Lec. 8 (10/18/2022)

One way to prove convexity of Spectrahedron is to use the property:

$\boxed{\text{If } f : \mathbb{R}^k \mapsto \mathbb{R}^m \text{ is affine, and } \mathcal{S} \text{ is a convex set, then inverse/pre-image of } \mathcal{S} \text{ under } f: \quad f^{-1}(\mathcal{S}) := \{\underline{x} \mid f(\underline{x}) \in \mathcal{S}\} \text{ is a convex set.}}$

Application: To show spectrahedron $\mathcal{X}$ is a convex set, write $\mathcal{X} = f^{-1}(\mathbb{S}^m_+)$ where $f : \mathbb{R}^n \mapsto \mathbb{S}^m$ is affine: $f(\underline{x}) := B - A(\underline{x})$

Convex functions

Defⁿ. (Zeroth order condition/characterization of function convexity)

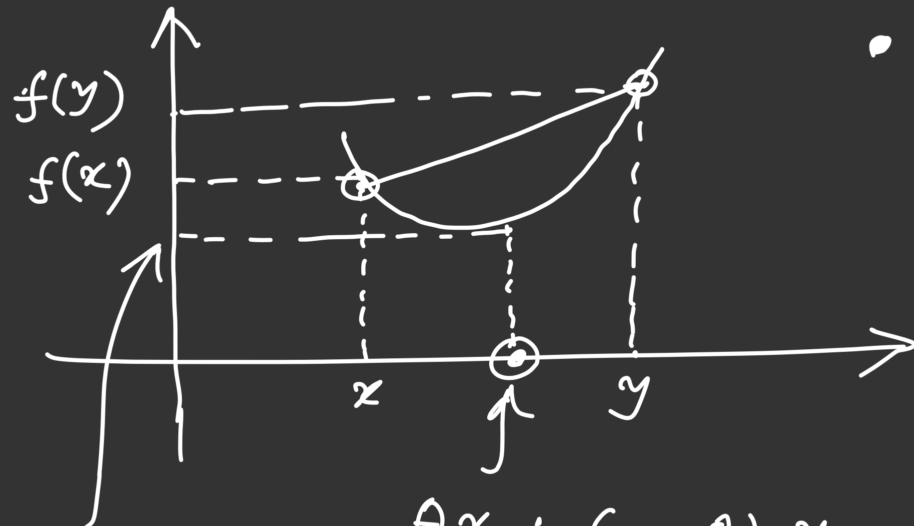
A function $f : \mathrm{dom}(f) \mapsto \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is $\boxed{\text{convex}}$ if

① $\mathrm{dom}(f)$ is a convex set,

② $\forall\ \underline{x}, \underline{y} \in \mathrm{dom}(f)$, and $\forall\, \theta$ s.t. $0 \leq \theta \leq 1$,

we have:

$$f(\theta \underline{x} + (1-\theta)\underline{y}) \leq \theta f(\underline{x}) + (1-\theta) f(\underline{y}).$$



$\theta \underline{x} + (1-\theta)\, \underline{y}$

$f(\theta \underline{x} + (1-\theta)\underline{y})$ $\quad \forall\, \theta \in [0, 1]$

- If instead of "$\leq$", we have "$<$" (strict inequality) then we say that the fn. f is $\boxed{\text{strictly convex}}$

- If the direction of inequality in condition 2 is opposite, then we say f is (strictly) concave.

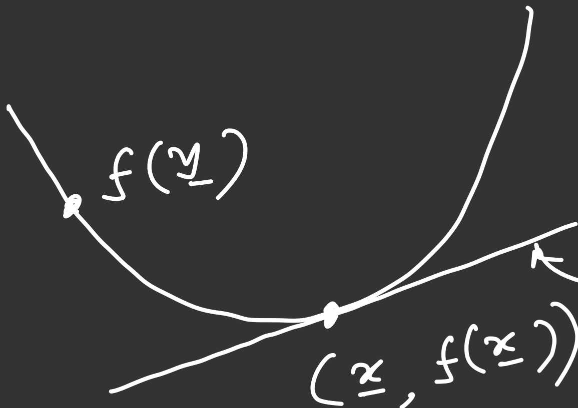
1^{st} order characterization/condition for function convexity :

