Separating and supporting hyperplanes

- Converse separating hyperplane theorems

Theorem 1. *Any two convex sets C and D, at least one of which is open, are disjoint if and only if there exists a separating hyperplane.*

Example: (Theorem of alternatives for strict linear inequalities) We derive the necessary and sufficient conditions for solvability of a system of strict linear inequalities $\boxed{\mathbf{A}\mathbf{x} < \mathbf{b}.}$

pair of alternatives

These inequalities are infeasible if and only if the (convex) sets

$$C = \{\mathbf{b} - \mathbf{A}\mathbf{x} | \mathbf{x} \in \mathbb{R}^n\}, \quad D = \mathbb{R}^m_{++} = \{\mathbf{y} \in \mathbb{R}^m | \mathbf{y} \succ \mathbf{0}\}$$

do not intersect. The set D is open, C is an affine set. Hence by the above theorem, C and D are disjoint iff there exists a separating hyperplane, i.e., a nonzero $\boldsymbol{\lambda} \in \mathbb{R}^m$ and $\mu \in \mathbb{R}$ such that $\boldsymbol{\lambda}^T\mathbf{y} \leq \mu$ on C and $\boldsymbol{\lambda}^T\mathbf{y} \geq \mu$ on D.

$\mu \leq 0$ and $\boldsymbol{\lambda} \geq \mathbf{0}$, $\boldsymbol{\lambda} \neq \mathbf{0}$. $\Rightarrow$ $\boxed{\exists \boldsymbol{\lambda} \text{ s.t. } \boldsymbol{\lambda} \neq \mathbf{0}, \boldsymbol{\lambda} \geq \mathbf{0}, \mathbf{A}^T\boldsymbol{\lambda} = \mathbf{0}, \boldsymbol{\lambda}^T\mathbf{b} \leq 0.}$

Separating and supporting hyperplanes

- Converse separating hyperplane theorems

Theorem 1 (Theorem of the Alternative (Fakas' Lemma)). *For* $\mathbf{A} \in \mathbb{R}^{m \times n}$ *and* $\mathbf{b} \in \mathbb{R}^m$ *the following are strong alternatives:*

1. $\exists \mathbf{x} \in \mathbb{R}^n_+$ *such that* $\mathbf{A}\mathbf{x} = \mathbf{b}$,

2. $\exists \mathbf{y} \in \mathbb{R}^m$ *such that* $\mathbf{A}^T\mathbf{y} \geq \mathbf{0}$ *and* $\mathbf{b}^T\mathbf{y} < 0$.

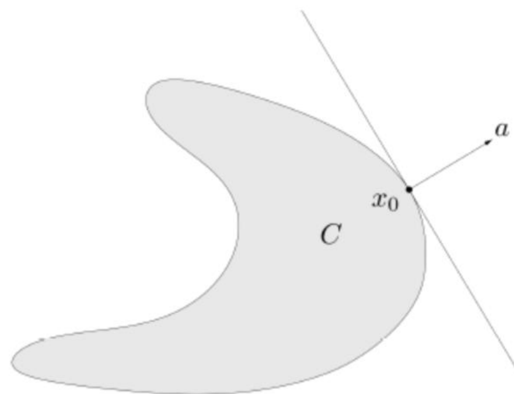
Proof. 1) $\Longrightarrow \neg$2): For $\mathbf{x} \in \mathbb{R}^n_+$ with $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{y} \in \mathbb{R}^m$ with $\mathbf{A}^T\mathbf{y} \geq 0$ we have $\mathbf{b}^T\mathbf{y} = \mathbf{x}^T\mathbf{A}^T\mathbf{y} \geq 0$.

