# Homework (17)

1. [Continued from Homework 16] Wite down the dual problems of the following problems and check whether the strong dualities hold.

$$
\begin{aligned}
\text{a)} \quad & \min_{\mathbf{x}} \; x_1^2 + x_2^2, \\
& s.t. \; -x_2 \le 0, \\
& \qquad x_1^3 - x_2 \le 0, \\
& \qquad x_1^3(x_2 - x_1^3) \le 0.
\end{aligned}
$$

b) Support vector machine:

$$
\begin{aligned}
& \min_{\mathbf{w}} \; \frac{1}{2}\|\mathbf{w}\|^2, \\
& s.t. \; y_i(\langle \mathbf{w}, \mathbf{x}_i \rangle + \beta) \ge 1, \quad i = 1, \cdots, m.
\end{aligned}
$$

$$
\begin{aligned}
\text{c)} \quad & \min_{x} \; x^2 + 1, \\
& s.t. \; (x-2)(x-4) \le 0.
\end{aligned}
$$

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2. Express the dual problem of

$$\begin{aligned} &\min_{\mathbf{x}} \ \mathbf{c}^T\mathbf{x} \\ &s.t. \ f(\mathbf{x}) \leq 0, \end{aligned}$$

with $\mathbf{c} \neq \mathbf{0}$, in terms of the conjugate $f^*$. Explain why the problem you give is convex. We do not assume $f$ is convex.

3. We consider the convex piecewise-linear minimization problem

$$\min_{\mathbf{x}} \max_{i=1,\cdots,m} (\mathbf{a}_i^T\mathbf{x} + b_i) \tag{1}$$

with variable $\mathbf{x} \in \mathbb{R}^n$.

(a) Derive a dual problem, based on the Lagrange dual of the equivalent problem

$$\begin{aligned} &\min_{\mathbf{x}} \ \max_{i=1,\cdots,m} y_i \\ &s.t. \ \mathbf{a}_i^T\mathbf{x} + b_i = y_i, \quad i = 1, \cdots, m, \end{aligned}$$

with variables $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$.

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(b) Formulate the piecewise-linear minimization problem (1) as an LP, and form the dual of the LP. Relate the LP dual to the dual obtained in part (a).

(c) Suppose we approximate the objective function in (2) by the smooth function

$$f_0(\mathbf{x}) = \log\left(\sum_{i=1}^{m} \exp(\mathbf{a}_i^T\mathbf{x} + b_i)\right),$$

and solve the unconstrained geometric program

$$\min_{\mathbf{x}} \log\left(\sum_{i=1}^{m} \exp(\mathbf{a}_i^T\mathbf{x} + b_i)\right). \tag{2}$$

Let $p^*_{pwl}$ and $p^*_{gp}$ be the optimal values of (1) and (2), respectively. Show that

$$0 \leq p^*_{gp} - p^*_{pwl} \leq \log m.$$

# Homework (17)

(d) Derive similar bounds for the difference between $p^*_{pwl}$ and the optimal value of

$$\min_{\mathbf{x}} (1/\gamma) \log \left( \sum_{i=1}^{m} \exp(\gamma(\mathbf{a}_i^T \mathbf{x} + b_i)) \right),$$

where $\gamma > 0$ is a parameter. What happens as we increase $\gamma$?