# Chapter 3: Convex Sets

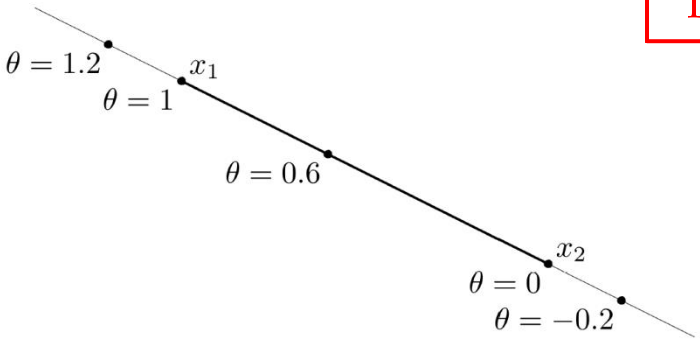
- Affine and convex sets
- Important examples
- Operators that preserve convexity
- Generalized inequalities
- Separating and supporting hyperplanes
- Dual cones and generalized inequalities

# Convex sets

- Affine sets

also defined for
infinite-dim spaces

Lines and line segments: $\mathbf{y} = \theta \mathbf{x}_1 + (1-\theta)\mathbf{x}_2$




A set is called an *affine subspace* iff it contains all the lines passing through any two points.

If $C$ is an affine subspace and $\mathbf{x}_0 \in C$, then the set

$$V = C - \mathbf{x}_0 = \{\mathbf{x} - \mathbf{x}_0 | \ \mathbf{x} \in C\}$$

is a linear subspace.

We define $\dim C = \dim V$.

Example: Solution set of linear equations.

# Convex sets

- Affine sets

The set of all affine combinations of points in some set $C$ is called the *affine hull* of $C$, and denoted aff $C$:

$$\mathrm{aff}C = \{\overbrace{\boxed{\theta_1 \mathbf{x}_1 + \ldots + \theta_k \mathbf{x}_k \mid \mathbf{x}_1, \ldots, \mathbf{x}_k \in C, \theta_1 + \ldots + \theta_k = 1}}^{\textit{affine combination}}\}.$$

The affine hull is the smallest affine set that contains $C$.

We define the *affine dimension* of a set $C$ as the dimension of its affine hull.

Example: unit circle in $\mathbb{R}^2$

If aff$C$ is not the whole space, we define the *relative interior* of the set $C$, denoted ri$C$, as its interior relative to aff$C$:

$$\mathrm{ri}C = \{\mathbf{x} \in C \mid B(\mathbf{x}, r) \cap \mathrm{aff}C \subseteq C \text{ for some } r > 0\},$$

We can then define the *relative boundary* of a set $C$ as $\bar{C} \setminus \mathrm{ri}C$.

Example: unit square in $\mathbb{R}^3$

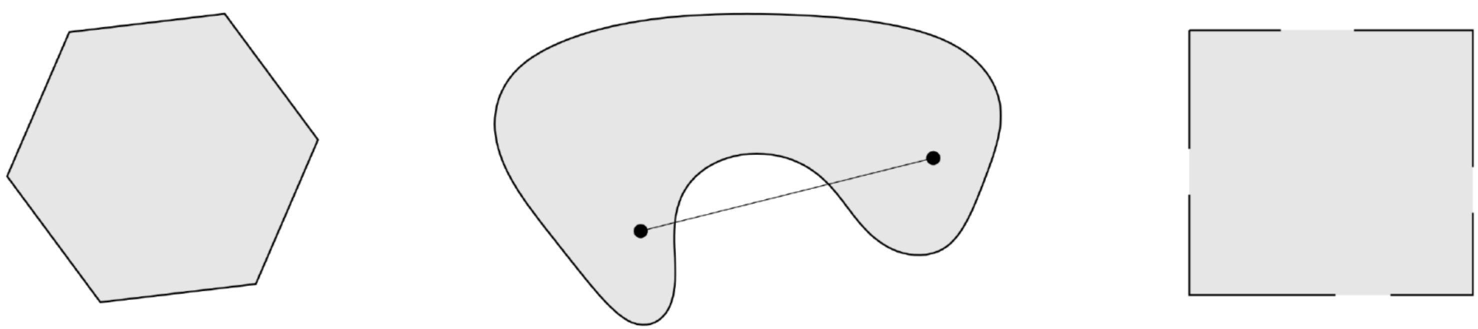
# Convex sets

- Convex sets

A set $C$ is *convex* if the line segment between any two points in $C$ lies in $C$, i.e., if for any $\mathbf{x}_1, \mathbf{x}_2 \in C$ and any $\theta$ with $0 \leq \theta \leq 1$, we have

$$\theta \mathbf{x}_1 + (1-\theta)\mathbf{x}_2 \in C.$$

Every two points can see each other.



