Basic properties and examples

- First-order conditions

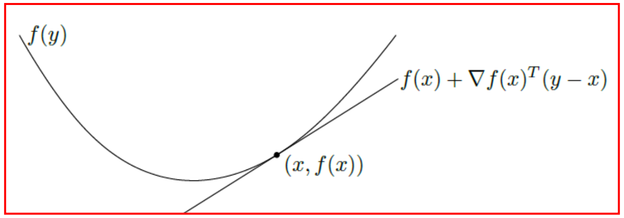
Suppose f is differentiable. Then f is convex iff $\mathrm{dom}\, f$ is convex and

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \tag{1}$$

holds for all $\mathbf{x}$, $\mathbf{y} \in \mathrm{dom}\, f$.

Proof. If f is convex, then $f((1-\alpha)\mathbf{x} + \alpha\mathbf{y}) \leq (1-\alpha) f(\mathbf{x}) + \alpha f(\mathbf{y})$, which can be rewritten as

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \frac{f(\mathbf{x} + \alpha(\mathbf{y} - \mathbf{x})) - f(\mathbf{x})}{\alpha},$$



Letting $\alpha \to 0^+$, we have (1). If (1) holds, we have

$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq f(\mathbf{x}) - (1-\alpha)\langle \nabla f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle,$$
$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq f(\mathbf{y}) + \alpha\langle \nabla f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle.$$

Multiplying the first inequality with α and the second with $(1-\alpha)$ and adding them together, we obtain $f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \leq \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y})$.

Basic properties and examples

If $\nabla f(\mathbf{x}) = \mathbf{0}$, then for all $\mathbf{y} \in \operatorname{dom} f$, $f(\mathbf{y}) \geq f(\mathbf{x})$, *i.e.*, $\mathbf{x}$ is a global minimizer of f.

Strictly convex:

$$f(\mathbf{y}) > f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle, \quad \text{if } \mathbf{y} \neq \mathbf{x}. \tag{1}$$

Proof. $f(\mathbf{y}) > f(\mathbf{x}) + \frac{f(\mathbf{x}+\alpha(\mathbf{y}-\mathbf{x}))-f(\mathbf{x})}{\alpha}, \quad \forall \alpha \in (0,1)$. For all $\alpha \in (0,1)$ by the convexity we have $f(\mathbf{x} + \alpha(\mathbf{y} - \mathbf{x})) - f(\mathbf{x}) \geq \alpha \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle$. Thus $\langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle = \inf_{\alpha \in (0,1)} \frac{f(\mathbf{x}+\alpha(\mathbf{y}-\mathbf{x}))-f(\mathbf{x})}{\alpha}$. If there exists $\alpha \in (0,1)$ such that $\frac{f(\mathbf{x}+\alpha(\mathbf{y}-\mathbf{x}))-f(\mathbf{x})}{\alpha} > \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle$, then (1) holds. Otherwise,

$$\frac{f(\mathbf{x} + \alpha(\mathbf{y} - \mathbf{x})) - f(\mathbf{x})}{\alpha} = \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle, \quad \forall \alpha \in (0,1).$$

So $f(\mathbf{x} + \alpha(\mathbf{y} - \mathbf{x}))$ is a linear function of $\alpha \in (0,1)$ and f cannot be strictly convex.

Strongly convex: $f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \frac{\mu}{2} \|\mathbf{y} - \mathbf{x}\|^2$.
Proof. Follow the proof of convexity.

Basic properties and examples

- Second-order conditions

Assume that f is twice differentiable. Then f is convex iff $\mathrm{dom}\, f$ is convex and its Hessian is positive semidefinite: for all $\mathbf{x} \in \mathrm{dom}\, f$,

$$\nabla^2 f(\mathbf{x}) \succeq \mathbf{0}.$$

For a function on $\mathbb{R}$, this reduces to the simple condition $f''(x) \geq 0$ (and $\mathrm{dom}\, f$ convex, *i.e.*, an interval), which means that the derivative is nondecreasing. The condition $\nabla^2 f(\mathbf{x}) \succeq \mathbf{0}$ can be interpreted geometrically as the requirement that the graph of the function have positive (upward) curvature at $\mathbf{x}$.

