Problem 1. [50 points] Lyapunov techniques for estimating ROA

Consider the following nonlinear system with state vector $(x_1, x_2) \in \mathbb{R}^2$, given by

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -x_1 - x_2 + x_1^3.$$

(a) [10 points] Fixed points

Find all isolated fixed points. Show all your calculations.

1. From $\dot{x_1} = x_2$, we know fixed points have $\dot{x_1} = 0 \rightarrow x_2 = 0$

2. Given $x_2 = 0$:

$$\begin{aligned} \dot{x_2} = -x_1 - x_2 + x_1^3 = x_1(x_1^2 - 1) \quad (1) \\ \rightarrow x_1 = 0, \pm 1 \quad (2) \end{aligned}$$

3. Therefore, isolated fixed points are: $(0,0), (0,1), (0,-1)$

(b) [10 + 10 = 20 points] Origin is AS

(i) Using the Lyapunov function $V(x_1, x_2) = \frac{1}{2}x_2^2 + \int_0^{x_1}(y - y^3)\mathrm{d}y$, **prove that** the origin is AS. You may need to use the LaSalle invariance.

1. We can perform the integration in V:

$$V = \frac{x_2^2}{2} + \frac{x_1^2}{2} - \frac{x_1^4}{4} \quad (3)$$

2. We can derive $\dot{V}$:

$$\begin{aligned} \dot{V}(x_1, x_2) &= \frac{\partial V}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial V}{\partial x_2}\frac{\partial x_2}{\partial t} \quad (4) \\ &= (x_1 - x_1^3)x_2 + x_2(x_1^3 - x_1 - x_2) \quad (5) \\ &= -x_2^2 \quad (6) \end{aligned}$$

3. From 1. we see that: $\dot{V}$ exists $\rightarrow V$ is C_1
4. We see that:

$$\begin{aligned} \frac{x_1^2}{2} - \frac{x_1^4}{4} \geq 0 \quad (7) \\ \frac{x_1^2}{2} \geq \frac{x_1^4}{4} \quad (8) \\ 2 \geq x_1^2 \quad (9) \\ \|x_1\| \leq \sqrt{2} \rightarrow \frac{x_1^2}{2} - \frac{x_1^4}{4} \geq 0 \quad (10) \end{aligned}$$

   But since $x_1 = 1, -1$ are other isolated fixed points, we cannot be A.S. if D includes them So Let $D = \left\{x \in R^2 : \|x_1\| < 1\right\}$

5. Then from 4. we see that over the set $D, V \geq 0, V(0) = 0, V(\neq 0) > 0$. V over D is positive definite function.

6. We can also see that in D, $\dot{V}$, which is not a function of x_1 so it makes no difference, $\dot{V} = -x_2^2 <= 0$

7. Let $S = \left\{x \in D | \dot{V}(x) = 0\right\}$

8. Then for $x \in S$, we see that:

$$\begin{aligned} x \in S \rightarrow -x_2^2 = 0 \rightarrow x_2 = 0 \quad (11) \\ \dot{x_1} = 0 \rightarrow \dot{x_1} = x_2 = 0 \quad (12) \\ \dot{x_2} = 0 \rightarrow \dot{x_2} = x_1^3 - x_1 = 0 \quad (13) \\ \rightarrow x_1 = \{-1, 0, 1\} \rightarrow x_1 = 0 \quad (14) \end{aligned}$$

   So, we see that in $S \in D$, only $(0,0)$ stays identically in S

9. Then by 5., 6., 7. + 8., and Lec3 pg14-15, $(0,0)$ **is A.S.**

(ii) **Make a 3d surface plot** of the above Lyapunov function V over the domain $[-2, 2]^2$ along with the contour/level set plots of V in the plane over the same domain. It will be helpful to make the surface plot somewhat transparent.

Contour plot

Contour plot

(c) [15 + 5 = 20 points] ROA estimation

(i) Use the Lyapunov function in part (b) to **estimate the ROA for the origin**.

1. Given the set D defined in (b line 5.), we know over D, V is positive definite.
2. From (b line 6.) we know that over D, $\dot{V} < 0$ for $x_2 \neq 0$
3. Given the set D defined in (b), we know $\|x_1\| < 1 \rightarrow x_1 = 1 - \epsilon, \quad 0 < \epsilon < 2$
4. By definition of positive invariance, we seek to find the bounds for (x_1, x_2) s.t.

$$\begin{aligned} \dot{x} \leq 0 \quad (15) \\ \rightarrow \dot{x_1} = x_2 \leq 0 \quad (16) \\ \rightarrow \dot{x_2} = -x_1 - x_2 + x_1^3 \leq 0 \quad (17) \\ \rightarrow x_1^3 - x_1 \leq x_2 \leq 0 \quad (18) \\ \rightarrow x_1(x_1^2 - 1) \leq x_2 \leq 0 \quad (19) \\ \rightarrow (1-\epsilon)((1-\epsilon)^2 - 1) \leq x_2 \leq 0 \quad (20) \\ \rightarrow 3\epsilon^2 - 2\epsilon - \epsilon^3 \leq x_2 \leq 0 \quad (21) \end{aligned}$$

