Lec. 11 (10/27/2022)

Convex Optimization Problems

Standard form:

$$\min_{x \in \mathcal{X}} f(x)$$

We say the above standard form is a convex optimization problem if and only if (iff)

- The objective $f(\cdot)$ is a $\boxed{\text{convex function}}$
- The feasible set $\mathcal{X}$ is a $\boxed{\text{convex set}}$

Some $\underbrace{\text{standard convex optimization problems}}_{\text{(standard forms)}}$:

Linear program (LP)

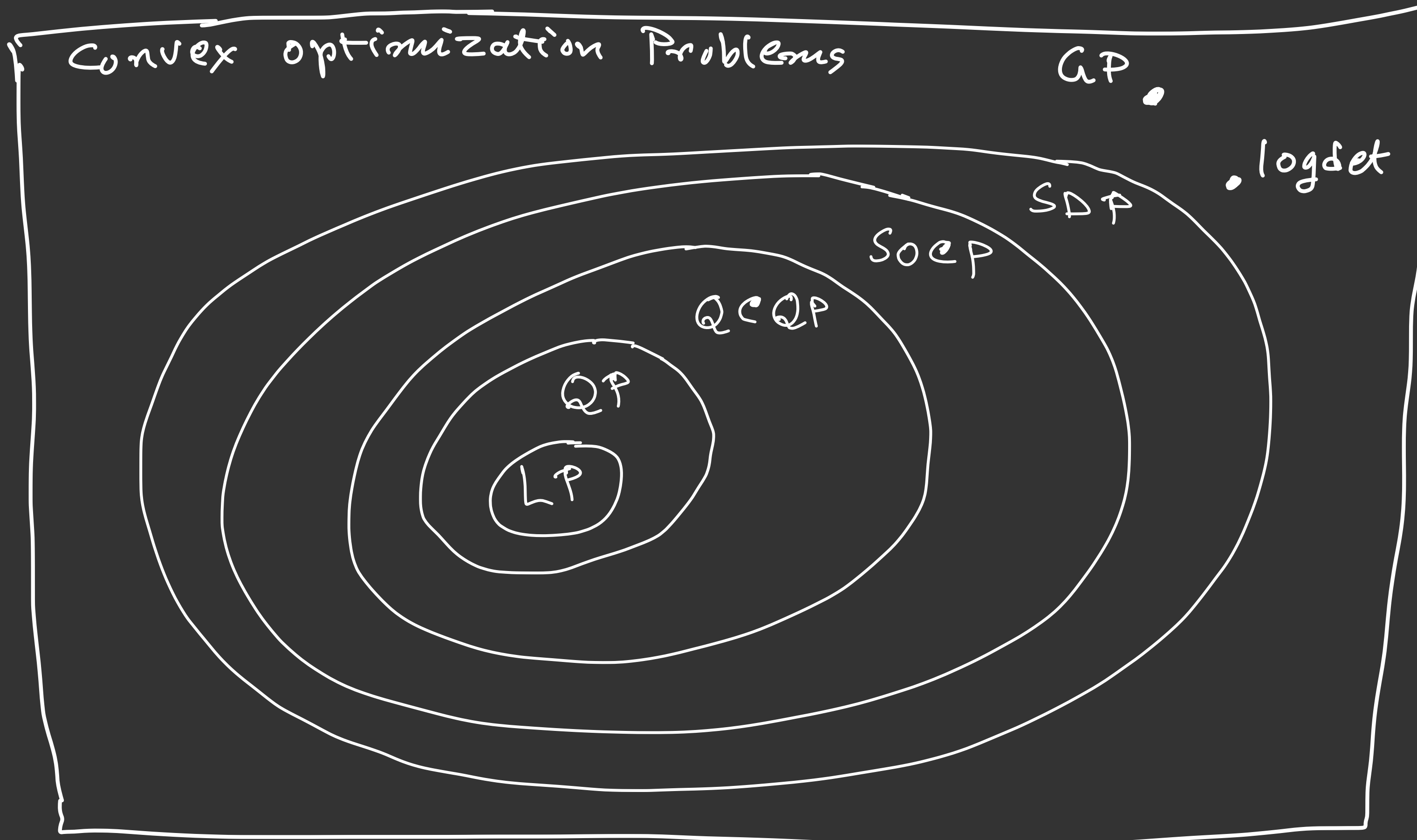
Quadratic program (QP)

