Problem 1 [60 points] Lyapunov analysis for unforced and forced systems

(a) [15 + 15 = 30 points] Asymptotic stability for the unforced system

(i) Consider the nonlinear system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -h_1(x_1) - h_2(x_2),$$

where for $i = 1, 2$, the functions $h_i(\cdot)$ are locally Lipschitz, $h_i(0) = 0$ and $yh_i(y) > 0 \; \forall \, y \in (-a, a)$ for some nonzero real constant a. Clearly, the origin is a fixed point but there may be more fixed points depending on the nonlinear functions $h_i(\cdot)$. **Prove that** the origin is AS.

To do this, motivated by the pendulum example in Lec. 3, p. 13-14 and Lec. 4, p. 1-2, consider the Lyapunov function $V(x_1, x_2) = \int_0^{x_1} h_1(y)\,\mathrm{d}y + \frac{1}{2}x_2^2$. This Lyapunov function can be thought of as a generalized energy: the intergral term is a generalized potential energy and the second summand is a generalized kinetic energy. You may need to use the LaSalle invariance.

(ii) Let us consider a specific instance of the above given by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\alpha x_1^3 - \beta x_2, \qquad \alpha, \beta > 0.$$

Prove that the origin for this system is in fact GAS.

(i)

1. $\int_0^{x_1} h_1(y)$ scales to any function $h_1(x_1)$ by a x_1 term, and we are told $x_1 h_1(0) = 0, x_1 h_1(x_1) > 0$. In addition $\frac{1}{2}x_2^2$ is pos def. So $V(x_1, x_2)$ is positive definite since $V(0) = 0, V(\geq 0) \geq 0$
2. Derive $\dot{V}$:

$$\begin{aligned}
\dot{V}(x_1, x_2) &= \begin{bmatrix} h_1(x_1) & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -h_1(x_1) - h_2(x_2) \end{bmatrix} && (1)\\
&= h_1(x_1)x_2 - h_1(x_1)x_2 - x_2 h_2(x_2) && (2)\\
&= -x_2 h_2(x_2) && (3)
\end{aligned}$$

3. $x_2 h_2(x_2) > 0 \Rightarrow \dot{V} = -x_2 h_2(x_2) < 0$
4. From 3., $\dot{V}(0) = 0, \dot{V}(\neq 0) < 0$, so $\dot{V}$ is **negative semi-definite**
5. Let the set $\mathcal{S} = \{x \quad s.t. \, \dot{V}(x) = -x_2 h_2(x_2) = 0\} = \{x \quad s.t. -x_2 = 0\}$
6. Since $\dot{x_1} = x_2, x \in \mathcal{S} \Rightarrow \dot{x_1} = 0$
7. Since $\dot{x_2} = -h_1(x_1) - h_2(x_2), x \in \mathcal{S} \Rightarrow \dot{x_2} = -h_1(x_1)$
8. From 7., $\dot{x_2} = 0 \Rightarrow x_1 = 0$
9. From 6. and 8., only $x \in \mathcal{S} \quad s.t. \, \dot{x} = 0 \Rightarrow x = (0, 0)$
10. From 9 and by LaSalle's Invariance Thm, we see that $(0, 0)$ is **A.S.**

(ii)

1. We see that $h_1(x_1) = \alpha x_1^3$
2. We can see $V(x_1, x_2) = \frac{\alpha x_1^4}{4} + \frac{x_2^2}{2}$
3. We can see also that $\lim_{\|x\|_2 \to +\infty} V = +\infty$, thus V is **radially unbounded**
4. From (i) we showed the origin is **A.S.**, and from Lecture 3 pg 9 and 3., we see the origin is also **G.A.S.**

(b) [30 points] Finite gain $\mathcal{L}_2$ stability for the forced system

Now consider the forced system in input-output form given by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -\alpha x_1^3 - \beta x_2 + u, \qquad \alpha, \beta > 0, \qquad y = x_2.$$

Use Lec. 10, p. 1-2 to **prove that** the above system is finite gain $\mathcal{L}_2$ stable **by deriving an upper bound on $\mathcal{L}_2$ gain in terms of parameters** α, β.

(**Hint:** to use the Hamilton-Jacobi inequality theorem in Lec. 10, p. 1-2, choose the function V to be a positive scaling of the Lyapunov function in part (a). This will lead to an inequality involving the gain upper bound γ, the scaling and the parameters. Requiring γ to be smallest will yield the optimal scaling.)