Convexity vs. closedness

Examples: $\emptyset$, $\{\mathbf{x}_0\}$, $\mathbb{R}^n$, and affine sets

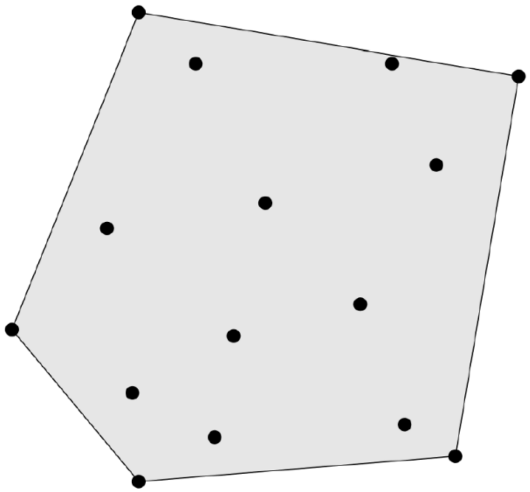
# Convex sets

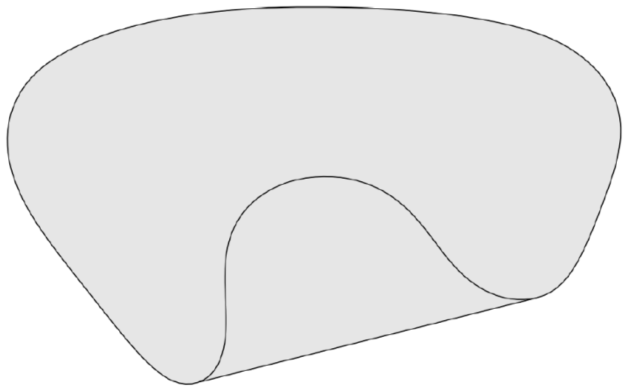
- Convex hull

The *convex hull* of a set $C$, denoted conv$C$, is the set of all convex combinations of points in $C$:

$$\text{conv}C = \{\overbrace{\boxed{\theta_1\mathbf{x}_1 + ... + \theta_k\mathbf{x}_k | \mathbf{x}_i \in C, \theta_i \geq 0, i = 1, ..., k, \theta_1 + ... + \theta_k = 1}}^{\textit{Convex combination}}\}.$$

conv$C$ is always convex. It is the smallest convex set that contains $C$.





Example: $\text{conv}\{\mathbf{e}_i\mathbf{e}_j^T, i = 1, \cdots, m, j = 1, \cdots, n\}$, $\text{conv}\{\mathbf{uv}^T | \|\mathbf{u}\| = 1, \|\mathbf{v}\| = 1\}$

# Convex sets

- General convex combination

Suppose $\theta_1, \theta_2, \ldots$ satisfy

$$\theta_i \geq 0, \quad i = 1, 2, \ldots, \quad \sum_{i=1}^{\infty} \theta_i = 1,$$

and $\mathbf{x}_1, \mathbf{x}_2, \ldots \in C$, where $C \subseteq \mathbb{R}^n$ is convex. Then

$$\sum_{i=1}^{\infty} \theta_i \mathbf{x}_i \in C,$$

if the series converges. More generally, suppose $p : \mathbb{R}^n \to \mathbb{R}$ satisfies $p(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in C$ and $\int_C p(\mathbf{x}) d\mathbf{x} = 1$, where $C \subseteq \mathbb{R}^n$ is convex. Then

$$\int_C p(\mathbf{x}) \mathbf{x} d\mathbf{x} \in C, \qquad \boxed{\mathbb{E}\mathbf{x} \in C}$$

if the integral exists.

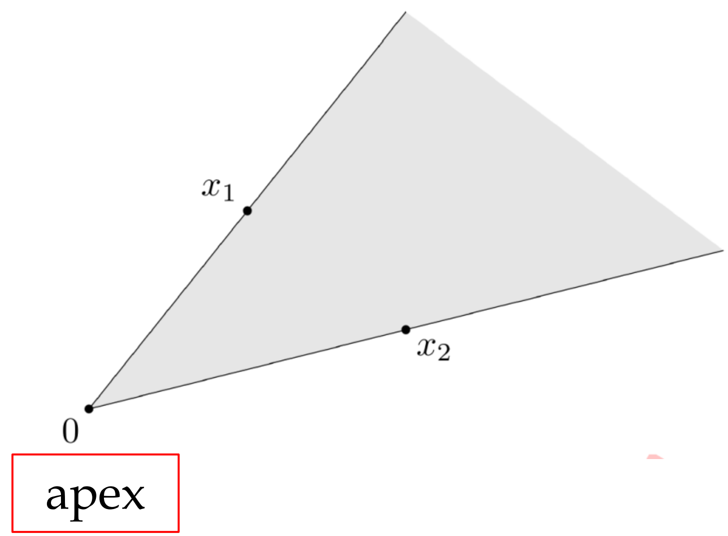
# Convex sets

- Cones

A set $C$ is called a *cone* if for every $\mathbf{x} \in C$ and $\theta \geq 0$ we have $\theta\mathbf{x} \in C$.

