Chapter 3: Convex Sets

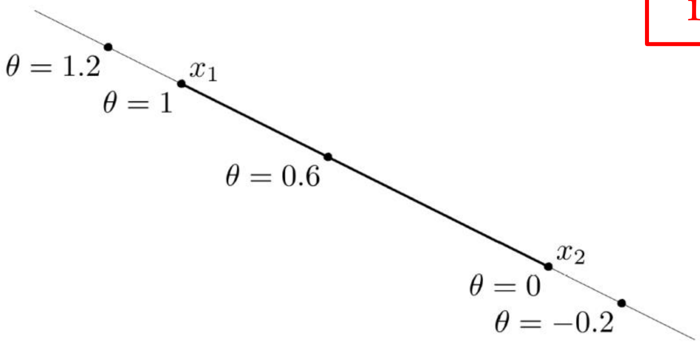
- Affine and convex sets
- Important examples
- Operators that preserve convexity
- Generalized inequalities
- Separating and supporting hyperplanes
- Dual cones and generalized inequalities

Convex sets

- Affine sets

Lines and line segments: $\mathbf{y} = \theta \mathbf{x}_1 + (1-\theta)\mathbf{x}_2$

also defined for infinite-dim spaces



A set is called an *affine subspace* iff it contains all the lines passing through any two points.

If C is an affine subspace and $\mathbf{x}_0 \in C$, then the set

$$V = C - \mathbf{x}_0 = \{\mathbf{x} - \mathbf{x}_0 | \ \mathbf{x} \in C\}$$

is a linear subspace.

We define $\dim C = \dim V$.

Example: Solution set of linear equations.

Convex sets

- Affine sets

The set of all affine combinations of points in some set C is called the *affine hull* of C, and denoted aff C:

$$\mathrm{aff}C = \{\underbrace{\theta_1\mathbf{x}_1 + ... + \theta_k\mathbf{x}_k |\ \mathbf{x}_1, ..., \mathbf{x}_k \in C, \theta_1 + ... + \theta_k = 1}_{\textit{affine combination}}\}.$$

The affine hull is the smallest affine set that contains C.

We define the *affine dimension* of a set C as the dimension of its affine hull.

Example: unit circle in $\mathbb{R}^2$

If aff$C$ is not the whole space, we define the *relative interior* of the set C, denoted ri$C$, as its interior relative to aff$C$:

$$\mathrm{ri}C = \{\mathbf{x} \in C |\ B(\mathbf{x}, r) \cap \mathrm{aff}C \subseteq C \text{ for some } r > 0\},$$

We can then define the *relative boundary* of a set C as $\bar{C} \setminus \mathrm{ri}C$.

Example: unit square in $\mathbb{R}^3$

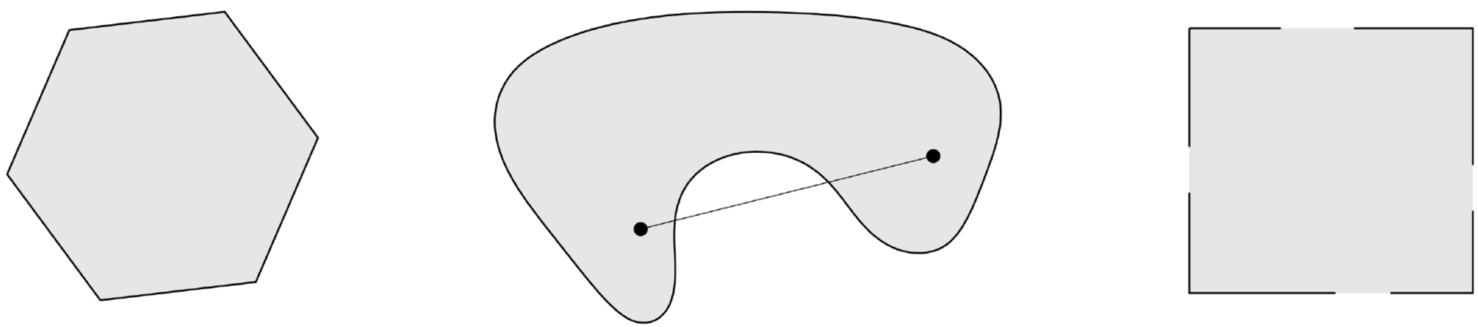
Convex sets

- Convex sets

A set C is *convex* if the line segment between any two points in C lies in C, i.e., if for any $\mathbf{x}_1, \mathbf{x}_2 \in C$ and any θ with $0 \leq \theta \leq 1$, we have

$$\theta \mathbf{x}_1 + (1 - \theta)\mathbf{x}_2 \in C.$$

Every two points can see each other.



