Lec. 12 (11/01/2022)

Another standard form (we call it (**)) for SDP:

$$(**)\quad \begin{cases} \min\limits_{\underline{x} \in \mathbb{R}^n} & \langle \underline{c}, \underline{x} \rangle \\ \text{s.t.} & \boxed{F(\underline{x}) \succcurlyeq 0} \end{cases}$$

LMI

spectrahedron

where $F(\underline{x}) := F_0 + x_1 F_1 + x_2 F_2 + \dots + x_n F_n$

and $F_0, F_1, F_2, \dots, F_n \in \mathbb{S}^m$ (see last three pages of Lec. 7)

So the feasible/constraint set

$$\mathcal{X} := \{ \underline{x} \in \mathbb{R}^n \mid F(\underline{x}) \succcurlyeq 0 \} \quad \text{(spectrahedron)}$$

Brief outline to go from (**) to (*):

Split vector $\underline{x} \in \mathbb{R}^n$ into its positive and negative parts:

$$\underline{x} = \underline{x}^+ - \underline{x}^- \quad \text{such that } \underline{x}^+, \underline{x}^- \geq 0.$$

Introduce "slack" variable $\tilde{X}$ (matrix) to convert inequality constraints to equality:

Write : $$F(\underline{x}) = \tilde{X} \succcurlyeq 0$$

$\therefore$ (**) becomes:

$$\min_{\underline{x}^+,\, \underline{x}^-,\, \tilde{X}} \quad \underline{c}^T \underline{x}^+ - \underline{c}^T \underline{x}^-$$

s.t.

- $\displaystyle\sum_{i=1}^{m} F_i \, \underline{x}_i^+ - \sum_{i=1}^{m} F_i \, \underline{x}_i^- - \tilde{X} = -F_0$

- $\tilde{X} \succcurlyeq 0$

- $\underline{x}^+ \geqslant \underline{0}$, $\underline{x}^- \geqslant \underline{0}$ (elementwise)

Then, let

$$X := \begin{pmatrix} \mathrm{diag}(\underline{x}^+) & & 0 \\ & \mathrm{diag}(\underline{x}^-) & \\ 0 & & \tilde{X} \end{pmatrix}$$

in form (*)

$$C := \begin{pmatrix} \mathrm{diag}(\underline{c}) & & 0 \\ & -\mathrm{diag}(\underline{c}) & \\ 0 & & 0 \end{pmatrix}$$

These two lines above define $\mathrm{tr}(C^T X)$ in (*).
Now write A_k, b_k in terms of $F_0, F_1, \dots, F_n$ (exercise).

Examples of convex optimization problems and their reduction to standard form

Example: **Chebyshev center of a polyhedron:**

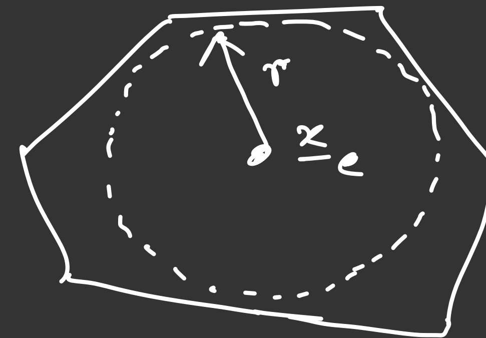
Given a polyhedron $P = \{ \underline{x} \in \mathbb{R}^n \mid \underline{a}_i^T \underline{x} \leq b_i,\ i = 1, \ldots, m \}$

Find $\underbrace{B}_{\text{ball}} := \underbrace{\{ \underline{x}_c + \underline{u} \mid \| \underline{u} \|_2 \leq r \}}_{\text{ball with center } \underline{x}_c \in \mathbb{R}^n \text{ and radius } r}$

s.t. B is the largest ball in P.

$$\Longleftrightarrow \quad \max_{(\underline{x}_e,\, r)} \; r$$

$$\text{s.t.} \quad \mathcal{B} \subseteq \mathcal{P}$$



Objective function $f(\underline{x}_e, r) = r$

Our decision variable $(\underline{x}_e, r) \in \mathbb{R}^n \times \mathbb{R}_{++}$

