}(s) = \overline{F}(s) + m\dot{x}(0) + \left(ms + c\right)x(0)} \quad (3)$$

The right hand side of equation (3) can be regarded as a generalised excitation.

In the first instance we take $\ddot{x}(0), \dot{x}(0)$ equal with zero, which is equivalent to ignoring the homogeneous solution.

From equation (3): $$\bar{Z}(s) = \frac{\bar{F}(s)}{\bar{x}(s)} = ms^2 + cs + k$$

Where $\bar{Z}(s)$ is known as *generalised impedance* of the system

The reciprocal function of $\bar{Z}(s)$ is known as the *admittance* or *transfer function:*

$$\bar{Y}(s) = \frac{1}{\bar{Z}(s)} = \frac{\bar{x}(s)}{\bar{F}(s)} = \frac{1}{ms^2 + cs + k} = \frac{1}{m\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

$$\Downarrow$$

$$\overbrace{\bar{x}(s) = \bar{Y}(s)\bar{F}(s)}$$

Therefore, the transfer function can be regarded as an algebraic operator that operates on the transformed force to yield the transformed response.

The Inverse Laplace Transform

To find the desired response, we have to take the inverse Laplace transform:

$$x(t) = \mathscr{L}^{-1}\bar{x}(s) = \mathscr{L}^{-1}\bar{Y}(s)\bar{F}(s)$$

In general, the operator $\mathscr{L}^{-1}$ involves a line integral in the complex domain. Fortunately we need not evaluate these integrals separately for each problem; such integrations have been carried out for various common forms of the function $F(t)$ and tabulated.

To find the solution, we usually look for ways of decomposing $\bar{x}(s)$ into a combination of simple functions whose inverse transformations are available in Laplace transform tables.

In the above discussion, we ignored the homogeneous solution by assuming the initial conditions as zero. We now consider the general solution by taking the initial conditions: $x(0) = x_0; \dot{x}(0) = \dot{x}_0$ From equation (3) we obtain:

$$\bar{x}(s) = \frac{\bar{F}(s)}{m\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} + \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}x_0 + \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}\dot{x}_0 \tag{4}$$

The solution is obtain by considering each term in Eq. (4) separately

We make use of the following useful formula:

$$\mathcal{L}^{-1}\bar{f}_1(s)\bar{f}_2(s)=\int_0^t f_1(\tau)\,f_2(t-\tau)\,d\tau$$

1) **<u>The 1st term:</u>** (from the right hand side of equation 4)

$$\mathcal{L}^{-1}\left[\frac{\bar{F}(s)}{m\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}\right]=\mathcal{L}^{-1}\bar{f}_1(s)\bar{f}_2(s) \qquad \begin{cases}\bar{f}_1(s)=\bar{F}(s)\\ \bar{f}_2(s)=1\Big/\left[m\left(s^2+2\zeta\omega_n s+\omega_n^2\right)\right]\end{cases}$$

$$\mathcal{L}^{-1}\bar{f}_1(s)\bar{f}_2(s)=\frac{1}{m\omega_d}\int_0^t F(\tau)\,e^{-\zeta\omega_n(t-\tau)}\sin\omega_d\,(t-\tau)\,d\tau$$

(eq. 14)

Use the attached table of Laplace transforms

2) **<u>The 2nd term:</u>**

$$\mathcal{L}^{-1}\left[\frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}\right]=\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_d t+\phi_1\right); \qquad \phi_1=\cos^{-1}(\zeta)$$

(eq. 16)

3) **<u>The 3rd term:</u>**

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}\right]=\frac{1}{\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t \qquad \text{(eq. 14)}$$

The general solution can be expressed as:

$$\boxed{\begin{aligned}\overline{x}(s) &= \frac{1}{m\omega_d}\int_0^t F(\tau)\, e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)\, d\tau + \\ &+ \frac{x_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_d t + \phi_1\right) + \frac{\dot{x}_0}{\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t\end{aligned}}$$

Example:

Find the response of a spring-mass-dumper system when z<1 and the system is subjected to the force:

$$F(t) = \begin{cases} F_0 & 0 \le t \le t_0 \\ 0 & t > t_0 \end{cases}$$

Solution:

By taking the Laplace transform of the governing differential equation (2) we obtain the equation (4). Using the table, with

$$\overline{F}(s) = \mathscr{L}F(t) = \frac{F_0\left(1 - e^{-t_0 s}\right)}{s}$$

Equation (4) can be written as:

$$\bar{x}(s) = \frac{F_0\left(1-e^{-t_0 s}\right)}{ms\left(s^2+2\zeta\omega_n s+\omega_n^2\right)} + \frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}x_0 + \frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}\dot{x}_0$$

$$\bar{x}(s) = \frac{F_0}{m\omega_n^2}\frac{1}{s\left(\frac{s^2}{\omega_n^2}+\frac{2\zeta s}{\omega_n}+1\right)} - \frac{F_0}{m\omega_n^2}\frac{e^{-t_0 s}}{s\left(\frac{s^2}{\omega_n^2}+\frac{2\zeta s}{\omega_n}+1\right)} + \\ + \frac{x_0}{\omega_n^2}\frac{s}{\left(\frac{s^2}{\omega_n^2}+\frac{2\zeta s}{\omega_n}+1\right)} + \left(\frac{2\zeta x_0}{\omega_n}+\frac{\dot{x}_0}{\omega_n^2}\right)\frac{1}{\left(\frac{s^2}{\omega_n^2}+\frac{2\zeta s}{\omega_n}+1\right)}$$

The inverse transform can be expressed by using the table with typical Laplece transforms:

$$x(t)=\frac{F_0}{m\omega_n\omega_d}\left[-e^{-\zeta\omega_n t}\sin\left(\omega_d t+\phi_1\right)\right]+e^{-\zeta\omega_n\left(t-t_0\right)}\sin\left[\omega_d\left(t-t_0\right)+\phi_1\right]-$$

$$-\frac{\omega_n x_0}{\omega_d}e^{-\zeta\omega_n t}\sin\left(\omega_d t-\phi_1\right)+\frac{2\zeta\omega_n x_0+\dot{x}_0}{\omega_d}e^{-\zeta\omega_n t}\sin\left(\omega_d t\right)$$

The previous equation can be put in the following format:

$$x(t)=\frac{F_0}{m\omega_n^2}\left[-\sin\left(\omega_n t+\frac{\pi}{2}\right)+\sin\left[\omega_n\left(t-t_0\right)+\frac{\pi}{2}\right]\right]-x_0\sin\left(\omega_n t-\frac{\pi}{2}\right)+\frac{\dot{x}_0}{\omega_n}\sin\omega_n t$$

$$=\frac{F_0}{k}\left[\cos\omega_n\left(t-t_0\right)-\cos\omega_n t\right]+x_0\cos\omega_n t+\frac{\dot{x}_0}{\omega_n}\sin\omega_n t$$

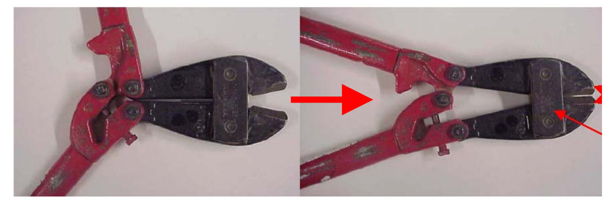
Laplace transform Pairs

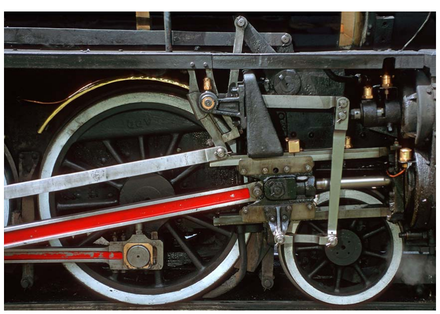
	Laplace domain $\bar{f}(s)=\int_0^\infty e^{-st}f(t)dt$	Time domain $f(t)$
1.	$c_1\bar{f}(s)+c_2\bar{g}(s)$	$c_1 f(s)+c_2 g(s)$
2.	$\bar{f}\left(\frac{s}{a}\right)$	$f(a\cdot t)a$
3.	$\bar{f}(s)\bar{g}(s)$	$\int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(\tau)g(t-\tau)d\tau$
4.	$s^n\bar{f}(s)-\sum_{j=1}^{n}s^{n-j}\frac{d^{j-1}}{dt^{j-1}}f(0)$	$\frac{d^n}{dt^n}f(t)$
5.	$\frac{1}{s^n}\bar{f}(s)$	$\underbrace{\int_0^t\ldots\int_0^t}_{n} f(\tau)d\tau\ldots d\tau$
6.	$\bar{f}(s+a)$	$e^{-at}f(t)$
7.	$\frac{a}{s(s+a)}$	$1-e^{-at}$
8.	$\frac{s+a}{s^2}$	$1+at$
9.	$\frac{a^2}{s^2(s+a)}$	$at-\left(1-e^{-at}\right)$
10.	$\frac{s+b}{s(s+a)}$	$\frac{b}{a}\left\{1-\left(1-\frac{a}{b}\right)e^{-at}\right\}$
11.	$\frac{a}{s^2+a^2}$	$\sin at$
12.	$\frac{s}{s^2+a^2}$	$\cos at$
13.	$\frac{a^2}{s\left(s^2+a^2\right)}$	$1-\cos at$
14.*	$\frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}$	$\frac{1}{\omega_n}e^{-\zeta\omega_n t}\sin\omega_d t$
15.*	$\frac{s}{s^2+2\zeta\omega_n s+\omega_n^2}$	$-\frac{\omega_d}{\omega_n}e^{-\zeta\omega_n t}\sin\left(\omega_d t-\phi_1\right)$
16.*	$\frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}$	$\frac{\omega_d}{\omega_n}e^{-\zeta\omega_n t}\sin\left(\omega_d t+\phi_1\right)$
17.*	$\frac{\omega_n^2}{s\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}$	$1-\frac{\omega_d}{\omega_n}e^{-\zeta\omega_n t}\sin\left(\omega_d t+\phi_1\right)$
18.*	$\frac{s+2\zeta\omega_n}{s\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}$	$e^{-\zeta\omega_n t}\sin\left(\omega_d t+\phi_1\right)$

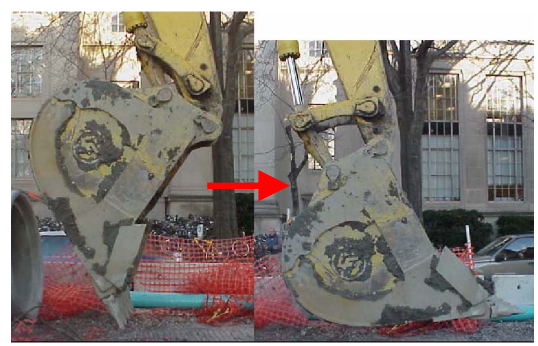
Mechanism Design & Analysis: The 4-Bar Mechanism

Goal:

- Find the minimum number of masses, which can simulate fairly accurately the dynamics of the real system with an acceptable computer time.







