Homework (2)

1. Judge the properties of the following sets (openess, closeness, boundedness, compactness) and give their interiors, closures, boundaries, and accumulation points:

a. $\mathcal{C}_1 = \emptyset$.

b. $\mathcal{C}_2 = \mathbb{R}^n$.

c. $\mathcal{C}_3 = \{x|0 \leq x < 1\} \cup \{x|2 \leq x \leq 3\} \cup \{x|4 < x \leq 5\}$.

d. $\mathcal{C}_4 = \{(x, y)^T|x \geq 0, y > 0\}$.

e. $\mathcal{C}_5 = \{k|k \in \mathbb{Z}\}$.

f. $\mathcal{C}_6 = \{k^{-1}|k \in \mathbb{Z}\}$.

g. $\mathcal{C}_7 = \{(1/k, \sin k)^T|k \in \mathbb{Z}\}$.

2. Prove that a set $\mathcal{C} \subseteq \mathbb{R}^n$ is closed iff (aka. if and only if) it contains the limit point of every convergent sequence in it.

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3. Prove that a point $\mathbf{x}$ is a boundary point of $\mathcal{C} \subseteq \mathbb{R}^n$ iff for $\forall \epsilon > 0$, there exists $\mathbf{y} \in \mathcal{C}$ and $\mathbf{z} \notin \mathcal{C}$ such that

$$\|\mathbf{y} - \mathbf{x}\|_2 \leq \epsilon, \quad \|\mathbf{z} - \mathbf{x}\|_2 \leq \epsilon.$$

4. Prove that $\mathcal{C} \subseteq \mathbb{R}^n$ is closed iff it contains its boundary, and is open iff it contains no boundary points.

5. Prove the following:

   a. $\overline{\mathcal{A} \cup \mathcal{B}} = \overline{\mathcal{A}} \cup \overline{\mathcal{B}}$; $\overline{\mathcal{A} \cap \mathcal{B}} \subseteq \overline{\mathcal{A}} \cap \overline{\mathcal{B}}$. Give an example showing that $\overline{\mathcal{A} \cap \mathcal{B}} \neq \overline{\mathcal{A}} \cap \overline{\mathcal{B}}$.

   b. $(\overline{\mathcal{A} \cap \mathcal{B}})^\circ = \mathcal{A}^\circ \cap \mathcal{B}^\circ$; $(\overline{\mathcal{A} \cup \mathcal{B}})^\circ \supseteq \mathcal{A}^\circ \cup \mathcal{B}^\circ$. Give an example showing that $(\overline{\mathcal{A} \cup \mathcal{B}})^\circ \neq \mathcal{A}^\circ \cup \mathcal{B}^\circ$.

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6. For each of the following sequences, determine the rate of convergence and the rate constant.

a. $x_k = 2^{-k}$, for $k = 1, 2, \cdots$.

b. $x_k = 1 + 5 \times 10^{-2k}$, for $k = 1, 2, \cdots$.

c. $x_k = 2^{-2^k}$.

d. $x_k = 3^{-k^2}$.

e. $x_k = 1 - 2^{-2^k}$ for k odd, and $x = 1 + 2^{-k}$ for k even.

7. Let $\{x_k\}$ and $\{c_k\}$ be convergent sequences, and assume that

$$\lim_{k \to \infty} c_k = c \neq 0.$$

Consider the sequence $\{y_k\}$ with $y_k = c_k x_k$. Can its convergence rate and rate constant be determined from those of $\{x_k\}$ and $\{c_k\}$?