5. We want to maximize $V = \frac{x_2^2}{2} + \frac{x_1^2}{2} - \frac{x_1^4}{4}$ that meets all the constraints so far.
6. If we take the derivative of the $\frac{x_1^2}{2} - \frac{x_1^4}{4}$ to find the x_1 to maximize V contribution

$$\begin{aligned} \frac{\partial}{\partial t}(\frac{x_1^2}{2} - \frac{x_1^4}{4}) = x_1 - x_1^3 = 0 \quad (22) \\ \rightarrow x_1(1 - x_1^2) = 0 \quad (23) \\ \rightarrow x_1 \in -1, 0, 1 \quad (24) \end{aligned}$$

7. So we see that the x_1 term is maximum $(= 1/4)$ when $x_1 \in -1, 1$, but since those fall outside the set D, it is inconclusive, so we switch to x_2
8. From 4., we see that $3\epsilon^2 - 2\epsilon - \epsilon^3 \leq x_2 \leq 0$, so we want to minimize the leftmost term to maximize $\frac{x_2^2}{2}$ term of V
9. Taking the derivative and solving for 0 will yield the ϵ that will minimize the leftmost term:

$$\begin{aligned} \frac{\partial}{\partial t}(3\epsilon^2 - 2\epsilon - \epsilon^3) = 3\epsilon^2 - 6\epsilon + 2 = 0 \quad (25) \\ \rightarrow \epsilon = 1 \pm \frac{\sqrt{3}}{3} \quad (26) \\ \rightarrow @\epsilon = 0.423 \rightarrow 3\epsilon^2 - 2\epsilon - \epsilon^3 = -0.385 \leq x_2 \quad (27) \\ \rightarrow x_2 = -0.385 \quad (28) \\ \rightarrow @\epsilon = 0.423, x_1 = 1 - \epsilon = 0.577 \quad (29) \\ \rightarrow @x_1 = 0.577, x_2 = -0.385, V = 0.213 \quad (30) \end{aligned}$$

10. So $V = 0.213 = c$ we can conclude is the maximal c s.t. $x \in D$, and $\dot{x} <= 0$ (positive invariant). We can than construct

$$\Omega_c = \left\{x \in R^2 | \quad V(x) \leq c = 0.213\right\} \quad (31)$$

    Then we see that Ω_c is closed and bounded, and $\Omega_c \subset D$
11. Then by Lec 7 pg 6-7, $\Omega_c \subseteq ROA$

(ii) **Make a 2d plot of your estimated ROA** in the state space.

State space

Problem 2. [50 points] Lyapunov stability for continuous time LTI system

Consider the LTI system $\dot{x} = Ax$ where $x \in \mathbb{R}^n$. Recall that the origin is GAS if and only if the matrix A is Hurwitz (i.e., all its eigenvalues lie in the open left-half of the complex plane). In this exercise, we revisit the LTI stability from the Lyapunov viewpoint.

(a) [10 + 10 = 20 points] Lyapunov matrix inequality and equation

(i) Use the Lyapunov function $V(x) = x^\top Px$ to **prove that** the LTI system is GAS provided the following two matrix inequalities are simultaneously satisfied:

$$P \succ 0, \quad \mathcal{L}(P) := A^\top P + PA \prec 0.$$

The latter inequality is called the "Lyapunov matrix inequality".

(i)

1. We can derive $\dot{V}$ as follows given $V = x^TPx, \quad \dot{x} = Ax$:

$$\begin{aligned} \dot{V} &= \dot{x}^TPx + x^TP\dot{x} \quad (32) \\ &= (Ax)^TPx + x^TPAx \quad (33) \\ &= x^TA^TPx + x^TPAx \quad (34) \\ &= x^T(A^TP + PA)x \quad (35) \\ &= x^T\mathcal{L}(P)x \quad (36) \end{aligned}$$

2. From 1. we see that if $x^T\mathcal{L}(P)x < 0$, then $\dot{V} < 0$, **negative definite**
3. If $P \succ 0$, then V is **positive definite function**.
4. We can show that V is **radially unbounded**.

$$\begin{aligned} Px = \lambda x, \quad \lambda \succ 0 \quad (37) \\ \rightarrow x^TPx = \lambda\|x\|^2 \quad (38) \\ \lim_{\|x\|^2 \rightarrow \infty} x^TPx = \lim_{\|x\|^2 \rightarrow \infty} \lambda\|x\|^2 = +\infty \quad (39) \end{aligned}$$

5. So by 2., 3., and 4., and Lec 3 pg 9 (Barbashin-Krasovski Thm.), the LTI system is **G.A.S**

(ii) **Argue that** the condition $\mathcal{L}(P) \prec 0$ is equivalent to the statement: for any $Q \succ 0$, there exists $P \succ 0$ that solves the linear matrix equation $\mathcal{L}(P) = -Q$. This equation is called the "Lyapunov matrix equation".