Now,

$$\boxed{\begin{gathered} \underline{a}_i^T \underline{x} \le b_i \;\; \forall\, \underline{x} \in \mathcal{B} \\ \Updownarrow \\ \mathcal{B} \subseteq \mathcal{P} \end{gathered}}$$

$$\therefore \quad \sup\{ \underline{a}_i^T(\underline{x}_c + \underline{u}) \mid \|\underline{u}\|_2 \le r \}$$

$$= \underline{a}_i^T \underline{x}_c + r \|\underline{a}_i\|_2 \le b_i$$

$$\forall\, i = 1, \dots, m$$

<u>why?</u>

if $> b_i$ then the ball spills out

<u>next pg.</u>

$$\iff \max_{(\underline{x}_c, r) \in \mathbb{R}^n \times \mathbb{R}_{++}} r$$

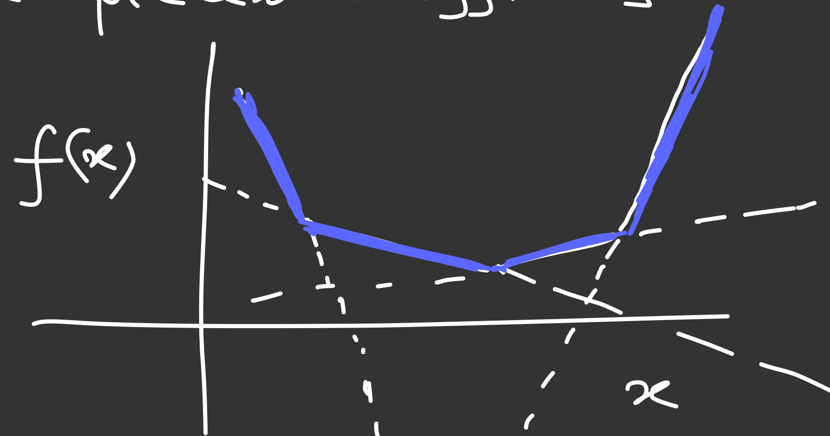
$$\underbrace{\text{s.t.} \quad \underline{a}_i^T \underline{x}_c + r \|\underline{a}_i\|_2 \leq b_i, \ \forall i = 1, \ldots, m}$$

This is linear in $(\underline{x}_c, r) \in \mathbb{R}^n \times \mathbb{R}_{++}$

$\therefore$ LP!

Example: minimize a piecewise affine function

$$\min_{\underline{x} \in \mathbb{R}^n} \underbrace{\max_{i=1,\dots,m} \left(\underline{a}_i^T \underline{x} + b_i \right)}_{\text{piecewise affine}}$$



Trick:

Standard convex problem:

$$\min_{\underline{x}} \; f_0(\underline{x}) \;, \quad \underline{x} \in \mathbb{R}^n$$

$$\text{s.t.} \quad f_i(\underline{x}) \leq 0, \; i = 1, \dots, m$$

$$h_j(\underline{x}) = 0 \;, \; j = 1, \dots, p$$

$\Longleftrightarrow$

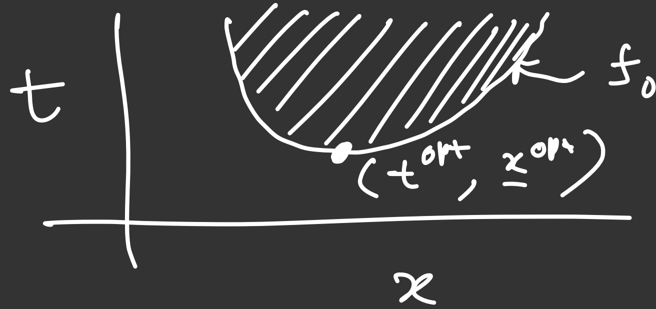
Epigraph form:

