Lec. 5 (10/06/2022)

Affine Set

$\mathcal{S} \subseteq \mathbb{R}^n$ is affine $\iff$

Affine/Linear Combination

If $\underline{x}_1, \underline{x}_2 \in \mathcal{S}$ then

$\theta \underline{x}_1 + (1-\theta)\underline{x}_2 \in \mathcal{S}$

$\forall \theta \in \mathbb{R}$

$\Updownarrow$

If $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_k \in \mathcal{S}$

then :

$\theta_1 \underline{x}_1 + \theta_2 \underline{x}_2 + \dots + \theta_k \underline{x}_k \in \mathcal{S}$

$\forall \theta_1, \theta_2, \dots, \theta_k \in \mathbb{R}$

s.t. $\theta_1 + \theta_2 + \dots + \theta_k = 1$

Examples

$\mathbb{R}^n$ (✓)

$\mathbb{R}^n_+$ (✗)

$\mathbb{S}^n$ (✓)

Defn: (Subspace) If S is an affine set and $\underline{x}_0 \in S$, then the set

$$V := S - \underline{x}_0 = \{\underline{x} - \underline{x}_0 \mid \underline{x} \in S\}$$

is called a subspace.

- Notice that a subspace V is closed under sum & scalar multiplication:

If $\underline{v}_1, \underline{v}_2 \in V$ then $\alpha \underline{v}_1 + \beta \underline{v}_2 \in V$

$\forall \alpha, \beta \in \mathbb{R}$.

$\therefore$ Affine set = Subspace + Offset

$$\mathcal{S} = \mathcal{V} + \underline{x}_0$$

$$= \{ \underline{v} + \underline{x}_0 \mid \underline{v} \in \mathcal{V} \}$$

__Example:__ Solution set of linear matrix-vector equations

$$\underbrace{A}_{\mathbb{R}^{m\times n}} \underline{x} = \underline{b}$$

$\underline{x} \in \mathbb{R}^n$, $\underline{b} \in \mathbb{R}^m$

__Question:__ Is $\mathcal{S} := \{ \underline{x} \in \mathbb{R}^n \mid A\underline{x} = \underline{b} \}$ affine?

Yes, because if $\boxed{\underline{x}_1, \underline{x}_2 \in S} \iff A\underline{x}_1 = \underline{b}, \; A\underline{x}_2 = \underline{b}$

then $\forall \theta \in \mathbb{R}$, we get :

$$A(\theta \underline{x}_1 + (1-\theta)\underline{x}_2) = \theta \underbrace{A\underline{x}_1} + (1-\theta)\underbrace{A\underline{x}_2}$$

$$= \theta \underline{b} + (1-\theta)\underline{b}$$

$$= \underline{b}$$

$\therefore \theta \underline{x}_1 + (1-\theta)\underline{x}_2 \in S \quad \forall \theta \in \mathbb{R}$

$\therefore S$ is an affine set.

$$\underbrace{\mathcal{N}}_{\text{subspace}} = \text{nullspace or kernel of matrix } A$$

$$= \{ \underline{x} \in \mathbb{R}^n \mid \underset{m\times n}{A} \; \underset{n\times 1}{\underline{x}} = \underset{m\times 1}{\underline{0}} \}$$

End of example.

Affine hull of S: Suppose $S \subseteq \mathbb{R}^n$ is NOT affine.

$$\underbrace{\mathrm{aff}(S)}_{\text{affine hull of } S} := \left\{ \theta_1 \underline{x}_1 + \dots + \theta_2 \underline{x}_k \;\middle|\; \underline{x}_1, \dots, \underline{x}_k \in S, \; \sum_{i=1}^{k} \theta_i = 1, \; \theta_1, \dots, \theta_k \in \mathbb{R} \right\}$$

