Homework (18)

1. Consider the equality constrained least-squares problem

$$\begin{aligned} &\min_{\mathbf{x}} \ \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2, \\ &s.t. \ \mathbf{G}\mathbf{x} = \mathbf{h}, \end{aligned}$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$ with rank $\mathbf{A} = n$, and $\mathbf{G} \in \mathbb{R}^{p \times n}$ with rank $\mathbf{G} = p$. Give the KKT conditions, and derive expressions for the primal solution $\mathbf{x}^*$ and the dual solution $\boldsymbol{\nu}^*$.

2. Show that the strong duality holds for the problem

$$\begin{aligned} &\min_{\mathbf{x}} \ -3x_1^2 + x_2^2 + 2x_3^2 + 2(x_1 + x_2 + x_3) \\ &s.t. \ x_1^2 + x_2^2 + x_3^2 = 1, \end{aligned}$$

even though the problem is not convex. Derive the KKT conditions. Find all solutions $\mathbf{x}$, $\boldsymbol{\nu}$ that satisfy the KKT conditions. Which pair corresponds to the optimum?

Homework (18)

3. Consider a convex problem with no equality constraints,

$$\min_{\mathbf{x}} \; f_0(\mathbf{x})$$
$$s.t. \; f_i(\mathbf{x}) \leq 0, \quad i = 1, \cdots, m.$$

Assume that $\mathbf{x}^* \in \mathbb{R}^n$ and $\boldsymbol{\lambda}^* \in \mathbb{R}^m$ satisfy the KKT conditions

$$\begin{aligned} f_i(\mathbf{x}^*) &\leq 0, \quad i = 1, \cdots, m \\ \lambda_i^* &\geq 0, \quad i = 1, \cdots, m \\ \lambda_i^* f_i(\mathbf{x}^*) &= 0, \quad i = 1, \cdots, m \end{aligned}$$

$$\nabla f_0(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i^* \nabla f_i(x^*) = 0.$$

Show that

$$\nabla f_0(\mathbf{x}^*)^T (\mathbf{x} - \mathbf{x}^*) \geq 0$$

for all feasible $\mathbf{x}$.

Homework (18)

4. Derive a dual problem for

$$\min_{\mathbf{x}} \sum_{i=1}^{N} \|\mathbf{A}_i\mathbf{x} + \mathbf{b}_i\|_2 + (1/2)\|\mathbf{x} - \mathbf{x}_0\|_2^2.$$

The problem data are $\mathbf{A}_i \in \mathbb{R}^{m_i \times n}$, $\mathbf{b}_i \in \mathbb{R}^{m_i}$, and $\mathbf{x}_0 \in \mathbb{R}^n$. First introduce new variables $\mathbf{y}_i \in \mathbb{R}^{m_i}$ and equality constraints $\mathbf{y}_i = \mathbf{A}_i\mathbf{x} + \mathbf{b}_i$.

5. Consider the optimization problem

$$\min_{x,y} \; e^{-x}$$

$$s.t. \; x^2/y \leq 0$$

with variables x and y, and domain $\mathcal{D} = \{f(x,y)|y > 0\}$.

Homework (18)

(a) Verify that this is a convex optimization problem. Find the optimal value.

(b) Give the Lagrange dual problem, and find the optimal solution λ^* and optimal value d^* of the dual problem. What is the optimal duality gap?

(c) Does Slater's condition hold for this problem?

(d) What is the optimal value $p^*(u)$ of the perturbed problem

$$\min_{u} \; e^{-x}$$
$$s.t. \; x^2/y \leq u,$$

as a function of u? Verify that the global sensitivity inequality

$$p^*(u) \geq p^*(0) - \lambda^* u$$

does not hold.