Definitions:

1. Multi-body dynamics is the study of motion of an assembly of components (or a cluster of objects), where the response of the system is affected by the interaction of its individual elements.

2. The field of study can be broken down to its constituent sub-divisions in terms of the behaviour of a system as:

   - *Kinematics*
   - *Statics and Quasi-statics*
   - *Full Dynamics*

Classification:

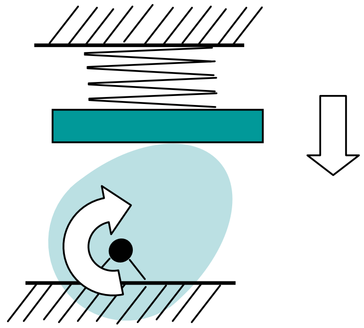
1. Kinematic Chain - an assemblage of links and joints, interconnected in a way to provide a controlled output in response to a supplied input motion

2. Mechanism - a kinematic chain in which at least one link has been grounded, or attached to a frame of reference

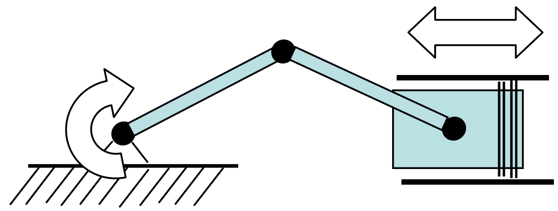
3. Machine - a collection of mechanisms arranged to transmit forces and do work

Principal Power Mechanisms

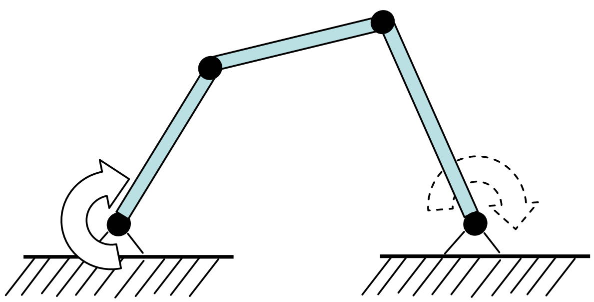
1. Cams



2. Slider-Crank



3. 4-Bar Linkage



Determining the Degrees of Freedom:

$$DOF = 3L - 2J - 3G$$

DOF - The number of inputs which need to be provided in order to create a predictable output

L - number of links

J - number of joints

G - number of grounded Links

Simplification of Gruebler's Equation

1. Multiple ground attachments are all to the same link
2. Since G is always one, this simplifies to:

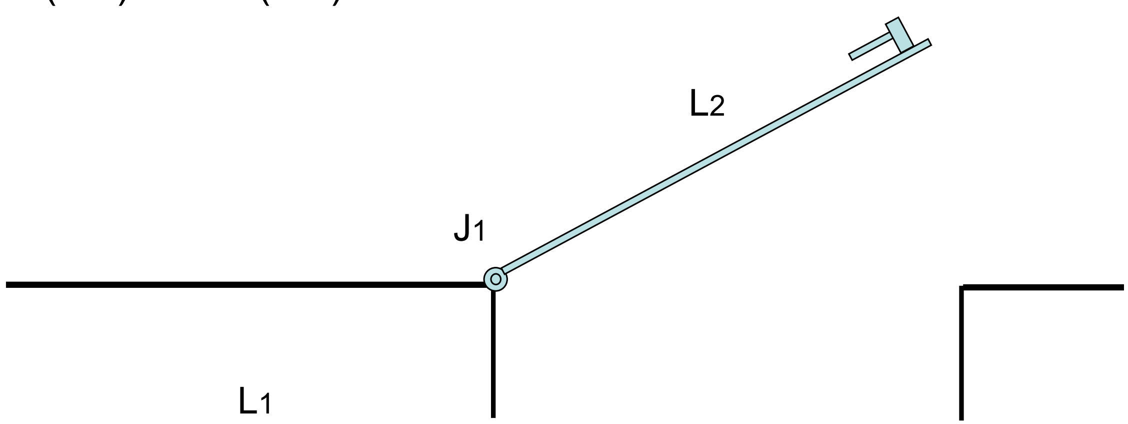
$$DOF = 3(L-1) - 2J$$

Example 1: Door Hinge

L=2

J=1

DOF = 3(L-1) - 2J=3(2-1)-2*1=1

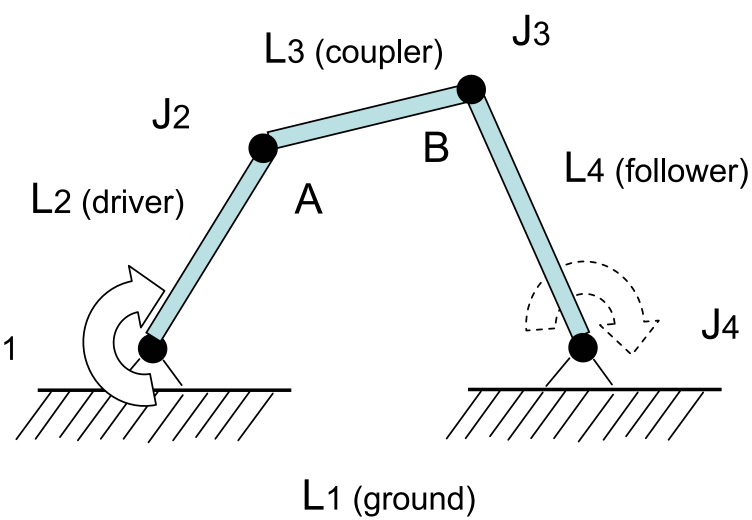


Example 2: The Four Bar Mechanism

L=4

J=4

DOF = 3(L-1) - 2J= 3(4-1) – 2*4= 9-8= 1



The Grashof Condition

1. A fourbar is Grashof if at least one link is capable of making a full revolution
2. A fourbar is non-Grashof if no link is capable of making a complete revolution

Calculating the Grashof Criteria

Let: S = length of shortest link

L = length of longest link

P = length of one remaining link

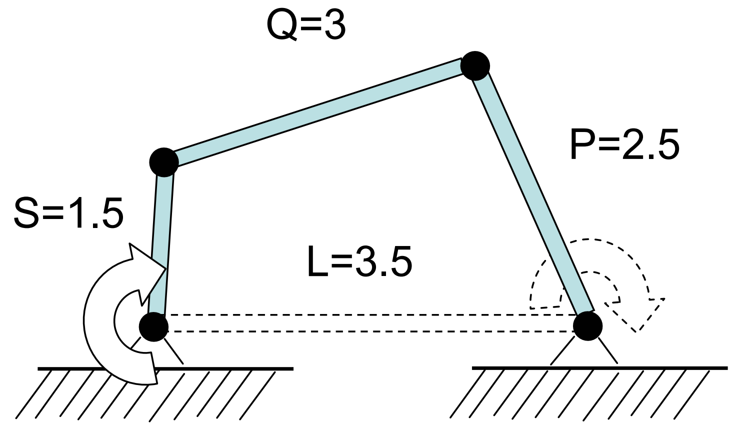
Q = length of other remaining link

Linkage is Grashof if:

$$S + L \leq P + Q$$

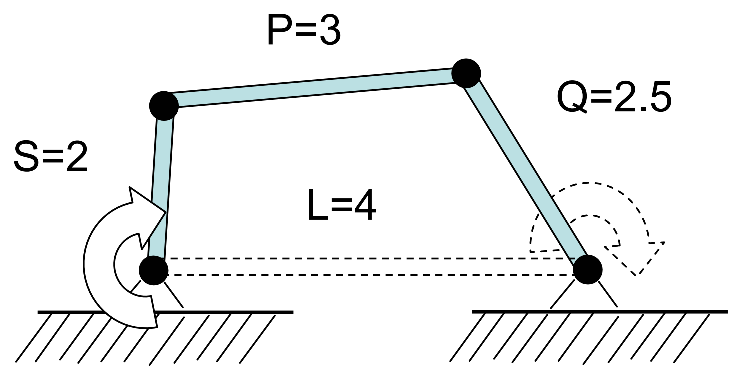
Example 1:

1. S + L = 5
2. P + Q = 5.5
3. S + L < P + Q,
4. *The Mechanism is Grashof*



Example 2:

1. S + L = 6
2. P + Q = 5.5
3. S + L > P + Q
4. *The Mechanism is Non-Grashof*



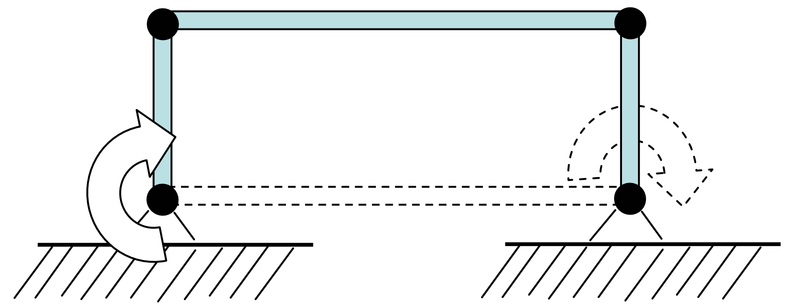
Special Case Grashof: S + L = P + Q

1. Linkage can form parallelogram or anti-parallelogram (crossed)
2. Often used to keep coupler parallel (drafting machine)

S = 2.5; L = 4.5;

P = 2.5 (or 4.5); Q = 4.5 (or 2.5)

S + L = P + Q

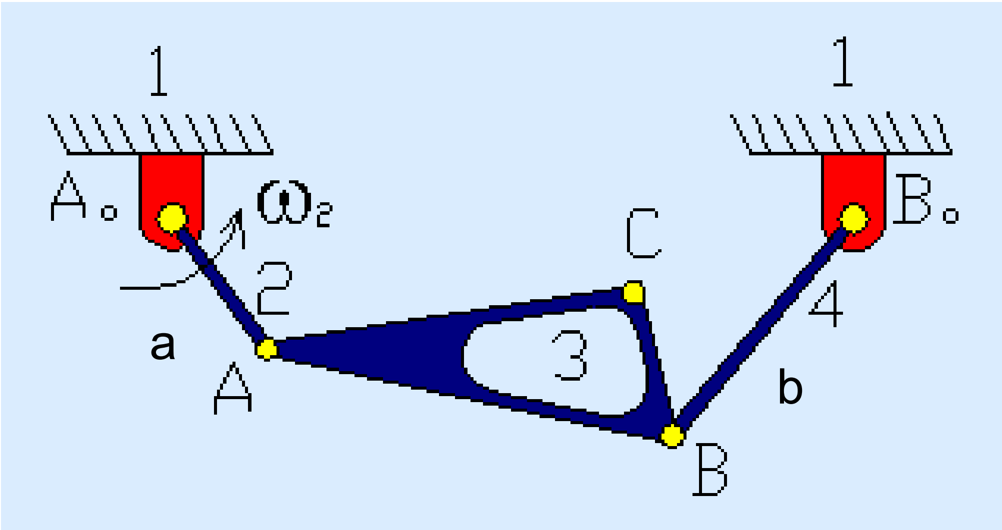


Problems w/ Special Grashof:

1. Behavior at change points is indeterminate
2. If used in continuous machine, must have some mechanism to “carry through” (steam engine)

Velocities of Points in Linkages

Considering the four-bar linkage shown in the figure, in which A_0B_0 is the fixed link, A_0A is the driver, B_0B is the follower and AB the coupler.