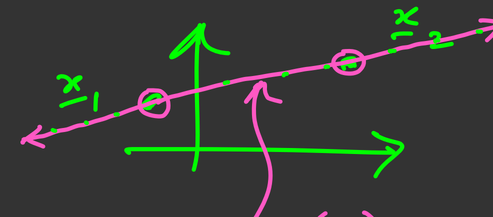
smallest affine set containing S

$$S \subseteq \mathrm{aff}(S)$$

Examples:

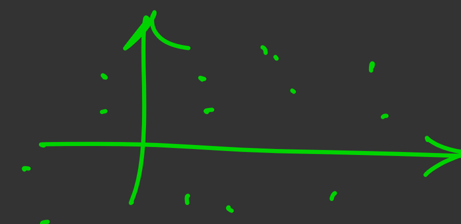
- $\mathcal{S} = \underbrace{\{\underline{x}_1, \underline{x}_2\}}_{\text{finite set}} \subset \mathbb{R}^2$

(collection of 2 points in $\mathbb{R}^2$)



aff($\mathcal{S}$), in this case, is the entire straight line connecting $\underline{x}_1$ & $\underline{x}_2$

- $\mathcal{S} = \underbrace{\{\underline{x}_1, \ldots, \underline{x}_k\}}_{k \geq 3 \text{ points in } \mathbb{R}^2} \subset \mathbb{R}^2$



$$\text{aff}(\mathcal{S}) = \begin{cases} \text{straight line if all points are collinear} \\ \mathbb{R}^2 \text{ if the points are noncollinear} \end{cases}$$

- $S = \{\underline{x}_1, \ldots, \underline{x}_k\} \subset \mathbb{R}^3$

$$\text{aff}(S) = \begin{cases} \text{the straight line} & \text{if points are collinear} \\ \mathbb{R}^2 & \text{if points are coplanar} \\ \mathbb{R}^3 & \text{else} \end{cases}$$

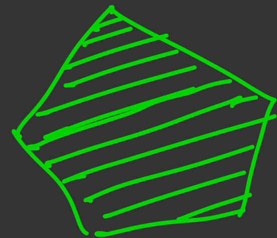
Convex set: (Def^n) If $\forall\, \underline{x}_1, \underline{x}_2 \in S$, $0 \leq \theta \leq 1$,

$$\theta \underline{x}_1 + (1-\theta)\underline{x}_2 \in S$$

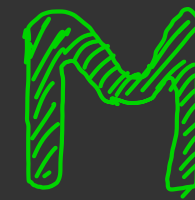
then S is a <u>convex set</u>.

$\theta \underline{x}_1 + (1-\theta)\underline{x}_2$ is an element of this line segment.

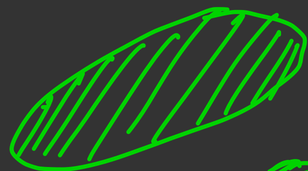
Examples :
(in $\mathbb{R}^2$)



Convex (✓)



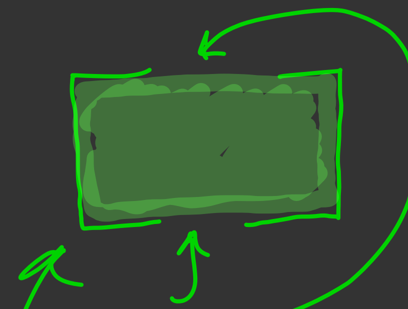
Nonconvex



Convex (✓)

Convex (✓)

Nonconvex



Missing boundary segments

Nonconvex

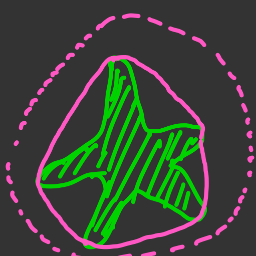
Convex combination of $\underline{x}_1, \ldots, \underline{x}_k$:

Any point $\underline{x} = \theta_1 \underline{x}_1 + \ldots + \theta_k \underline{x}_k$

s.t. $\theta_i \geq 0$, and $\sum_{i=1}^{k} \theta_i = 1$

is called a convex combination of $\{\underline{x}_1, \ldots, \underline{x}_k\}$.