Convexity vs. closedness

Examples: $\emptyset$, $\{\mathbf{x}_0\}$, $\mathbb{R}^n$, and affine sets

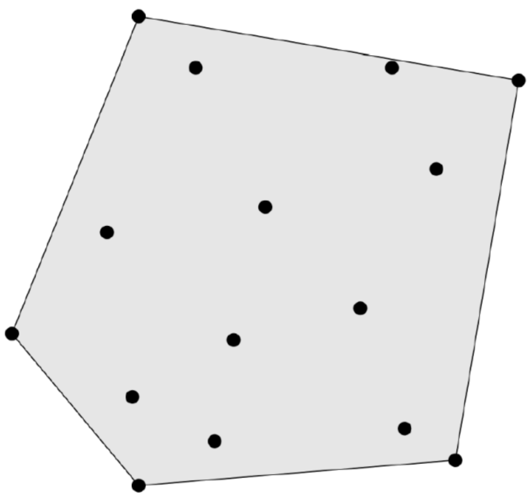
Convex sets

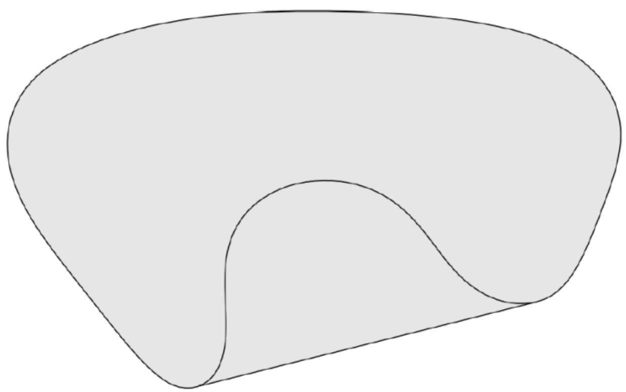
- Convex hull

The *convex hull* of a set C, denoted $\mathrm{conv}C$, is the set of all convex combinations of points in C:

$$\mathrm{conv}C = \{\overbrace{\boxed{\theta_1 \mathbf{x}_1 + ... + \theta_k \mathbf{x}_k | \mathbf{x}_i \in C, \theta_i \geq 0, i = 1, ..., k, \theta_1 + ... + \theta_k = 1}}^{\textit{Convex combination}}\}.$$

$\mathrm{conv}C$ is always convex. It is the smallest convex set that contains C.





Example: $\mathrm{conv}\{\mathbf{e}_i \mathbf{e}_j^T, i = 1, \cdots, m, j = 1, \cdots, n\}$, $\mathrm{conv}\{\mathbf{u}\mathbf{v}^T | \|\mathbf{u}\| = 1, \|\mathbf{v}\| = 1\}$

Convex sets

- General convex combination

Suppose $\theta_1, \theta_2, \ldots$ satisfy

$$\theta_i \geq 0, \quad i = 1, 2, \ldots, \quad \sum_{i=1}^{\infty} \theta_i = 1,$$

and $\mathbf{x}_1, \mathbf{x}_2, \ldots \in C$, where $C \subseteq \mathbb{R}^n$ is convex. Then

$$\sum_{i=1}^{\infty} \theta_i \mathbf{x}_i \in C,$$

if the series converges. More generally, suppose $p : \mathbb{R}^n \to \mathbb{R}$ satisfies $p(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in C$ and $\int_C p(\mathbf{x}) d\mathbf{x} = 1$, where $C \subseteq \mathbb{R}^n$ is convex. Then

$$\int_C p(\mathbf{x}) \mathbf{x} d\mathbf{x} \in C, \qquad \boxed{\mathbb{E}\mathbf{x} \in C}$$

if the integral exists.