Let f be differentiable (∇f exists $\forall \underline{x} \in dom(f)$)

Then, f is convex $\boxed{\text{if and only if}}$

① $dom(f)$ is a convex set

② $$f(\underline{y}) \geq f(\underline{x}) + \underbrace{\langle \nabla f(\underline{x}), \underline{y} - \underline{x} \rangle}_{(\nabla f(\underline{x}))^T (\underline{y} - \underline{x})} \quad \forall \underline{x}, \underline{y} \in dom(f)$$



$$\underbrace{f(\underline{x}) + \langle \nabla f(\underline{x}), \underline{y} - \underline{x} \rangle}_{\text{first order Taylor series approximation of } f \text{ about the point } \underline{x}}$$

$\iff$ In its entire dom(f), the function f lies above its linear approximation/tangent hyperplane

$\therefore$ Convex $\iff$ the 1^{st} order Taylor approximation is a global underestimator

2nd order condition/characterization of function convexity :

Let f be twice differentiable in $\text{dom}(f)$.

$\iff \underbrace{\nabla^2 f}$ exists at all $\underline{x} \in \text{dom}(f)$.

Hessian of f

(Jacobian of gradient)

Then,

f is convex (concave) $\boxed{\text{if and only if}}$ ① $\text{dom}(f)$ is a convex set

② $\nabla^2 f \succcurlyeq 0 \;\; \forall \underline{x} \in \text{dom}(f)$

$(\preccurlyeq)$

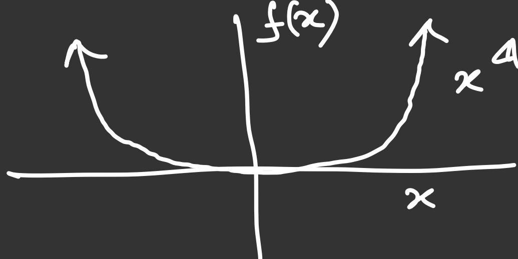
If $\nabla^2 f \succ 0$ ($\Leftrightarrow \nabla^2 f \in S^n_{++}$) $\forall\, \underline{x} \in \mathrm{dom}(f)$,

then f is strictly convex.

BUT the converse fails:

Counterexample: $f : \mathbb{R} \mapsto \overline{\mathbb{R}}$, $\underbrace{f(x) = x^4}_{}$

strictly convex function but



$f''(x) = 12x^2$

vanishes @ $x = 0$

Examples of convex functions:

① (Affine f^n): $f(\underline{x}) = \langle \underline{a}, \underline{x} \rangle + b$

$\underbrace{\qquad\qquad\qquad\qquad}_{\text{both convex and concave}}$

② (Quadratic f^n): $f : \mathbb{R}^n \mapsto \overline{\mathbb{R}}$

$$f(\underline{x}) = \frac{1}{2} \underline{x}^T A \underline{x} + \langle \underline{b}, \underline{x} \rangle + c$$

where $A \in S^n$
$\underline{b} \in \mathbb{R}^n$
$c \in \mathbb{R}$

$$\nabla^2 f = A$$

$\therefore f$ is convex $\Leftrightarrow A \succeq 0 \; (A \in S^n_+)$

concave $\Leftrightarrow A \preceq 0 \; (A \in S^n_-)$

Strictly convex $\Leftrightarrow A \succ 0 \; (A \in S^n_{++})$

Strictly concave $\Leftrightarrow A \prec 0 \quad (A \in S^n_{--})$

Note that :

$$f(\underline{x}) = \frac{1}{2}\underline{x}^T A \underline{x} + \langle \underline{b}, \underline{x} \rangle + c$$