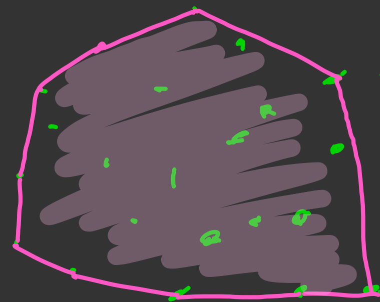
Convex hull = conv($\mathcal{S}$) ← possibly nonconvex to begin with



= Smallest convex set that contains the original set $\mathcal{S}$

$\mathcal{S} \subseteq \text{conv}(\mathcal{S})$

Example:



S = Finite collection of points in $\mathbb{R}^2$

$\leftarrow \text{conv}(S)$

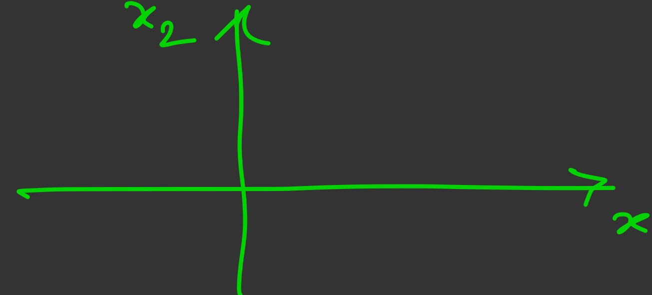
Cones:

We say a set S is a cone if $\forall \underline{x} \in S$ and $\theta \geq 0$, we have $\boxed{\theta \underline{x} \in S}$

$\iff$ Cone is a set that is closed under nonneg. scaling.

Examples:

- $\mathbb{R}^2_+ \longrightarrow$ cone (✓)
  convex (✓)



- $\underbrace{\underbrace{\mathbb{R}^2_+}_{1^{st}\text{ quadrant}} \cup \underbrace{\mathbb{R}^2_-}_{3^{rd}\text{ quadrant}}}$

Nonconvex cone $\begin{cases} \text{cone } (✓) \\ \text{convex } (✗) \end{cases}$

Convex cone:

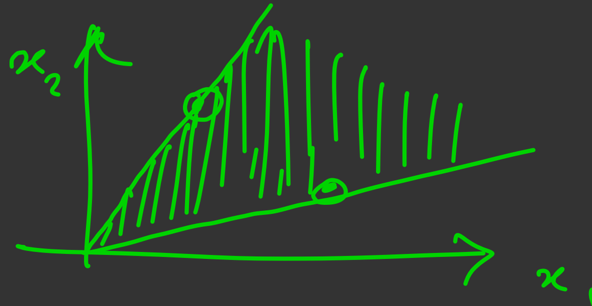
Cone that is also convex.

$\Updownarrow$

$\forall \underline{x}_1, \underline{x}_2 \in S, \quad \theta_1 \underline{x}_1 + \theta_2 \underline{x}_2 \in S$

$\forall \theta_1, \theta_2 \geq 0.$

Examples:

- 

  2D pie/pizza slice

  $\rightarrow$ convex cone

- $\mathbb{R}^2_-$ $\begin{cases} \text{convex } (\checkmark) \\ \text{cone } (\checkmark) \end{cases}$

