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Elliptic surfaces with four singular fibres

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Already at the beginning of the sixties, elliptic surfaces were considered by Kodaira [6]; Kas embedded them in a projective bundle over the base curve B [5]; Hunt and Meyer introduced an estimate for the Euler number which depended on the genus of the base curve and the number of singular fibres [4]; for elliptic surfaces with three singular fibres and section over $\mathbb{P}_1\mathbb{C}$, Schmickler-Hirzebruch proved that there are only 36 combinations of singular fibres, subdivided in 12 cases [13].

When studying elliptic surfaces with four singular fibres, section and nonconstant $\mathscr{J}$-invariant over $\mathbb{P}_1\mathbb{C}$, as presented here, it is practical to distinguish two sets:

$$T^+ = \{I_n\ (n \geqq 0), II, III, IV\} \quad \text{and} \quad T^- = \{I_n^*\ (n \geqq 0), IV^*, III^*, II^*\},$$

where I_0 is a regular fibre. At least one fibre is then of type I_n, $n>0$, or I_n^*, $n>0$. By a suitable choice of the homological invariant $\mathscr{G}$ belonging to the $\mathscr{J}$-invariant, all possible fibre combinations can be reduced by "twisting" such that at most one fibre is in T^-, see p. 324 or [9, p. 203].

Theorem 6 summarises the results. Table 3 shows all fibre combinations and Weierstrass models. The proof will be given by example. The notation is taken from Kodaira [6] or Barth et al. [1].

Naruki [11], Miranda, and Persson [9, 10, 12] achieved similar results using different methods.

For an elliptic surface $\pi: E \to B$, where E is a two-dimensional compact complex analytic manifold, B is a compact Riemann surface of genus g and π is a proper holomorphic mapping, $E_b := \pi^{-1}(b)$ is a nonsingular curve of genus 1 for all $b \in B_0$, $B_0 := B - P$, $P := \{\varrho_1, \varrho_2, \ldots, \varrho_n\}$, $\varrho_i \in B$, $i = 1, \ldots, n$. From now on it will be assumed that E is minimal and admits a section, i.e. E has no exceptional curves of the first kind in the fibres. All singular fibres are simple, because there is a section.

The monodromy representation of $\pi: E \to B$ is a homomorphism

$$\chi: \pi_1(B_0, b) \to \mathrm{SL}(2, \mathbb{Z}), \qquad b \in B_0,$$

which is unique up to conjugation in $SL(2,\mathbb{Z})$. The image of $\pi_1(B_0, b)$ is called the monodromy group. Elements of this group are the monodromy matrices A_{β_i} corresponding to the closed paths β_i around ϱ_i, $i=1,\dots,n$.

For each type of singular fibre F_i over ϱ_i there is one $SL(2,\mathbb{Z})$-conjugate class of monodromy matrices. In Table 1 they are listed in normal and general form for the singular fibres.

The homological invariant $\mathcal{G}$, a sheaf over B, is equivalent to the monodromy representation. In a base point ϱ with the monodromy matrix A the stalk $\mathcal{G}_\varrho$ is isomorphic to $\{x\in\mathbb{Z}^2 \mid Ax=x\}$.

Each regular fibre E_ϱ of an elliptic surface $\pi: E\to B$ is isomorphic to $\mathbb{C}/\omega(\varrho)\mathbb{Z}\oplus\mathbb{Z}$. $\omega: \tilde{B}_0\to\mathbb{H}$ with $\omega(\tilde{\beta}(\tilde{b}))=A_\beta(\omega(\tilde{b}))$ is a unique holomorphic function. Here A_β is the monodromy in $SL(2,\mathbb{Z})$ of the closed path β in B_0, $\sigma: \tilde{B}_0\to B_0$ is the universal covering of B_0, $\mathbb{H}$ the upper halfplane, $\sigma(\tilde{b})=b$ and

$$\begin{aligned}\pi_1(B_0)&\to \mathrm{Aut}(\tilde{B}_0)\\ \beta&\mapsto\tilde{\beta}\end{aligned}$$

is the deck transformation.

There is a mapping $\mathcal{J}: B_0\to SL(2,\mathbb{Z})\backslash\mathbb{H}$, which allows the diagram to commute:

$$\begin{array}{ccc} & \tilde{B}_0 \xrightarrow{\ \omega\ } & \mathbb{H} \\ & \sigma\big\downarrow & \big\downarrow \tilde{\mathcal{J}} \\ \dfrac{\tilde{B}_0}{\pi_1(B_0,b)} = & B_0 \xrightarrow{\ \mathcal{J}\ } & SL(2,\mathbb{Z})\backslash\mathbb{H}\cong\mathbb{C}, \end{array}$$

where $\tilde{\mathcal{J}}$ is the elliptic modular function.

The functional invariant of E is defined as the holomorphic continuation of $\mathcal{J}$ on B in $SL(2,\mathbb{Z})\backslash\mathbb{H}^*\cong\mathbb{P}_1\mathbb{C}$, $\mathbb{H}^*=\mathbb{H}\cup\mathbb{P}_1\mathbb{Q}$. The values of $\mathcal{J}$ in $\varrho_i\in P$, depending on the type of the singular fibre over ϱ_i, are 0, 1 or ∞, except for I_0^*.

Let $P:=\{\varrho_i\in B \mid i=1,\dots,n\}$ $n\geqq 2$ be the exceptional set and

$$\chi: \pi_1(\mathrm{B}_0, *)\to \mathrm{Aut}^+(H_1(E_*,\mathbb{Z}))\cong SL(2,\mathbb{Z})$$

the monodromy representation of the fundamental group, where

$$\pi_1(B_0,*)\cong\left\langle a_i, b_i, c_j \ {}^{i=1,\dots,g}_{j=1,\dots,n} \,\middle|\, \prod_{i=1}^{g}[a_i,b_i]\prod_{j=1}^{n}c_j\right\rangle, \quad \text{with} \quad [a_i,b_i]=a_ib_ia_i^{-1}b_i^{-1}.$$

$\mathcal{J}$ is the functional invariant of the elliptic surface $E\to B$.

The extension of the homological invariant $\mathcal{G}_0$ over B_0 to $\mathcal{G}$ over B is uniquely given by the monodromy representation χ, which is determined by the $\mathcal{J}$-invariant except for its sign, i.e. there are 2^{2g+n-1} different homological invariants, depending on choice of sign for the matrices $A_i=\chi(a_i)$, $B_i=\chi(b_i)$, and $C_j=\chi(c_j)$, $i=1,\dots,g$; $j=1,\dots,n$, in the product $\prod_{i=1}^{g}[A_i,B_i]\prod_{j=1}^{n}C_j=1$.

Definition. Two elliptic surfaces $\pi: E\to B$ and $\pi': E'\to B'$ are *isomorphic*, if there are biholomorphic mappings f, g, so that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\ f\ } & E' \\ \pi\big\downarrow & & \big\downarrow\pi' \\ B & \xrightarrow{\ g\ } & B' \end{array}$$

commutes.

Let $\mathscr{F}(\mathscr{J},\mathscr{G})$ be the family of isomorphism classes of elliptic surfaces over B with only simple singular fibres with functional invariant $\mathscr{J}$ and homological invariant $\mathscr{G}$. For each such family $\mathscr{F}(\mathscr{J},\mathscr{G})$ Kodaira constructed a basic member $\mathscr{B}$, which is defined by a global holomorphic section $\sigma: B \to E$ [6, Sect. 8], and proved the following [6, Sects. 9, 10]:

Theorem 1. *Let $\pi: E \to B$ be an elliptic surface with a global section, belonging to the family $\mathscr{F}(\mathscr{J},\mathscr{G})$. Then E is isomorphic to the uniquely determined basic member $\mathscr{B}$ of the family $\mathscr{F}(\mathscr{J},\mathscr{G})$.*

Kas described this using the Weierstrass model [5].

Let $\pi: E \to B$ be a minimal elliptic surface. $K(E)$ and $K(B)$ are the function fields of E and of B respectively. π induces a homomorphism $\pi^*: K(B) \to K(E)$, and $K(E)$ is a transcendental extension of $K(B)$ of transcendence degree and genus one. The section $\sigma: B \to E$ determines a rational point. E is birationally equivalent to the subscheme E^* in $\mathrm{Proj}(\mathcal{O} \oplus \mathcal{O}(2L) \oplus \mathcal{O}(3L))$, which is given by the equation

$$V^2 W = 4U^3 - g_2 U W^2 - g_3 W^3,$$

where $\mathcal{O}$ is the structure sheaf of B, L is a line bundle and where $g_2 \in H^0(B, \mathcal{O}(4L))$ and $g_3 \in H^0(B, \mathcal{O}(6L))$ are sections with $\Delta = g_2^3 - 27g_3^2 \not\equiv 0$.

Theorem 2 (Kas). *E^* is an algebraic surface with rational double points as the only singularities. E is the minimal resolution of E^*. E^* is determined by g_2, g_3 up to $\mathbb{C}^*$-operation*

$$(g_2, g_3) \to (\lambda^4 g_2, \lambda^6 g_3), \quad \lambda \in \mathbb{C}^* = \mathbb{C} - \{0\}.$$

g_2, g_3 satisfy

(i) $\Delta = g_2^3 - 27g_3^2 \not\equiv 0$,

(ii) $\min(3v_p(g_2), 2v_p(g_3)) < 12$ *for all* $p \in B$,

where $v_p(g_2)$, $v_p(g_3)$, and $v_p(\Delta)$ are the order of the zeroes of g_2, g_3 and Δ in p. The singular fibre in E^* over p consists of the minimal resolution of the rational double point and the rational curve, which is defined by the section σ. The type of rational double point and thereby the type of the singular fibre determines $v_p(g_2)$, $v_p(g_3)$, and $v_p(\Delta)$. E^* is called the Weierstrass model of the elliptic surface.

The $\mathscr{J}$-invariant of the model is $\mathscr{J} = \dfrac{g_2^3}{\Delta}$.