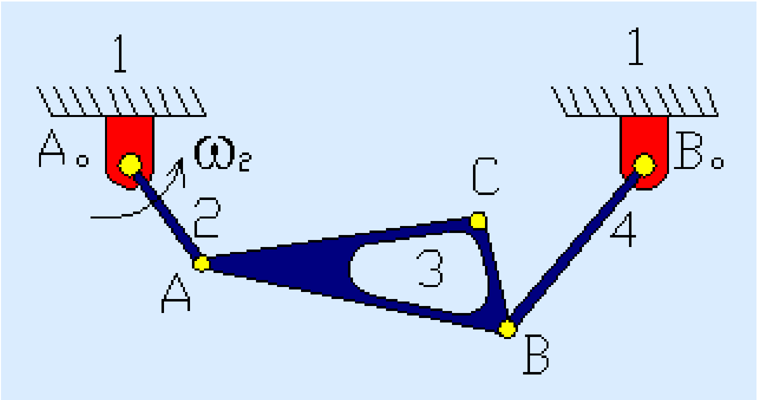


- Point A follows a path of radius *a.*

-Point B follows a path of radius *c*.

Both links are in pure rotation, whereas the displacement of link AB has both translation and rotation.

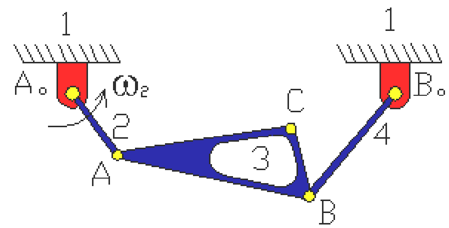
Given the four bar linkage $\mathbf{A_O ABB_O}$ shown in the figure. The crank rotates at a given angular velocity ω_2. Find the velocity of point **C**.



The relative velocity equation for the velocity of point **C** wrt point A is:

$$\mathbf{V_C = V_A + V_{CA}}$$

Can you solve this equation?

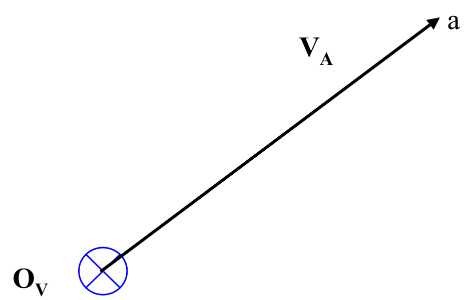


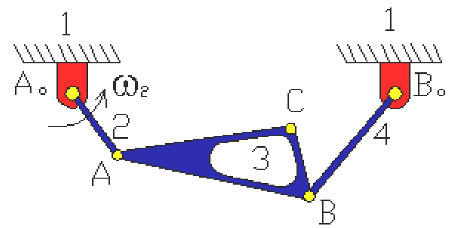
Draw a space diagram of the mechanism to a chosen space scale. Then set up the vector equation for the velocity of slider **B** relative to slider **A**:

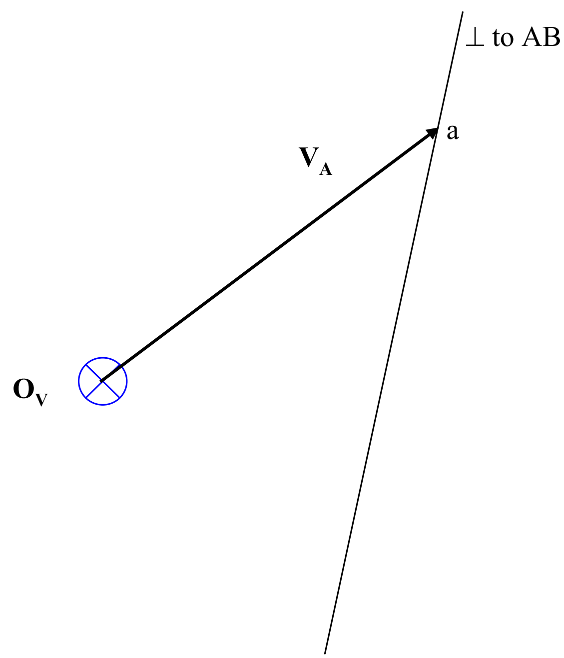
$$\mathbf{V_B = V_A + V_{BA}}$$

Both magnitude and direction of V_A are known. Directions of V_B and V_{BA} are also known. Hence this equation can be solved.

Select an origin, O_V and a velocity scale factor. Draw the velocity vector $\mathbf{V_A}$ starting at the origin.



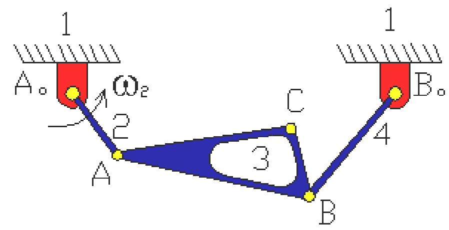




$\mathbf{V_B = V_A + V_{BA}}$

Direction of $\mathbf{V_{BA}}$ is known. Since $\mathbf{V_{BA}}$ is to be added to $\mathbf{V_A}$, vector $\mathbf{V_{BA}}$ passes through the head of vector $\mathbf{V_A}$.

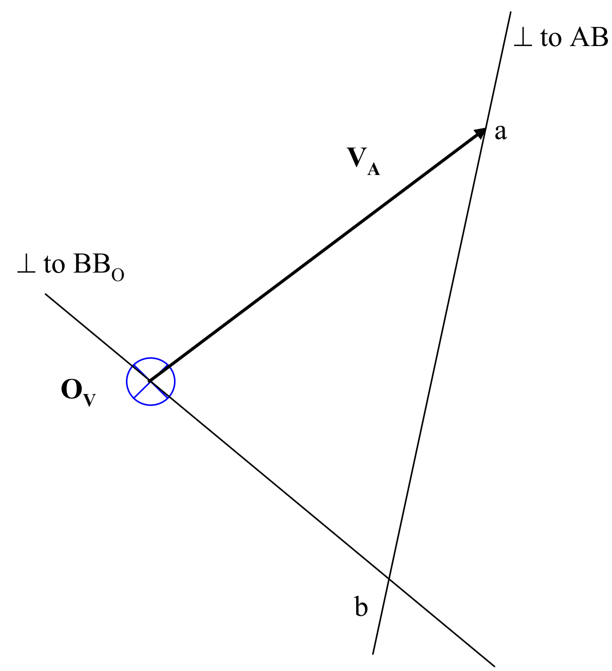
Draw a line through point a perpendicular to link AB.

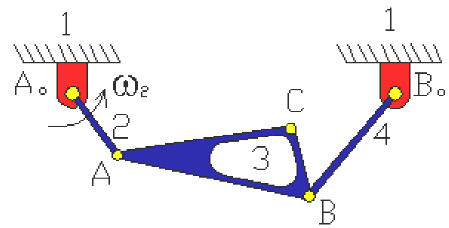


$\mathbf{V_B} = \mathbf{V_A} + \mathbf{V_{BA}}$

Direction of $\mathbf{V_B}$ is known. Since $\mathbf{V_B}$ equals the sum of vectors $\mathbf{V_A}$ and $\mathbf{V_{BA}}$, vector $\mathbf{V_B}$ passes through the origin $\mathbf{O_V}$ and its direction is perpendicular to BB_O.

Draw a line through the origin and perpendicular to BB_O.



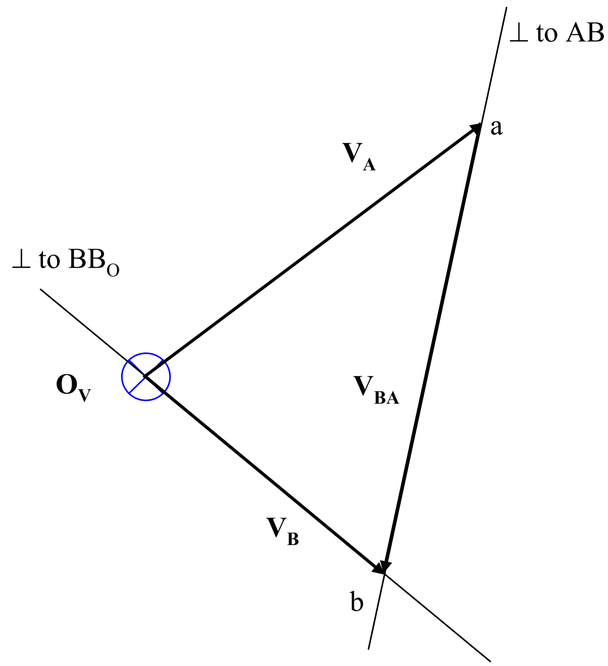


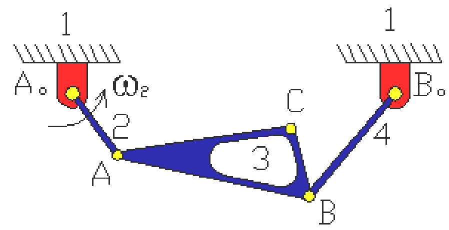
$\mathbf{V_B = V_A + V_{BA}}$

The intersection of the two lines that indicate the directions of $\mathbf{V_B}$ and $\mathbf{V_{BA}}$ determines the magnitudes of vectors $\mathbf{V_B}$ and $\mathbf{V_{BA}}$.

The vector from the origin to point b is the velocity of point B. The vector from a to b is the relative velocity of point B wrt point A.

Measure the magnitudes of these two vectors and determine velocities $\mathbf{V_B}$ and $\mathbf{V_{BA}}$ using the scale factor.