Strictly convex: $\nabla^2 f(\mathbf{x}) \succ \mathbf{0}$.
Strongly convex: $\nabla^2 f(\mathbf{x}) \succeq \mu \mathbf{I}$.

Basic properties and examples

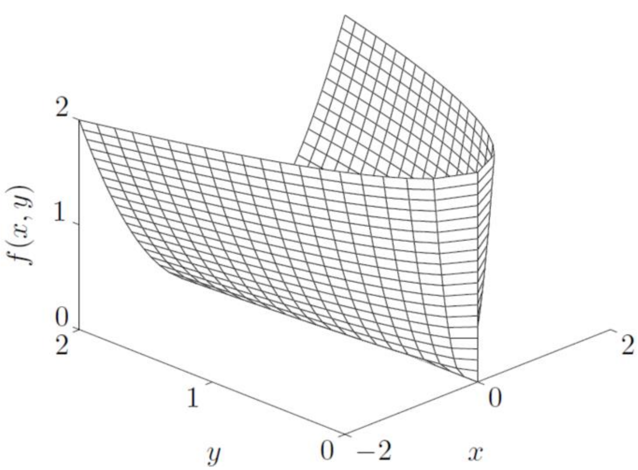
- Examples

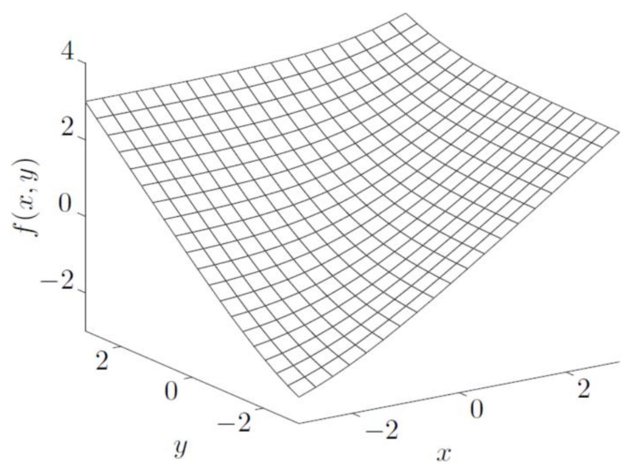
- *Exponential.* e^{ax} is convex on $\mathbb{R}$, for any $a \in \mathbb{R}$.
- *Powers.* x^a is convex on $\mathbb{R}_{++}$ when $a \geq 1$ or $a \leq 0$, and concave for $0 \leq a \leq 1$.
- *Powers of absolute value.* $|x|^p$, for $p \geq 1$, is convex on $\mathbb{R}$.
- *Logarithm.* $\log x$ is concave on $\mathbb{R}_{++}$.
- *Negative entropy.* $x \log x$ (either on $\mathbb{R}_{++}$, or on $\mathbb{R}_+$, defined as 0 for $x = 0$) is convex.

Basic properties and examples

- Examples

- *Norms.* Every norm on $\mathbb{R}^n$ is convex.
- *Max function.* $f(\mathbf{x}) = \max\{x_1, \ldots, x_n\}$ is convex on $\mathbb{R}^n$.
- $f(x, y) = x^2/y$ with $\mathbf{dom}\, f = \mathbb{R} \times \mathbb{R}_{++} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ is convex.
- *Log-sum-exp.* $f(\mathbf{x}) = \log(e^{x_1} + \cdots + e^{x_n})$ is convex on $\mathbb{R}^n$.
- *Geometric mean.* $f(\mathbf{x}) = (\prod_{i=1}^n x_i)^{1/n}$ is concave on $\mathbf{dom}\, f = \mathbb{R}^n_{++}$.