$\neg$1) $\Longrightarrow$ 2): $C := \mathit{cone}(\mathbf{A})$ is a closed convex cone which does not contain the vector $\mathbf{b}$: by the Separating Hyperplane Theorem there exists a $\mathbf{y} \in \mathbb{R}^m$ with $\langle \mathbf{y}, \mathbf{x} \rangle \geq 0 > \langle \mathbf{y}, \mathbf{b} \rangle$ for all $\mathbf{x} \in C$, in particular $\mathbf{A}_i^T\mathbf{y} = \langle \mathbf{y}, \mathbf{A}_i \rangle \geq 0$, $\forall i$, that is, $\mathbf{A}^T\mathbf{y} \geq \mathbf{0}$.

Separating and supporting hyperplanes

- Supporting hyperplanes

Suppose $C \subseteq \mathbb{R}^n$, and $\mathbf{x}_0$ is a point in its boundary ∂C. If $\mathbf{a} \neq \mathbf{0}$ satisfies $\mathbf{a}^T\mathbf{x} \leq \mathbf{a}^T\mathbf{x}_0$ for all $\mathbf{x} \in C$, then the hyperplane $\{\mathbf{x} |\ \mathbf{a}^T\mathbf{x} = \mathbf{a}^T\mathbf{x}_0\}$ is called a *supporting hyperplane* to C at the point $\mathbf{x}_0$.



Theorem 1 (Supporting Hyperplane Theorem)**.** *For any nonempty convex set C, and any $\mathbf{x}_0 \in \partial C$, there exists a supporting hyperplane to C at $\mathbf{x}_0$.*

Proof: Two cases: $C^\circ \neq \emptyset$ and $C^\circ = \emptyset$.

Dual cones and generalized inequalities

- Dual cones

Let K be a cone. The set

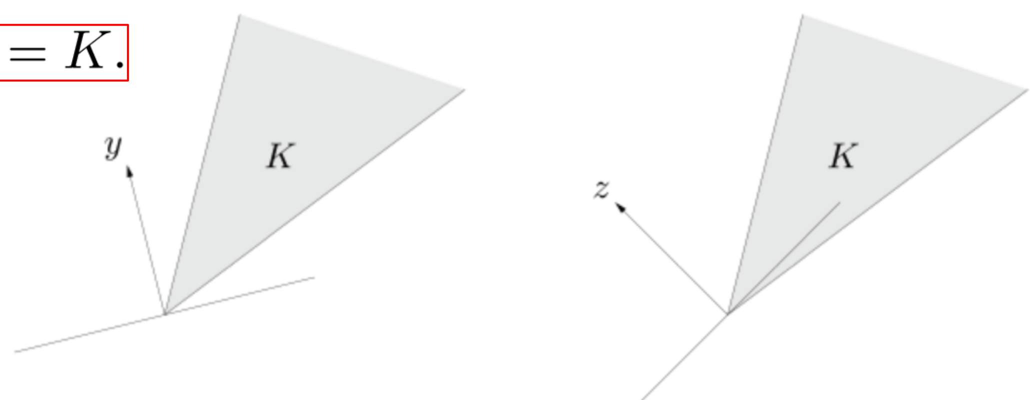
$$K^* = \{\mathbf{y} |\ \mathbf{x}^T\mathbf{y} \geq 0 \text{ for all } \mathbf{x} \in K\}$$

is called the dual cone of K.

K^* is a cone, and is always convex, even when the original cone K is not.

Geometrically, $\mathbf{y} \in K^*$ if and only if $-\mathbf{y}$ is the normal of a hyperplane that supports K at the origin.

$\boxed{\textit{Self-dual cone}\text{: } K^* = K.}$



Example: subspace, nonnegative orthant, positive semidefinite cone, norm cone

Dual cones and generalized inequalities

- Properties of dual cones

- K^* is closed and convex.

- $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$.

- If K has nonempty interior, then K^* is pointed.

- If the closure of K is pointed then K^* has nonempty interior.

- K^{**} is the closure of the convex hull of K. (Hence if K is convex and closed, $K^{**} = K$.)

These properties show that if K is a proper cone, then so is its dual K^*, and moreover, that $K^{**} = K$.