Meyer proved the following [7]: For each locally trivial fibre bundle $E \to X$ it is possible to compute the signature of E as the signature of the E_2-term of the Leray spectral sequence of the fibration, i.e. for an elliptic fibration $E \to B$:

Let $B_0 := B - \bigcup_{i=1}^{n} D_i$, with D_i being disjoint small disks around the base points ϱ_i of the singular fibres. $E_0 = E_{|B_0}$ is called the "smooth" part and $E_s := E - E_0$ the "singular" part of E. The signature τ of the fibration is

$$\tau(E) = \tau(E_0) + \tau(E_s).$$

Let F_i be the singular fibre over $\varrho_i \in B$, then:

$$\tau(E_s) = \sum_{i=1}^{n} \tau(F_i),$$

with $\tau(F_i) = \tau(E_{|D_i})$.

Table 1

Singular fibre	Euler number	Monodromy matrix: Normal form A	Monodromy matrix: Conjugate form TAT^{-1}, $T=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z})$	Orders of zeroes: $\nu_\varrho(g_2)$	Orders of zeroes: $\nu_\varrho(g_3)$	Orders of zeroes: $\nu_\varrho(\Delta)$	Value of $\mathcal{J}(\varrho)$	Signature of the singular fibre: $\tau(F)$	Signature of the singular fibre: $\phi(F)$
I_0	0	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	0	0	0	$\neq 0, 1, \infty$	0	0
I_n $n>0$	n	$\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} =: P^n$	$\begin{bmatrix} 1-acn & a^2n \\ -c^2n & 1+acn \end{bmatrix}$ a, c relatively prime	0	0	n	Pole of order n	$1-n$	$1-\frac{n}{3}$
II	2	$\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} =: S$	$\begin{bmatrix} ad-bd-ac & a^2+b^2-ab \\ cd-c^2-d^2 & bd+ac-bc \end{bmatrix}$	$\geqq 1$	1	2	0	0	$\frac{4}{3}$
III	3	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} =: J$	$\begin{bmatrix} -bd-ac & a^2+b^2 \\ -c^2-d^2 & ac+bd \end{bmatrix}$	1	$\geqq 2$	3	1	-1	1
IV	4	$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = S^2$	$\begin{bmatrix} bc-bd-ac & a^2+b^2-ab \\ cd-c^2-d^2 & bd+ac-ad \end{bmatrix}$	$\geqq 2$	2	4	0	-2	$\frac{2}{3}$
I_0^*	6	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	2	>3	6	1	-4	0
				>2	3	6	0		
				2	3	6	$\neq 0, 1, \infty$		
I_n^* $n>0$	$n+6$	$\begin{bmatrix} -1 & -n \\ 0 & -1 \end{bmatrix} = -P^n$	$\begin{bmatrix} -1+acn & -a^2n \\ c^2n & -1-acn \end{bmatrix}$ a, c relatively prime	2	3	$n+6$	Pole of order n	$-n-4$	$-\frac{n}{3}$
II^*	10	$\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = S^{-1} = -S^2$	$\begin{bmatrix} ac+bd-bc & ab-a^2-b^2 \\ c^2+d^2-cd & ad-bd-ac \end{bmatrix}$	$\geqq 4$	5	10	0	-8	$-\frac{4}{3}$
III^*	9	$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = J^{-1} = -J$	$\begin{bmatrix} bd+ac & -a^2-b^2 \\ c^2+d^2 & -ac-bd \end{bmatrix}$	3	$\geqq 5$	9	1	-7	-1
IV^*	8	$\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} = -S = S^{-2}$	$\begin{bmatrix} ac+bd-ad & ab-a^2-b^2 \\ c^2+d^2-cd & bc-bd-ac \end{bmatrix}$	$\geqq 3$	4	8	0	-6	$-\frac{2}{3}$

There exists a uniquely determined mapping

$$\phi : \mathrm{SL}(2,\mathbb{Z}) \to \tfrac{1}{3}\mathbb{Z},$$

so that

$$\tau(E_0) = -\sum_{i=1}^{n} \phi(\gamma_i);$$

where γ_i is the monodromy of a closed path around ϱ_i. Then:

$$\tau(E) = \sum_{i=1}^{n} (\tau(F_i) - \phi(\gamma_i)).$$

The values of $\tau(F_i)$ and $\phi(\gamma_i)$ are listed in Table 1:

$$\tau(F_i) + e(F_i) = \begin{cases} 1 & \text{if} \quad F_i \text{ has type } I_n,\ n>0, \\ 2 & \text{else} \end{cases}$$

where $e(F_i)$ is the Euler number of the singular fibre F_i.

Furthermore:

Lemma 3 (Hunt [3]). *For the signature of the "smooth" part is*

$$|\tau(E_0)| \leqq 4g - 4 + 2n,$$

where g is the genus of the base curve.

It is known that for each minimal elliptic surface

$$\tau(E) = -\tfrac{2}{3} e(E).$$

Noethers formula implies that for compact complex surfaces S

$$\chi(S) = \frac{\tau(S) + e(S)}{4},$$

where $\chi(S)$ is the arithmetic genus of S and for an elliptic surface E

$$\chi(E) = \tfrac{1}{12} e(E).$$

Calculation of the possible fibre combinations

With the above notation, let $E \to \mathbb{P}_1\mathbb{C}$ be a minimal elliptic surface with a section σ and nonconstant $\mathcal{J}$-invariant. The singular fibres F_i are over ϱ_i, $\varrho_i \neq \varrho_j$ for $i \neq j$. Let

$$P := \{\varrho_1, \varrho_2, \varrho_3, \varrho_4\} \quad \text{and} \quad \chi : \pi_1(\mathbb{P}_1\mathbb{C} - P, *) \to \mathrm{SL}(2,\mathbb{Z})$$

be the monodromy representation of the fundamental group

$$\pi_1(\mathbb{P}_1\mathbb{C} - P, *) = \langle a_1, a_2, a_3, a_4 \mid a_1 a_2 a_3 a_4 = 1 \rangle,$$

where a_i is a closed path around ϱ_i and $A_i := \chi(a_i)$, $i = 1, \ldots, 4$, is a monodromy matrix. The homological invariant $\mathcal{G}$ of the elliptic surface E is determined by A_i with

$$A_1 A_2 A_3 A_4 = 1, \tag{1}$$

where A_i, $i=1,\ldots,4$, is conjugate to a matrix in $M=M^+\cup M^-$ with

$$M^+:=\{\mathrm{Id}, P^n\ (n>0),\ S, J, S^2\}$$

and

$$M^-:=\{-\mathrm{Id}, -P^n\ (n>0),\ -S, -J, -S^2\} \quad \text{(see Table 1)}.$$

$\mathscr{G}$ belongs to the functional invariant $\mathscr{J}$. For each functional invariant $\mathscr{J}$ and associated homological invariant $\mathscr{G}$ there is exactly one elliptic surface $\mathscr{B}$ over $\mathbb{P}_1\mathbb{C}$ with section. $\mathscr{B}$ is the basic member of $\mathscr{F}(\mathscr{J},\mathscr{G})$.

Its' Weierstrass model E^* will be calculated as follows: Let $G_2=g_2$, $g_2\in H^0(\mathbb{P}_1\mathbb{C}, \mathcal{O}(4L))$; $G_3=3\sqrt{3}g_3$, $g_3\in H^0(\mathbb{P}_1\mathbb{C}, \mathcal{O}(6L))$, where g_2, g_3 are the sections which determine the Weierstrass model. The matrices $\tilde{A}_i=\varepsilon_i A_i$, $i=1,\ldots,4$, $\varepsilon_i=\pm 1$, with $\tilde{A}_1\tilde{A}_2\tilde{A}_3\tilde{A}_4=1$ and therefore $\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4=1$, determine the homological invariant $\tilde{\mathscr{G}}$. The model $\tilde{E}^*$ of the basic member $\tilde{\mathscr{B}}\in\mathscr{F}(\mathscr{J},\tilde{\mathscr{G}})$ can easily be calculated from E^* by "transfering of *" and "*" pairs.

"*" a fibre over ϱ_i corresponds to multiplying the monodromy matrix A_i with $-\mathrm{Id}$. I_n, II, III, and IV change to I_n^*, IV^*, III^*, and II^* respectively and vice versa (see Table 1).

The Euler number of the singular fibre increases or decreases by six respectively. In the Weierstrass model the polynomials G_2, G_3, and Δ are multiplied with $(X-\varrho_i Y)^{2,\,3,\,\text{and }6\text{ resp.}}$, if $A_i\in M^+$ and $\varepsilon_i=-1$, or divided by the same expression, if $A_i\in M^-$ and $\varepsilon_i=-1$.

In the "transfer of *" of the singular fibre $F_i\in T^-$ over ϱ_i to the singular fibre $F_j\in T^+$ over ϱ_j (in short, from ϱ_i to ϱ_j), the monodromy matrices A_i and A_j will be multiplied with $-\mathrm{Id}$, the polynomials G_2, G_3, and Δ with $\dfrac{(X-\varrho_j Y)^{2,\,3,\,\text{and }6\text{ resp.}}}{(X-\varrho_i Y)^{2,\,3,\,\text{and }6\text{ resp.}}}$.

So it suffices to restrict the calculation to the canonical basic member $\mathscr{B}\in\mathscr{F}(\mathscr{J},\mathscr{G})$. $\mathscr{G}$ is determined by $A_1A_2A_3A_4=1$, where at most one A_i is conjugate to a matrix in M^-. At least one singular fibre has to be of type I_n or I_n^*. For the classification of surfaces, which have one fibre of type I_0, a regular fibre, see [13].

Lemma 4. *The Euler number of an elliptic surface with four singular fibres in T^+ over $\mathbb{P}_1\mathbb{C}$ is twelve.*

Proof. The Euler number of an elliptic surface, which depends on the Euler number of the singular fibres only, is:

$$e(E)=12\chi(E)=3(e(E_s)+\tau(E_s)+\tau(E_0))\leqq 3(2n+4g-4+2n),$$

where n is the number of singular fibres and g is the genus of the base curve. For $g=0$, this is

$$0<e(E)\leqq 12(n-1).$$

Let n be even and all singular fibres of E in T^+. "*" all these fibres gives E^* with

$$e(E^*)=e(E)+6n\leqq 12(n-1).$$

Therefore

$$e(E)\leqq 6(n-2).$$

For $n=4$ this proofs the lemma.