$$\min_{(\underline{x}, t)} \; t$$

$$\text{s.t.} \quad f_0(\underline{x}) - t \leq 0 \,,$$

$$f_i(\underline{x}) \leq 0 \,, \; i = 1, \dots, m$$

$$h_j(\underline{x}) = 0 \,, \; j = 1, \dots, p$$



In epigraph form, our piecewise affine minimization problem (which we agree, is a convex problem) becomes:

$$\boxed{\begin{aligned} &\min_{(\underline{x},t)} \quad t \\ &\text{s.t.} \quad \max_{i=1,\ldots,m} \left(\underline{a}_i^T \underline{x} + b_i\right) \le t \end{aligned}}$$

$$\Updownarrow$$

LP $\longrightarrow$ $\min_{(\underline{x},t)} \; t \quad \text{s.t.} \quad \underline{a}_i^T \underline{x} + b_i \le t \;\; \forall i=1,\ldots,m$

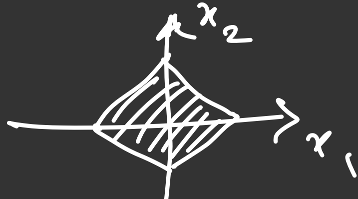
Problems involving ℓ_1 and ℓ_∞ norms:

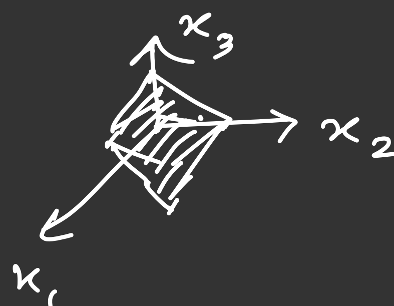
$$\min_{\underline{x} \in \mathbb{R}^n} \langle \underline{c}, \underline{x} \rangle$$

s.t. $\underbrace{\|\underline{x}\|_1 \le \alpha}$

$$\Updownarrow$$

$$|x_1| + |x_2| + \dots + |x_n| \le \alpha$$





$$\min_{\underline{x} \in \mathbb{R}^n} \langle \underline{c}, \underline{x} \rangle$$

s.t. $\|\underline{x}\|_\infty \le \alpha$

$$\Updownarrow$$

$$\max_{i=1,\dots,n} |x_i| \le \alpha$$

LP

Example showing LP reduction:

$$\min_{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2} \left\| \begin{matrix} 2x + 3y - 1 \\ y \end{matrix} \right\|_1$$

$$= \min_{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2} \left\{ |2x+3y-1| + |y| \right\}$$

Trick: Let $t_1 \geq 0$, $t_2 \geq 0$

s.t. $|2x + 3y - 1| \leq t_1$

and $|y| \leq t_2$

$\Longleftrightarrow$

$$\min_{\begin{pmatrix} x \\ y \\ t_1 \\ t_2 \end{pmatrix} \in \mathbb{R}^4} t_1 + t_2 \quad \left(= \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} x \\ y \\ t_1 \\ t_2 \end{pmatrix} \right\rangle \right)$$

s.t. $-t_1 \leq 2x + 3y - 1 \leq t_1$

$-t_2 \leq y \leq t_2$

LP in standard form

Same story:

$$\min_{\underline{x} \in \mathbb{R}^n} \| A\underline{x} - \underline{b} \|_1$$

LP

$$\Updownarrow$$

$$\left.\begin{aligned} &\min_{(\underline{x}, \underline{t})} \quad \mathbb{1}^T \underline{t} \\ &\text{s.t.} \quad A\underline{x} - \underline{b} \preceq \underline{t} \\ &\qquad\; -(A\underline{x} - \underline{b}) \preceq \underline{t} \\ &\qquad\; t_1, t_2, \ldots, t_n \geq 0 \end{aligned}\right\} \text{LP}$$

as norm:

$$\min_{\binom{x}{y} \in \mathbb{R}^2} \left\| \begin{pmatrix} 2x + 3y - 1 \\ y \end{pmatrix} \right\|_\infty$$

$$= \min_{\binom{x}{y} \in \mathbb{R}^2} \max \left\{ |2x + 3y - 1|, |y| \right\}$$

