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When is the derivative of an eta quotient another eta quotient? ☆

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A B S T R A C T

In this paper we use techniques from the theory of modular forms to determine all eta quotients whose derivative is also an eta quotient of levels up to 36. In addition, we present an algorithm that determines all eta quotients in $M_{2k}(\Gamma_0(N))$. We also discuss some applications of these results. In particular, we evaluate a number of integrals in terms of algebraic constants.

1. Introduction

Let $\mathbb{H}$ denote the complex upper half plane and throughout the paper we let $\tau \in \mathbb{H}$ and $q = e^{2\pi i\tau}$. Let $\Gamma_0(N)$ denote the congruence subgroup of level N defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1,\ c \equiv 0 \pmod N \right\}.$$

A set of representatives from each equivalence class of cusps for $\Gamma_0(N)$ is given by [13, Proposition 2.6]

$$R(\Gamma_0(N)) = \bigcup_{c|N} \left\{ \frac{a}{c} : 1 \le a \le N,\ \gcd(a,c) = 1 \text{ and } a \equiv a' \pmod{\gcd(c, N/c)} \text{ iff } a = a' \right\}. \tag{1.1}$$

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We use $M_k(\Gamma_0(N))$ to denote the space of all modular forms of weight k for $\Gamma_0(N)$.

The Dedekind eta function is defined by the infinite product

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi in\tau}) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$

and an eta quotient of level N is defined to be of the form

$$f(\tau) = \prod_{0<\delta|N}\eta(\delta\tau)^{r_\delta}$$

where δ runs over all positive divisors of the integer N and the exponents r_δ are integers. The weight attached to this eta quotient is $k = \frac{1}{2}\sum_{0<\delta|N} r_\delta$. We use

$$\eta_N[r_1,\dots,r_\delta,\dots,r_N](\tau) = \prod_{0<\delta|N}\eta(\delta\tau)^{r_\delta} \tag{1.2}$$

as a shorthand notation for an eta quotient of level N. Many interesting examples of modular forms or generating functions can be expressed as eta quotients. For example,

$$\eta(\tau)^{24} = \eta_1[24](\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n$$

is a modular form of weight 12 and level 1. The function $\tau(n)$ is called the Ramanujan tau function, in honor of Ramanujan who conjectured and proved [18] many properties satisfied by these coefficients. Deligne's proof of Ramanujan's conjectured bounds was an important breakthrough in the field. (See [16, p. 122] for more details.) A second example is

$$\eta(\tau)\eta(3\tau)\eta(5\tau)\eta(15\tau) = \eta_{15}[1,1,1,1](\tau),$$

a weight 2 modular form for $\Gamma_0(15)$ that is associated with the elliptic curve $y^2+xy+y = x^3+x^2$ [17]. Our third and most important example for our purpose is the generating function for the number of representations of an integer as a sum of four squares.

$$\begin{aligned}\sum_{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4} q^{x_1^2+x_2^2+x_3^2+x_4^2} &= \frac{\eta(2\tau)^{20}}{\eta(\tau)^8\eta(4\tau)^8}\\ &= \eta_4[-8,20,-8](\tau).\end{aligned}$$

Jacobi [14] showed that the number of such representations of n is equal to eight times the sum of all the positive divisors of n which are not divisible by 4. In modern notation,

$$\begin{aligned}\sum_{(x_1,x_2,x_3,x_4)\in\mathbb{Z}^4} q^{x_1^2+x_2^2+x_3^2+x_4^2} &= 1+8\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} - 8\sum_{n=1}^{\infty}\frac{4nq^{4n}}{1-q^{4n}}\\ &= 8\left(E_2(\tau)-4E_2(4\tau)\right),\end{aligned} \tag{1.3}$$

where

$$E_2(\tau) = -\frac{1}{24} + \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}$$

is the Eisenstein series of weight 2. (Remark: we follow [21] by normalizing the Eisenstein series such that the coefficient of q is 1.)

It is straightforward to show that

$$q\frac{d}{dq}\log(\eta(\tau)) = \frac{1}{24} - \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} = -E_2(\tau).$$

Thus it is possible to write Jacobi's result above as

$$q\frac{d}{dq}\log\left(\frac{\eta(4\tau)^8}{\eta(\tau)^8}\right) = \frac{\eta(2\tau)^{20}}{\eta(\tau)^8\eta(4\tau)^8}. \tag{1.4}$$

In other words, Jacobi's identity can be expressed as an identity where the logarithmic derivative of an eta quotient equals another eta quotient. Another identity of the same type is

$$q\frac{d}{dq}\log\left(\frac{\eta(9\tau)^3}{\eta(\tau)^3}\right) = \frac{\eta(3\tau)^{10}}{\eta(\tau)^3\eta(9\tau)^3}. \tag{1.5}$$

This identity was discovered through symbolic computation by Borwein and Garvan [5], who subsequently deduced a proof with results from Ramanujan's notebooks, and utilized it to produce a ninth order iteration to $1/\pi$. Identities (1.4) and (1.5) involve eta quotients of levels 4 and 9 respectively. Other identities of levels 6, 8 and 12 were also studied by Fine [11, pp. 78-91]. We list below two of Fine's identities:

$$q\frac{d}{dq}\log(\eta_8[-4,2,-2,4](\tau)) = \eta_8[-4,6,6,-4](\tau), \tag{1.6}$$

$$q\frac{d}{dq}\log(\eta_{12}[-4,4,4,-4,-4,4](\tau)) = \eta_{12}[-4,10,-4,-4,10,-4](\tau). \tag{1.7}$$

An interesting question arises naturally. Does there exist infinitely many identities of these type? In this paper, we use the theory of modular forms to attempt to find a complete list of eta quotients whose logarithmic derivatives are constant multiples of eta quotients. In our search, we found a total of 203 *distinct* identities from level 4 to 36. (We shall explain in the next section what we mean by distinct.) For each of the levels our list is complete. No other distinct identities were found for other levels $N < 100$. We conjecture that our list is complete. Table 1 shows the number of distinct identities for each level.

Table 1
Number of distinct identities for each level.

Level	4	6	8	9	12	16	18	20	24	36
No. of Identities	3	10	4	1	100	4	12	12	32	25

We shall explain in the next section our approach in finding the identities and how these identities are classified. In Section 3, we use elementary arguments to determine the sum of orders of an eta quotient. This will help us determine all eta quotients in $M_{2k}(\Gamma_0(N))$. Section 4 is devoted to three algorithms based on the results of the two previous sections. The table listing all the 203 distinct identities for $N \leq 36$ is given in Section 5. In the final section, we discuss applications of these identities and other related work.

2. Preliminaries

We first explain what we mean by distinct identities in our tabulations in Table 1. Suppose $f(\tau)$ is an eta quotient that is mapped to a constant multiple of another eta quotient under the differential operator $\frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}$. Then it is clear that any constant multiple or any integral power of $f(\tau)$ will also be mapped to some constant multiple of another eta quotient under the differential operator. For this reason, we prefer the logarithmic derivative since

$$q\frac{d}{dq}\log(f(\tau)^j) = \frac{j}{2\pi i}\cdot\frac{\frac{df(\tau)}{d\tau}}{f(\tau)}.$$

Thus our identities with the logarithmic differential operator are unique up to constant multiples. For brevity, we shall henceforth omit the phrase "a constant multiple of" when referring to eta quotients.

Secondly, any eta quotient of level N is automatically an eta quotient of level mN for any positive integer m. Hence we only consider identities which are *primitive* in the following sense. If $f(\tau)$ and $g(\tau)$ are eta quotients, and α a constant such that

$$q\frac{d}{dq}\log(f(\tau)) = \alpha g(\tau),$$

then except for constant multiples, there does not exist any eta quotients $F(\tau)$, $G(\tau)$ and integer $m > 1$ such that

$$q\frac{d}{dq}\log(F(\tau)) = G(\tau), \quad f(\tau) = F(m\tau) \quad \text{and} \quad g(\tau) = G(m\tau).$$

There is a third sense in which identities can be equivalent. As discussed in Fine [11, p. 86], by applying Atkin-Lehner involutions, one may obtain (2.2) below from (2.1).

$$q\frac{d}{dq}\log\left(\eta_6[3,-3,-9,9](\tau)\right) = \eta_6[3,3,-1,-1](\tau), \tag{2.1}$$

$$q\frac{d}{dq}\log\left(\eta_6[-3,3,1,-1](\tau)\right) = 3\eta_6[-1,-1,3,3](\tau). \tag{2.2}$$

We have chosen to keep these cases as distinct in Table 2 to assist readers who may be searching for specific identities in the list. Identities (2.1) and (2.2) are labelled respectively as $f_{6,4a}$ and $f_{6,4b}$ in Table 2, as these form the fourth set of identities from level 6. The suffix means that these are equivalent up to some involution. Jacobi's identity (1.4) is labelled $f_{4,2}$ and the absence of suffixes means that it is invariant under Atkin-Lehner involutions.

