Homework (9)

New rule of grading:

1. If you finished required number of problems correctly you can get the full score 100.
2. Those did more than required can get extra scores, in proportional to the base number.

Note: in order not to incur too much pressure on TA, I cannot leave too many problems unanswered.

1. Derive the conjugates of the following functions. (Choose four)

   (a) Sum of largest elements. $f(\mathbf{x}) = \sum_{i=1}^{r} x_{[i]}$ *on* $\mathbb{R}^n$.

   (b) Piecewise-linear function on $\mathbb{R}$. $f(x) = \max_{i=1,\ldots,n}(a_i x + b_i)$ *on* $\mathbb{R}$. *You can assume that the* a_i *are sorted in increasing order, i.e.,* $a_1 \leq \ldots \leq a_m$, *and that none of the functions* $a_i x + b_i$ *is redundant, i.e., for each* k *there is at least one* x *with* $f(x) = a_k x + b_k$.

   (c) Power function. $f(x) = x^p$ *on* $\mathbb{R}_{++}$, *where* $p > 1$. *Repeat for* $p < 0$.

   (d) Geometric mean. $f(\mathbf{x}) = -(\prod x_i)^{1/n}$ *on* $\mathbb{R}^n_{++}$.

   (e) Negative generalized logarithm for second-order cone. $f(\mathbf{x}, t) = -\log(t^2 - \mathbf{x}^T\mathbf{x})$ *on* $\{(\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R} \mid \|\mathbf{x}\|_2 < t\}$.

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2. Properties of conjugate functions. (Choose two)

   (a) Conjugate of convex plus affine function. *Define* $g(\mathbf{x}) = f(\mathbf{x}) + \mathbf{c}^T\mathbf{x} + d$, *where* f *is convex. Express* g^* *in terms of* f^* *(and* $\mathbf{c}, d$*).*

   (b) Conjugate of perspective. *Express the conjugate of the perspective of a convex function* f *in terms of* f^**.*

   (c) Conjugate and minimization. *Let* $f(\mathbf{x}, \mathbf{z})$ *be convex in* $(\mathbf{x}, \mathbf{z})$ *and define* $g(\mathbf{x}) = \inf_z f(\mathbf{x}, \mathbf{z})$*. Express the conjugate* g^* *in terms of* f^**. As an application, express the conjugate of* $g(\mathbf{x}) = \inf\{h(\mathbf{z}) \mid \mathbf{A}\mathbf{z} + \mathbf{b} = \mathbf{x}\}$, *where* h *is convex, in terms of* h^*, $\mathbf{A}$, *and* $\mathbf{b}$.

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3. Find the proximal mappings of the following functions. (Choose three)

   (a) $\|\mathbf{x}\|_1$.

   (b) $\|\mathbf{x}\|_2$.

   (c) $\sum_{i=1}^{n} \log x_i$.

   (d) $\frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^t\mathbf{x} + \mathbf{c}$.

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(Choose three)

4. Show that the conjugate of $f(\mathbf{X}) = \mathbf{tr}(\mathbf{X}^{-1})$ with $\mathbf{dom}\, f = \mathbb{S}^n_{++}$ is given by

$$f^*(\mathbf{Y}) = -2\mathbf{tr}(-\mathbf{Y})^{1/2}, \quad \mathbf{dom} f^* = -\mathbb{S}^n_+.$$

5. Young's inequality. Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function, with $f(0) = 0$, and let g be its inverse. Define F and G as

$$F(x) = \int_0^x f(a)da, G(y) = \int_0^y g(a)da.$$

Show that F and G are conjugates. Give a simple graphical interpretation of Young's inequality,

$$xy \le F(x) + G(y).$$

6. Find the projection onto the second order cone.

7. Let $d(\mathbf{x})$ be the distance from $\mathbf{x}$ to a closed convex set $\mathcal{C}$: $d(\mathbf{x}) = \inf_{\mathbf{y}\in\mathcal{C}} \|\mathbf{x}-\mathbf{y}\|_2$.

Find the proximal mappings of $d(\mathbf{x})$ and $\frac{1}{2}d^2(\mathbf{x})$.