Homework (6)

1. Show that $f(\mathbf{x}) = \sum_{i=1}^{r} \alpha_i x_{[i]}$ is a convex function of $\mathbf{x}$, where $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_r \geq 0$, and $x_{[i]}$ denotes the ith largest component of $\mathbf{x}$.
2. Show that $f(\mathbf{x}) = \mathbf{tr}\,(\mathbf{A}_0 + x_1\mathbf{A}_1 + ... + x_n\mathbf{A}_n)^{-1}$ is convex on $\{\mathbf{x} \mid \mathbf{A}_0 + x_1\mathbf{A}_1 + ... + x_n\mathbf{A}_n \succ \mathbf{0}\}$, where $\mathbf{A}_i \in \mathbb{S}^m$.
3. Show that $f(\mathbf{x}) = -\log\left(-\log\left(\sum_{i=1}^{m} e^{\mathbf{a}_i^T\mathbf{x}+b_i}\right)\right)$ is convex on $\mathbf{dom}\, f = \{\mathbf{x} \mid e^{\mathbf{a}_i^T\mathbf{x}+b_i} < 1\}$.
4. Show that the Minkowski function:

$$M_C(\mathbf{x}) = \inf\{t > 0 \mid t^{-1}\mathbf{x} \in C\},$$

is convex, where C is a convex set. Find its conjugate function.
5. Derive the conjugates of the following functions.

   (a) Sum of largest elements. $f(\mathbf{x}) = \sum_{i=1}^{r} x_{[i]}$ *on* $\mathbb{R}^n$.

   (b) Negative generalized logarithm for second-order cone. $f(\mathbf{x}, t) = -\log(t^2 - \mathbf{x}^T\mathbf{x})$ *on* $\{(\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R} \mid \|\mathbf{x}\|_2 < t\}$.

Note: Computing a conjugate function needs to specify its domain as well.

Homework (6)

6. Show that the conjugate of $f(\mathbf{X}) = \mathbf{tr}(\mathbf{X}^{-1})$ with $\mathbf{dom}\, f = \mathbb{S}^n_{++}$ is given by

$$f^*(\mathbf{Y}) = -2\mathbf{tr}(-\mathbf{Y})^{1/2}, \quad \mathbf{dom} f^* = -\mathbb{S}^n_+.$$

7. Find the projection onto the second order cone.
8. Let $d(\mathbf{x})$ be the distance from $\mathbf{x}$ to a closed convex set $\mathcal{C}$: $d(\mathbf{x}) = \inf_{\mathbf{y}\in\mathcal{C}} \|\mathbf{x}-\mathbf{y}\|_2$.

Find the proximal mappings of $d(\mathbf{x})$ and $\frac{1}{2}d^2(\mathbf{x})$.
9. Find the proximal mappings of the following functions.

(a) $\|\mathbf{x}\|_1$.

(b) $\|\mathbf{x}\|_2$.

(c) $\sum_{i=1}^{n} \log x_i$.