- S^n_+ convex $(\checkmark)$

  cone $(\checkmark)$

$$\underline{v}^T\left(\theta X_1 + (1-\theta)X_2\right)\underline{v}$$

$$= \theta \underbrace{\underline{v}^T X_1 \underline{v}} + (1-\theta)\underbrace{\underline{v}^T X_2 \underline{v}}$$

$$= \theta(\geq 0) + (1-\theta)(\geq 0)$$

$$= \geq 0$$

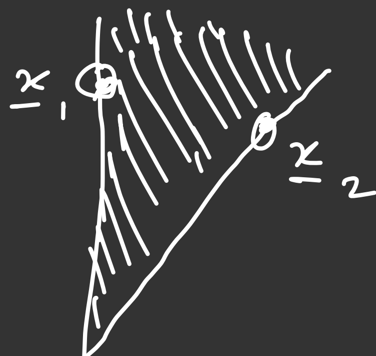
$$\boxed{\theta \in [0,1]}$$

$\therefore$ S^n_+ is a convex cone

Conic Combination:

Nonnegative combination of points $\underline{x}_1, \underline{x}_2$ of the form:

$$\underline{x} = \theta_1 \underline{x}_1 + \theta_2 \underline{x}_2 \quad \text{where } \theta_1, \theta_2 \geq 0.$$

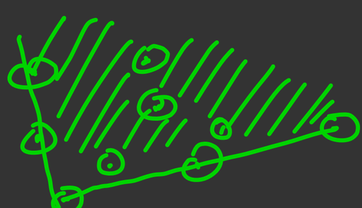


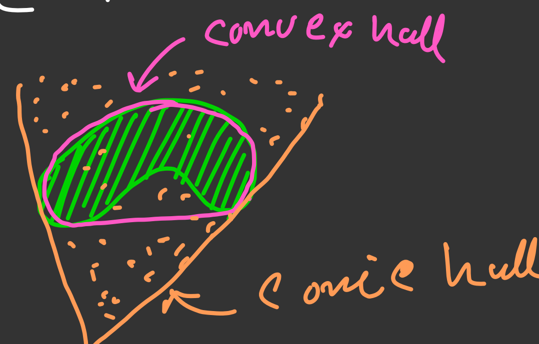
Conic hull:

Smallest (convex) cone containing S

$$\iff \{\theta_1 \underline{x}_1 + \ldots + \theta_k \underline{x}_k \mid \underline{x}_i \in S,\ \theta_i \geq 0,\ \text{for all } i = 1, \ldots, k\}$$

2D Examples:





Example: **Euclidean / 2-norm cone:**

$$\mathcal{S} := \left\{ (\underline{x}, t) \in \mathbb{R}^{n+1} \;\middle|\; \|\underline{x}\|_2 \leq t \right\}$$

$$= \left\{ \begin{pmatrix} \underline{x} \\ t \end{pmatrix} \;\middle|\; \begin{pmatrix} \underline{x} \\ t \end{pmatrix}^T \begin{bmatrix} I_{n\times n} & 0_{n\times 1} \\ 0_{1\times n} & \underbrace{-1}_{1\times 1} \end{bmatrix} \begin{pmatrix} \underline{x} \\ t \end{pmatrix} \leq 0, \; t \geq 0 \right\}$$

is a cone because

if $\begin{pmatrix} \underline{x} \\ t \end{pmatrix} \in \mathcal{S}$ then $\begin{pmatrix} \underline{y} \\ \tau \end{pmatrix} := \begin{pmatrix} \theta \underline{x} \\ \theta t \end{pmatrix} \in \mathcal{S}$.

$\forall \theta \geq 0$.

$$\left.\begin{pmatrix} \underline{y} \\ t \end{pmatrix} = \theta \begin{pmatrix} \underline{x}_1 \\ t_1 \end{pmatrix} + (1-\theta) \begin{pmatrix} \underline{x}_2 \\ t_2 \end{pmatrix}, \forall \theta \geq 0 \right\} \Longrightarrow \boxed{\text{convex}}$$

$\in \mathcal{S}$ whenever $\begin{pmatrix} \underline{x}_1 \\ t_1 \end{pmatrix}, \begin{pmatrix} \underline{x}_2 \\ t_2 \end{pmatrix} \in \mathcal{S}$.