# Homework (6)

1. Suppose $\mathbf{A} \in \mathbb{R}^{m\times n},\ \mathbf{b} \in \mathbb{R}^m$, with $\mathbf{b} \in \text{Range}(\mathbf{A})$. Show that $\exists \mathbf{x}$ satisfying

$$\mathbf{x} > \mathbf{0},\ \ \mathbf{A}\mathbf{x} = \mathbf{b}$$

iff there exists no $\boldsymbol{\lambda}$ with

$$\mathbf{A}^T\boldsymbol{\lambda} \geq \mathbf{0},\ \ \mathbf{A}^T\boldsymbol{\lambda} \neq 0,\ \ \mathbf{b}^T\boldsymbol{\lambda} \leq 0.$$

2.

(a) Express $\{\mathbf{x} \in \mathbb{R}^2_+ | \mathbf{x}_1\mathbf{x}_2 \geq 1\}$ as an intersection of halfspaces.

(b) Let $C = \{\mathbf{x} \in \mathbb{R}^n |\ \|\mathbf{x}\|_\infty \leq 1\}$, the $\ell_\infty$-norm unit ball in $\mathbb{R}^n$, and let $\mathbf{x} \in \partial C$. Identify the supporting hyperplanes of $C$ at $\mathbf{x}$ explicitly.

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3. Prove the following.

(a) $(K^*)^\circ = \{\mathbf{y} |\ \mathbf{y}^T\mathbf{x} > 0 \text{ for all } \mathbf{x} \in K\}$.

(b) If $K$ has nonempty interior then $K^*$ is pointed.

(c) If the closure of $K$ is pointed then $K^*$ has nonempty interior.

4. Find the dual cone of $\{\mathbf{Ax} |\ \mathbf{x} \geq \mathbf{0}\}$, where $\mathbf{A} \in \mathbb{R}^{m \times n}$.
5. We define the monotone nonnegative cone as

$$K_{m+} = \{\mathbf{x} \in \mathbb{R}^n |\ x_1 \geq x_2 \geq ... \geq x_n \geq 0\}.$$

i.e., all nonnegative vectors with components sorted in nonincreasing order.

(a) Show that $K_{m+}$ is a proper cone.

(b) Find the dual cone $K_{m+}^*$.

# Homework (6)

6. A matrix $\mathbf{X} \in \mathbb{S}^n$ is called copositive if $\mathbf{z}^T\mathbf{X}\mathbf{z} \geq 0$ for all $\mathbf{z} \geq \mathbf{0}$. Verify that the set of copositive matrices is a proper cone. Find its dual cone.

7. A square matrix $\mathbf{A}$ is called conditionally negative definite (c.n.d.), if for all $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x}^T\mathbf{1} = 0$, we have

$$\mathbf{x}^T\mathbf{A}\mathbf{x} \leq 0.$$

Is the set of c.n.d. matrices a proper cone? Find its dual cone.

8. Let $K$ and $\tilde{K}$ be two convex cones whose interiors are nonempty and disjoint. Show that there is a nonzero $\mathbf{y}$ such that $\mathbf{y} \in K^*, -\mathbf{y} \in \tilde{K}^*$.

# Homework (6)

9. For each of the following functions determine whether it is convex or concave.

   (a) $f(x) = e^x - 1$ on $\mathbb{R}$.

   (b) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$.

   (c) $f(x_1, x_2) = 1/(x_1 x_2)$ on $\mathbb{R}^2_{++}$.

   (d) $f(x_1, x_2) = x_1/x_2$ on $\mathbb{R}^2_{++}$.

   (e) $f(x_1, x_2) = x_1^2/x_2$ on $\mathbb{R} \times \mathbb{R}_{++}$.

   (f) $f(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, where $0 \le \alpha \le 1$ on $\mathbb{R}^2_{++}$.

10. Prove that $f : \mathbb{R}^n \to \mathbb{R}$ is convex (resp., stictly convex, strongly convex) iff for every $\mathbf{x}, \mathbf{y} \in \operatorname{dom} f$, the function $g(t) = f(t\mathbf{x} + (1 - t)\mathbf{y})$ is a convex (resp., stictly convex, strongly convex) function on $[0, 1]$.