Dual cones and generalized inequalities

- Dual generalized inequalities

Suppose that the convex cone K is proper, so it induces a generalized inequality $\preceq_K$. Then its dual cone K^* is also proper, and therefore induces a generalized inequality. We refer to the generalized inequality $\preceq_{K^*}$ as the dual of the generalized inequality $\preceq_K$. Some important properties relating a generalized inequality and its dual are:

- $\mathbf{x} \preceq_K \mathbf{y}$ if and only if $\boldsymbol{\lambda}^T\mathbf{x} \leq \boldsymbol{\lambda}^T\mathbf{y}$ for all $\boldsymbol{\lambda} \succeq_{K^*} \mathbf{0}$.
- $\mathbf{x} \prec_K \mathbf{y}$ if and only if $\boldsymbol{\lambda}^T\mathbf{x} < \boldsymbol{\lambda}^T\mathbf{y}$ for all $\boldsymbol{\lambda} \succeq_{K^*} \mathbf{0}, \boldsymbol{\lambda} \neq \mathbf{0}$.

Since $K = K^{**}$, the dual generalized inequality associated with $\preceq_{K^*}$ is $\preceq_K$, so these properties hold if the generalized inequality and its dual are swapped. As a specific example, we have $\boldsymbol{\lambda} \preceq_{K^*} \boldsymbol{\mu}$ if and only if $\boldsymbol{\lambda}^T\mathbf{x} \leq \boldsymbol{\mu}^T\mathbf{x}$ for all $\mathbf{x} \succeq_K \mathbf{0}$.

Dual cones and generalized inequalities

- Theorem of alternatives for linear strict generalized inequalities

Suppose $K \subseteq \mathbb{R}^m$ is a proper cone. Consider the strict generalized inequality

$$\mathbf{A}\mathbf{x} \prec_K \mathbf{b}, \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$. Then the inequality systems (1) and

$$\exists \boldsymbol{\lambda} \text{ s.t. } \boldsymbol{\lambda} \neq \mathbf{0}, \boldsymbol{\lambda} \succeq_{K^*} \mathbf{0},\ \mathbf{A}^T \boldsymbol{\lambda} = 0, \boldsymbol{\lambda}^T \mathbf{b} \leq 0. \tag{2}$$

are alternatives.

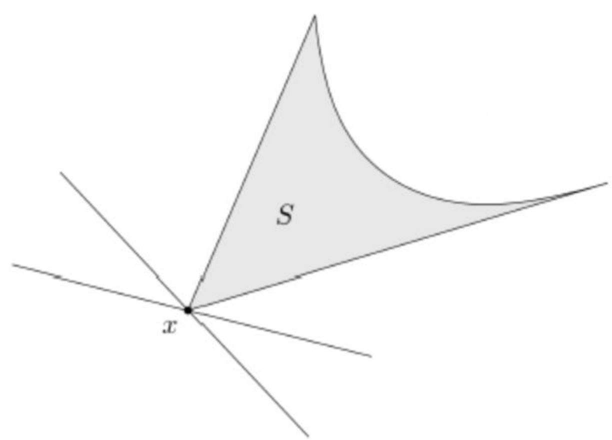
Dual cones and generalized inequalities

- Minimum and minimal elements via dual inequalities
  - Dual characterization of minimum element

$\mathbf{x}$ is the minimum element of S, with respect to the generalized inequality $\preceq_K$, iff for all $\boldsymbol{\lambda} \succ_{K^*} \mathbf{0}$, $\mathbf{x}$ is the unique minimizer of $\boldsymbol{\lambda}^T\mathbf{z}$ over $\mathbf{z} \in S$. Geometrically, this means that for any $\boldsymbol{\lambda} \succ_{K^*} \mathbf{0}$, the hyperplane

$$\{\mathbf{z} | \boldsymbol{\lambda}^T(\mathbf{z} - \mathbf{x}) = 0\}$$

is a strict supporting hyperplane to S at $\mathbf{x}$.