Trick: Let $t \geq 0$ be such that

$$|2x + 3y - 1| \leq t$$

and $|y| \leq t$, $t \geq 0$

next pg.

$$\therefore \quad \min_{\underline{x} \in \mathbb{R}^n} \|A\underline{x} - \underline{b}\|_\infty$$

$$\Updownarrow$$

LP $\left\{ \begin{array}{ll} \min\limits_{(\underline{x}, t)} & t \\ \text{s.t.} & A\underline{x} - \underline{b} \preceq t\mathbb{1} \\ & -(A\underline{x} - \underline{b}) \preceq t\mathbb{1} \\ & t \geq 0 \end{array} \right.$

$\mathbb{1}$ is all ones vector

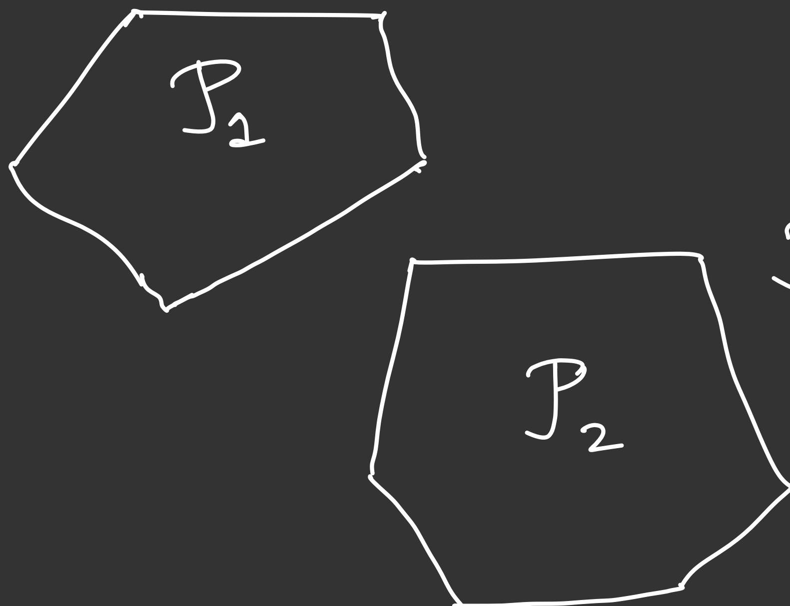
Exercise :

$$\begin{cases} \min\limits_{\underline{x} \in \mathbb{R}^n} & \| A\underline{x} - \underline{b} \|_1 \\ \text{s.t.} & \| \underline{x} \|_\infty \leq 1 \end{cases}$$

$\Longleftrightarrow$ LP

$$\begin{cases} \min\limits_{(\underline{x},\, \underline{y})} & \mathbb{1}^T \underline{y} \\ \text{s.t.} & -\underline{y} \leq A\underline{x} - \underline{b} \leq +\underline{y} \\ & -\mathbb{1} \leq \underline{x} \leq +\mathbb{1} \end{cases}$$

Examples of QP: (Compute distance between two polyhedra)



$$\mathcal{P}_1 = \{ \underline{x} \in \mathbb{R}^n \mid A_1 \underline{x} \leq \underline{b}_1 \}$$

$$\mathcal{P}_2 = \{ \underline{x} \in \mathbb{R}^n \mid A_2 \underline{x} \leq \underline{b}_2 \}$$

$$\min_{(\underline{x}_1, \underline{x}_2)} \quad \| \underline{x}_1 - \underline{x}_2 \|_2$$

$$\text{s.t.} \quad \underline{x}_1 \in \mathcal{P}_1, \quad \underline{x}_2 \in \mathcal{P}_2$$

$$\iff \quad \min_{\underline{x}_1, \underline{x}_2 \in \mathbb{R}^n} \; \|\underline{x}_1 - \underline{x}_2\|_2^2$$

$$\text{s.t.} \quad A_1 \underline{x}_1 \leq \underline{b}_1$$

$$A_2 \underline{x}_2 \leq \underline{b}_2$$

QP