Nash Equilibria for Quadratic Voting

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Abstract

A group of N individuals must choose between two collective alternatives. Under Quadratic Voting (QV), agents buy votes in favor of their preferred alternative from a clearing house, paying the square of the number of votes purchased, and the sum of all votes purchased determines the outcome. Each agent is assumed to have a private value that determines her utility; these values are assigned by simple random sampling from a probability distribution F with a smooth density on a compact interval $[\underline{u}, \overline{u}]$. Under these assumptions, the structure of the Bayes-Nash equilibrium is described when N is large. The results imply that the quadratic voting mechanism is asymptotically efficient.

Keywords: social choice, collective decisions, large markets, costly voting, vote trading, Bayes-Nash equilibrium.

1 Introduction and Main Results

Consider a binary collective-decision problem in which a group of N individuals must choose between two alternatives. Each individual i has a privately known value u_i that determines her willingness to pay for one alternative over the other; positive values indicate affinity for outcome +1, negative values for outcome -1. Quadratic Voting (QV) is a simple and detail-free mechanism designed to maximize

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utilitarian efficiency in this setting.¹ In this system, individuals buy votes (either negative or positive, depending on which alternative is favored) from a clearing house, paying the square of the number of votes purchased. The sum of all votes purchased then determines the outcome. The utility (payoff) of the outcome to an individual with value u is +u if outcome + is adopted, but -u if outcome - is adopted.²

The heuristic rationale for QV is quite simple. The marginal benefit to a voter of an additional vote is her value multiplied by her *marginal pivotality* (roughly, the perceived probability that an additional unit of vote will sway the decision). She maximizes utility by equating this marginal utility to the linear marginal cost of a vote. Therefore, if voters share the same marginal pivotality, they will buy votes in proportion to their values, thus bringing about utilitarian efficiency. Furthermore, the quadratic cost function is the *unique* cost function with this property. This argument is explained in further detail in Subsection 1.3.

Variations of this rationale have been used to justify quadratic mechanisms in a number of related collective-decision-making problems. However, to our knowledge, this heuristic rationale has never been translated into a rigorous argument for efficiency in the sort of non-cooperative, incomplete information game theoretic model in which mechanisms for allocating private goods have been studied at least since the work of Myerson [11]. In fact, as we will show, in the modified setting of quadratic voting that we will consider the crucial *ansatz* of this rationale – that in equilibrium all voters will have the same marginal pivotality – is false. As a result, formal equilibrium analysis is a far more subtle task than the heuristic argument of the previous paragraph might suggest. Nevertheless, we will show that for voters with values in the "bulk" of the distribution, the marginal pivotality is approximately constant.³

To our knowledge the use of quadratic pricing for collective decision-making was first suggested by Groves and Ledyard [6], who proposed it as a Nash implementation of the optimal level of continuous public goods under complete information that avoids the fragility of previously suggested efficient mechanisms. Hylland and Zeckhauser [8] provided the first variant of the heuristic rationale above to uniquely justify quadratic pricing mechanism and proposed an iterative

¹Clearly many other objectives are possible for this problem, and many involve distributional considerations. However, we focus on a utilitarian objective because it is the one most extensively studied in the literature [2, 5].

²Our results will apply to a modified version of the problem in which the utility is "smoothed" in such a way that each voter's utility is a continuous function of the vote total. See section 1.1 for details.

³Theorems of Kahn et al. [9] and Al-Najjar et al. [1] imply marginal pivotality must converge to zero as $N \to \infty$. Our results will show that, with probability approaching 1, the *ratio* between the marginal pivotalities of two randomly chosen voters will be close to 1.

procedure that they conjectured would converge to [6]'s complete information optimum in the presence of private information. In a previous version of this paper, Weyl [13] first proposed the use of QV for binary collective decision problems, and conjectured that it would lead to asymptotically efficient decisions in the environment we consider based on an (independently discovered) extension of [8]'s heuristic rationale. Goeree and Zhang [4] independently suggested using a detail-based, approximately direct variant of QV in the special case where values are sampled from zero-mean normal distributions, and derived an equilibrium in the case N=2.

1.1 Model Assumptions

We consider an independent symmetric private values environment with N voters $i=1,\ldots,N$. Each voter i is characterized by a value, u_i ; these values are drawn independently from a continuous probability distribution F with C^{∞} , strictly positive density f supported by a finite closed interval $[\underline{u},\overline{u}]$. Each individual knows her own value, but not the values of any of the other N-1 voters; however, the sampling distribution F is known to all. For normalization, we assume the numeraire has been scaled so that $\min(-\underline{u},\overline{u}) \geq 1$. We denote by μ , σ^2 , and μ_3 , respectively, respectively the mean, variance, and raw third moments of u under F.

We consider a variant of the payoff described above, in which the utility of the outcome is "smoothed". ⁵ Each voter i chooses a number of votes $v_i \in \mathbb{R}$ to buy, and pays v_i^2 dollars for these. The payoff to voter i is then

$$\Psi(V)u_i, \quad \text{where } V = \sum_{i=1}^N v_i$$
 (1)

is the vote total and $\Psi:\mathbb{R}\to[-1,1]$ is an odd, nondecreasing, C^∞ function such

⁴The assumption that the density f is positive at the endpoints $\underline{u}, \overline{u}$ is of critical importance for our main results, as "extremists" play a crucial role in the Bayes-Nash equilibria for the game. Our methods would extend to densities f that vanish at one or both of the endpoints, but the nature of the Bayes-Nash equilibria changes in these cases.

⁵Although both the discrete binary choice set-up of [13] and the continuous public goods model of [8] helped inspire this model, it differs from both. Consequently, our results have no direct implications for those models. It differs from [13]'s model in that the outcome is smoothed rather than jumping discontinuously at 0. It differs in a variety of respects from [8]'s, notably in that utility is linear in the common and bounded outcome, whereas [8] assume strictly concave preferences with heterogeneous ideal points and an outcome that may take values in the full real space. [8] also consider a multidimensional issue space with no access to transfers and an iterative procedure to converge to this outcome, none of which feature in our model.

that for some $\delta > 0$,

$$\Psi(x) = \operatorname{sgn}(x) \quad \text{for all} \quad |x| \ge \delta;$$
 (2)

$$\psi(x) := \Psi'(x) > 0 \quad \text{for all } x \in (-\delta, \delta); \tag{3}$$

$$\psi'(x) > 0$$
 for all $x \in (-\delta, 0)$; and (4)

$$\psi$$
 has a single inflection point in $(-\delta, 0)$. (5)

We shall refer to Ψ as the *payoff function*, because it determines the quantity by which values u_i are multiplied to obtain the allocative component of each individual's utility.⁶ Conditional on the values $\{v_i\}$, individual i earns expected utility

$$\Psi(V)u_i - v_i^2. \tag{6}$$

Thus, in a *type-symmetric Bayes-Nash equilibrium*⁷, a voter with value u will maximize

$$E[u\Psi(V_{-1}+v)]-v^2,$$
 (7)

where $V_{-1} := \sum_{i \neq 1} v_i$ is the *one-out vote total*, the sum of all votes cast by all but a single individual. For notational convenience we will henceforth write n = N - 1 for the number of voters minus one and $S_n = V_{-1}$ for the vote total of the first n voters. For brevity, we shall refer to *type-symmetric Bayes-Nash equilibria* as *Nash equilibria*.

We define the expected inefficiency as

$$EI \equiv \frac{1}{2} - \frac{E[U\Psi(V)]}{2E[|U|]} \in [0, 1],$$

where $U \equiv \sum_i u_i$. This measure is the unique negative monotone linear functional of aggregate utility realized $U\Psi(V)$ that is normalized to lie in the unit interval.

1.2 Existence of Equilibria

Proposition 1. For any N > 1 a monotone increasing, type-symmetric Bayes-Nash Equilibrium v exists.

 $^{^6}$ The assumptions on the payoff function Ψ are primarily for mathematical convenience. However, there are some circumstances where a smoothing of the payoff for vote totals near 0 might be natural: for instance, (i) in some close elections, it might be necessary for the winning side to form a coalition with some of the losers to form a functioning majority; or (ii) for vote totals near 0, a recount might be necessary, leading to the possibility that the winning side might be overturned.

⁷ See section 2 for the definition and a proof that Bayes-Nash equilibria use non-randomized strategies. Roughly, a type-symmetric equilibrium is a function v(u) such that, if all players use the rule $u \mapsto v(u)$ for buying votes then no player could improve her expected utility by defecting from the strategy.

This result follows directly from [12], Theorem 4.5. All of [12]'s conditions can easily be checked, so we highlight only the less obvious ones. Continuity of payoffs as functions of the actions v_i follows from the continuity and boundedness of Ψ . Type-conditional utility is only bounded from above, not below, but boundedness from below can easily be restored by simply deleting for each value type u votes of magnitude greater $\sqrt{2|u|}$. The existence of a monotone best-response follows from the obvious super-modularity of payoffs in value and votes.

Although Nash equilibria always exist, they need not be unique. Indeed, we will show that in some circumstances (cf. Theorem 3) Nash equilibria have points u_* of discontinuity; at any such point, there are at least two distinct pure-strategy Nash equilibria, one with $v(u_*) = v(u_*+)$, the other with $v(u_*) = v(u_*-)$. We conjecture, however, that at least when N is large, non-uniqueness of Nash equilibria can only occur for this trivial reason: in particular, we conjecture that if v_1 and v_2 are distinct Nash equilibria then $v_1(u) = v_2(u)$ for all but at most one value u.

1.3 Rationale for QV

Formally differentiating expression (7) with respect to v (see section 3.1 for a formal proof) yields the following first-order condition for maximization:

$$uE\left[\psi\left(V_{-1}+v\right)\right] = 2v \implies v(u) = \underbrace{\frac{E\left[\psi\left(V_{-1}+v(u)\right)\right]}{2}}_{\text{marginal pivotality}} u. \tag{8}$$

The marginal benefit of an additional unit of vote is thus twice the individual's value multiplied by the influence this extra vote has on the chance the alternative is adopted, the vote's *marginal pivotality*. The marginal cost of a vote is twice the number of votes already purchased.

When the number N of voters is large, most would reason that their votes v(u) will have a negligible effect on the vote total $V_{-1}+v(u)$. Taking this logic to an extreme, if voters acted as if marginal pivotality p were constant across the population, then an individual with value u would buy v(u)=pu votes. This voting strategy would imply $V=p\sum_i u_i$; that is, the vote total would be exactly proportional to the sum of the values, and consequently the expected inefficiency would be 0. Clearly, this argument holds only for a quadratic cost function, because only quadratic functions have linear derivatives.

Our main results will show, however, that the marginal pivotality is not constant; in fact, when the mean μ of the sampling distribution F is non-zero the marginal pivotality can have large jump discontinuities in the tail of the distribution. Thus,

voters do not buy votes strictly in proportion to their values, and so in general the vote total will not be a scalar multiple of the aggregate value $\sum_i u_i$.

1.4 Main Results

Our main results concern the structure of equilibria in the game described in the previous section when the number N of agents is large, and the implications for the efficiency of QV.

1.4.1 Characterization of equilibrium in the zero mean case

The structure of a Nash equilibrium differs radically depending on whether $\mu=0$ or $\mu\neq 0$. The case $\mu=0$, although non-generic, is of particular interest because in some elections – for instance, when two candidates are vying for an elected office – the alternatives may be tailored so that an approximate population balance is achieved [10].

Theorem 1. For any sampling distribution F with mean $\mu = 0$ that satisfies the hypotheses above, there exist constants $\epsilon_N \to 0$ such that in any Nash equilibrium, v(u) is C^{∞} and strictly increasing on $[\underline{u}, \overline{u}]$ and satisfies the following approximate proportionality rule:

$$\left| \frac{v(u)}{p_N u} - 1 \right| \le \epsilon_N \quad \text{where} \quad p_N = \frac{1}{2^{\frac{3}{4}} \sqrt{\sigma} \sqrt[4]{\pi(N-1)}}. \tag{9}$$

Furthermore, there exist constants $\alpha_N, \beta_N \to 0$ such that in any equilibrium the vote total $V = V_N$ and expected inefficiency satisfy

$$|E[V]| \le \alpha_N \sqrt{\operatorname{var}(V)}$$
 and (10)

$$EI < \beta_N. \tag{11}$$

The proof will be given in section 9.

Thus, in any equilibrium, agents buy votes approximately in proportion to their values u_i , which corresponds to their behavior under price-taking, as described in the previous section. Given this approximate proportionality, it is not difficult to understand why the number of votes a typical voter buys should be of order $N^{-1/4}$. If the vote function v(u) in a Bayes-Nash equilibrium follows a proportionality rule $v(u) \approx \beta u$, the constant β must be the consensus marginal pivotality. On the other hand, by the local limit theorem of probability (see [3], ch. XVI), if $\beta = CN^{-\alpha}$ for some constants $C \neq 0$ and $\alpha \in \mathbb{R}$, the chance that $V \in [-\delta, \delta]$ would be of order $N^{\alpha-\frac{1}{2}}$, and so α must be 1/4.

Although the relation (9) asserts the ratio v(u)/u is approximately constant, it is not exactly constant: in fact, v(u) is a genuinely nonlinear function of u. Thus, even though E[U]=0, it need not be the case that E[V]=0. To establish the asymptotic efficiency assertion (11), we must establish assertion (10), namely, that the nonlinearities vanish rapidly enough that the bias created by non-linearity is smaller than the sampling variation in u. This will require a rather subtle application of the Edgeworth expansion of the distribution of V_{-1} . If it were the case that E[V]=0, and if the distribution of V_{-1} were exactly normal, a standard Taylor expansion and the $N^{-1/4}$ decay of v(u)/u could be used directly to show that non-linearities vanish with N^{-1} even relative to the leading term of v(u)/u. A detailed analysis of this argument leads us to conjecture that, under the hypotheses of Theorem 1, the inefficiency of QV decays like $\mu_3^2/(16\sigma^6N)$.

1.4.2 Characterization of equilibrium in the non-zero mean case

When μ is not zero, the nature of equilibrium can be quite different: in particular, if the payoff function is sufficiently sharp (the support of its derivative is sufficiently small) then for sufficiently large N, any type-symmetric Bayes-Nash equilibrium has a large discontinuity in the extreme tail of the sampling distribution. Nevertheless, in all cases the quadratic voting mechanism is asymptotically efficient, as the following theorem shows.

Theorem 2. Assume that the sampling distribution F has mean $\mu > 0$ and that F and Ψ satisfy the hypotheses above. Then there exist constants $\beta_N \to 0$ such that in any type-symmetric Bayes-Nash equilibrium v(u),

$$EI < \beta_N.$$
 (12)

Furthermore, there exist constants $\alpha \geq \delta$ and $\beta > 0$ depending on the sampling distribution F and the payoff function Ψ but not on N such that in any equilibrium v(u), for any $\epsilon > 0$,

$$\sup_{\underline{u}+\beta N^{-3/2} \le u \le \overline{u}} |v(u) - \alpha \mu^{-1} u/N| < \alpha_N/N \quad \text{and hence}$$
 (13)

$$P\{|V_N - \alpha| > \epsilon\} \le \epsilon_N,\tag{14}$$

where $\epsilon_N, \alpha_N \to 0$ are constants that depend only on the sample size N, and not on the particular equilibrium.

This theorem allows for two cases. In the first, where $\alpha = \delta$, the vote total is near δ with high probability for large N. This case occurs for large δ and thus relatively smooth payoff functions. In the second, $\alpha > \delta$, so that with high probability the vote total is outside $[-\delta, \delta]$ for large N. This case arises when $\delta > 0$ is small. In

both cases, the approximate proportionality rule (13) holds except possibly in the extreme lower tail of the value distribution F.

