Lec. 7 (10/13/2022)

Alternative way to describe a polyhedron:

$$\mathcal{P} = \left\{ \theta_1 \underline{v}_1 + \dots + \theta_k \underline{v}_k \;\middle|\; \begin{array}{l} \theta_1 + \theta_2 + \dots + \theta_m = 1, \\ \theta_i \geq 0 \;\; \forall\, i = 1, \dots, k, \\ m \leq k \end{array} \right\}$$

= Non-negative linear combination of vectors $\underline{v}_i$ but only the first m coefficients/weights sum to unity (1)

= Convex hull of points $\underline{v}_1, \dots, \underline{v}_m$ and conic hull of points $\underline{v}_{m+1}, \underline{v}_{m+2}, \dots, \underline{v}_k$.

Example:

Unit 1-norm ball in $\mathbb{R}^n$ = $\operatorname{conv}\Big(\underbrace{\{\underline{e}_1, -\underline{e}_1, \dots, \underline{e}_n, -\underline{e}_n\}}_{\text{plus and minus standard basis unit vectors}}\Big)$

$\underbrace{\hphantom{xxxxxxxxxxxxxxxxxxxx}}_{2n \text{ vertices}}$

Unit ∞-norm ball in $\mathbb{R}^n$ = $\underbrace{\operatorname{conv}\left(\{-1, +1\}^n\right)}_{2^n \text{ vertices}}$

Calculus of convex sets :
(Operations preserving set convexity)

① Affine transformation :

$$A \underbrace{\mathcal{X}}_{\text{convex set}} + \underline{b}$$

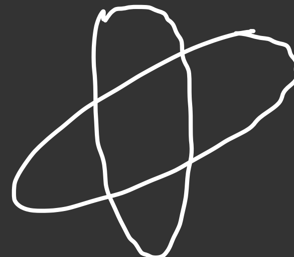
$\underbrace{\text{(e.g., projection, scaling, translation, rotation)}}_{\text{any combination of them}}$

② Intersection : (countable or uncountable)

Suppose $\mathcal{X}_t$ is convex set for fixed t

Then $\bigcap_{a \le t \le b} \mathcal{X}_t$ is also convex

Union of convex sets is nonconvex in general



If S_1 & S_2 are convex then

$$S_1 \cup S_2$$

in general, nonconvex.

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③ Cartesian product: (preserves convexity)

$$\underbrace{S_1}_{\text{convex}} \times \underbrace{S_2}_{\text{convex}} := \underbrace{\{ (\underline{x}, \underline{y}) \mid \underline{x} \in S_1, \underline{y} \in S_2 \}}_{\text{convex}}$$

$S_1 \times S_2 \times \dots \times S_m$

preserves convexity

Application: Set sum / Minkowski sum

$$\underbrace{\mathcal{S}_1}_{\text{convex}} + \underbrace{\mathcal{S}_2}_{\text{convex}} := \underbrace{\left\{ \underline{x} + \underline{y} \;\middle|\; \underline{x} \in \mathcal{S}_1,\ \underline{y} \in \mathcal{S}_2 \right\}}_{\text{convex}}$$

$$= \underline{x} + \underline{y} = \underbrace{\left[I \mid I \right] \begin{pmatrix} \underline{x} \\ \underline{y} \end{pmatrix}}_{\text{linear image of Cartesian product } \mathcal{S}_1 \times \mathcal{S}_2}$$

4 Perspective function:

$$p : \mathbb{R}^n \times \mathbb{R}_{++} \longmapsto \mathbb{R}^n$$

$$(\mathbb{R}_{>0})$$

$$p(\underline{z}, t) := \underline{z}/t .$$

$\underline{z} \in \mathbb{R}^n$, $t \in \mathbb{R}_{++}$

$$\underline{x} \equiv \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} \longmapsto p(\underline{x}) \equiv \begin{pmatrix} x_1/x_n \\ x_2/x_n \\ \vdots \\ x_{n-1}/x_n \end{pmatrix}$$

Example: If $n = 3$,

$$\underline{z} = \begin{pmatrix} 1 \\ -2 \\ 1.5 \end{pmatrix}$$

$$t = 4$$

$$p(\underline{z}, t) = \begin{pmatrix} 1/4 \\ -1/2 \\ 3/8 \end{pmatrix}$$

Physical application/interpretation:

