# Homework (10)

1. Use Moreau decomposition to find the proximal mapping of

   (a) $\|\mathbf{x}\|_1$.

   (b) $\|\mathbf{X}\|_*$.

2. Use Moreau decomposition to prove that $\mathbf{x} = P_L(\mathbf{x}) + P_{L^\perp}(\mathbf{x})$, where $L$ is a subspace and $L^\perp$ is its orthogonal complement.
3. Show that the function $f(\mathbf{X}) = \mathbf{X}^{-1}$ is matrix convex on $\mathbb{S}^n_{++}$.
4. *Schur complement.* Suppose $\mathbf{X} \in \mathbb{S}^n$ partitioned as

$$\mathbf{X} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix},$$

where $\mathbf{A} \in \mathbb{S}^k$. The Schur complement of $\mathbf{X}$ (with respect to $\mathbf{A}$) is $\mathbf{S} = \mathbf{C} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$. Show that the Schur complement, viewed as function from $\mathbb{S}^n$ into $\mathbb{S}^{n-k}$, is matrix concave on $\mathbb{S}^n_{++}$.

(3&4 choose one)

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5. *Sublevel sets and epigraph of $K$-convex functions.* Let $K \subseteq \mathbb{R}^m$ be a proper convex cone with associated generalized inequality $\preceq_K$, and let $f : \mathbb{R}^n \to \mathbb{R}^m$. For $\boldsymbol{\alpha} \in \mathbb{R}^m$, the $\boldsymbol{\alpha}$-sublevel set of $f$ (with respect to $\preceq_K$) is defined as

$$C_{\boldsymbol{\alpha}} = \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) \preceq_K \boldsymbol{\alpha}\}.$$

The epigraph of $f$, with respect to $\preceq_K$, is defined as the set

$$\mathbf{epi}_K\, f = \{(\mathbf{x}, \mathbf{t}) \in \mathbb{R}^{n+m} \mid f(\mathbf{x}) \preceq_K \mathbf{t}\}.$$

Show the following:

(a) If $f$ is $K$-convex, then its sublevel sets $C_{\boldsymbol{\alpha}}$ are convex for all $\boldsymbol{\alpha}$.

(b) $f$ is $K$-convex iff $\mathbf{epi}_K\, f$ is a convex set.

# Homework (10)

(6-9 choose 3)

6. Let $\pi_{\mathcal{C}}$ be the projection operator onto a convex set $\mathcal{C}$. Prove:

$$\langle \pi_{\mathcal{C}}(\mathbf{y}) - \mathbf{x}, \pi_{\mathcal{C}}(\mathbf{y}) - \mathbf{y} \rangle \leq 0.$$

Further show that

$$\|\pi_{\mathcal{C}}(\mathbf{y}) - \mathbf{x}\|^2 + \|\pi_{\mathcal{C}}(\mathbf{y}) - \mathbf{y}\rangle\|^2 \leq \|\mathbf{x} - \mathbf{y}\|^2.$$

7. If $f$ is an $L$-smooth function (a short for "$\nabla f$ is Lipschitz continuous with a Lipschitz constant $L$"), prove

$$f\left(\mathbf{x} - \frac{1}{\beta}\nabla f(\mathbf{x})\right) - f(\mathbf{x}) \leq -\frac{1}{2\beta}\|\nabla f(\mathbf{x})\|^2.$$

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8. Let $f$ satisfy

$$0 \leq f(\mathbf{x}) - f(\mathbf{y}) - \langle \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle \leq \frac{\beta}{2} \|\mathbf{x} - \mathbf{y}\|^2, \quad \forall \mathbf{x}, \mathbf{y}.$$

Prove that

$$f(\mathbf{x}) - f(\mathbf{y}) \leq \langle \nabla f(\mathbf{x}), \mathbf{x} - \mathbf{y} \rangle \leq \frac{1}{2\beta} \|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\|^2, \quad \forall \mathbf{x}, \mathbf{y}.$$

9. Let $f$ be $L$-smooth and $\mu$-strongly convex on $\mathbb{R}^n$. Then

$$\langle \nabla f(\mathbf{x}) - \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle \geq \frac{L\mu}{L + \mu} \|\mathbf{x} - \mathbf{y}\|^2 + \frac{1}{L + \mu} \|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\|^2, \quad \forall \mathbf{x}, \mathbf{y}.$$