

# Homework (4)

1. Let  $C \subseteq \mathbb{R}^n$  be a convex set, with  $\mathbf{x}_1, \dots, \mathbf{x}_k \in C$ , and let  $\theta_1, \dots, \theta_k \in \mathbb{R}$  satisfy  $\theta_i \geq 0$ ,  $\theta_1 + \dots + \theta_k = 1$ . Show that  $\theta_1 \mathbf{x}_1 + \dots + \theta_k \mathbf{x}_k \in C$ .
2. A set  $C$  is midpoint convex if whenever two points  $\mathbf{a}, \mathbf{b}$  are in  $C$ , the average or midpoint  $(\mathbf{a} + \mathbf{b})/2$  is in  $C$ . Prove that if  $C$  is closed and midpoint convex, then  $C$  is convex.
3. Which of the following sets  $S$  are polyhedra? If possible, express  $S$  in the form  $S = \{\mathbf{x} | \mathbf{Ax} \leq \mathbf{b}, \mathbf{Fx} = \mathbf{g}\}$ .
  - (a)  $S = \{y_1 \mathbf{a}_1 + y_2 \mathbf{a}_2 | -1 \leq y_1 \leq 1, -1 \leq y_2 \leq 1\}$ , where  $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^n$ .
  - (b)  $S = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} \succeq \mathbf{0}, \mathbf{1}^T \mathbf{x} = 1, \sum_{i=1}^n x_i a_i = b_1, \sum_{i=1}^n x_i a_i^2 = b_2\}$ , where  $a_1, \dots, a_n \in \mathbb{R}$  and  $b_1, b_2 \in \mathbb{R}$ .
  - (c)  $S = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} \succeq \mathbf{0}, \mathbf{x}^T \mathbf{y} \leq 1 \text{ for all } \mathbf{y} \text{ with } \|\mathbf{y}\|_2 = 1\}$ .
  - (d)  $S = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} \succeq \mathbf{0}, \mathbf{x}^T \mathbf{y} \leq 1 \text{ for all } \mathbf{y} \text{ with } \sum_{i=1}^n |y_i| = 1\}$ .

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4. Which of the following sets are convex?

(a) A slab, i.e., a set of the form  $\{\mathbf{x} \in \mathbb{R}^n \mid \alpha \leq \mathbf{a}^T \mathbf{x} \leq \beta\}$ .

(b) A wedge, i.e.,  $\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{a}_1^T \mathbf{x} \leq b_1, \mathbf{a}_2^T \mathbf{x} \leq b_2\}$ .

(c) The set of points closer to a given point than a given set, i.e.,  $\{\mathbf{x} \mid \|\mathbf{x} - \mathbf{x}_0\|_2 \leq \|\mathbf{x} - \mathbf{y}\|_2 \text{ for all } \mathbf{y} \in S\}$  where  $S \subseteq \mathbb{R}^n$ .

(d) The set of points closer to one set than another, i.e.,  $\{\mathbf{x} \mid \mathbf{dist}(\mathbf{x}, S) \leq \mathbf{dist}(\mathbf{x}, T)\}$ , where  $S, T \subseteq \mathbb{R}^n$ , and

$$\mathbf{dist}(\mathbf{x}, S) = \inf\{\|\mathbf{x} - \mathbf{z}\|_2 \mid \mathbf{z} \in S\}.$$

(e) The set  $\{\mathbf{x} \mid \mathbf{x} + S_2 \subseteq S_1\}$ , where  $S_1, S_2 \subseteq \mathbb{R}^n$  with  $S_1$  convex.

(f) The set of points whose distance to  $\mathbf{a}$  does not exceed a fixed fraction  $\theta$  of the distance to  $\mathbf{b}$ , i.e., the set  $\{\mathbf{x} \mid \|\mathbf{x} - \mathbf{a}\|_2 \leq \theta \|\mathbf{x} - \mathbf{b}\|_2\}$  ( $\mathbf{a} \neq \mathbf{b}$  and  $0 \leq \theta \leq 1$ ).

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5. Find the convex hull of the set  $\{\mathbf{u}\mathbf{u}^T \mid \|\mathbf{u}\| = 1\}$ .
6. Consider the set of rank- $k$  outer products, defined as  $\{\mathbf{X}\mathbf{X}^T \mid \mathbf{X} \in \mathbb{R}^{n \times k}, \text{rank}\mathbf{X} = k\}$ . Describe its conic hull in simple terms.
7. Give an expression  $\bigcap_{\alpha \in \mathcal{A}} S_\alpha$  for the unit ball  $\{\mathbf{X} \mid \|\mathbf{X}\|_2 \leq 1\}$ .