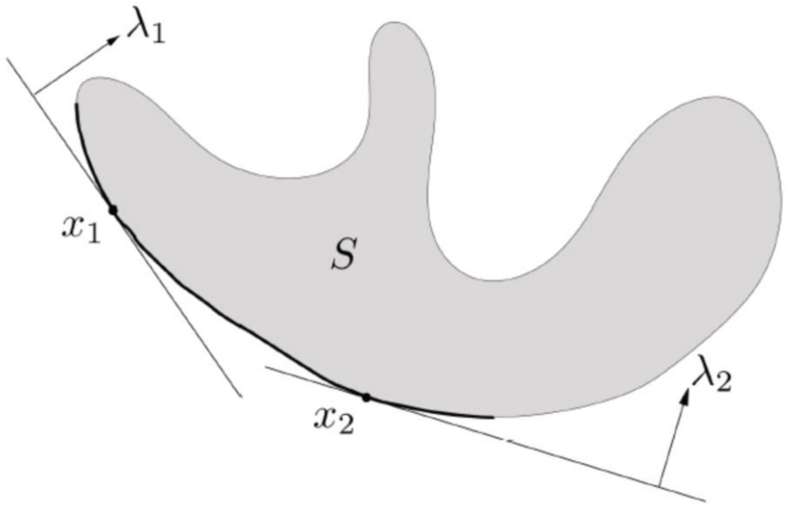


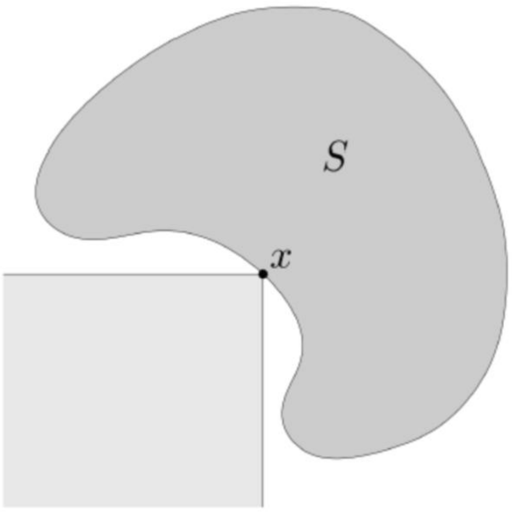
Dual cones and generalized inequalities

- Minimum and minimal elements via dual inequalities
  - Dual characterization of minimal element

only sufficient!

If $\boldsymbol{\lambda} \succ_{K^*} \mathbf{0}$ and $\mathbf{x}$ minimizes $\boldsymbol{\lambda}^T\mathbf{z}$ over $\mathbf{z} \in S$, then $\mathbf{x}$ is minimal.





Convexity matters!

