Homework (3)

1. Let $C \subseteq \mathbb{R}^n$ be a convex set, with $\mathbf{x}_1, ..., \mathbf{x}_k \in C$, and let $\theta_1, ..., \theta_k \in \mathbb{R}$ satisfy $\theta_i \geq 0$, $\theta_1 + ... + \theta_k = 1$. Show that $\theta_1 \mathbf{x}_1 + ... + \theta_k \mathbf{x}_k \in C$.
2. A set C is midpoint convex if whenever two points $\mathbf{a}, \mathbf{b}$ are in C, the average or midpoint $(\mathbf{a} + \mathbf{b})/2$ is in C. Prove that if C is closed and midpoint convex, then C is convex.
3. Which of the following sets S are polyhedra? If possible, express S in the form $S = \{\mathbf{x} | \mathbf{A}\mathbf{x} \leq \mathbf{b},\ \mathbf{F}\mathbf{x} = \mathbf{g}\}$.

(a) $S = \{y_1 \mathbf{a}_1 + y_2 \mathbf{a}_2 | -1 \leq y_1 \leq 1, -1 \leq y_2 \leq 1\}$, where $\mathbf{a}_1,\ \mathbf{a}_2 \in \mathbb{R}^n$.

(b) $S = \{\mathbf{x} \in \mathbb{R}^n |\ \mathbf{x} \geq \mathbf{0},\ \mathbf{1}^T \mathbf{x} = 1, \sum_{i=1}^n x_i a_i = b_1, \sum_{i=1}^n x_i a_i^2 = b_2\}$, where $a_1, ..., a_n \in \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}$.

(c) $S = \{\mathbf{x} \in \mathbb{R}^n |\ \mathbf{x} \succeq \mathbf{0},\ \mathbf{x}^T \mathbf{y} \leq 1$ for all $\mathbf{y}$ with $\|\mathbf{y}\|_2 = 1\}$.

(d) $S = \{\mathbf{x} \in \mathbb{R}^n |\ \mathbf{x} \succeq \mathbf{0},\ \mathbf{x}^T \mathbf{y} \leq 1$ for all $\mathbf{y}$ with $\sum_{i=1}^n |y_i| = 1\}$.

Homework (3)

4. Which of the following sets are convex?

(a) A slab, i.e., a set of the form $\{\mathbf{x} \in \mathbb{R}^n |\ \alpha \leq \mathbf{a}^T\mathbf{x} \leq \beta\}$.

(b) A wedge, i.e., $\{\mathbf{x} \in \mathbb{R}^n |\ \mathbf{a}_1^T\mathbf{x} \leq b_1,\ \mathbf{a}_2^T\mathbf{x} \leq b_2\}$.

(c) The set of points closer to a given point than a given set, i.e., $\{\mathbf{x} | \|\mathbf{x}-\mathbf{x}_0\|_2 \leq \|\mathbf{x}-\mathbf{y}\|_2 \text{ for all } \mathbf{y} \in S\}$ where $S \subseteq \mathbb{R}^n$.

(d) The set of points closer to one set than another, i.e., $\{\mathbf{x} | \mathbf{dist}(\mathbf{x}, S) \leq \mathbf{dist}(\mathbf{x}, T)\}$, where $S, T \subseteq \mathbb{R}^n$, and

$$\mathbf{dist}(\mathbf{x}, S) = \inf\{\|\mathbf{x}-\mathbf{z}\|_2 |\ \mathbf{z} \in S\}.$$

(e) The set $\{\mathbf{x} | \mathbf{x} + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbb{R}^n$ with S_1 convex.

(f) The set of points whose distance to $\mathbf{a}$ does not exceed a fixed fraction θ of the distance to $\mathbf{b}$, i.e., the set $\{\mathbf{x} | \|\mathbf{x}-\mathbf{a}\|_2 \leq \theta\|\mathbf{x}-\mathbf{b}\|_2\}$ ($\mathbf{a} \neq \mathbf{b}$ and $0 \leq \theta \leq 1$).

Homework (3)

5. Find the convex hull of the set $\{\mathbf{u}\mathbf{u}^T | \|\mathbf{u}\| = 1\}$.
6. Consider the set of rank-k outer products, defined as $\{\mathbf{X}\mathbf{X}^T | \mathbf{X} \in \mathbb{R}^{n\times k}, \text{rank}\mathbf{X} = k\}$. Describe its conic hull in simple terms.
7. Give an expression $\bigcap_{\alpha\in\mathcal{A}} S_\alpha$ for the unit ball $\{\mathbf{X} | \|\mathbf{X}\|_2 \leq 1\}$.

Homework (3)

8. Give an explicit description of the positive semidefinite cone $\mathbb{S}^n_+$, in terms of the matrix coefficients and ordinary inequalities, for $n = 1, 2, 3$. To describe a general element of $\mathbb{S}^n$, for $n = 1, 2, 3$, use the notation

$$x_1, \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}, \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix}.$$

9. Suppose $K \subseteq \mathbb{R}^2$ is a closed convex cone.

(a) Give a simple description of K in terms of the polar coordinates of its elements ($\mathbf{x} = r(\cos\phi, \sin\phi)^T$ with $r \geq 0$).

(b) When is K pointed?

(c) When is K proper (hence, defines a generalized inequality)? Draw a plot illustrating what $\mathbf{x} \preceq_K \mathbf{y}$ means when K is proper.