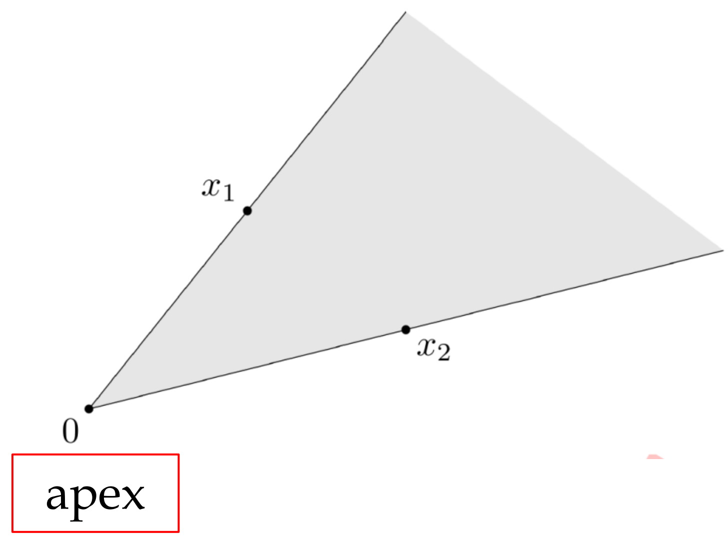
Convex sets

- Cones

A set C is called a *cone* if for every $\mathbf{x} \in C$ and $\theta \geq 0$ we have $\theta\mathbf{x} \in C$.

A set C is a *convex cone* if it is convex and a cone, which means that for any $\mathbf{x}_1, \mathbf{x}_2 \in C$ and $\theta_1, \theta_2 \geq 0$, we have

$$\theta_1\mathbf{x}_1 + \theta_2\mathbf{x}_2 \in C.$$



Convex sets

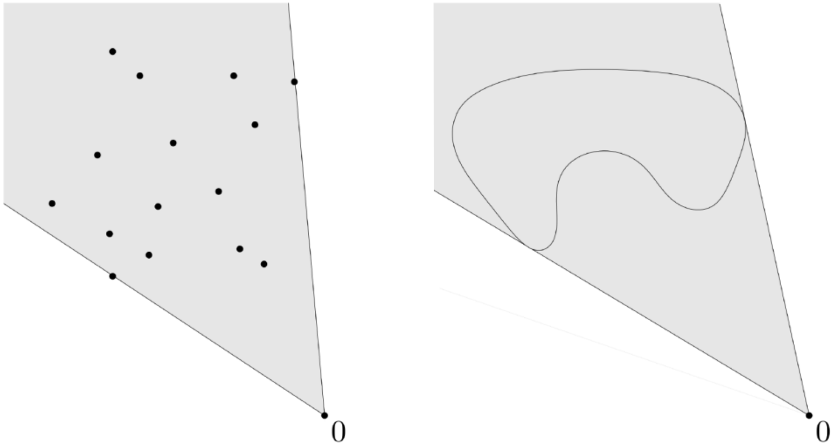
- Conic hull

The *conic hull* of a set C is the set of all conic combinations of points in C, i.e.,

Conic combination

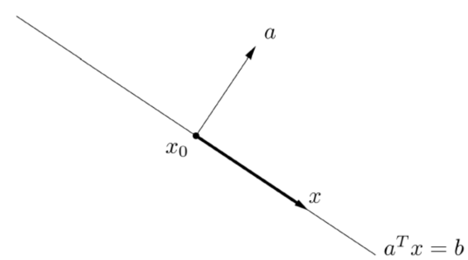
$$\{\boxed{\theta_1 \mathbf{x}_1 + \ldots + \theta_k \mathbf{x}_k \mid \mathbf{x}_i \in C, \theta_i \geq 0, i = 1, \ldots, k}\}.$$

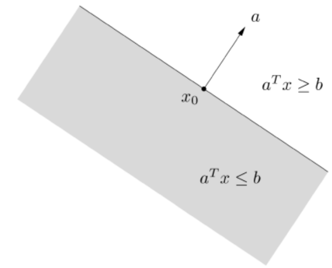
It is the smallest convex cone that contains C.

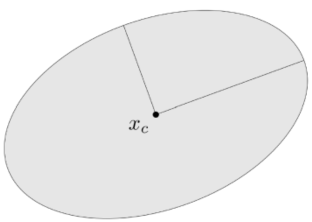


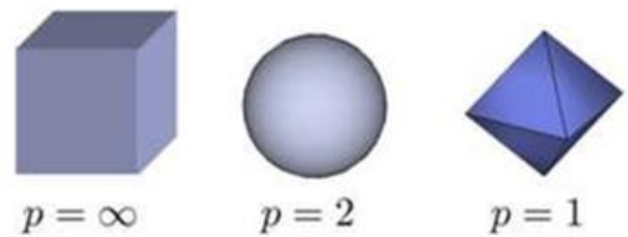
Important examples

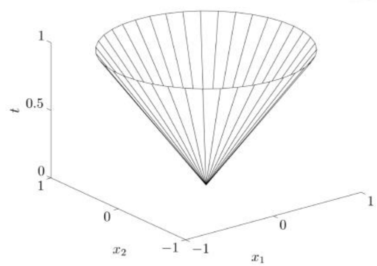
- line, line segment, ray, subspace
- hyperplanes and halfspaces
- Euclidean balls and ellipsoids
- norm balls and norm cones
- polyhedra, nonnegative orthant
- simplexes
- positive semidefinite cone