Finally, in addition to Atkin-Lehner involutions, it is also possible to obtain a new identity through the transformation $q \mapsto -q$ because of the elementary property

$$\prod_{n=1}^{\infty}(1-(-q)^n) = \prod_{n=1}^{\infty}\frac{(1-q^{2n})^3}{(1-q^n)(1-q^{4n})}. \tag{2.3}$$

Not only is every eta-quotient mapped to another eta-quotient, the same holds for Eisenstein series.

$$\sum_{n=1}^{\infty}\frac{n(-q)^n}{1-(-q)^n} = -\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n} + \sum_{n=1}^{\infty}\frac{6nq^{2n}}{1-q^{2n}} - \sum_{n=1}^{\infty}\frac{4nq^{4n}}{1-q^{4n}}. \tag{2.4}$$

For example, applying the transformation $q \mapsto -q$ to Jacobi's identity (1.4) or $f_{4,2}$ yields

$$q\frac{d}{dq}\log\left(\eta_4[8,-24,16](\tau)\right) = \eta_4[8,-4,0](\tau), \tag{2.5}$$

which is labelled as $f_{4,1a}$ in Table 2. We again chose to retain both cases and record these additional relations in Table 3. We use the notation

$$f_{4,1a} \sim f^-_{4,2}$$

to indicate the relation between Jacobi's identity (1.4) and identity (2.5). Clearly, $f_{4,2} \sim f^-_{4,1a}$ also holds but only one out of the two possible relations is recorded in Table 3. All together, 200 out of the 203 distinct identities were paired up to give 100 relations. The remaining three identities were invariant under the transformation. For example, we have

$$\eta_4[-2,3,-1](\tau)\,\big|_{q\mapsto -q} = \eta_4[2,-3,1](\tau).$$

We recorded the above relation as $f_{4,1b} \sim f^-_{4,1b}$ and the other two are $f_{12,18b} \sim f^-_{12,18b}$ and $f_{12,18d} \sim f^-_{12,18d}$.

Suppose now that $f(\tau)$ is an eta quotient of level N and weight k. We have

$$f(\tau) = \prod_{0<\delta|N} \eta^{r_\delta}(\delta\tau) \quad \text{where} \quad \sum_{0<\delta|N} r_\delta = 2k.$$

Applying logarithmic differentiation,

$$\begin{aligned} q\frac{d}{dq}\log f(\tau) &= -\sum_{0<\delta|N} r_\delta \delta E_2(\delta\tau) \\ &= -2kE_2(\tau) + \sum_{0<\delta|N} r_\delta \left(E_2(\tau) - \delta E_2(\delta\tau)\right) \\ &= -2kE_2(\tau) + \sum_{1<\delta|N} r_\delta L_\delta(\tau), \end{aligned} \tag{2.6}$$

where we used the notation

$$L_\delta(\tau) = E_2(\tau) - \delta E_2(\delta\tau), \tag{2.7}$$

which is in $M_2(\Gamma_0(N))$ whenever δ is a positive divisor of N [21, Theorem 5.8].

We now prove an important lemma which will help us classify the eta quotients whose logarithmic derivatives are eta quotients.

Lemma 2.1. *Let $f(\tau) = \prod_{0<\delta|N} \eta^{r_\delta}(\delta\tau)$ be an eta quotient of level N and weight k. Then $\frac{1}{2\pi i}\frac{d}{d\tau}f(\tau)$ is an eta quotient only if $k = 0$. Furthermore, if $k = 0$, then $\frac{1}{2\pi i}\frac{d}{d\tau}f(\tau)$ is an eta quotient if and only if $\sum_{1<\delta|N} r_\delta L_\delta(\tau)$ is an eta quotient.*

Proof. Let $f(\tau)$ be as given. Assume that $g(\tau) = \frac{1}{2\pi i}\frac{d}{d\tau}f(\tau)$ is an eta quotient of level N_2 and weight k_2. Then $\dfrac{g(\tau)}{f(\tau)}$ is an eta quotient of level $\mathrm{lcm}(N, N_2)$. It is well known that for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, $M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \Gamma_0(N_2)$ and $M_3 = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \in \Gamma_0(\mathrm{lcm}(N, N_2))$ we have

$$\begin{aligned} f(M\tau) &= \chi_{M,f}(c\tau+d)^k f(\tau), \\ g(M_2\tau) &= \chi_{M_2,g}(c_2\tau+d_2)^{k_2} g(\tau), \\ \frac{g(M_3\tau)}{f(M_3\tau)} &= \chi_{M_3,g/f}(c_3\tau+d_3)^{k_2-k}\frac{g(\tau)}{f(\tau)}, \end{aligned} \tag{2.8}$$

where $\chi_{M,f}$, $\chi_{M_2,g}$ and $\chi_{M_3,g/f}$ are 24th roots of unity. By (2.6) we have

$$\frac{g(\tau)}{f(\tau)} = -2kE_2(\tau) + \sum_{1<\delta|N} r_\delta L_\delta(\tau). \tag{2.9}$$

Then since $M_3 \in \Gamma_0(\mathrm{lcm}(N, N_2))$, $\Gamma_0(N)$ and $\Gamma_0(N_2)$ we have

$$\begin{aligned} \frac{g(M_3\tau)}{f(M_3\tau)} &= -2kE_2(M_3\tau) + \sum_{1<\delta|N} r_\delta L_\delta(M_3\tau) \\ &= -2k\left((c_3\tau+d_3)^2E_2(\tau) - \frac{6ic_3(c_3\tau+d_3)}{\pi}\right) + (c_3\tau+d_3)^2\sum_{1<\delta|N} r_\delta L_\delta(\tau) \\ &= (c_3\tau+d_3)^2\left(-2kE_2(\tau) + \sum_{1<\delta|N} r_\delta L_\delta(\tau)\right) + 2k\frac{6ic_3(c_3\tau+d_3)}{\pi} \\ &= (c_3\tau+d_3)^2\frac{g(\tau)}{f(\tau)} + 2k\frac{6ic_3(c_3\tau+d_3)}{\pi}. \end{aligned}$$

This contradicts with (2.8), unless $k = 0$. $\square$

We combine (2.6) with Lemma 2.1 to summarize the above discussion as the following classification theorem.

Theorem 2.1. *Let $f(\tau) = \prod_{0<\delta|N} \eta^{r_\delta}(\delta\tau)$ be an eta quotient of level N and weight k. Then $\frac{1}{2\pi i}\frac{d}{d\tau}f(\tau)$ is an eta quotient if and only if $k = 0$ and $\sum_{1<\delta|N} r_\delta L_\delta(\tau)$ is an eta quotient.*

Thus the problem of determining eta quotients whose derivatives are also eta quotients is reduced to that of finding eta quotients in $M_2(\Gamma_0(N))$ which are also in the space spanned by $L_\delta(\tau)$. We shall denote this space as

$$E_N = \left\{ \sum_{1<\delta|N} u_\delta L_\delta(\tau) \,\middle|\, u_\delta \in \mathbb{Q} \right\}. \tag{2.10}$$

Corollary 2.1. *Let $f(\tau)$ be an eta quotient in E_N, i.e.*

$$f(\tau) = \prod_{0<\delta|N} \eta^{s_\delta}(\delta\tau) = \sum_{1<\delta|N} u_\delta L_\delta(\tau)$$

where $u_\delta \in \mathbb{Q}$. Denote $u_1 = -\sum_{1<\delta|N} u_\delta$ and let $c \in \mathbb{Z}$ be such that $cu_\delta \in \mathbb{Z}$ for all $\delta \mid N$. Then we have

$$\frac{1}{2\pi i}\frac{d}{d\tau}\prod_{0<\delta|N}\eta^{cu_\delta}(\delta\tau)=c\prod_{0<\delta|N}\eta^{s_\delta+cu_\delta}(\delta\tau).$$

3. Eta quotients in $M_{2k}(\Gamma_0(N))$

In this section we use elementary methods to deduce a formula for the sum of the order of vanishing of eta quotients at cusps of $\Gamma_0(N)$. If the eta quotient is holomorphic, the formula will give an upper bound for vanishing at each cusp. This upper bound will be used in the next section where we describe an algorithm to determine all the eta quotients in $M_{2k}(\Gamma_0(N))$. This section is a refined version of Section 4.2 of Aygin's unpublished PhD thesis [3]. We shall use $\phi(n)$ to denote the Euler totient function and all sums in this section are taken over positive divisors of N. We begin with several lemmas.

