Homework (11)

1. With $f(\mathbf{x}) := x_1^2 + x_2^2$ for $\mathbf{x} \in \mathbb{R}^2$ consider

$$(P) \begin{cases} \min\limits_{\mathbf{x}} f(\mathbf{x}) \\ -x_2 \leq 0 \\ x_1^3 - x_2 \leq 0 \\ x_1^3(x_2 - x_1^3) \leq 0 \end{cases}.$$

a) Check whether SCQ holds.

b) Find its dual problem. Check whether strong duality holds.

2. Find the point $\mathbf{x} \in \mathbb{R}^2$ that lies closest to the point $\mathbf{p} := (2, 3)^T$ under the constraints $g_1(\mathbf{x}) := x_1 + x_2 \leq 0$ and $g_2(\mathbf{x}) := x_1^2 - 4 \leq 0$.

a) Check whether the problem fulfills SCQ.

b) Find its dual problem. Check whether the strong duality holds.

Homework (11)

3. Given a support vector machine:

$$\min_{\mathbf{w}} \frac{1}{2}\|\mathbf{w}\|^2,$$
$$s.t.\ y_i(\langle \mathbf{w}, \mathbf{x}_i \rangle + \beta) \geq 1, (i = 1, \cdots, m).$$

   a) Check whether the problem fulfills SCQ. What does SCQ mean in this scenario?

   b) Find its dual problem.

4. Express the dual problem of

$$\min_{\mathbf{x}} \mathbf{c}^T \mathbf{x}$$
$$s.t.\ f(\mathbf{x}) \leq 0,$$

with $\mathbf{c} \neq \mathbf{0}$, in terms of the conjugate f^*. Explain why the problem you give is convex. We do not assume f is convex.

Homework (11)

5. The following problems arise in experiment design.

   (a) D-optimal design.

$$\min_{\mathbf{x}} \ \log\det\left(\sum_{i=1}^{p} x_i \mathbf{v}_i \mathbf{v}_i^T\right)^{-1}, \quad s.t.\ \mathbf{x} \geq \mathbf{0}, \mathbf{1}^T\mathbf{x} = 1.$$

   (b) A-optimal design.

$$\min_{\mathbf{x}} \ \mathrm{tr}\left(\sum_{i=1}^{p} x_i \mathbf{v}_i \mathbf{v}_i^T\right)^{-1}, \quad s.t.\ \mathbf{x} \geq \mathbf{0}, \mathbf{1}^T\mathbf{x} = 1.$$

   The domain of both problems is $\left\{\mathbf{x} \middle| \sum_{i=1}^{p} x_i \mathbf{v}_i \mathbf{v}_i^T \succ \mathbf{0}\right\}$.

   Derive dual problems by first introducing a new variable $\mathbf{X} \in \mathbb{S}^n$ and an equality constraint $\mathbf{X} = \sum_{i=1}^{p} x_i \mathbf{v}_i \mathbf{v}_i^T$, and then applying Lagrange duality. Simplify the dual problems as much as you can.

Homework (11)

6. We consider the convex piecewise-linear minimization problem

$$\min_{\mathbf{x}} \max_{i=1,\cdots,m} (\mathbf{a}_i^T \mathbf{x} + b_i) \tag{1}$$

with variable $\mathbf{x} \in \mathbb{R}^n$.

(a) Derive a dual problem, based on the Lagrange dual of the equivalent problem

$$\begin{aligned} &\min_{\mathbf{x}} \max_{i=1,\cdots,m} y_i \\ &s.t.\ \mathbf{a}_i^T \mathbf{x} + b_i = y_i, \quad i = 1, \cdots, m, \end{aligned}$$

with variables $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$.

(b) Formulate the piecewise-linear minimization problem (1) as an LP, and form the dual of the LP. Relate the LP dual to the dual obtained in part (a).

Homework (11)

(c) Suppose we approximate the objective function in (1) by the smooth function

$$f_0(\mathbf{x}) = \log\left(\sum_{i=1}^{m} \exp(\mathbf{a}_i^T\mathbf{x} + b_i)\right),$$

and solve the unconstrained geometric program

$$\min_{\mathbf{x}} \log\left(\sum_{i=1}^{m} \exp(\mathbf{a}_i^T\mathbf{x} + b_i)\right). \tag{2}$$

Let p_{pwl}^* and p_{gp}^* be the optimal values of (1) and (2), respectively. Show that

$$0 \leq p_{gp}^* - p_{pwl}^* \leq \log m.$$

Homework (11)

(d) Derive similar bounds for the difference between p^*_{pwl} and the optimal value of

$$\min_{\mathbf{x}} (1/\gamma) \log \left(\sum_{i=1}^{m} \exp(\gamma(\mathbf{a}_i^T \mathbf{x} + b_i)) \right),$$

where $\gamma > 0$ is a parameter. What happens as we increase γ?