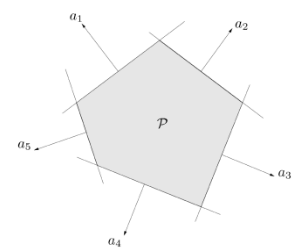


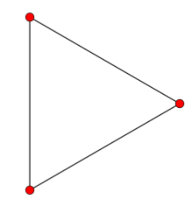


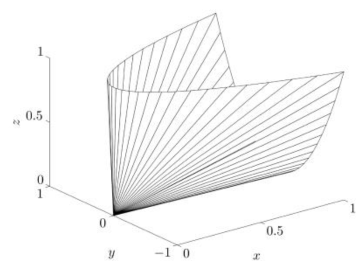












Operations that preserve convexity

- Intersection

If S_α is convex for every $\alpha \in \mathcal{A}$, then $\cap_{\alpha \in \mathcal{A}} S_\alpha$ is convex.

Examples: 1. $\mathbb{S}^n_+ = \bigcap_{\mathbf{z} \neq \mathbf{0}} \{\mathbf{X} \in \mathbb{S}^n | \ \mathbf{z}^T \mathbf{X} \mathbf{z} \geq 0\}$.

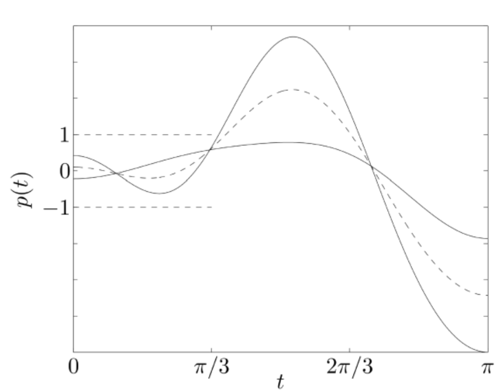
2. Consider the set

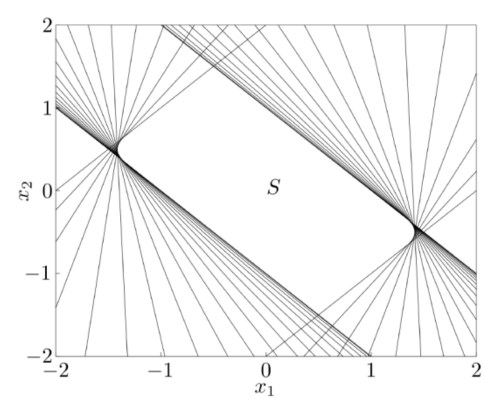
$$S = \{\mathbf{x} \in \mathbb{R}^m | \ |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\},$$

where $p(t) = \sum_{k=1}^m x_k \cos kt$. The set S can be expressed as the intersection of an infinite number of slabs: $S = \cap_{|t| \leq \pi/3} S_t$, where

$$S_t = \{\mathbf{x} | -1 \leq (\cos t, ..., \cos mt)^T \mathbf{x} \leq 1\},$$

and so is convex.





Operations that preserve convexity

- Intersection

3. A closed convex set S is the intersection of all halfspaces that contain it:

$$S = \bigcap \{\mathcal{H} \mid \mathcal{H} \text{ halfspace}, S \subseteq \mathcal{H}\}.$$

Operations that preserve convexity

- Affine functions

Suppose $S \subseteq \mathbb{R}^n$ is convex and $f : \mathbb{R}^n \to \mathbb{R}^m$ is an *affine function*: $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$. Then the *image* of S under f,

$$f(S) = \{f(\mathbf{x}) | \mathbf{x} \in S\},$$

is convex. Similarly, if $f : \mathbb{R}^k \to \mathbb{R}^n$ is an affine function, the *inverse image* of S under f,

$$f^{-1}(S) = \{\mathbf{x} | f(\mathbf{x}) \in S\},$$

is convex.