Lemma 3.1. *Let $N\in\mathbb{Z}^+$ and p be a prime such that $p\parallel N$. If $\lambda\mid\frac{N}{p}$, then for all $c\mid\frac{N}{p}$, we have*

$$\frac{\phi(\gcd(c,\frac{N}{c}))\cdot\gcd(c,\lambda)}{\gcd(c,\frac{N}{c})\cdot\operatorname{lcm}(c,\lambda)}=\frac{\phi(\gcd(cp,\frac{N}{cp}))\cdot\gcd(cp,p\lambda)}{\gcd(cp,\frac{N}{cp})\cdot\operatorname{lcm}(cp,p\lambda)}\tag{3.1}$$

and

$$\frac{\phi(\gcd(cp,\frac{N}{cp}))\cdot\gcd(cp,\lambda)}{\gcd(cp,\frac{N}{cp})\cdot\operatorname{lcm}(cp,\lambda)}=\frac{\phi(\gcd(c,\frac{N}{c}))\cdot\gcd(c,p\lambda)}{\gcd(c,\frac{N}{c})\cdot\operatorname{lcm}(c,p\lambda)}.\tag{3.2}$$

Proof. Since $c\mid\frac{N}{p}$ and $p\parallel N$, we have $\gcd(c,p)=\gcd(p,\frac{N}{cp})=1$. We can then conclude

$$\gcd\left(cp,\frac{N}{cp}\right)=\gcd\left(c,\frac{N}{cp}\right)\cdot\gcd\left(p,\frac{N}{cp}\right)=\gcd\left(c,\frac{N}{cp}\right)=\gcd\left(c,\frac{N}{c}\right).$$

Then (3.1) follows from $\gcd(cp,p\lambda)=p\cdot\gcd(c,\lambda)$ and $\operatorname{lcm}(cp,p\lambda)=p\cdot\operatorname{lcm}(c,\lambda)$. Equation (3.2) can be proved similarly by utilizing $\gcd(c,p\lambda)=\gcd(c,\lambda)$ and $\operatorname{lcm}(c,p\lambda)=p\cdot\operatorname{lcm}(c,\lambda)$ whenever $\gcd(c,p)=1$. □

Lemma 3.2. *Let $N\in\mathbb{Z}^+$ and p be a prime such that $p^s\parallel N$ for some $s>1$. If $\lambda\mid\frac{N}{p}$, then for all $c\mid\frac{N}{p^s}$, we have*

$$\begin{aligned}\frac{\phi(\gcd(c,\frac{N}{c}))\cdot\gcd(c,\lambda)}{\gcd(c,\frac{N}{c})\cdot\operatorname{lcm}(c,\lambda)}=&\frac{\phi(\gcd(c,\frac{N}{c}))\cdot\gcd(c,p\lambda)}{\gcd(c,\frac{N}{c})\cdot\operatorname{lcm}(c,p\lambda)}\\&+\frac{\phi(\gcd(cp,\frac{N}{cp}))\cdot\gcd(cp,p\lambda)}{\gcd(cp,\frac{N}{cp})\cdot\operatorname{lcm}(cp,p\lambda)}\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}\frac{\phi(\gcd(cp^s,\frac{N}{cp^s}))\cdot\gcd(cp^s,p\lambda)}{\gcd(cp^s,\frac{N}{cp^s})\cdot\operatorname{lcm}(cp^s,p\lambda)}=&\frac{\phi(\gcd(cp^{s-1},\frac{N}{cp^{s-1}}))\cdot\gcd(cp^{s-1},\lambda)}{\gcd(cp^{s-1},\frac{N}{cp^{s-1}})\cdot\operatorname{lcm}(cp^{s-1},\lambda)}\\&+\frac{\phi(\gcd(cp^s,\frac{N}{cp^s}))\cdot\gcd(cp^s,\lambda)}{\gcd(cp^s,\frac{N}{cp^s})\cdot\operatorname{lcm}(cp^s,\lambda)}.\end{aligned}\tag{3.4}$$

Furthermore, for $1\le i<s-1$, we have

$$\frac{\phi(\gcd(cp^i,\frac{N}{cp^i}))\cdot\gcd(cp^i,\lambda)}{\gcd(cp^i,\frac{N}{cp^i})\cdot\operatorname{lcm}(cp^i,\lambda)}=\frac{\phi(\gcd(cp^{i+1},\frac{N}{cp^{i+1}}))\cdot\gcd(cp^{i+1},p\lambda)}{\gcd(cp^{i+1},\frac{N}{cp^{i+1}})\cdot\operatorname{lcm}(cp^{i+1},p\lambda)}.\tag{3.5}$$

Proof. Since $c \mid \frac{N}{p^s}$ with $s > 1$ and $p^s \parallel N$, we have $\gcd(c, p) = 1$ and $\gcd(p, \frac{N}{cp}) = p$. It then follows that

$$\gcd\left(cp, \frac{N}{cp}\right) = \gcd\left(c, \frac{N}{cp}\right) \cdot \gcd\left(p, \frac{N}{cp}\right) = p \cdot \gcd\left(c, \frac{N}{cp}\right) = p \cdot \gcd\left(c, \frac{N}{c}\right). \tag{3.6}$$

Also since $\gcd\left(p, \gcd\left(c, \frac{N}{c}\right)\right) = 1$, we have

$$\phi\left(\gcd\left(cp, \frac{N}{cp}\right)\right) = \phi\left(p \cdot \gcd\left(c, \frac{N}{c}\right)\right) = (p-1) \cdot \phi\left(\gcd\left(c, \frac{N}{c}\right)\right). \tag{3.7}$$

Hence (3.3) follows from (3.6) and (3.7).

The proofs of the other two identities are similar and hinge on the fact that if $1 \le i \le s$, we have

$$\gcd\left(cp^i, \frac{N}{cp^i}\right) = \begin{cases} p^i \gcd(c, \frac{N}{c}) & \text{if } i < \frac{s}{2}; \\ p^{s-i} \gcd(c, \frac{N}{c}) & \text{if } i \ge \frac{s}{2}. \end{cases} \tag{3.8}$$

Furthermore, for all $t \ge 1$,

$$\frac{\phi(p^{t+1})}{p^{t+1}} = \frac{p-1}{p} = \frac{\phi(p^t)}{p^t}.$$

Combining the above observation with (3.8) proves (3.5).