To see how the dichotomy arises, suppose that for some $\alpha \geq \delta$ there were a value $w \in (-\delta,0)$ such that

$$(1 - \Psi(w)) |\underline{u}| > (\alpha - w)^2;$$
 (15)

then an agent with value u_i near the lower extreme \underline{u} , knowing that with high probability the one-out vote total $V_{-i} = \sum_{j \neq i} v_j$ is near α , would find it worthwhile to buy $-\alpha + w$ votes and thus single-handedly move the vote total to w. Consequently, there can be no equilibrium in which V_{-i} concentrates strictly below α if such a w exists, as this would lead a large number of individuals to act as extremists, contradicting the concentration of the vote total. Therefore, in any equilibrium the voters with positive values u_j must buy enough votes to guarantee that the vote total concentrates at or above α . The minimal value $\alpha \geq \delta$ at which the advantage of "extremist" behavior in the extreme lower tail disappears thus determines the equilibrium behavior (13). This will be at $\alpha = \delta$ unless there is a solution to the following problem.

Optimization Problem. Determine $\alpha > \delta$ and a matching real number $w \in [-\delta, 0]$ such that

$$(1 - \Psi(w)) |\underline{u}| = (\alpha - w)^2 \quad \text{and}$$

$$(1 - \Psi(w')) |\underline{u}| \le (\alpha - w')^2 \quad \text{for all } w' \in (-\delta, \delta) \setminus \{w\}$$

Proposition 2. If $\delta < 1/\sqrt{2}$ then there exists a unique pair $\alpha > \delta$ and $w \in [-\delta, 0]$ that satisfy the Optimization Problem (16).

The proof will be given in section 10. When the Optimization Problem has a solution, Nash equilibria take a rather interesting form in which extremists must appear, but with vanishing probability, as the following theorem shows.

Theorem 3. Assume that the sampling distribution F has mean $\mu > 0$ and that F and Ψ satisfy the hypotheses above. Assume further that the Optimization Problem (16) has a unique solution (α, w) . Then there exists a constant $\zeta > 0$ depending on F such that for any $\epsilon > 0$ and any type-symmetric Bayes-Nash equilibrium v(u), when N is sufficiently large,

- (i) v(u) has a single discontinuity at u_* , where $|u_* + |\underline{u}| \zeta N^{-2}| < \epsilon N^{-2}$;
- (ii) $|v(u) + \alpha w| < \epsilon$ for $u \in [\underline{u}, u_*)$; and
- (iii) the approximate proportionality rule (13) holds for all $u \in [u_*, \overline{u}]$.

Theorems 2 and 3 will be proved in section 8.

Theorem 3 implies that an agent with value u will buy approximately $\alpha\mu^{-1}u/N$ votes unless u is in the extreme lower tail of F. Since such exceptional agents occur only with probability $\approx \zeta N^{-1}f(\underline{u})$, it follows by the law of large numbers that with probability $\approx 1-\zeta N^{-1}f(\underline{u})$, the vote total will be very near α . If, on the other hand, the sample contains an agent with value less than u_* then this agent will buy approximately $\alpha-w\approx-\sqrt{|\underline{u}|}$ votes, enough to move the overall vote total close to w. Agents of the first type will be called *moderates*, and agents of the second kind *extreme contrarians* or *extremists* for short. Because the tail region in which extremists reside has F-probability on the order N^{-2} , the sample of agents will contain an extremist with probability only on the order N^{-1} , and will contain two or more extremists with probability on the order N^{-2} . Given that the sample contains no extremists, the conditional probability that $|V-\alpha|>\epsilon$ is $O(e^{-\varrho n})$ for some $\varrho>0$, by standard large deviations estimates, and so the event that V<0 essentially coincides with the event that the sample contains an extremist.

Why does equilibrium take the somewhat counter-intuitive form described in Theorem 3? Following is a brief heuristic explanation. For an agent i with value u_i in the "bulk" of the sampling distribution F, there is very little information about the vote total V in the agent's value u_i , and so for most such agents the marginal pivotality $E\psi\left(V_{-i}+v(u_i)\right)$ will be approximately $E\psi(V)$. Consequently, v(u) will be approximately linear in u except possibly in the extreme tails of the distribution, and so by the law of large numbers, the vote total will, with high probability, be near $\frac{1}{2}N\mu E\psi(V)$.

Because $\mu>0$, agents with negative values will, with high probability, be on the losing side of the election. However, if $\frac{1}{2}N\mu E\psi(V)$ were small, then an agent with even moderately negative value could increase her expected utility by buying more than $\frac{1}{2}N\mu E\psi(V)$ votes; since many voters with negative values would find it beneficial to adopt such a strategy, the vote total would, with high probability, be negative, in contradiction to the fact that it must be concentrated near $\frac{1}{2}N\mu E\psi(V)$. Therefore, $NE\psi(V)\mu$ must remain bounded away from 0.

On the other hand, if $\frac{1}{2}N\mu E\psi(V)$ were too large, then no individual could profitably act as an extremist, so except with exponentially small probability V would be bounded away from $[-\delta,\delta]$. But this would force $\mathbb{E}\psi(V)$ to be exponentially small, which is impossible. Thus, the aggregate number of votes must concentrate near a constant value, and so most voters must buy on the order of 1/N votes. For this scenario to occur, $E\psi(V)$ must decay as $\frac{1}{N}$. But the primary contribution to this expectation must come from the event in which an extremist exists, and so the probability of this event must decay as $\frac{1}{N}$.

1.5 Plan of the paper

The remainder of the paper will be devoted to the proofs of Theorems 1–3 and Proposition 2. Because essentially nothing (other than monotonicity) is known a priori about the nature of Nash equilibria, information must be teased out in steps, each relying on the previous steps. In section 4, a weak form of the approximate proportionality rule will be proved for agents in the bulk of the distribution F. Using this weak approximate proportionality rule, we will, in section 5, use an anticoncentration inequality for sums of i.i.d. random variables to derive bounds for Nash equilibria. In section 6 we will show that any discontinuities must be large, and that except at discontinuities any Nash equilibrium must be smooth. We will then be able to deduce, in section 7, that approximate proportionality holds except in the extreme tails of F. The proofs of Theorems 2–3 will be given in section 8, and the proof of Theorem 1 in section 9. Finally, the proof of Proposition 2 will be given in section 10.

2 Terminology and Notation

Under the model assumptions we have stipulated, it would never make sense for an agent to purchase more than $\sqrt{2\overline{u}}$ or fewer than $-\sqrt{2|\underline{u}|}$ votes. Consequently, we shall restrict attention to strategies under which, with probability one, the number of votes purchased by any agent falls in the interval $[-\sqrt{2\underline{u}},\sqrt{2\overline{u}}]$. A pure strategy is a Borel measurable function $v:[\underline{u},\overline{u}]\to[-\sqrt{2|\underline{u}|},\sqrt{2\overline{u}}]$; when a pure strategy v is adopted, each agent buys v(u) votes, where u is the agent's utility. A mixed strategy is a Borel measurable function $\pi_V:[\underline{u},\overline{u}]\to\Pi$, where Π is the collection of Borel probability measures on $[-\sqrt{2|\underline{u}|},\sqrt{2\overline{u}}]$; when a mixed strategy π_V is adopted, each agent i will buy a random number V_i of votes, where V_1,V_2,\ldots are conditionally independent given the utilities U_1,U_2,\ldots and V_i has conditional distribution $\pi_V(U_i)$. Clearly, the set of mixed strategies contains the pure strategies.

A *best response* for an agent with utility u to a strategy (either pure or mixed) is a value v such that

$$E\Psi(v+S_n)u-v^2 = \sup_{\tilde{v}} E\Psi(\tilde{v}+S_n)u-\tilde{v}^2, \tag{17}$$

where S_n is the sum of the votes of the other n agents when these agents all play the specified strategy and E denotes expectation. (Thus, under E, the random

⁸The space of Borel probability measures on $[-\sqrt{2|\underline{u}|},\sqrt{2\overline{u}}]$ is given the topology of weak convergence; Borel measurability of a function with range Π is relative to the Borel field induced by this topology. Proposition 5 below implies that in the Quadratic Voting game only pure strategies are relevant, so measurability issues play no role in this paper.

variables V_i of the n other voters are distributed in accordance with the strategy and the sampling rule for utility values U_i described above.) Because Ψ is continuous and bounded, equation (17) and the dominated convergence theorem imply that for each u the set of best responses is closed and hence has well-defined maximal and minimal elements $v_+(u), v_-(u)$.

A mixed strategy π_V is a *type-symmetric Bayes-Nash equilibrium*, or *Nash equilibrium* for short, if for every $u \in [\underline{u}, \overline{u}]$ the measure $\pi_V(u)$ is supported by the set of best responses to π_V for an agent with utility u. Two strategies will be called *equivalent* if they coincide for all u except in a set of Lebesgue measure 0. Because the values U_i of the various agents are sampled from a distribution that is absolutely continuous, if two strategies are equivalent then with probability one they result in exactly the same actions.

Notation. The symbols $\Psi, \psi, \delta, F, \mu, \zeta, \underline{u}, \overline{u}$ will be reserved for the functions and constants specified in Subsection 3.1 of the text, and the letters N, n will be used only for the sample size and sample size minus one. The symbols $\alpha, \beta, \gamma, \epsilon, \varrho$ and C will be used for generic constants whose values might change from one lemma to the next. Because many of the arguments to follow will involve the values of the equilibrium vote function v at points near one of the endpoints $\underline{u}, \overline{u}$, we will use the following shorthand notation, for any $0 < \epsilon < 1$:

$$\overline{u}_{\epsilon} = \overline{u} - \epsilon$$
 and $\underline{u}_{\epsilon} = \underline{u} + \epsilon$.

3 Nash Equilibria: Basic Properties

3.1 Necessary Condition for a Nash Equilibrium

Let π_V be a mixed-strategy Nash equilibrium, and let S_n be the sum of the votes of n agents with utilities U_i obtained by random sampling from F, all acting in accordance with the strategy π_V . For an agent with utility u, a best response v must satisfy equation (17), and so in particular for every $\Delta > 0$, if u > 0 then

$$E\left\{\Psi(S_n + v + \Delta) - \Psi(S_n + v)\right\} u \le 2\Delta v + \Delta^2 \quad \text{and} \quad E\left\{\Psi(S_n + v - \Delta) - \Psi(S_n + v)\right\} u \le -2\Delta v + \Delta^2$$

Similarly, if u < 0 and $\Delta > 0$ then

$$E\left\{\Psi(S_n + v - \Delta) - \Psi(S_n + v)\right\} u \le -2\Delta v + \Delta^2 \quad \text{and} \quad E\left\{\Psi(S_n + v + \Delta) - \Psi(S_n + v)\right\} u \le 2\Delta v + \Delta^2$$

Because Ψ is C^{∞} and its derivative ψ has compact support, differentiation under the expectation is permissible. Thus, we have the following necessary condition.

Proposition 3. If π_V is a mixed-strategy Nash equilibrium then for every u a best response v must satisfy

$$E\psi(S_n + v)u = 2v. (18)$$

Consequently, every pure-strategy Nash equilibrium v(u) must satisfy the functional equation

$$E\psi(S_n + v(u))u = 2v(u). \tag{19}$$

3.2 Monotonicity Properties of Nash Equilibria

Lemma 4. Let π_V be a mixed-strategy Nash equilibrium, and let v, \tilde{v} be best responses for agents with utilities u, \tilde{u} , respectively. If u = 0 then v = 0, and if $u < \tilde{u}$, then $v \leq \tilde{v}$. Consequently, any pure-strategy Nash equilibrium v(u) is a nondecreasing function of u and therefore has at most countably many discontinuities and is differentiable almost everywhere.

Proof. It is obvious that the only best response for an agent with u=0 is v=0, and the monotonicity of the payoff function Ψ implies a best response v for an agent with utility u must be of the same sign as u. If v, \tilde{v} are best responses for agents with utilities $0 \le u < \tilde{u}$, then by definition

$$E\Psi(\tilde{v}+S_n)\tilde{u}-\tilde{v}^2 \ge E\Psi(v+S_n)\tilde{u}-v^2 \quad \text{and} \quad E\Psi(v+S_n)u-v^2 > E\Psi(\tilde{v}+S_n)u-\tilde{v}^2,$$

and so, after re-arrangement of terms,

$$(E\Psi(\tilde{v}+S_n) - E\Psi(v+S_n))\tilde{u} \ge \tilde{v}^2 - v^2 \quad \text{and} \quad (E\Psi(\tilde{v}+S_n) - E\Psi(v+S_n))u \le \tilde{v}^2 - v^2.$$

Hence,

$$(E\Psi(\tilde{v}+S_n)-E\Psi(v+S_n))(\tilde{u}-u)\geq 0.$$

The monotonicity of Ψ implies that if $0 \le \tilde{v} < v$ then $E\Psi(\tilde{v}+S_n) \le E\Psi(v+S_n)$, and so it follows that the two expectations must be equal, because $\tilde{u}-u>0$. But if the two expectations were equal v could not possibly be a best response at u, because an agent with utility u could obtain the same expected payoff $E\Psi(v+S_n)u$ at a lower vote cost by purchasing \tilde{v} votes. This argument proves that if $0 \le u < \tilde{u}$ best responses v, \tilde{v} for agents with utilities u, \tilde{u} must satisfy $0 \le v \le \tilde{v}$. A similar argument shows that if $u < \tilde{u} \le 0$ best responses v, \tilde{v} for agents with utilities u, \tilde{u} must satisfy $v \le \tilde{v} \le 0$.

Proposition 5. If a mixed strategy π_V is a Nash equilibrium, the set of utility values $u \in [\underline{u}, \overline{u}]$ for which more than one best response (and hence the set of values u such that $\pi_V(u)$ is not supported by just a single point v(u)) is at most countable.

Proof of Proposition 5. For each u denote by $v_-(u)$ and $v_+(u)$ the minimal and maximal best responses at u. Lemma 4 implies that if $u < \tilde{u}$ then $v_+(u) \le v_-(\tilde{u})$. Consequently, for any $\epsilon > 0$ the set of utilities values u at which $v_+(u) - v_-(u) \ge \epsilon$ must be finite, because otherwise $v_+(u) \to \infty$ as $u \to \overline{u}$, which is impossible because best responses must take values between $-\sqrt{2|\underline{u}|}$ and $\sqrt{2\overline{u}}$.

Because by hypothesis the values U_i are sampled from a distribution F that is absolutely continuous with respect to Lebesgue measure, the probability that one of the votes i will have utility value U_i equal to one of the countably many values where more than one best response exists is zero. Consequently, for every Nash equilibrium an equivalent pure-strategy Nash equilibrium v(u) exists. Henceforth, we consider only pure-strategy Nash equilibria; whenever we refer to a Nash equilibrium we mean a pure-strategy Nash equilibrium.

Lemma 6. If v(u) is a Nash equilibrium then $v(u) \neq 0$ for all $u \neq 0$.

Proof. If v(u)=0 for some u>0 then by Lemma 4 v(u')=0 for all $u'\in(0,u)$. Because the density f(u) of the value distribution F is strictly positive on $[\underline{u},\overline{u}]$, it follows that the probability p that every agent in the sample casts vote $V_i=0$ is strictly positive. But then an agent with utility u could improve her expectation by buying $\varepsilon>0$ votes, where $\varepsilon\ll u\psi(0)p$, because the expected utility gain would be at least

$$u\Psi(\varepsilon)p \sim u\psi(0)p\varepsilon$$

at a cost of ε^2 . Because by hypothesis $\psi(0) > 0$, the expected utility gain would overwhelm the increased vote cost for small $\varepsilon > 0$.