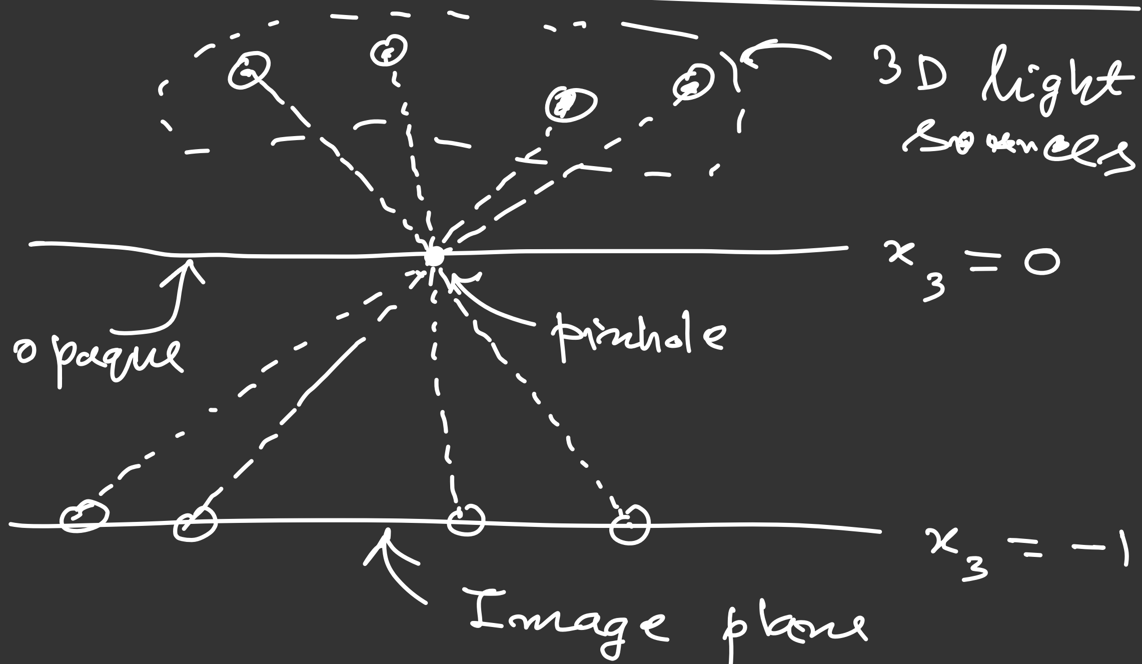


Image of a point $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

appears at

$$y = -p(\underline{x})$$

$$= \begin{pmatrix} -x_1/x_3 \\ -x_2/x_3 \end{pmatrix}$$

Image and inverse image of a convex set under the perspective transformation is convex.

Image: Let $S \subseteq \mathbb{R}^n \times \mathbb{R}_{>0}$ is convex.

Then the set $p(S) := \{ p(\underline{x}) \mid \underline{x} \in S \}$ is also convex.

Inverse image/Pre-image:

Let $S \subseteq \mathbb{R}^n$ is convex set

Then the lifting $p^{-1}(S) := \{ (\underline{x}, t) \in \mathbb{R}^{n+1} \mid x/t \in S, \ t > 0 \}$ is also convex set.

(5) LFT (linear fractional transformation)

$$f(\underline{x}) = \frac{A\underline{x} + \underline{b}}{\langle \underline{c}, \underline{x} \rangle + d}, \quad \left.\begin{array}{l} A \in \mathbb{R}^{m \times n} \\ \underline{b} \in \mathbb{R}^{m} \\ \underline{c} \in \mathbb{R}^{n} \\ d \in \mathbb{R} \end{array}\right\} \underline{x} \in \mathbb{R}^{n}$$

$$f : \mathbb{R}^n \mapsto \mathbb{R}^m$$

$$\mathrm{dom}(f) = \{ \underline{x} \in \mathbb{R}^n \mid \langle \underline{c}, \underline{x} \rangle + d > 0 \}$$

<u>Result:</u> Image & inverse image/pre-image of a convex set S under LFT is convex.

See textbook example 2.13:

Set of all conditional probability vectors is convex.

- **Separating Hyperplane Theorem:**

Statement: Let $\mathcal{C}, \mathcal{D} \subset \mathbb{R}^n$ such that both $\mathcal{C}, \mathcal{D}$ are convex sets and $\mathcal{C} \cap \mathcal{D} = \phi$.