Basic properties and examples

- Examples
- *Log-determinant.* $f(\mathbf{X}) = \log\det\mathbf{X}$ is concave on $\operatorname{dom} f = \mathbb{S}^n_{++}$.

The Hessian of f at $\mathbf{X}$ is a fourth-order tensor $\mathcal{T}$. We have shownthat $\mathcal{T}(\Delta\mathbf{X}) = -\mathbf{X}^{-1}\Delta\mathbf{X}\mathbf{X}^{-1}$.

$$\langle\mathcal{T}(\Delta\mathbf{X}), \Delta\mathbf{X}\rangle = -\operatorname{tr}\left[(\mathbf{X}^{-1}\Delta\mathbf{X}\mathbf{X}^{-1})\Delta\mathbf{X}\right] = -\operatorname{tr}\left[\mathbf{X}^{-1}(\Delta\mathbf{X}\mathbf{X}^{-1}\Delta\mathbf{X})\right] \leq 0.$$

Basic properties and examples

- Sublevels

The *α-sublevel set* of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$C_\alpha = \{\mathbf{x} \in \operatorname{dom} f \mid f(\mathbf{x}) \le \alpha\}.$$

Sublevel sets of a convex function are convex, for any value of α.

The converse is not true: a function can have all its sublevel sets convex, but not be a convex function. Such functions are called *quasi-convex functions*.

Quasi-convex:

$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{y}) \le \max\{f(\mathbf{x}), f(\mathbf{y})\}, \quad \alpha \in [0,1].$$

Basic properties and examples

- Sublevels

Example: The geometric and arithmetic means of $\mathbf{x} \in \mathbb{R}^n_+$ are, respectively,

$$G(\mathbf{x}) = \left(\prod_{i=1}^{n} x_i\right)^{1/n}, \quad A(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i.$$

The arithmetic-geometric mean inequality states that $G(\mathbf{x}) \leq A(\mathbf{x})$.

Suppose $0 \leq \alpha \leq 1$, and consider the set

$$\{\mathbf{x} \in \mathbb{R}^n_+ \mid G(\mathbf{x}) \geq \alpha A(\mathbf{x})\},$$

i.e., the set of vectors with geometric mean at least as large as a factor α times the arithmetic mean. This set is convex, since it is the 0-superlevel set of the function $G(\mathbf{x}) - \alpha A(\mathbf{x})$, which is concave. In fact, the set is positively homogeneous, so it is a convex cone.

Basic properties and examples

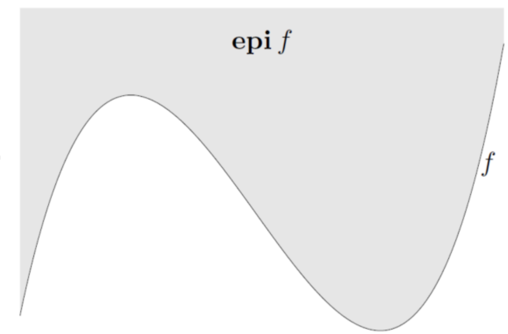
- Epigraph

The *epigraph* of a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\operatorname{epi} f = \{(\mathbf{x}, t) \mid \mathbf{x} \in \operatorname{dom} f, f(\mathbf{x}) \leq t\},$$

which is a subset of $\mathbb{R}^{n+1}$.

A function is convex iff its epigraph is a convex set.



Example: $f(\mathbf{x}, \mathbf{Y}) = \mathbf{x}^T \mathbf{Y}^{-1} \mathbf{x}$ is convex on $\operatorname{dom} f = \mathbb{R}^n \times \mathbb{S}^n_{++}$.