Let n be odd and at least $n-1$ singular fibres of E in T^+. By "*" all these fibres this gives

$$e(E) \leqq 6(n-1).$$

Theorem 5. *Let $E \to \mathbb{P}_1\mathbb{C}$ be a minimal elliptic surface with section, nonconstant $\mathcal{J}$-invariant and four singular fibres, of which at most one is in T^-. Up to permutation and "transfering of *" there are only those combinations of singular fibres which are listed in Table 3.*

(i) *If one singular fibre is in T^-, three in T^+, the Weierstrass model depends on a parameter. Given four different base points, in the case $I_1^* I_1 I_1$ III there exist four, in the cases $I_1^* I_1 I_2 II$ and $I_1 I_1 II$ IV^* there exist two elliptic surfaces, depending on the $\mathcal{J}$-invariant, and for all other fibre combinations there exists precisely one elliptic surface.*

(ii) *If all four singular fibres are in T^+, then the Weierstrass models are determined uniquely up to isomorphism, except for one combination. In the case $I_1 I_6 II$ III there are two nonisomorphic models.*

In Table 3 the Weierstrass models including the $\mathcal{J}$-invariant and cross ratio of the base points for $\Delta = G_2^3 - 27G_3^2$ are listed.

Corollary 6. *<u>All</u> elliptic surfaces with four singular fibres can be deduced from Table 3 by the following methods:*

(i) *"*" the singular fibres in pairs;*

(ii) *"transfering of *" of singular fibres.*

Proof. 1. If one singular fibre is of type I_0^*, a surface with three singular fibres is obtained by "transfering of *". So these elliptic surfaces are easily calculated [13, p. 120ff., cases 6–12].

(In the following it is assumed that $n > 0$ for all fibres of type I_n or I_n^*.)

2. Determination of all possible fibre combinations.

Because of (1), it follows for the monodromy matrices $A_i \in \mathrm{SL}(2, \mathbb{Z})$, $i = 1, \ldots, 4$ that:

$$\operatorname{trace}(A_1 A_2) = \operatorname{trace}((A_3 A_4)^{-1}) = \operatorname{trace}(A_3 A_4). \tag{2}$$

The trace is preserved under conjugation. So let A_2 and A_4 be in normal form, A_1 and A_3 be conjugate to $\pm P^n, S, J, S^2$ (see Table 1). Table 2 lists the $\operatorname{trace}(A_i A_{i+1})$ for different fibre combinations.

In the following the calculation will be separate according to the occurrence of a fibre $F_1 \in T^-$ and the number of fibres of type I_n.

2.1. One singular fibre in T^-. Assume that this fibre F_1 is of type I_n^*.

2.1.1. F_3 of type I_n, $n > 0$; $F_2, F_4 \in T^+ - \{I_n\}$. See Table 2. There is $\operatorname{trace}(A_1 A_2) \geqq 0$ and $\operatorname{trace}(A_3 A_4) \leqq 0$ with "=" exactly for $F_2 = F_4 = II$. It follows that

$$-1 + n_1(a_1^2 + a_1 c_1 + c_1^2) = 1 - n_3(a_3^2 + a_3 c_3 + c_3^2).$$

Because of $a_i^2 + a_i c_i + c_i^2 > 0$, we have $n_1 = n_3 = 1$ and the combination is $I_1^* I_1 II$ II.

2.1.2. $F_2, F_3, F_4 \in T^+ - \{I_n\}$. Table 2 shows that the Eq. (2) cannot be satisfied.

Table 2

Singular fibre	Trace (A_iA_{i+1})	
$I_{n_i}I_{n_{i+1}}$	$2-c_i^2n_in_{i+1}\leqq 2$	
$I_{n_i}II$	$1-n_i(a_i^2+a_ic_i+c_i^2)\leqq 0$	a_i, c_i relatively prime
$I_{n_i}III$	$-n_i(a_i^2+c_i^2)\leqq -1$	a_i, c_i relatively prime
$I_{n_i}IV$	$-[1+n_i(a_i^2+a_ic_i+c_i^2)]\leqq -2$	a_i, c_i relatively prime
$II\,II$	$-[(b_i-\frac{1}{2}a_i+\frac{1}{2}d_i)^2+(c_i+\frac{1}{2}a_i-\frac{1}{2}d_i)^2+\frac{1}{2}(a_i^2+d_i^2)]\leqq -1$	
$II\,III$	$-(a_i^2-a_ib_i+b_i^2+c_i^2-c_id_i+d_i^2)\leqq -2$	
$II\,IV$	$-[(a_i-\frac{1}{2}b_i+\frac{1}{2}c_i)^2+(d_i+\frac{1}{2}b_i-\frac{1}{2}c_i)^2+\frac{1}{2}(b_i^2+c_i^2)]\leqq -2$	
$III\,III$	$-(a_i^2+b_i^2+c_i^2+d_i^2)\leqq -2$	
$III\,IV$	$-(a_i^2+a_ic_i+c_i^2+b_i^2+b_id_i+d_i^2)\leqq -2$	
$IV\,IV$	$-[(b_i-\frac{1}{2}a_i+\frac{1}{2}d_i)^2+(c_i+\frac{1}{2}a_i-\frac{1}{2}d_i)^2+\frac{1}{2}(a_i^2+d_i^2)]\leqq -1$	
$I^*_{n_i}I_{n_{i+1}}$	$-2+c_i^2n_in_{i+1}\geqq -2$	
$I^*_{n_i}II$	$-1+n_i(a_i^2+a_ic_i+c_i^2)\geqq 0$	a_i, c_i relatively prime
$I^*_{n_i}III$	$n_i(a_i^2+c_i^2)\geqq 1$	a_i, c_i relatively prime
$I^*_{n_i}IV$	$1+n_i(a_i^2+a_ic_i+c_i^2)\geqq 2$	a_i, c_i relatively prime

For F_2, F_3, F_4 of type I_n, $n>0$, or F_2, F_3 of type I_n, $n>0$, $F_4\in T^+-\{I_n\}$ similar calculations as above show that the only possible fibre combinations are those which are listed in Table 3.

2.2. Four singular fibres in T^+.

2.2.1. F_1, F_2, F_3, F_4 of type I_n, $n>0$. Eq. (2) is equivalent to $c_1^2n_1n_2=c_3^2n_3n_4$, see Table 2. If $c_1=c_3=0$, one easily deduces a contradiction $A_1A_2A_3A_4\neq 1$ to Eq. (1). So Eq. (2) is now equivalent to

$$n_1n_2n_3n_4=\frac{c_3^2}{c_1^2}n_3^2n_4^2.$$

So $\prod_{i=1}^{4} n_i$ is a square and $\sum_{i=1}^{4} n_i=12$. Only the fibre combinations, which are listed in Table 3, exist up to permutation.

2.2.2. F_1, F_3 of type I_n, $n>0$; $F_2, F_4\in T^+-\{I_n\}$. Lemma 5 shows

$$n_1+n_3=12-e(F_2)-e(F_4).$$

$I_5II\,I_3II$ and $I_3II\,I_3IV$ are excluded, because of

$$0\equiv 5(a_1^2+a_1c_1+c_1^2)\not\equiv 3(a_3^2+a_3c_3+c_3^2)\,\text{modulo}\,5$$

and

$$0\equiv 3(a_1^2+a_1c_1+c_1^2)\not\equiv 3(a_3^2+a_3c_3+c_3^2)+2\,\text{modulo}\,3,$$

see Table 2 and (2). The remaining fibre combinations, up to permutation of the fibres, are those which are listed in Table 3, and the combination I_3I_1IVIV. Explicit calculation of the Weierstrass model shows, that the last combination is impossible.

2.2.3. F_1 of type I_n, $n>0$; $F_2, F_3, F_4 \in T^+ - \{I_n\}$. Lemma 5 shows that the Euler number is twelve. Only the combinations listed in Table 3 and $I_1 III\, IV\, IV$, $I_2 II\, IV\, IV$ can be possible up to permutation. In the last two cases G_2 and G_3 must have the degree $\geqq 5$ and $\geqq 6$ or $\geqq 5$ and 5 respectively (see Table 1). This, however, is impossible.

2.2.4. If there are three fibres of type I_n, one gets all combinations of Table 3 and $I_4 I_3 I_1 IV$ as in 2.2.2. This fibre combination can be excluded by explicit calculation.

3. Calculation of the polynomials G_2, G_3, and Δ in homogeneous coordinates (X, Y) of $\mathbb{P}_1\mathbb{C}$

Equation $\Delta = G_2^3 - G_3^2$ gives a nonlinear system of equations for the coefficients of G_2, G_3. Common factors of G_2^3, G_3^2, and Δ will be cancelled.

Note. Let $\bar{\Delta} = \dfrac{G_2^3 - G_3^2}{\gcd(G_2^3, G_3^2)} = \dfrac{\Delta}{\gcd(G_2^3, G_3^2)}$ (see Table 1) and let C_i be the coefficient of $X^{k-i}Y^i$ in $\bar{\Delta}$, where k is the sum of the n_j over the numbers of the fibres of types I_{n_j} and $I_{n_j}^*$ of the surface with $0 \leqq i \leqq k$. The base points are written as quadruple $(\varrho_1, \varrho_2, \varrho_3, \varrho_4)$.

3.1. One singular fibre in T^-.

$$I_1^* I_1 II\, II.$$

It may be assumed that the singular fibres are over $(0, \infty, 1, \varrho_4)$. The orders of zeroes at the base points have to be:

ϱ	$v_\varrho(G_2)$	$v_\varrho(G_3)$	$v_\varrho(\Delta)$
0	2	3	7
∞	0	0	1
1	$\geqq 1$	1	2
ϱ_4	$\geqq 1$	1	2
Sum	$\geqq 4$	5	12

The equation $\Delta = G_2^3 - G_3^2$ with

$$G_2(X, Y) = \mu X^2 (X - Y)(X - \varrho_4 Y),$$
$$G_3(X, Y) = \nu X^3 (X - Y)(X - \varrho_4 Y)(X + BY),$$
$$\Delta(X, Y) = \sigma \mu^3 X^7 Y (X - Y)^2 (X - \varrho_4 Y)^2,$$

where $\mu, \nu, \sigma \in \mathbb{C}^*$ produces the following system of equations with $\mu^3 - \nu^2 = 0$:

$$C_1 = -(\varrho_4 + 1 + 2B) = \sigma,$$
$$C_2 = \varrho_4 - B^2 = 0.$$

It follows that $\varrho_4 = B^2 \neq 0$ and $\sigma = -(B+1)^2$. Consequently one gets

$$G_2(X, Y) = \mu X^2(X-Y)(X-B^2 Y),$$
$$G_3(X, Y) = \nu X^3(X-Y)(X-B^2 Y)(X+BY),$$
$$\Delta(X, Y) = -(B+1)^2 \mu^3 X^7 Y(X-Y)^2(X-B^2 Y)^2,$$

where $B \neq -1, 1$.

The cross ratio of the base points $\mathrm{CR}(I_1^* I_1 | II\, II)$ is $\frac{1}{B^2}$. If the base points are given, there exist two different Weierstrass models, depending on the choice of the $\mathscr{J}$-invariant. Let $\tilde{\Delta} = 27\Delta$ and $\tilde{G}_2 = 3G_2$. Table 3 lists the surface for $\mu = 1$, $\nu = 1$, $\tilde{\Delta}$ as Δ and $\tilde{G}_2$ as G_2 in abuse of the notation. G_2 and G_3 are uniquely determined up to a transformation $(G_2, G_3) \to (h^4 G_2, h^6 G_3)$, $h \in \mathbb{C}^*$. Consequently in this calculation, as in the following ones, there are values given for μ and ν, so that one arrives at the polynomials G_2, G_3, and Δ as above which are listed in Table 3.

3.2. Four singular fibres in T^+.

3.2.1. Calculation by using the common divisor of G_2, G_3, and Δ.

$$I_4 I_4 I_2 I_2.$$

The orders of zeroes have to be:

ϱ	$\nu_\varrho(G_2)$	$\nu_\varrho(G_3)$	$\nu_\varrho(\Delta)$
ϱ_1	0	0	4
ϱ_2	0	0	4
ϱ_3	0	0	2
ϱ_4	0	0	2
Sum	0	0	12

Therefore

$$-\bar{\Delta}^2 = \Delta = G_2^3 - G_3^2 \tag{4}$$

with $\bar{\Delta}$, $G_3 \in H^0(\mathbb{P}_1\mathbb{C}, \mathcal{O}(6L))$, $G_2 \in H^0(\mathbb{P}_1\mathbb{C}, \mathcal{O}(4L))$. (4) is equivalent to

$$G_2^3 = G_3^2 - \bar{\Delta}^2 = (G_3 - \bar{\Delta})(G_3 + \bar{\Delta}).$$

It follows from (3) that G_2, G_3, and $\bar{\Delta}$ are relatively prime. Therefore are

$$H_1^3 = G_3 - \bar{\Delta}$$

and

$$H_2^3 = G_3 + \bar{\Delta}$$

with $H_1, H_2 \in H^0(\mathbb{P}_1\mathbb{C}, \mathcal{O}(2L))$ relatively prime. It is

$$\bar{\Delta} = \tfrac{1}{2}(H_2^3 - H_1^3) = \tfrac{1}{2}(H_2 - H_1)(\eta^2 H_2 - \eta H_1)(\eta H_2 - \eta^2 H_1)$$

with $\eta = e^{\frac{2\pi i}{3}}$. Because of (3) $\bar{\Delta}$ has two double zeroes at ϱ_1 and ϱ_2 and two single at ϱ_3 and ϱ_4. Let

$$J_1 := H_2 - H_1,$$
$$J_2 := \eta^2 H_2 - \eta H_1,$$
$$J_3 := \eta H_2 - \eta^2 H_1$$

with $J_i \in H^0(\mathbb{P}_1\mathbb{C}, \mathcal{O}(2L))$ $i=1,2,3$. J_1, J_2, J_3 are relatively prime.

It can be assumed that the double zeroes of $\bar{\Delta}$ are in J_2 at $\varrho_1 = 0$ and in J_3 at $\varrho_2 = \infty$, therefore $J_2 = X^2$ and $J_3 = Y^2$. It follows:

$$J_1 = -(X^2 + Y^2),$$
$$H_1 = \frac{1}{1-\eta}(X^2 - \eta Y^2),$$
$$H_2 = \frac{1}{\eta^2 - 1}(X^2 - \eta^2 Y^2),$$
$$\bar{\Delta} = -\tfrac{1}{2} X^2 Y^2 (X^2 + Y^2)$$

and

$$G_2(X, Y) = -\tfrac{1}{3}(X^4 + X^2 Y^2 + Y^4),$$
$$G_3(X, Y) = \frac{1}{6\eta(\eta - 1)}(2X^6 + 3X^4 Y^2 - 3X^2 Y^4 - 2Y^6),$$
$$\Delta(X, Y) = -\tfrac{1}{4} X^4 Y^4 (X^2 + Y^2)^2.$$

The cross ratio is $\mathrm{CR}(I_4 I_4 | I_2 I_2) = -1$. Table 3 gives the surface after the transformations $(X, Y) \to (X, iY)$ and $(G_2, G_3) \to (-12G_2, 24i\sqrt{3}G_3)$.

Using this method, it is also possible to calculate the Weierstrass models of

$$I_3 I_3 I_3 I_3, \quad I_4 I_4 II\, II, \quad I_3 I_3 III\, III \quad [2].$$

3.2.2. All other fibre combinations are calculated using the same method as in 3.1 [2].

$$I_7 I_1 I_1 III, \quad I_6 I_2 I_1 III, \quad \text{and} \quad I_6 II\, I_1 III.$$

An $\mathrm{Aut}(\mathbb{P}_1\mathbb{C})$-operation transforms the singular fibres over the base points $(\infty, \varrho_2, \varrho_3, 0)$. The fibres over ∞, ϱ_2 are either of type $I_7 I_1$ or of type $I_6 I_2$ and $I_6 II$ respectively, therefore the three calculations differ by a common factor of $G_2^3 - G_3^2$ and Δ only.

Let

$$i := \begin{cases} 0 & \text{at} \quad I_7 I_1 I_1 III \\ 1 & \text{at} \quad I_6 I_2 I_1 III \text{ and } I_6 II\, I_1 III. \end{cases}$$

The orders of zeroes have to be:

ϱ	$v_\varrho(G_2)$	$v_\varrho(G_3)$	$i=0$ $v_\varrho(\Delta)$	$i=1$ $v_\varrho(\Delta)$
∞	0	0	7	6
ϱ_2	$0(\geqq 1)$	$0(1)$	1	2
ϱ_3	0	0	1	1
0	1	$\geqq 2$	3	3
Sum	$1(\geqq 2)$	$\geqq 2(3)$	12	12

The orders of zero for the fibre over ϱ_2 in $T^+ - \{I_n\}$ are in brackets.

The equation $\Delta = G_2^3 - G_3^2$ with

$$G_2(X, Y) = \mu X(X^3 + A_1X^2Y + A_2XY^2 + A_3Y^3),$$
$$G_3(X, Y) = \nu X^2(X^4 + B_1X^3Y + B_2X^2Y^2 + B_3XY^3 + B_4Y^4),$$
$$\Delta(X, Y) = \sigma\mu^3X^3Y^{7-i}(X - \varrho_2Y)^{1+i}(X - \varrho_3Y),$$

where $\mu, \nu, \sigma \in \mathbb{C}^*$; $i = 0, 1$ produces the following system of equations with $\mu^3 - \nu^2 = 0$:

$$C_1 = 3A_1 - 2B_1 = 0,$$
$$C_2 = 3A_1^2 + 3A_2 - B_1^2 - 2B_2 = 0,$$
$$C_3 = A_1^3 + 3A_3 + 6A_1A_2 - 2B_3 - 2B_1B_2 = 0,$$
$$C_4 = 3A_2^2 + 3A_1^2A_2 + 6A_1A_3 - B_2^2 - 2B_4 - 2B_1B_3 = 0,$$
$$C_5 = 3A_1A_2^2 + 3A_1^2A_3 + 6A_2A_3 - 2B_1B_4 - 2B_2B_3 = 0,$$
$$C_6 = A_2^3 + 3A_3^2 + 6A_1A_2A_3 - B_3^2 - 2B_2B_4 = \begin{cases} 0 & i=0 \\ \sigma & i=1, \end{cases}$$
$$C_7 = 3A_1A_3^2 + 3A_2^2A_3 - 2B_3B_4 = \begin{cases} \sigma & i=0 \\ -(2\varrho_2 + \varrho_3)\sigma & i=1, \end{cases}$$
$$C_8 = 3A_2A_3^2 - B_4^2 = \begin{cases} -(\varrho_2 + \varrho_3)\sigma & i=0 \\ (2\varrho_2\varrho_3 + \varrho_2^2)\sigma & i=1, \end{cases}$$
$$C_9 = A_3^3 = \begin{cases} \varrho_2\varrho_3\sigma & i=0 \\ -\varrho_2^2\varrho_3\sigma & i=1. \end{cases}$$

To fulfil $C_1 = C_2 = C_3 = 0$, let

$$A_1 = 2\alpha, \qquad B_1 = 3\alpha,$$
$$A_2 = 2\beta - \alpha^2, \qquad B_2 = 3\beta,$$
$$A_3 = 2\gamma, \qquad B_3 = 3\gamma - 2\alpha^3 + 3\alpha\beta,$$

where $\alpha, \beta, \gamma \in \mathbb{C}$.

From $C_4 = 0$ it follows that:

$$B_4 = \tfrac{3}{2}[(\beta - \alpha^2)^2 + 2\alpha\gamma]$$

and from $C_5 = 0$:

$$3(\beta - \alpha^2)[2\gamma - \alpha(\beta - \alpha^2)] = 0.$$

Assume:

1) $\beta - \alpha^2 = 0.$

The result is

$$C_6=3\gamma^2=\begin{cases}0 & i=0\\ \sigma & i=1,\end{cases}$$

$$C_7=6\alpha\gamma^2=\begin{cases}\sigma & i=0\\ -(2\varrho_2+\varrho_3)\sigma & i=1,\end{cases}$$

$$C_8=3\alpha^2\gamma^2=\begin{cases}-(\varrho_2+\varrho_3)\sigma & i=0\\ (2\varrho_2\varrho_3+\varrho_2^2)\sigma & i=1,\end{cases}$$

$$C_9=8\gamma^3=\begin{cases}\varrho_2\varrho_3\sigma & i=0\\ -\varrho_2^2\varrho_3\sigma & i=1.\end{cases}$$

γ may not equal zero, so $i=1$. Consequently the only solution is

$$\gamma=\tfrac{1}{18}\alpha^3,$$
$$\varrho_2=-\tfrac{1}{3}\alpha,$$
$$\varrho_3=-\tfrac{4}{3}\alpha,$$
$$\sigma=\tfrac{1}{108}\alpha^3.$$

If $\alpha=-3$:

$$G_2(X,Y)=\mu X(X^3-6X^2Y+9XY^2-3Y^3),$$
$$G_3(X,Y)=\tfrac{1}{2}\nu X^2(2X^4-18X^3Y+54X^2Y^2-63XY^3+27Y^4),$$
$$\Delta(X,Y)=\tfrac{27}{4}\mu^3X^3Y^6(X-Y)^2(X-4Y)$$

with $\mu^3-\nu^2=0$, $\mu,\nu\in\mathbb{C}^*$.

The cross ratio is $\mathrm{CR}(I_6I_2\,|\,I_1III)=-\frac{1}{3}$. Table 3 shows the surface for $\mu=4$, $\nu=8$.

2) $\beta-\alpha^2\neq 0$.
$C_5=0$ leads to

$$\gamma=\tfrac{1}{2}\alpha(\beta-\alpha^2).$$

After the substitution of $\delta=\beta-\alpha^2$, it follows for C_6, C_7, C_8, and C_9 that:

$$\begin{array}{l|c|c}
 & i=0 & i=1\\
\hline
C_6=-\delta^2(\delta-\frac{3}{4}\alpha^2)= & 0 & \sigma\\
C_7=-\frac{3}{2}\alpha\delta^2(\delta-\alpha^2)= & \sigma & -(2\varrho_2+\varrho_3)\sigma\\
C_8=-\frac{3}{4}\delta^2(3\delta+\alpha^2)(\delta-\alpha^2)= & -(\varrho_2+\varrho_3)\sigma & (2\varrho_2\varrho_3+\varrho_2^2)\sigma\\
C_9=\alpha^3\delta^3= & \varrho_2\varrho_3\sigma & -\varrho_2^2\varrho_3\sigma.
\end{array} \tag{5}$$

(i) $i=0$.
Because $\sigma \neq 0$, this gives:

$$\delta=\tfrac{3}{4}\alpha^2,$$

$$C_7=\frac{3^3}{2^7}\alpha^7=\sigma,$$

$$C_8=\frac{3^3\cdot 13}{2^{10}}\alpha^8=-(\varrho_2+\varrho_3)\sigma,$$

$$C_9=\frac{3^3}{2^6}\alpha^9=\varrho_2\varrho_3\sigma,$$

$$\varrho_2+\varrho_3=-\frac{13}{2^3}\alpha,$$

$$\varrho_2\cdot\varrho_3=2\alpha^2,$$

and

$$\varrho_2=\frac{1}{8}\alpha\left(\frac{1\pm i\sqrt{7}}{2}\right)^7,$$

$$\varrho_3=\frac{1}{8}\alpha\left(\frac{1\mp i\sqrt{7}}{2}\right)^7.$$

If $\alpha=2$, we get:

$$G_2(X,Y)=\mu X(X^3+4X^2Y+10XY^2+6Y^3),$$
$$G_3(X,Y)=\tfrac{1}{2}\nu X^2(2X^4+12X^3Y+42X^2Y^2+70XY^3+63Y^4),$$
$$\Delta(X,Y)=\tfrac{27}{4}\mu^3X^3Y^7(4X^2+13XY+32Y^2)$$

with $\mu^3-\nu^2=0$ $\mu,\nu\in\mathbb{C}^*$.

The cross ratio is $\mathrm{CR}(I_7I_1|I_1III)=\dfrac{(1+i\sqrt{7})^7}{(1+i\sqrt{7})^7-(1-i\sqrt{7})^7}$.

Table 3 shows the surface for $\mu=4$, $\nu=8$.

(ii) $i=1$.

There is a double zero of Δ at ϱ_2. The discriminant of

$$-\tfrac{1}{4}\delta^2[(4\delta-3\alpha^2)X^3+6\alpha(\delta-\alpha^2)X^2Y+3(3\delta^2-2\alpha^2\delta-\alpha^4)XY^2-4\alpha^3\delta Y^3]$$

vanishes [see (5)], i.e.

$$-\tfrac{27}{16}\delta(\delta^2-\alpha^2\delta+\tfrac{1}{3}\alpha^4)^3=0.$$

Because $\delta=\frac{1}{2}\alpha^2(1\pm\frac{1}{3}\omega)$ with $\omega=i\sqrt{3}$, it follows from (5) that:

$$C_6=\tfrac{1}{72}\alpha^6(1\pm\omega)(3\mp 2\omega)=\sigma,$$
$$C_7=\tfrac{1}{24}\alpha^7(1\pm\omega)(3\mp\omega)=-(2\varrho_2+\varrho_3)\sigma,$$
$$C_8=\tfrac{1}{48}\alpha^8(1\pm\omega)(9\mp\omega)=(2\varrho_2\varrho_3+\varrho_2^2)\sigma,$$
$$C_9=\tfrac{1}{36}\alpha^9(1\pm\omega)(3\pm\omega)=-\varrho_2^2\varrho_3\sigma.$$

The result of the system of equations is:

$$\varrho_2 = \frac{\alpha(9+\omega)}{2(2\omega-3)},$$

$$\varrho_3 = -\frac{4\alpha\omega}{2\omega-3}.$$

Let $\alpha = -1+\omega$. After the transformation $(X, Y) \to (X, \frac{1}{2}Y)$:

$$G_2(X, Y) = \tfrac{1}{6}\mu X(X-Y)[6X^2 + 6\omega XY - (3+\omega)Y^2],$$

$$G_3(X, Y) = \tfrac{1}{4}\nu X^2(X-Y)[4X^3 - 2(1-3\omega)X^2Y - 4(2+\omega)XY^2 + (5-\omega)Y^3],$$

$$\Delta(X, Y) = -\frac{1}{2^3 \cdot 3^2}\mu^3 X^3 Y^6 (X-Y)^2[(9+\omega)X + 8\omega Y]$$

with $\mu^3 - \nu^2 = 0$.

The cross ratio is $\mathrm{CR}(I_6 II | I_1 III) = \frac{3-2\omega}{9}$. After the transformation

$$(G_2, G_3) \to (\tfrac{1}{9}G_2, \tfrac{1}{27}G_3)$$

Table 3 shows the surface for $\mu = 36$, $\nu = 216$.

Notes to Table 3. Table 3 lists the Weierstrass models of the fibre combinations with the base points $(\varrho_1, \varrho_2, \varrho_3, \varrho_4)$ of the fibres. G_2 and G_3 appear as follows: The discriminant is $\Delta = G_2^3 - 27G_3^2$. All polynomials without parameters can be chosen to have integer coefficients except for the combination $I_6 I_1 II\ III$. (G_2, G_3) are determined up to $(\lambda^4 G_2, \lambda^6 G_3)$ $\lambda \in \mathbb{C}^*$ only.

If there is a singular fibre in T^-, then Table 3 lists in addition those values of the cross ratio

$$\mathrm{CR}(\varrho_1\varrho_2 | \varrho_3\varrho_4) = \frac{\varrho_1 - \varrho_3}{\varrho_2 - \varrho_3} : \frac{\varrho_1 - \varrho_4}{\varrho_2 - \varrho_4},$$

which are excluded.

All surfaces with four singular fibres, section and nonconstant $\mathcal{J}$-invariant $\mathcal{J} = \frac{G_2^3}{\Delta}$ can easily be calculated from the models by "transfering of *" and "*" the fibres (see p. 324). They are uniquely determined up the operation of $\mathrm{Aut}(\mathbb{P}_1\mathbb{C})$.

Table 3

Fibre combination	Weierstrass model, $\mathcal{J}$-invariant and cross ratio of the base points
$I_4 I_1 I_1 I_0^*$	$G_2 = 3(X - \varrho_4 Y)^2(X^2 + 14XY + Y^2)$
$(1, \infty, 0, \varrho_4)$	$G_3 = (X - \varrho_4 Y)^3(X^3 - 33X^2Y - 33XY^2 + Y^3)$
	$\Delta = 2^2 \cdot 3^6 XY(X - \varrho_4 Y)^6(X-Y)^4$
	$\mathcal{J} = \frac{1}{108}\frac{(X^2 + 14XY + Y^2)^3}{XY(X-Y)^4}$
	$\mathrm{CR} = \frac{1}{1-\varrho_4} \neq 0, 1, \infty$

Table 3 (continued)

$I_2 I_2 I_2 I_0^*$ $(1, \infty, 0, \varrho_4)$	$G_2 = 12(X-\varrho_4 Y)^2(X^2-XY+Y^2)$ $G_3 = 4(X-\varrho_4 Y)^3(2X^3-3X^2Y-3XY^2+2Y^3)$ $\Delta = 2^4 \cdot 3^6 X^2 Y^2 (X-\varrho_4 Y)^6 (X-Y)^2$ $\mathscr{J} = \frac{4}{27} \frac{(X^2-XY+Y^2)^3}{X^2Y^2(X-Y)}$ $\mathrm{CR} = \frac{1}{1-\varrho_4} \neq 0, 1, \infty$
$I_1 I_1 I_1 I_3^*$ $(\omega_1, \omega_2, \infty, 0)$	$G_2 = 12X^2(X^2+2\alpha XY+Y^2), \quad \alpha \neq -2, -\frac{5}{3}, 1$ $G_3 = 4X^3(2X^3+3(\alpha^2+1)X^2Y+6\alpha XY^2+2Y^3)$ $\Delta = -2^4 \cdot 3^3 (\alpha-1)^2 X^9 Y[12X^2+3(3\alpha^2+6\alpha-1)XY+4(\alpha+2)Y^2]$ $\mathscr{J} = -\frac{4}{(\alpha-1)^2} \frac{(X^2+2\alpha XY+Y^2)^3}{X^3Y[12X^2+3(3\alpha^2+6\alpha-1)XY+4(\alpha+2)Y^2]}$ $\mathrm{CR} = \frac{\omega_2}{\omega_1} \neq 0, 1, \infty, \quad \omega_{1,2} = -\frac{1}{8}[3\alpha^2+6\alpha-1 \pm \sqrt{\frac{1}{3}(\alpha-1)(3\alpha+5)^3}]$
$I_1 I_1 I_2 I_2^*$ $(\omega_1, \omega_2, \infty, 0)$	$G_2 = 12X^2(X^2+\alpha XY+Y^2)$ $G_3 = 4X^3(2X^3+3\alpha X^2Y+3\alpha XY^2+2Y^3)$ $\Delta = 2^4 \cdot 3^3 (2-\alpha)^2 X^8 Y^2 [3X^2+2(2\alpha-1)XY+3Y^2]$ $\mathscr{J} = \frac{4}{(2-\alpha)^2} \frac{(X^2+\alpha XY+Y^2)^3}{X^2Y^2[3X^2+2(2\alpha-1)XY+3Y^2]}, \quad \alpha \neq -1, 2$ $\mathrm{CR} = \frac{\omega_2}{\omega_1} \neq 0, 1, \infty, \quad \omega_{1,2} = -\frac{1}{3}(2\alpha-1 \pm 2\sqrt{\alpha^2-\alpha-2})$
$I_3 I_1 II\, I_0^*$ $(0, \infty, 1, \varrho_4)$	$G_2 = 3(X-\varrho_4 Y)^2(X-Y)(X-9Y)$ $G_3 = (X-\varrho_4 Y)^3(X-Y)(X^2+18XY-27Y^2)$ $\Delta = -2^6 \cdot 3^3 X^3 Y (X-Y)^2 (X-\varrho_4 Y)^6$ $\mathscr{J} = -\frac{1}{64} \frac{(X-Y)(X-9Y)^3}{X^3Y}$ $\mathrm{CR} = \frac{1}{\varrho_4} \neq 0, 1, \infty$
$I_2 I_1 III\, I_0^*$ $(0, \infty, 1, \varrho_4)$	$G_2 = 3(X-\varrho_4 Y)^2(X-Y)(X-4Y)$ $G_3 = (X-\varrho_4 Y)^3(X-Y)^2(X+8Y)$ $\Delta = -3^6 X^2 Y (X-Y)^3 (X-\varrho_4 Y)^6$ $\mathscr{J} = -\frac{1}{27} \frac{(X-4Y)^3}{X^2Y}$ $\mathrm{CR} = \frac{1}{\varrho_4} \neq 0, 1, \infty$
$I_1 I_1 IV\, I_0^*$ $(0, \infty, 1, \varrho_4)$	$G_2 = 3(X-\varrho_4 Y)^2(X-Y)^2$ $G_3 = (X-\varrho_4 Y)^3(X-Y)^2(X+Y)$ $\Delta = -108XY(X-Y)^4(X-\varrho_4 Y)^6$ $\mathscr{J} = -\frac{1}{4} \frac{(X-Y)^2}{XY}$ $\mathrm{CR} = \frac{1}{\varrho_4} \neq 0, 1, \infty$

Table 3 (continued)

$I_1 I_1 I_1 III^*$ $(\omega_1, \omega_2, \infty, 0)$	$G_2 = 3X^3(X+\alpha Y)$ $G_3 = X^5(X+Y)$ $\Delta = 27X^9 Y[(3\alpha-2)X^2+(3\alpha^2-1)XY+\alpha^3 Y^2]$ $\mathcal{J} = \dfrac{(X+\alpha Y)^3}{Y[(3\alpha-2)X^2-(3\alpha^2-1)XY+\alpha^3 Y]}, \quad \alpha \neq -\tfrac{1}{3}, 0, \tfrac{2}{3}, 1$ $\mathrm{CR} = \dfrac{\omega_2}{\omega_1} \neq 0, 1, \infty, \quad \omega_{1,2} = -\dfrac{1}{6\alpha-4}[3\alpha^2-1 \pm \sqrt{(3\alpha+1)(1-\alpha)^3}]$
$I_1 I_1 I_2 IV^*$ $(\omega_1, \omega_2, \infty, 0)$	$G_2 = 3X^3(X+2\alpha Y)$ $G_3 = X^4(X^2+3\alpha XY+Y^2)$ $\Delta = 27X^8 Y^2[(3\alpha^2-2)X^2+2\alpha(4\alpha^2-3)XY-Y^2]$ $\mathcal{J} = \dfrac{X(X+2\alpha Y)^3}{Y^2[(3\alpha^2-2)X^2+2\alpha(4\alpha^2-3)XY-Y^2]}, \quad \alpha \neq 0, \pm\sqrt{\tfrac{1}{2}}, \pm\sqrt{\tfrac{2}{3}}$ $\mathrm{CR} = \dfrac{\omega_2}{\omega_1} \neq -1, 0, \tfrac{1}{2}, 1, 2, \infty, \quad \omega_{1,2} = -\dfrac{1}{3\alpha^2-2}[\alpha(4\alpha^2-3) \pm \sqrt{2(2\alpha^2-1)^3}]$
$I_1 II\, III\, I_0^*$ $(\infty, 0, 1, \varrho_4)$	$G_2 = 3X(X-\varrho_4 Y)^2(X-Y)$ $G_3 = X(X-\varrho_4 Y)^3(X-Y)^2$ $\Delta = 27X^2 Y(X-Y)^3(X-\varrho_4 Y)^6$ $\mathcal{J} = \dfrac{X}{Y}$ $\mathrm{CR} = \varrho_4 \neq 0, 1, \infty$
$I_2 II\, II\, I_0^*$ $(\infty, 0, 1, \varrho_4)$	$G_2 = 12X(X-\varrho_4 Y)^2(X-Y)$ $G_3 = 4X(X-\varrho_4 Y)^3(X-Y)(2X-Y)$ $\Delta = -2^4 \cdot 3^3 X^2 Y^2(X-Y)^2(X-\varrho_4 Y)^6$ $\mathcal{J} = -4\dfrac{X(X-Y)}{Y^2}$ $\mathrm{CR} = \varrho_4 \neq 0, 1, \infty$
$I_1 I_1 II\, IV^*$ $(0, \infty, 1, \alpha^2)$	$G_2 = 3(X-Y)(X-\alpha^2 Y)^3$ $G_3 = (X-Y)(X-\alpha^2 Y)^4(X+\alpha Y)$ $\Delta = -27(\alpha+1)^2 XY(X-Y)^2(X-\alpha^2 Y)^8$ $\mathcal{J} = -\dfrac{1}{(\alpha+1)^2}\dfrac{(X-Y)(X-\alpha^2 Y)}{XY}$ $\mathrm{CR} = \dfrac{1}{\alpha^2} \neq 0, 1, \infty, \quad \alpha \neq -1, 0, 1, \infty$
$I_1 I_1 I_1 I_9$ $(1, \eta, \eta^2, \infty)$	$G_2 = 3X(9X^3-8Y^3)$ $G_3 = 27X^6-36X^3Y^3+8Y^6$ $\Delta = 2^6 \cdot 3^3 Y^9(X^3-Y^3)$ $\mathcal{J} = \dfrac{1}{64}\dfrac{X^3(9X^3-8Y^3)^3}{Y^9(X^3-Y^3)}$ $\mathrm{CR} = -\eta, \quad \eta = e^{\frac{2\pi i}{3}}$

Table 3 (continued)

$I_1I_1I_2I_8$ $(-1,1,0,\infty)$	$G_2=3(16X^4-16X^2Y^2+Y^4)$ $G_3=64X^6-96X^4Y^2+30X^2Y^4+Y^6$ $\Delta=2^2\cdot3^6X^2Y^8(X+Y)(X-Y)$ $\mathcal{J}=\frac{1}{108}\frac{(16X^4-16X^2Y^2+Y^4)^3}{X^2Y^8(X+Y)(X-Y)}$ $\mathrm{CR}=-1$
$I_1I_2I_3I_6$ $(4,-\frac{1}{2},0,\infty)$	$G_2=12(X^4-4X^3Y+2XY^3+Y^4)$ $G_3=4(2X^6-12X^5Y+12X^4Y^2+14X^3Y^3+3X^2Y^4+6XY^5+2Y^6)$ $\Delta=2^4\cdot3^6X^3Y^6(2X+Y)^2(X-4Y)$ $\mathcal{J}=\frac{4}{27}\frac{(X^4-4X^3Y+2XY^3+Y^4)^3}{X^3Y^6(2X+Y)^2(X-4Y)}$ $\mathrm{CR}=-8$
$I_1I_1I_5I_5$ $(\omega_1,\omega_2,0,\infty)$	$G_2=3(X^4-12X^3Y+14X^2Y^2+12XY^3+Y^4)$ $G_3=X^6-18X^5Y+75X^4Y^2+75X^2Y^4+18XY^5+Y^6$ $\Delta=2^6\cdot3^6X^5Y^5(X^2-11XY-Y^2)$ $\mathcal{J}=\frac{1}{2^6\cdot3^3}\frac{(X^4-12X^3Y+14X^2Y^2+12XY^3+Y^4)^3}{X^5Y^5(X^2-11XY-Y^2)}$ $\mathrm{CR}=\left(\frac{1+\sqrt{5}}{1-\sqrt{5}}\right)^5,\quad \omega_{1,2}=\left(\frac{1\pm\sqrt{5}}{2}\right)^5$
$I_2I_2I_4I_4$ $(-1,1,0,\infty)$	$G_2=12(X^4-X^2Y^2+Y^4)$ $G_3=4(2X^6-3X^4Y^2-3X^2Y^4+2Y^6)$ $\Delta=2^4\cdot3^6X^4Y^4(X+Y)^2(X-Y)^2$ $\mathcal{J}=\frac{4}{27}\frac{(X^4-X^2Y^2+Y^4)^3}{X^4Y^4(X+Y)^2(X-Y)^2}$ $\mathrm{CR}=-1$
$I_3I_3I_3I_3$ $(1,\eta,\eta^2,\infty)$	$G_2=3Y(8X^3+Y^3)$ $G_3=8X^6+20X^3Y^3-Y^6$ $\Delta=-2^6\cdot3^3X^3(X^3-Y^3)^3$ $\mathcal{J}=-\frac{1}{64}\frac{Y^3(8X^3+Y^3)^3}{X^3(X^3-Y^3)^3}$ $\mathrm{CR}=-\eta,\quad \eta=e^{\frac{2\pi i}{3}}$
$I_1I_1I_8II$ $(\omega_1,\omega_2,\infty,0)$	$G_2=12X(X^3-6X^2Y+15XY^2-12Y^3)$ $G_3=4X(2X^5-18X^4Y+72X^3Y^2-144X^2Y^3+135XY^4-27Y^5)$ $\Delta=-2^4\cdot3^6X^2Y^8(3X^2-14XY+27Y^2)$ $\mathcal{J}=-\frac{4}{27}\frac{X(X^3-6X^2Y+15XY^2-12Y^3)^3}{Y^8(3X^2-14XY+27Y^2)}$ $\mathrm{CR}=\left(\frac{1-i\sqrt{2}}{1+i\sqrt{2}}\right)^4,\quad \omega_{1,2}=-\frac{1}{3}(1\pm i\sqrt{2})^4$

Table 3 (continued)

$I_1I_2I_7II$ $(-\frac{9}{4}, -\frac{8}{9}, \infty, 0)$	$G_2 = 12X(9X^3+36X^2Y+42XY^2+14Y^3)$ $G_3 = 12X(18X^5+108X^4Y+234X^3Y^2+222X^2Y^3+87XY^4+8Y^5)$ $\Delta = -2^4\cdot 3^3X^2Y^7(9X+8Y)^2(4X+9Y)$ $\mathcal{J} = -4\dfrac{X(9X^3+36X^2Y+42XY^2+14Y^3)^3}{Y^7(9X+8Y)^2(4X+9Y)}$ $\mathrm{CR} = \frac{32}{81}$
$I_1I_4I_5II$ $(-10, 0, \infty, \frac{1}{8})$	$G_2 = 3(8X-Y)(8X^3+87X^2Y+96XY^2-64Y^3)$ $G_3 = (8X-Y)(64X^5+2^4\cdot 5\cdot 13X^4Y+5^2\cdot 157X^3Y^2+100X^2Y^3$ $+2^7\cdot 5^2XY^4-2^9Y^5)$ $\Delta = -2^3\cdot 3^{15}X^4Y^5(8X-Y)^2(X+10Y)$ $\mathcal{J} = -\dfrac{1}{2^3\cdot 3^{12}}\dfrac{(8X-Y)(8X^3+87X^2Y+96XY^2-64Y^3)^3}{X^4Y^5(X+10Y)}$ $\mathrm{CR} = \frac{1}{81}$
$I_2I_3I_5II$ $(-\frac{5}{9}, 0, \infty, 3)$	$G_2 = 3(X-3Y)(81X^3-9X^2Y-53XY^2-27Y^3)$ $G_3 = (X-3Y)(3^6X^5-3^5\cdot 5X^4Y-2\cdot 3^3\cdot 5^2X^3Y^2-350X^2Y^3$ $-3^3\cdot 5^2XY^4-243Y^5)$ $\Delta = -2^{14}\cdot 3^4X^3Y^5(X-3Y)^2(9X+5Y)^2$ $\mathcal{J} = -\dfrac{1}{2^{14}\cdot 3}\dfrac{(X-3Y)(81X^3-9X^2Y-53XY^2-27Y^3)^3}{X^3Y^5(9X+5Y)^2}$ $\mathrm{CR} = \frac{27}{32}$
$I_1I_1I_7III$ $(\omega_1, \omega_2, \infty, 0)$	$G_2 = 12X(X^3+4X^2Y+10XY^2+6Y^3)$ $G_3 = 4X^2(2X^4+12X^3Y+42X^2Y^2+70XY^3+63Y^4)$ $\Delta = 2^4\cdot 3^6X^3Y^7(4X^2+13XY+32Y^2)$ $\mathcal{J} = \dfrac{4}{27}\dfrac{(X^3+4X^2Y+10XY^2+6Y^3)^3}{Y^7(4X^2+13XY+32Y^2)}$ $\mathrm{CR} = \left(\dfrac{1-i\sqrt{7}}{1+i\sqrt{7}}\right)^7, \quad \omega_{1,2} = \dfrac{1}{4}\left(\dfrac{1\pm i\sqrt{7}}{2}\right)^7$
$I_1I_2I_6III$ $(4, 1, \infty, 0)$	$G_2 = 12X(X^3-6X^2Y+9XY^2-3Y^3)$ $G_3 = 4X^2(2X^4-18X^3Y+54X^2Y^2-63XY^3+27Y^4)$ $\Delta = 2^4\cdot 3^6X^3Y^6(X-Y)^2(X-4Y)$ $\mathcal{J} = \dfrac{4}{27}\dfrac{(X^3-6X^2Y+9XY^2-3Y^3)^3}{Y^6(X-Y)^2(X-4Y)}$ $\mathrm{CR} = \frac{1}{4}$
$I_1I_3I_5III$ $(-\frac{25}{3}, 0, \infty, \frac{1}{5})$	$G_2 = 75(5X-Y)(5X^3+45X^2Y+39XY^2-25Y^3)$ $G_3 = 25(5X-Y)^2(25X^4+340X^3Y+2\cdot 3\cdot 181X^2Y^2+100X^3Y+5^4Y^4)$ $\Delta = -2^{14}\cdot 3^6\cdot 5^4X^3Y^5(5X-Y)^3(3X+25Y)$ $\mathcal{J} = -\dfrac{25}{2^{14}\cdot 3^3}\dfrac{(5X^3+45X^2Y+39XY^2-25Y^3)^3}{X^3Y^5(3X+25Y)}$ $\mathrm{CR} = \frac{3}{128}$

Table 3 (continued)

$I_2I_3I_4III$ $(-\frac{1}{3}, 0, \infty, 1)$	$G_2=3(X-Y)(16X^3-3XY^2-Y^3)$ $G_3=(X-Y)^2(64X^4+32X^3Y+6X^2Y^2+5XY^3+Y^4)$ $\Delta=2^2\cdot 3^6X^3Y^4(X-Y)^3(3X+Y)^2$ $\mathscr{J}=\frac{1}{108}\frac{(16X^3-3XY^2-Y^3)^3}{X^3Y^4(3X+Y)^2}$ $\mathrm{CR}=\frac{3}{4}$
$I_1I_1I_6IV$ $(1, -1, \infty, 0)$	$G_2=3X^2(9X^2-8Y^2)$ $G_3=X^2(27X^4-36X^2Y^2+8Y^4)$ $\Delta=2^6\cdot 3^3X^4Y^6(X-Y)(X+Y)$ $\mathscr{J}=\frac{1}{64}\frac{X^2(9X^2-8Y^2)^3}{Y^6(X-Y)(X+Y)}$ $\mathrm{CR}=-1$
$I_1I_2I_5IV$ $(-\frac{27}{4}, -\frac{1}{2}, \infty, 0)$	$G_2=12X^2(X^2+8XY+10Y^2)$ $G_3=4X^2(2X^4+24X^3Y+78X^2Y^2+56XY^3+27Y^4)$ $\Delta=-2^4\cdot 3^6X^4Y^5(2X+Y)^2(4X+27Y)$ $\mathscr{J}=-\frac{4}{27}\frac{X^2(X^2+8XY+10Y^2)^3}{Y^5(2X+Y)^2(4X+27Y)}$ $\mathrm{CR}=\frac{2}{27}$
$I_3I_3I_2IV$ $(\infty, 0, -1, 1)$	$G_2=3(X-Y)^2(9X^2+14XY+9Y^2)$ $G_3=(X-Y)^2(27X^4+36X^3Y+2X^2Y^2+36XY^3+27Y^4)$ $\Delta=-2^{12}\cdot 3^3X^3Y^3(X-Y)^4(X+Y)^2$ $\mathscr{J}=-\frac{1}{2^{12}}\frac{(X-Y)^2(9X^2+14XY+9Y^2)^3}{X^3Y^3(X+Y)^2}$ $\mathrm{CR}=-1$
$I_1I_7II\,II$ $(0, \infty, \omega_1, \omega_2)$	$G_2=3(X^2-13XY+49Y^2)(X^2-5XY+Y^2)$ $G_3=(X^2-13XY+49Y^2)(X^4-14X^3Y+63X^2Y^2-70XY^3-7Y^4)$ $\Delta=-2^6\cdot 3^6XY^7(X^2-13XY+49Y^2)^2$ $\mathscr{J}=-\frac{1}{2^6\cdot 3^3}\frac{(X^2-13XY+49Y^2)(X^2-5XY+Y^2)^3}{XY^7}$ $\mathrm{CR}=\left(\frac{-1+3i\sqrt{3}}{-1-3i\sqrt{3}}\right)^2, \quad \omega_{1,2}=-\frac{1}{4}(-1\pm 3i\sqrt{3})^2$
$I_2I_6II\,II$ $(0, \infty, 1, -1)$	$G_2=3(X-Y)(X+Y)(9X^2-Y^2)$ $G_3=(X-Y)(X+Y)(27X^4-18X^2Y^2-Y^4)$ $\Delta=-2^6\cdot 3^3X^2Y^6(X-Y)^2(X+Y)^2$ $\mathscr{J}=-\frac{1}{64}\frac{(X-Y)(X+Y)(9X^2-Y^2)^3}{X^2Y^6}$ $\mathrm{CR}=-1$
$I_4I_4II\,II$ $(\omega_1, \omega_2, \frac{1}{2}, -4)$	$G_2=12XY(2X-Y)(X+4Y)$ $G_3=2(2X-Y)(X+4Y)(X^4+4X^3Y+8XY^3-4Y^4)$ $\Delta=-108(2X-Y)^2(X+4Y)^2(X^2+2XY-2Y^2)^4$ $\mathscr{J}=-16\frac{X^3Y^3(2X-Y)(X+4Y)}{(X^2+2XY-2Y^2)^4}$ $\mathrm{CR}=(-2+\sqrt{3})^3, \quad \omega_{1,2}=-1\pm\sqrt{3}$

Table 3 (continued)

$I_1 I_6 II\ III$ $(\omega, \infty, 1, 0)$	$G_2 = 2X(X-Y)[6X^2+6\zeta XY-(3+\zeta)Y^2]$ $G_3 = 2X^2(X-Y)[4X^3-2(1-3\zeta)X^2Y-4(2+\zeta)XY^2+(5-\zeta)Y^3]$ $\Delta = 24X^3Y^6(X-Y)^2[(9+\zeta)X+8\zeta Y]$ $\mathscr{J} = \dfrac{1}{3}\dfrac{(X-Y)\,[6X^2+6\zeta XY-(3+\zeta)Y^2]^3}{Y^6[(9+\zeta)X+8\zeta Y]}$ $\mathrm{CR} = \frac{3}{8}(3-\zeta), \quad \omega = -\frac{2}{7}(3\zeta+1),\ \zeta = \pm i\sqrt{3}$
$I_2 I_5 II\ III$ $(\frac{125}{14}, \infty, 0, \frac{27}{2})$	$G_2 = 3X(2X-27Y)(2X^2-35XY+140Y^2)$ $G_3 = X(2X-27Y)^2(2X^3-39X^2Y+222XY^2-250Y^3)$ $\Delta = 2^2\cdot 3^6X^2Y^5(2X-27Y)^3(14X-125Y)^2$ $\mathscr{J} = \dfrac{1}{108}\dfrac{X(2X^2-35XY+140Y^2)^3}{Y^5(14X-125Y)^2}$ $\mathrm{CR} = -\frac{125}{64}$
$I_3 I_4 II\ III$ $(0, \infty, -27, 1)$	$G_2 = 3(X-Y)(X+27Y)(16X^2+80XY-243Y^2)$ $G_3 = (X-Y)^2(X+27Y)(64X^3+2^5\cdot 43X^2Y+2\cdot 3^5XY^2+3^9Y^3)$ $\Delta = -2^2\cdot 3^6\cdot 7^7X^3Y^4(X-Y)^3(X+27Y)^2$ $\mathscr{J} = -\dfrac{1}{2^2\cdot 3^3\cdot 7^7}\dfrac{(X+27Y)(16X^2+80XY-243Y^2)^3}{X^3Y^4}$ $\mathrm{CR} = -27$
$I_1 I_5 II\ IV$ $(-\frac{16}{3}, \infty, 3, 0)$	$G_2 = 3X^2(X-3Y)(X+5Y)$ $G_3 = X^2(X-3Y)(X^3+6X^2Y-3XY^2-32Y^3)$ $\Delta = -2^6\cdot 3^3X^4Y^5(X-3Y)^2(3X+16Y)$ $\mathscr{J} = -\dfrac{1}{64}\dfrac{X^2(X-3Y)(X+5Y)^3}{Y^5(3X+16Y)}$ $\mathrm{CR} = \frac{25}{16}$
$I_2 I_4 II\ IV$ $(\frac{1}{9}, \infty, 1, 0)$	$G_2 = 36X^2(X-Y)(3X-Y)$ $G_3 = 4X^2(X-Y)(54X^3-54X^2Y+9XY^2-Y^3)$ $\Delta = -2^4\cdot 3^3X^4Y^4(X-Y)^2(9X-Y)^2$ $\mathscr{J} = -108\dfrac{X^2(X-Y)(3X-Y)^3}{Y^4(9X-Y)^2}$ $\mathrm{CR} = -8$
$I_1 I_5 III\ III$ $(-\frac{11}{2}, \infty, i, -i)$	$G_2 = 3(X^2+Y^2)(X^2+6XY+4Y^2)$ $G_3 = (X^2+Y^2)^2(X^2+9XY+19Y^2)$ $\Delta = -3^6Y^5(X^2+Y^2)^3(2X+11Y)$ $\mathscr{J} = -\dfrac{1}{27}\dfrac{(X^2+6XY+4Y^2)^3}{Y^5(2X+11Y)}$ $\mathrm{CR} = \left(\dfrac{1+2i}{1-2i}\right)^3$
$I_2 I_4 III\ III$ $(0, \infty, 1, -1)$	$G_2 = 3(X-Y)(X+Y)(4X^2-Y^2)$ $G_3 = (X-Y)^2(X+Y)^2(8X^2+Y^2)$ $\Delta = 3^6X^2Y^4(X-Y)^3(X+Y)^3$ $\mathscr{J} = \dfrac{1}{27}\dfrac{(4X^2-Y^2)^3}{X^2Y^4}$ $\mathrm{CR} = -1$

Table 3 (continued)

$I_3 I_3 III\, III$ $(\omega_1, \omega_2, 0, \infty)$	$G_2 = 3XY(X^2+6XY-3Y^2)$ $G_3 = 6X^2Y^2(X^2+3Y^2)$ $\Delta = 27X^3Y^3(X^2-6XY-3Y^2)^3$ $\mathscr{J} = \frac{(X^2+6XY-3Y^2)^3}{(X^2-6XY-3Y^2)^3}$ $\mathrm{CR} = -(2+\sqrt{3})^2, \quad \omega_{1,2} = 3 \pm 2\sqrt{3}$
$I_1 I_4 III\, IV$ $(-\frac{27}{5}, \infty, 1, 0)$	$G_2 = 12X^2(X-Y)(X+5Y)$ $G_3 = 4X^2(X-Y)^2(2X^2+16XY+27Y^2)$ $\Delta = 2^4 \cdot 3^6 X^4 Y^4 (5X+27Y)(X-Y)^3$ $\mathscr{J} = \frac{4}{27} \frac{X^2(X+5Y)^3}{Y^4(5X+27Y)}$ $\mathrm{CR} = \frac{32}{27}$
$I_2 I_3 III\, IV$ $(\frac{1}{5}, \infty, 1, 0)$	$G_2 = 3X^2(X-Y)(9X-5Y)$ $G_3 = X^2(X-Y)^2(27X^2-9XY+2Y^2)$ $\Delta = 108X^4Y^3(5X-Y)^2(X-Y)^3$ $\mathscr{J} = \frac{1}{4} \frac{X^2(9X-5Y)^3}{Y^3(5X-Y)^2}$ $\mathrm{CR} = -4$
$I_2 I_2 IV\, IV$ $(0, \infty, 1, -1)$	$G_2 = 3(X-Y)^2(X+Y)^2$ $G_3 = (X-Y)^2(X+Y)^2(X^2+Y^2)$ $\Delta = -108X^2Y^2(X-Y)^4(X+Y)^4$ $\mathscr{J} = -\frac{1}{4} \frac{(X-Y)^2(X+Y)^2}{X^2Y^2}$ $\mathrm{CR} = -1$
$I_2 IV\, III\, III$ $(\infty, 0, 1, -1)$	$G_2 = 3X^2(X-Y)(X+Y)$ $G_3 = X^2(X-Y)^2(X+Y)^2$ $\Delta = 27X^4Y^2(X-Y)^3(X+Y)^3$ $\mathscr{J} = \frac{X^2}{Y^2}$ $\mathrm{CR} = -1$
$I_3 III\, III\, III$ $(\infty, \eta, \eta^2, 1)$	$G_2 = 3X(X^3-Y^3)$ $G_3 = (X^3-Y^3)^2$ $\Delta = 27Y^3(X^3-Y^3)^3$ $\mathscr{J} = \frac{X^3}{Y^3}$ $\mathrm{CR} = -\eta^2, \quad \eta = e^{\frac{2\pi i}{3}}$
$I_3 II\, III\, IV$ $(\infty, -3, 1, 0)$	$G_2 = 3X^2(X-Y)(X+3Y)$ $G_3 = X^2(X-Y)^2(X+3Y)(X+2Y)$ $\Delta = 108X^4Y^3(X+3Y)^2(X-Y)^3$ $\mathscr{J} = \frac{1}{4} \frac{X^2(X+3Y)}{Y^3}$ $\mathrm{CR} = \frac{3}{4}$

Table 3 (continued)

$I_4 IV\, II\, II$ $(\infty, 0, 1, -1)$	$G_2 = 12X^2(X-Y)(X+Y)$ $G_3 = 4X^2(X-Y)(X+Y)(2X^2-Y^2)$ $\Delta = -2^4 \cdot 3^3 X^4 Y^4 (X-Y)^2 (X+Y)^2$ $\mathscr{J} = -4\dfrac{X^2(X-Y)(X+Y)}{Y^4}$ $\mathrm{CR} = -1$
$I_4 II\, III\, III$ $(\infty, -5, \omega_1, \omega_2)$	$G_2 = 3(X^2+2Y^2)(X+5Y)(X+Y)$ $G_3 = (X^2+2Y^2)^2(X+5Y)(X+4Y)$ $\Delta = -3^6 Y^4 (X+5Y)^2 (X^2+2Y^2)^3$ $\mathscr{J} = -\dfrac{1}{27}\dfrac{(X+5Y)(X+Y)^3}{Y^4}$ $\mathrm{CR} = \left(\dfrac{1+i\sqrt{2}}{1-i\sqrt{2}}\right)^3, \quad \omega_{1,2} = \pm i\sqrt{2}$
$I_5 III\, II\, II$ $(\infty, 0, \omega_1, \omega_2)$	$G_2 = 3X(X^2+11XY+64Y^2)(X+3Y)$ $G_3 = X^2(X^2+11XY+64Y^2)(X^2+10XY+45Y^2)$ $\Delta = 2^6 \cdot 3^6 X^3 Y^5 (X^2+11XY+64Y^2)^2$ $\mathscr{J} = \dfrac{1}{2^6 \cdot 3^3}\dfrac{(X^2+11XY+64Y^2)(X+3Y)^3}{Y^5}$ $\mathrm{CR} = \left(\dfrac{1-i\sqrt{15}}{1+i\sqrt{15}}\right)^3, \quad \omega_{1,2} = \frac{1}{8}(1 \pm i\sqrt{15})^3$
$I_6 II\, II\, II$ $(\infty, \eta, \eta^2, 1)$	$G_2 = 12X(X^3-Y^3)$ $G_3 = 4(X^3-Y^3)(2X^3-Y^3)$ $\Delta = -2^4 \cdot 3^3 Y^6 (X^3-Y^3)^2$ $\mathscr{J} = -4\dfrac{X^3(X^3-Y^3)}{Y^6}$ $\mathrm{CR} = -\eta^2, \quad \eta = e^{\frac{2\pi i}{3}}$

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