Corollary 7. Any Nash equilibrium v(u) is strictly increasing on $[\underline{u}, \overline{u}]$.

Proof. By Lemma 6 and the necessary condition (19),

$$E\psi(S_n + v(u)) > 0 \quad \text{for every } u \in [\underline{u}, \overline{u}] \setminus \{0\}.$$
 (20)

Differentiating in (19) with respect to u shows that

$$E\psi(S_n + v(u)) = (2 - E\psi'(S_n + v(u)))v'(u)$$
(21)

at every $u \in D$, where D is the set of values u where v(u) is differentiable. Since v(u) is nondecreasing, by Lemma 4, it follows that $2 \ge E\psi'(S_n + v(u))$ at all points

of D; since $E\psi(S_n + v(u)) > 0$, it follows that v'(u) > 0 at every $u \in D$. But since v is nondecreasing, D^c has Lebesgue measure 0, so for any two values $u_1 < u_2$ in $[\underline{u}, \overline{u}]$,

$$v(u_2) - v(u_1) \ge \int_{[u_1, u_2] \cap D} v'(u) du > 0.$$

4 Weak Consensus Estimates

According to Lemma 3, in any Nash equilibrium the number of votes v(u) an agent with utility u purchases must satisfy the necessary condition (19). It is natural to expect that when the sample size n+1 is large the effect of adding a single vote v to the aggregate total S_n should be small, and so one might expect that the function v(u) should satisfy the approximate proportionality rule

$$2v(u) \approx E\psi(S_n)u$$
.

As we will show later, this naive approximation can fail badly for utility values u in the extreme tails of the distribution F; however, in the bulk of the value distribution approximate proportionality does indeed hold (cf. Proposition 20 in Sec. 7.1). In this section, we will prove the following weaker version of Proposition 20.

Lemma 8. For every $\epsilon > 0$ and every $\alpha > 0$, if n is sufficiently large then in any Nash equilibrium $v(\cdot)$,

$$E\psi(v(u) + S_n) \ge (1 - \alpha)E\psi(v(u') + S_n) \tag{22}$$

for any two values u, u' not within distance ϵ of either \underline{u} , or \overline{u} , or 0.

The proof will require the following *a priori* lower bound.

Lemma 9. There exists $\gamma > 0$ such that for all sufficiently large n, in any Nash equilibrium,

$$\max(v(\overline{u}_{1/n}), -v(\underline{u}_{1/n}) \ge \gamma/n. \tag{23}$$

Proof. Suppose inequality (23) did not hold; then with probability $\approx (1 - (f(\underline{u}) + f(\overline{u}))/n)^n \approx \exp\{-f(\underline{u}) - f(\overline{u})\} := p$, the values of all agents would lie in the interval $[\underline{u}_{1/n}, \overline{u}_{1/n}]$, and so the vote total would be no more than γ in absolute value. But if γ were sufficiently small, then an agent with value u = 1 would find it advantageous to defect from the equilibrium strategy by buying 3γ votes, at cost $9\gamma^2$, thus raising her expected payoff by at least $p(\Psi(2\gamma) - \Psi(\gamma)) \approx p\psi(\gamma)\gamma \gg 9\gamma^2$.

Proof of Lemma 8. Let J_1, J_2, \ldots, J_k be any partition of the interval $[\underline{u}, \overline{u}]$ into non-overlapping Borel sets of positive Lebesgue measure, and for each index i let M_i be the number of agents (in the entire sample of size N=n+1) with values in the set J_i . The random vector (M_1, M_2, \ldots, M_k) has the multinomial distribution

$$P(M_i = m_i \text{ for each } i \le k) = \frac{(n+1)!}{m_1! m_2! \cdots m_k!} \prod_{i=1}^k p_i^{m_i}$$

where

$$p_i = P(U_1 \in J_i) = \int_{J_i} f(u) \, du.$$

Conditional on the event that $M_i = m_i$ for each $i \leq k$, the sample $\{U_1, U_2, \dots, U_n\}$ has the same distribution as a stratified random sample gotten by choosing m_i elements from the set J_i according to the density $f\mathbf{1}_{J_i}/p_i$ for each $i \leq k$. Consequently, for any choice of index $i \leq k$,

$$E[\psi(v(U_{n+1}) + S_n) | U_{n+1} \in J_i] = \sum_{m_1, m_2, \dots, m_k} \frac{n!}{m_1! m_2! \cdots m_k!} \left(\prod_{i=1}^k p_i^{m_i} \right) E_*(m_1, m_2, \dots, m_{i-1}, m_i + 1, m_{i+1}, \dots m_k)$$
 (24)

where

$$E_*(m_1, m_2, \dots, m_k) = E(\psi(S_{n+1}) \mid M_i = m_i \ \forall i \le k)$$

for any set of nonnegative integers m_j that sum to n+1. Note that the expectations $E_*(m_1, m_2, \ldots, m_k)$ are all bounded above by $\|\psi\|_{\infty}$.

The proof of the lemma will be based on systematic exploitation of equation (24). There are eight separate cases, depending on whether u and u' are positive or negative, and on which of $v(\overline{u}_{1/n})$ or $-v(\underline{u}_{i/n})$ exceeds γ/n (cf. Lemma 9). Since all of these are similar, we shall consider only the case u, u' > 0 and $v(\overline{u}_{1/n}) \ge \gamma/n$.

To relate the conditional expectation (24) to the expectation in the necessary condition (19), we appeal to the monotonicity of $v(\cdot)$. Fix $u \in (\epsilon, \overline{u}]$ and let $J = [u - \alpha u, u]$ and $J' = [u, u + \alpha u]$, where $\alpha > 0$ is small enough that $J \subset [\epsilon, \overline{u}]$; then by (19) and the monotonicity of v,

$$(1-\alpha)E\psi(v(u')+S_n) \leq E\psi(v(u)+S_n)$$
 for all $u' \in J$ and $(1+\alpha)E\psi(v(u'')+S_n) \geq E\psi(v(u)+S_n)$ for all $u'' \in J'$.

Consequently,

$$(1 - \alpha)E[\psi(v(U_{n+1}) + S_n) \mid U_{n+1} \in J] \le E\psi(v(u) + S_n) \quad \text{and}$$

$$(1 + \alpha)E[\psi(v(U_{n+1}) + S_n) \mid U_{n+1} \in J'] \ge E\psi(v(u) + S_n).$$
(25)

Thus, to prove (22), it suffices to prove analogous inequalities for conditional expectations. For this we will need the following crude lower bound on Nash equilibria, which we will deduce from Lemma 9.

Claim 10. For every $\epsilon > 0$ there exists $C = C_{\epsilon} > 0$ such that for all sufficiently large n and any Nash equilibrium $v(\cdot)$,

$$v(u) \ge \frac{C}{n^{3/2}}$$
 for every $u \in [\epsilon, \overline{u}]$.

Proof of Claim 10. By (25) and the necessary condition (19), it suffices to prove that for some interval $J_1 = [u - \alpha u, u]$ and some C > 0,

$$E\psi(v(U_{n+1}) + S_n) | U_{n+1} \in J_1 | \ge Cn^{-3/2}$$

for all n sufficiently large and any Nash equilibrium v. Fix $\beta>0$, and let $J_2=J_2(\beta)=[\overline{u}_{\beta n^{-3/2}},\overline{u}]$, and let J_3 be the complement of $J_1\cup J_2$ in $[\underline{u},\overline{u}]$. (Observe that J_2 has F-probability $p_2\sim\beta f(\overline{u})/n^{3/2}$; however, J_1 is fixed for all n, as is its F-probability p_1 .) By Lemma 9, inequality (25), and the monotonicity of v, we have

$$E\psi(v(U_{n+1}) + S_n) | U_{n+1} \in J_2] \ge \gamma' n^{-1}$$

for some $\gamma' > 0$ not depending on n. Thus, it suffices to show that for all sufficiently large n and any Nash equilibrium v,

$$\frac{E\psi(v(U_{n+1}) + S_n) | U_{n+1} \in J_1]}{E\psi(v(U_{n+1}) + S_n) | U_{n+1} \in J_2]} \ge \frac{\beta f(\overline{u})}{8n^{1/2}}.$$
(26)

For brevity, we shall denote the numerator by E_1 and the denominator by E_2 . Both E_1 and E_2 can be expressed as sums of the form (24). In both sums the same factors $E_*(m_1, m_2, m_3)$ occur, with these exceptions: in E_1 only factors $E_*(m_1, m_2, m_3)$ with $m_1 \geq 1$ occur, but in E_2 only factors $E_*(m_1, m_2, m_3)$ with $m_2 \geq 1$ occur. The contribution to either expectation E_i from terms with factors $E_*(m_1, m_2, m_3)$ with $m_1 \leq np_1/2$ is negligible, because by standard results in large deviation theory (e.g., Hoeffding's inequality, cf. [7]) the probability that $M_1 \leq np_1/2$ decays exponentially in n. The contribution to E_1 from terms with factors $E_*(m_1, m_2, m_3)$ where $m_2 \geq 4$ is $O(n^{-5})$ (because this is the chance of having 4 or more elements in I_2 in a sample of size I_2 in a similarly the contribution to I_2 from terms with factors I_2 from terms with factors

Now for any triple (m_1, m_2, m_3) with $m_1 \ge 1$ and $m_2 \ge 1$, terms with factor $E_*(m_1, m_2, m_3)$ occur, but with different coefficients

$$\frac{n!}{(m_1-1)!m_2!m_3!}p_1^{m_1-1}p_2^{m_2}p_3^{m_3} \quad \text{in E_1,} \quad \text{and} \\ \frac{n!}{m_1!(m_2-1)!m_3!}p_1^{m_1}p_2^{m_2-1}p_3^{m_3} \quad \text{in E_2.}$$

The ratio of these coefficients (E_1 to E_2) is

$$\frac{m_1}{m_2} \frac{p_2}{p_1} \sim \frac{m_1 f(\overline{u})}{p_1 m_2 n^{3/2}}$$

By the remarks above, terms with $m_1 \le np_1/2$ or $m_2 \ge 4$ can be ignored; for the rest, the coefficient ratio is, for all large n, at least

$$\frac{\beta f(\overline{u})}{4n^{1/2}}.$$

Therefore, $E_1/E_2 \ge f(\overline{u})/(8n^{1/2})$ for all large n, proving the Claim.

The proof of inequality (22) follows a similar line, but requires the result of Claim 10 to justify ignoring terms of exponentially small size. By inequalities (25), it suffices to show that for any $\alpha > 0$ and any two non-overlapping intervals J_1, J_2 of positive length contained in $[\epsilon, \overline{u} - \epsilon]$,

$$\frac{E[\psi(v(U_{n+1}) + S_n) \mid U_{n+1} \in J_1]}{E[\psi(v(U_{n+1}) + S_n) \mid U_{n+1} \in J_2]} \ge 1 - \alpha$$

provided n is sufficiently large. As in the proof of the Claim, denote the numerator and denominator by E_1 and E_2 , respectively.

Each of the expectations E_1 , E_2 has a representation (24) with k=3, where J_3 is the complement of $J_1 \cup J_2$ in $[\underline{u}, \overline{u}]$. For any $\beta>0$, the contribution to either E_1 or E_2 from terms of (24) for which $|m_i-np_i|\geq n\beta p_i$ is exponentially small, by Hoeffding's inequality, and hence can be ignored. For any triple (m_1, m_2, m_3) with $m_i\geq 1$ and $m_2\geq 1$, terms with factor $E_*(m_1,m_2,m_3)$ occur in both E_1 and E_2 , with the same coefficients as in the proof of the Claim. The ratio of these coefficients (E_1 to E_2) is

$$\frac{m_1}{m_2} \frac{p_2}{p_1}$$

Since only those triples with $|m_i - np_i| < n\beta p_i$ contribute substantially to the expectations, it follows that for large n,

$$\frac{E_1}{E_2} \ge \frac{1+2\beta}{1-2\beta}.$$

Clearly, if $\beta > 0$ is sufficiently small then the lower bound will exceed $1 - \alpha$. This completes the proof of Lemma 8.

The proof of Claim 10 – in particular, the inequalities (25) and (26) – also yields the following result, which will be used in the proof of Lemma 14 below.

Corollary 11. For any $\beta > 0$, all sufficiently large n and every Nash equilibrium v,

$$\begin{split} &\frac{E\psi(v(u)+S_n)}{E\psi(v(\overline{u}_{\beta n^{-3/2}})+S_n)} \geq \frac{\beta f(\overline{u})}{8n^{1/2}} \quad \text{and} \\ &\frac{E\psi(v(u)+S_n)}{E\psi(v(\underline{u}_{\beta n^{-3/2}})+S_n)} \geq \frac{\beta f(\underline{u})}{8n^{1/2}}. \end{split}$$

5 Concentration and size constraints

Because the vote total S_n is the sum of independent, identically distributed random variables $v(U_i)$ (albeit with unknown distribution), its distribution is subject to concentration restrictions, such as those imposed by the following lemma.

Lemma 12. For any $\epsilon > 0$ there exists $\gamma = \gamma(\epsilon) < \infty$ such that for all sufficiently large values of n and any Nash equilibrium v(u), if

$$||v||_{\infty} \ge \epsilon \tag{27}$$

then

$$|v(u)| \le \frac{\gamma}{\sqrt{n}} \quad \text{for all } u \in [\underline{u}_{\epsilon}, \overline{u}_{\epsilon}].$$
 (28)

We will deduce Lemma 12 from the following general fact about sums of independent, identically distributed random variables.

Lemma 13. Fix $\alpha > 0$. For any $\epsilon > 0$ and any $C < \infty$ there exists $C' = C'(\epsilon, C) > 0$ and $n' = n'(\epsilon, C) < \infty$ such that the following statement is true: if $n \ge n'$ and Y_1, Y_2, \ldots, Y_n are independent random variables such that

$$E|Y_1 - EY_1|^3 \le C \text{var}(Y_1)^{3/2}$$
 and $\text{var}(Y_1) \ge C'/n$ (29)

then for every interval $J \subset \mathbb{R}$ of length α or greater,

$$P\left\{\sum_{i=1}^{n} Y_i \in J\right\} \le \epsilon |J|/\alpha. \tag{30}$$

The proof of this lemma, a routine exercise in the use of Fourier methods, is relegated to the end of this section.

Proof of Lemma 12. By Lemma 8, there exists a constant $\alpha>0$ such that for any point $u\in [\underline{u}_{\epsilon},\overline{u}_{\epsilon}]\setminus [-2\epsilon,2\epsilon]$ the ratio v(u)/u is at least $\alpha v(\overline{u}_{\epsilon})/\overline{u}_{\epsilon}$. Because the density f is bounded below, it follows that for suitable constants $0< C<\infty$ and $\frac{1}{2}>p>0$, for all sufficiently large n and every Nash equilibrium v(u) there are intervals

$$J_{+} = [u_{+}, \overline{u}_{\epsilon}] \subset (0, \overline{u})$$
 and $J_{-} = [u_{-}, \underline{u}_{\epsilon}] \subset (\underline{u}, 0),$

both with F-probability p, such that

$$\max(v(\overline{u}_{\epsilon}), -v(\underline{u}_{\epsilon})) \leq C \min(v(u_{+}), -v(u_{-})).$$

Consequently, if $g = f \mathbf{1}_{J_+ \cup J_-}/2p$ is the conditional density of a value U given that $U \in J_+ \cup J_-$, then g has variance at least $\min(v(u_+)^2, v(u_-)^2)/4$, and its third moment obeys the restriction (29).

