Homework (12)

1. Consider the equality constrained least-squares problem

$$\min_{\mathbf{x}} \ \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2,$$
$$s.t. \ \mathbf{G}\mathbf{x} = \mathbf{h},$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$ with rank $\mathbf{A} = n$, and $\mathbf{G} \in \mathbb{R}^{p \times n}$ with rank $\mathbf{G} = p$. Give the KKT conditions, and derive expressions for the primal solution $\mathbf{x}^*$ and the dual solution $\boldsymbol{\nu}^*$.

2. Show that the strong duality holds for the problem

$$\min_{\mathbf{x}} \ -3x_1^2 + x_2^2 + 2x_3^2 + 2(x_1 + x_2 + x_3)$$
$$s.t. \ x_1^2 + x_2^2 + x_3^2 = 1,$$

even though the problem is not convex. Derive the KKT conditions. Find all solutions $\mathbf{x}$, $\boldsymbol{\nu}$ that satisfy the KKT conditions. Which pair corresponds to the optimum?

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3. Consider a convex problem with no equality constraints,

$$\begin{aligned} &\min_{\mathbf{x}} \; f_0(\mathbf{x}) \\ &s.t. \; f_i(\mathbf{x}) \leq 0, \quad i = 1, \cdots, m. \end{aligned}$$

Assume that $\mathbf{x}^* \in \mathbb{R}^n$ and $\boldsymbol{\lambda}^* \in \mathbb{R}^m$ satisfy the KKT conditions

$$\begin{aligned} f_i(\mathbf{x}^*) \leq 0, &\quad i = 1, \cdots, m \\ \lambda_i^* \geq 0, &\quad i = 1, \cdots, m \\ \lambda_i^* f_i(\mathbf{x}^*) = 0, &\quad i = 1, \cdots, m \end{aligned}$$

$$\nabla f_0(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*) = 0.$$

Show that

$$\nabla f_0(\mathbf{x}^*)^T (\mathbf{x} - \mathbf{x}^*) \geq 0$$

for all feasible $\mathbf{x}$.

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4. In the subsection "Nonstrict Inequalities", the lecture note only gives the proof of primal infeasibility implying dual feasibility. This is insufficient to claim the strong alternative. Please make the proof complete.
5. Consider the linear equations $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{m \times n}$. From linear algebra we know that this equation has a solution if and only $\mathbf{b} \in \mathcal{R}(\mathbf{A})$, which occurs if and only if $\mathbf{b} \perp \mathcal{N}(\mathbf{A}^T)$. In other words, $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution if and only if there exists no $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{A}^T\mathbf{y} = \mathbf{0}$ and $\mathbf{b}^T\mathbf{y} \neq 0$. Derive this result from the theorems of alternatives.
6. Let $\mathbf{P} \in \mathbb{R}^{n \times n}$ be a matrix that satisfies

$$P_{ij} \geq 0, \quad i, j = 1, \cdots, n, \quad \mathbf{P}^T\mathbf{1} = \mathbf{1},$$

i.e., the coefficients are nonnegative and the columns sum to one. Use Farkas' lemma to prove there exists a $\mathbf{y} \in \mathbb{R}^n$ such that

$$\mathbf{P}\mathbf{y} = \mathbf{y}, \mathbf{y} \geq \mathbf{0}, \mathbf{1}^T\mathbf{y} = 1.$$

(We can interpret $\mathbf{y}$ as an equilibrium distribution of the Markov chain with n states and transition probability matrix $\mathbf{P}$.)