Problem 1 [100 points] Feedback Linearization

Consider the dynamics of one-link robotic arm, given by

$$J_1\ddot{q}_1 + F_1\dot{q}_1 + \frac{K}{N}\left(q_2 - \frac{q_1}{N}\right) = \tau,$$
$$J_2\ddot{q}_2 + F_2\dot{q}_2 + K\left(q_2 - \frac{q_1}{N}\right) = -mgd\cos q_2,$$

where q_1, q_2 denote angular positions (in rad), and τ denotes the actuation torque. The parameters $\{J_i, F_i\}_{i=1}^{2}$ respectively denote the moments of inertia and rotational damping coeffcients of the actuator shaft (for $i = 1$) and the link (for $i = 2$), respectively. The parameters K, N, m, d denote the torsional spring constant, the transmission gear ratio, mass of the link, and distance of the center of gravity of the link from a fixed frame of reference, respectively. The **state vector** is $x := (q_1, q_2, \dot{q}_1, \dot{q}_2)^\top \in [0, 2\pi)^2 \times \mathbb{R}^2$ and the **control input** is $u := \tau \in \mathbb{R}$.

(a) [(5+5) + 20 = 30 points] Deciding feedback linearizability

(i) Write the dynamics of the robotic arm in **standard control-affine form** $\dot{x} = f(x) + g(x)u$, i.e., **explicitly write** the drift vector field $f(x)$ and the input vector field $g(x)$, both of size 4×1, in terms of the state components x_1, x_2, x_3, x_4 and the parameters.

1. We can do some algebra on each equation to isolate the $\ddot{q}_1$ and $\ddot{q}_2$ terms on the RHS, and given $\tau = u$, we can express the dynamics as:

$$\dot{x} = \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}\dot{q}_1\\ \dot{q}_2\\ \ddot{q}_1\\ \ddot{q}_2\end{bmatrix} = \begin{bmatrix} x_3\\ x_4\\ \frac{-F_1\dot{q}_1 - \frac{K\left(q_2 - \frac{q_1}{N}\right)}{N}}{J_1}\\ \frac{-F_2\dot{q}_2 - K\left(q_2 - \frac{q_1}{N}\right) - dgm\cos q_2}{J_2}\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \frac{1}{J_1}\\ 0\end{bmatrix} u \tag{1}$$

$$= \begin{bmatrix} x_3\\ x_4\\ -\frac{F_1x_3}{J_1} - \frac{Kx_2}{J_1N} + \frac{Kx_1}{J_1N^2}\\ -\frac{F_2x_4}{J_2} - \frac{Gdm\cos(x_2)}{J_2} - \frac{Kx_2}{J_2} + \frac{Kx_1}{J_2N}\end{bmatrix} (f(x)) + \begin{bmatrix}0\\ 0\\ \frac{1}{J_1}\\ 0\end{bmatrix} (g(x))\, u \tag{2}$$

(ii) Determine the region in the state space $[0, 2\pi)^2 \times \mathbb{R}^2$, over which this system is full state static feedback linearizable. (**Hint:** use Lec. 16 p. 11, also Step 1 in p. 12.)

1. From Lec 16 pg 11, condition (i) we check is that the M matrix has full rank. We see that:

$$M = \begin{bmatrix} g(x^0) & adj_f g(x^0) & adj_f^2 g(x^0) & adj_f^3 g(x^0)\end{bmatrix} \tag{3}$$

$$= \begin{bmatrix} 0 & -\frac{1}{J_1} & -\frac{F_1}{J_1^2} & -\frac{F_1^2}{J_1^3} - \frac{K}{J_1^2N^2}\\ 0 & 0 & 0 & -\frac{K}{J_1J_2N}\\ \frac{1}{J_1} & \frac{F_1}{J_1^2} & \frac{F_1^2}{J_1^3} + \frac{K}{J_1^2N^2} & \frac{F_1^3}{J_1^4} + \frac{2F_1K}{J_1^3N^2}\\ 0 & 0 & \frac{K}{J_1J_2N} & \frac{K(F_1J_2+F_2J_1)}{J_1^2J_2^2N}\end{bmatrix} \tag{4}$$