Alternatively:

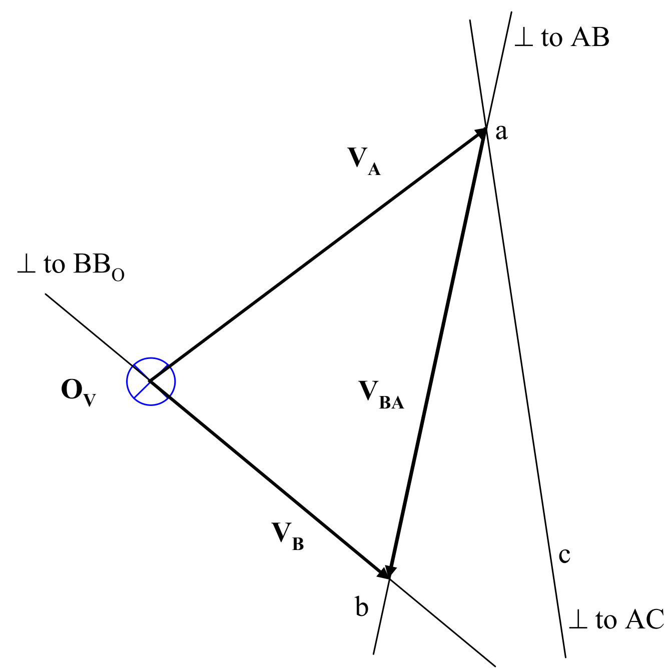
Using the relative velocity equations

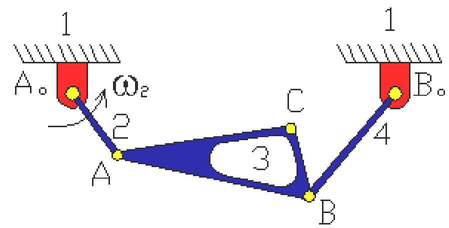
$\mathbf{V_C = V_A + V_{CA}}$

$\mathbf{V_C = V_B + V_{CB}}$

and simultaneously solving them we can determine the position of point c.

Draw a line passing through a and perpendicular to AC.

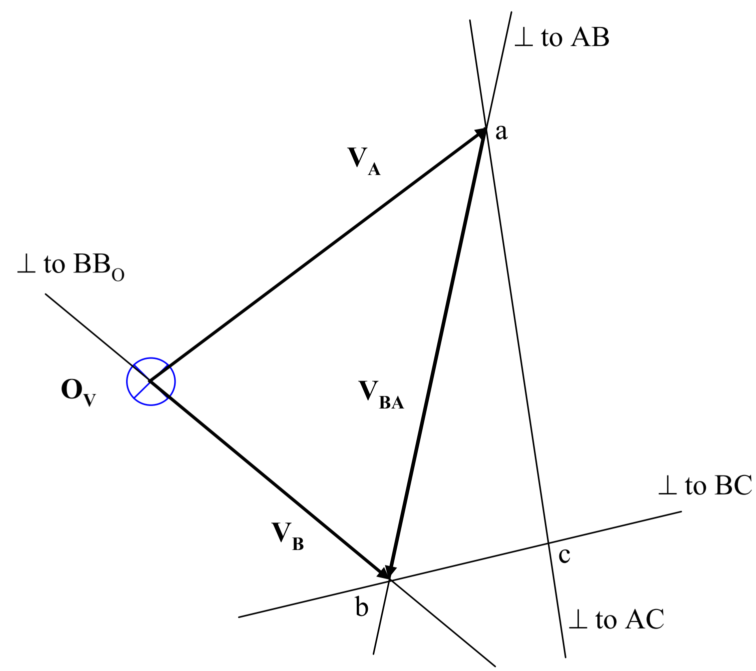


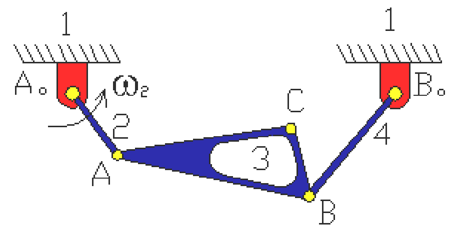


$$\mathbf{V}_C = \mathbf{V}_A + \mathbf{V}_{CA}$$

$$\mathbf{V}_C = \mathbf{V}_B + \mathbf{V}_{CB}$$

Then draw a line passing through b and perpendicular to BC. The intersection of these two lines determines the position of point c.



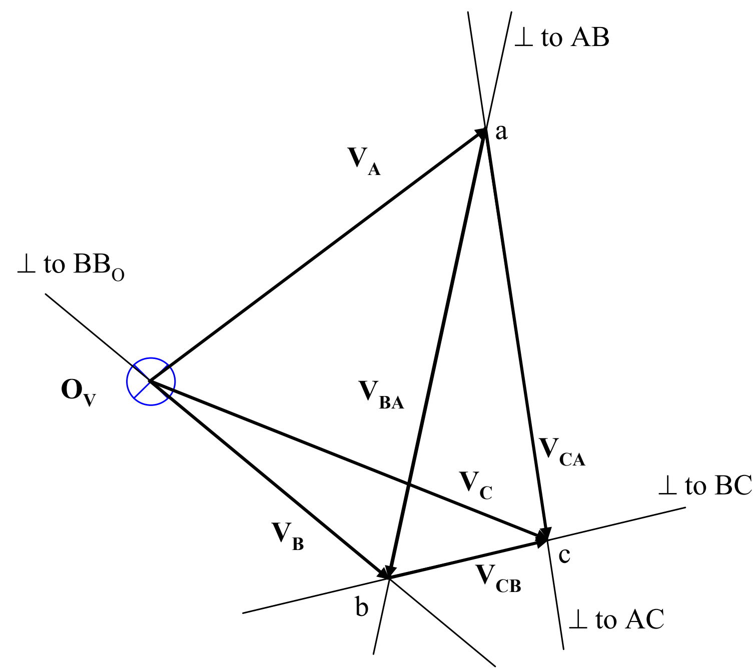


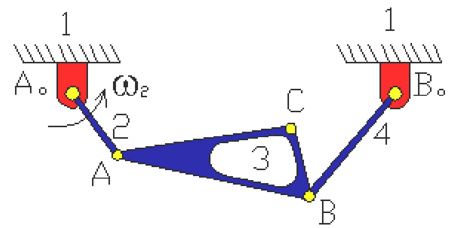
Then draw a vector from the origin, $\mathbf{O_V}$ to point c. This is the absolute velocity of point C on the coupler (link 3).

The relative velocities of $\mathbf{V_{CA}}$ and $\mathbf{V_{CB}}$ are the vectors from a to c and b to c respectively, satisfying the relative velocity equations considered:

$$\mathbf{V_C} = \mathbf{V_A} + \mathbf{V_{CA}}$$

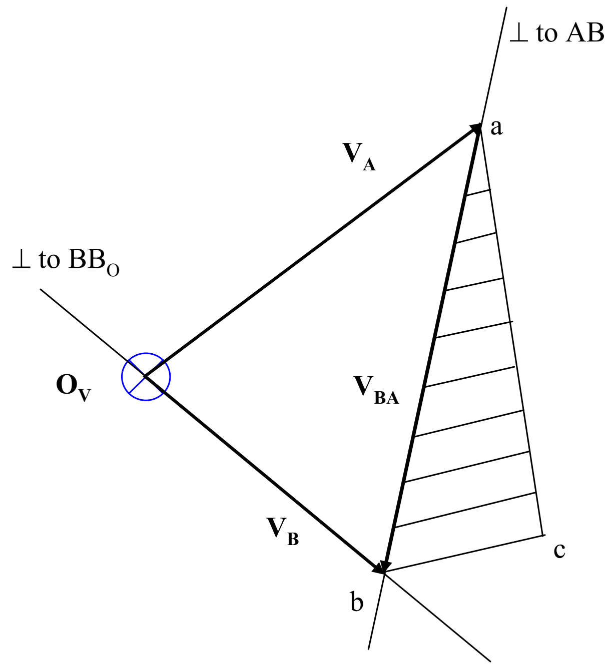
$$\mathbf{V_C} = \mathbf{V_B} + \mathbf{V_{CB}}$$

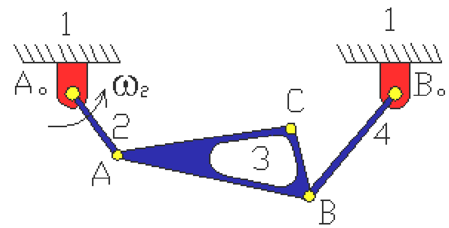




The velocity of point C can be found using the 'velocity image' concept.

Draw the velocity image of triangle ABC on the velocity diagram (triangle abc) to determine the location of point c.

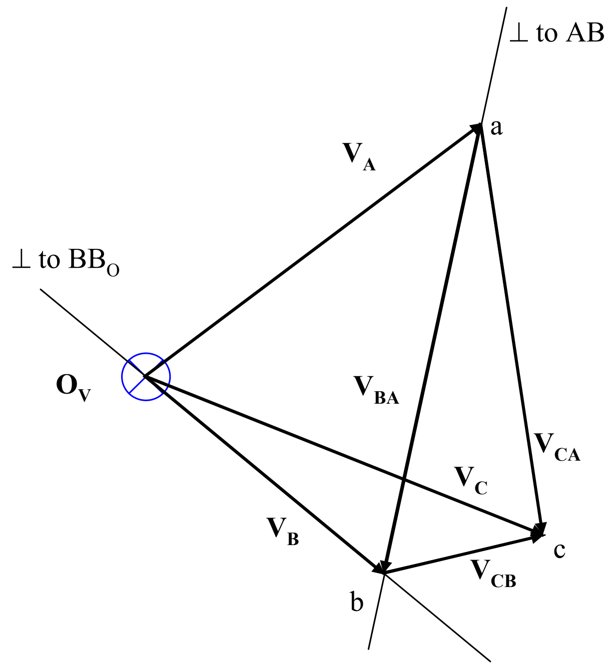


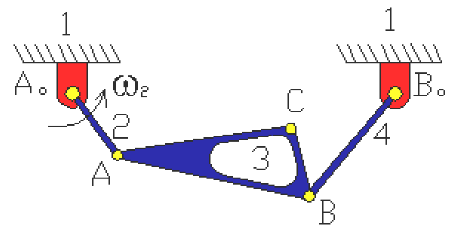


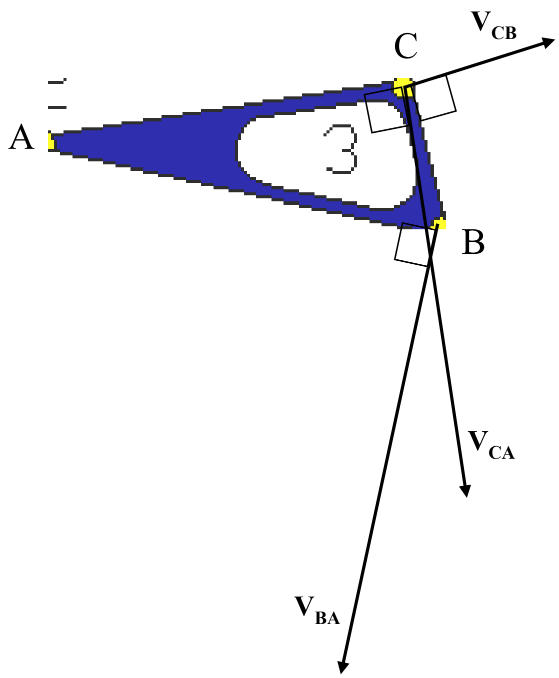
Then draw a vector from the origin, $\mathbf{O_V}$ to point c. This is the absolute velocity of point C on the coupler (link 3).