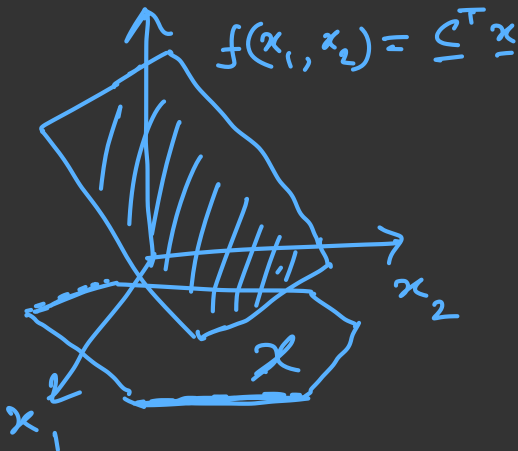
Quadratically constrained quadratic program (QCQP)

Second order cone program (SOCP)

Semidefinite program (SDP)

Geometric program (GP), det/logdet maximization (logdet)


Convex optimization Problems
GP
logdet
SDP
SOCP
QCQP
QP
LP



Linear Program (LP)

$$\min \underbrace{\langle \underline{c}, \underline{x} \rangle}$$

vector inner product $\underline{c}^T \underline{x}$

$\underline{x} \in \mathbb{R}^n$

s.t. $\quad A\underline{x} \leq \underline{b}$

(elementwise vector inequalities)

- Objective function: $f(\underline{x}) = \underbrace{\langle \underline{c}, \underline{x} \rangle}_{\text{linear function in } \underline{x}}$

- Constraint set: $\mathcal{X}$ = Polyhedron (intersection of finite number of $\boxed{\text{linear inequalities}}$ and $\boxed{\text{equalities}}$)

Lec. 6 (p. 5-6)

halfspaces

hyperplanes

Quadratic Program (QP)

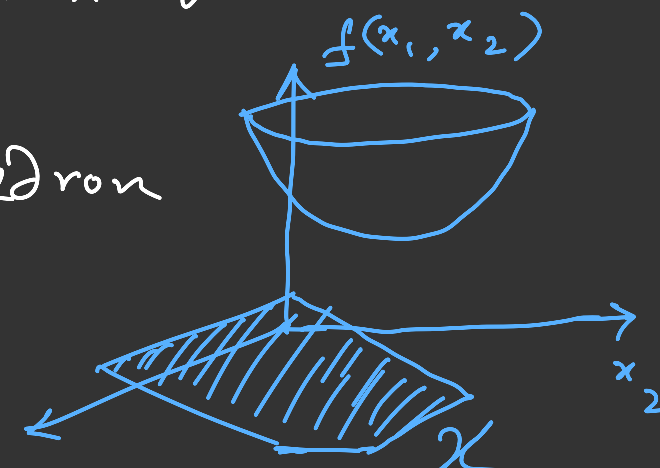
$$\min_{\underline{x} \in \mathbb{R}^n} \ \frac{1}{2} \underline{x}^T A \underline{x} + \langle \underline{b}, \underline{x} \rangle + c$$

$$\text{s.t.} \quad P \underline{x} \leq \underline{q}$$

where $A \in S^n_+ \Leftrightarrow A \succeq 0$

- Objective function : $f(\underline{x})$ is a convex quadratic function

- Constraint set : $\mathcal{X}$ = Polyhedron



Quadratically Constrained Quadratic Program (QCQP)

$$\min_{\underline{x} \in \mathbb{R}^n} \frac{1}{2} \underline{x}^T A \underline{x} + \langle \underline{b}, \underline{x} \rangle + c, \quad A \succcurlyeq 0$$

$$\text{s.t.} \quad \frac{1}{2} \underline{x}^T M_i \underline{x} + \langle \underline{n}_i, \underline{x} \rangle + r_i \leq 0 \ \forall i = 1, \dots, m$$