2. $rank(M) = 4 = n$, it is **full rank** over $[0, 2\pi)^2 \times \mathbb{R}^2$ since **no entries are a function of any element of the state**, they are all constant parameters.
3. We also need to check condition (ii) from the same reference, we see that:

$$\begin{bmatrix} g(x^0) & adj_f g(x^0)\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \tag{5}$$

$$\begin{bmatrix} g(x^0) & adj_f^2 g(x^0)\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \tag{6}$$

$$\begin{bmatrix} adj_f g(x^0) & adj_f^2 g(x^0)\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \tag{7}$$

4. For each of the lie brackets in 3., we see that the lie bracket does **not** add rank, and each 2 pair of $g(x^0)$, $adj_f g(x^0)$, $adj_f^2 g(x^0)$ are **not** full rank (they are rank(2)). Combining each pair of $g(x^0)$, $adj_f g(x^0)$, $adj_f^2 g(x^0)$ with the pair's lie bracket would still be **rank(2) < rank(3)**.
5. Then by definition the span is closed under Lie bracket, **is involutive** over $[0, 2\pi)^2 \times \mathbb{R}^2$
6. Then from Lec 16 pg 11, we've checked for the 2 constructive conditions, and from 16 pg 12, Step 1, we've verified the **system is feedback linearizable**

(b) [10 + 10 = 20 points] Finding $\lambda(x)$ and r

(i) **Show that** the scalar field $\lambda(x)$ in the Theorem given in Lec. 16 p. 10, for this system, can be taken as $\lambda(x) = x_2$. (**Hint:** use Lec. 16 p. 13, Step 2.)

1. We can see that for $\lambda(x) = x_2$, $\frac{\partial\lambda}{\partial x} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$
2. We also can see that since $g(x^0)[2] = adj_f g(x^0)[2] = adj_f^2 g(x^0)[2] = 0$:

$$\mathcal{L}_g\lambda(x) = \begin{bmatrix}0 & 0 & \frac{1}{J_1} & 0\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = 0 \tag{8}$$

$$\mathcal{L}_{adj_f g(x^0)}\lambda(x) = \begin{bmatrix}-\frac{1}{J_1} & 0 & \frac{F_1}{J_1^2} & 0\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = 0 \tag{9}$$

$$\mathcal{L}_{adj_f^2 g(x^0)}\lambda(x) = \begin{bmatrix}-\frac{F_1}{J_1^2} & 0 & \frac{F_1^2}{J_1^3} + \frac{K}{J_1^2N^2} & \frac{K}{J_1J_2N}\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = 0 \tag{10}$$

3. We can also see that since $adj_f^3 g(x^0)[2] \neq 0$:

$$\mathcal{L}_{adj_f^3 g(x^0)}\lambda(x) = \begin{bmatrix}-\frac{F_1^2}{J_1^3} - \frac{K}{J_1^2N^2} & -\frac{K}{J_1J_2N} & \frac{F_1^3}{J_1^4} + \frac{2F_1K}{J_1^3N^2} & \frac{K(F_1J_2+F_2J_1)}{J_1^2J_2^2N}\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = -\frac{K}{J_1J_2N} \neq 0 \tag{11}$$

4. Then from Lec 16 pg 13, Step 2, we see that $\lambda(x) = x_2$ **satisfies the desired system of PDEs**.

(ii) **By directly computing the relative degree** r (a positive integer), **prove that** the state equation derived in part (a)(i), augmented with the output equation $y = \lambda(x) = x_2$, indeed has relative degree 4, that is, satisfies the $r = n$ condition in the region determined in part (a)(ii). (**Hint:** use Lec. 15 p. 12 for computing the relative degree)

1. We can show first that for our system, $y = h(x) = \lambda(x) = x_2$, and:

$$\mathcal{L}_g\mathcal{L}_f^0 h = \mathcal{L}_g h = \mathcal{L}_g\lambda(x) = \begin{bmatrix}0 & 0 & \frac{1}{J_1} & 0\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = 0 \tag{12}$$

$$\mathcal{L}_g\mathcal{L}_f^1 h = \mathcal{L}_g\left\langle\frac{\partial h}{\partial x}, f\right\rangle = \mathcal{L}_g\langle\begin{bmatrix}0 & 1 & 0 & 0\end{bmatrix}, f\rangle = \mathcal{L}_g x_4 = \begin{bmatrix}0 & 0 & \frac{1}{J_1} & 0\end{bmatrix}\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = 0 \tag{13}$$