A set $C$ is a *convex cone* if it is convex and a cone, which means that for any $\mathbf{x}_1, \mathbf{x}_2 \in C$ and $\theta_1, \theta_2 \geq 0$, we have

$$\theta_1\mathbf{x}_1 + \theta_2\mathbf{x}_2 \in C.$$



# Convex sets

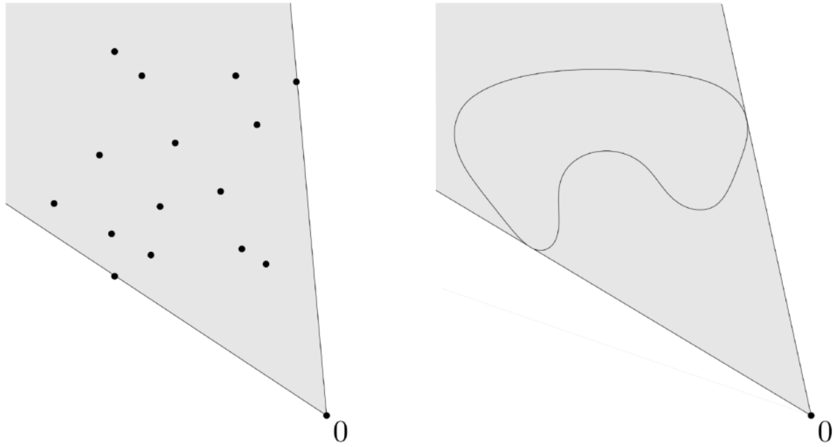
- Conic hull

The *conic hull* of a set $C$ is the set of all conic combinations of points in $C$, i.e.,

$$\{\underbrace{\theta_1 \mathbf{x}_1 + \ldots + \theta_k \mathbf{x}_k \mid \mathbf{x}_i \in C, \theta_i \geq 0, i = 1, \ldots, k}_{}\}.$$

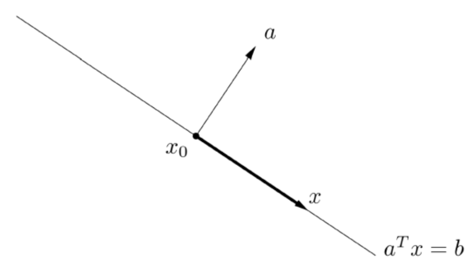
*Conic combination*

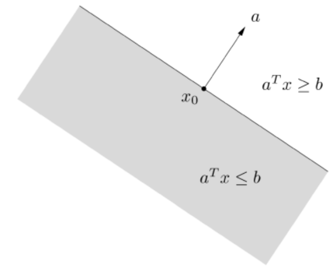
It is the smallest convex cone that contains $C$.

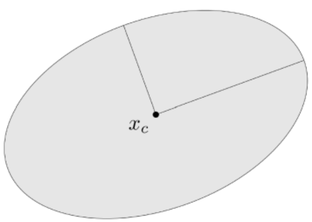


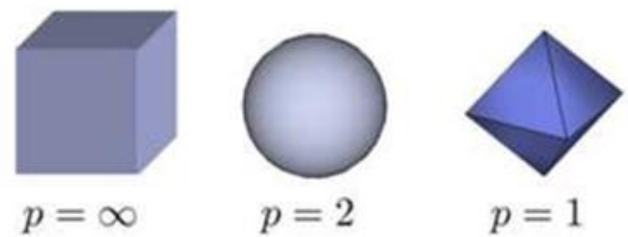
# Important examples

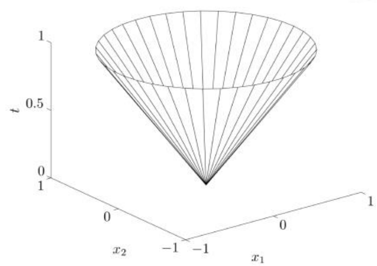
- line, line segment, ray, subspace
- hyperplanes and halfspaces
- Euclidean balls and ellipsoids
- norm balls and norm cones
- polyhedra, nonnegative orthant
- simplexes
- positive semidefinite cone

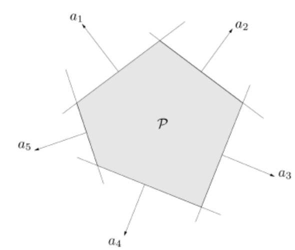


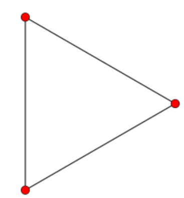



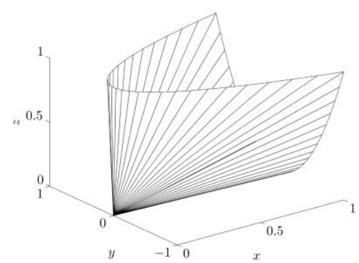





# Operations that preserve convexity

- Intersection

If $S_\alpha$ is convex for every $\alpha \in \mathcal{A}$, then $\cap_{\alpha \in \mathcal{A}} S_\alpha$ is convex.

