Lec. 14 (11/08/2022)

Duality (Ch. 5)

Any (possibly nonconvex) optimization problem:

$$p^* := \min_{\underline{x} \in \mathbb{R}^n} f_0(\underline{x})$$

$$\text{s.t.} \quad f_i(\underline{x}) \leq 0, \quad i = 1, \ldots, m$$

$$h_j(\underline{x}) = 0, \quad j = 1, \ldots, p$$

We say, this problem is the "$\underbrace{\text{primal}}_{\text{(original)}}$ problem"

Construct Lagrangian L where:

$$L : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \mapsto \mathbb{R}$$

$\underline{x} \in \mathbb{R}^n$

$\underline{\lambda} \in \mathbb{R}^m$

$\underline{\nu} \in \mathbb{R}^p$

defined as:

$$L(\underline{x}, \underline{\lambda}, \underline{\nu})$$

Lagrange multipliers

$$:= f_0(\underline{x}) + \langle \underline{\lambda}, \underline{f}(\underline{x}) \rangle + \langle \underline{\nu}, \underline{h}(\underline{x}) \rangle$$

$$\underline{f}(\underline{x}) = \begin{pmatrix} f_1(\underline{x}) \\ \vdots \\ f_m(\underline{x}) \end{pmatrix}, \quad \underline{h}(\underline{x}) = \begin{pmatrix} h_1(\underline{x}) \\ \vdots \\ h_p(\underline{x}) \end{pmatrix}$$

Lagrange dual (function):

$$g : \mathbb{R}^m \times \mathbb{R}^p \longmapsto \mathbb{R}$$

defined as the $\boxed{\text{unconstrained}}$ minimum of L over $\underline{x} \in \mathbb{R}^n$

$$\therefore \; g(\underbrace{\underline{\lambda}}_{\in \mathbb{R}^m}, \underbrace{\underline{\nu}}_{\in \mathbb{R}^p}) := \inf_{\underline{x} \in \mathbb{R}^n} L(\underline{x}, \underline{\lambda}, \underline{\nu})$$

- If L is unbounded below, then $g = -\infty$
- $g(\underline{\lambda}, \underline{\nu})$ is always <u>CONCAVE Function</u> in $(\underline{\lambda}, \underline{\nu})$

Why CONCAVE Function: because g, by def$^{\underline{n}}$, is pointwise inf of affine. (Lec. 9, p. 11)

Notice that even when the primal problem is nonconvex, still $g(\underline{\lambda}, \underline{\nu})$ is concave in $(\underline{\lambda}, \underline{\nu})$.

Relation between $g(\underline{\lambda}, \underline{\nu})$ and p^* :

Statement: $\forall\ \underline{\lambda} \in \mathbb{R}^m_{\geq 0}$ and $\forall\ \underline{\nu} \in \mathbb{R}^p$,

we have:

$$\underbrace{g(\underline{\lambda}, \underline{\nu}) \leq p^*}_{\text{Lower bound on the answer of the primal problem}}$$

Proof: Let $\underline{\tilde{x}}$ be a feasible point for the primal problem and $\underline{\lambda} \in \mathbb{R}^m_{\geq 0}$.

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Then:

$$\underbrace{\langle \underline{\lambda}, \underline{f}(\tilde{\underline{x}}) \rangle}_{\leq 0} + \underbrace{\langle \underline{\nu}, \underline{h}(\tilde{\underline{x}}) \rangle}_{= 0} \leq 0$$

Adding $f_0(\tilde{\underline{x}})$ to both sides:

$$\Leftrightarrow \boxed{L(\tilde{\underline{x}}, \underline{\lambda}, \underline{\nu}) \leq f_0(\tilde{\underline{x}})}$$

$$\Rightarrow \underbrace{\inf_{\underline{x} \in \mathbb{R}^n} L(\underline{x}, \underline{\lambda}, \underline{\nu})}_{=:\, g(\underline{\lambda}, \underline{\nu})} \leq \boxed{L(\tilde{\underline{x}}, \underline{\lambda}, \underline{\nu}) \leq f_0(\tilde{\underline{x}})}$$

Since $g(\underline{\lambda}, \underline{\nu}) \leq f_0(\underline{\tilde{x}}) \ \forall\ \underline{\tilde{x}}$ feasible

therefore, $g(\underline{\lambda}, \underline{\nu}) \leq p^*$ (Proved.)

If $g(\underline{\lambda}, \underline{\nu}) = -\infty$, then $-\infty \leq p^*$.

