Homework (8)

1. Compute the subgradients of $\|\mathbf{X}\|_{2,1}$ and $\|\mathbf{D}\,\mathrm{Diag}(\mathbf{x})\|_*$.
2. Prove that $\mathrm{conv}\{\boldsymbol{\alpha}\boldsymbol{\alpha}^T | \|\boldsymbol{\alpha}\| = 1\} = \{\mathbf{W} | \mathbf{W} \succcurlyeq \mathbf{0}, \mathrm{tr}\,\mathbf{W} = 1\}$.
3. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Show that $\partial f : \mathbb{R}^n \to \mathbb{R}^n$ is a monotone mapping, i.e.,

$$\langle \mathbf{g}_1 - \mathbf{g}_2, \mathbf{x}_1 - \mathbf{x}_2 \rangle \geq 0, \quad \forall \mathbf{g}_i \in \partial f(\mathbf{x}_i), i = 1, 2. \tag{2}$$

Further, if f is μ-strongly convex, then the above inequality can be strengthened as

$$\langle \mathbf{g}_1 - \mathbf{g}_2, \mathbf{x}_1 - \mathbf{x}_2 \rangle \geq \mu \|\mathbf{x}_1 - \mathbf{x}_2\|^2, \quad \forall \mathbf{g}_i \in \partial f(\mathbf{x}_i), i = 1, 2. \tag{3}$$

4. Show that $f(\mathbf{x}) = \sum_{i=1}^r \alpha_i x_{[i]}$ is a convex function of $\mathbf{x}$, where $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_r \geq 0$, and $x_{[i]}$ denotes the ith largest component of $\mathbf{x}$.
5. Show that $f(\mathbf{x}) = \mathbf{tr}\,(\mathbf{A}_0 + x_1\mathbf{A}_1 + ... + x_n\mathbf{A}_n)^{-1}$ is convex on $\{\mathbf{x} \mid \mathbf{A}_0 + x_1\mathbf{A}_1 + ... + x_n\mathbf{A}_n \succ \mathbf{0}\}$, where $\mathbf{A}_i \in \mathbb{S}^m$.
6. Show that $f(\mathbf{x}) = -\log\left(-\log\left(\sum_{i=1}^m e^{\mathbf{a}_i^T\mathbf{x}+b_i}\right)\right)$ is convex on $\mathbf{dom}\, f = \{\mathbf{x} \mid e^{\mathbf{a}_i^T\mathbf{x}+b_i} < 1\}$.

Homework (8)

7. Show that the Minkowski function:

$$M_C(\mathbf{x}) = \inf\{t > 0 \mid t^{-1}\mathbf{x} \in C\},$$

is convex, where C is a convex set. Find its conjugate function.
8. Derive the conjugates of Max function: $f(\mathbf{x}) = \max_{i=1,\ldots,n} x_i$ on $\mathbb{R}^n$.
Note: Computing a conjugate function needs to specify its domain as well.