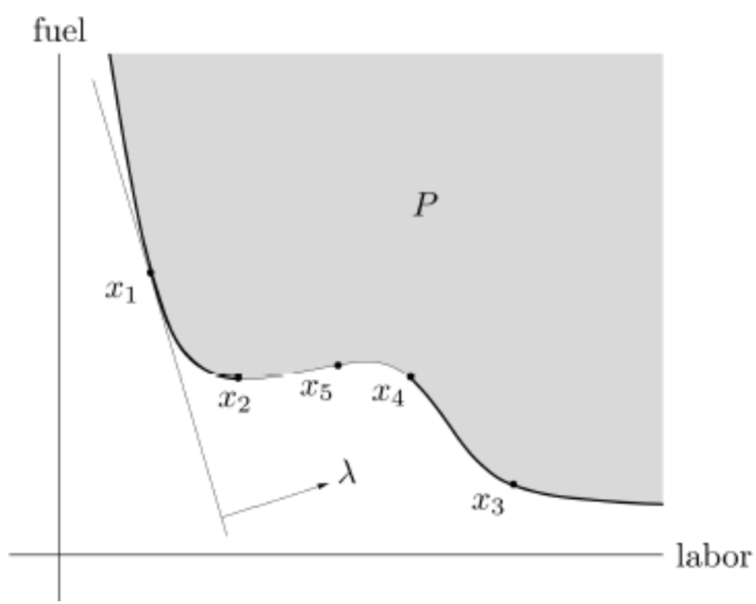
Dual cones and generalized inequalities

- Example: Pareto optimal production frontier

Minimize

$$\boldsymbol{\lambda}^T \mathbf{x} = \boldsymbol{\lambda}_1 \mathbf{x}_1 + ... + \boldsymbol{\lambda}_n \mathbf{x}_n$$

over the set P of production vectors, using any $\boldsymbol{\lambda} > \mathbf{0}$.



Chapter 4: Convex Functions

- Basic properties and examples
- Operations that preserve convexity
- The conjugate function
- A little about nonconvex analysis

Basic properties and examples

A function $f : \mathbb{R}^n \to \mathbb{R}$ is *convex* if $\operatorname{dom} f$ is a convex set and if for all $\mathbf{x}$, $\mathbf{y} \in \operatorname{dom} f$, and θ with $0 \leq \theta \leq 1$, we have

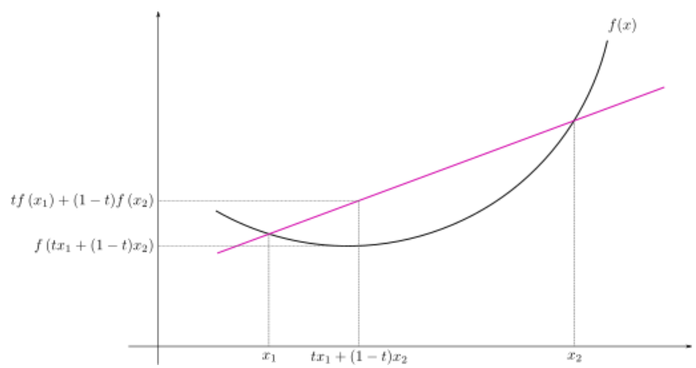
$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y}). \tag{1}$$

A function f is *strictly convex* if strict inequality holds in (1) whenever $\mathbf{x} \neq \mathbf{y}$ and $0 < \theta < 1$.

A function f is *strongly convex* if

$$f(\theta\mathbf{x} + (1-\theta)\mathbf{y}) \leq \theta f(\mathbf{x}) + (1-\theta) f(\mathbf{y}) - \frac{\theta(1-\theta)\overset{\boxed{modulus}}{\boxed{\mu}}}{2}\|\mathbf{y}-\mathbf{x}\|^2, \quad \forall\theta \in [0,1]. \tag{2}$$

f is *concave*, *strictly concave*, *strongly concave* if $-f$ is convex, strictly convex, strongly convex. A function is both convex and concave iff it is an affine function.



Basic properties and examples

A convex function is continuous on the relative interior of its domain; it can have discontinuities only on its relative boundary.

Theorem 1 (Rademacher's Theorem)**.** *A convex function is differentiable almost everywhere on the relative interior of its domain.*

- Extended-value extensions

If f is convex we define its *extended-value extension* $\tilde{f} : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ by

$$\tilde{f}(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \mathbf{x} \in \operatorname{dom} f \\ \infty, & \mathbf{x} \notin \operatorname{dom} f. \end{cases}$$

We will use the same symbol to denote a convex function and its extension.