1. We can see for this system (picking $V = K * V_{part(a)}$ for some $K > 0$:

$$\begin{aligned}
h(*) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} && (4)\\
g(*) &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} && (5)\\
\frac{\partial V}{\partial x} &= \begin{bmatrix} K\alpha x_1^3 & K x_2 \end{bmatrix} && (6)\\
\frac{\partial V}{\partial t} &= \frac{\partial V}{\partial x} f(*) = -\beta K x_2^2 && (7)
\end{aligned}$$

2. Constructing and solving the H-J inequality:

$$\begin{aligned}
\frac{\partial V}{\partial t} + \frac{1}{2\gamma^2}\frac{\partial V}{\partial x} g g^T \frac{\partial V}{\partial x}^T + \frac{1}{2}h^T h &\leq 0 && (8)\\
= -\beta K x_2^2 + \frac{1}{2\gamma^2}K^2 x_2^2 + \frac{1}{2}x_2^2 &\leq 0 && (9)\\
= -\beta K + \frac{1}{2\gamma^2}K^2 + \frac{1}{2} &\leq 0 && (10)\\
\Rightarrow -\beta K + \frac{1}{2\gamma^2}K^2 + \frac{1}{2} &\leq 0 && (11)\\
\Rightarrow -2\gamma^2 \beta K + K^2 + \gamma^2 &\leq 0 && (12)\\
\Rightarrow K^2 &\leq 2\gamma^2 \beta K - \gamma^2 && (13)\\
\Rightarrow K^2 &\leq (2\beta K - 1)\gamma^2 && (14)\\
\Rightarrow \gamma^2 &\geq \frac{K^2}{2\beta K - 1} && (15)\\
\Rightarrow \gamma &\geq \sqrt{\frac{K^2}{2\beta K - 1}} && (16)
\end{aligned}$$

3. We can see that the RHS in 2. will be positive as long as the denominator is positive. To achieve this, we can select K s.t.

$$\begin{aligned}
2\beta K - 1 &\geq 0 && (17)\\
\Rightarrow K &\geq \frac{1}{2\beta} && (18)\\
\Rightarrow Let K &= \frac{1}{\beta} && (19)\\
\Rightarrow \gamma &\geq \sqrt{\frac{K^2}{2 - 1}} && (20)\\
\Rightarrow \gamma &\geq \sqrt{K^2} && (21)\\
\Rightarrow \gamma &\geq K && (22)\\
\Rightarrow \gamma &\geq \frac{1}{\beta} && (23)
\end{aligned}$$

4. From 3. (RHS is positive, thus $\gamma > 0$), we've shown there exists such an upper bound, therefore the above system is $\mathcal{L}_2$ **stable**

Problem 2 [40 points] $\mathcal{L}_p$ gain for composition

In Lec. 10, p. 3-4, we stated that the finite gain $\mathcal{L}_p$ stability is preserved under series and parrallel compositions of the subsystems. **Prove the same by deriving $\mathcal{L}_p$ gain upper bound for the overall system in terms of the $\mathcal{L}_p$ gain upper bounds for the subsystems**.

1. Let system 1 be $\mathcal{L}_p$ stable, then by definition $\|y_1\|_2 \leq \gamma_1 \|u_1\|_2$.
2. Let system 2 be likewise $\mathcal{L}_p$ stable, $\|y_2\|_2 \leq \gamma_1 \|u_2\|_2$
3. In series by definition $u_2 == y_1$, so:

$$\begin{aligned}
\|y_2\|_2 \leq \gamma_2 \|u_2\|_2 &= \gamma_2 \|y_1\|_2 && (24)\\
\Rightarrow \|y_2\|_2 \leq \gamma_2 \|y_1\|_2 &\leq \Rightarrow \gamma_2\gamma_1 \|u_1\|_2 && (25)\\
\Rightarrow \|y_2\|_2 &\leq \gamma_2\gamma_1 \|u_1\|_2 && (26)\\
\Rightarrow \frac{\|y_2\|_2}{\|u_1\|_2} &\leq \gamma_2\gamma_1 && (27)
\end{aligned}$$

   We see the RHS is a product of positive terms, so the LHS is also positive
   So system 1 in series with system 2 is $\mathcal{L}_p$ **stable with upper bound** $\gamma_2\gamma_1$
4. Likewise, we can show if system 1 is parallel to system 2:

$$\begin{aligned}
\|y\|_2 &= \|y_1\|_2 + \|y_2\|_2 && (28)\\
\Rightarrow \|y\|_2 &\leq \gamma_1 \|u_1\|_2 + \gamma_2 \|u_2\|_2 && (29)\\
\Rightarrow \|y\|_2 &\leq \gamma_1 \|u\|_2 + \gamma_2 \|u\|_2 && (30)\\
\Rightarrow \|y\|_2 &\leq (\gamma_1 + \gamma_2)\|u\|_2 && (31)\\
\Rightarrow \frac{\|y\|_2}{\|u\|_2} &\leq \gamma_2 + \gamma_1 && (32)
\end{aligned}$$

   We see the RHS is a product of positive terms, so the LHS is also positive
   So system 1 parallel with system 2 is $\mathcal{L}_p$ **stable with upper bound** $\gamma_2 + \gamma_1$