Homework (10)

1.

a) Solve the optimization problem

$$\min_{\mathbf{x}} f(x_1, x_2) := 2x_1 + 3x_2, \quad s.t. \quad \sqrt{x_1} + \sqrt{x_2} = 5,$$

using Lagrange multipliers.

b) Visualize the contour lines of f as well as the set of feasible points, and mark the optimal solution $\mathbf{x}^*$.

c) Find all its KKT points. Do they all correspond to local minima?

d) Find all the saddle points of its Lagrangian function. Do they all correspond to local minima?

Homework (10)

2. With $f(\mathbf{x}) := x_1^2 + x_2^2$ for $\mathbf{x} \in \mathbb{R}^2$ consider

$$(P)\begin{cases} \min\limits_{\mathbf{x}} f(\mathbf{x}) \\ -x_2 \leq 0 \\ x_1^3 - x_2 \leq 0 \\ x_1^3(x_2 - x_1^3) \leq 0 \end{cases} .$$

a) Determine the linearizing cone, the tangent cone and the feasible direction cones at the (strict global) minimal point $\mathbf{x}_0 := (0,0)^T$.

b) Find all its KKT points. Do they all correspond to local minima?

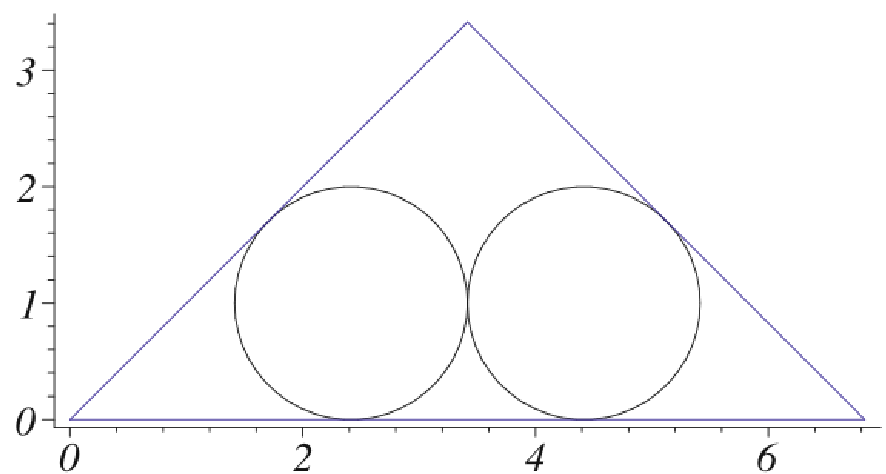
c) Find all the saddle points of its Lagrangian function. Do they all correspond to local minima?

Homework (10)

3. Determine a triangle with minimal area containing two disjoint disks with radius 1. Without loss of generalization, let $(0,0)$, $(x_1, 0)$ and (x_2, x_3) with $x_1, x_3 \geq 0$ be the vertices of the triangle; (x_4, x_5) and (x_6, x_7) denote the centers of the disks.

a) Formulate this problem as a minimization problem in terms of seven variables and nine constraints.

b) $\mathbf{x}^* = (4 + 2\sqrt{2}, 2 + \sqrt{2}, 2 + \sqrt{2}, 1 + \sqrt{2}, 1, 3 + \sqrt{2}, 1)^T$ is a solution of this problem; calculate the corresponding Lagrange multipliers $\boldsymbol{\lambda}^*$, such that the KKT conditions are fulfilled.



Homework (10)

4. Find local extremizers (either minimizer or maximizer) for the following optimization problems:

$$
\begin{aligned}
1)\ \min_{\mathbf{x}} \quad & x_1^2 + 2x_1x_2 + 3x_2^2 + 4x_1 + 5x_2 + 6x_3 \\
\text{s.t.} \quad & x_1 + 2x_2 = 3 \\
& 4x_1 + 5x_3 = 6.
\end{aligned}
$$

$$
\begin{aligned}
2)\ \max_{\mathbf{x}} \quad & 4x_1 + x_2^2 \\
\text{s.t.} \quad & x_1^2 + x_2^2 = 9.
\end{aligned}
$$

$$
\begin{aligned}
3)\ \min_{\mathbf{x}} \quad & x_1x_2 \\
\text{s.t.} \quad & x_1^2 + 4x_2^2 = 1.
\end{aligned}
$$

Homework (10)

5. Let $g : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{x}_0 \in \mathbb{R}^n$ be given, where $g(\mathbf{x}_0) \geq 0$. Consider the problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|\mathbf{x} - \mathbf{x}_0\|^2 \\ \text{subject to} & g(\mathbf{x}) \leq 0. \end{array}$$

Suppose that $\mathbf{x}^*$ is a solution to the problem and $g \in \mathcal{C}^1$. Use the KKT theorem to decide which of the following equations/inequalities hold:

1. $g(\mathbf{x}^*) \leq 0$.
2. $g(\mathbf{x}^*) = 0$.
3. $(\mathbf{x}^* - \mathbf{x}_0)^T \nabla g(\mathbf{x}^*) \leq 0$.
4. $(\mathbf{x}^* - \mathbf{x}_0)^T \nabla g(\mathbf{x}^*) = 0$.
5. $(\mathbf{x}^* - \mathbf{x}_0)^T \nabla g(\mathbf{x}^*) \geq 0$.

Homework (10)

6. Consider the problem with equality constraint

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & \mathbf{h}(\mathbf{x}) = \mathbf{0} \end{aligned}$$

We can convert the above into the equivalent optimization problem

$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & \frac{1}{2}\|\mathbf{h}(\mathbf{x})\|^2 \leq 0. \end{aligned}$$

Write down the KKT condition for the equivalent problem (with inequality constraint) and explain why the KKT theorem cannot be applied in this case.