Response to the Comments on "Fundamental Limits of Reconstruction-Based Superresolution Algorithms under Local Translation"

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Abstract—Wang and Feng [1] pointed out that the deduction in [2] overlooked the validity of the perturbation theorem used in [2]. In this paper, we show that, when the perturbation theorem is invalid, the probability of successful superresolution is very low. Therefore, we only have to derive the limits under the condition that validates the perturbation theorem, as done in [2].

Index Terms—Superresolution, reconstruction-based algorithms, perturbation theory.

WANG and Feng [1] pointed out that the perturbation theorem cited by [2] is only valid when $\kappa\varepsilon_{\mathbf{P}} < 1$ (due to page limits, please refer to [2] for notations), but this condition was overlooked by [2]. Therefore, they doubted the validity of the fundamental limits of reconstruction-based superresolution (SR) algorithms claimed by Lin and Shum [2]. In the following, we will show that, when $\kappa\varepsilon_{\mathbf{P}} \geq 1$, the probability of successful SR will be very low. Therefore, basically, we need not consider the case where $\kappa\varepsilon_{\mathbf{P}} \geq 1$.

When $\kappa\varepsilon_{\mathbf{P}} \geq 1$, the right-hand side of (7) in [2] is indeed no longer a valid bound for $||\delta\mathbf{H}||$. Fortunately, Theorem 5.3.1 of [3] is still of help to bound $||\delta\mathbf{H}||$. Using the notations in [2], the theorem can be written as:

Theorem. *If* $\sin\theta = \frac{||\mathbf{r}||}{||\mathbf{L}-\mathbf{E}||} \neq 1$, *then*

$$||\delta\mathbf{H}|| \leq \left[\hat{\varepsilon}\left(\frac{2\kappa}{\cos\theta} + \tan\theta\cdot\kappa^2\right) + O\big(\hat{\varepsilon}^2\big)\right]||\mathbf{H}||,$$
$$\text{where } \hat{\varepsilon} = \max\left\{\varepsilon_{\mathbf{P}}, \frac{||\delta\mathbf{E}||}{||\mathbf{L}-\mathbf{E}||}\right\}.$$

In our case, $\mathbf{r} \equiv \mathbf{0}$ (see footnote 3 of [2]). Hence, this theorem applies and

$$||\delta\mathbf{H}|| \leq \left[2\kappa\hat{\varepsilon} + O\big(\hat{\varepsilon}^2\big)\right]||\mathbf{H}||. \tag{1}$$

Using the notations in [2], $||\delta\mathbf{H}|| = \delta_h N_h$ and $||\mathbf{H}|| = \sigma_h N_h$, where N_h is the square root of the size of $\mathbf{H}$. As we have argued in [2], δ_h should be below a threshold T so that SR can be successful, and this threshold T is usually very small. A reference upper bound of T is 18.55, as Fig. 5 of [2] exemplifies.

As $\delta\mathbf{H}$ depends on the independent random matrix/vector $\delta\mathbf{P}$ and $\delta\mathbf{E}$, it must also be a random vector in the ball $B_R = \{\mathbf{x} : ||\mathbf{x}|| \leq RN_h\}$, where $R > 2\kappa\hat{\varepsilon}\sigma_h$ thanks to (1). Note that σ_h cannot be very small (please refer to Section 4 of [2], where a reference lower bound of σ_h is 15). Otherwise, the ground-truth high resolution image is of very low contrast. Hence, SR is not possible. Therefore, if $\kappa\varepsilon_{\mathbf{P}} \geq 1$, then it is guaranteed that $T \ll R$. (Please compare the reference bounds of T and σ_h. Also, note that $\kappa\hat{\varepsilon} \geq \kappa\varepsilon_{\mathbf{P}}$.) Then, we can choose an R' such that $T < R' < R$ and R' is close to T. As T is small, the distribution of $\delta\mathbf{H}$ should be close to uniform in the small ball $B_{R'}$. Moreover, $\delta\mathbf{H}$ cannot concentrate around $\mathbf{0}$ (otherwise, $\delta\mathbf{P}$ and $\delta\mathbf{E}$ will be close to correlated, see Section 2.2 of [2]). These two aspects ensure that the probability of $\delta_h \leq T$ can be estimated as:

$$P \leq \frac{P(\delta_h \leq T)}{P(\delta_h \leq R')} = \frac{P(||\delta\mathbf{H}|| \leq TN_h)}{P(||\delta\mathbf{H}|| \leq R'N_h)} \approx \frac{\text{volume of } B_T}{\text{volume of } B_{R'}} = \left(\frac{T}{R'}\right)^{N_h^2}.$$

As $N_h^2 \gg 1$ when doing SR, though $\frac{T}{R'}$ may not be far below 1, P will still be very low.

Note that the above arguments agree with Theorem 3 of [4]. It states that the volume of possible solutions grows at a power of N_h^2. Therefore, the probability of finding acceptable solutions should decrease at a power of N_h^2. Again, that $N_h^2 \gg 1$ plays the key role to ensure the low probability of successful SR.

Now, we also use matrices (4) and (6) of [2], whose $\kappa\varepsilon_{\mathbf{P}} = 7.49$ [1], to give a casual example to show that SR is nearly impossible when $\kappa\varepsilon_{\mathbf{P}} \geq 1$. We choose $\mathbf{H} = (10, 20, 30, \cdots, 90)^t$, $\mathbf{L} = \mathrm{round}(\mathbf{PH})$, and $\delta\mathbf{E} = \mathbf{E} = \mathbf{L} - \mathbf{PH}$, then we have

$$\begin{aligned}\tilde{\mathbf{H}} = (&-115.82, 124.91, -162.86, 153.36,\\ &-39.99, 222.05, -64.12, 176.57, -89.61)^t\end{aligned}$$

and $\delta_h = 137.76$. This δ_h is very large and $\tilde{\mathbf{H}}$ is also an undesired SR result.

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