$$\nabla f(\underline{x}) = \frac{1}{2} \cdot \underbrace{\nabla(\underline{x}^T A \underline{x})} + \underbrace{\nabla(\cancel{\underline{b}^T \underline{x}})}_{\underline{b}} + \nabla \cancel{c}^{\,0}$$

$$= \frac{1}{2} \cdot (A + \underbrace{\cancel{A^T}}_{A})\,\underline{x}$$

$$= \frac{1}{\cancel{2}} \cdot \cancel{2} A \underline{x} + \underline{b}$$

$$= A \underline{x} + \underline{b}$$

$$\nabla^2 f(\underline{x}) = A$$

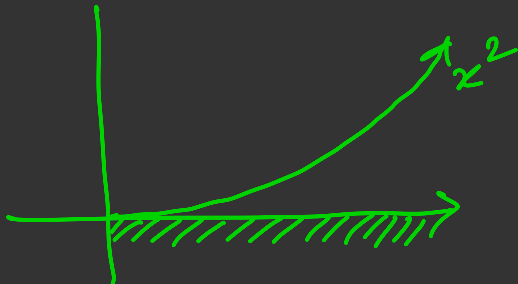
③ Power functions:

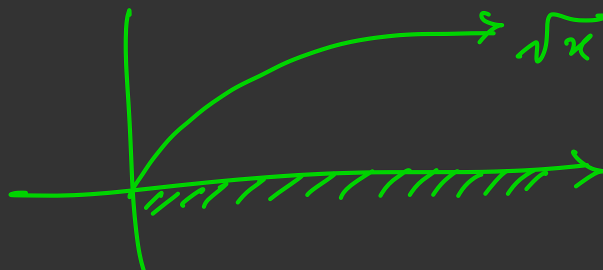
$f : \mathbb{R}_{>0} \mapsto \mathbb{R}$

$$\underbrace{f(x) = x^{\alpha}}$$

Convex for $\underline{\alpha \geq 1}$, $\alpha \leq 0$

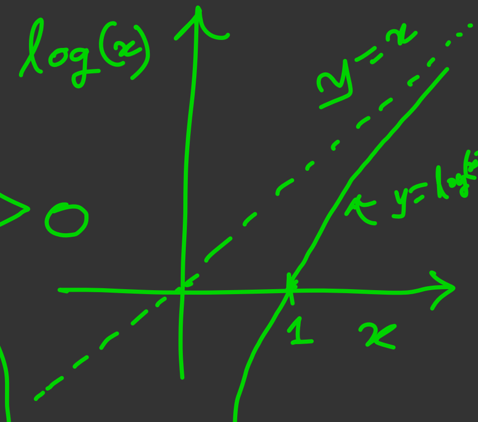
$$\boxed{\text{Concave for } 0 \leq \alpha \leq 1}$$





④ Logarithm:

$x \in dom(f) = \mathbb{R}_{>0}$



$f(x) = \log(x)$ is concave $\Longleftrightarrow$ $-f(x) = -\log(x) = \log(1/x)$ convex

(5) **Norms**: $f : \mathbb{R}^n \mapsto \mathbb{R}$, $f(\underline{x}) = \|\underline{x}\|_p$, $p \geq 1$

$$\|\theta \underline{x} + (1-\theta)\underline{y}\|_p \leq \|\theta \underline{x}\|_p + \|(1-\theta)\underline{y}\|_p$$

$$= \theta \underbrace{\|\underline{x}\|_p}_{f(\underline{x})} + (1-\theta) \underbrace{\|\underline{y}\|_p}_{f(\underline{y})}$$

$\therefore$ $f(\cdot)$ is a convex function over $\mathrm{dom}(f) = \mathbb{R}^n$.