Example: Indicator function of a convex set

$$\begin{aligned} &\min_{\mathbf{x}} \ f(\mathbf{x}), \\ &s.t.\ \mathbf{x} \in \mathcal{C}. \end{aligned} \quad \Longleftrightarrow \quad \min_{\mathbf{x}} \ f(\mathbf{x}) + \tilde{I}_{\mathcal{C}}(\mathbf{x}).$$

Basic properties and examples

- First-order conditions

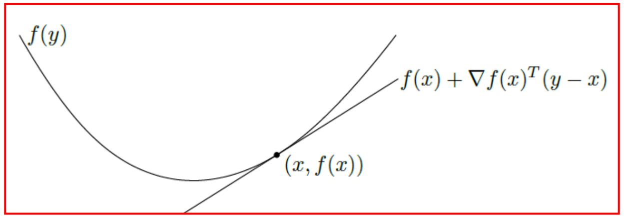
Suppose f is differentiable. Then f is convex iff $\mathrm{dom}\, f$ is convex and

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle \tag{1}$$

holds for all $\mathbf{x}$, $\mathbf{y} \in \mathrm{dom}\, f$.

Proof. If f is convex, then $f((1-\alpha)\mathbf{x} + \alpha \mathbf{y}) \leq (1-\alpha) f(\mathbf{x}) + \alpha f(\mathbf{y})$, which can be rewritten as

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \frac{f(\mathbf{x} + \alpha(\mathbf{y} - \mathbf{x})) - f(\mathbf{x})}{\alpha},$$



Letting $\alpha \to 0^+$, we have (1). If (1) holds, we have

$$f(\alpha \mathbf{x} + (1-\alpha)\mathbf{y}) \leq f(\mathbf{x}) - (1-\alpha)\langle \nabla f(\alpha \mathbf{x} + (1-\alpha)\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle,$$
$$f(\alpha \mathbf{x} + (1-\alpha)\mathbf{y}) \leq f(\mathbf{y}) + \alpha \langle \nabla f(\alpha \mathbf{x} + (1-\alpha)\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle.$$

Multiplying the first inequality with α and the second with $(1-\alpha)$ and adding them together, we obtain $f(\alpha \mathbf{x} + (1-\alpha)\mathbf{y}) \leq \alpha f(\mathbf{x}) + (1-\alpha) f(\mathbf{y})$.

Basic properties and examples

If $\nabla f(\mathbf{x}) = \mathbf{0}$, then for all $\mathbf{y} \in \operatorname{dom} f$, $f(\mathbf{y}) \geq f(\mathbf{x})$, *i.e.*, $\mathbf{x}$ is a global minimizer of f.

Strictly convex:

$$f(\mathbf{y}) > f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle, \quad \text{if } \mathbf{y} \neq \mathbf{x}. \tag{1}$$

Proof. $f(\mathbf{y}) > f(\mathbf{x}) + \frac{f(\mathbf{x}+\alpha(\mathbf{y}-\mathbf{x}))-f(\mathbf{x})}{\alpha}, \quad \forall \alpha \in (0,1)$. For all $\alpha \in (0,1)$ by the convexity we have $f(\mathbf{x} + \alpha(\mathbf{y} - \mathbf{x})) - f(\mathbf{x}) \geq \alpha \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle$. Thus $\langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle = \inf_{\alpha \in (0,1)} \frac{f(\mathbf{x}+\alpha(\mathbf{y}-\mathbf{x}))-f(\mathbf{x})}{\alpha}$. If there exists $\alpha \in (0,1)$ such that $\frac{f(\mathbf{x}+\alpha(\mathbf{y}-\mathbf{x}))-f(\mathbf{x})}{\alpha} > \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle$, then (1) holds. Otherwise,

$$\frac{f(\mathbf{x} + \alpha(\mathbf{y} - \mathbf{x})) - f(\mathbf{x})}{\alpha} = \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle, \quad \forall \alpha \in (0,1).$$

So $f(\mathbf{x} + \alpha(\mathbf{y} - \mathbf{x}))$ is a linear function of $\alpha \in (0,1)$ and f cannot be strictly convex.

Strongly convex: $f(\mathbf{y}) \geq f(\mathbf{x}) + \langle \nabla f(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \frac{\mu}{2} \|\mathbf{y} - \mathbf{x}\|^2$.
Proof. Follow the proof of convexity.