When $i = s - 1$,

$$\gcd\left(cp^{s-1}, \frac{N}{cp^{s-1}}\right) = p \cdot \gcd\left(c, \frac{N}{c}\right) = p \cdot \gcd\left(cp^s, \frac{N}{cp^s}\right). \tag{3.9}$$

Since the largest power of p that divides λ is at most $s - 1$, we have

$$\gcd(cp^s, \lambda) = \frac{1}{p} \gcd(cp^s, p\lambda),$$
$$\operatorname{lcm}(cp^s, \lambda) = \operatorname{lcm}(cp^s, p\lambda).$$

The right side of (3.4) is then

$$\begin{aligned}
&\frac{\phi(\gcd(cp^{s-1}, \frac{N}{cp^{s-1}})) \cdot \gcd(cp^{s-1}, \lambda)}{\gcd(cp^{s-1}, \frac{N}{cp^{s-1}}) \cdot \operatorname{lcm}(cp^{s-1}, \lambda)} + \frac{\phi(\gcd(cp^s, \frac{N}{cp^s})) \cdot \gcd(cp^s, \lambda)}{\gcd(cp^s, \frac{N}{cp^s}) \cdot \operatorname{lcm}(cp^s, \lambda)} \\
&= \frac{\phi(p \cdot \gcd(c, \frac{N}{c})) \cdot \frac{1}{p} \gcd(cp^s, p\lambda)}{p \cdot \gcd(c, \frac{N}{c}) \cdot \frac{1}{p} \operatorname{lcm}(cp^s, p\lambda)} + \frac{\phi(\gcd(c, \frac{N}{c})) \cdot \frac{1}{p} \gcd(cp^s, p\lambda)}{\gcd(c, \frac{N}{c}) \cdot \operatorname{lcm}(cp^s, p\lambda)} \\
&= \frac{(p-1)\phi(\gcd(c, \frac{N}{c})) \cdot \gcd(cp^s, p\lambda)}{p \cdot \gcd(c, \frac{N}{c}) \cdot \operatorname{lcm}(cp^s, p\lambda)} + \frac{\phi(\gcd(c, \frac{N}{c})) \cdot \gcd(cp^s, p\lambda)}{p \cdot \gcd(c, \frac{N}{c}) \cdot \operatorname{lcm}(cp^s, p\lambda)} \\
&= \frac{\phi(\gcd(c, \frac{N}{c})) \cdot \gcd(cp^s, p\lambda)}{\gcd(c, \frac{N}{c}) \cdot \operatorname{lcm}(cp^s, p\lambda)} \\
&= \frac{\phi(\gcd(cp^s, \frac{N}{cp^s})) \cdot \gcd(cp^s, p\lambda)}{\gcd(cp^s, \frac{N}{cp^s}) \cdot \operatorname{lcm}(cp^s, p\lambda)}. \quad \square
\end{aligned}$$

From Lemmas 3.1 and 3.2, we deduce the following.

Lemma 3.3. *Let $N \in \mathbb{Z}^+$ and p be a prime divisor of N. If $\lambda \mid \frac{N}{p}$, we have*

$$\sum_{c|N} \frac{\phi(\gcd(c, \frac{N}{c})) \cdot \gcd(c, \lambda)}{\gcd(c, \frac{N}{c}) \cdot \operatorname{lcm}(c, \lambda)} = \sum_{c|N} \frac{\phi(\gcd(c, \frac{N}{c})) \cdot \gcd(c, p\lambda)}{\gcd(c, \frac{N}{c}) \cdot \operatorname{lcm}(c, p\lambda)}. \tag{3.10}$$

Lemma 3.4. *Let $N \in \mathbb{Z}^+$. If $\delta \mid N$, we have*

$$\sum_{c|N} \frac{\phi(\gcd(c, N/c))}{\gcd(c, N/c) \cdot c} = \sum_{c|N} \frac{\phi(\gcd(c, N/c)) \cdot \gcd(c, \delta)}{\gcd(c, N/c) \cdot \operatorname{lcm}(c, \delta)}. \tag{3.11}$$

Proof. Let $\delta = p_1^{i_1} \cdots p_r^{i_r}$ be a divisor of N. Then taking $\lambda = \delta/p_1^j$ in Lemma 3.3 for j from 1 to i_1, we have

$$\begin{aligned}
\sum_{c|N} \frac{\phi(\gcd(c, N/c)) \cdot \gcd(c, \delta)}{\gcd(c, N/c) \cdot \operatorname{lcm}(c, \delta)} &= \sum_{c|N} \frac{\phi(\gcd(c, N/c)) \cdot \gcd(c, \delta/p_1)}{\gcd(c, N/c) \cdot \operatorname{lcm}(c, \delta/p_1)} \\
&= \sum_{c|N} \frac{\phi(\gcd(c, N/c)) \cdot \gcd(c, \delta/p_1^2)}{\gcd(c, N/c) \cdot \operatorname{lcm}(c, \delta/p_1^2)} \\
&\ \ \vdots \\
&= \sum_{c|N} \frac{\phi(\gcd(c, N/c)) \cdot \gcd(c, \delta/p_1^{i_1})}{\gcd(c, N/c) \cdot \operatorname{lcm}(c, \delta/p_1^{i_1})}.
\end{aligned}$$

Inductively, we have

$$\begin{aligned}
\sum_{c|N} \frac{\phi(\gcd(c, N/c)) \cdot \gcd(c, \delta)}{\gcd(c, N/c) \cdot \operatorname{lcm}(c, \delta)} &= \sum_{c|N} \frac{\phi(\gcd(c, N/c)) \cdot \gcd(c, \delta/p_1^{i_1})}{\gcd(c, N/c) \cdot \operatorname{lcm}(c, \delta/p_1^{i_1})} \\
&= \sum_{c|N} \frac{\phi(\gcd(c, N/c)) \cdot \gcd(c, \delta/(p_1^{i_1}p_2^{i_2}))}{\gcd(c, N/c) \cdot \operatorname{lcm}(c, \delta/(p_1^{i_1}p_2^{i_2}))} \\
&\ \ \vdots \\
&= \sum_{c|N} \frac{\phi(\gcd(c, N/c)) \cdot \gcd(c, \delta/\delta)}{\gcd(c, N/c) \cdot \operatorname{lcm}(c, \delta/\delta)}. \qquad \square
\end{aligned}$$

We are now ready to prove an important theorem which allows us to determine all eta quotients in $M_{2k}(\Gamma_0(N))$.

Theorem 3.1. *Let $N, k \in \mathbb{Z}^+$. Suppose $f(\tau) = \prod_{0<\delta|N} \eta(\delta\tau)^{r_\delta}$ is an eta quotient of level N and weight $2k$. Let $v_r(f(\tau))$ be the order of vanishing of $f(\tau)$ at the cusp $r \in R(\Gamma_0(N))$. Then we have*

$$\sum_{r \in R(\Gamma_0(N))} v_r(f(\tau)) = \frac{4kN}{24} \sum_{c|N} \frac{\phi(\gcd(c, N/c))}{\gcd(c^2, N)}. \tag{3.12}$$

Furthermore, if $f(\tau) \in M_{2k}(\Gamma_0(N))$, we have

$$0 \le v_r(f(\tau)) \le \frac{4kN}{24} \sum_{c|N} \frac{\phi(\gcd(c, N/c))}{\gcd(c^2, N)}. \tag{3.13}$$

Proof. Note that, given $r = \frac{a}{c}$ with $\gcd(a, c) = 1$,

$$v_r(f(z)) = \frac{N}{24\gcd(c^2, N)} \sum_{\delta|N} \frac{\gcd(\delta, c)^2 \cdot r_\delta}{\delta},$$

depends only on the denominator of r. Using Lemma 3.3 we obtain

$$\begin{aligned}
\sum_{r\in R(\Gamma_0(N))} v_r(f(z)) &= \sum_{r\in R(\Gamma_0(N))} \frac{N}{24\gcd(c^2, N)} \sum_{\delta|N} \frac{\gcd(\delta, c)^2 \cdot r_\delta}{\delta} \\
&= \sum_{1\le\delta|N} \frac{N}{24} \sum_{r\in R(\Gamma_0(N))} \frac{\gcd(\delta, c)^2 \cdot r_\delta}{\gcd(c^2, N)\cdot\delta} \\
&= \frac{N}{24} \sum_{\delta|N} r_\delta \sum_{c|N} \frac{\phi(\gcd(c, N/c)) \cdot \gcd(\delta, c)^2}{\gcd(c^2, N)\cdot\delta} \\
&= \frac{N}{24} \sum_{\delta|N} r_\delta \sum_{c|N} \frac{\phi(\gcd(c, N/c))}{\gcd(c^2, N)} \\
&= \frac{N}{24} \sum_{c|N} \frac{\phi(\gcd(c, N/c))}{\gcd(c^2, N)} \sum_{\delta|N} r_\delta \\
&= \frac{4kN}{24} \sum_{c|N} \frac{\phi(\gcd(c, N/c))}{\gcd(c^2, N)}.
\end{aligned}$$

When $f(\tau) \in M_{2k}(\Gamma_0(N))$, then we have $v_r(f(z)) \ge 0$ for all $r \in R(\Gamma(N))$. Thus the inequality (3.13) follows. $\square$

4. Algorithms

In this section we present three algorithms. Theorem 3.1 provides an upper bound for the orders of vanishing of eta quotients at all the cusps. The first algorithm uses this upper bound to determine all eta quotients in $M_{2k}(\Gamma_0(N))$. We define the sum of the t-th powers of the divisors function,

$$\sigma_t(n) = \sum_{0<d|n} d^t. \tag{4.1}$$

Algorithm 4.1. *This algorithm determines all eta quotients in* $M_{2k}(\Gamma_0(N))$.