By its epigraph:

$$\begin{aligned} \operatorname{epi} f &= \{(\mathbf{x}, \mathbf{Y}, t) \mid \mathbf{Y} \succ \mathbf{0},\, \mathbf{x}^T \mathbf{Y}^{-1} \mathbf{x} \leq t\} \\ &= \left\{ (\mathbf{x}, \mathbf{Y}, t) \,\middle|\, \begin{bmatrix} \mathbf{Y} & \mathbf{x} \\ \mathbf{x}^T & t \end{bmatrix} \succeq \mathbf{0},\, \mathbf{Y} \succ \mathbf{0} \right\}. \end{aligned}$$

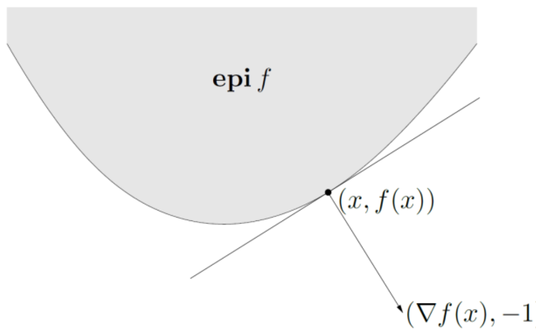
The last condition is a linear matrix inequality in $(\mathbf{x}, \mathbf{Y}, t)$, and therefore $\operatorname{epi} f$ is convex.

Basic properties and examples

- Epigraph

Many results for convex functions can be proved (or interpreted) geometrically using epigraphs, and applying results for convex sets. As an example, consider the first-order condition for convexity:

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle .$$



If $(\mathbf{y}, t) \in \operatorname{epi} f$, then

$$t \geq f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle .$$

We can express this as:

$$(\mathbf{y}, t) \in \operatorname{epi} f \Longrightarrow \begin{bmatrix} \nabla f(\mathbf{x}) \\ -1 \end{bmatrix}^T \left(\begin{bmatrix} \mathbf{y} \\ t \end{bmatrix} - \begin{bmatrix} \mathbf{x} \\ f(\mathbf{x}) \end{bmatrix} \right) \leq 0.$$

This means that the hyperplane defined by $(\nabla f(\mathbf{x}), -1)$ supports $\operatorname{epi} f$ at the boundary point $(\mathbf{x}, f(\mathbf{x}))$.

Basic properties and examples

- Proper function

f is called *proper* if $f(\mathbf{x}) < \infty$ for at least one $\mathbf{x} \in \mathcal{X}$ and $f(\mathbf{x}) > -\infty$ for all $\mathbf{x} \in \mathcal{X}$, and we say that f is *improper* if it is not proper. In words, a function is proper iff its epigraph is nonempty and does not contain a verfical line.

Basic properties and examples

- Jensen's inequality and extensions

$$f\left(\theta \mathbf{x}+(1-\theta) \mathbf{y}\right) \leq \theta f(\mathbf{x})+(1-\theta) f(\mathbf{y}).$$

$$\Updownarrow$$

$$f(\theta_1 \mathbf{x}_1+\cdots+\theta_k \mathbf{x}_k) \leq \theta_1 f(\mathbf{x}_1)+\cdots+\theta_k f(\mathbf{x}_k).$$

$$f\left(\int_S p(\mathbf{x}) \mathbf{x}\, d \mathbf{x}\right) \leq \int_S f(\mathbf{x}) p(\mathbf{x})\, d \mathbf{x}.$$

$$\Updownarrow$$

$$f(\mathbb{E}\, \mathbf{x}) \leq \mathbb{E}\, f(\mathbf{x}).$$

Basic properties and examples

- Inequality

Arithmetic-geometric mean inequality:

$$\sqrt{ab} \leq (a+b)/2.$$

Hölder's inequality: for $p, q > 1$, $1/p + 1/q = 1$, and $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$,

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \left(\sum_{i=1}^{n} |y_i|^q \right)^{1/q}.$$