The relative velocities of $\mathbf{V_{CA}}$

and $\mathbf{V_{CB}}$ are the vectors from points a to c and b to c respectively.



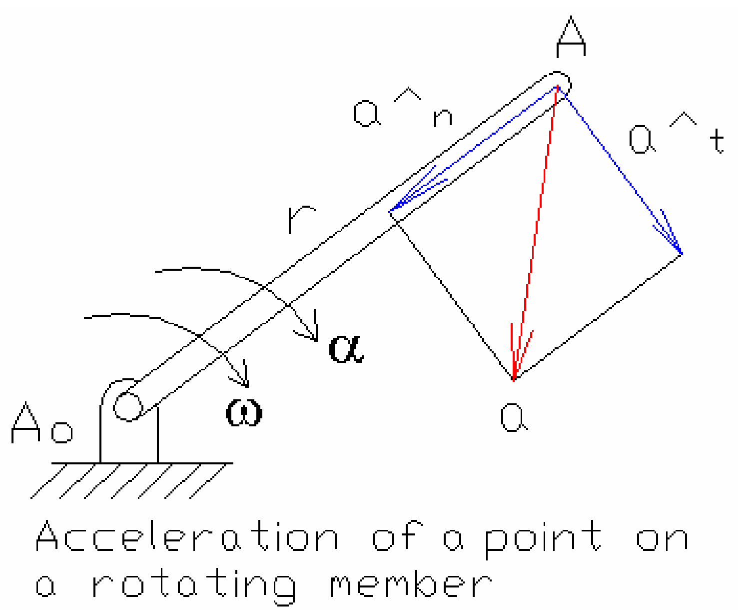




You can use either of the three relative velocity equations to determine the direction and magnitude of the angular velocity of link 3.

$\omega_3 = \mathbf{V_{CB}}/CB = \mathbf{V_{CA}}/CA = \mathbf{V_{BA}}/BA$

Graphical Acceleration Analysis



If a rotating member has no angular acceleration (**a^t = 0**) the resulting acceleration is purely the value of **a^n**.

Tangential Acceleration, a^t :

this is the rate of change of the tangential velocity of the crank pin A, and its magnitude is equal to the angular acceleration **a**, multiplied by the distance **r**.

$a^t = \alpha" r$

Normal Acceleration, a^n :

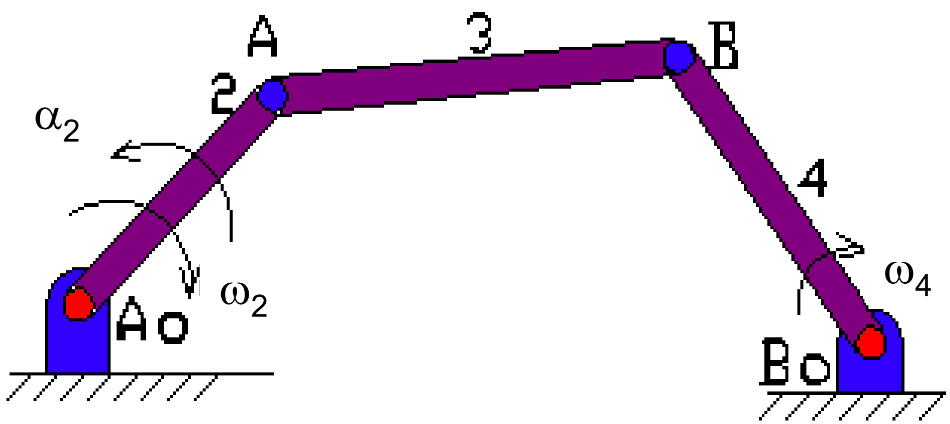
is the acceleration directed towards the centre of the rotating member. The magnitude of the normal acceleration is equivalent to**:**

$a^n = \omega^2 r = V_A^2 / r$

The resultant acceleration will be**:**

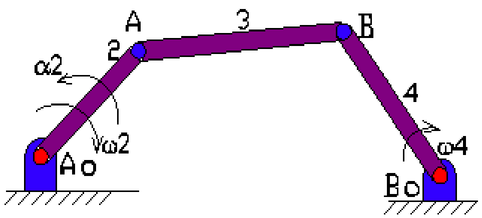
$\mathbf{a = a^t + a^n}$

Example:

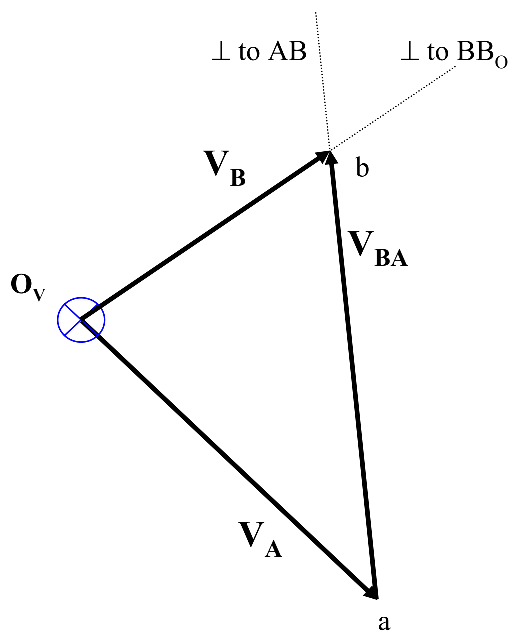


Example: Given the four-bar linkage, the angular velocity of crank 2, (ω_2) is 12 rad/s (CW) and its angular acceleration (α''_2) is 60 rad/s^2 (CCW).

Find the angular acceleration of link 3, and the angular velocity and acceleration of link 4.

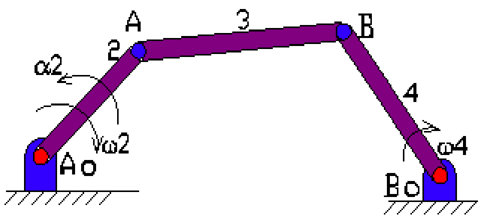


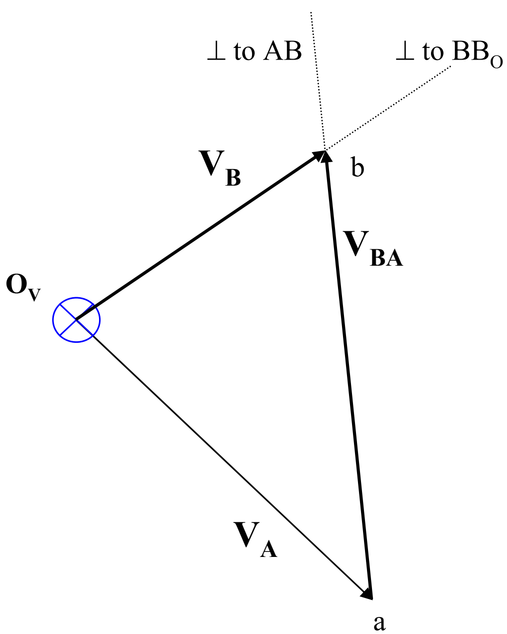
First draw a scaled diagram of the given mechanism, then construct the velocity diagram for it.



$$\mathbf{V_B = V_A + V_{BA}}$$

Remember, before attempting the acceleration analysis you must complete the velocity analysis.





Acceleration equation to be solved is:

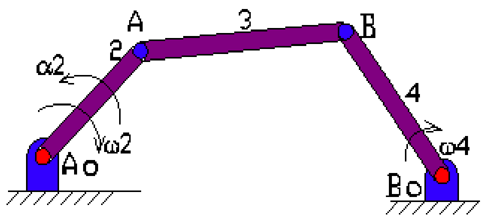
$$\mathbf{a_B = a_A + a_{BA}}$$

$\mathbf{a_A}$ is the acceleration of point A. Since the angular acceleration of link 2 is **not** zero, there will be both normal ($\mathbf{a^n_A}$) and tangential ($\mathbf{a^t_A}$) components of this acceleration.

$\mathbf{a_B}$ and $\mathbf{a_{BA}}$ are the absolute acceleration of point B and the relative acceleration of point B wrt A, respectively. Both may have a normal and a tangential component each.

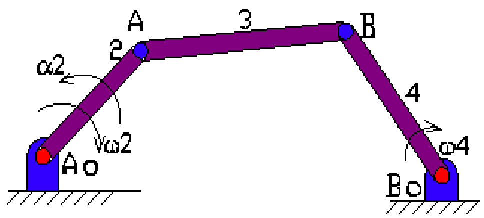
Therefore the above acceleration equation becomes:

$$\mathbf{a^n_B + a^t_B = a^n_A + a^t_A + a^n_{BA} + a^t_{BA}}$$



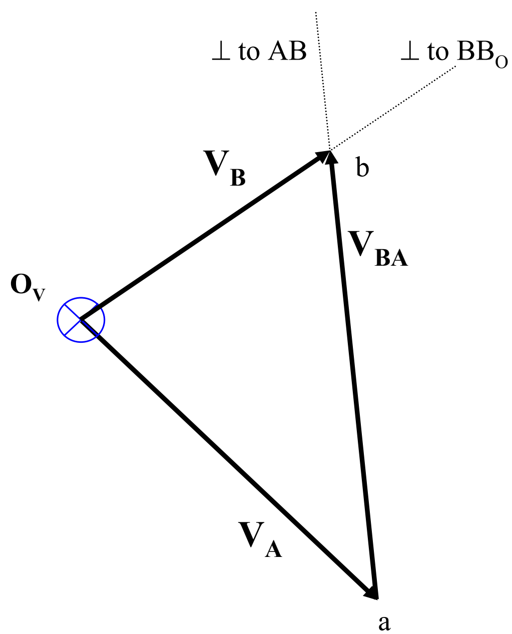
$$\underline{\underline{\mathbf{a^n_B}}} + \underline{\mathbf{a^t_B}} = \underline{\underline{\mathbf{a^n_A}}} + \underline{\underline{\mathbf{a^t_A}}} + \underline{\underline{\mathbf{a^n_{BA}}}} + \underline{\mathbf{a^t_{BA}}}$$

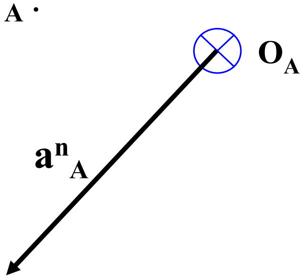
$a^n_A = \omega_2^2\, r_2$	both ω_2 and r_2 are known, hence the magnitude is known; direction is from A towards A_O
$a^t_A = \alpha''_2\, r_2$	magnitude is known since = α_2 is given; direction is perpendicular to that of a^n_A or perpendicular to AA_O
$a^n_{BA} = \omega_3^2\, r_{AB}$	both ω_3 and r_{AB} are known, hence the magnitude is known; direction is from B towards A
$a^t_{BA} = \alpha''_3\, r_{AB}$	direction is perpendicular to that of a^n_{BA} or perpendicular to BA
$a^n_B = \omega_4^2\, r_4$	both ω_4 and r_4 are known, hence the magnitude is known; direction is from A towards A_O
$a^t_B = \alpha_4\, r_4$	direction is perpendicular to that of a^n_B or perpendicular to AA_O