$$\mathcal{L}_g\mathcal{L}_f^2 h = \mathcal{L}_g\left\langle\frac{\partial}{\partial x}\mathcal{L}_f^1 h, f\right\rangle = \mathcal{L}_g\left\langle\frac{\partial}{\partial x}x_4, f\right\rangle = \mathcal{L}_g\left\langle\begin{bmatrix}0\\0\\0\\1\end{bmatrix}, f\right\rangle \tag{14}$$

$$= \mathcal{L}_g f[3] = \left\langle\frac{\partial}{\partial x}f[3], g\right\rangle = \left\langle\begin{bmatrix}\frac{K}{J_2N} & \frac{-K+dgm\sin(x_2)}{J_2} & 0 & -\frac{F_2}{J_2}\end{bmatrix}, \begin{bmatrix}0\\0\\\frac{1}{J_1}\\0\end{bmatrix}\right\rangle = 0 \tag{15}$$

$$\mathcal{L}_g\mathcal{L}_f^3 h = \mathcal{L}_g\left\langle\frac{\partial}{\partial x}\mathcal{L}_f^2 h, f\right\rangle = \mathcal{L}_g\left\langle\frac{\partial}{\partial x}f[3], f\right\rangle \tag{16}$$

$$= \left\langle\frac{\partial}{\partial x}\mathcal{L}_f^3 h, g\right\rangle = \frac{K}{J_1J_2N} \neq 0 \tag{17}$$

2. From 1., we see that $\mathcal{L}_g\mathcal{L}_f^0 h = \mathcal{L}_g\mathcal{L}_f^1 h = \mathcal{L}_g\mathcal{L}_f^2 h = 0$, and $\mathcal{L}_g\mathcal{L}_f^3 h \neq 0$.
3. This means t hat it will at $r = 3 + 1$, $y = \lambda(x) = x_2$, $y^r = \mathcal{L}_f^4\lambda(x) + \mathcal{L}_g\mathcal{L}_f^3\lambda(x)u$ has a **nonzero u** coefficient.
4. From 2. and Lec 15 pg 12, we prove that this SISO system has relative degree $3 + 1 = 4 = n$

(c) [(5+5+5+5) + 5 + 25 = 50 points] Normal form and closed loop simulation

(i) Use the Steps 3 and 4 in Lec. 16 p. 14-15 to **derive the feedback linearizing tuple** $(\tau(\cdot), \alpha(\cdot), \beta(\cdot))$. **Also clearly write down** the control system in the feedback linearized coordinates with the new 4×1 state vector z and the new control $v \in \mathbb{R}$, where $z := \tau(x)$ and $u = \alpha(x) + \beta(x)v$.

1. We can derive $\alpha(\cdot)$:

$$\alpha(x) = \frac{-\left\langle\frac{\partial}{\partial x}\mathcal{L}_f^3 h, f\right\rangle}{\left\langle\frac{\partial}{\partial x}\mathcal{L}_f^3 h, g\right\rangle} \tag{18}$$

$$= F_1x_3 + \frac{F_2^3J_1Nx_4}{J_2^2K} + \frac{F_2^2GJ_1Ndm\cos(x_2)}{J_2^2K} + \frac{F_2^2J_1Nx_2}{J_2^2} - \frac{F_2^2J_1x_1}{J_2^2} \tag{19}$$

$$+ \frac{2F_2GJ_1Ndmx_4\sin(x_2)}{J_2K} - \frac{2F_2J_1Nx_4}{J_2} + \frac{F_2J_1x_3}{J_2} \tag{20}$$

$$+ \frac{G^2J_1Nd^2m^2\sin(x_2)\cos(x_2)}{J_2K} - \frac{GJ_1Ndm\,\mathrm{x_4}^2(t)\cos(x_2)}{K} \tag{21}$$

$$+ \frac{GJ_1Ndmx_2\sin(x_2)}{J_2} - \frac{GJ_1Ndm\cos(x_2)}{J_2} - \frac{GJ_1dmx_1\sin(x_2)}{J_2} \tag{22}$$

$$- \frac{J_1KNx_2}{J_2} + \frac{J_1Kx_1}{J_2} + \frac{Kx_2}{N} - \frac{Kx_1}{N^2} \tag{23}$$