Basic properties and examples

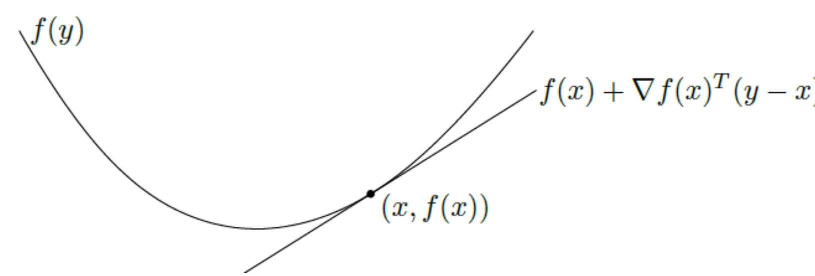
- Bregman distance

Given a differentiable strictly convex function $f : C \to \mathbb{R}$, where $C \subset \mathbb{R}^n$ is a convex set, the Bregman distance is defined as:

$$B_f(\mathbf{y}, \mathbf{x}) = f(\mathbf{y}) - f(\mathbf{x}) - \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle. \tag{1}$$

It it clear that $B_f(\mathbf{y}, \mathbf{x}) \geq 0$ for all $\mathbf{x}, \mathbf{y} \in C$ due to the convexity of f. However, the Bregman distance may not be symmetric: $B_f(\mathbf{y}, \mathbf{x}) \neq B_f(\mathbf{x}, \mathbf{y})$.

Examples:



- $f(\mathbf{x}) = \|\mathbf{x}\|^2$.
- $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \mathbf{P} \mathbf{x}$.
- $f(\mathbf{x}) = \sum_i x_i \log x_i$.

Basic properties and examples

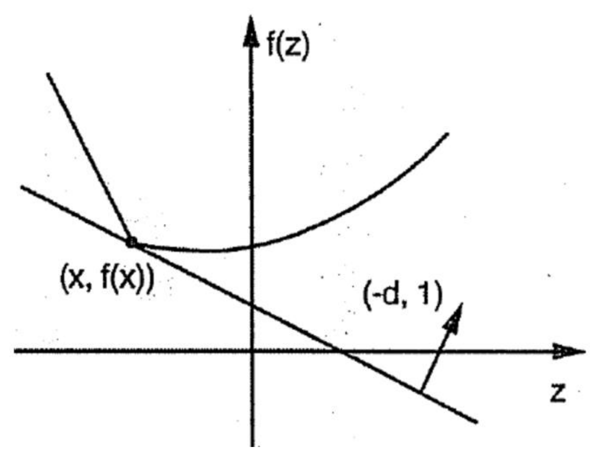
- Subgradient

$$\partial f(\mathbf{x}) = \{\mathbf{g} | f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \mathbf{g}, \mathbf{y} - \mathbf{x} \rangle, \forall \mathbf{x}, \mathbf{y} \in \operatorname{dom} f\}.$$

Subgradient can be identified with a non-vertical supporting hyperplane to the epigraph of f at $(\mathbf{x}, f(\mathbf{x}))$.

Proposition 1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a proper convex function. The subgradient $\partial f(\mathbf{x})$ is nonempty, convex, and compact for all $\mathbf{x} \in (\operatorname{dom} f)^\circ$.*

$\partial f(\mathbf{x})$ may be empty when $\mathbf{x} \in \partial(\operatorname{dom} f)$. Example?



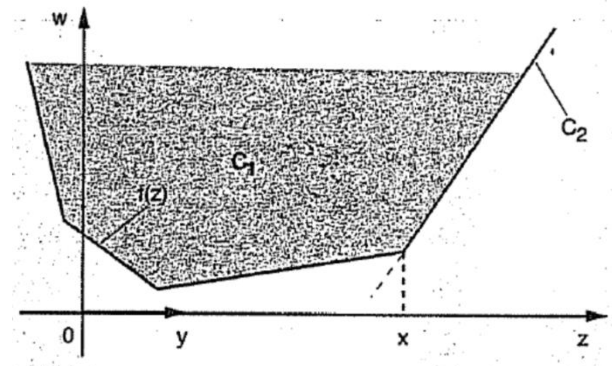
Basic properties and examples

- Subgradient

Proposition 1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. For every $\mathbf{x} \in \mathbb{R}^n$, we have*

$$f'(\mathbf{x}; \mathbf{y}) = \max_{\mathbf{g} \in \partial f(\mathbf{x})} \langle \mathbf{y}, \mathbf{g} \rangle, \quad \forall \mathbf{y} \in \mathbb{R}^n. \tag{1}$$

In particular, f is differentiable at $\mathbf{x}$ with gradient $\nabla f(\mathbf{x})$ iff it has $\nabla f(\mathbf{x})$ as its unique subgradient at $\mathbf{x}$.