Let M be the number of points U_i in the sample U_1, U_2, \ldots, U_n that fall in $J_+ \cup J_-$, and let S_n^* be the sum of the votes $v(U_i)$ for those agents i whose values U_i fall in this range. By construction, M has the binomial-(n, 2p) distribution. Moreover, conditional on the event M=m and $S_n-S_n^*=w$, the random variable S_n^* is the sum of m independent random variables Y_i with density g. Consequently, by Lemma 13, if either $v(u_\epsilon)\sqrt{n}$ or $v(\underline{u}_\epsilon)\sqrt{n}$ is sufficiently large then the conditional probability, given $M=m\geq np$ and $S_n-S_n^*=w$, that S_n^* lies in any interval of length 4δ is bounded above by $\epsilon/2$. Therefore, for every $v\in\mathbb{R}$,

$$E\psi(S_n + v) \le \|\psi\|_{\infty} \left(P\{M \le np\} + \epsilon/2\right).$$

Since $P\{M \leq np\}$ decays exponentially as $n \to \infty$, the lemma follows, by the necessary condition (19).

Lemma 12 implies that for any $\epsilon > 0$, if n is sufficiently large then for any Nash equilibrium v(u), the absolute value |v(u)| can assume large values only at utility values u within distance ϵ of one of the endpoints $\underline{u}, \overline{u}$. The following proposition improves this bound to the extreme tails of the distribution.

Lemma 14. For any $0 < \epsilon < \infty$ there exists $\beta = \beta(\epsilon) > 0$ such that for all sufficiently large n, every Nash equilibrium v(u) satisfies the inequality

$$|v(u)| \le \epsilon \tag{31}$$

for all u at distance greater than $\beta n^{-3/2}$ from both endpoints $\underline{u}, \overline{u}$.

Proof. Lemma 12 implies that for any $\epsilon > 0$ there exists $\gamma = \gamma(\epsilon) > 0$ such that if n is sufficiently large and $\|v\|_{\infty} \geq \epsilon$ then $2v(u) \leq \gamma/\sqrt{n}$ for all $u \in [\underline{u}_{\epsilon}, \overline{u}_{\epsilon}]$. Hence, by the necessary condition (19),

$$E\psi(v(\overline{u}_{\epsilon}) + S_n)\overline{u}_{\epsilon} \le \frac{\gamma}{\sqrt{n}}.$$

But by Corollary 11, for any $\beta > 0$, if n is sufficiently large then for every Nash equilibrium v,

$$\frac{E\psi(v(u) + S_n)}{E\psi(v(\overline{u}_{\beta_n - 3/2}) + S_n)} \ge \frac{\beta f(\overline{u})}{8\sqrt{n}}$$

for all u, and in particular for $u = \overline{u}_{\epsilon}$. The last two displayed inequalities now combine to yield

$$E\psi(v(\overline{u}_{\beta n^{-3/2}}) + S_n) \le \frac{8\gamma}{\beta f(\overline{u})\overline{u}} \implies 2v(\overline{u}_{\beta n^{-3/2}}) \le \frac{8\gamma}{\beta f(\overline{u})};$$

thus, the inequality $2v(u)>\epsilon$ can hold at some $u=u_{\beta n^{-3/2}}$ only if

$$\beta < \frac{8\gamma}{\epsilon f(\overline{u})\overline{u}}.$$

Proof of Lemma 13. It suffices to prove this for intervals of length δ , because any interval of length $n\delta$ can be partitioned into n pairwise disjoint intervals each of length δ . Without loss of generality, $EY_1=0$ and $\delta=1$ (if not, translate and rescale). Let g be a nonnegative, even, C^{∞} function with $\|g\|_{\infty}=1$ that takes the value 1 on $[-\frac{1}{2},\frac{1}{2}]$ and is identically zero outside [-1,1]. It is enough to show that for any $x\in\mathbb{R}$,

$$Eg(S_n + x) \le \epsilon.$$

Because g is C^{∞} , even, and has compact support, its Fourier transform is real-valued and integrable, so the Fourier inversion theorem implies

$$Eg(S_n + x) = \frac{1}{2\pi} \int \hat{g}(\theta) \varphi(-\theta)^n e^{-i\theta x} d\theta,$$

where $\varphi(\theta)=Ee^{i\theta Y_1}$ is the characteristic function of Y_1 . Because $EY_1=0$, the derivative of the characteristic function at $\theta=0$ is 0, and hence φ has Taylor expansion

$$|1 - \varphi(\theta) - \frac{1}{2}EY_1^2\theta^2| \le \frac{1}{6}E|Y_1|^3|\theta|^3.$$

Consequently, if the hypotheses (29) hold then for any $\gamma > 0$, if n is sufficiently large,

$$|\varphi(\theta)^n| \le e^{-\beta^2 \theta^2/4}$$

for all $|\theta| \leq \gamma$. This bound implies (because $|\hat{q}| \leq 2$) that

$$Eg(S_n + x) \le \frac{1}{\pi} \int_{|\theta| < \gamma} e^{-\beta^2 \theta^2/4} d\theta + \frac{1}{2\pi} \int_{|\theta| > \gamma} |\hat{g}(\theta)| d\theta.$$

Because \hat{g} is integrable, the constant γ can be chosen so that the second integral is less that $\epsilon/2$, and if β is sufficiently large then the first integral is bounded by $\epsilon/2$.

6 Discontinuities and Smoothness

6.1 Discontinuities

Because any Nash equilibrium v(u) is monotone in the utility u, it can have at most countably many discontinuities. Moreover, because any Nash equilibrium is bounded in absolute value by $\sqrt{2\max\left(\left|\underline{u}\right|,\overline{u}\right)}$ (because no agent will pay more for votes than she could gain in expected utility) the sum of the jumps is bounded by $\sqrt{2\max\left(\left|\underline{u}\right|,\overline{u}\right)}$. We will now establish a *lower* on the size of |v| at a discontinuity.

Lemma 15. Let v(u) be a Nash equilibrium. If v is discontinuous at $u \in (\underline{u}, \overline{u})$ then

$$E\psi'(\tilde{v}+S_n)u=2\tag{32}$$

for some $\tilde{v} \in [v_-, v_+]$, where v_- and v_+ are the left and right limits of v(u') as $u' \to u$.

Proof. The necessary condition (19) holds at all u' in a neighborhood of u, so by monotonicity of v and continuity of ψ , Equation (19) must hold when v(u) is replaced by either of v_{\pm} , that is,

$$2v_+ = E\psi(v_+ + S_n)u \quad \text{and} \quad$$

$$2v_- = E\psi(v_- + S_n)u. \quad$$

Subtracting one equation from the other and using the differentiability of ψ we obtain

$$2v_{+} - 2v_{-} = uE \int_{v}^{v_{+}} \psi'(t + S_{n}) dt = u \int_{v}^{v_{+}} E\psi'(t + S_{n}) dt.$$

The result then follows from the mean value theorem of calculus.

Lemma 16. There exists $\Delta > 0$ such that for all sufficiently large n, at any point u_* of discontinuity of a Nash equilibrium,

$$v_{+} := \lim_{u \to u_{*}+} v(u) \ge \Delta \quad \text{if } u_{*} \ge 0 \quad \text{and}$$

$$v_{-} := \lim_{u \to u_{*}-} v(u) \le -\Delta \quad \text{if } u_{*} \le 0.$$

$$(33)$$

Consequently, there exists $\beta < \infty$ such that if n is sufficiently large n then no Nash equilibrium $v(\cdot)$ has a discontinuity at a point u at distance greater than $\beta n^{-3/2}$ from one of the endpoints $\underline{u}, \overline{u}$.

Proof. Because the function ψ has support contained in the interval $[-\delta, \delta]$, equation (32) implies v can have a discontinuity at u_* only if the distribution of S_n is highly concentrated: specifically,

$$P\{S_n + \tilde{v} \in [-\delta, \delta]\} \ge \frac{2}{\|\psi'\| \max(|\underline{u}|, \overline{u})}$$

for some $\tilde{v} \in [v_-, v_+]$, where v_- and v_+ are the right and left limits of v at u_* . In fact, because ψ' vanishes at the endpoints of $[-\delta, \delta]$, there exists $0 < \delta' < \delta$ such that

$$P\{S_n + \tilde{v} \in [-\delta', \delta']\} \ge \frac{1}{\|\psi'\| \max(|\underline{u}|, \overline{u})}.$$

Hence, because the function ψ is bounded away from 0 on the interval $[-\delta', \delta']$,

$$E\psi(\tilde{v} + S_n) \ge \frac{\min_{u \in [-\delta'\delta']} \psi(u)}{\|\psi'\|_{\infty} \max(|\underline{u}|, \overline{u})}.$$
(34)

Lemma 12 asserts that such strong concentration of the distribution of S_n can occur only if |v(u)| is vanishingly small in the interior of the interval $[\underline{u}, \overline{u}]$. In particular, if $\epsilon < (\|\psi'\| \max{(|\underline{u}|, \overline{u})})^{-1}$ and n is sufficiently large then $|v(u)| < \gamma_{\epsilon}/\sqrt{n}$ for all $u \in [\underline{u}_{\epsilon}, \overline{u}_{\epsilon}]$. But v(u) must satisfy the necessary condition (19) at all such u, so

$$E\psi(v(u)+S_n)|u| < 2\gamma_{\epsilon}/\sqrt{n}$$

for all $u \in [\underline{u}_{\epsilon}, \overline{u}_{\epsilon}]$, and in particular for $u = \overline{u}_{\epsilon}$ and $u = \underline{u}_{\epsilon}$. But by the mean value theorem,

$$|E\psi(\tilde{v}+S_n) - E\psi(v(\underline{u}_{\epsilon}) + S_n)| \le ||\psi'||_{\infty} |\tilde{v} - v(\underline{u}_{\epsilon})| \quad \text{and} \quad |E\psi(\tilde{v}+S_n) - E\psi(v(\overline{u}_{\epsilon}) + S_n)| \le ||\psi'||_{\infty} |\tilde{v} - v(\overline{u}_{\epsilon})|,$$

so when n is large the inequality (34) can only occur if $|\tilde{v}| > \Delta$, where

$$2\Delta = \frac{\min_{u \in [-\delta'\delta']} \psi(u)}{\|\psi'\|_{\infty}^2 \max(|\underline{u}|, \overline{u})}$$

Lemma 14 now implies that any such discontinuities can occur only within distance $\beta n^{-3/2}$ of one of the endpoints $\underline{u}, \overline{u}$.

6.2 Smoothness

Because Nash equilibria are monotone, by Lemma 4, they are necessarily differentiable almost everywhere. We will show that in fact differentiability must hold at *every* u, except near the endpoints \underline{u} , \overline{u} .

Lemma 17. *If* v(u) *is a Nash equilibrium then at every* u *where* v *is differentiable,*

$$E\psi(S_n + v(u)) + E\psi'(S_n + v(u))uv'(u) = 2v'(u).$$
(35)

Proof. Given the smoothness of the function ψ , the result follows from the chain and product rules.

Equation (35) can be rewritten as a first-order differential equation:

$$v'(u) = \frac{E\psi(S_n + v(u))}{2 - E\psi'(S_n + v(u))u}.$$
(36)

This differential equation becomes singular at any point where the denominator approaches 0, but is regular in any interval where $E\psi'(S_n+v(u))u\leq 1$. The following lemma implies regularity on any interval where |v(u)| remains sufficiently small.

Lemma 18. For any $\alpha > 0$ there exists $\beta = \beta(\alpha) > 0$ such that for any Nash equilibrium v(u), any $\tilde{v} \in \mathbb{R}$, any $u \in [u, \overline{u}]$, and all n,

$$E|\psi'(\tilde{v}+S_n)u| \ge \alpha \implies E\psi(\tilde{v}+S_n)|u| \ge \beta \text{ and}$$

$$E|\psi''(\tilde{v}+S_n)u| \ge \alpha \implies E\psi(\tilde{v}+S_n)|u| \ge \beta.$$
(37)

Proof. Recall that $\psi/2$ is a C^{∞} probability density with support $[-\delta, \delta]$, and that ψ is *strictly* positive in the open interval $(-\delta, \delta)$. Consequently, on any interval $J \subset (-\delta, \delta)$ where $|\psi'|$ (or $|\psi''|$) is bounded below by a positive number, so is ψ .

Fix $\epsilon > 0$ so small that $\epsilon \max{(\underline{u}, \overline{u})} < \alpha/2$. In order that $E|\psi'(\tilde{v} + S_n)u| \ge \alpha$, it must be the case that the event $\{|\psi'(\tilde{v} + S_n)| \ge \epsilon\}$ contributes at least $\alpha/2$ to the expectation; hence,

$$P\{|\psi'(\tilde{v}+S_n)| \ge \epsilon\} \ge \frac{\alpha}{2\|\psi'\|_{\infty} \max(\underline{u}, \overline{u})}.$$

But on this event the random variable $\psi(\tilde{v} + S_n)$ is bounded below by a positive number $\eta = \eta_{\epsilon}$, so it follows that

$$E\psi(\tilde{v}+S_n)|u| \ge \frac{\eta\alpha}{2\|\psi'\|_{\infty} \max(u,\overline{u})}.$$

A similar argument proves the corresponding result for ψ'' .

Lemma 19. There are constants $C, \alpha > 0$ such that for all sufficiently large n, any Nash equilibrium v(u) is continuously differentiable on any interval where $|v(u)| \leq C$ (and therefore, by Lemma 14, on $(\underline{u} + \beta n^{-3/2}, \overline{u} - \beta n^{-3/2})$, for some $\beta > 0$), and the derivative satisfies

$$\alpha \le \frac{v'(u)}{E\psi(v(u) + S_n)} \le \alpha^{-1}.$$
(38)

Proof. The function v(u) is differentiable almost everywhere, by Lemma 4, and at every point u where v(u) is differentiable the differential equation (36) holds. By Lemma 16, the sizes of discontinuities are bounded below, and so if C>0 is sufficiently small then a Nash equilibrium v(u) can have no discontinuities on any interval where $|v(u)| \leq C$. Furthermore, if C>0 is sufficiently small then by Lemma 18 and the necessary condition (19), we must have $E\psi'(v(u)+S_n) \leq 1$ on any interval where $|v(u)| \leq C$. Because the functions $v \mapsto E\psi(S_n+v)$ and $v \mapsto E\psi'(S_n+v)$ are continuous (by dominated convergence), it now follows from Equation (36) that if C>0 is sufficiently small then on any interval where $|v(u)| \leq C$ the function v'(u) extends to a continuous function. Finally, because the denominator in equation (36) is at least 1 and no larger than $2+\|\psi'\|_{\infty}$, the inequalities (38) follow.

Similar arguments show that Nash equilibria have derivatives of higher orders provided the sample size is sufficiently large. The analysis below will require information about the second derivative v''(u), which can be obtained by differentiating under the expectations in (36):

$$v''(u) = \frac{E\psi'(v(u) + S_n)v'(u)}{2 - E\psi'(S_n + v(u))u} + \frac{E\psi(v(u) + S_n)(E\psi''(v(u) + S_n)v'(u)u + E\psi'(v(u) + S_n)}{(2 - E\psi'(S_n + v(u)u))^2}.$$
 (39)

A repetition of the proof of Lemma 19 now shows that for suitable constants $C, \beta > 0$ and all sufficiently large n, any Nash equilibrium v(u) is twice continuously differentiable on any interval where $|v(u)| \leq C$ and for some $\tilde{\alpha} > 0$ satisfies the inequalities

$$\tilde{\alpha} \le \frac{v''(u)}{E\psi(v(u) + S_n)} \le \tilde{\alpha}^{-1}.$$
(40)

7 Approximate Proportionality

7.1 The approximate proportionality rule

The information that we now have about the form of Nash equilibria can be used to sharpen the heuristic argument given in Subsection 4 to support the "approximate proportionality rule". Recall that any Nash equilibrium $v(\cdot)$ must satisfy the necessary condition $2v(u) = E\psi(v(u) + S_n)u$. We have shown in Lemma 14 that for any Nash equilibrium, v(u) must be small except in the extreme tails of the distribution (in particular, for all u at distance much more than $n^{-3/2}$ from both endpoints $\underline{u}, \overline{u}$). Because ψ is uniformly continuous, it follows that the expectation $E\psi(v(u) + S_n)$ cannot differ by very much from $E\psi(S_n)$.