Input: $N, k \in \mathbb{Z}^+$.

(1) Compute: $L(N) = \frac{4kN}{24} \sum_{c|N} \frac{\phi(\gcd(c, N/c))}{\gcd(c^2, N)}$.

(2) Set up the following system of $\sigma_0(N)$ linear equations for $r = \frac{a}{c}$, $c \mid N$.

$$v_r = \frac{N}{24\gcd(c^2, N)} \sum_{\delta|N} \frac{\gcd(\delta, c)^2 \cdot r_\delta}{\delta}.$$

(3) For all $v_r \in \mathbb{Z}$ and $0 \leq v_r \leq L(N)$ such that

$$\sum_{r \in R(N)} v_r = L(N),$$

solve the above system to obtain r_δ. If $r_\delta \in \mathbb{Z}$ for all $\delta \mid N$, $\sum_{\delta/p \in \mathbb{Z}^+} \nu_p(\delta) r_\delta$ is even for all prime $p \mid N$, and $\sum r_\delta = 4k$, then save $[r_1, \ldots, r_\delta, \ldots, r_N]$ in an array $\mathbf{A}$. Here $\nu_p(\delta)$ denotes the power of p that divides δ.

Output: $\{\eta_N[r_1, \ldots, r_\delta, \ldots, r_N](z) : [r_1, \ldots, r_\delta, \ldots, r_N] \in \mathbf{A}\}$.

We can modify Algorithm 4.1 to find all eta quotients in $M_{2k}(\Gamma_0(N), \chi)$ where χ is a quadratic character. To do this one needs to let v_r take half integer values, and put appropriate parity conditions for $\sum_{\delta/p \in \mathbb{Z}^+} \nu_p(\delta) r_\delta$. For our purposes in this paper, we only require the case when $k = 1$. The second algorithm determines which of the eta quotients in $M_2(\Gamma_0(N))$ are in the space E_N. We do this by attempting to express the first m Fourier coefficients of each eta quotient found by Algorithm 4.1 in terms of the Fourier coefficients of the basis elements $L_\delta(\tau)$ of E_N. Here m is the Sturm bound [22].

Algorithm 4.2. *This algorithm determines the eta quotients in E_N.*

Input: $N \in \mathbb{Z}^+$ and array $\mathbf{A}$ from Algorithm 4.1 with inputs N and $k = 1$.

(1) Put the divisors of N in an array **Divs**.
(2) Set up an $m \times \sigma_0(N)$ matrix $\mathbf{M}$, whose $(i+1, j)th$ entry is

$$\mathbf{M}[i,j] = \begin{cases} \frac{\mathbf{Divs}[j]-1}{24}, \text{ if } i = 0, \\ \sigma_1(i) - \mathbf{Divs}[j]\sigma_1\left(\dfrac{i}{\mathbf{Divs}[j]}\right), \text{ if } 1 \leq i < m. \end{cases}$$

(3) For each $\mathbf{a}$ in $\mathbf{A}$, let $f(\tau) = \prod \eta^{\mathbf{a}[j]}(\mathbf{Divs}[j] \cdot \tau)$ be an eta quotient with Fourier expansion $f(\tau) = \sum_{n \geq 0} b_n q^n$. Put $\mathbf{b} = [b_0, b_1, \ldots, b_{m-1}]$ and solve

$$\mathbf{M} \cdot \mathbf{x} = \mathbf{b}.$$

If a solution $\mathbf{x}$ exists, save $[\mathbf{a}, \mathbf{x}]$ in an array $\mathbf{S}$.

Output: Eta quotients in E_N in terms of $L_\delta(\tau)$ where $\delta \mid N$:

$$\eta_N \mathbf{S}[i][1](\tau) = \sum_{l=1}^{\sigma_0(N)} \mathbf{S}[i][2][l-1] L_{\mathbf{Divs}[l]}(\tau).$$

Finally, the third algorithm implements the computation in Corollary 2.1 to determine the antiderivative of each eta quotient in E_N.

Algorithm 4.3. *This algorithm determines the antiderivatives of the eta quotients in E_N.*

Input: $N \in \mathbb{Z}^+$ and array $\mathbf{S}$ from Algorithm 4.2.

(1) If all entries of $\mathbf{S}[i][2] \in \mathbb{Z}$ then put $\mathbf{C} = 1$, otherwise put $\mathbf{C}$ as the lowest common multiple of denominators of entries of $\mathbf{S}[i][2]$.

(2) Define

$$\mathbf{T}[i] = \left[-\mathbf{C} \cdot \sum_{l=1}^{\sigma_0(N)} \mathbf{S}[i][2][l-1], \mathbf{C} \cdot \mathbf{S}[i][2][1], \ldots, \mathbf{C} \cdot \mathbf{S}[i][2][\sigma_0(N)-1] \right].$$

Output: Eta quotients whose logarithmic derivatives are eta quotients:

$$\frac{1}{2\pi i} \frac{d}{d\tau} \log(\eta_N \mathbf{T}[i](\tau)) = \mathbf{C} \eta_N \mathbf{S}[i][1](\tau).$$

5. Results

We used the algorithms given in Section 4 to find all eta quotients whose logarithmic derivatives are eta quotients in $M_2(\Gamma_0(N))$ for $N \leq 36$. A total of 203 *distinct* eta quotients and their respective logarithmic derivatives are listed in Table 2. Note that in this list we chose $c \in \mathbb{Z}$ in Corollary 2.1 to be the smallest positive constant such that $cu_\delta \in \mathbb{Z}$ for all $\delta \mid N$. We have suppressed the argument (τ) in the eta quotients so as to improve readability.

As mentioned in Section 2, we can apply transformation $q \mapsto -q$ to all the 203 identities in Table 2. With the exception of three identities which are invariant, every other identity can be mapped to another (possibly of different level) in the list. Table 3 lists all the 103 relations.

The three identities of level 4 are all equivalent up to Atkin-Lehner involution or the transformation $q \mapsto -q$. Each can be interpreted as series expansions of the product of four theta functions. Using Ramanujan's notation for theta functions,

$$\varphi(q) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \tag{5.1}$$

and

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}, \tag{5.2}$$

identities $f_{4,1a}$, $f_{41,b}$ and $f_{4,2}$ correspond respectively to representations of $\varphi(-q)^4$, $q\psi(q^2)^4$ and $\varphi(q)^4$. These representations are classical and known to Jacobi. The Eisenstein series representation for $\varphi(q)^4$ was given in equation (1.3). The other two may be deduced from our identities. One may refer to [9, p. 196] for alternatives proofs of these representations.

Remark: The anonymous referee pointed out that since

$$\eta_4[8, -24, 16](\tau) \times (\eta_4[-2, 3, -1](\tau))^8 = \eta_4[-8, 0, 8](\tau),$$

we can recover another of Jacobi's identities, namely

$$\varphi(-q)^4 + 16q\psi(q^2)^4 = \varphi(q)^4.$$

Identities equivalent to all ten entries of level 6 can be found in [9, pp. 375-378]. If we define

$$z_a = \eta_6[6, -3, -2, 1](\tau), z_b = \eta_6[-3, 6, 1, -2](\tau), z_c = \eta_6[-2, 1, 6, -3](\tau)$$

and

Table 2
Logarithmic derivatives of eta quotients which are also eta quotients.