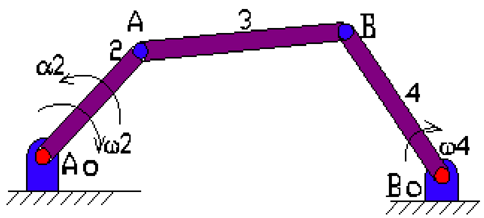


$\mathbf{a^n_B + a^t_B = a^n_A + a^t_A + a^n_{BA} + a^t_{BA}}$

Choose a scale factor and an origin for the acceleration diagram. You can now begin to construct the acceleration diagram. First draw the normal component of the absolute acceleration of point A, $\mathbf{a^n_A}$:

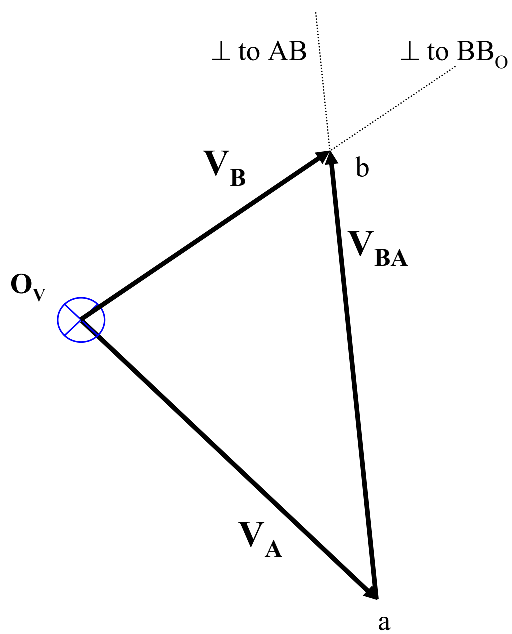


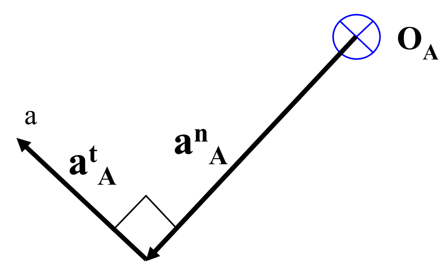


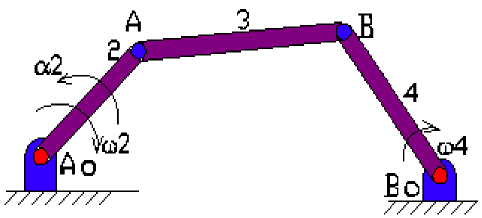


$$\mathbf{a}^n_B + \mathbf{a}^t_B = \mathbf{a}^n_A + \mathbf{a}^t_A + \mathbf{a}^n_{BA} + \mathbf{a}^t_{BA}$$

Then draw tangential component of the absolute acceleration of point A, and $\mathbf{a}^t_A$:

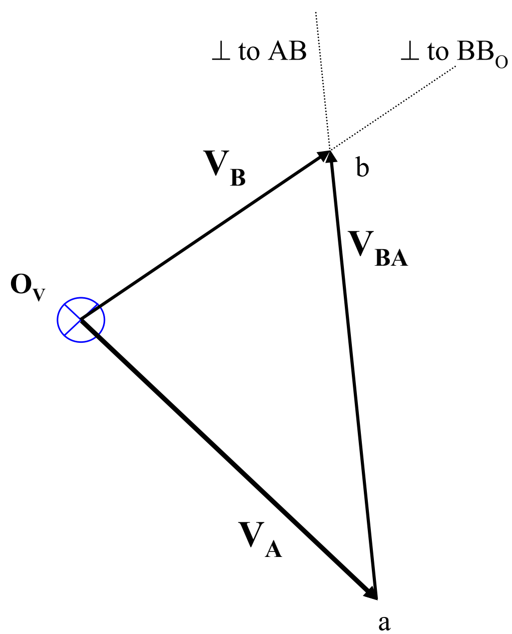


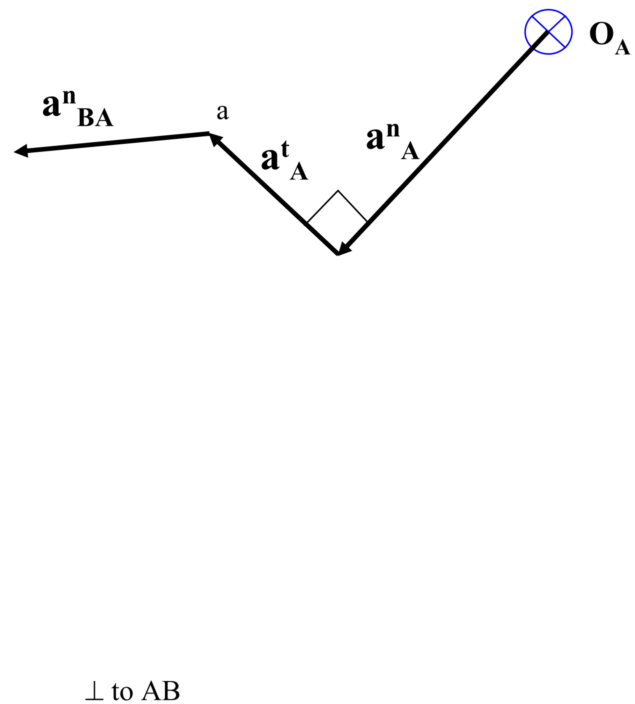


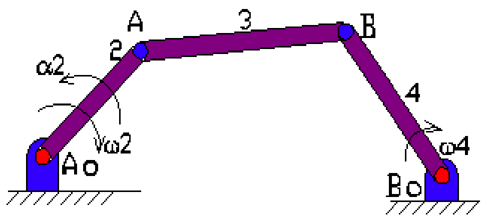


$$\mathbf{a^n_B + a^t_B = a^n_A + a^t_A + a^n_{BA} + a^t_{BA}}$$

Then draw the normal component of the relative acceleration of point B wrt A, $\mathbf{a^n_{BA}}$, parallel to BA pointing from B to A :

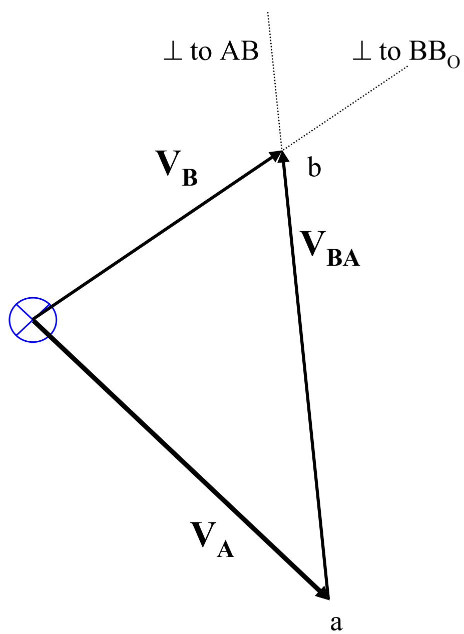


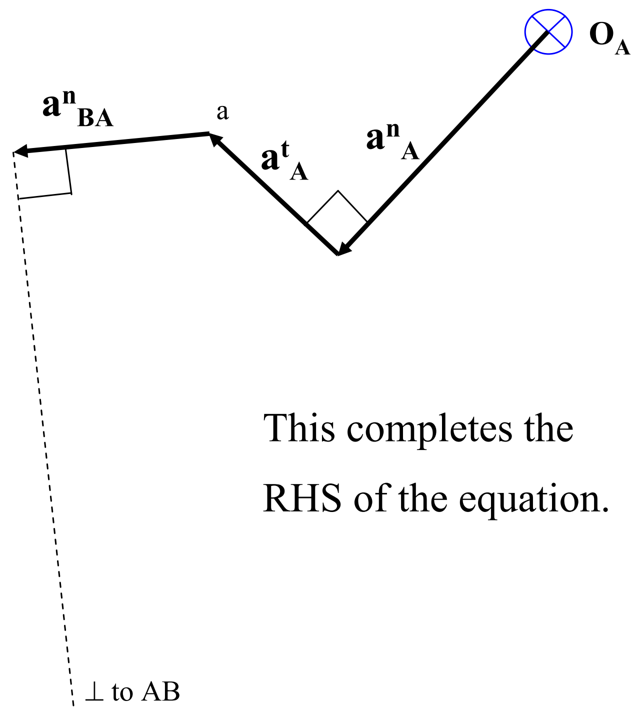




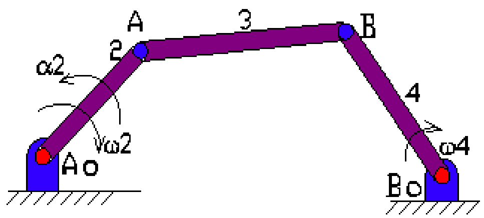
$$\mathbf{a^n_B} + \mathbf{a^t_B} = \mathbf{a^n_A} + \mathbf{a^t_A} + \mathbf{a^n_{BA}} + \mathbf{a^t_{BA}}$$

Then draw a line perpendicular to this vector at its head to indicate the direction of the tangential component of the acceleration of point B wrt A, $\mathbf{a^t_{BA}}$.