The *projection* of a convex set onto some of its coordinates is convex: if $S \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is convex, then

$$T = \{\mathbf{x}_1 \in \mathbb{R}^m | (\mathbf{x}_1, \mathbf{x}_2) \in S \text{ for some } \mathbf{x}_2 \in \mathbb{R}^n\}$$

is convex.

Operations that preserve convexity

- Sum, Cartesian product, and partial sum

If S_1 and S_2 are convex, then their *sum* $S_1 + S_2 = \{\mathbf{x} + \mathbf{y} | \mathbf{x} \in S_1,\ \mathbf{y} \in S_2\}$ is convex, so is their *Cartesian product* $S_1 \times S_2 = \{(\mathbf{x}_1, \mathbf{x}_2) |\ \mathbf{x}_1 \in S_1, \mathbf{x}_2 \in S_2\}$.

If S_1 and S_2 are convex sets in $\mathbb{R}^n \times \mathbb{R}^m$, then the *partial sum*

$$S = \{(\mathbf{x}, \mathbf{y}_1 + \mathbf{y}_2) | \mathbf{x} \in S_1,\ \mathbf{y}_1, \mathbf{y}_2 \in S_2\}$$

is convex.

Operations that preserve convexity

- Examples

1. The polyhedron $\{\mathbf{x} | \mathbf{A}\mathbf{x} \preceq \mathbf{b}, \mathbf{C}\mathbf{x} = \mathbf{d}\}$ can be expressed as the inverse image of the Cartesian product of the nonnegative orthant and the origin under the affine function $f(\mathbf{x}) = (\mathbf{b} - \mathbf{A}\mathbf{x},\ \mathbf{d} - \mathbf{C}\mathbf{x})$:

$$\{\mathbf{x} | \mathbf{A}\mathbf{x} \preceq \mathbf{b}, \mathbf{C}\mathbf{x} = \mathbf{d}\} = \{\mathbf{x} | f(\mathbf{x}) \in \mathbb{R}_+^m \times \{0\}\}.$$

2. The condition

$$A(\mathbf{x}) = x_1 \mathbf{A}_1 + ... + x_n \mathbf{A}_n \preceq \mathbf{B},$$

where $\mathbf{B}, \mathbf{A}_i \in \mathbb{S}^m$, is called a *linear matrix inequality (LMI)* in $\mathbf{x}$.

The solution set of a linear matrix inequality, $\{\mathbf{x} | A(\mathbf{x}) \preceq \mathbf{B}\}$, is convex. Indeed, it is the inverse image of the positive semidefinite cone under the affine function $f : \mathbb{R}^n \to \mathbb{S}^m$ given by $f(\mathbf{x}) = \mathbf{B} - A(\mathbf{x})$.

Operations that preserve convexity

- Examples

3. The set

$$\{\mathbf{x}|\mathbf{x}^T\mathbf{P}\mathbf{x} \leq (\mathbf{c}^T\mathbf{x})^2,\ \mathbf{c}^T\mathbf{x} \geq 0\}$$

where $\mathbf{P} \in \mathbb{S}_+^n$ and $\mathbf{c} \in \mathbb{R}^n$, is convex, since it is the inverse image of the second-order cone,

$$\{(\mathbf{z}, t)|\ \mathbf{z}^T\mathbf{z} \leq t^2, t \geq 0\},$$

under the affine function $f(\mathbf{x}) = (\mathbf{P}^{1/2}\mathbf{x},\ \mathbf{c}^T\mathbf{x})$.

4. The ellipsoid

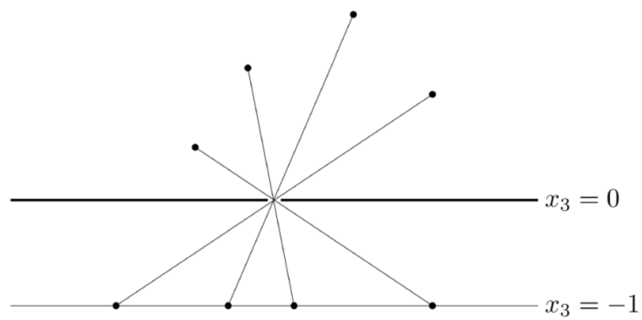
$$\epsilon = \{\mathbf{x}|(\mathbf{x} - \mathbf{x}_c)^T\mathbf{P}^{-1}(\mathbf{x} - \mathbf{x}_c) \leq 1\},$$

where $\mathbf{P} \in \mathbb{S}_{++}^n$, is the image of the unit Euclidean ball $\{\mathbf{u}|\ \|\mathbf{u}\|_2 \leq 1\}$ under the affine mapping $f(\mathbf{u}) = \mathbf{P}^{1/2}\mathbf{u} + \mathbf{x}_c$. (It is also the inverse image of the unit ball under the affine mapping $g(\mathbf{x}) = \mathbf{P}^{-1/2}(\mathbf{x} - \mathbf{x}_c)$.)

