Homework (4)

1. Prove the Separating Hyperplane Theorem under the case that the distance between two convex sets C and D is 0.
2. Suppose $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, with $\mathbf{b} \in \text{Range}(\mathbf{A})$. Show that $\exists \mathbf{x}$ satisfying

$$\mathbf{x} > \mathbf{0}, \ \mathbf{A}\mathbf{x} = \mathbf{b}$$

iff there exists no $\boldsymbol{\lambda}$ with

$$\mathbf{A}^T\boldsymbol{\lambda} \geq \mathbf{0}, \ \mathbf{A}^T\boldsymbol{\lambda} \neq 0, \ \mathbf{b}^T\boldsymbol{\lambda} \leq 0.$$

3.

(a) Express $\{\mathbf{x} \in \mathbb{R}^2_+ | \mathbf{x}_1\mathbf{x}_2 \geq 1\}$ as an intersection of halfspaces.

(b) Let $C = \{\mathbf{x} \in \mathbb{R}^n | \, \|\mathbf{x}\|_\infty \leq 1\}$, the ℓ_∞-norm unit ball in $\mathbb{R}^n$, and let $\mathbf{x} \in \partial C$. Identify the supporting hyperplanes of C at $\mathbf{x}$ explicitly.

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4. Prove the following.

(a) $(K^*)^\circ = \{\mathbf{y} |\ \mathbf{y}^T\mathbf{x} > 0 \text{ for all } \mathbf{x} \in K\}$.

(b) If K has nonempty interior then K^* is pointed.

(c) If the closure of K is pointed then K^* has nonempty interior.

5. Find the dual cone of $\{\mathbf{A}\mathbf{x} |\ \mathbf{x} \geq \mathbf{0}\}$, where $\mathbf{A} \in \mathbb{R}^{m\times n}$.

6. We define the monotone nonnegative cone as

$$K_{m+} = \{\mathbf{x} \in \mathbb{R}^n |\ x_1 \geq x_2 \geq ... \geq x_n \geq 0\}.$$

i.e., all nonnegative vectors with components sorted in nonincreasing order.

(a) Show that K_{m+} is a proper cone.

(b) Find the dual cone K_{m+}^*.

Homework (4)

7. A matrix $\mathbf{X} \in \mathbb{S}^n$ is called copositive if $\mathbf{z}^T\mathbf{X}\mathbf{z} \geq 0$ for all $\mathbf{z} \geq \mathbf{0}$. Verify that the set of copositive matrices is a proper cone. Find its dual cone.
8. A square matrix $\mathbf{A}$ is called conditionally negative definite (c.n.d.), if for all $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x}^T\mathbf{1} = 0$, we have

$$\mathbf{x}^T\mathbf{A}\mathbf{x} \leq 0.$$

Is the set of c.n.d. matrices a proper cone? Find its dual cone.
9. Let K and $\tilde{K}$ be two convex cones whose interiors are nonempty and disjoint. Show that there is a nonzero $\mathbf{y}$ such that $\mathbf{y} \in K^*, -\mathbf{y} \in \tilde{K}^*$.

Homework (4)

10. For each of the following functions determine whether it is convex or concave (you don't have to give details).

(a) $f(x) = e^x - 1$ on $\mathbb{R}$.

(b) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$.

(c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbb{R}^2_{++}$.

(d) $f(x_1, x_2) = x_1/x_2$ on $\mathbb{R}^2_{++}$.

(e) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbb{R} \times \mathbb{R}_{++}$.

(f) $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, where $0 \leq \alpha \leq 1$ on $\mathbb{R}^2_{++}$.

11. Prove that $f : \mathbb{R}^n \to \mathbb{R}$ is convex (resp., stictly convex, strongly convex) iff for every $\mathbf{x}, \mathbf{y} \in \operatorname{dom} f$, the function $g(t) = f(t\mathbf{x} + (1-t)\mathbf{y})$ is a convex (resp., stictly convex, strongly convex) function on $[0, 1]$.