Homework (5)

1. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be the linear-fractional function

$$f(\mathbf{x}) = (\mathbf{A}\mathbf{x} + \mathbf{b})/(\mathbf{c}^T\mathbf{x} + d), \ \mathbf{dom}\ f = \{\mathbf{x}|\mathbf{c}^T\mathbf{x} + d > 0\}.$$

For each of the following sets $C \subseteq \mathbb{R}^n$, give a simple description of $f^{-1}(C)$.

(a) The halfspace $C = \{\mathbf{y}|\ \mathbf{g}^T\mathbf{y} \leq h\}$ (with $g \neq 0$).

(b) The polyhedron $C = \{\mathbf{y}|\ \mathbf{G}\mathbf{y} \preceq \mathbf{h}\}$.

(c) The ellipsoid $\{\mathbf{y}|\ \mathbf{y}^T\mathbf{P}^{-1}\mathbf{y} \leq 1\}$ (where $\mathbf{P} \in \mathbb{S}^n_{++}$).

(d) The solution set of a linear matrix inequality,$C = \{\mathbf{y}|\ \mathbf{y}_1\mathbf{A}_1 + ... + \mathbf{y}_n\mathbf{A}_n \preceq \mathbf{B}\}$, where $\mathbf{A}_1, ...,\ \mathbf{A}_n, \mathbf{B} \in \mathbb{S}^p$.

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2. Give an explicit description of the positive semidefinite cone $\mathbb{S}^n_+$, in terms of the matrix coefficients and ordinary inequalities, for $n = 1, 2, 3$. To describe a general element of $\mathbb{S}^n$, for $n = 1, 2, 3$, use the notation

$$x_1, \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}, \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix}.$$

3. Suppose $K \subseteq \mathbb{R}^2$ is a closed convex cone.

(a) Give a simple description of K in terms of the polar coordinates of its elements ($\mathbf{x} = r(\cos\phi, \sin\phi)^T$ with $r \geq 0$).

(b) When is K pointed?

(c) When is K proper (hence, defines a generalized inequality)? Draw a plot illustrating what $\mathbf{x} \preceq_K \mathbf{y}$ means when K is proper.

Homework (5)

4. A matrix $\mathbf{X} \in \mathbb{S}^n$ is called copositive if $\mathbf{z}^T\mathbf{X}\mathbf{z} \geq 0$ for all $\mathbf{z} \geq \mathbf{0}$. Verify that the set of copositive matrices is a proper cone.

5. Prove the Separating Hyperplane Theorem under the case that the distance between two convex sets C and D is 0.

6. Suppose that C and D are disjoint subsets of $\mathbb{R}^n$. Consider the set of $(\mathbf{a}, b) \in \mathbb{R}^{n+1}$ for which $\mathbf{a}^T\mathbf{x} \leq b$ for all $\mathbf{x} \in C$, and $\mathbf{a}^T\mathbf{x} \geq b$ for all $\mathbf{x} \in D$. Show that this set is a convex cone (which is the singleton $\{\mathbf{0}\}$ if there is no hyperplane that separates C and D).