Unfortunately, this argument only shows that the approximation $2v(u) \approx E\psi(S_n)u$ is valid up to an error of size $\epsilon_n|u|$ where $\epsilon_n \to 0$ as $n \to \infty$. However, as $n \to \infty$ the expectation $E\psi(S_n) \to 0$, and so the error in the approximation above might be considerably larger than the approximation itself. Proposition 20 makes the stronger assertion that when n is large the relative error in the approximate proportionality rule is small.

Proposition 20. For any $\epsilon > 0$ there exists constants $n_{\epsilon} < \infty$ and $\beta < \infty$ such that if $n \ge n_{\epsilon}$ then for any Nash equilibrium v(u) and for all $u \in [\underline{u}_{\beta n^{-3/2}}, \overline{u}_{\beta n^{-3/2}}]$,

$$(1 - \epsilon)E\psi(S_n)|u| \le |2v(u)| \le (1 + \epsilon)E\psi(S_n)|u|. \tag{41}$$

Furthermore, for all sufficiently large n any Nash equilibrium v(u) with no discontinuities must satisfy (41) for all $u \in [\underline{u}, \overline{u}]$.

Proof of Proposition 20. Because ψ is C^{∞} and has compact support, the function $v \mapsto E\psi(v+S_n)$ is differentiable with derivative $E\psi'(v+S_n)$. Consequently, by Taylor's theorem, for every u there exists $\tilde{v}(u)$ intermediate between 0 and v(u) such that

$$2v(u) = E\psi(v(u) + S_n)u = E\psi(S_n)u + E\psi'(\tilde{v}(u) + S_n)v(u)u.$$
(42)

We will argue that for all C>0 sufficiently small, if $|2v(u)| \le C$ then the expectation $E\psi'(\tilde{v}(u)+S_n)$ remains below ϵ in absolute value, provided n is sufficiently large. Lemma 14 will then imply that for some $\beta<\infty$ independent of n the inequalities (41) hold for all $u\in(\underline{u},\overline{u})$ at distance greater than $\beta n^{-3/2}$ from the endpoints $\underline{u},\overline{u}$.

If $|2v(u)| \leq C$ then $|E\psi(v(u) + S_n)| \leq C/\max(|\underline{u}|, \overline{u})$, by the necessary condition (19). If $C < \Delta$, where Δ is the discontinuity threshold established in Lemma 16, then v(u) is continuous on any interval $[0, u_C]$ where $|v(u)| \leq C$, so for each u in this interval there exists $u' \in [0, u]$ such that $\tilde{v}(u) = v(u')$. Consequently, $|E\psi(\tilde{v}(u) + S_n)| \leq C/\max(|\underline{u}|, \overline{u})$. But Lemma 18 implies that for any

 $\epsilon > 0$, if C > 0 is sufficiently small then for all large n and any Nash equilibrium v(u),

$$|E\psi'(\tilde{v}(u) + S_n)| < \epsilon$$

on any interval $[0,u_C]$ where $|v(u)| \leq C$. Thus, the error in the approximation (42) will be small when n is large and |2v(u)| < C, for u > 0. A similar argument applies for $u \leq 0$. This proves that (41) holds for all $u \in (\underline{u}, \overline{u})$ at distance greater than $\beta n^{-3/2}$ from the endpoints $\underline{u}, \overline{u}$.

Finally, suppose that v(u) is a Nash equilibrium with no discontinuities. By Lemma 14, for any C>0 there exists $\beta<\infty$ exists such that $|v(u)|\leq C/2$ except at arguments u within distance $\beta/n^{3/2}$ of one of the endpoints. Moreover, by Lemma 19, if C is sufficiently small then on any interval where $|v(u)|\leq C$ the function v is differentiable, with derivative $v'(u)\leq C'$ for some constant $C'<\infty$ not depending on v or on the particular Nash equilibrium. Because v(u) is continuous up to \overline{u} , if $v(u)\geq C$ for some $v>\overline{u}-\beta n^{-3/2}$, then by the intermediate value theorem there would a smallest $v'\in [\overline{u}-\beta n^{-3/2},\overline{u}]$ at which v'(u')=C. But then v would be differentiable all the way up to v', with derivative bounded above by v', and so

$$C = v(u') = v(\overline{u} - \beta n^{-3/2}) + \int_{\overline{u} - \beta n^{-3/2}}^{u'} v'(u) du$$

$$\leq C/2 + \int_{\overline{u} - \beta n^{-3/2}}^{u'} v'(u) du$$

$$\leq C/2 + C'\beta n^{-3/2},$$

which is impossible for large n. A similar argument shows that for large n, if v(u) has no discontinuities then v cannot attain the value -C near \underline{u} . Therefore, for all sufficiently large n, if v has no discontinuities then $|v(u)| \leq C$ for all $u \in [\underline{u}, \overline{u}]$, and so by the preceding argument it follows that v(u) must satisfy the proportionality relations (41) for all $u \in [\underline{u}, \overline{u}]$.

7.2 Consequences of Proposition 20

Proposition 20 puts strong constraints on the distribution of the vote total S_n in a Nash equilibrium. According to this proposition, the approximate proportionality rule (41) holds for all $u \in [\underline{u}, \overline{u}]$ except those values u within distance $\beta n^{-3/2}$ of one of the endpoints $\underline{u}, \overline{u}$. Fix $\beta = \beta(\epsilon) > 0$ as in Proposition 20, and call $[\overline{u}_{\beta n^{-3/2}}, \overline{u}] \cup [\underline{u}, \underline{u}_{\beta n^{-3/2}}]$ the *extremist range*. Denote by G the event that the sample U_1, U_2, \ldots, U_n contains no values in the extremist range. By Proposition 20, on the event G the approximate proportionality rule (41) will apply for each agent; furthermore, for Nash equilibria with no discontinuities, (41) holds for all $u \in [\underline{u}, \overline{u}]$. Thus, *conditional*

on the event G (or, for continuous Nash equilibria, *unconditionally*) the random variables $v(U_i)$ are (at least for sufficiently large n) bounded above and below by $2E\psi(S_n)\overline{u}$ and $2E\psi(S_n)\underline{u}$, and so Hoeffding's inequality applies.

Corollary 21. Let G be the event that the sample U_i contains no values u in the extremist range. Then for all sufficiently large n and any Nash equilibrium v(u),

$$P(|S_n - E(S_n|G)| \ge tE\psi(S_n) | G) \le \exp\{-t^2/2nE\psi(S_n)^2(\overline{u} - \underline{u})^2\};$$
(43)

and for any Nash equilibrium with no discontinuities,

$$P(|S_n - ES_n| \ge tE\psi(S_n)) \le \exp\{-t^2/2nE\psi(S_n)^2(\overline{u} - \underline{u})^2\}.$$

Proposition 20 also implies uniformity in the normal approximation to the distribution of S_n , because the proportionality rule (41) guarantees that the ratio of the third moment to the 3/2 power of the variance of $v(U_i)$ is uniformly bounded. Hence, by the Berry-Esseen theorem, we have the following corollary.

Corollary 22. There exists $\kappa < \infty$ such that for all sufficiently large n and any Nash equilibrium v(u), the vote total S_n satisfies

$$\sup_{t \in \mathbb{R}} |P(S_n - E(S_n|G) \le t\sqrt{\operatorname{var}(S_n|G)} |G) - \Phi(t)| \le \kappa n^{-1/2},$$

and consequently, since $1 - P(G) = O(n^{-1/2})$, there exists $\kappa' < \infty$ such that

$$\sup_{t\in\mathbb{R}} |P\{S_n - E(S_n|G) \le t\sqrt{\operatorname{var}(S_n|G)}\} - \Phi(t)| \le \kappa' n^{-1/2}.$$

Here Φ denotes the standard normal cumulative distribution function.

8 Unbalanced Populations: Proofs of Theorems 2–3

8.1 Concentration of the vote total

Lemma 23. If $\mu > 0$ then for all large n no Nash equilibrium v(u) has a discontinuity at a nonnegative value of u. Moreover, for any $\epsilon > 0$, if n is sufficiently large then in any Nash equilibrium the vote total S_n must satisfy

$$\delta - \epsilon \le ES_n \le \delta + \epsilon + \sqrt{2|\underline{u}|}$$
 and (44)

$$P\{|S_n - ES_n| > \epsilon\} < \epsilon. \tag{45}$$

In addition, for any $\epsilon > 0$ there exists $\gamma > 0$ such that if n is sufficiently large and v(u) is a Nash equilibrium with no discontinuities, then

$$P\{|S_n - ES_n| > \epsilon\} < e^{-\gamma n}. \tag{46}$$

Proof. By Lemma 16, a Nash equilibrium v(u) can have no discontinuities at distance greater than $Cn^{-3/2}$ of one of the endpoints $\underline{u}, \overline{u}$. Agents with such utilities are designated *extremists*; if G is the event that the sample U_1, U_2, \ldots, U_n contains no extremists, then

$$P(G^c) \sim C(f(\underline{u}) + f(\overline{u}))/\sqrt{n}$$
.

Since not even an extremist would ever buy more than $\max(\sqrt{2|\underline{u}|}, \sqrt{2\overline{u}})$ votes, it follows that the extremist contribution to ES_n is of size at most $O(n^{-1/2})$.

By Proposition 20, Nash equilibria v(u) obey the approximate proportionality rule (41) except in the extremist range. Consequently, if S'_n is the vote total of the non-extremists among the first n voters, then for any $\epsilon > 0$, if n is large,

$$E\psi(S_n)\mu(1-\epsilon) \le ES_n'/n \le E\psi(S_n)\mu(1+\epsilon). \tag{47}$$

Since $\mu > 0$ and $ES_n - ES_n' = O(n^{-1/2})$, it follows that $ES_n \ge C' n^{-1/2}$ for some $C' < \infty$ not depending on n.

Suppose now that $ES_n < \delta - 2\epsilon'$ for some small $\epsilon' > 0$. Since the constant $\epsilon > 0$ in (47) can be chosen arbitrarily small relative to ϵ' , it follows that $nE\psi(S_n)\mu \leq \delta - \epsilon'$. But then (47), together with Hoeffding's inequality (43) and the fact that $P(G^c) = O(n^{-1/2})$, implies that $-\delta/2 \leq S_n \leq \delta - \epsilon'/2$ with probability tending to 1 as $n \to \infty$. We would then have

$$E\psi(S_n) \ge (1 - \epsilon) \min_{v \in [-\delta/2, \delta - \epsilon'/2]} \psi(v).$$

(Recall that ψ is bounded away from 0 on any compact sub-interval of $(-\delta, \delta)$. This, however, would contradict the hypothesis that $nE\psi(S_n)<\delta-\epsilon'/2$. This proves that for any $\epsilon>0$, if n is sufficiently large then in any Nash equilibrium,

$$ES_n \geq \delta - \epsilon$$
.

Next, suppose that $ES_n > \delta + \sqrt{2|\underline{u}|} + 4\epsilon'$ for some $\epsilon' > 0$. The proportionality rule (41) (applied with some $\epsilon > 0$ small relative to ϵ') then implies that $nE\psi(S_n) > \delta + \sqrt{2|\underline{u}|} + 3\epsilon'$, provided n is large. Hence, by the Hoeffding inequality (43), there exists $\gamma = \gamma(\epsilon') > 0$ such that

$$P(S_n \le \delta + \sqrt{2|\underline{u}|} + \epsilon' | G) \le e^{-\gamma n},$$

because on the event $S_n \leq \delta + \sqrt{2|\underline{u}|}$ the sum S_n must deviate from its expectation by more than $nE\psi(S_n)\epsilon'$. Hence, for all $v \geq -\sqrt{2|\underline{u}|}$,

$$E\psi(v+S_n) \le e^{-\gamma n} \|\psi\|_{\infty} + P(G^c) \|\psi\|_{\infty}.$$

Thus, v(u) must be vanishingly small (of order no greater than $O(n^{-1/2})$) for all $u \in [\underline{u}, \overline{u}]$, and so by Lemma 16 the function $v(\cdot)$ can have no discontinuities in $[\underline{u}, \overline{u}]$. But then the proportionality rule (41) would hold for all $u \in [\underline{u}, \overline{u}]$, and so another application of Hoeffding's inequality implies

$$P(S_n \le \delta + \sqrt{2|\underline{u}|}) \le e^{-\gamma n} \implies E\psi(S_n) \le e^{-\gamma n} \|\psi\|_{\infty},$$

which contradicts Lemma 9. This proves that for every $\epsilon > 0$, if n is sufficiently large then for every Nash equilibrium,

$$ES_n \le \delta + \sqrt{2|\underline{u}|} + \epsilon,$$

thus establishing assertion (44).

Because ES_n is now bounded away from 0 and ∞ , it follows as before that $nE\psi(S_n)$ is bounded away from 0 and ∞ , and so the proportionality rule (41) implies the conditional variance of S_n given the event G is $O(n^{-1})$. The assertion (45) therefore follows from Chebyshev's inequality and the bound $P(G^c) = O(n^{-1/2})$. Given (44) and (45), we can now conclude that there can be no discontinuities at nonnegative values of u, because in view of Lemma 16, the monotonicity of Nash equilibria, and the necessary condition (19), such discontinuities would entail that

$$E\psi(v(\overline{u}) + S_n)\overline{u} \ge 2\Delta,$$

which is incompatible with (44) and (45), because for small $\epsilon > 0$ the function $\psi(w)$ is uniformly small for $w \geq \delta - 2\epsilon$.

Finally, if v is a Nash equilibrium with no discontinuities then Corollary 21 implies the exponential bound (46).

8.2 Identification of the concentration point

Lemma 24. Assume that $\mu > 0$, and let (α, w) be the solution of the Optimization Problem (cf. section 1.4.2), if one exists, or let $\alpha = \delta$ if not. Then for any $\epsilon > 0$, if n is sufficiently large then in every Nash equilibrium,

$$\left| \frac{1}{2} E \psi(S_n) - \alpha \mu^{-1} \right| < \epsilon. \tag{48}$$

Proof. The lemma is equivalent to the assertion that $|ES_n - \alpha| \to 0$, by the proportionality rule (41). We will prove this in two steps, by first showing that for sufficiently large n the expectation ES_n cannot be smaller than $\alpha - 3\epsilon$, and then that it cannot be larger than $\alpha + 3\epsilon$.

If $\alpha = \delta$ then Lemma 23 implies that $ES_n < \alpha - \epsilon$ is impossible for large n, so to prove that $ES_n \ge \alpha - 3\delta$ for large n it suffices to consider the case where $\alpha > \delta$. Suppose that $ES_n < \alpha - 3\epsilon$, where $\epsilon > 0$ is small enough that $\alpha - 4\epsilon > \delta$; then by Lemma 23,

$$P\{\delta \leq S_n \leq \alpha - 2\epsilon\} \longrightarrow 1 \text{ as } n \to \infty.$$

But since (α, w) satisfies (16), for all sufficiently small $\epsilon' > 0$ we must have

$$(1 - \Psi(w))|u| > (\alpha - \epsilon - w)^2$$

for all u in a neighborhood $[\underline{u},\underline{u}+\varrho]$, where $\varrho>0$. Hence, a voter with value u in this neighborhood could with probability near 1, improve her expected utility payoff from u to $\Psi(w)u$, at a cost of $(\alpha-\epsilon-w)^2$, and so all such voters would defect from the equilibrium strategy. This is a contradiction; hence, we conclude that for all large n, in any equilibrium, $ES_n \geq \alpha - 3\epsilon$.