Identity	Weight 0 eta quotient	Logarithmic derivative
$f_{4,1a}$	$\eta_4[8,-24,16]$	$\eta_4[8,-4,0]$
$f_{4,1b}$	$\eta_4[-2,3,-1]$	$2\eta_4[0,-4,8]$
$f_{4,2}$	$\eta_4[-8,0,8]$	$\eta_4[-8,20,-8]$
$f_{6,1a}$	$\eta_6[12,-48,36,0]$	$\eta_6[12,-6,-4,2]$
$f_{6,1b}$	$\eta_6[-12,3,0,9]$	$2\eta_6[-6,12,2,-4]$
$f_{6,1c}$	$\eta_6[-12,0,-4,16]$	$3\eta_6[-4,2,12,-6]$
$f_{6,1d}$	$\eta_6[0,-3,4,-1]$	$6\eta_6[2,-4,-6,12]$
$f_{6,2a}$	$\eta_6[-1,5,-5,1]$	$\eta_6[7,-5,-5,7]$
$f_{6,2b}$	$\eta_6[-5,1,-1,5]$	$\eta_6[-5,7,7,-5]$
$f_{6,3a}$	$\eta_6[4,-8,-4,8]$	$\eta_6[4,-2,4,-2]$
$f_{6,3b}$	$\eta_6[-2,1,2,-1]$	$2\eta_6[-2,4,-2,4]$
$f_{6,4a}$	$\eta_6[3,-3,-9,9]$	$\eta_6[3,3,-1,-1]$
$f_{6,4b}$	$\eta_6[-3,3,1,-1]$	$3\eta_6[-1,-1,3,3]$
$f_{8,1a}$	$\eta_8[4,-10,2,4]$	$\eta_8[4,-6,10,-4]$
$f_{8,1b}$	$\eta_8[-2,-1,5,-2]$	$2\eta_8[-4,10,-6,4]$
$f_{8,2}$	$\eta_8[-4,2,-2,4]$	$\eta_8[-4,6,6,-4]$
$f_{8,3}$	$\eta_8[-2,7,-7,2]$	$2\eta_8[4,-2,-2,4]$
$f_{9,1}$	$\eta_9[-3,0,3]$	$\eta_9[-3,10,-3]$
$f_{12,1a}$	$\eta_{12}[10,-36,18,8,0,0]$	$\eta_{12}[10,-7,-6,1,9,-3]$
$f_{12,1b}$	$\eta_{12}[-18,0,-10,0,36,-8]$	$3\eta_{12}[-6,9,10,-3,-7,1]$
$f_{12,1c}$	$\eta_{12}[4,-18,0,5,0,9]$	$4\eta_{12}[1,-7,-3,10,9,-6]$
$f_{12,1d}$	$\eta_{12}[0,0,-4,-9,18,-5]$	$12\eta_{12}[-3,9,1,-6,-7,10]$
$f_{12,2a}$	$\eta_{12}[-2,4,6,0,-16,8]$	$\eta_{12}[-2,7,6,-3,-5,1]$
$f_{12,2b}$	$\eta_{12}[6,-16,-2,8,4,0]$	$\eta_{12}[6,-5,-2,1,7,-3]$
$f_{12,2c}$	$\eta_{12}[-4,8,0,-3,-2,1]$	$4\eta_{12}[1,-5,-3,6,7,-2]$
$f_{12,2d}$	$\eta_{12}[0,-2,-4,1,8,-3]$	$4\eta_{12}[-3,7,1,-2,-5,6]$
$f_{12,3a}$	$\eta_{12}[2,0,-6,-8,12,0]$	$\eta_{12}[2,5,2,-3,-3,1]$
$f_{12,3b}$	$\eta_{12}[6,-12,-2,0,0,8]$	$3\eta_{12}[2,-3,2,1,5,-3]$
$f_{12,3c}$	$\eta_{12}[-4,0,0,1,6,-3]$	$4\eta_{12}[-3,5,1,2,-3,2]$
$f_{12,3d}$	$\eta_{12}[0,-6,4,3,0,-1]$	$12\eta_{12}[1,-3,-3,2,5,2]$
$f_{12,4a}$	$\eta_{12}[6,-12,-18,24,0,0]$	$\eta_{12}[6,3,-2,-3,-1,1]$
$f_{12,4b}$	$\eta_{12}[-6,0,2,0,-4,8]$	$3\eta_{12}[-2,-1,6,1,3,-3]$
$f_{12,4c}$	$\eta_{12}[-12,6,0,-3,0,9]$	$4\eta_{12}[-3,3,1,6,-1,-2]$
$f_{12,4d}$	$\eta_{12}[0,0,-4,3,2,-1]$	$12\eta_{12}[1,-1,-3,-2,3,6]$
$f_{12,5a}$	$\eta_{12}[9,-30,9,12,0,0]$	$\eta_{12}[9,-6,-3,3,2,-1]$
$f_{12,5b}$	$\eta_{12}[-3,0,-3,0,10,-4]$	$3\eta_{12}[-3,2,9,-1,-6,3]$
$f_{12,5c}$	$\eta_{12}[12,-30,0,9,0,9]$	$4\eta_{12}[3,-6,-1,9,2,-3]$
$f_{12,5d}$	$\eta_{12}[0,0,-4,-3,10,-3]$	$12\eta_{12}[-1,2,3,-3,-6,9]$
$f_{12,6a}$	$\eta_{12}[-8,-2,-8,0,26,-8]$	$\eta_{12}[-8,16,8,-6,-8,2]$
$f_{12,6b}$	$\eta_{12}[8,-26,8,8,2,0]$	$\eta_{12}[8,-8,-8,2,16,-6]$
$f_{12,6c}$	$\eta_{12}[4,-13,0,4,1,4]$	$2\eta_{12}[2,-8,-6,8,16,-8]$
$f_{12,6d}$	$\eta_{12}[0,-1,-4,-4,13,-4]$	$2\eta_{12}[-6,16,2,-8,-8,8]$
$f_{12,7a}$	$\eta_{12}[-4,6,12,4,-30,12]$	$\eta_{12}[-4,14,4,-6,-6,2]$
$f_{12,7b}$	$\eta_{12}[-2,-3,-6,2,15,-6]$	$2\eta_{12}[-6,14,2,-4,-6,4]$
$f_{12,7c}$	$\eta_{12}[12,-30,-4,12,6,4]$	$3\eta_{12}[4,-6,-4,2,14,-6]$
$f_{12,7d}$	$\eta_{12}[-6,15,-2,-6,-3,2]$	$6\eta_{12}[2,-6,-6,4,14,-4]$
$f_{12,8a}$	$\eta_{12}[-1,-3,-9,-5,27,-9]$	$\eta_{12}[-9,23,3,-10,-9,6]$
$f_{12,8b}$	$\eta_{12}[-10,-6,-18,-2,54,-18]$	$\eta_{12}[-10,23,6,-9,-9,3]$
$f_{12,8c}$	$\eta_{12}[18,-54,10,18,6,2]$	$3\eta_{12}[6,-9,-10,3,23,-9]$
$f_{12,8d}$	$\eta_{12}[9,-27,1,9,3,5]$	$3\eta_{12}[3,-9,-9,6,23,-10]$
$f_{12,9a}$	$\eta_{12}[-6,6,18,18,-54,18]$	$\eta_{12}[-6,21,2,-9,-7,3]$
$f_{12,9b}$	$\eta_{12}[-9,-3,-9,3,27,-9]$	$\eta_{12}[-9,21,3,-6,-7,2]$
$f_{12,9c}$	$\eta_{12}[-3,9,-3,-3,-1,1]$	$3\eta_{12}[3,-7,-9,2,21,-6]$
$f_{12,9d}$	$\eta_{12}[6,-18,-2,6,2,6]$	$3\eta_{12}[2,-7,-6,3,21,-9]$
$f_{12,10a}$	$\eta_{12}[-12,-12,-36,-12,108,-36]$	$\eta_{12}[-12,30,4,-12,-10,4]$
$f_{12,10b}$	$\eta_{12}[12,-36,4,12,4,4]$	$3\eta_{12}[4,-10,-12,4,30,-12]$
$f_{12,11a}$	$\eta_{12}[-1,6,-9,4,0,0]$	$\eta_{12}[9,-4,-3,-1,0,3]$
$f_{12,11b}$	$\eta_{12}[-9,0,-1,0,6,4]$	$3\eta_{12}[-3,0,9,3,-4,-1]$
$f_{12,11c}$	$\eta_{12}[-4,-6,0,1,0,9]$	$4\eta_{12}[-1,-4,3,9,0,-3]$
$f_{12,11d}$	$\eta_{12}[0,0,-4,9,-6,1]$	$12\eta_{12}[3,0,-1,-3,-4,9]$
$f_{12,12a}$	$\eta_{12}[7,-21,3,8,3,0]$	$\eta_{12}[7,-7,-5,4,9,-4]$
$f_{12,12b}$	$\eta_{12}[8,-21,0,7,3,3]$	$2\eta_{12}[4,-7,-4,7,9,-5]$
$f_{12,12c}$	$\eta_{12}[-3,-3,-7,0,21,-8]$	$3\eta_{12}[-5,9,7,-4,-7,4]$
$f_{12,12d}$	$\eta_{12}[0,-3,-8,-3,21,-7]$	$6\eta_{12}[-4,9,4,-5,-7,7]$
$f_{12,13a}$	$\eta_{12}[-1,4,-1,-4,2,0]$	$\eta_{12}[5,-2,1,-1,-2,3]$
$f_{12,13b}$	$\eta_{12}[1,-2,1,0,-4,4]$	$\eta_{12}[1,-2,5,3,-2,-1]$