Then, $\exists\ \underline{a} \neq \underline{0} \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $\langle \underline{a}, \underline{x} \rangle \leq b \ \forall\ \underline{x} \in \mathcal{C}$

and $\langle \underline{a}, \underline{x} \rangle \geq b \ \forall\ \underline{x} \in \mathcal{D}$

Proof is in book

Picture :

We say, $\mathcal{C}$ and $\mathcal{D}$ are $\boxed{\text{separable}}$ by the hyperplane $\underline{a}^T \underline{x} = b$

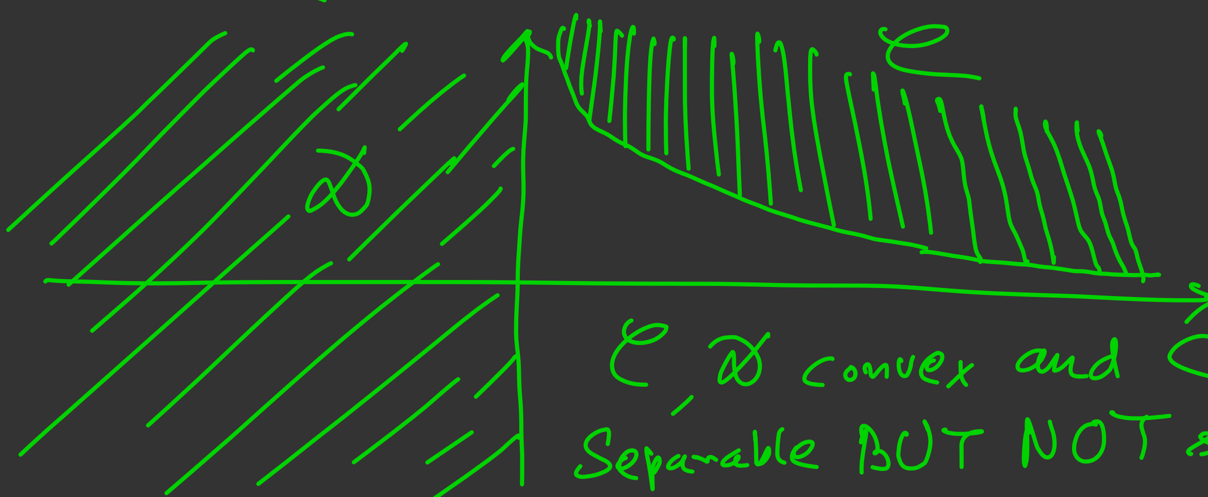
We say, $\mathcal{C}$ and $\mathcal{D}$ are $\boxed{\text{strictly separable}}$ if

$$\underline{a}^T \underline{x} < b \ \forall\, \underline{x} \in \mathcal{C} \text{ and } \underline{a}^T \underline{x} > b \ \forall\, \underline{x} \in \mathcal{D}$$

Counter-example: disjoint convex sets but NOT strictly separable (in 2D)

$$\mathcal{C} := \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x_1 x_2 \geq 1 \text{ and } x_1, x_2 > 0 \right\}$$

$$\mathcal{D} := \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x_1 \leq 0 \right\}$$



$\mathcal{C}, \mathcal{D}$ convex and $\mathcal{C} \cap \mathcal{D} = \emptyset$.

Separable BUT NOT strictly separable.

- Converse of the separating hyperplane theorem is NOT true in general :

i.e., existence of a hyperplane $\underline{a}^T \underline{x} = b$

s.t. $\underline{a}^T \underline{x} \leq b \;\; \forall \; \underline{x} \in \mathcal{C}$ convex

and $\underline{a}^T \underline{x} \geq b \;\; \forall \; \underline{x} \in \mathcal{D}$ convex

$$\not\Rightarrow \quad \mathcal{C} \cap \mathcal{D} = \emptyset .$$

(unless, we add an extra condition that at least one of the sets $\mathcal{C}$ or $\mathcal{D}$ be open)

- **Supporting hyperplane:** (Defⁿ)

Suppose $\mathcal{C} \subseteq \mathbb{R}^n$ and $\underline{x}_0 \in \underbrace{\text{boundary}(\mathcal{C})}_{\text{Textbook Appendix A.2.1}}$

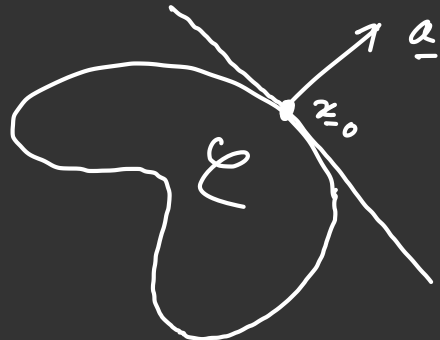
If $\underline{a} \neq \underline{0} \in \mathbb{R}^n$ satisfies $\underline{a}^T\underline{x} \leq \underline{a}^T\underline{x}_0 \ \forall\, \underline{x} \in \mathcal{C}$

then the hyperplane $\{\underline{x} \in \mathbb{R}^n \mid \underline{a}^T\underline{x} = \underline{a}^T\underline{x}_0\}$

is called a <u>supporting hyperplane</u> to $\mathcal{C}$ at $\underline{x}_0$.

