Homework (5)

1. Judge which of the following functions are strongly convex or concave and find their moduli.

   (a) $f(x) = e^x - 1$ on $\mathbb{R}$.

   (b) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$.

   (c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbb{R}^2_{++}$.

   (d) $f(x_1, x_2) = x_1/x_2$ on $\mathbb{R}^2_{++}$.

   (e) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbb{R} \times \mathbb{R}_{++}$.

   (f) $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, where $0 < \alpha < 1$ on $\mathbb{R}^2_{++}$.

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2. Suppose $p < 1, p \neq 0$. Show that the function

$$f(\mathbf{x}) = \left(\sum_{i=1}^{n} \mathbf{x}_i^p \right)^{1/p}$$

with $\operatorname{dom} f = \mathbb{R}^n_{++}$ is concave.

3. Prove that

(a) $f(\mathbf{X}) = \operatorname{tr}(\mathbf{X}^{-1})$ is convex on $\mathbf{dom}\, f = \mathbb{S}^n_{++}$.

(b) $f(\mathbf{X}) = (\det \mathbf{X})^{1/n}$ is convex on $\mathbf{dom}\, f = \mathbb{S}^n_{++}$.

4. Suppose $f : \mathbf{b}^n \to \mathbb{R}$ is convex with $\mathbf{dom}\, f = \mathbb{R}^n$, and bounded above on $\mathbb{R}^n$. Show that f is constant.

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5. Does the Bregman distance satisfy the triangular inequality?
6. Prove the identity:

$$\langle \nabla f(\mathbf{x}) - \nabla f(\mathbf{y}), \mathbf{x} - \mathbf{z} \rangle = B_f(\mathbf{z}, \mathbf{x}) + B_f(\mathbf{x}, \mathbf{y}) - B_f(\mathbf{z}, \mathbf{y}).$$

7. Prove that $B_f(\mathbf{x}, \mathbf{y})$ is convex in $\mathbf{x}$ but not necessarily in $\mathbf{y}$.
8. Compute the subgradients of $f(\mathbf{x}) = \frac{1}{2}x_1^2 + |x_2|$, $\|\mathbf{x}\|_2$, $\|\mathbf{x}\|_\infty$, $\|\mathbf{X}\|_{2,1}$, and $\|\mathbf{D}\,\mathrm{Diag}(\mathbf{x})\|_*$.
9. Prove that $\mathrm{conv}\{\boldsymbol{\alpha}\boldsymbol{\alpha}^T | \|\boldsymbol{\alpha}\| = 1\} = \{\mathbf{W} | \mathbf{W} \succcurlyeq \mathbf{0}, \mathrm{tr}\,\mathbf{W} = 1\}$.
10. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Show that $\partial f : \mathbb{R}^n \to \mathbb{R}^n$ is a monotone mapping, i.e.,

$$\langle \mathbf{g}_1 - \mathbf{g}_2, \mathbf{x}_1 - \mathbf{x}_2 \rangle \geq 0, \quad \forall \mathbf{g}_i \in \partial f(\mathbf{x}_i), i = 1, 2. \tag{1}$$

Further, if f is μ-strongly convex, then the above inequality can be strengthened as

$$\langle \mathbf{g}_1 - \mathbf{g}_2, \mathbf{x}_1 - \mathbf{x}_2 \rangle \geq \mu \|\mathbf{x}_1 - \mathbf{x}_2\|^2, \quad \forall \mathbf{g}_i \in \partial f(\mathbf{x}_i), i = 1, 2. \tag{2}$$