Table 2 (*Continued*)

Identity	Weight 0 eta quotient	Logarithmic derivative
$f_{12,13c}$	$\eta_{12}[0,-2,4,1,-4,1]$	$4\eta_{12}[3,-2,-1,1,-2,5]$
$f_{12,13d}$	$\eta_{12}[-4,4,0,-1,2,-1]$	$4\eta_{12}[-1,-2,3,5,-2,1]$
$f_{12,14a}$	$\eta_{12}[5,-12,-3,4,6,0]$	$\eta_{12}[5,-4,1,3,0,-1]$
$f_{12,14b}$	$\eta_{12}[-3,6,5,0,-12,4]$	$3\eta_{12}[1,0,5,-1,-4,3]$
$f_{12,14c}$	$\eta_{12}[-4,12,0,-5,-6,3]$	$4\eta_{12}[3,-4,-1,5,0,1]$
$f_{12,14d}$	$\eta_{12}[0,-6,-4,3,12,-5]$	$12\eta_{12}[-1,0,3,1,-4,5]$
$f_{12,15a}$	$\eta_{12}[-7,0,-3,1,12,-3]$	$\eta_{12}[-7,14,5,-3,-6,1]$
$f_{12,15b}$	$\eta_{12}[-1,0,3,7,-12,3]$	$\eta_{12}[-3,14,1,-7,-6,5]$
$f_{12,15c}$	$\eta_{12}[3,-12,-1,3,0,7]$	$3\eta_{12}[1,-6,-3,5,14,-7]$
$f_{12,15d}$	$\eta_{12}[-3,12,-7,-3,0,1]$	$3\eta_{12}[5,-6,-7,1,14,-3]$
$f_{12,16a}$	$\eta_{12}[-1,3,3,-8,3,0]$	$\eta_{12}[3,5,-1,-4,-3,4]$
$f_{12,16b}$	$\eta_{12}[-8,3,0,-1,3,3]$	$2\eta_{12}[-4,5,4,3,-3,-1]$
$f_{12,16c}$	$\eta_{12}[-3,-3,1,0,-3,8]$	$3\eta_{12}[-1,-3,3,4,5,-4]$
$f_{12,16d}$	$\eta_{12}[0,-3,8,-3,-3,1]$	$6\eta_{12}[4,-3,-4,-1,5,3]$
$f_{12,17a}$	$\eta_{12}[-1,-2,-5,-1,14,-5]$	$\eta_{12}[-7,16,5,-7,-8,5]$
$f_{12,17b}$	$\eta_{12}[5,-14,1,5,2,1]$	$\eta_{12}[5,-8,-7,5,16,-7]$
$f_{12,18a}$	$\eta_{12}[-1,3,3,-2,-9,6]$	$\eta_{12}[-1,5,3,0,-3,0]$
$f_{12,18b}$	$\eta_{12}[-2,3,6,-1,-9,3]$	$2\eta_{12}[0,5,0,-1,-3,3]$
$f_{12,18c}$	$\eta_{12}[-3,9,1,-6,-3,2]$	$3\eta_{12}[3,-3,-1,0,5,0]$
$f_{12,18d}$	$\eta_{12}[-6,9,2,-3,-3,1]$	$6\eta_{12}[0,-3,0,3,5,-1]$
$f_{12,19a}$	$\eta_{12}[-3,6,9,-3,-18,9]$	$\eta_{12}[-3,12,1,-3,-4,1]$
$f_{12,19b}$	$\eta_{12}[-3,6,1,-3,-2,1]$	$3\eta_{12}[1,-4,-3,1,12,-3]$
$f_{12,20a}$	$\eta_{12}[-4,-1,0,0,1,4]$	$2\eta_{12}[-2,-2,6,6,-2,-2]$
$f_{12,20b}$	$\eta_{12}[0,-1,4,-4,1,0]$	$2\eta_{12}[6,-2,-2,-2,-2,6]$
$f_{12,21a}$	$\eta_{12}[4,-9,0,2,-3,6]$	$2\eta_{12}[2,-4,2,6,0,-2]$
$f_{12,21b}$	$\eta_{12}[-2,9,-6,-4,3,0]$	$2\eta_{12}[6,-4,-2,2,0,2]$
$f_{12,21c}$	$\eta_{12}[-6,3,-2,0,9,-4]$	$6\eta_{12}[-2,0,6,2,-4,2]$
$f_{12,21d}$	$\eta_{12}[0,-3,4,6,-9,2]$	$6\eta_{12}[2,0,2,-2,-4,6]$
$f_{12,22a}$	$\eta_{12}[4,-6,-12,8,6,0]$	$\eta_{12}[4,2,-4,-2,6,-2]$
$f_{12,22b}$	$\eta_{12}[-4,3,0,-2,-3,6]$	$2\eta_{12}[-2,2,-2,4,6,-4]$
$f_{12,22c}$	$\eta_{12}[-12,6,4,0,-6,8]$	$3\eta_{12}[-4,6,4,-2,2,-2]$
$f_{12,22d}$	$\eta_{12}[0,-3,-4,6,3,-2]$	$6\eta_{12}[-2,6,-2,-4,2,4]$
$f_{12,23a}$	$\eta_{12}[12,-33,0,12,9,0]$	$2\eta_{12}[6,-6,-2,6,2,-2]$
$f_{12,23b}$	$\eta_{12}[0,-3,-4,0,11,-4]$	$6\eta_{12}[-2,2,6,-2,-6,6]$
$f_{12,24a}$	$\eta_{12}[1,0,-3,-1,0,3]$	$\eta_{12}[1,2,-3,1,6,-3]$
$f_{12,24b}$	$\eta_{12}[-3,0,1,3,0,-1]$	$3\eta_{12}[-3,6,1,-3,2,1]$
$f_{12,25a}$	$\eta_{12}[-1,1,-1,-1,-1,3]$	$\eta_{12}[-1,1,-5,2,13,-6]$
$f_{12,25b}$	$\eta_{12}[2,-2,-6,2,2,2]$	$\eta_{12}[2,1,-6,-1,13,-5]$
$f_{12,25c}$	$\eta_{12}[-1,-1,-1,3,1,-1]$	$\eta_{12}[-5,13,-1,-6,1,2]$
$f_{12,25d}$	$\eta_{12}[-6,2,2,2,-2,2]$	$\eta_{12}[-6,13,2,-5,1,-1]$
$f_{12,26a}$	$\eta_{12}[-2,6,6,-10,-6,6]$	$\eta_{12}[-2,11,-2,-5,3,-1]$
$f_{12,26b}$	$\eta_{12}[-5,3,3,-1,-3,3]$	$\eta_{12}[-5,11,-1,-2,3,-2]$
$f_{12,26c}$	$\eta_{12}[-3,3,5,-3,-3,1]$	$3\eta_{12}[-1,3,-5,-2,11,-2]$
$f_{12,26d}$	$\eta_{12}[-6,6,2,-6,-6,10]$	$3\eta_{12}[-2,3,-2,-1,11,-5]$
$f_{12,27}$	$\eta_{12}[-4,4,4,-4,-4,4]$	$\eta_{12}[-4,10,-4,-4,10,-4]$
$f_{12,28a}$	$\eta_{12}[3,-7,-1,2,1,2]$	$\eta_{12}[3,-5,-1,4,7,-4]$
$f_{12,28b}$	$\eta_{12}[-1,1,3,2,-7,2]$	$\eta_{12}[-1,7,3,-4,-5,4]$
$f_{12,28c}$	$\eta_{12}[-2,-1,-2,1,7,-3]$	$2\eta_{12}[-4,7,4,-1,-5,3]$
$f_{12,28d}$	$\eta_{12}[-2,7,-2,-3,-1,1]$	$2\eta_{12}[4,-5,-4,3,7,-1]$
$f_{12,29a}$	$\eta_{12}[-3,2,1,-1,-2,3]$	$\eta_{12}[-3,4,1,1,4,-3]$
$f_{12,29b}$	$\eta_{12}[-1,2,3,-3,-2,1]$	$\eta_{12}[1,4,-3,-3,4,1]$
$f_{12,30}$	$\eta_{12}[-2,5,2,-2,-5,2]$	$2\eta_{12}[2,-2,2,2,-2,2]$
$f_{16,1a}$	$\eta_{16}[2,-5,2,-1,2]$	$\eta_{16}[2,-5,8,1,-2]$
$f_{16,1b}$	$\eta_{16}[-2,1,-2,5,-2]$	$2\eta_{16}[-2,1,8,-5,2]$
$f_{16,2}$	$\eta_{16}[-2,1,0,-1,2]$	$\eta_{16}[-2,1,6,1,-2]$
$f_{16,3}$	$\eta_{16}[-2,5,0,-5,2]$	$2\eta_{16}[2,-5,10,-5,2]$
$f_{18,1a}$	$\eta_{18}[3,-6,-2,2,3,0]$	$\eta_{18}[3,-3,2,4,-1,-1]$
$f_{18,1b}$	$\eta_{18}[-6,3,2,-2,0,3]$	$2\eta_{18}[-3,3,4,2,-1,-1]$
$f_{18,1c}$	$\eta_{18}[-3,0,2,-2,-3,6]$	$3\eta_{18}[-1,-1,2,4,3,-3]$
$f_{18,1d}$	$\eta_{18}[0,-3,-2,2,6,-3]$	$6\eta_{18}[-1,-1,4,2,-3,3]$
$f_{18,2a}$	$\eta_{18}[-1,2,2,-6,3,0]$	$\eta_{18}[1,1,6,-4,-3,3]$
$f_{18,2b}$	$\eta_{18}[2,-1,-6,2,0,3]$	$2\eta_{18}[1,1,-4,6,3,-3]$
$f_{18,2c}$	$\eta_{18}[-3,0,-2,6,1,-2]$	$3\eta_{18}[-3,3,6,-4,1,1]$
$f_{18,2d}$	$\eta_{18}[0,-3,6,-2,-2,1]$	$6\eta_{18}[3,-3,-4,6,1,1]$
$f_{18,3a}$	$\eta_{18}[-2,1,1,-1,-1,2]$	$\eta_{18}[-2,1,3,3,1,-2]$
$f_{18,3b}$	$\eta_{18}[-1,2,1,-1,-2,1]$	$\eta_{18}[1,-2,3,3,-2,1]$
$f_{18,4a}$	$\eta_{18}[1,-2,0,0,-1,2]$	$\eta_{18}[1,-2,2,4,1,-2]$
$f_{18,4b}$	$\eta_{18}[-2,1,0,0,2,-1]$	$2\eta_{18}[-2,1,4,2,-2,1]$