Proof: Apply Separating Hyperplane Theorem to

$$C_1 = \{(\mathbf{z}, w) | f(\mathbf{z}) < w\},$$

and

$$C_2 = \{(\mathbf{z}, w) | \mathbf{z} = \mathbf{x} + \alpha \mathbf{y}, w = f(\mathbf{x}) + \alpha f'(\mathbf{x}; \mathbf{y}), \alpha \geq 0\}.$$

Basic properties and examples

- Subgradient

Example: $|x|$, $\max\{0, \frac{1}{2}(x^2 - 1)\}$, $I_{\mathcal{C}}(\mathbf{x})$.

Basic properties and examples

- Subgradient

Proposition 1. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a convex function.*

(a) If $\mathcal{X}$ *is a bounded set, then the set* $\cup_{\mathbf{x}\in\mathcal{X}}\partial f(\mathbf{x})$*is bounded.*

(b) If a sequence $\{\mathbf{x}_k\}$ *converges to a vector* $\mathbf{x} \in \mathbb{R}^n$ *and* $\mathbf{g}_k \in \partial f(\mathbf{x}_k)$ *for all* k*, then the sequence* $\{\mathbf{g}_k\}$ *is bounded and each of its accumulation points is a subgradient of* f *at* $\mathbf{x}$*.*

Proposition 2. *Let* $f_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, \cdots, m$, *be convex functions and let* $f = f_1 + \cdots + f_m$. *Then*

$$\partial f(\mathbf{x}) = \partial f_1(\mathbf{x}) + \cdots + \partial f_m(\mathbf{x}).$$

Basic properties and examples

- Subgradient

Proposition 1 (Chain Rule). *(a) Let $f : \mathbb{R}^m \to \mathbb{R}$ be a convex function, and let $\mathbf{A}$ be an $m \times n$ matrix. Then the subgradient of the function F, defined by $F(\mathbf{x}) = f(\mathbf{A}\mathbf{x})$, is given by*

$$\partial F(\mathbf{x}) = \mathbf{A}^T \partial f(\mathbf{A}\mathbf{x}).$$

(b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $h : \mathbb{R} \to \mathbb{R}$ be a differentiable scalar function. Then the function F, defined by $F(\mathbf{x}) = h(f(\mathbf{x}))$, is directionally differentiable at all $\mathbf{x}$, given by

$$F'(\mathbf{x}; \mathbf{y}) = h'(f(\mathbf{x})) f'(\mathbf{x}; \mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

Furthermore, if h is convex and monotonically nondecreasing, then F is convex and its subgradient is given by

$$\partial F(\mathbf{x}) = \partial h(f(\mathbf{x})) \partial f(\mathbf{x}) = \{g\mathbf{g} | g \in \partial h(f(\mathbf{x})), \mathbf{g} \in \partial f(\mathbf{x})\}, \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

Basic properties and examples

- Subgradient

Theorem 1 (Subgradient of norms)**.** *Let $\mathcal{H}$ be a real Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$ and a norm $\|\cdot\|$. Then $\partial\|\mathbf{x}\| = \{\mathbf{y} | \langle \mathbf{y}, \mathbf{x} \rangle = \|\mathbf{x}\|$ and $\|\mathbf{y}\|^* \leq 1\}$, where $\| \cdot \|^*$ is the dual norm of $\| \cdot \|$.*

