Lecture # 2 (09/27/2022)

Notations and Nomenclature

Optimization $\equiv$ Programming

Math notations:

Scalar	Vector	Matrix	Set
x	$\underline{x}$	X	$\mathcal{X}$

For us, all vectors are by default column vectors

So row vectors will be denoted as: $\underline{x}^T$

Notations:

$:=$ (defined as)

sup $\leftrightarrow$ max
inf $\leftrightarrow$ min

$\max\limits_{0<x<3} x \rightarrow$ maximum does NOT exist

$\sup\limits_{0<x<3} x \rightarrow$ supremum is 3

$\min\limits_{0<x<3} x \rightarrow$ minimum does NOT exist

$\inf\limits_{0<x<3} x \rightarrow$ infimum is 0

Shorthands:

$\forall$ (for all), iff (if and only if), $\Longleftrightarrow$ (equivalent to / if and only if)

Common sets:

Scalar sets:

(set of) reals

$$\mathbb{R} := (-\infty, +\infty)$$

extended reals

$$\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\} = [-\infty, +\infty]$$

$\mathbb{R}_+$ ← set of non-negative reals

$$:= [0, +\infty)$$

$$\mathbb{R}_{++} := (0, +\infty)$$

← set of positive reals

Vector sets :

$$\mathbb{R}^n := (-\infty, +\infty)^n$$

$$\underline{x} \in \mathbb{R}^n, \qquad \underline{x}^T = \underbrace{(x_1 \; - - - \; x_n)}_{\text{row}}$$

$$\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\text{column}}$$

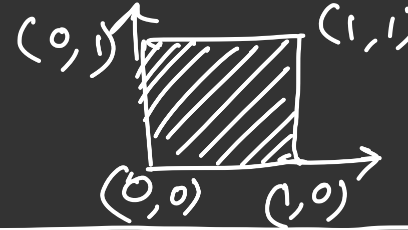
$$\overline{\mathbb{R}}^n := [-\infty, +\infty]^n$$

$$\underbrace{\mathbb{R}^n_+}_{\text{non-negative orthant}} := [0, \infty)^n$$

$$\mathbb{R}^n_{++} := (0, \infty)^n \longleftarrow \text{positive orthant}$$

For $n = 2$ $\quad \overbrace{\mathbb{R}^2_+}^{\text{First quadrant}}$

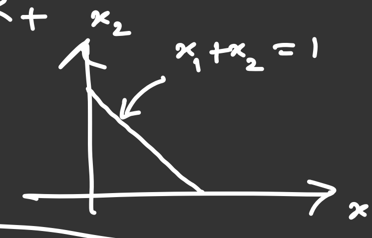
$[0, 1]^n$ n-dimensional unit cube $\subset \mathbb{R}^n_+$



$\Delta^{n-1} \subset \mathbb{R}^n_+$

n-dimensional standard simplex

If $n=2$, $\Delta^1 \subset \mathbb{R}^2_+$



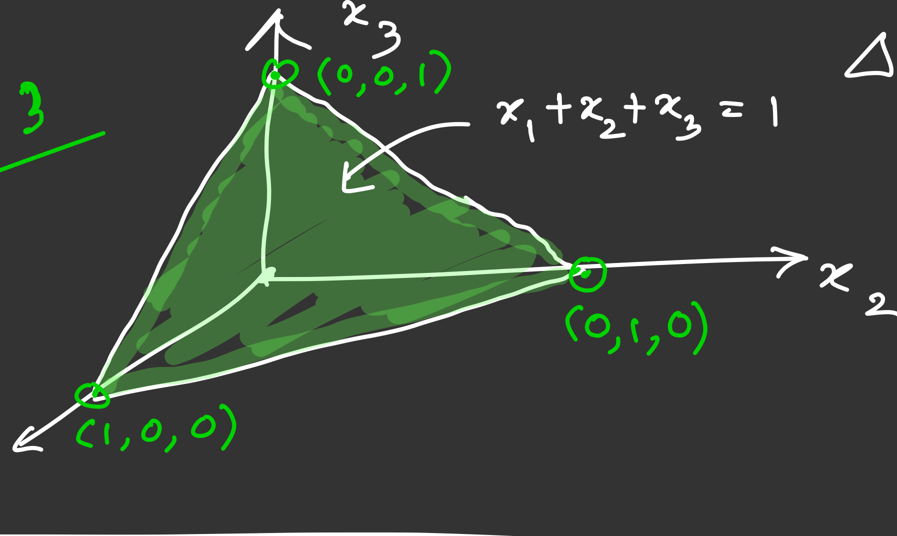
$$\underline{x} \in \Delta^{n-1} := \left\{ \underline{x} \in \mathbb{R}^n_+ \;\middle|\; \sum_{i=1}^{n} x_i = 1 \right\}$$

$$= \left\{ \underline{x} \in \mathbb{R}^n \;\middle|\; 0 \leq x_i \leq 1, \; x_1 + \ldots + x_n = 1 \right\}$$

probability simplex

For example:

when $n = 3$



$\Delta^2 \subset \mathbb{R}^3_+$

Matrix sets:

$\mathbb{R}^{m \times n}$ ← set of all $m \times n$ matrices whose entries are real numbers

$\mathbb{R}^{n \times n}$ ← set of all square matrices of size $n \times n$ with real entries