2. We can derive $\beta(\cdot)$:

$$\beta(x) = \frac{1}{\mathcal{L}_g\mathcal{L}_f^3\lambda(x)} = \frac{J_1J_2N}{K} \tag{24}$$

3. We can derive $\tau(\cdot)$:

$$z = \tau(x) = \begin{pmatrix}\lambda(x)\\ \mathcal{L}_f\lambda(x)\\ \mathcal{L}_f^2\lambda(x)\\ \mathcal{L}_f^3\lambda(x)\end{pmatrix} \tag{25}$$

$$\begin{bmatrix}z_1\\z_2\\z_3\\z_4\end{bmatrix} = \begin{bmatrix} x_2\\ x_4\\ -\frac{F_2x_4}{J_2} - \frac{Gdm\cos(x_2)}{J_2} - \frac{Kx_2}{J_2} + \frac{Kx_1}{J_2N}\\ -\frac{F_2\left(-\frac{F_2x_4}{J_2} - \frac{Gdm\cos(x_2)}{J_2} - \frac{Kx_2}{J_2} + \frac{Kx_1}{J_2N}\right)}{J_2} + \left(\frac{Gdm\sin(x_2)}{J_2} - \frac{K}{J_2}\right)x_4 + \frac{Kx_3}{J_2N}\end{bmatrix} \tag{26}$$

4. Finally, for the new control system in new coordinates:

$$\begin{pmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\\ \dot{z}_4\end{pmatrix} = \begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\begin{pmatrix}z_1\\z_2\\z_3\\z_4\end{pmatrix} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix} v \tag{27}$$

(ii) Give physical interpretation of the components of the vector z (think of their units).

1. We see that for the elements of $z = \tau(x)$
2. z_1 is joint position
3. z_2 is velocity of that joint
4. z_3 is the acceleration of that joint, as we derived in (a)(i) for $\ddot{q}_2$
5. z_4 is the Lie derivative of $z_3 = \ddot{q}_2$ along f, intuitively it is how the acceleration term $z_3 = \ddot{q}_2$ changes as we move along the system dynamics vector field f.

(iii) **Use a simple pole placement controller** $v = k^\top z$ where $k := (k_1, k_2, k_3, k_4)^\top \in \mathbb{R}^4$ to stabilize the feedback linearized states to the origin. For this stablizing controller, **submit the plots** for x_1, x_2, x_3, x_4, u versus time $t = 0 : 0.01 : 5$ with initial condition $x(t = 0) = (\pi/6, \pi/3, 1, 2)^\top$. (**Hint:** recall that over the region in which the system is feedback linearizable, the map τ must be a diffeomorphism, and $x = \tau^{-1}(z)$)

1. We can define $x = \tau^{-1}(z)$:

$$\tau^1(z) = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{bmatrix}\frac{N(F_2z_2+J_2z_3+Kz_1+dgm\cos(z_1))}{K}\\ z_1\\ \frac{N(F_2z_3+J_2z_4+Kz_2-dgmz_2\sin(z_1))}{K}\\ z_2\end{bmatrix} \tag{28}$$

2. We can use τ to translate the initial state in $x(t = 0)$ to $z(t = 0)$, and then use the z coordinate closed-loop dynamics ($v = k^\top z$) to integrate out $z(t)$:

$$\begin{pmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\\ \dot{z}_4\end{pmatrix} = \begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\begin{pmatrix}z_1\\z_2\\z_3\\z_4\end{pmatrix} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\begin{pmatrix}k_1 & k_2 & k_3 & k_4\end{pmatrix}\begin{pmatrix}z_1\\z_2\\z_3\\z_4\end{pmatrix} \tag{29}$$

$$= \begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\k_1&k_2&k_3&k_4\end{bmatrix}\begin{pmatrix}z_1\\z_2\\z_3\\z_4\end{pmatrix} \tag{30}$$

3. Using the matrix in 2. (the companion matrix), we can see that if we want the eigenvalues to be at -1, -2, -3, -4, then we set the k values = the negative characteristic polynomial coefficients: $k = [-24, -50, -35, -10]$, and we can plot a simulator output as below, one simulating by z, the other using u and original system dynamics (as expected they are equivalent). Derivation in **derivation_1c.py**, simulation in **1c.py**.

In [1]:
```
from IPython.display import Image
Image(filename='plot.png')
```

Out[1]: