SBER Quant Problemset

Ivan Lomikovskiy

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1 Problem 1.

1. $\tau < \infty$ *a.s.*

   Proof. Let $\{\xi_n\}_{n=1}^{\infty}$ be a sequence of *i.i.d.* random variables, defined as follows

   $$\xi_n = \begin{cases} 1, & \mathbb{P} = \dfrac{1}{2} \\ -1, & \mathbb{P} = \dfrac{1}{2} \end{cases}.$$

   Definition 1. Define stopping moment

   $$\tau \stackrel{def}{=} inf\ \{k : \xi_k = 1\}$$

   Suppose that $\tau = \infty$. It means that $\forall k \geq 1 \quad \xi_k = -1$. The probability of such an event is

   $$\mathbb{P}(\forall k \geq 1 \quad \xi_k = -1) = \mathbb{P}\left(\bigcap_{i=1}^{\infty} \xi_i = -1\right) = \lim_{n\to\infty} \mathbb{P}\left(\bigcap_{i=1}^{n} \xi_i = -1\right) = \lim_{n\to\infty} \left(\mathbb{P}\left(\xi_1 = -1\right)\right)^n = \lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0.$$

   Therefore, $\tau < \infty$ *a.s.* □

2. Find $\mathbb{E}[\pi_\tau]$

   Proof. Let us note that profit (debt) can be expressed in terms of $\{\xi_n\}_{n=1}^{\infty}$ and $\{V_t\}_{t=1}^{\infty}$. For instance,

   $$\pi_0 = 0 \tag{1}$$
   $$\pi_t = \sum_{i=1}^{t} V_i \xi_i = \sum_{i=1}^{t} 2^{i-1} \xi_i. \tag{2}$$

   In our case,

   $$\pi_\tau = \sum_{i=1}^{\tau} 2^{i-1} \xi_i.$$

   However, since we already know that τ is the stopping moment, $\forall 1 \leq t < \tau \quad \xi_t = -1$ and $\xi_\tau = 1$ *a.s.* by the definition of τ. Therefore,

   $$\pi_\tau = \sum_{i=1}^{\tau} 2^{i-1} \xi_i = 2^{\tau-1} - (2^{\tau-2} + \cdots + 2^0) = 2^{\tau-1} - (2^{\tau-1} - 1) = 1.$$

   Hence, $\pi_\tau = 1$ *a.s.* and $\mathbb{E}[\pi_\tau] = 1$ □

3. Find $\mathbb{E}[\pi_t]$

Proof. Consider the natural filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$, associated with the sequence of random variables $\{\xi_n\}_{n=1}^{\infty}$, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

Claim 1. *π_t is a martingale w.r.t. filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$*

Proof.

(a) π_t is $\mathcal{F}_t$ – measurable
Obviously, π_0 is $\mathcal{F}_0$ – measurable. It is clear that π_t is $\mathcal{F}_t$ – measurable, since π_t is a linear combination of $\xi_1, \dots, \xi_t$ and $\mathcal{F}_t = \sigma(\xi_1, \dots, \xi_t)$.

(b) $\mathbb{E}[\pi_{t+1}|\mathcal{F}_t] = \pi_t$

$$\mathbb{E}[\pi_{t+1}|\mathcal{F}_t] = \mathbb{E}[\pi_t + V_{t+1}\xi_{t+1}|\mathcal{F}_t] = \pi_t + \mathbb{E}[V_{t+1}\xi_{t+1}|\mathcal{F}_t] = \pi_t + V_{t+1}\mathbb{E}[\xi_{t+1}] = \pi_t.$$

Since π_t is $\mathcal{F}_t$ – measurable, V_{t+1} is $\mathcal{F}_t$ – measurable and $\mathbb{E}[\xi_{t+1}] = 0$.

□

Since we have proven that π_t is in fact a martingale, we can utilize useful properties of them. For instance, we know that for martingales the following property holds

$$\mathbb{E}[\pi_t] = \mathbb{E}[\pi_0] = 0.$$

Therefore, $\mathbb{E}[\pi_t] = 0$. □

4. Compare results. Explain this in terms of Doob's optional sampling theorem.

Proof. Result in 2. can be interpreted as follows: if we had infinitely many resources, i.e. infinitely large capital for making bets, then eventually we would end up winning the amount of money we bet on the first turn, in our case it is 1. As for the result in 3., it means that for any finite number of steps the expected profit (debt) is zero. We shall explain this fenomena with the use of *Doob's optional sampling theorem*, also known as *Doob's optional stopping theorem.*

Theorem 1 (Doob's optional sampling theorem)**.** *Let $\{X_t\}_{t=0}^{\infty}$ be a martingale w.r.t. filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$. Suppose that S, T are stopping times with $S \leq T$ a.s. Then*

$$\mathbb{E}[X_T|\mathcal{F}_S] = X_S.$$

It follows that if stopping time τ is bounded above by some constant c, then $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$. However, we do not have the said property in our case.

Nevertheless, we can apply the theorem to case 3, if we interpret t as a fixed stopping time. More particularly, let us say that we wish to calculate our profit (debt) up until moment t. Then, as the theorem suggests, we have $\mathbb{E}[\pi_t] = \mathbb{E}[\pi_0] = 0$, as we have already seen.

Finally, let us note that the theorem implies that there exists no winning strategy for the fair lottery game with fixed stopping time t. In fact, in our calculations for $\mathbb{E}[\pi_t]$ the only property of V_t that we used is it's measurability. So, we may take arbitrary sequence $\{V_t\}_{t=1}^{\infty}$, such that each V_t is fully defined by all previous steps $V_1, \dots, V_{t-1}$, plug it into our calculations and end up with some new sequence $\{\pi_t\}_{t=0}^{\infty}$, which will be a martingale *w.r.t.* same filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$. By the *Doob's optional sampling theorem*, $\mathbb{E}[\pi_t] = \mathbb{E}[\pi_0] = 0$.

Morale of the story, in real life (with limited resources), whatever strategy it is, average win in fair lottery equals 0.

□

2 Problem 2.

1. $\mathcal{U}[a,b]$

   Proof. First, note that if $x_{(i)} \sim \mathcal{U}[a,b]$, then $x_{(i)} \in [a,b]$ *a.s.*

   Uniform distribution is fully defined by interval limits, therefore we need to estimate parameters a, b. It is natural to retrieve information about limits of an interval as minimum and maximum values of drawn samples.

   In particular, define

$$L_1 : \ell_1 = 1, \ell_2 = \dots \ell_n = 0 \tag{3}$$
$$L_2 : \ell_1 = \dots \ell_{n-1} = 0, \ell_n = 1. \tag{4}$$

   $L_1(x)$ is an estimation of a, and $L_2(x)$ is an estimation of b. □

2. $\mathcal{N}(\mu, \sigma)$

   First, note that $\mathcal{N}(\mu, \sigma)$ is symmetrical *w.r.t.* 0, therefore $\mu = median(\mathcal{N}(\mu, \sigma))$. Keeping this result in mind, we shall estimate μ as follows

$$\hat{\mu} = median(x) = \begin{cases} \frac{1}{2}(x_{(\lfloor n/2 \rfloor)} + x_{(\lfloor n/2 \rfloor + 1)}), & n \text{ is even} \\ x_{((n-1)/2)}, & n \text{ is odd} \end{cases}.$$

   As for the standard deviation, it is normally computed as empirical standard deviation, which is clearly not in our grasp, since we need to make estimations using linear combinations of order statistics. Under these circumstances one shall develop a more sophisticated construction.

   We will use the method of so-called L - *moments*.

   Definition 2 (L - moment)**.** Define r-th sample L - *moment* to be

$$\ell_r \stackrel{def}{=} \binom{n}{r}^{-1} \sum_{1 \le i_1 \le \dots \le i_r \le n} r^{-1} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} x_{(i_{r-k})}$$

   We are mostly interested in ℓ_2 sample L - *moment*, which will be used for our estimation. In particular

$$\ell_2 = \frac{1}{2} \binom{n}{2}^{-1} \sum_{1 \le j < i \le n} (x_{(i)} - x_{(j)}).$$

   It is known, that $\sqrt{\pi}\ell_2$ is a 98% efficient estimator of standard deviation of a normal distribution.

   Let us further explore the underlying sum in ℓ_2 definition and rearrange the terms

$$\sum_{i>j} (x_{(i)} - x_{(j)}) = \sum_{i>j} x_{(i)} - \sum_{i>j} x_{(j)} \tag{5}$$
$$= \sum_{1 \le j < i \le n} x_{(i)} - \sum_{1 \le j < i \le n} x_{(j)} \tag{6}$$
$$= \sum_{i=2}^{n} (i-1) x_{(i)} - \sum_{j=1}^{n-1} (n-j) x_{(j)} \tag{7}$$
$$= \sum_{i=1}^{n} (i-1) x_{(i)} - \sum_{i=1}^{n} (n-i) x_{(i)} \tag{8}$$
$$= \sum_{i=1}^{n} (2i - n - 1) x_{(i)}. \tag{9}$$

Finally

$$\ell_2 = \frac{1}{2}\binom{n}{2}^{-1}\sum_{i=1}^{n}(2i-n-1)x_{(i)}.$$

And our final estimation of distribution parameters is

$$\hat{\sigma} = \sqrt{\pi}\ell_2 = \sqrt{\pi}\frac{1}{2}\binom{n}{2}^{-1}\sum_{i=1}^{n}(2i-n-1)x_{(i)}. \tag{10}$$

$$\hat{\mu} = median(x) = \begin{cases} \frac{1}{2}(x_{(\lfloor n/2 \rfloor)} + x_{(\lfloor n/2 \rfloor + 1)}), & n \text{ is even} \\ x_{((n-1)/2)}, & n \text{ is odd} \end{cases}. \tag{11}$$

It is clear that these estimations can be represented with $L - estimators$.

3. Pois(λ)

Proof. $\mathbb{E}[\text{Pois}(\lambda)] = \lambda$. Therefore, we can take empirical mean and take it as an estimation of distribution parameter.

$$\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} x_{(i)}.$$

□

To sum up, our estimator is as follows

$$L : \ell_1 = \cdots = \ell_n = \frac{1}{n}.$$

3 Problem 3.

1. Pricing Model

Proof. Let us first list our model's parameters

- S — current FX spot rate
- K — strike price
- r_d — domestic risk-free interest rate
- r_f — foreign risk-free interest rate
- T — maturity
- σ — volatility of FX rate

We will assume that our domestic currency is **USD** and foreign currency is **EUR**. In order to use Garman-Kohlhagen model we need to determine values for parameters. For the current FX spot rate S we use spot rates for **USD/EUR** pair, provided by the European Central Bank (ECB) at the day of solving this task (6 May 2025). Therefore, we set $S = 1.1325$. Other sources can be used as well, however, their rates are almost identical and we have chosen ECB for it's reputation and credibility.

Once S is fixed we can proceed with calculating strike prices. As was mentioned in the task description, we are in possession of two European Call Options A_1, A_2 with corresponding strikes K_1, K_2, which are defined as follows

$$K_1 = 0.9S = 0.9 \cdot 1.1325 = 1.01925 \tag{12}$$

$$K_2 = 1.1S = 1.1 \cdot 1.1325 = 1.24575 \tag{13}$$

Now we proceed with determining risk-free interests rates. Indeed, quite a few approaches could be chosen. The usual approach is to set up interest rate as a yield to maturity of zero-coupon bonds. For instance, as for the **USD** currency zero-coupon treasury bonds yield to maturity could be used as a proxy for interest rate. However, in practice traders do not use bonds rates as a risk-free interst rates proxy, due to the fact that bonds rates are highly influenced by taxes in regulatory factors. Instead, it is common to use risk-free reference rates, created from overnignt rates, in valuing derivatives. We shall take the same approach and calculate r_d using SOFR and r_f using ESTER, which are specifically reference risk-free rates for United States and Eurozone markers, respectively.

$$r_d = 0.0433 \tag{14}$$

$$r_f = 0.02167 \tag{15}$$

Rate r_d for domestic currency was published by Federal Reserve Bank of New York on 5 May 2025, and rate r_f for foreign currency was published by ECB on 6 May 2025.

Maturity T is given in task description and equals 1 year.

Finally, we are left with determining volatility σ. Despite all previous parameters, volatility is not available "as-it-is" and should be predicted by some kind of a model. For instance, we shall use 1 year volatility prediction by GARCH model, published by Volatility Laboratory at NYU Stern. GARCH model has proved to be quite effective in derivative valuing, and is commonly used.

$$\sigma = 0.0895$$

When all parameters are determined, we can use Garman-Kohlhagen model to determine values of options

$$C_1 = 0.1354, \tag{16}$$

$$C_2 = 0.0118. \tag{17}$$

□

2. Optimal proportions

Proof. Suppose N is fixed, define

$$n_i = \frac{N_i}{N}$$

to be the proportion of the corresponding asset. We are interested in forming portfolio, consisting of buying n_1 options of the first type, selling n_2 options of the second type and n_3 of underlying asset, which we can either sell or buy. For instance, if $n_3 < 0$ we are selling $|n_3|$ **EUR**, if $n_3 \geq 0$ we are buying $|n_3|$ **EUR**.

Before we continue any further, we need to define the so-called *Greeks*

Definition 3. Suppose that C is the price of derivative over underlying asset S. Then Δ, Γ, Θ of C are defined as follows

$$\Delta C = \frac{\partial C}{\partial S} \tag{18}$$
$$\Gamma C = \frac{\partial C^2}{\partial S^2} \tag{19}$$
$$\Theta C = \frac{\partial C}{\partial t} \tag{20}$$

Garman-Kohlhagen model is an extension of Black-Scholes model, and it is well known what Greeks are equal to, when applied to Black-Scholes pricing of European call options. We can utilize this knowledge to determine Δ, Γ, Θ values for options, listed in the task.

$$\Delta_1 = 0.90845, \Gamma_1 = 1.31964, \Theta_1 = -0.02316 \tag{21}$$
$$\Delta_2 = 0.21346, \Gamma_2 = 2.84469, \Theta_2 = -0.01923 \tag{22}$$

As for the portfolio Π, it's Δ can be calculated as follows

$$\Delta\Pi = n_1\Delta_1 - n_2\Delta_2 + n_3.$$

Since Δ of the underlying asset always equals 1.

Similarly, Γ of the portfolio Π is

$$\Gamma\Pi = n_1\Gamma_1 - n_2\Gamma_2.$$

Since Γ of the underlying asset always equals 0.

Finally, to create delta-neutral and gamma-neutral portfolio we need to solve system of equations

$$\begin{cases} n_1\Delta_1 - n_2\Delta_2 + n_3 & = 0 \\ n_1\Gamma_1 - n_2\Gamma_2 & = 0 \\ n_1 + n_2 + n_3 & = 1 \end{cases} \tag{23}$$

Solving with respect to n_1, n_2, n_3, we obtain

$$n_1 = 1.5279, \tag{24}$$
$$n_2 = 0.7088, \tag{25}$$
$$n_3 = -1.2367. \tag{26}$$

□

3. Capital management

Suppose we are in possession of $N_{usd} = 100000$ **USD**, and we want to invest this capital using calculated optimal proportions. First, note that in order to obtain N_1 options of the first type we need to pay C_1N_1 **USD**. Similarly, we receive C_2N_2 **USD** for the options of second type. Finally, since $n_3 < 0$, we need to have $S|N_3|$ dollars in our portfolio after all operations with options. Keeping that in mind, we end up with the equation

$$N_{usd} - C_1N_1 + C_2N_2 = |N_3|S.$$

Due to the definiton of n_1, n_2, n_3 we have the following line of equalities:

$$\frac{n_1}{N_1} = \frac{n_2}{N_2} = \frac{n_3}{N_3} = \frac{1}{N_1 + N_2 + N_3} = \frac{1}{N}.$$

Therefore

$$N_1 = \frac{n_1}{n_2} N_2 \tag{27}$$

$$|N_3| = \frac{|n_3|}{n_2} N_2 \tag{28}$$

Substituting $N_2, |N_3|$ in the above equation, we obtain

$$N_{usd} - C_1 \frac{n_1}{n_2} N_2 + C_2 N_2 = \frac{|n_3|}{n_2} N_2 S$$

Therefore

$$N_2 = \frac{N_{usd}}{\frac{|n_3|}{n_2} S - C_2 + \frac{n_1}{n_2} C1}$$

Having calculated this value, we can determine $N_1, |N_3|$ values as well. In our case, the result is

$$N_1 = 95546.6852, \tag{29}$$

$$N_2 = 44323.7792, \tag{30}$$

$$|N_3| = 77338.1386. \tag{31}$$

It means, that we need to enter long position with 95546.6852 call options of the first type, short position with 44323.7792 call options of the second type and short position with 77338.1386 euros.

Now, in order to calculate Θ value of the portfolio we proceed with same approach as for Γ value of the portfolio, since Θ value of the underlying asset is 0. Therefore

$$\Theta\Pi = N_1\Theta_1 - N_2\Theta_2.$$

In our case

$$\Theta\Pi = -1356.9227.$$

It means that when time passes our portfolio's value is decreasing. That is due to the fact when all other parameters are constant, options tend to become less valuable. Note, that this value represents value loss over the course of one year. If we are interested in day loss, we should divide this number by number of days. The usual approach is to use total number of days in one year or number of trading days.

4. Assumptions

   Since Garman-Kohlhagen model is an extension of Black-Scholes-Merton model, it requires similar assumptions to be made.

   (a) **Constant Interest Rates:**
   The domestic and foreign risk-free interest rates are known, constant, and continuously compounded.

   (b) **Lognormal Exchange Rate:**
   The spot exchange rate follows a geometric Brownian motion with constant drift and volatility.

   (c) **No Arbitrage:**
   Markets are arbitrage-free, allowing the construction of a riskless hedging portfolio.

   (d) **Frictionless Markets:**
   No transaction costs, taxes, or restrictions on short selling. Continuous trading is possible, enabling instantaneous portfolio adjustments.

(e) **Constant Volatility:**
The volatility of the exchange rate is constant over the option's life and known in advance.

(f) **European Exercise:**
Options can only be exercised at expiration.

(g) **Credit Risk Absence:**
Counterparties are assumed to have no default risk, obligations are always fulfilled.

However, in real life situation is not so perfect and these assumptions tend to be broken.

(a) **Constant Interest Rates:**
In real life interest rates might be considered constant only for a short period of time. *How to fix it:* Use stochastic interest rate models, use interest rate caps and floors to hedge rate risk, treat interest rates as variables and additional source of risk.

(b) **Lognormal Exchange Rate:**
Exchange rates often exhibit fat tails, skewness, and jumps, for example during hyperinflation of currency crisises. *How to fix it:* Hedging using OTM options, account for volatility smiles.

(c) **No Arbitrage and Frictionless Markets:**
Model specifically does not account any transactional costs, which are very present in real life. Due to this, continuous hedging becomes too costly. *How to fix it:* Adjust the model for transactional costs, consider discrete hedging instead of continuous rebalancing.

(d) **Constant Volatility:**
Volatility is stochastic. *How to fix it:* Stochastic volatility models, construct implied volatility surfaces using market data.

(e) **European Exercise:**
Many FX options are American-style, therefore model can't be applied to pricing them. *How to fix it:* Use Monte Carlo simulations for early exercise.

(f) **No Credit Risk:**
Counterparty default risk exists. *How to fix it:* Incorporate Credit Valuation Adjustment, hedge risks using swaps.