Proof. Let $S = \{\mathbf{y} | \langle \mathbf{y}, \mathbf{x} \rangle = \|\mathbf{x}\|$ and $\|\mathbf{y}\|^* \leq 1\}$.
For every $\mathbf{y} \in \partial\|\mathbf{x}\|$, we have

$$\|\mathbf{w} - \mathbf{x}\| \geq \|\mathbf{w}\| - \|\mathbf{x}\| \geq \langle \mathbf{y}, \mathbf{w} - \mathbf{x} \rangle, \quad \forall\ \mathbf{w} \in \mathcal{H}. \tag{1}$$

Choosing $\mathbf{w} = 0$ and $\mathbf{w} = 2\mathbf{x}$ for the second inequality above, which results from the convexity of norm $\| \cdot \|$, we can deduce that

$$\|\mathbf{x}\| = \langle \mathbf{y}, \mathbf{x} \rangle . \tag{2}$$

Basic properties and examples

- Subgradient

On the other hand, (1) gives

$$\|\mathbf{w}-\mathbf{x}\| \geq \langle \mathbf{y}, \mathbf{w}-\mathbf{x} \rangle, \quad \forall\, \mathbf{w} \in \mathcal{H}. \tag{3}$$

So

$$\left\langle \mathbf{y}, \frac{\mathbf{w}-\mathbf{x}}{\|\mathbf{w}-\mathbf{x}\|} \right\rangle \leq 1, \quad \forall\, \mathbf{w} \neq \mathbf{x}.$$

Therefore $\|\mathbf{y}\|^* \leq 1$. Thus $\partial\|\mathbf{x}\| \subset S$.

For every $\mathbf{y} \in S$, we have

$$\langle \mathbf{y}, \mathbf{w}-\mathbf{x} \rangle = \langle \mathbf{y}, \mathbf{w} \rangle - \langle \mathbf{y}, \mathbf{x} \rangle = \langle \mathbf{y}, \mathbf{w} \rangle - \|\mathbf{x}\| \leq \|\mathbf{y}\|^* \|\mathbf{w}\| - \|\mathbf{x}\| \leq \|\mathbf{w}\| - \|\mathbf{x}\|, \quad \forall\, \mathbf{w} \in \mathcal{H}, \tag{4}$$

where the second equality utilizes $\langle \mathbf{y}, \mathbf{x} \rangle = \|\mathbf{x}\|$ and the first inequality is by the definition of dual norm. Thus, $\mathbf{y} \in \partial\|\mathbf{x}\|$. So $S \subset \partial\|\mathbf{x}\|$.

Basic properties and examples

- Subgradient

Theorem 1 (Danskin's Theorem). *Let $\mathcal{Z}$ be a compact subset of $\mathbb{R}^m$, and let $\phi : \mathbb{R}^n \times \mathcal{Z} \to \mathbb{R}$ be continuous and such that $\phi(\cdot, \mathbf{z}) : \mathbb{R}^n \to \mathbb{R}$ is convex for each $\mathbf{z} \in \mathcal{Z}$. Define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(\mathbf{x}) = \max_{\mathbf{z} \in \mathcal{Z}} \phi(\mathbf{x}, \mathbf{z})$ and*

$$\mathcal{Z}(\mathbf{x}) = \left\{ \bar{\mathbf{z}} \,\middle|\, \phi(\mathbf{x}, \bar{\mathbf{z}} = \max_{\mathbf{z} \in \mathcal{Z}} \phi(\mathbf{x}, \mathbf{z})) \right\}.$$

If $\phi(\cdot, \mathbf{z})$ is differentiable for all $\mathbf{z} \in \mathcal{Z}$ and $\nabla_x \phi(\mathbf{x}, \cdot)$ is continuous on $\mathcal{Z}$ for each $\mathbf{x}$, then

$$\partial f(\mathbf{x}) = conv\{\nabla_x \phi(\mathbf{x}, \mathbf{z}) | \mathbf{z} \in \mathcal{Z}(\mathbf{x})\}, \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

Basic properties and examples

- Subgradient

Example: $\partial\|\mathbf{X}\|_*$, $\partial\|\mathbf{X}\|_2$.