1. For the equivalence first direction:
   $\forall Q \succ 0$, there exists $P \succ 0$ that solves $\mathcal{L}(P) = -Q \Rightarrow \mathcal{L}(P) \prec 0$
2. For the other direction: we want to construct a $P \succ 0$ s.t. $A^TP + PA = -Q$ for $Q < 0$
3. Let $P = \int_0^\infty e^{tA^T}Qe^{tA}$ for some $Q \succ s.t. Q \succ 0$ We can show that $P \succ 0$:

$$\begin{aligned} x \neq 0, \quad x^TPx = 0 \Rightarrow \int_0^\infty x^Te^{tA^T}Qe^{tA}x = 0 \quad (40) \\ \Rightarrow e^{At}x = 0 \Rightarrow x = 0 \quad (e^{At} \neq 0) \quad (41) \end{aligned}$$

   This contradiction shows $x^TPx \succ 0 \Leftrightarrow x \neq 0, x^TPx = 0 \Leftrightarrow x = 0$, P is **positive definite**.
4. We can now evaluate $PA + A^TP$:

$$\begin{aligned} PA + A^TP &= \int_0^\infty Ae^{tA^T}Qe^{tA} + \int_0^\infty A^Te^{tA^T}Qe^{tA} \quad (42) \\ &= \int_0^\infty Ae^{tA^T}Qe^{tA} + A^Te^{tA^T}Qe^{tA} \quad (43) \\ &= \int_0^\infty \left(\frac{\partial}{\partial t}e^{tA^T}Qe^{tA}\right) \quad (44) \\ &= e^{tA^T}Qe^{tA}|_0^\infty \quad (45) \\ &= -Q \quad (46) \end{aligned}$$

5. So we've shown by construction that for all $Q \succ 0$, we can construct a P based on that Q s.t. $\mathcal{L}(P) = -Q$
6. So from 3., 4., 5., we know that if $\mathcal{L}(P) < 0$, by definition $\mathcal{L}(P) = PA + A^TP = -Q$ for some $Q > 0$, and in 3. and 4. we show how this is done.

(b) [10 + 10 = 20 points] A Hurwitz $\Rightarrow$ Unique solution for Lyapunov matrix equation

You have shown in part (a) that existence of solution for the Lyapunov matrix equation (equivalently, Lyapunov matrix inequality) $\Rightarrow$ GAS $\Leftrightarrow$ A Hurwitz. **Now prove the converse:** if A is Hurwitz then for any $Q \succ 0$, there exists **unique** $P \succ 0$ that solves $\mathcal{L}(P) = -Q$.

(**Hint:** prove existence by construction. Prove uniqueness by contradiction.)

1. Let's say that P is **not unique**, then there is another $\bar{P} \neq P$ s.t. $\mathcal{L}(\bar{P}) = -Q$
2. $\mathcal{L}(P) - \mathcal{L}(\bar{P}) = -Q + Q = 0 = (P - \bar{P})A + A^T(P - \bar{P})$ And furthermore:

$$\begin{aligned} (P - \bar{P})A + A^T(P - \bar{P}) = 0 \quad (47) \\ \Rightarrow e^{A^Tt}(P - \bar{P})A + A^T(P - \bar{P})e^{At} = 0 \quad (48) \\ \Rightarrow \frac{\partial}{\partial t}e^{A^Tt}(P - \bar{P})e^{At} = 0 \quad (49) \end{aligned}$$

3. So we know from 2. that $f(t) = e^{A^Tt}(P - \bar{P})e^{At}$ is **constant**, thus:

$$\begin{aligned} f(0) = f(\infty) \quad (50) \\ f(0) = P - \bar{P} \quad (51) \\ f(\infty) = 0 \quad (52) \\ P - \bar{P} = 0 \quad (53) \\ P = \bar{P} \quad (54) \end{aligned}$$

4. We see from 1. and 3. that there is a contradiction, therefore **P must be unique**

(c) [10 points] Monotonicity of the solution of Lyapunov matrix equation

For an LTI system with A Hurwitz, **prove that** if $Q_1 \succ Q_2$ then $P_1 \succ P_2$. (The converse is false but you don't need to prove that).

1. From 2.a.ii above, $P = \int_0^\infty e^{tA^T}Qe^{tA}$, then:

$$\begin{aligned} Q_1 \succ Q_2 \quad (55) \\ \Rightarrow e^{A^Tt}Q_1e^{At} \succ e^{A^Tt}Q_2e^{At} \quad (56) \\ \Rightarrow \int_0^\infty e^{A^Tt}Q_1e^{At} \succ \int_0^\infty e^{A^Tt}Q_2e^{At} \quad (57) \\ \Rightarrow P_1 \succ P_2 \quad (58) \end{aligned}$$

   This direct proof shows the claim.