Examples: 1. $\mathbb{S}^n_+ = \bigcap_{\mathbf{z} \neq \mathbf{0}} \{\mathbf{X} \in \mathbb{S}^n | \ \mathbf{z}^T \mathbf{X} \mathbf{z} \geq 0\}$.

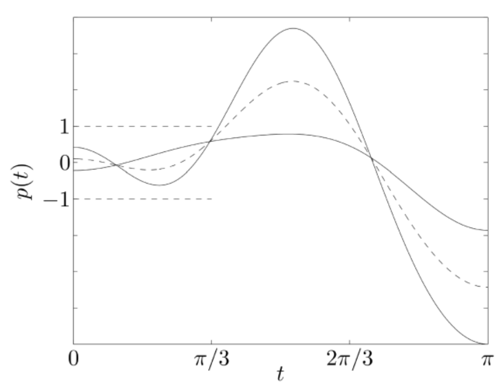
2. Consider the set

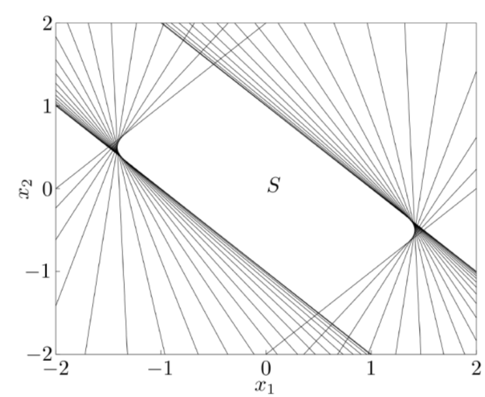
$$S = \{\mathbf{x} \in \mathbb{R}^m | \ | \ p(t)| \leq 1 \text{ for } |t| \leq \pi/3\},$$

where $p(t) = \sum_{k=1}^m x_k \cos kt$. The set $S$ can be expressed as the intersection of an infinite number of slabs: $S = \cap_{|t| \leq \pi/3} S_t$, where

$$S_t = \{\mathbf{x} | -1 \leq (\cos t, ..., \cos mt)^T \mathbf{x} \leq 1\},$$

and so is convex.



# Operations that preserve convexity

- Intersection

3. $\{\mathbf{X} \mid \|\mathbf{X}\|_* \leq 1\} = \bigcap_{\mathbf{U}^T\mathbf{U}=\mathbf{I}, \mathbf{V}^T\mathbf{V}=\mathbf{I}} \{\mathbf{X} \mid \langle \mathbf{U}\mathbf{V}^T, \mathbf{X} \rangle \leq 1\}.$
4. A closed convex set $S$ is the intersection of all halfspaces that contain it:

$$S = \bigcap \{\mathcal{H} \mid \mathcal{H} \text{ halfspace}, S \subseteq \mathcal{H}\}.$$

# Operations that preserve convexity

- Affine functions

Suppose $S \subseteq \mathbb{R}^n$ is convex and $f : \mathbb{R}^n \to \mathbb{R}^m$ is an *affine function*: $f(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$. Then the *image* of $S$ under $f$,

$$f(S) = \{f(\mathbf{x}) | \mathbf{x} \in S\},$$

is convex. Similarly, if $f : \mathbb{R}^k \to \mathbb{R}^n$ is an affine function, the *inverse image* of $S$ under $f$,

$$f^{-1}(S) = \{\mathbf{x} | f(\mathbf{x}) \in S\},$$

is convex.

The *projection* of a convex set onto some of its coordinates is convex: if $S \subseteq \mathbb{R}^m \times \mathbb{R}^n$ is convex, then

$$T = \{\mathbf{x}_1 \in \mathbb{R}^m | (\mathbf{x}_1, \mathbf{x}_2) \in S \text{ for some } \mathbf{x}_2 \in \mathbb{R}^n\}$$

is convex.

# Operations that preserve convexity

- Sum and Cartesian product

If $S_1$ and $S_2$ are convex, then their *sum* $S_1 + S_2 = \{\mathbf{x} + \mathbf{y} | \mathbf{x} \in S_1,\ \mathbf{y} \in S_2\}$ is convex, so is their *Cartesian product* $S_1 \times S_2 = \{(\mathbf{x}_1, \mathbf{x}_2) |\ \mathbf{x}_1 \in S_1, \mathbf{x}_2 \in S_2\}$.