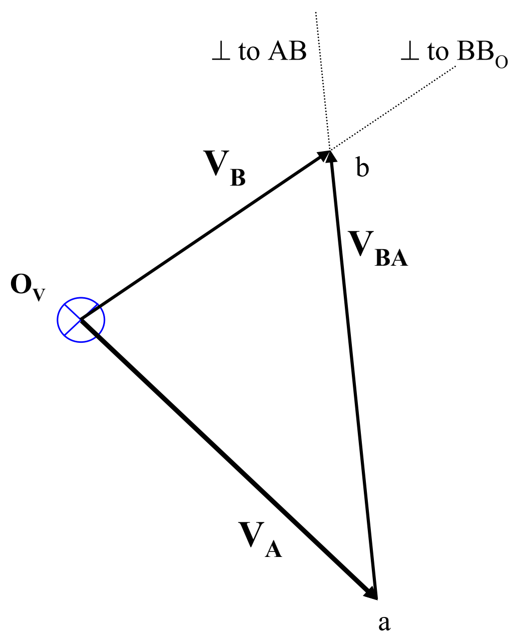


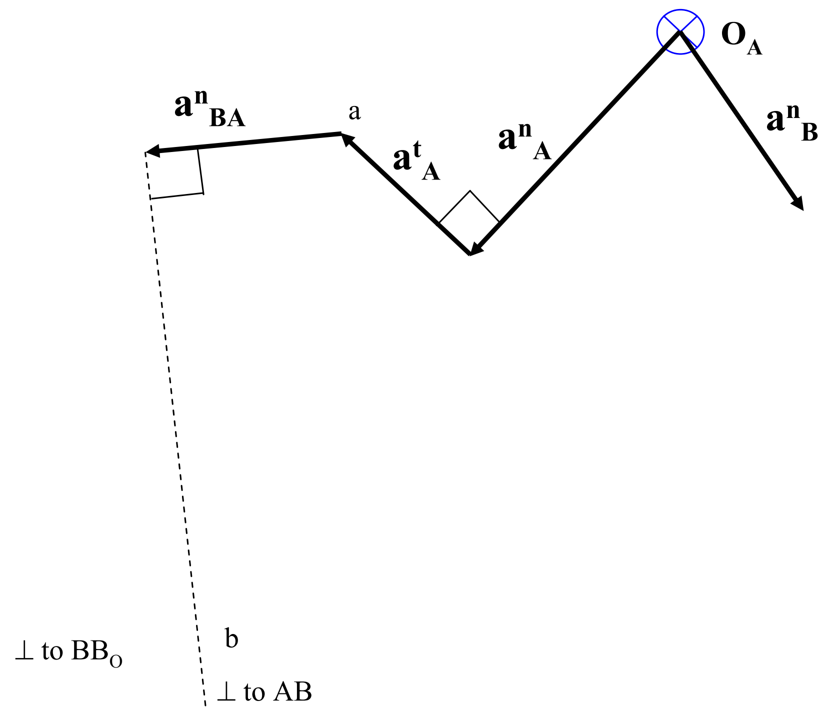
This completes the RHS of the equation.

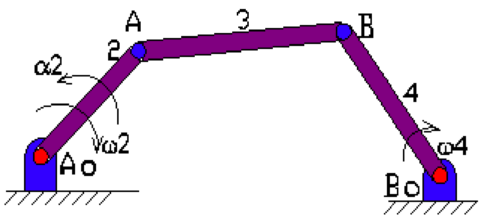


$$a^n_B + a^t_B = a^n_A + a^t_A + a^n_{BA} + a^t_{BA}$$

Then draw the normal component, at the origin, of the absolute acceleration of point B, $\mathbf{a^n_B}$, parallel to BB_O pointing from B to B_O :

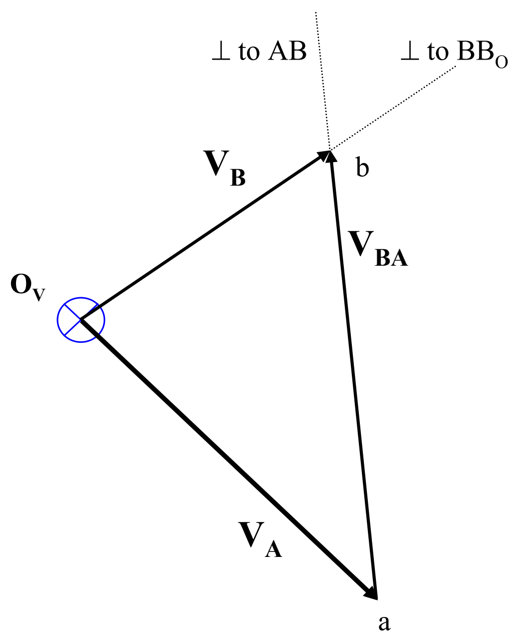


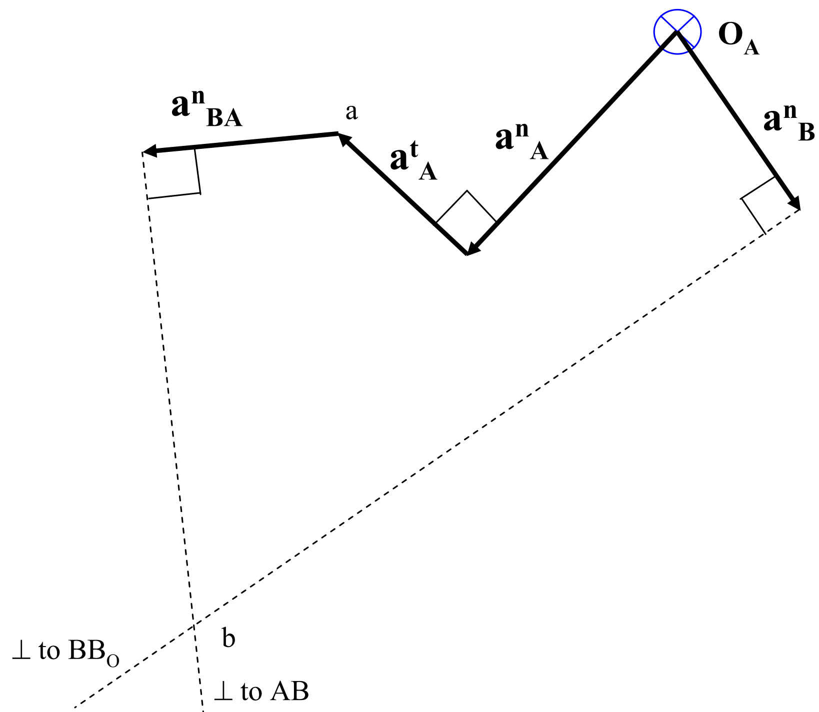


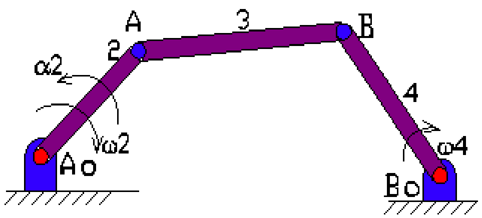


$$a^n_B + a^t_B = a^n_A + a^t_A + a^n_{BA} + a^t_{BA}$$

Then draw a line perpendicular to this vector at its head to indicate the direction of the tangential component of the acceleration of point B, $\mathbf{a^t_B}$.

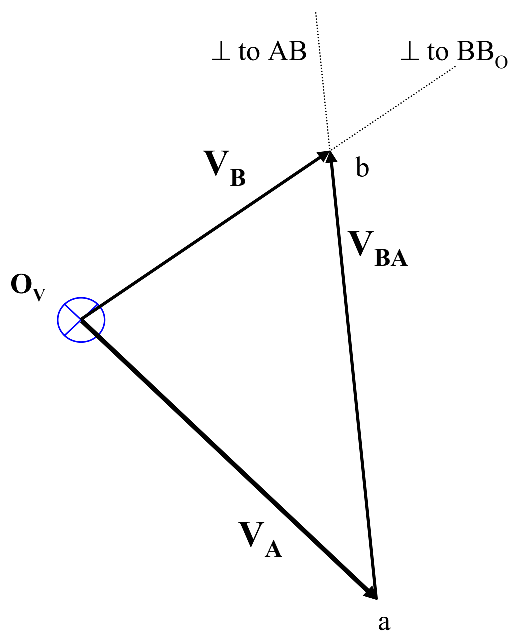


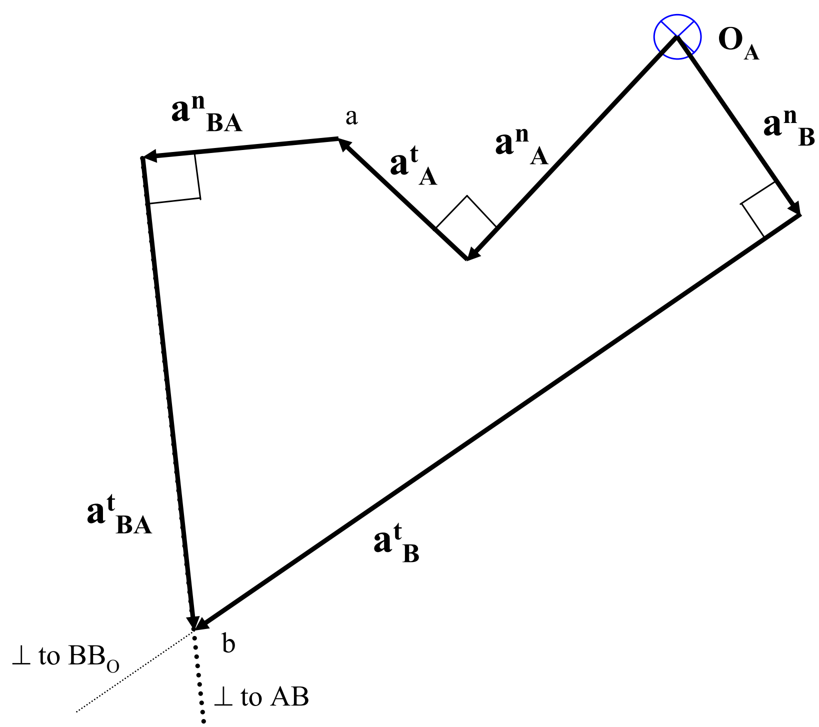


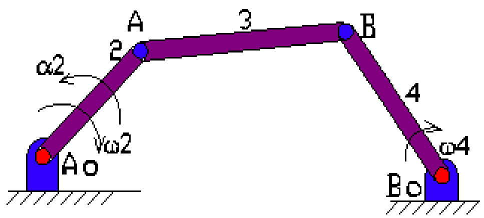


$$\mathbf{a^n_B + a^t_B = a^n_A + a^t_A + a^n_{BA} + a^t_{BA}}$$

The intersection of these two lines completes the acceleration polygon, determining the unknown magnitudes, $\mathbf{a^t_B}$ and $\mathbf{a^t_{BA}}$.