Now suppose that $ES_n > \alpha + 3\epsilon$. Then by Lemma 23, the one-out vote total S_n would exceed $\alpha - 2\epsilon$ with probability near 1, for large n. But by (16), for all $w \leq \delta$,

$$(\alpha + 2\epsilon - w)^2 \ge 4\epsilon^2 + (1 - \Psi(w))|\underline{u}|,$$

so it would be sub-optimal for a voter with value $u=\underline{u}$ to buy more than $\Delta/2$ negative votes, where Δ is the discontinuity threshold. Thus, by Lemmas 16 and 23, all Nash equilibria are continuous when n is sufficiently large. But then assertion (46) would imply

$$P\{S_n \le \alpha - 2\epsilon\} < e^{-\varrho n}$$

which in turn would ensure that

$$E\psi(S_n) \le \|\psi\|_{\infty} e^{-\varrho n}.$$

This is impossible, because the proportionality rule (41) would then imply that for some constant $C < \infty$ not depending on n or the particular equilibrium, $\|v\|_{\infty} \le Ce^{-\varrho n}$, contradicting Lemma 9.

8.3 Proof of Theorem 2

Proof of assertion (12). The asymptotic efficiency of quadratic voting in the unbalanced case $\mu > 0$ is a direct and easy consequence of Lemma 23. This implies that for any $\epsilon > 0$, the probability that the vote total $S_N = S_n + v(U_{n+1})$ will fall below $\delta - 2\epsilon$ is less than ϵ for all large n, and so by the continuity of the payoff function Ψ , for any $\epsilon > 0$

$$P\{\Psi(S_N) \le 1 - \epsilon\} < \epsilon$$

for all sufficiently large N and all Nash equilibria. Moreover, the law of large numbers guarantees that for large N,

$$P\{|N^{-1}U - \mu| \ge \epsilon\} < \epsilon \text{ where } U := \sum_{i=1}^{N} U_i.$$

Because the random variables U and $\Psi(S_N)$ are bounded, it therefore follows that for any $\epsilon > 0$, if N is sufficiently large then in any equilibrium

$$\left| \frac{E[U\Psi(S_N)]}{2E|U|} - 1 \right| < \epsilon.$$

Proof of assertions (13)–(14). The second assertion (14) will follow immediately from the first, by the law of large numbers for the sequence U_1, U_2, \ldots , and the assertion (14) will follow directly from the approximate proportionality rule (41) and Lemma 24.

8.4 Proof of Theorem 3

Assume now that $\mu>0$ and that the Optimization Problem (16) has a unique solution (α,w) . We will first prove that for large n, every Nash equilibrium has a discontinuity u_* near \underline{u} , and then we will argue that this discontinuity must occur very near $\underline{u}+\zeta n^{-2}$ for a constant $\zeta>0$ depending only on the payoff function Ψ and the sampling distribution F.

We have shown, in Lemma 24, that for any $\epsilon > 0$, every Nash equilibrium must satisfy $|ES_n - \alpha| < \epsilon$ when n is sufficiently large. In addition, we have shown in Lemma 23 that if n is large and v(u) is a Nash equilibrium with no discontinuities, then $P\{|S_n - ES_n| > \epsilon\}$ decays exponentially fast in n. Because $\alpha > \delta$, we may choose $\epsilon > 0$ so small that $\alpha - 3\epsilon > \delta$. It then follows that for a suitable constant $\gamma > 0$, if n is large and v(u) is a Nash equilibrium with no discontinuities, then

$$P\{S_n \le \delta\} \le e^{-\gamma n}.$$

But as in the proof of Lemma 24, this would imply that $E\psi(S_n)$ decays exponentially with n, which is impossible in view of the proportionality rule and Lemma 9. Therefore, for large n every Nash equilibrium v(u) has a discontinuity. Lemma 23 asserts that there are no discontinuities at points $u \in [0, \overline{u}]$, so any discontinuity

must be located in $[\underline{u}, 0)$. Lemma 16 implies any such discontinuity must occur at a point within distance $O(n^{-3/2})$ of \underline{u} .

Let v(u) be a Nash equilibrium, and let u_* be the rightmost point of discontinuity of v. By Lemma 16, the size of any discontinuity is at least Δ , so $v(u) < -\Delta$ for every $u < u_*$. Obviously, the expected payoff for an agent with utility u must exceed the expected payoff under the alternative strategy of buying no votes. The latter expectation is approximately \underline{u} , because S_n is highly concentrated near $ES_n > \alpha - \epsilon$ and so $E\Psi(S_n) \approx 1$. On the other hand, the expected payoff at $u < u_*$ for an agent playing the Nash strategy is approximately

$$\Psi(\alpha - v(u))(\underline{u}) - v(u)^2$$
,

an improvement over the alternative strategy of buying no votes of about

$$(1 - \Psi(\alpha - v(u))|u| - v(u)^2.$$

In order that this difference be nonnegative, it must be the case that $|v(u)| \approx \alpha - w$, because by hypothesis, (α, w) is the unique pair such that relations (11) hold. Because this approximation is valid for all $u \in [\underline{u}, u_*)$, it follows by Lemma 16 that v(u) cannot have another discontinuity in the interval $[\underline{u}, u_*)$. This also proves assertion (ii) of Theorem 3, that

$$v(\underline{u}) = -(\alpha - w).$$

Because u_* must be within distance $Cn^{-3/2}$ of \underline{u} , the probability that an extremist exists in the sample U_1, U_2, \ldots, U_n is of order $nf(\underline{u})(u_* - \underline{u}) = O(n^{-1/2})$, and the conditional probability that more than one extremist exists given that at least one does is of order $O(n^{-1/2})$. Consequently, because the distribution of S_n is highly concentrated near α (cf. Lemma 23), where $\psi=0$, the major contribution to the expectation $E\psi(S_n)$ comes from samples with exactly one extremist; hence,

$$E\psi(S_n + v(\underline{u})) \approx n\psi(w)f(\underline{u})(u_* + |\underline{u}|) + O(n^{-1/2}(u_* + |\underline{u}|)).$$

On the other hand, because $ES_n \approx \alpha$, the proportionality rule (41) implies $nE\psi(S_n) \approx \alpha$, and so

$$n^2 \mu \psi(w) f(\underline{u}) (u_* + |\underline{u}|) \approx \alpha \implies u_* - \underline{u} \sim \zeta n^{-2},$$

where ζ is the unique solution of the equation $\alpha = \zeta \psi(w) f(\underline{u})$.

9 Balanced Populations: Proof of Theorem 1

9.1 Continuity of Nash equilibria

Proposition 25. If $\mu = 0$, then for all sufficiently large values of n no Nash equilibrium v(u) has a discontinuity in $[\underline{u}, \overline{u}]$. Moreover, for any $\epsilon > 0$, if n is sufficiently large every Nash equilibrium v(u) satisfies

$$||v||_{\infty} \le \epsilon. \tag{49}$$

Proof. The size of any discontinuity is bounded below by a positive constant Δ , by Lemma 16, so it suffices to prove the assertion (49). Fix $\epsilon > 0$, and suppose that in some Nash equilibrium there is a value $u_* \in [\underline{u}, \overline{u}]$ (necessarily in the extremist range $[\underline{u}, \overline{u}] \setminus [\underline{u}_{\beta n^{-3/2}}, \overline{u}_{\beta n^{-3/2}}]$, by Lemma 14) such that $|v(u_*)| \geq \epsilon$; then $E\psi(v(u_*) + S_n)|u| \geq 2\epsilon$, by the necessary condition (19), and so

$$P\{S_n + v(u_*) \in [-\delta, \delta]\} \ge \frac{2\epsilon}{\|\psi\|_{\infty} \max(|\underline{u}|, \overline{u})}.$$
 (50)

We will show that if n is large then this leads to a contradiction. Assume for convenience that $v(u_*)>0$ (and hence that $u_*\geq \overline{u}_{\beta n^{-3/2}}$, where $\beta=\beta(\epsilon)$ is as in Lemma 14); the case $v(u_*)<0$ can be argued in similar fashion.

By Proposition 12, there exists $\gamma=\gamma(\epsilon)>0$ such that if n is sufficiently large then any Nash equilibrium v(u) satisfying $\|v\|_{\infty}>\epsilon$ must also satisfy $|v(u)|\leq \gamma/\sqrt{n}$ for all u not within distance ϵ of one of the endpoints $\underline{u},\overline{u}$. Hence, the approximate proportionality relation (41) implies

$$E\psi(S_n) \le \frac{C}{\sqrt{n}} \tag{51}$$

for a suitable $C=C(\gamma)<\infty$ independent of n. It now follows that $v(u_*)\geq \delta/2$, since otherwise (50) would be incompatible with (51). Moreover, the relation (41) holds for all u not within distance $\beta n^{-3/2}$ of one of the endpoints, so for some C'<2C

$$|v(u)| \le C'/\sqrt{n}$$
 for all $u \in [\underline{u}_{\beta n^{-3/2}}, \overline{u}_{\beta n^{-3/2}}]$ (52)

consequently,

$$var(S_n | G) \le (2C)^2 E U_1^2 := C''$$

where G is the event that the sample U_1, U_2, \ldots, U_n contains no extremists (voters with values within distance $\beta n^{-3/2}$ of $\{\underline{u}, \overline{u}\}$).

Recall that the Berry-Essen theorem (cf. Corollary 22) implies that

$$|P\{S_n - E(S_n|G) \le t\sqrt{\operatorname{var}(S_n|G)}\} - \Phi(t)| \le \kappa n^{-1/2}$$
 for all $t \in \mathbb{R}$.

Therefore, in order that inequality (50) hold, it must be that for a suitable constant $C'''' < \infty$ independent of n,

$$|v(u) - E(S_n|G)| \le C''' \sqrt{\operatorname{var}(S_n|G)},$$

and hence, since $\operatorname{var}(S_n|G)$ remains bounded as $n\to\infty$, so must the conditional expectation $E(S_n|G)$. To complete the proof, we will show that if $\mu=0$ then $\operatorname{var}(S_n|G)$ must remain bounded *below* by a positive constant independent of n; the Berry-Esseen bound will then imply that $P\{S_n\in[-\delta/2,\delta/2]\}$ remains bounded below, which in turn implies that $E\psi(S_n)$ is bounded below, in contradiction to (51).

To show that $var(S_n|G)$ remains bounded below we use the necessary condition (19) and Taylor's theorem to obtain

$$2v(u) = E\psi(S_n)u + E\psi'(S_n)v(u)u + \frac{1}{2}E\psi''(S_n)v(u)^2u + \frac{1}{6}E\psi'''(S_n + \tilde{v}(u))v(u)^3u$$

for some $\tilde{v}(u)$ intermediate between 0 and v(u). Since $E\psi(S_n)=O(n^{-1/2})$, by (51), Lemma 18 implies that both $E\psi'(S_n)$ and $E\psi''(S_n)$ are bounded by constants $\epsilon_n\to 0$. Since $|v(u)|\leq C'/\sqrt{n}$ for all u not within distance $\beta n^{-3/2}$ of the endpoints, it follows that for all such u,

$$2v(u) = E\psi(S_n)u + E\psi'(S_n)v(u)u + R_n(u)$$

= $E\psi(S_n)u + \frac{1}{2}E\psi'(S_n)E\psi(S_n)u^2 + R_n^*(u)$ (53)

where $|R_n^*(u)| \le 2\epsilon_n/n$. Using the hypothesis that $\mu = 0$ and the fact that $P(G) = O(n^{-1/2})$, we infer that for some sequence $\epsilon'_n \to 0$

$$|2E(S_n|G) - \frac{n}{2}E\psi'(S_n)E\psi(S_n)\sigma^2| \le \epsilon'_n \quad \text{and}$$
$$|\operatorname{var}(S_n|G) - \frac{n}{4}(E\psi(S_n))^2| \le \epsilon'_n.$$

Fix $\alpha>0$ small, and suppose that $\sqrt{n}E\psi(S_n)<\alpha$. Since $E(S_n|G)>\delta/2$, we would then have $\sqrt{n}E\psi'(S_n)>\delta/\alpha$ for all sufficiently large n. Now by our standing model assumptions (2)–(5), the function ψ' is odd and positive on $(-\delta,0)$, so in order that $\sqrt{n}E\psi'(S_n)>\delta/\alpha$ we must have

$$P(S_n < 0) \ge \delta/(\alpha \|\psi'\|_{\infty} \sqrt{n}).$$

But by Hoeffding's inequality (cf. Corollary 21), for all large n,

$$P(S_n < 0) \le P(G^c) + P(|S_n - E(S_n|G)| > \delta/2)$$

$$\le \beta'/\sqrt{n} + P(|S_n - E(S_n|G)| > \sqrt{n}\delta E\psi(S_n)/2\alpha|G)$$

$$\le \beta'/\sqrt{n} + \exp\{-n\delta^2/2(\overline{u} - \underline{u})^2\},$$

where $\beta'=2\beta(f(\underline{u})+f(\overline{u}))$. Now $\beta=\beta(\epsilon)$ is fixed, so if $\alpha>0$ is sufficiently small then the opposing inequalities for $P(S_n<0)$ are incompatible. Thus, there exists some constant $\alpha>0$ such that $\mathrm{var}(S_n|G)\geq \alpha$ for all large n.

Because $||v||_{\infty}$ is small for any Nash equilibrium v, the distribution of the vote total S_n cannot be too highly concentrated. This in turn implies the proportionality constant $E\psi(S_n)$ in (41) cannot be too small.

Lemma 26. For any $C < \infty$ a $n_C < \infty$ exists such that for all $n \ge n_C$ and every Nash equilibrium,

$$nE\psi(S_n) \ge C. \tag{54}$$

Proof. By the approximate proportionality rule (41) and the necessary condition (19), for any $\epsilon > 0$ and all sufficiently large n,

$$|ES_n| \le n\epsilon E\psi(S_n)E|U|.$$

Thus, by Hoeffding's inequality (Corollary 21), if $nE\psi(S_n) < C$ then the distribution of S_n must be highly concentrated in a neighborhood of 0. But if this were so we would have, for all large n,

$$E\psi(S_n) \approx \psi(0) > 0,$$

which is a contradiction.

9.2 Edgeworth expansions

For the analysis of the case $\mu_U=0$ refined estimates of the errors in the approximate proportionality rule (41) will be necessary. We derive these from the Edgeworth expansion for the density of a sum of independent, identically distributed random variables (cf. [3], Ch. XVI, sec. 2, Th. 2). The relevant summands here are the random variables $v(U_i)$, and because the function v(u) depends on the particular Nash equilibrium (and hence also on n), we must employ a version of the Edgeworth expansion in which the error is precisely quantified. The following variant of Feller's Theorem 2 (which can be proved in the same manner as in [3]) will suffice for our purposes.

Proposition 27. Let Y_1, Y_2, \ldots, Y_n be independent, identically distributed random variables with mean $EY_1 = 0$, variance $EY_1^2 = 1$, and finite 2rth moment $E|Y_1|^{2r} = \mu_{2r} \le m_{2r}$. Assume the distribution of Y_1 has a density $f_1(y)$ whose Fourier transform \hat{f}_1 satisfies $|\hat{f}_1(\theta)| \le g(\theta)$, where g is a C^{2r} function such that $g \in L^{\nu}$ for some $\nu \ge 1$ and such that for every $\epsilon > 0$,

$$\sup_{|\theta| > \epsilon} g(\theta) < 1. \tag{55}$$

Then for some sequence $\epsilon_n \to 0$ depending only on m_{2r} and on the function g, the density $f_n(y)$ of $\sum_{i=1}^n Y_i/\sqrt{n}$ satisfies

$$\left| f_n(x) - \frac{e^{-x^2/2}}{\sqrt{2\pi n}} \left(1 + \sum_{k=3}^{2r} n^{-(k-2)/2} P_k(x) \right) \right| \le \frac{\epsilon_n}{n^{-r+1}}$$
 (56)

for all $x \in \mathbb{R}$, where $P_k(x) = C_k H_k(x)$ is a multiple of the kth Hermite polynomial $H_k(x)$, and C_k is a continuous function of the moments $\mu_3, \mu_4, \dots, \mu_k$ of Y_1 .