# Operations that preserve convexity

- Examples

1. The polyhedron $\{\mathbf{x}|\mathbf{A}\mathbf{x} \preceq \mathbf{b}, \mathbf{C}\mathbf{x} = \mathbf{d}\}$ can be expressed as the inverse image of the Cartesian product of the nonnegative orthant and the origin under the affine function $f(\mathbf{x}) = (\mathbf{b} - \mathbf{A}\mathbf{x},\ \mathbf{d} - \mathbf{C}\mathbf{x})$:

$$\{\mathbf{x}|\mathbf{A}\mathbf{x} \preceq \mathbf{b}, \mathbf{C}\mathbf{x} = \mathbf{d}\} = \{\mathbf{x}|f(\mathbf{x}) \in \mathbb{R}_+^m \times \{0\}\}.$$

2. The condition

$$A(\mathbf{x}) = x_1\mathbf{A}_1 + ... + x_n\mathbf{A}_n \preceq \mathbf{B},$$

where $\mathbf{B}, \mathbf{A}_i \in \mathbb{S}^m$, is called a *linear matrix inequality (LMI)* in $\mathbf{x}$.

The solution set of a linear matrix inequality, $\{\mathbf{x}|A(\mathbf{x}) \preceq \mathbf{B}\}$, is convex. Indeed, it is the inverse image of the positive semidefinite cone under the affine function $f : \mathbb{R}^n \to \mathbb{S}^m$ given by $f(\mathbf{x}) = \mathbf{B} - A(\mathbf{x})$.

# Operations that preserve convexity

- Examples

3. The set

$$\{\mathbf{x} | \mathbf{x}^T \mathbf{P} \mathbf{x} \leq (\mathbf{c}^T \mathbf{x})^2, \ \mathbf{c}^T \mathbf{x} \geq 0\}$$

where $\mathbf{P} \in \mathbb{S}_+^n$ and $\mathbf{c} \in \mathbb{R}^n$, is convex, since it is the inverse image of the second-order cone,

$$\{(\mathbf{z}, t) | \ \mathbf{z}^T \mathbf{z} \leq t^2, t \geq 0\},$$

under the affine function $f(\mathbf{x}) = (\mathbf{P}^{1/2}\mathbf{x}, \ \mathbf{c}^T \mathbf{x})$.

4. The ellipsoid

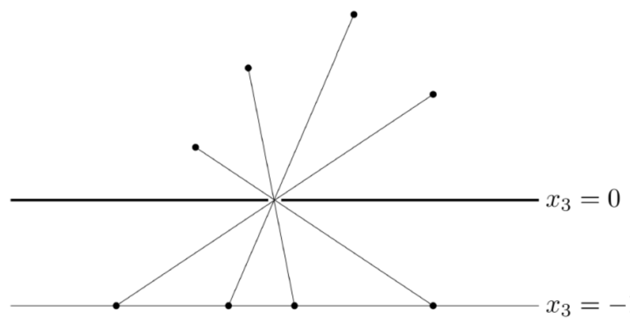
$$\epsilon = \{\mathbf{x} | (\mathbf{x} - \mathbf{x}_c)^T \mathbf{P}^{-1} (\mathbf{x} - \mathbf{x}_c) \leq 1\},$$

where $\mathbf{P} \in \mathbb{S}_{++}^n$, is the image of the unit Euclidean ball $\{\mathbf{u} | \ \|\mathbf{u}\|_2 \leq 1\}$ under the affine mapping $f(\mathbf{u}) = \mathbf{P}^{1/2}\mathbf{u} + \mathbf{x}_c$. (It is also the inverse image of the unit ball under the affine mapping $g(\mathbf{x}) = \mathbf{P}^{-1/2}(\mathbf{x} - \mathbf{x}_c)$.)

# Operations that preserve convexity

- Perspective functions

We define the *perspective function* $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$, with domain $\mathbf{dom}\ \ P = \mathbb{R}^n \times \mathbb{R}_{++}$, as $P(\mathbf{z}, t) = \mathbf{z}/t$.




The inverse image of a convex set under the perspective function is also convex: if $C \subseteq \mathbb{R}^n$ is convex, then

$$P^{-1}(C) = \{(\mathbf{x}, t) \in \mathbb{R}^{n+1} | \mathbf{x}/t \in C, t > 0\}$$

is convex.

Question: If function $f$ preserves convexity: if $C_1$ is convex then $f(C_1)$ is also convex, does $f^{-1}$ also perserve convexity?

# Operations that preserve convexity

- Linear-fractional functions

A linear-fractional function is formed by composing the perspective function with an affine function. Suppose $g : \mathbb{R}^n \to \mathbb{R}^{m+1}$ is affine, i.e.,

$$g(\mathbf{x}) = \begin{bmatrix} \mathbf{A} \\ \mathbf{c}^T \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{b} \\ d \end{bmatrix},$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$, and $d \in \mathbb{R}$. The function $f : \mathbb{R}^n \to \mathbb{R}^m$ given by $f = P \circ g$, i.e.,