$\Leftrightarrow$ The hyperplane is tangent to $\mathcal{C}$ at $\underline{x}_0$

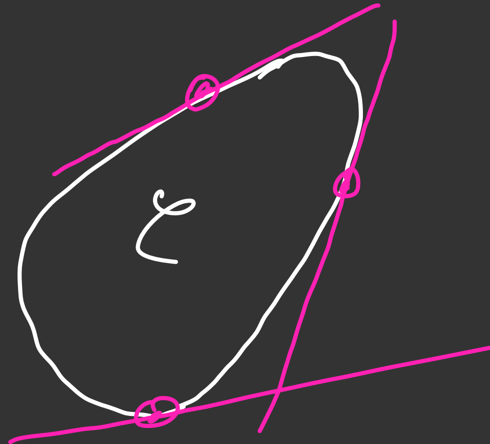
and the half-space $\{\underline{x} \in \mathbb{R}^n \mid \underline{a}^T\underline{x} \leq \underline{a}^T\underline{x}_0\}$ contains $\mathcal{C}$.

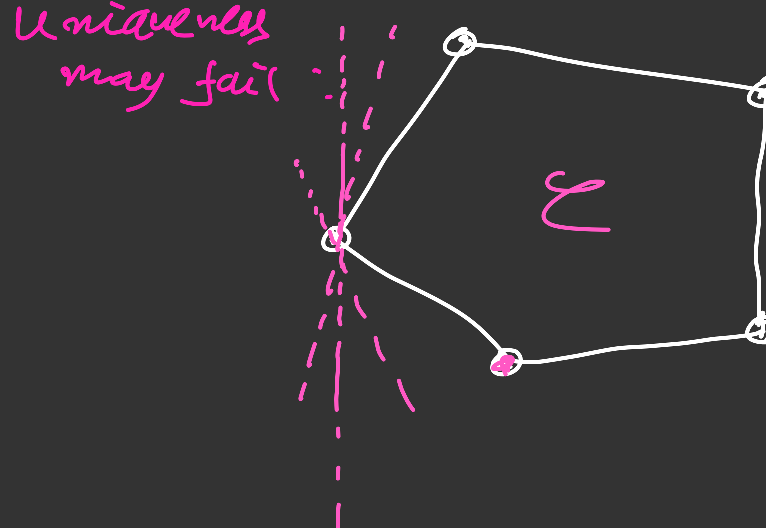


$\leftarrow$ supporting hyp. plane to $\mathcal{C}$ @ $\underline{x}_0$.

Supporting Hyperplane Theorem:

For any nonempty convex set $\mathcal{C} \subseteq \mathbb{R}^n$ and any $\underline{x}_0 \in \text{boundary}(\mathcal{C})$, there exists (may not be unique) a supporting hyperplane to $\mathcal{C}$ at $\underline{x}_0$





Partial converse :

If the set $\mathcal{C}$ is
- → closed
- → has nonempty interior
- → has supporting hyperplane at every $x_0 \in$ boundary$(\mathcal{C})$

Then $\mathcal{C}$ is a convex set.

An important convex set:

Spectrahedron: $\quad F : \mathbb{R}^n \mapsto \mathbb{S}^m_+$

Let $\underline{x} := \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

$$F(\underline{x}) := F_0 + x_1 F_1 + x_2 F_2 + \dots + x_n F_n \succcurlyeq 0$$

where $F_0, F_1, \dots, F_n \in \mathbb{S}^m$

The set $\{\underline{x} \in \mathbb{R}^n \mid F(\underline{x}) \succcurlyeq 0\}$ is called a <u>spectrahedron</u>.

This set is also called the (solution set of) <u>linear matrix inequality</u> (LMI)

Let us rewrite the condition:

$$F(\underline{x}) \succcurlyeq 0 \iff F_0 + x_1 F_1 + \dots + x_n F_n \succcurlyeq 0$$

$$\iff x_1 F_1 + \dots + x_n F_n \succcurlyeq -F_0$$

$$\iff x_1 \underbrace{(-F_1)} + \dots + x_n \underbrace{(-F_n)} \preccurlyeq \underbrace{F_0}$$

$$\iff \boxed{x_1 A_1 + \dots + x_n A_n \preccurlyeq B}$$

new names: $-F_1 := A_1, \dots, -F_n := A_n, F_0 \equiv B$

linear ineq. / halfspace looks like:

$$\underline{a}^T \underline{x} \leq b$$

$$\Leftrightarrow a_1 x_1 + \dots + a_n x_n \leq b$$

LMI: $x_1 A_1 + \dots + x_n A_n \preceq B$

The set $\mathcal{X} := \{ \underline{x} \in \mathbb{R}^n \mid A(\underline{x}) := x_1 A_1 + \dots + x_n A_n \preceq B \}$

is spectrahedron

is $\boxed{\text{convex set.}}$