$\{X \in \mathbb{R}^{n \times n} \mid \det(X) \neq 0\}$ ← $GL(n) \subset \mathbb{R}^{n \times n}$ (set of all $n \times n$ invertible matrices)

$$SL(n) := \{ X \in GL(n) \mid \det(X) = +1 \}$$

$$\underbrace{O(n)}_{\text{Orthogonal matrices}} := \{ X \in \mathbb{R}^{n \times n} \mid X X^T = X^T X = I_n \}$$

I_n: $n \times n$ identity matrices

$$= \{ X \in \mathbb{R}^{n \times n} \mid \det(X) = \pm 1 \}$$

$$S^n := \underbrace{\{ X \in \mathbb{R}^{n \times n} \mid X = X^T \}}_{\text{Symmetric matrices}}$$

next pg.

$\mathbb{S}^n_+ \leftarrow$ set of all symmetric positive semi-definite matrices

$\mathbb{S}^n_{++} \leftarrow$ set of all symmetric positive definite matrices

$$\boxed{\mathbb{S}^n_{++} \subset \mathbb{S}^n_+ \subset \mathbb{S}^n}$$

$$\mathbb{S}^n_+ := \left\{ X \in \mathbb{S}^n \,\middle|\, \underline{v}^T X \underline{v} \geq 0 \text{ for all } \underline{v} \in \mathbb{R}^n \setminus \{\underline{0}\} \right\}$$

$$\mathbb{S}^n_{++} := \left\{ X \in \mathbb{S}^n \,\middle|\, \underline{v}^T X \underline{v} > 0 \text{ for all } \underline{v} \in \mathbb{R}^n \setminus \{\underline{0}\} \right\}$$

We say, $X \in \mathbb{S}^n_+$ or $X \succcurlyeq 0_{n \times n}$

Similarly, $X \in \mathbb{S}^n_{++}$ or $X \succ 0_{n \times n}$

Löwner partial order:

$$X \succcurlyeq Y \iff X - Y \succcurlyeq 0_{n\times n}$$

$$\iff (X-Y) \in S_+^n$$

Similarly,

$$X \preccurlyeq Y \iff X - Y \preccurlyeq 0_{n\times n}$$

$$\iff Y - X \succcurlyeq 0_{n\times n}$$

$$\iff (Y-X) \in S_+^n$$

Example:

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \in S^3$$

$$\underline{v}^T X \underline{v} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$= 1 \cdot v_1^2 + 2 \cdot v_2^2 + 3 \cdot v_3^2 > 0$$

$$\forall \, (v_1, v_2, v_3) \neq (0, 0, 0)$$

$$\boxed{\therefore \; X \in S^3_{++}}$$

Example:

$$X = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \in S^3$$

$$\underline{v}^T X \underline{v} = (v_1 \quad v_2 \quad v_3) \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$= (v_1 \quad v_2 \quad v_3) \begin{pmatrix} 2v_1 - v_2 \\ -v_1 + 2v_2 - v_3 \\ -v_2 + 2v_3 \end{pmatrix}$$

$$= 2v_1^2 + 2v_2^2 + 2v_3^2 - 2v_1v_2 - 2v_2v_3$$

$$= (v_1 - v_2)^2 + (v_2 - v_3)^2 + v_3^2 + v_1^2 \cancel{\geq} 0 \quad > 0$$

sum of squares is zero iff $v_1 = v_2 = v_3 = 0$

$\therefore \boxed{X \in S^3_{++}}$ (X is a positive definite matrix)

Alternative characterizations of positive (semi) definite matrices :

#1 : in terms of eig. values :

$$X \in \mathbb{S}^n_+ \iff \lambda_i(X) \geq 0 \text{ for all } i=1,\ldots,n$$

$$X \in \mathbb{S}^n_{++} \iff \lambda_i(X) > 0 \text{ for all } i=1,\ldots,n$$

Proof :

$$X\underline{v} = \lambda \underline{v}$$

$$\Rightarrow \underbrace{\underline{v}^T X \underline{v}}_{\geq (>) 0} = \underline{v}^T \lambda \underline{v} = \lambda(\underline{v}^T \underline{v})$$

$$= \underbrace{\lambda \|\underline{v}\|_2^2}_{\geq (>) 0}$$

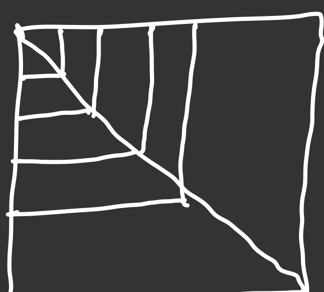
Since $\|\underline{v}\|_2^2 > 0 \;\forall\, \underline{v} \neq \underline{0}$, $\therefore \lambda \geq (>) 0$. Done

#2 : in terms principal minors :

$X \in \mathbb{S}^n_{++} \iff X \in \mathbb{S}^n$ and all leading principal minors of X are > 0.

$X \in \mathbb{S}^n_{+} \iff X \in \mathbb{S}^n$ and all principal minors of X are ≥ 0.

Leading principal minors : determinants of



- upper left 1×1 corner of X
- " " 2×2 " " X
- " " 3×3 " " X
- $\vdots$
- " " $n \times n$ " " X

(1, 1) entry

det(X) itself