The following lemma ensures that in any Nash equilibrium the sums $S_n = \sum_{i=1}^n v(U_i)$, after suitable renormalization, meet the requirements of Proposition 27.

Lemma 28. There exist constants $0 < \sigma_1 < \sigma_2 < m_{2r} < \infty$ and a function $g(\theta)$ satisfying the hypotheses of Proposition 27 (with r=4) such that for all sufficiently large n and any Nash equilibrium v(u) the following statement holds. If $w(u) = 2v(u)/E\psi(S_n)$

- (a) $\sigma_1^2 < \text{var}(w(U_i)) < \sigma_2^2$;
- (b) $E|w(U_i) Ew(U_i)|^{2r} \le m_{2r}$; and
- (c) the random variables $w(U_i)$ have density $f_W(w)$ whose Fourier transform is bounded in absolute value by g.

Proof. These statements are consequences of the proportionality relations (41) and the smoothness of Nash equilibria. By Proposition 25, Nash equilibria are continuous on $[\underline{u}, \overline{u}]$ and for large n satisfy $\|v\|_{\infty} < \epsilon$, where $\epsilon > 0$ is any small constant. Consequently, by Proposition 20, the proportionality relations (41) hold on the entire interval $[\underline{u}, \overline{u}]$. Because $EU_1 = 0$, it follows that for any $\epsilon > 0$, if n is sufficiently large then $|Ew(U_i)| < \epsilon$, and so assertions (a)–(b) follow routinely from (41).

The existence of the density $f_W(w)$ follows from the smoothness of Nash equilibria, which was established in Subsection 6.2. In particular, by Lemma 19, inequalities (40), and the proportionality relations (41), if the sample size n is sufficiently large and v is any continuous Nash equilibrium then v is twice continuously differentiable on $[\underline{u}, \overline{u}]$, and constants $\alpha, \beta > 0$ exist not depending on n or on the particular Nash equilibrium such that the derivatives satisfy

$$\alpha \le \frac{v'(u)}{E\psi(S_n)} \le \alpha^{-1} \quad \text{and} \quad \beta \le \frac{v''(u)}{E\psi(S_n)} \le \beta^{-1}$$
 (57)

for all $u \in [\underline{u}, \overline{u}]$. Consequently, if U is a random variable with density f(u) the random variable $W := 2v(U)/E\psi(S_n)$ has density

$$f_W(w) = f(u)E\psi(S_n)/(2v'(u))$$
 where $w = 2v(u)/E\psi(S_n)$. (58)

Furthermore, since by Lemma 25 $||v||_{\infty}$ and therefore also $E\psi(S_n)$ are small, Lemma 19 and inequalities (40), together with the proportionality relations (41), imply that the density $f_W(w)$ is continuously differentiable, and its derivative

$$f'_W(w) = \frac{f'(u)(E\psi(S_n))^2}{4v'(u)^2} - \frac{f(u)(E\psi(S_n))^2v''(u)}{4v'(u)^3}$$

satisfies

$$|f_W'(w)| \le \kappa \tag{59}$$

where $\kappa < \infty$ is a constant that does not depend on either n or on the choice of Nash equilibrium.

The last step is to prove the existence of a dominating function $g(\theta)$ for the Fourier transform of f_W . We do this in three pieces: (i) for values $|\theta| \leq \gamma$, where $\gamma > 0$ is a small fixed constant; (ii) for values $|\theta| \geq K$, where K is a large but fixed constant; and (iii) for $\gamma < |\theta| < K$. Region (i) is easily dealt with, in view of the bounds (a)–(b) on the second and third moments and the estimate $|Ew(U)| < \epsilon'$, as these together with Taylor's theorem imply that for all $|\theta| < 1$,

$$|\hat{f}_W(\theta) - (1 + i\theta Ew(U) - \theta^2 \operatorname{var}(w(U))/2| \le m_3 |\theta|^3.$$

Next consider region (ii), where $|\theta|$ is large. Integration by parts shows that

$$\hat{f}_W(\theta) = \int_{w(\underline{u})}^{w(\overline{u})} f_W(w) e^{i\theta w} dw = -\int_{w(\underline{u})}^{w(\overline{u})} \frac{e^{i\theta w}}{i\theta} f'_W(w) dw + \frac{e^{i\theta w}}{i\theta} f_W(w) \Big|_{wu}^{w(\overline{u})};$$

because $f_W(w)$ is uniformly bounded at $w(\underline{u})$ and $w(\overline{u})$, by (57) and (58), and because $|f_W'(w)| \le \kappa$, by (59), it follows that there is a constant $C < \infty$ such that for all sufficiently large n and all Nash equilibria,

$$|\hat{f}_W(\theta)| \le C/|\theta| \quad \forall \ \theta \ne 0.$$

Thus, setting $g(\theta) = C/|\theta|$ for all $|\theta| \ge 2C$, we have a uniform bound for the Fourier transforms $\hat{f}_W(\theta)$ in the region (ii).

Finally, to bound $|\hat{f}_W(\theta)|$ in the region (iii) of intermediate θ -values, we use the proportionality rule once again in the form $|w(u) - u| < \epsilon$, valid for all $u \in [\underline{u}, \overline{u}]$.

This implies

$$\hat{f}_W(\theta) = \int_{\underline{u}}^{\overline{u}} e^{i\theta w(u)} f(u) du$$

$$= \int_{\underline{u}}^{\overline{u}} e^{i\theta u} f(u) du + \int_{\underline{u}}^{\overline{u}} (e^{i\theta w(u)} - e^{i\theta u}) f(u) du$$

$$= \hat{f}_U(\theta) + R(\theta)$$

where $|R(\theta)| < \epsilon'$ uniformly for $|\theta| \le C$ and $\epsilon' \to 0$ as $\epsilon \to 0$. Because \hat{f}_U is the Fourier transform of an absolutely continuous probability density, its absolute value is bounded away from 1 on the complement of $[-\gamma, \gamma]$, for any $\gamma > 0$. Since $\epsilon > 0$ can be made arbitrarily small (cf. Proposition 20), it follows that a continuous, positive function $g(\theta)$ that is bounded away from 1 on $|\theta| \in [\gamma, C]$ exists such that $|\hat{f}_W)\theta| \le g(\theta)$ for all $|\theta| \in [\gamma, C]$. The extension of g to the whole real line can now be done by smoothly interpolating at the boundaries of regions (i), (ii), and (iii). \square

9.3 Proof of Theorem 1

Because the function ψ is smooth and has compact support, differentiation under the expectation in the necessary condition $2v(u) = E\psi(v(u) + S_n)u$ is permissible, and so for every $u \in [-\underline{u}, \overline{u}]$ a $\tilde{v}(u)$ exists intermediate between 0 and v(u) such that

$$2v(u) = E\psi(S_n)u + E\psi'(S_n + \tilde{v}(u))v(u)u.$$
(60)

The proof of Theorem 1 will hinge on the use of the Edgeworth expansion (Proposition 27) to approximate each of the two expectations in (60) precisely.

As in Lemma 28, let $w(u) = 2v(u)/E\psi(S_n)$. We have already observed, in the proof of Lemma 28, that for any $\epsilon > 0$, if n is sufficiently large then for any Nash equilibrium, $|Ew(U)| < \epsilon$. It therefore follows from the proportionality rule that

$$\left| \frac{4\operatorname{var}(v(U))}{(E\psi(S_n))^2\sigma_U^2} - 1 \right| \le \epsilon \quad \text{and} \quad \left| \frac{E|v(u) - Ev(u)|^k}{(E\psi(S_n))^k E|U|^k} \right| < \epsilon \quad \forall \, k \le 8. \tag{61}$$

Moreover, Lemma 28 and Proposition 27 imply the distribution of S_n has a density with an Edgeworth expansion, and so for any continuous function $\varphi : [-\delta, \delta] \to \mathbb{R}$,

$$E\varphi(S_n) = \int_{-\delta}^{\delta} \varphi(x) \frac{e^{-y^2/2}}{\sqrt{2\pi n}\sigma_V} \left(1 + \sum_{k=3}^m n^{-(k-2)/2} P_k(y) \right) dx + r_n(\varphi)$$
 (62)

where

$$\sigma_V^2 := \operatorname{var}(v(U)),$$

$$y = y(x) = (x - ES_n) / \sqrt{\operatorname{var}(S_n)},$$

and $P_k(y) = C_k H_3(y)$ is a multiple of the kth Hermite polynomial. The constants C_k depend only on the first k moments of w(U), and consequently are uniformly bounded by constants C_k' not depending on n or on the choice of Nash equilibrium. The error term $r_n(\varphi)$ satisfies

$$|r_n(\varphi)| \le \frac{\epsilon_n}{n^{(m-2)/2}} \int_{-\delta}^{\delta} \frac{|\varphi(x)|}{\sqrt{2\pi \operatorname{var}(S_n)}} dx.$$
 (63)

In the special case $\varphi = \psi$, (62) and the remainder estimate (63) (with m = 4) imply

$$E\psi(S_n) \le \frac{1}{\sqrt{2\pi n}\sigma_V} \int_{-\delta}^{\delta} \psi(x) \, dx + o(n^{-1}\sigma_V^{-1}).$$

Because $4\sigma_V^2 \approx (E\psi(S_n))^2\sigma_U^2$ for large n, this implies that for a suitable constant $\kappa < \infty$,

$$E\psi(S_n) \le \frac{\kappa}{\sqrt[4]{n}}. (64)$$

Claim 29. There exist constants $\alpha_n \to \infty$ such that in every Nash equilibrium,

$$|ES_n| \le \alpha_n^{-1} \sqrt{\operatorname{var}(S_n)}$$
 and (65)

$$\operatorname{var}(S_n) \ge \alpha_n^2. \tag{66}$$

Proof of Theorem 1: Conclusion. Before we begin the proof of the claim, we indicate how it will imply Theorem 1. If (65) and (66) hold, then for every $x \in [-\delta, \delta]$,

$$|y(x)| \le (1+2\delta)/\alpha_n \to 0.$$

Consequently, the dominant term in the Edgeworth expansion (62) for $\varphi = \psi$ (with m=4), is the first, and so for any $\epsilon > 0$, if n is sufficiently large,

$$E\psi(S_n) = \frac{1}{\sqrt{2\pi n}\sigma_V} \int_{-\delta}^{\delta} \psi(x) \, dx (1 \pm \epsilon).$$

(Here the notation $(1 \pm \epsilon)$ means the ratio of the two sides is bounded above and below by $(1 \pm \epsilon)$.) Thus $4 \sigma_V^2 \approx (E\psi(S_n))^2 \sigma_U^2$ implies

$$\sqrt{\pi n/2}\sigma_U(E\psi(S_n))^2 = \int_{-\delta}^{\delta} \psi(x) \, dx (1 \pm \epsilon) = 2 \pm 2\epsilon,$$

proving the assertion (10).

Proof of Claim 29. First we deal with the remainder term $r_n(\varphi)$ in the Edgeworth expansion (62). By Lemma 26, the expectation $E\psi(S_n)$ is at least C/n for large n, and so by (61) the variance of S_n must be at least C'/n. Consequently, by (63), the remainder term $r_n(\varphi)$ in (62) satisfies

$$|r_n(\varphi)| \le C'' \frac{\epsilon_n \|\varphi\|_1}{n^{(m-2)/2} \sqrt{\operatorname{var}(S_n)}} \le C''' \frac{\epsilon_n \|\varphi\|_1}{n^{(m-3)/2}}.$$

Suitable choice of m will make this bound small compared to any desired monomial n^{-A} , and so we may ignore the remainder term in the arguments to follow.

Suppose there were a constant $C < \infty$ such that along some sequence $n \to \infty$ Nash equilibria existed satisfying $\text{var}(S_n) \le C$. By (61), this would force $C/n \le E\psi(S_n) \le C'/\sqrt{n}$, which in turn would imply that

$$C''\operatorname{var}(S_n)\log n \ge |ES_n|^2 \ge C'''\operatorname{var}(S_n)\log n,\tag{67}$$

because otherwise the dominant term in the Edgeworth series for $E\psi(S_n)$ would be either too large or too small asymptotically (along the sequence $n \to \infty$) to match the requirement that $C/n \le E\psi(S_n) \le C'/\sqrt{n}$. (Observe that because the ratio $|ES_n|^2/\text{var}(S_n)$ is bounded above by $C'''\log n$, the terms $e^{-y^2/2}P_k(y)$ in the integral (62) are of size at most $(\log n)^A$ for some A depending on m, and so the first term in the Edgeworth series is dominant.) We will show that (67) leads to a contradiction.

Suppose $ES_n > 0$ (the case $ES_n < 0$ is similar). The Taylor expansion (60) for v(u) and the hypothesis EU = 0 implies

$$2Ev(U) = E\psi'(S_n + \tilde{v}(U))v(U)U. \tag{68}$$

The Edgeworth expansion (62) for $E\psi'(S_n + \tilde{v}(u))$ together with the independence of S_n and U and the inequalities (67), implies that for any $\epsilon > 0$, if n is sufficiently large then

$$E\psi'(S_n + \tilde{v}(u))$$

$$= \frac{1}{\sqrt{2\pi \operatorname{var}(S_n)}} \int_{-\delta}^{\delta} \psi'(x) \exp\{-(x + \tilde{v}(u) - ES_n)^2 / 2\operatorname{var}(S_n)\} dx (1 \pm \epsilon). \quad (69)$$

Now because ψ and ψ' have support $[-\delta, \delta]$, integration by parts yields

$$\int_{-\delta}^{\delta} \psi'(x) \exp\{-(x+\tilde{v}(u)-ES_n)^2/2\operatorname{var}(S_n)\} dx$$

$$= \int_{-\delta}^{\delta} \psi(x) \exp\{-(x+\tilde{v}(u)-ES_n)^2/2\operatorname{var}(S_n)\} \frac{x+\tilde{v}(u)-ES_n}{\operatorname{var}(S_n)} dx, \quad (70)$$

and because $x + \tilde{v}(u)$ is of smaller order of magnitude than ES_n , it follows that for large n

$$E\psi'(S_n + \tilde{v}(u)) = -\frac{ES_n}{\operatorname{var}(S_n)} E\psi(S_n) (1 \pm \epsilon). \tag{71}$$

But it now follows from the Taylor series for $2Ev(U_i)$ (by summing over i) that

$$2ES_n = -n\frac{ES_n}{\operatorname{var}(S_n)}E\psi(S_n)Ev(U)U(1 \pm \epsilon), \tag{72}$$

which is a contradiction, because the right side is negative and the left side positive. This proves the assertion (66).

The proof of inequality (65) is similar. Suppose for some C>0 Nash equilibria existed along a sequence $n\to\infty$ for which $ES_n\geq C\sqrt{\mathrm{var}(S_n)}$. In view of (66), this hypothesis implies in particular that $ES_n\to\infty$, and also that $|y(x)|\geq C/2$ for all $x\in [-\delta,\delta]$. Thus, the Edgeworth approximation (69) remains valid, as does the integration by parts identity (70). Because $ES_n\to\infty$, the terms $x+\tilde{v}(u)$ are of smaller order of magnitude that ES_n , and so once again (71) and therefore (72) follow. Again we have a contradiction, because the right side of (72) is negative while the left side diverges to $+\infty$.