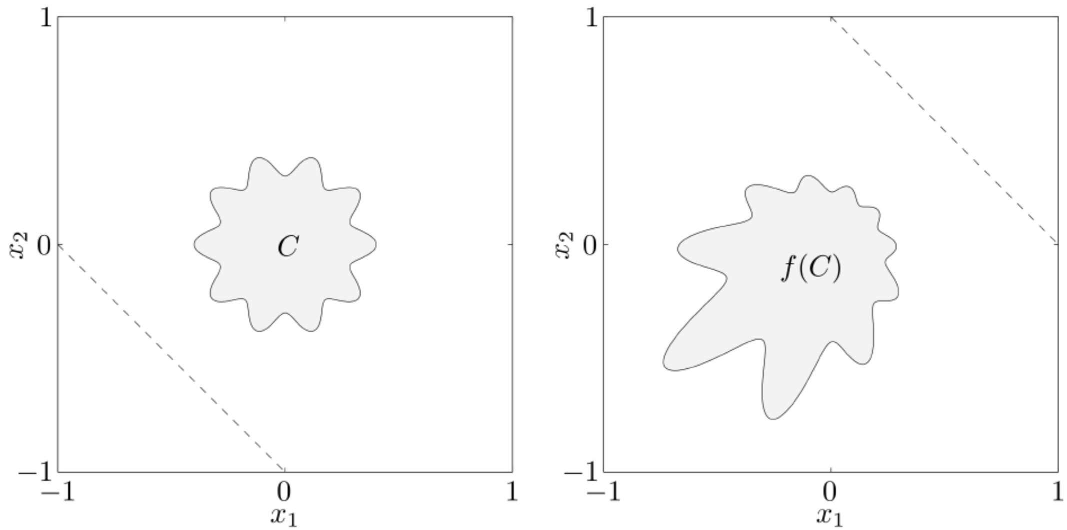
$$f(\mathbf{x}) = (\mathbf{A}\mathbf{x} + \mathbf{b})/(\mathbf{c}^T\mathbf{x} + d), \mathrm{dom}\, f = \{\mathbf{x} | \mathbf{c}^T\mathbf{x} + d > 0\},$$

is called a *linear-fractional* (or projective) function.

# Operations that preserve convexity

- Linear-fractional functions

$$f(\mathbf{x}) = \frac{1}{\mathbf{x}_1 + \mathbf{x}_2 + 1}\mathbf{x}, \ \mathbf{dom}\ f = \{(x_1, x_2) | x_1 + x_2 + 1 > 0\}.$$




*Conditional probabilities:* Let $p_{ij} = \mathbb{P}(u = i, v = j)$. Then the conditional probability

$$f_{ij} = \frac{p_{ij}}{\sum_{k=1}^{n} p_{kj}}$$

is obtained by a linear-fractional mapping from $\mathbf{p}$.

# Generalized inequalities

- Proper cones and generalized inequalities

A cone $K \subseteq \mathbb{R}^n$ is called a *proper cone* if it satisfies the following:

- $K$ is convex.
- $K$ is closed.
- $K$ is solid, which means it has nonempty interior.
- $K$ is pointed, which means that it contains no line (or equivalently, $\mathbf{x} \in K, -\mathbf{x} \in K \implies \mathbf{x} = 0$).

We associate the proper cone $K$ with the partial ordering on $\mathbb{R}^n$ defined by

$$\mathbf{x} \preceq_K \mathbf{y} \iff \mathbf{y} - \mathbf{x} \in K.$$

We also write $\mathbf{x} \succeq_K \mathbf{y}$ for $\mathbf{y} \preceq_K \mathbf{x}$. Similarly, we define an associated strict partial ordering by

$$\mathbf{x} \prec_K \mathbf{y} \iff \mathbf{y} - \mathbf{x} \in K^\circ,$$

and write $\mathbf{x} \succ_K \mathbf{y}$ for $\mathbf{y} \prec_K \mathbf{x}$.

# Generalized inequalities

- Examples

1. When $K = \mathbb{R}_+$, the partial ordering $\preceq_K$ is the usual ordering $\leq$ on $\mathbb{R}$, and the strict partial ordering $\prec_K$ is the same as the usual strict ordering $<$ on $\mathbb{R}$.

2. *Nonnegative orthant and componentwise inequality:* The nonnegative orthant $K = \mathbb{R}_+^n$ is a proper cone. The associated generalized inequality $\preceq_K$ corresponds to componentwise inequality between vectors: $\mathbf{x} \preceq_K \mathbf{y}$ means that $\mathbf{x}_i \leq \mathbf{y}_i, i = 1, \ldots, n$. The associated strict inequality corresponds to componentwise strict inequality: $\mathbf{x} \prec_K \mathbf{y}$ means that $\mathbf{x}_i < \mathbf{y}_i, i = 1, \ldots, n$.