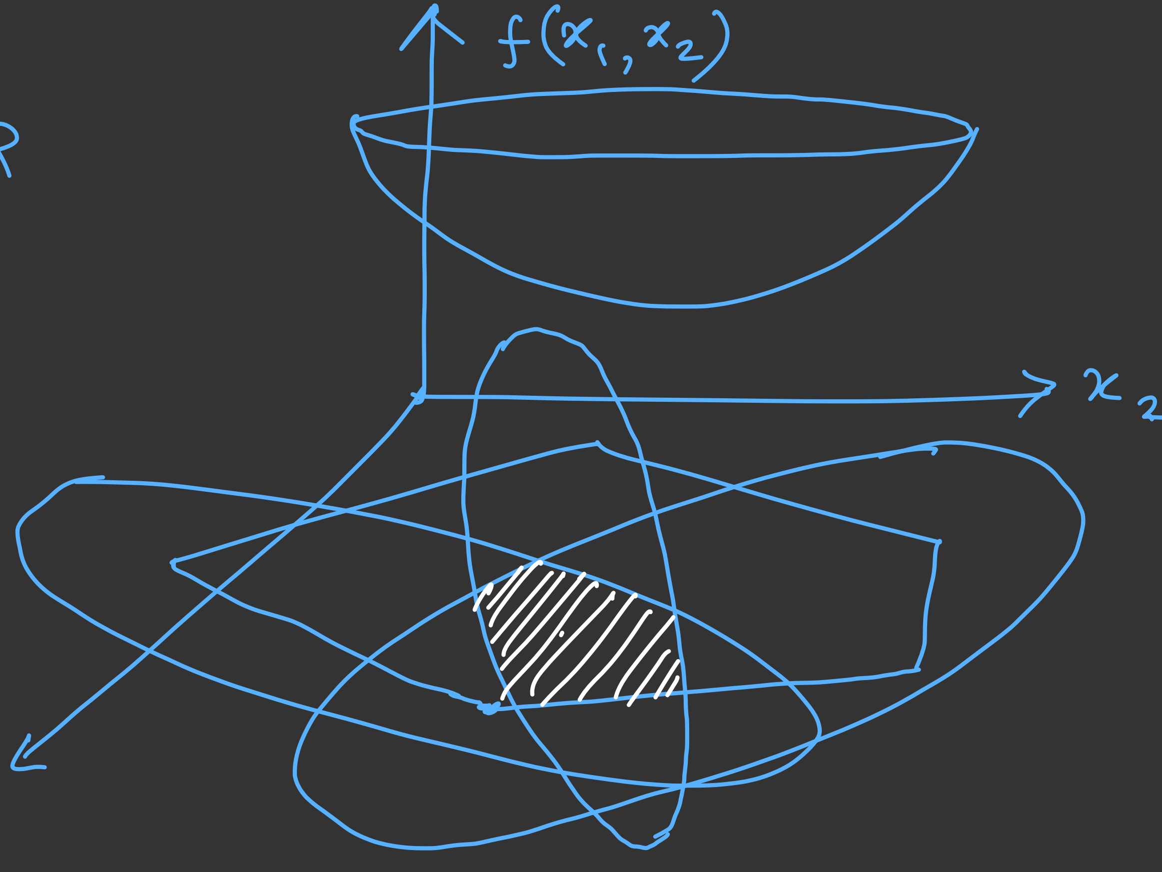
where $M_i \succ 0 \ \forall i$

$$P \underline{x} \preccurlyeq \underline{q}$$

$\Leftrightarrow M_i \in \mathbb{S}^n_{++}$

- Objective function : $f(\underline{x})$ is a convex quadratic function
- Constraint set : $\mathcal{X}$ = Intersection of m ellipsoids with polyhedron

QCQP



Second Order Cone Programs
(SOCP)

$$\min_{\underline{x} \in \mathbb{R}^n} \langle \underline{f}, \underline{x} \rangle$$

(Sometimes can consider convex quadratic in the objective)

$$\text{s.t.} \quad \| A_i \underline{x} + \underline{b}_i \|_2 \leq \langle \underline{c}_i, \underline{x} \rangle + d_i$$

$$\forall i = 1, \ldots, m$$

$$F \underline{x} \leq \underline{g} \quad \text{where } A_i \in \mathbb{R}^{n_i \times n}$$

$$F \in \mathbb{R}^{p \times n}$$

- Objective function : linear
- Constraint set : $\mathcal{K}$ = Intersection of polyhedron with . . .

Second order cone

where $(A_i \underline{x} + \underline{b}_i, \langle \underline{c}_i, \underline{x} \rangle + d_i)$ defines the i^{th} second order cone in $\mathbb{R}^{n_i+1}$

Lec. 5 (p. 14)
Lec. 6 (p. 1)

- If $A_i = 0 \ \forall i = 1, \dots, m$, then SOCP $\rightsquigarrow$ LP
- If $c_i = 0 \ \forall i = 1, \dots, m$ then SOCP $\rightsquigarrow$ QCQP

Semidefinite Programs (SDP)

Two different standard forms

Form (*) :

$$\underbrace{\begin{aligned} &\min \ \mathrm{tr}(C^T X) \\ &X \succcurlyeq 0 \\ &\text{s.t.} \ \mathrm{tr}(A_k^T X) \le b_k \\ &\qquad \forall k = 1, \dots, m \end{aligned} \quad \left\{ \begin{aligned} &\min \ \langle C, X \rangle \\ &X \succcurlyeq 0 \\ &\text{s.t.} \quad \langle A_k, X \rangle \le b_k \\ &\qquad \forall k = 1, \dots, m \end{aligned} \right.}_{\text{Looks like LP}}$$

- Objective function : linear
- Constraint set : $\mathcal{X}$ is intersection of pos. semi definite cone with half spaces and hyperplanes

affine slice of pos. semi definite cone