10 Proof of Proposition 2

We shall assume throughout that $\delta < 1/\sqrt{2}$, and that the function $\psi = \Psi'$ satisfies the standing assumptions of section 1.1. Thus, $\psi/2$ is an even, C^{∞} probability density with support $[-\delta, \delta]$; it has positive derivative ψ' on $(-\delta, 0)$ (and hence negative derivative on $(0, \delta)$); and it has a single point of inflection in the interval $(-\delta, 0)$.

Define

$$H(\alpha, w) = (1 - \Psi(w))|\underline{u}| - (\alpha - w)^2. \tag{73}$$

Proposition 2 asserts that, under the assumption $\delta < 1/\sqrt{2}$, there is a unique value $\alpha > \delta$ such that (i) the maximum value of the function $w \mapsto H(\alpha, w)$ on $w \in \mathbb{R}$ is 0, and (ii) this maximum is attained at precisely two values of w, one at $w = \alpha$, the other at a point $w \in (-\delta, \delta)$. Local maxima and minima of smooth functions are attained only at *critical points*, that is, roots of the equation

$$\frac{\partial H}{\partial w}(\alpha, w) = -\psi(w)|\underline{u}| + 2(\alpha - w) = 0. \tag{74}$$

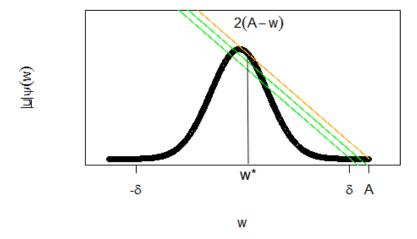


Figure 1: Possible configurations of the optimization problem faced by individuals with extreme negative utilities depending on the size of δ .

Lemma 30. There exists $A \in (\delta, \infty)$ such that

- (i) $\alpha > A \implies$ a unique critical point exists at $w = \alpha$;
- (ii) $\alpha = A \implies 2$ critical points exist, at $w = \alpha$ and at $w_* \in (-\delta, \delta)$; and
- (iii) $\delta \leq \alpha < A \implies$ 3 critical points exist, at $w = \alpha$ and at distinct points $w_{-}(\alpha), w_{+}(\alpha) \in (-\delta, \delta)$.

For $\alpha \in [\delta, A)$ the two critical points $w_{-}(\alpha) < w_{+}(\alpha)$ vary continuously with α ; the function $w_{-}(\alpha)$ is increasing in α , while $w_{+}(\alpha)$ is decreasing. Moreover, as $\alpha \to A_{-}$, the critical points $w_{\pm}(\alpha) \to w_{*}$.

The basic idea can be seen clearly graphically, as shown in Figure 1, which pictures the different arrangements that are possible.

Proof. It is clear that for every $\alpha \geq \delta$ the equation (74) holds at $w = \alpha$, because $\psi = 0$ outside of the interval $(-\delta, \delta)$. All other critical points are points where the straight line of slope -2 through $(\alpha, 0)$ intersects the graph of $y = |\underline{u}|\psi(w)$. Because $\psi(w) \neq 0$ only for $w \in (-\delta, \delta)$, critical points not equal to α must be located in the interval $(-\delta, \delta)$.

(1) We first show that for $\alpha \geq \delta$ near δ at least two such intersection points exist, one in each of the intervals $(-\delta,0)$ and $(0,\delta)$. To see this, observe that because $\psi/2$ is a probability density with support $(-\delta,\delta) \subset (-1/\sqrt{2},1/\sqrt{2})$, its maximum value, which is assumed at w=0, must exceed $1/\sqrt{2}$, and so $\psi(0)>\sqrt{2}$. But the line $y=2(\delta-w)$ intersects the y-axis at $y=2\delta<\sqrt{2}$, so it must cross the graph of

 $y = |\underline{u}|\psi(w)$ at least once in each of the intervals $(0, \delta)$ and $(-\delta, 0)$. Similarly, if $\alpha - \delta$ is sufficiently small then the line $y = 2(\alpha - w)$ must also cross the graph twice, once on each side of 0.

(2) Next, we show that for any $\alpha \geq \delta$ at most two roots of (74) in $(-\delta, \delta)$ exist, and that two exist if and only if both intersections are *transversal*. Assume at least two distinct roots exist; let $w_+(\alpha) \in (0, \delta)$ be the largest, and let $w_-(\alpha)$ be the second largest. We claim that at $w = w_+(\alpha)$ the intersection between the line $y = 2(\delta - w)$ and the graph of $y = |\underline{u}|\psi(w)$ is *transversal*, that is,

$$|\underline{u}|\psi'(w) \neq -2. \tag{75}$$

It is impossible for $|\underline{u}|\psi'(w_+(\alpha)) > -2$, because $2(\delta - w) > |\underline{u}|\psi(w)$ for all $w_+(\alpha) < w < \alpha$, so we must only show that the line $y = 2(\delta - w)$ cannot be tangent to the graph at $w = w_+(\alpha)$. But if this were the case then $|\underline{u}|\psi''(w_+(\alpha)) < 0$, once again because the line lies above the graph in the interval $w_+(\alpha) < w < \alpha$; because ψ has only a single inflection point in $(0, \delta)$, it would then follow that $w_+(\alpha)$ is the *only* root of (74) in $(-\delta, \delta)$, contrary to our hypothesis. This proves (75) for $w = w_+(\alpha)$.

It now follows from (75) that the intersection between the line $y=2(\delta-w)$ and the graph of $y=|\underline{u}|\psi(w)$ at $w_-(\alpha)$ is also transversal. To see this, note first that it suffices to consider the case where $w_-(\alpha)>0$, because at any $w\in(-\delta,0]$ the slope of the graph of $|\underline{u}|\psi(w)$ is nonnegative. Next, observe that by the Mean Value Theorem a maximal $w_*\in[w_-(\alpha),w_+(\alpha)]$ at which $|\underline{u}|\psi'(w_*)=-2$ exists. This point w_* cannot be $w_+(\alpha)$, because at $w=w_+(\alpha)$ we have (75); furthermore, because $|\underline{u}|\psi'(w_+(\alpha))<-2$, it must be the case that

$$|\underline{u}|\psi'(w) < -2$$
 for all $w \in (w_*, w_+(\alpha))$,

and so by the Fundamental Theorem of calculus,

$$|\underline{u}|\psi(w_*) > 2(\delta - w_*).$$

This implies $w_* > w_-(\alpha)$. Now because the slope of the graph at w_* is larger than at $w_+(\alpha)$, the second derivative $|\underline{u}|\psi''(w)$ must be negative at all $0 < w \le w_*$, because of the standing hypothesis that ψ has only one inflection point in $(0, \delta)$. Finally, because $0 < w_-(\alpha) < w_*$, it follows that

$$|\underline{u}|\psi'(w_{-}(\alpha)) < |\underline{u}|\psi'(w_{*}) = -2.$$

Thus, both intersections, at $w = w_{-}(\alpha)$ and $w = w_{+}(\alpha)$, are transversal provided $w_{-}(\alpha) < w_{+}(\alpha)$.

To complete the proof that at most two roots of (74) in $(-\delta, \delta)$ exist, consider the cases $w_{-}(\alpha) > 0$ and $w_{-}(\alpha) \leq 0$ separately. In the first case, the intersection

at $w_-(\alpha)$ is transversal, and $|\underline{u}|\psi'(w)<-2$ at all $w\in[0,w_-(\alpha)]$. Hence, the line $y=2(\delta-w)$ is above the graph of $y=|\underline{u}|\psi(w)$ at w=0, and because the line has constant negative slope, it must remain above the graph at all w<0, because w=0 is the unique point where $\psi(w)$ attains its maximum. Thus, there can be no points of intersection to the left of $w_-(\alpha)$. The second case, where $w_-(\alpha)\leq 0$, is even easier: because ψ is increasing on $(-\delta,0]$, for any $w< w_-(\alpha)$ we must have

$$|\underline{u}|\psi(w) < |\underline{u}|\psi(w_{-}(\alpha)) = 2(\alpha - w_{-}(\alpha)) < 2(\alpha - w),$$

and so once again there can be no no points of intersection to the left of $w_{-}(\alpha)$.

(3) Because equation (74) has the form $G(\alpha,w)=0$ with G continuously differentiable, the Implicit Function Theorem guarantees that solutions vary continuously in a neighborhood of any solution where $\partial G/\partial w\neq 0$, i.e., where $|\underline{u}|\psi(w)\neq -2$. We have shown in point (2) above that this is the case for both $w_\pm(\alpha)$, as long as $w_+(\alpha)\neq w_-(\alpha)$; thus, $w_\pm(\alpha)$ have continuous extensions to an open interval with left endpoint δ . We must show that the maximal interval on which the functions w_\pm can be continuously and monotonically continued is such that at the right endpoint A the two roots coalesce, and the intersection becomes non-transversal.

Clearly, the linear function $2(\alpha - w)$ is increasing in α for each w. Consequently, if $A > \delta$ is any point where the line y = 2(A - w) intersects the graph of $|\underline{u}|\psi(w)$ non-transversally at some point w_* , then because this would be the *only* point of intersection in $[-\delta, \delta]$ (by point (2)), the graph would lie entirely below the line and there would be no solutions to equation (74) with $\alpha > A$. Thus, to complete the proof it suffices to establish the following claim.

Claim: The function $w_{-}(\alpha)$ increases continuously up to the smallest $\alpha = A$ such that $w_{-}(\alpha) = w_{+}(\alpha) := w_{*}$, where w_{*} is the unique point in $(0, \delta)$ at which the graph of $|\underline{u}|\psi$ has tangent line of slope -2, and lies entirely below its tangent line.

Proof of the Claim. First, observe that because $|\underline{u}|\psi$ is increasing on $[-\delta,0]$ and decreasing on $[0,\delta]$, the function $w_+(\alpha)$ is decreasing in α , and $w_-(\alpha)$ is increasing as long as $w_-(\alpha) \leq 0$. Let $\alpha_0 = |\underline{u}|\psi(0)/2$; at this value, the line $y = 2(\alpha_0 - w)$ intersects the graph of $|\underline{u}|\psi$ transversally at w = 0. Thus, $\alpha_0 < A$, and the functions $w_\pm(\alpha)$ are continuous and monotone on $[\delta,\alpha_0]$, and $w_+(\alpha_0) > 0$.

To see that a w_* at which the graph of $|\underline{u}|\psi$ has tangent line of slope -2 exists, observe that the intersections of the line $y=2(\alpha_0-w)$ with the graph of $|\underline{u}|\psi(w)$ are both transversal: at $w_-(\alpha_0)=0$ the slope of the graph is >-2, and at $W_+(\alpha_0)$ the slope is <-2. Consequently, by the mean value theorem, there must be a point $w_*\in(0,w_+(\alpha_0))$ at which $|\underline{u}|\psi'(w_*)=-2$. The tangent line at this point has the form y=2(A-w) for some $A>\delta$. Because the intersection at the point of tangency is non-transversal, it must be the *unique* point of intersection of this line with the

graph, and so it follows that the rest of the graph lies *below* the line, as claimed. Hence, the derivative $|\underline{u}|\psi'(w)$ is non-increasing at $w=w_*$; because ψ has only one inflection point in $(0,\delta)$, it follows that

$$|u|\psi'(w) > -2$$
 for all $0 \le w < w_*$. (76)

For each $w \in [0, w_*)$, the line of slope -2 through the point $(w, |\underline{u}|\psi(w))$ intersects the w-axis at a point $\alpha(w) > \delta$. Clearly, the mapping $w \mapsto \alpha(w)$ is continuous, and by (76) it is also increasing in w, with positive derivative. Furthermore, because the intersection of this line with the graph of $|\underline{u}|\psi(w)$ is transversal, there must be a second point of intersection to the right of w. Thus, by point (2),

$$w_{-}(\alpha(w)) = w.$$

This proves that $w_{-}(\alpha)$ is continuous and increasing in α on the interval $[\alpha_0, A]$, where A is defined to the unique point where $w_{-}(A) = w_*$.

It remains to prove that $w_+(A)=w_*$. Recall that w_+ is continuous and decreasing as long as $w_+>w_*$, because the intersections are transversal at all such points. By point (2), we cannot have $w_+(\alpha)=w_*$ for any $\alpha< A$, because at any such α a distinct second transversal intersection at $w_-(\alpha)< w_*$ exists, while the intersection at $w_+(\alpha)=w_*$ would be non-transversal. Similarly, we cannot have $w_+(A)>w_*$, because this intersection would be non-transversal, whereas the intersection at $w_*=w_-(A)$ would be transversal. Therefore, $w_+(A)=w_-(A)=w_*$.

Proof of Proposition 2. First, note that for any $\alpha \geq \delta$ the function $w \mapsto H(\alpha, w)$ satisfies

$$\lim_{w \to -\infty} H(\alpha, w) = \lim_{w \to +\infty} H(\alpha, w) = -\infty.$$

Consequently, for any $\alpha \in [\delta, A]$ neither of the critical points $w = \alpha$ nor $w_{-}(\alpha)$ can be a local *minimum* of $w \mapsto H(\alpha, w)$. It is easily verified that if $\alpha > \delta$ then $w = \alpha$ is a local maximum, because $\Psi = 1$ in a neighborhood of α and so $\partial^2 H/\partial w^2 = -2$ at all $w > \delta$.

Now consider the behavior of $H(\alpha, w)$ in a neighborhood of $w = w_{-}(\alpha)$. Because $\delta < 1/\sqrt{2}$, for all α near δ , we have

$$H(\alpha, -\delta) = (1 - \Psi(-\delta))|\underline{u}| - (\alpha + \delta)^2$$
$$= 2|\underline{u}| - (\alpha + \delta)^2$$
$$> 2 - (\alpha + \delta)^2 > 0,$$

because $(\alpha + \delta) \approx 2\delta$, and so the maximum value of $H(\alpha, w)$ for $w \in \mathbb{R}$ must be positive. Because $H(\alpha, \alpha) = 0$, it follows that the global maximum must be attained

at one of the other two critical points $w_{\pm}(\alpha)$, and because $w = \alpha$ is a *local* maximum, it must be that $w_{+}(\alpha)$ is a local minimum and $w_{-}(\alpha)$ the global maximum. Thus, $H(\alpha, w_{-}(\alpha)) = \max_{w} H(\alpha, w)$ for all $\alpha \in [\delta, A]$ such that $H(\alpha, w_{-}(\alpha)) > 0$.

Next, observe that for any fixed $w \leq \delta$ the function $H(\alpha,w)$ is *decreasing* in α . Hence, $h(\alpha) := \max_{w \in [-\delta,\delta]} H(\alpha,w)$ is decreasing in α . Because $h(\alpha) = H(\alpha,w_-(\alpha))$ for all α such that $H(\alpha,w_-(\alpha)) \geq 0$, it follows that $h(\alpha)$ decreases *continuously* with α up to the first point α_* where $h(\alpha_*) = 0$, if such a point exists. But it cannot be the case that $h(\alpha) > 0$ for all $\alpha > \delta$, because clearly the definition (73) of H forces $h(\alpha) \to -\infty$ as $\alpha \to \infty$. Finally, α_* cannot be larger than A, because for all $\alpha < \alpha_*$ the global minimum of $w \mapsto H(\alpha,w)$ is attained in $(-\delta,\delta)$, and so at least 2 critical points exist for every such α .

The point $\alpha = \alpha_*$ is the unique point where a solution to the Optimization Problem (16) exists, and $w_-(\alpha_*)$ is the unique matching real number in $[-\delta, \delta]$ where (11) holds.

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