Homework (7)

1. *Sublevel sets and epigraph of K-convex functions.* Let $K \subseteq \mathbb{R}^m$ be a proper convex cone with associated generalized inequality $\preceq_K$, and let $f : \mathbb{R}^n \to \mathbb{R}^m$. For $\boldsymbol{\alpha} \in \mathbb{R}^m$, the $\boldsymbol{\alpha}$-sublevel set of f (with respect to $\preceq_K$) is defined as

$$C_{\boldsymbol{\alpha}} = \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) \preceq_K \boldsymbol{\alpha}\}.$$

The epigraph of f, with respect to $\preceq_K$, is defined as the set

$$\mathbf{epi}_K\, f = \{(\mathbf{x}, \mathbf{t}) \in \mathbb{R}^{n+m} \mid f(\mathbf{x}) \preceq_K \mathbf{t}\}.$$

Show the following:

(a) If f is K-convex, then its sublevel sets $C_{\boldsymbol{\alpha}}$ are convex for all $\boldsymbol{\alpha}$.

(b) f is K-convex iff $\mathbf{epi}_K\, f$ is a convex set.

2. Show that the function $f(\mathbf{X}) = \mathbf{X}^{-1}$ is matrix convex on $\mathbb{S}^n_{++}$.

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3. *Schur complement.* Suppose $\mathbf{X} \in \mathbb{S}^n$ partitioned as

$$\mathbf{X} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix},$$

where $\mathbf{A} \in \mathbb{S}^k$. The Schur complement of $\mathbf{X}$ (with respect to $\mathbf{A}$) is $\mathbf{S} = \mathbf{C} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$. Show that the Schur complement, viewed as function from $\mathbb{S}^n$ into $\mathbb{S}^{n-k}$, is matrix concave on $\mathbb{S}^n_{++}$.

4. Let $\pi_{\mathcal{C}}$ be the projection operator onto a convex set $\mathcal{C}$. Prove:

$$\langle \pi_{\mathcal{C}}(\mathbf{y}) - \mathbf{x}, \pi_{\mathcal{C}}(\mathbf{y}) - \mathbf{y} \rangle \leq 0.$$

Further show that

$$\|\pi_{\mathcal{C}}(\mathbf{y}) - \mathbf{x}\|^2 + \|\pi_{\mathcal{C}}(\mathbf{y}) - \mathbf{y}\rangle\|^2 \leq \|\mathbf{x} - \mathbf{y}\|^2.$$

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5. If f is an L-smooth function (a short for "∇f is Lipschitz continuous with a Lipschitz constant L"), prove

$$f\left(\mathbf{x}-\frac{1}{\beta}\nabla f(\mathbf{x})\right)-f(\mathbf{x})\leq-\frac{1}{2\beta}\|\nabla f(\mathbf{x})\|^2.$$

6. Let f satisfy

$$0\leq f(\mathbf{x})-f(\mathbf{y})-\langle\nabla f(\mathbf{y}),\mathbf{x}-\mathbf{y}\rangle\leq\frac{\beta}{2}\|\mathbf{x}-\mathbf{y}\|^2,\quad\forall\mathbf{x},\mathbf{y}.$$

Prove that

$$f(\mathbf{x})-f(\mathbf{y})\leq\langle\nabla f(\mathbf{x}),\mathbf{x}-\mathbf{y}\rangle\leq\frac{1}{2\beta}\|\nabla f(\mathbf{x})-\nabla f(\mathbf{y})\|^2,\quad\forall\mathbf{x},\mathbf{y}.$$

7. Let f be L-smooth and μ-strongly convex on $\mathbb{R}^n$. Then

$$\langle\nabla f(\mathbf{x})-\nabla f(\mathbf{y}),\mathbf{x}-\mathbf{y}\rangle\geq\frac{L\mu}{L+\mu}\|\mathbf{x}-\mathbf{y}\|^2+\frac{1}{L+\mu}\|\nabla f(\mathbf{x})-\nabla f(\mathbf{y})\|^2,\quad\forall\mathbf{x},\mathbf{y}.$$

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8. Suppose f is strongly convex with $m\mathbf{I} \preceq \nabla^2 f(\mathbf{x}) \preceq M\mathbf{I}$. Let $\Delta\mathbf{x}$ be a descent direction at $\mathbf{x}$. Show that the backtracking stopping condition holds for

$$0 < t \leq -\frac{\nabla f(\mathbf{x})^T \Delta\mathbf{x}}{M\|\Delta\mathbf{x}\|_2^2}.$$

Use this to give an upper bound on the number of backtracking iterations.

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9. Consider the problem for computing the analytic center of the set of linear inequalities:

$$\min_{\mathbf{x}} f(\mathbf{x}) = -\sum_{i=1}^{m} \log(1 - \mathbf{a}_i^T \mathbf{x}) - \sum_{i=1}^{n} \log(1 - x_i^2),$$

with variable $\mathbf{x} \in \mathbb{R}^n$, and dom $f = \{\mathbf{x} | \mathbf{a}_i^T \mathbf{x} < 1, i = 1, \cdots, m, |x_i| < 1, i = 1, \cdots, n\}$.

We can choose $\mathbf{x}^{(0)} = \mathbf{0}$ as our initial point. You can generate instances of this problem by choosing $\mathbf{a}^i$ from some distribution on $\mathbb{R}^n$. Use the gradient method to solve the problem, using reasonable choices for the backtracking parameters, and a stopping criterion of the form $\|\nabla f(\mathbf{x})\|_2 \leq \eta$. Plot the objective function and step length versus iteration number. (Once you have determined p^* to high accuracy, you can also plot $f - p^*$ versus iteration.) Experiment with the backtracking parameters α and β to see their effect on the total number of iterations required. Hand in your code and a report showing how the parameters are chosen and the figures.