Operations that preserve convexity

- Perspective functions

We define the *perspective function* $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$, with domain $\mathbf{dom}\ \ P = \mathbb{R}^n \times \mathbb{R}_{++}$, as $P(\mathbf{z}, t) = \mathbf{z}/t$.



The inverse image of a convex set under the perspective function is also convex: if $C \subseteq \mathbb{R}^n$ is convex, then

$$P^{-1}(C) = \{(\mathbf{x}, t) \in \mathbb{R}^{n+1} | \mathbf{x}/t \in C, t > 0\}$$

is convex.

Question: If function f preserves convexity: if C_1 is convex then $f(C_1)$ is also convex, does f^{-1} also perserve convexity?

Operations that preserve convexity

- Linear-fractional functions

A linear-fractional function is formed by composing the perspective function with an affine function. Suppose $g : \mathbb{R}^n \to \mathbb{R}^{m+1}$ is affine, i.e.,

$$g(\mathbf{x}) = \begin{bmatrix} \mathbf{A} \\ \mathbf{c}^T \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{b} \\ d \end{bmatrix},$$

where $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$, and $d \in \mathbb{R}$. The function $f : \mathbb{R}^n \to \mathbb{R}^m$ given by $f = P \circ g$, i.e.,

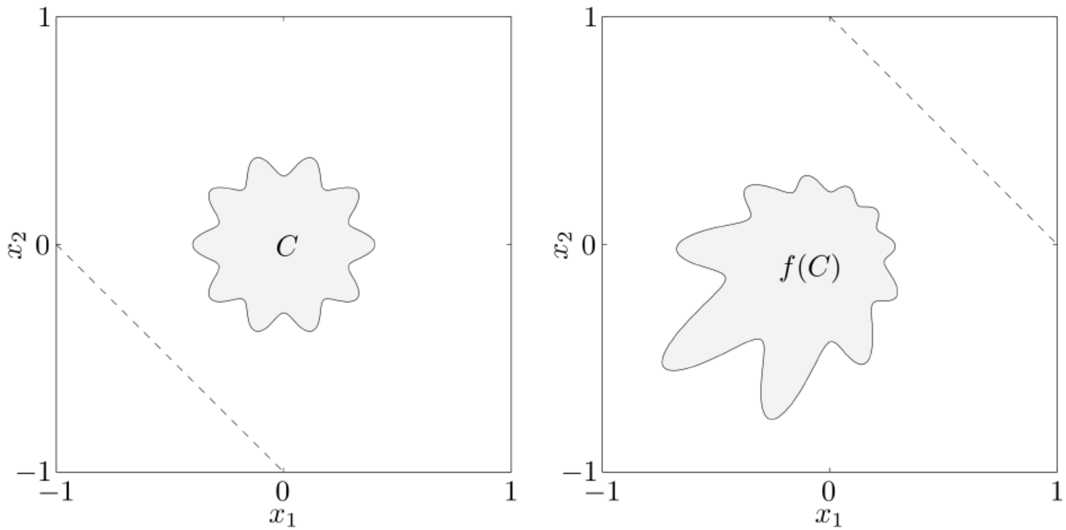
$$f(\mathbf{x}) = (\mathbf{A}\mathbf{x} + \mathbf{b})/(\mathbf{c}^T\mathbf{x} + d), \mathrm{dom}\, f = \{\mathbf{x} | \mathbf{c}^T\mathbf{x} + d > 0\},$$

is called a *linear-fractional* (or projective) function.

Operations that preserve convexity

- Linear-fractional functions

$$f(\mathbf{x}) = \frac{1}{\mathbf{x}_1 + \mathbf{x}_2 + 1}\mathbf{x}, \ \ \mathbf{dom}\ f = \{(x_1, x_2) | x_1 + x_2 + 1 > 0\}.$$



Conditional probabilities: Let $p_{ij} = \mathbb{P}(u = i, v = j)$. Then the conditional probability

$$f_{ij} = \frac{p_{ij}}{\sum_{k=1}^{n} p_{kj}}$$

is obtained by a linear-fractional mapping from $\mathbf{p}$.