For simplicity, we write $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{x} < \mathbf{y}$ instead of $\mathbf{x} \preceq_{\mathbb{R}_+^n} \mathbf{y}$ and $\mathbf{x} \prec_{\mathbb{R}_{++}^n} \mathbf{y}$

3. *Positive semidefinite cone and matrix inequality*: The positive semidefinite cone $S_+^n$ is a proper cone in $S^n$. The associated generalized inequality $\preceq_K$ is the usual matrix inequality: $\mathbf{X} \preceq_K \mathbf{Y}$ means $\mathbf{Y} - \mathbf{X}$ is positive semidefinite. The interior of $\mathbb{S}_+^n$ (in $\mathbb{S}^n$) consists of the positive definite matrices, so the strict generalized inequality also agrees with the usual strict inequality between symmetric matrices: $\mathbf{X} \prec_K \mathbf{Y}$ means $\mathbf{Y} - \mathbf{X}$ is positive definite.

For simplicity, we write $\mathbf{X} \preceq \mathbf{Y}$ and $\mathbf{X} \prec \mathbf{Y}$ instead of $\mathbf{X} \preceq_{\mathbb{S}_+^n} \mathbf{Y}$ and $\mathbf{X} \prec_{\mathbb{S}_+^n} \mathbf{Y}$

# Generalized inequalities

- Examples

4. *Cone of polynomials nonnegative on [0, 1]*: Let K be defined as

$$K = \{\mathbf{c} \in \mathbb{R}^n |\ c_1 + c_2 t + ... + c_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\},$$

i.e., $K$ is the cone of (coefficients of) polynomials of degree $n-1$ that are nonnegative on the interval [0, 1]. It can be shown that $K$ is a proper cone, its interior is the set of coefficients of polynomials that are positive on the interval [0, 1].

Two vectors $\mathbf{c},\ \mathbf{d} \in \mathbb{R}^n$ satisfy $\mathbf{c} \prec_K \mathbf{d}$ if and only if

$$c_1 + c_2 t + ... + c_n t^{n-1} \leq d_1 + d_2 t + ... + d_n t^{n-1}$$

for all $t \in [0, 1]$.

# Generalized inequalities

- Properties of generalized inequalities

- $\preceq_K$ is preserved under addition: if $\mathbf{x} \preceq_K \mathbf{y}$ and $\mathbf{u} \preceq_K \mathbf{v}$, then $\mathbf{x} + \mathbf{u} \preceq_K \mathbf{y} + \mathbf{v}$.
- $\preceq_K$ is transitive: if $\mathbf{x} \preceq_K \mathbf{y}$ and $\mathbf{y} \preceq_K \mathbf{z}$ then $\mathbf{x} \preceq_K \mathbf{z}$.
- $\preceq_K$ is preserved under nonnegative scaling: if $\mathbf{x} \preceq_K \mathbf{y}$ and $\alpha \geq 0$ then $\alpha\mathbf{x} \preceq_K \alpha\mathbf{y}$.
- $\preceq_K$ is reflexive: $\mathbf{x} \preceq_K \mathbf{x}$.
- $\preceq_K$ is antisymmetric: if $\mathbf{x} \preceq_K \mathbf{y}$ and $\mathbf{y} \preceq_K \mathbf{x}$, then $\mathbf{x} = \mathbf{y}$.
- $\preceq_K$ is preserved under limits: if $\mathbf{x}_i \preceq_K \mathbf{y}_i$ for $i = 1, 2, ..., \mathbf{x}_i \to \mathbf{x}$ and $\mathbf{y}_i \to \mathbf{y}$ as $i \to \infty$, then $\mathbf{x} \preceq_K \mathbf{y}$.

# Generalized inequalities

- Properties of generalized inequalities

- if $\mathbf{x} \prec_K \mathbf{y}$ then $\mathbf{x} \preceq_K \mathbf{y}$.
- if $\mathbf{x} \prec_K \mathbf{y}$ and $\mathbf{u} \preceq_K \mathbf{v}$ then $\mathbf{x} + \mathbf{u} \prec_K \mathbf{y} + \mathbf{v}$.
- if $\mathbf{x} \prec_K \mathbf{y}$ and $\alpha > 0$ then $\alpha\mathbf{x} \prec_K \alpha\mathbf{y}$.
- $\mathbf{x} \not\prec_K \mathbf{x}$.
- if $\mathbf{x} \prec_K \mathbf{y}$, then for $\mathbf{u}$ and $\mathbf{v}$ small enough, $\mathbf{x} + \mathbf{u} \prec_K \mathbf{y} + \mathbf{v}$.

# Homework (3)

1. Let $C \subseteq \mathbb{R}^n$ be a convex set, with $\mathbf{x}_1, ..., \mathbf{x}_k \in C$, and let $\theta_1, ..., \theta_k \in \mathbb{R}$ satisfy $\theta_i \geq 0$, $\theta_1 + ... + \theta_k = 1$. Show that $\theta_1 \mathbf{x}_1 + ... + \theta_k \mathbf{x}_k \in C$.
2. A set $C$ is midpoint convex if whenever two points $\mathbf{a}, \mathbf{b}$ are in $C$, the average or midpoint $(\mathbf{a} + \mathbf{b})/2$ is in $C$. Prove that if $C$ is closed and midpoint convex, then $C$ is convex.
3. Which of the following sets $S$ are polyhedra? If possible, express $S$ in the form $S = \{\mathbf{x} | \mathbf{A}\mathbf{x} \leq \mathbf{b},\ \mathbf{F}\mathbf{x} = \mathbf{g}\}$.