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⑥ Quadratic - over - linear:

$$f : \mathbb{R} \times \mathbb{R}_{>0} \mapsto \overline{\mathbb{R}}$$

$$f(x, y) := \frac{x^2}{y}, \quad dom(f) = \underbrace{\mathbb{R} \times \mathbb{R}_{>0}}_{\text{convex set}}$$

$$\nabla^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$$

$$= \frac{2}{y^3} \underbrace{\begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix}}_{2 \times 2}$$

$$= \frac{2}{y^3} \underbrace{\begin{pmatrix} y \\ -x \end{pmatrix}}_{2 \times 1} \underbrace{\begin{pmatrix} y & -x \end{pmatrix}}_{1 \times 2} \succcurlyeq 0 \quad \forall (x, y) \in dom(f)$$

outer product

$\therefore f$ is a convex function

⑦ GM (Geometric mean):

$$f(\underline{x}) := GM(\underline{x}) := \left(\prod_{i=1}^{n} x_i \right)^{1/n}, \quad \mathrm{dom}(f) = \mathbb{R}^n_{>0}$$

$f(\cdot)$ is concave f^n on $\mathrm{dom}(f) = \mathbb{R}^n_{>0}$

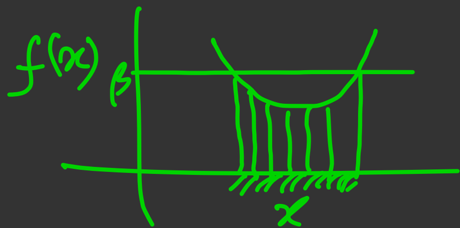
p. 74 of the textbook.

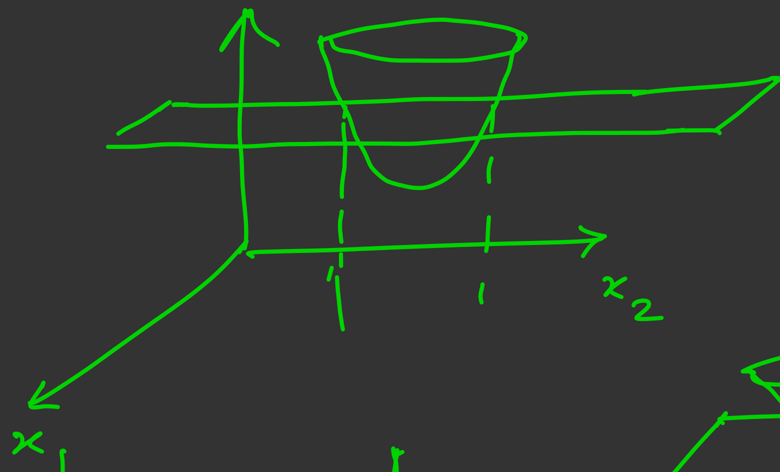
⑧ Sublevel and superlevel sets of a f^n:

$$S_\beta := \{ \underline{x} \in \mathrm{dom}(f) \mid f(\underline{x}) \leq \beta \} \leftarrow \beta \text{ sublevel set of } f$$

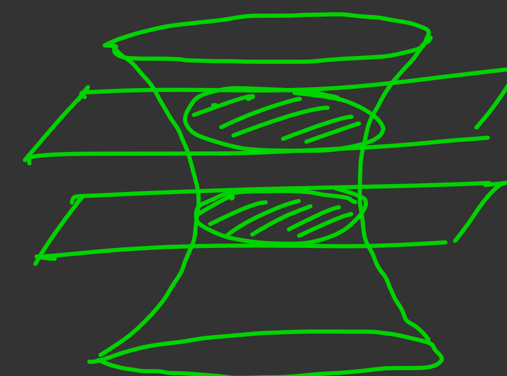
$$\{ \underline{x} \in \mathrm{dom}(f) \mid f(\underline{x}) \geq \beta \} \leftarrow \beta \text{ superlevel set of } f$$

Claim: Sublevel set of a **convex f^n** is (for any β) a **convex set**





Converse fails:



If f is concave then β-superlevel sets are convex.

Exercise: Choose $0 \leq \alpha \leq 1$.

Prove that the set

$$\mathcal{X} := \{ \underline{x} \in \mathbb{R}^n_{\geq 0} \mid \underbrace{GM(\underline{x})}_{\text{geometric mean}} \geq \alpha \underbrace{AM(\underline{x})}_{\text{arithmetic mean}} \}$$

is convex.