Example:

$$\boxed{\begin{array}{ll} \min\limits_{\underline{x} \in \mathbb{R}^n} & \underline{x}^T \underline{x} \\ \text{s.t.} & A\underline{x} = \underline{b} \end{array}} \rightarrow \text{QP}$$

where $A \in \mathbb{R}^{p \times n}$

$\underline{b} \in \mathbb{R}^p$

Lagrangian

$$L(\underline{x}, \underline{\nu}) = \underbrace{\underline{x}^T \underline{x}}_{f_0(\underline{x})} + \langle \underline{\nu}, \underbrace{A\underline{x} - \underline{b}}_{\underline{h}(\underline{x})} \rangle$$

$$= \underline{x}^T \underline{x} + \underline{\nu}^T (A\underline{x} - \underline{b})$$

$$\mathrm{dom}(L) = \mathbb{R}^n \times \mathbb{R}^p$$

$$g(\underline{\nu}) = \underbrace{\inf_{\underline{x} \in \mathbb{R}^n} \underbrace{L(\underline{x}, \underline{\nu})}_{\text{convex quadratic } f^n \text{ in } \underline{x}}}_{\text{unconstrained minimization}}$$

"Set the derivative of L w.r.t. $\underline{x}$ $= 0$" and solve for minimizer $\underline{x}^{opt}$:

$$\nabla_{\underline{x}} L(\underline{x}, \underline{\nu}) \Big|_{\underline{x} = \underline{x}^{opt}} = 2\underline{x}^{opt} + A^T \underline{\nu} = 0$$

$$\Rightarrow \underline{x}^{opt} = -\tfrac{1}{2} A^T \underline{\nu}$$

$\therefore$ Substitute back :

$$g(\underline{\nu}) = L\left(\underline{x} = -\tfrac{1}{2}A^T\underline{\nu},\ \underline{\nu}\right)$$

$$= \underbrace{-\frac{1}{4}\underline{\nu}^T(AA^T)\underline{\nu} - \underline{b}^T\underline{\nu}}$$

indeed concave quadratic over $\mathbb{R}^p$

$\therefore$ Our lower bound, for this problem, becomes:

$$\boxed{-\frac{1}{4}\underline{\nu}^T(AA^T)\underline{\nu} - \underline{b}^T\underline{\nu} \le p^* \quad \forall \underline{\nu} \in \mathbb{R}^p}$$

End of example.

So far, we know:

$$g(\underline{\lambda}, \underline{\nu}) \leq p^* \quad \forall\ \underline{\lambda} \in \mathbb{R}^m_{\geq 0}$$
$$\forall\ \underline{\nu} \in \mathbb{R}^p$$

$\therefore$ Tightest lower bound:

$$\underbrace{\underbrace{\underbrace{\sup_{\substack{\underline{\lambda} \in \mathbb{R}^m_{\geq 0} \\ \underline{\nu} \in \mathbb{R}^p}} g(\underline{\lambda}, \underline{\nu})}_{\text{call this scalar/answer } d^*}}_{\text{convex optimization problem!!}}}_{\text{This problem is called the } \boxed{\text{Lagrange dual problem}}} \leq p^*$$

$\therefore$ We always have the inequality:

$$d^* \leq p^*$$

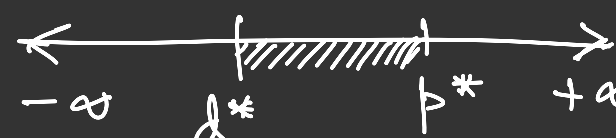
optimal value of the Lagrange dual (convex) problem

optimal value of the primal (possibly nonconvex) problem

<u>Weak duality Theorem</u>: (Primal may be nonconvex)

Always, $d^* \leq p^*$

Duality gap $= p^* - d^*$



We say "strong duality" holds when $\boxed{d^* = p^*}$

Sufficient conditions for strong duality :

If primal problem is convex + $\underbrace{(- - \cdot - \cdot)}_{\text{constraint qualification}}$

Then strong duality holds.

One "constraint qualification" condition is called the "Slater's condition":

<u>Primal problem:</u>

$$\min_{\underline{x} \in \mathbb{R}^n} f_0(\underline{x})$$
$$\text{s.t.} \quad f_i(\underline{x}) \le 0 \quad \forall i = 1, \dots, m$$
$$h_j(\underline{x}) = 0 \quad \forall j = 1, \dots, p$$

$$\left.\begin{array}{l} \exists\, \underline{x} \in \text{relative interior}(\text{dom}) \\ \text{s.t.} \quad f_i(\underline{x}) < 0 \quad \forall i = 1, \dots, m \\ \quad (\text{provided } f_i \text{ are nonlinear}) \\ \quad \text{strict feasibility} \end{array}\right\} \text{Slater's condition}$$

If "convex primal" + "Slater's condition"

then $d^* = p^*$.

- If $f_i(\underline{x})$ are linear in $\underline{x}$, then Slater's condition $\Longleftrightarrow$ primal feasibility

<u>Corollary:</u> LPs & QPs have strong duality.

Notice that even when the primal problem is convex, the dual problem is a different convex problem with different dimension