(a) $S = \{y_1 \mathbf{a}_1 + y_2 \mathbf{a}_2 | -1 \leq y_1 \leq 1, -1 \leq y_2 \leq 1\}$, where $\mathbf{a}_1,\ \mathbf{a}_2 \in \mathbb{R}^n$.

(b) $S = \{\mathbf{x} \in \mathbb{R}^n |\ \mathbf{x} \geq \mathbf{0},\ \mathbf{1}^T \mathbf{x} = 1, \sum_{i=1}^n x_i a_i = b_1, \sum_{i=1}^n x_i a_i^2 = b_2\}$, where $a_1, ..., a_n \in \mathbb{R}$ and $b_1, b_2 \in \mathbb{R}$.

(c) $S = \{\mathbf{x} \in \mathbb{R}^n |\ \mathbf{x} \succeq \mathbf{0},\ \mathbf{x}^T \mathbf{y} \leq 1$ for all $\mathbf{y}$ with $\|\mathbf{y}\|_2 = 1\}$.

(d) $S = \{\mathbf{x} \in \mathbb{R}^n |\ \mathbf{x} \succeq \mathbf{0},\ \mathbf{x}^T \mathbf{y} \leq 1$ for all $\mathbf{y}$ with $\sum_{i=1}^n |y_i| = 1\}$.

# Homework (3)

4. Which of the following sets are convex?

(a) A slab, i.e., a set of the form $\{\mathbf{x} \in \mathbb{R}^n | \ \alpha \leq \mathbf{a}^T\mathbf{x} \leq \beta\}$.

(b) A wedge, i.e., $\{\mathbf{x} \in \mathbb{R}^n | \ \mathbf{a}_1^T\mathbf{x} \leq b_1, \ \mathbf{a}_2^T\mathbf{x} \leq b_2\}$.

(c) The set of points closer to a given point than a given set, i.e., $\{\mathbf{x} | \|\mathbf{x} - \mathbf{x}_0\|_2 \leq \|\mathbf{x} - \mathbf{y}\|_2 \text{ for all } \mathbf{y} \in S\}$ where $S \subseteq \mathbb{R}^n$.

(d) The set of points closer to one set than another, i.e., $\{\mathbf{x} | \mathbf{dist}(\mathbf{x}, S) \leq \mathbf{dist}(\mathbf{x}, T)\}$, where $S, T \subseteq \mathbb{R}^n$, and

$$\mathbf{dist}(\mathbf{x}, S) = \inf\{\|\mathbf{x} - \mathbf{z}\|_2 | \ \mathbf{z} \in S\}.$$

(e) The set $\{\mathbf{x} | \mathbf{x} + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbb{R}^n$ with $S_1$ convex.

(f) The set of points whose distance to $\mathbf{a}$ does not exceed a fixed fraction $\theta$ of the distance to $\mathbf{b}$, i.e., the set $\{\mathbf{x} | \|\mathbf{x} - \mathbf{a}\|_2 \leq \theta\|\mathbf{x} - \mathbf{b}\|_2\}$ ($\mathbf{a} \neq \mathbf{b}$ and $0 \leq \theta \leq 1$).

# Homework (3)

5. Find the convex hull of the set $\{\mathbf{u}\mathbf{u}^T | \|\mathbf{u}\| = 1\}$.
6. Consider the set of rank-$k$ outer products, defined as $\{\mathbf{X}\mathbf{X}^T | \mathbf{X} \in \mathbb{R}^{n\times k}, \mathrm{rank}\mathbf{X} = k\}$. Describe its conic hull in simple terms.
7. Give an expression $\bigcap\limits_{\alpha\in\mathcal{A}} S_\alpha$ for the unit ball $\{\mathbf{X} | \|\mathbf{X}\|_2 \leq 1\}$.

# Homework (3)

8. Give an explicit description of the positive semidefinite cone $\mathbb{S}^n_+$, in terms of the matrix coefficients and ordinary inequalities, for $n = 1, 2, 3$. To describe a general element of $\mathbb{S}^n$, for $n = 1, 2, 3$, use the notation

$$x_1, \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}, \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix}.$$

9. Suppose $K \subseteq \mathbb{R}^2$ is a closed convex cone.

(a) Give a simple description of $K$ in terms of the polar coordinates of its elements ($\mathbf{x} = r(\cos\phi, \sin\phi)^T$ with $r \geq 0$).

(b) When is $K$ pointed?

(c) When is $K$ proper (hence, defines a generalized inequality)? Draw a plot illustrating what $\mathbf{x} \preceq_K \mathbf{y}$ means when $K$ is proper.