Hint: Consider the function:

$$f(\underline{x}) := GM(\underline{x}) - \alpha AM(\underline{x})$$

$$= \underbrace{\left(\prod_{i=1}^{n} x_i \right)^{1/n}}_{\text{concave fn}} - \underbrace{\alpha \left(\sum_{i=1}^{n} x_i / n \right)}_{\text{concave fn}}$$

Our set $\mathcal{X}$ = 0-superlevel set of this concave f^n^ $f(\underline{x})$.

$\therefore$ used f^n^ concavity to show set convexity.

Function convexity via Set Convexity:

Graph of a function:

Given $f: \mathbb{R}^n \mapsto \overline{\mathbb{R}}$, the graph of f is

$$\{(\underline{x}, f(\underline{x})) \mid \underline{x} \in \text{dom}(f)\} \subseteq \mathbb{R}^{n+1}$$

Two types of graphs:

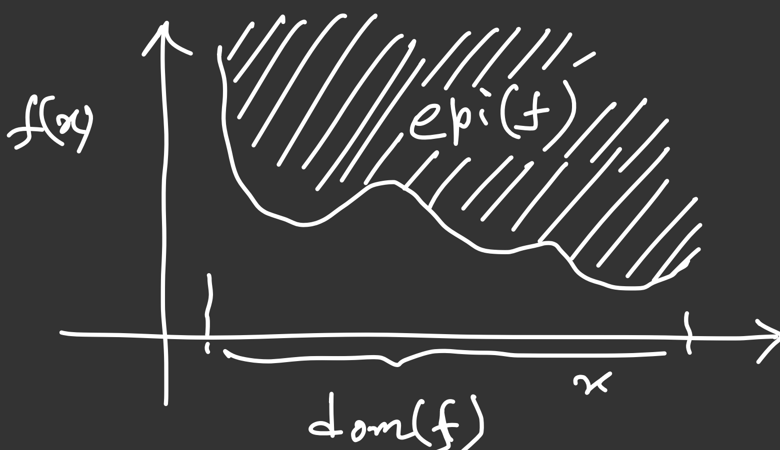
Epigraph	Hypograph
Epi ≡ above	Hypo ≡ below

Epigraph

(Above the graph)

$$f : \mathbb{R}^n \mapsto \overline{\mathbb{R}}$$

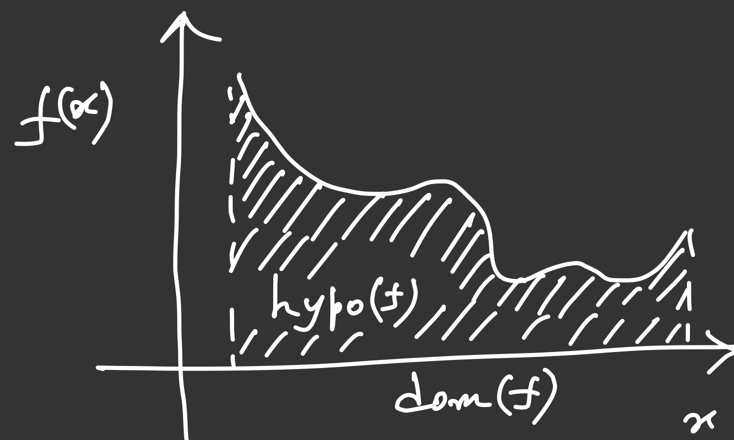
$$\operatorname{epi}(f) := \{ (\underline{x}, t) \mid \underline{x} \in \operatorname{dom}(f), \; f(\underline{x}) \leq t \} \subset \mathbb{R}^{n+1}$$



Hypograph

(Below the graph)

$$\operatorname{hypo}(f) := \{ (\underline{x}, t) \mid \underline{x} \in \operatorname{dom}(f), \; f(x) \geq t \} \subset \mathbb{R}^{n+1}$$



Result:

Function f is convex $\iff$ epi(f) is a convex set

" f " concave $\iff$ hypo(f) " " " "