Homework (13)

1. [Nonsingularity of the KKT matrix.] Consider the KKT matrix

$$\begin{bmatrix} \mathbf{P} & \mathbf{A}^T \\ \mathbf{A} & \mathbf{0} \end{bmatrix},$$

where $\mathbf{P} \in \mathbb{S}^n_+$, $\mathbf{A} \in \mathbb{R}^{p \times n}$, and $\operatorname{rank} \mathbf{A} = p < n$. Show that each of the following statements is equivalent to nonsingularity of the KKT matrix:

- $\mathcal{N}(\mathbf{P}) \cap \mathcal{N}(\mathbf{A}) = \{\mathbf{0}\}$.
- $\mathbf{A}\mathbf{x} = \mathbf{0}$, $\mathbf{x} \neq \mathbf{0} \Longrightarrow \mathbf{x}^T\mathbf{P}\mathbf{x} > 0$.
- $\mathbf{F}^T\mathbf{P}\mathbf{F} \succ \mathbf{0}$, where $\mathbf{F} \in \mathbb{R}^{n \times (n-p)}$ is a matrix for which $\mathcal{R}(\mathbf{F}) = \mathcal{N}(\mathbf{A})$.
- $\mathbf{P} + \mathbf{A}^T\mathbf{Q}\mathbf{A} \succ \mathbf{0}$ for some $\mathbf{Q} \succeq \mathbf{0}$.

Homework (13)

2. [Projected gradient method.]

- Let $(\mathbf{v}, \mathbf{w})$ be the unique solution of

$$\begin{bmatrix} \mathbf{I} & \mathbf{A}^T \\ \mathbf{A} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} -\nabla f(\mathbf{x}) \\ \mathbf{0} \end{bmatrix}.$$

  Show that

$$\mathbf{v} = \underset{\mathbf{Au}=\mathbf{0}}{\operatorname{argmin}} \|-\nabla f(\mathbf{x}) - \mathbf{u}\|_2, \quad \mathbf{w} = \underset{\mathbf{y}}{\operatorname{argmin}} \|\nabla f(\mathbf{x}) + \mathbf{A}^T\mathbf{y}\|_2.$$

- What is the relationship between the projected negative gradient $\mathbf{v}$ and the negative gradient of the reduced unconstrained problem $f(\mathbf{F}\mathbf{z} + \tilde{\mathbf{x}})$, assuming $\mathbf{F}^T\mathbf{F} = \mathbf{I}$?

Homework (13)

3. [Equality constrained entropy maximumization.] We consider the equality constrained entropy maximization problem

$$\min_{\mathbf{x}} f(\mathbf{x}) = \sum_{i=1}^{n} x_i \log x_i, \quad s.t. \quad \mathbf{A}\mathbf{x} = \mathbf{b}.$$

Use the following methods to solve it at $p = 100$ and $n = 500$ (you may generate $\mathbf{A}$ and $\mathbf{b}$ randomly):

- Direct projected gradient, with inexact line search.
- Elimiate equality constraint.
- Dual approach.
- Newton's method.

Write a report to describe your settings and compare their performance (numerical accuracy vs. iteration number). Codes should also be handed in.