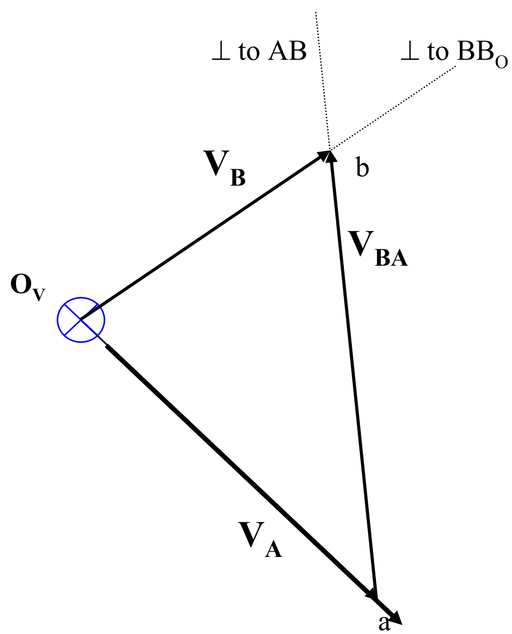


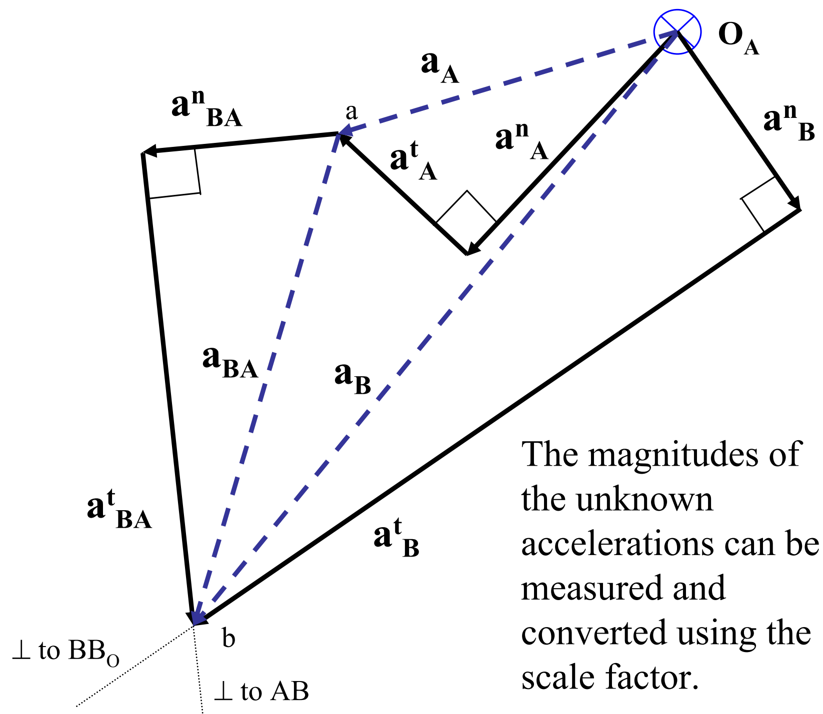


$$a^n{}_B + a^t{}_B = a^n{}_A + a^t{}_A + a^n{}_{BA} + a^t{}_{BA}$$

Note that the dashed lined vectors represent the original acceleration equation:

$$a_B = a_A + a_{BA}$$

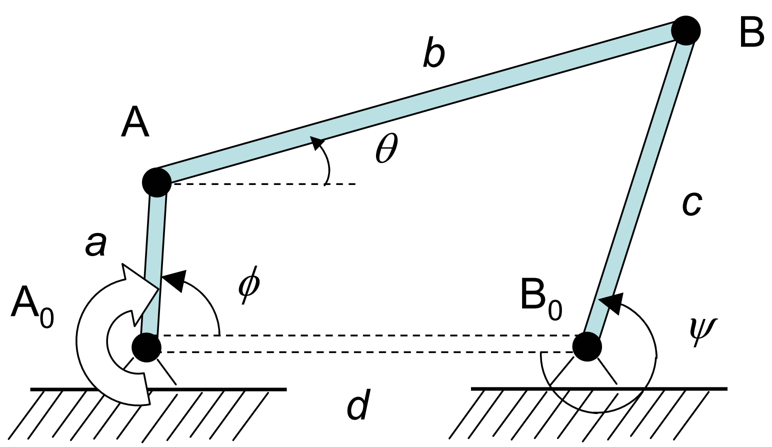




The magnitudes of the unknown accelerations can be measured and converted using the scale factor.

Analysis of a four-bar linkage:

- The 4-bar linkage can be analysed graphically to determine the velocities and accelerations. (method discussed before)
  - *Drawbacks*: Accuracy is limited and many diagrams are required
- In many cases it is advisable to express the problem analytically:
- There are two possible cases:
  1. Analysis: For a given input (ϕ) we can solve for the output (ψ) and obtain the velocity and acceleration.
  2. Synthesis: Calculate the link ratios for a given input-output relationship



- Projections on the two axes give:

$$\begin{matrix} x-axis: \\ y-axis: \end{matrix} \quad \begin{cases} a\cos\phi + b\cos\theta - c\cos(\psi - 180^\circ) - d = 0 \\ a\sin\phi + b\sin\theta - c\sin(\psi - 180^\circ) = 0 \end{cases} \longrightarrow \begin{cases} a\cos\phi + b\cos\theta + c\cos\psi = d \\ a\sin\phi + b\sin\theta + c\sin\psi = 0 \end{cases}$$

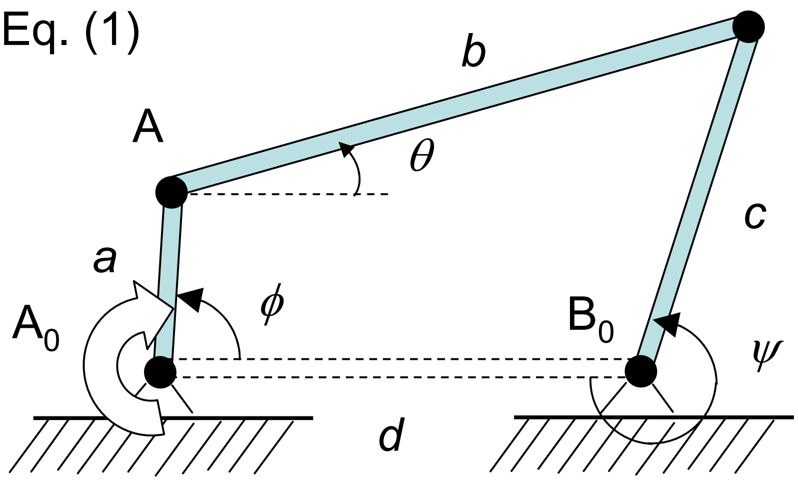
- We are looking for a relationship between the input (ϕ) and the output (ψ):

$$\begin{cases} b\cos\theta = d - \left(a\cos\phi + c\cos\psi\right) \\ b\sin\theta = -\left(a\sin\phi + c\sin\psi\right) \end{cases}$$

- We square and the equations. The resulting equation can be written more simply as:

$$\boxed{\begin{array}{l} K_1\cos\phi + K_2\cos\psi - K_3 = \cos\left(\phi - \psi\right) \\ where: \\ K_1 = \dfrac{d}{c} \qquad K_2 = \dfrac{d}{a} \qquad K_3 = \dfrac{a^2 - b^2 + c^2 + d^2}{2ac} \end{array}}$$

Eq. (1)



- This equation is fundamental in the design of 4-bar mechanisms!

"Freudenstein's equation"

1. Differentiate eq. (1) implicitly with time:

$$-(K_1 \sin\phi)\dot{\phi} - (K_2 \cos\psi)\dot{\psi} = -(\dot{\phi} - \dot{\psi})\sin(\phi - \psi) \qquad \text{Eq. (2)}$$

– $\dot{\varphi}$ is the angular velocity of the input
– $\dot{\psi}$ is the angular velocity of the output

2. Solving for $\dot{\psi}$ yields: $\boxed{\dfrac{\dot{\psi}}{\dot{\phi}} = \dfrac{\sin(\phi-\psi) - K_1 \sin\phi}{\sin(\phi-\psi) - K_2 \sin\phi}}$ Eq. (3)

3. We can derive the expression of ψ as a function of ϕ by rewriting Eq (1):

$$A\sin\psi + B\cos\psi = C \rightarrow \begin{cases} A = \sin\phi \\ B = \cos\phi - K_2 \\ C = K_1\cos\phi - K_3 \end{cases}$$

$$\leftarrow \begin{cases} \sin\psi = \dfrac{2\tan(\psi/2)}{1+\tan^2(\psi/2)} \\ and \\ \cos\psi = \dfrac{1-\tan^2(\psi/2)}{1+\tan^2(\psi/2)} \end{cases}$$

Eq. (4) $$\boxed{\psi = 2\arctan\left(\frac{A \pm \sqrt{A^2 + B^2 - C^2}}{B + C}\right)}$$

Example 1:

In the design of a special purpose machine incorporating a 4-bar linkage, the following link ratios were obtained:

$$\frac{a}{d} = -0.6601; \quad \frac{b}{d} = 1.4553; \quad \frac{c}{d} = -0.5042$$

If d=150 mm, calculate the *output position* and the *angular velocity ratio* for input values of 20^{o},40^{0} and 65^{o}.

The functional relationship* was $\psi = 240 + 0.095\phi^{1.5}$

---

* This is not the actual relationship between ψ and ϕ. This is the function which we wished to generate with the mechanism, before the mechanism was designed. The actual function between ψ and ϕ should be computed with Eq. (4)

1. Equation (3) gives the ratio $\left(\dot{\psi}/\dot{\varphi}\right)$ if we know ψ
2. ψ will be computed with equation (4). However, Eq. (4) gives 2 values for ψ. Before we can proceed, we have to decide which one to use.
3. From the functional relationship the total angular movement of the output is:

$$\Delta\psi = \psi\left(65^\circ\right) - \psi\left(20^\circ\right) = 240 + 0.095 \cdot 65^{1.5} - \left(240 + 0.095 \cdot 20^{1.5}\right) = 41.287^\circ$$

4. One root of equation (4) will give a value close to 41.287°, and the other one will not.
5. From Eq. (1) $K_1 = \frac{d}{c} = \frac{1}{\text{-0.5042}} = \text{-1.9833}; \quad K_2 = \frac{d}{a} = \text{-1.5149}; \quad K_3 = -0.6429$
6. Compute A,B and C.
7. Compute Eq. (4) for ϕ=20° and ϕ=65° and $\Delta\psi=\psi(65^\circ)-\psi(20^\circ)$
8. For positive root we have $\Delta\psi^+$=7.14° and for the negative root $\Delta\psi^-$=41.41°
9. The <u>*negative root*</u> is the correct one

$$-\left(K_1 \sin\phi\right)\dot{\phi} - \left(K_2 \cos\psi\right)\dot{\psi} = -\left(\dot{\phi} - \dot{\psi}\right)\sin\left(\phi - \psi\right) \qquad (2)$$

$$\dot{\psi}/\dot{\phi} = \left(\sin\left(\phi - \psi\right) - K_1 \sin\phi\right) / \left(\sin\left(\phi - \psi\right) - K_2 \sin\phi\right) \qquad (3)$$

$$\psi = 2\arctan\left(\left(A \pm \sqrt{A^2 + B^2 - C^2}\right) / (B + C)\right) \qquad (4)$$

10. Use equation 4 and 3 to compute the required values:

ϕ	ψ	$(\dot{\psi}/\dot{\varphi})$
20	-111.57	0.6613
40	-95.96	0.8947
65	-70.16	1.1748

$$-(K_1 \sin\phi)\dot{\phi} - (K_2 \cos\psi)\dot{\psi} = -(\dot{\phi} - \dot{\psi})\sin(\phi - \psi) \qquad (2)$$

$$\dot{\psi}/\dot{\phi} = (\sin(\phi-\psi) - K_1 \sin\phi)/(\sin(\phi-\psi) - K_2 \sin\phi) \qquad (3)$$

$$\psi = 2\arctan\left(\left(A \pm \sqrt{A^2 + B^2 - C^2}\right)/(B+C)\right) \qquad (4)$$