Table 2 (*Continued*)

Identity	Weight 0 eta quotient	Logarithmic derivative
$f_{20,1a}$	$\eta_{20}[-7,1,1,-5,15,-5]$	$\eta_{20}[-7,16,-5,3,-4,1]$
$f_{20,1b}$	$\eta_{20}[-1,-1,7,5,-15,5]$	$\eta_{20}[-5,16,-7,1,-4,3]$
$f_{20,1c}$	$\eta_{20}[5,-15,5,-1,-1,7]$	$5\eta_{20}[1,-4,3,-5,16,-7]$
$f_{20,1d}$	$\eta_{20}[-5,15,-5,-7,1,1]$	$5\eta_{20}[3,-4,1,-7,16,-5]$
$f_{20,2a}$	$\eta_{20}[-1,4,-8,5,0,0]$	$\eta_{20}[5,1,-2,-1,-1,2]$
$f_{20,2b}$	$\eta_{20}[-8,4,-1,0,0,5]$	$4\eta_{20}[-2,1,5,2,-1,-1]$
$f_{20,2c}$	$\eta_{20}[-5,0,0,1,-4,8]$	$5\eta_{20}[-1,-1,2,5,1,-2]$
$f_{20,2d}$	$\eta_{20}[0,0,-5,8,-4,1]$	$20\eta_{20}[2,-1,-1,-2,1,5]$
$f_{20,3a}$	$\eta_{20}[7,-20,8,5,0,0]$	$\eta_{20}[7,-5,2,-3,5,-2]$
$f_{20,3b}$	$\eta_{20}[8,-20,7,0,0,5]$	$4\eta_{20}[2,-5,7,-2,5,-3]$
$f_{20,3c}$	$\eta_{20}[-5,0,0,-7,20,-8]$	$5\eta_{20}[-3,5,-2,7,-5,2]$
$f_{20,3d}$	$\eta_{20}[0,0,-5,-8,20,-7]$	$20\eta_{20}[-2,5,-3,2,-5,7]$
$f_{24,1a}$	$\eta_{24}[-1,5,3,-11,-3,4,3,0]$	$\eta_{24}[-1,9,-1,-6,-1,2,4,-2]$
$f_{24,1b}$	$\eta_{24}[4,-11,0,5,3,-1,-3,3]$	$2\eta_{24}[2,-6,-2,9,4,-1,-1,-1]$
$f_{24,1c}$	$\eta_{24}[-3,3,1,-3,-5,0,11,-4]$	$3\eta_{24}[-1,-1,-1,4,9,-2,-6,2]$
$f_{24,1d}$	$\eta_{24}[0,-3,-4,3,11,-3,-5,1]$	$6\eta_{24}[-2,4,2,-1,-6,-1,9,-1]$
$f_{24,2a}$	$\eta_{24}[-2,5,6,-4,-15,4,6,0]$	$\eta_{24}[-2,9,2,-2,-5,-1,4,-1]$
$f_{24,2b}$	$\eta_{24}[6,-15,-2,6,5,0,-4,4]$	$3\eta_{24}[2,-5,-2,4,9,-1,-2,-1]$
$f_{24,2c}$	$\eta_{24}[-4,4,0,-5,-6,2,15,-6]$	$4\eta_{24}[-1,-2,-1,9,4,-2,-5,2]$
$f_{24,2d}$	$\eta_{24}[0,-6,-4,15,4,-6,-5,2]$	$12\eta_{24}[-1,4,-1,-5,-2,2,9,-2]$
$f_{24,3a}$	$\eta_{24}[1,2,-3,-10,6,4,0,0]$	$\eta_{24}[1,6,1,-5,-4,2,5,-2]$
$f_{24,3b}$	$\eta_{24}[-3,6,1,0,2,0,-10,4]$	$3\eta_{24}[1,-4,1,5,6,-2,-5,2]$
$f_{24,3c}$	$\eta_{24}[-4,10,0,-2,0,-1,-6,3]$	$4\eta_{24}[2,-5,-2,6,5,1,-4,1]$
$f_{24,3d}$	$\eta_{24}[0,0,-4,-6,10,3,-2,-1]$	$12\eta_{24}[-2,5,2,-4,-5,1,6,1]$
$f_{24,4a}$	$\eta_{24}[2,-1,-6,-2,3,4,0,0]$	$\eta_{24}[2,3,-2,0,1,-1,2,-1]$
$f_{24,4b}$	$\eta_{24}[-6,3,2,0,-1,0,-2,4]$	$3\eta_{24}[-2,1,2,2,3,-1,0,-1]$
$f_{24,4c}$	$\eta_{24}[-4,2,0,1,0,-2,-3,6]$	$4\eta_{24}[-1,0,-1,3,2,2,1,-2]$
$f_{24,4d}$	$\eta_{24}[0,0,-4,-3,2,6,1,-2]$	$12\eta_{24}[-1,2,-1,1,0,-2,3,2]$
$f_{24,5a}$	$\eta_{24}[-1,2,3,-2,-6,4,0,0]$	$\eta_{24}[1,4,1,1,-2,-2,-1,2]$
$f_{24,5b}$	$\eta_{24}[3,-6,-1,0,2,0,-2,4]$	$3\eta_{24}[1,-2,1,-1,4,2,1,-2]$
$f_{24,5c}$	$\eta_{24}[-4,2,0,-2,0,1,6,-3]$	$4\eta_{24}[-2,1,2,4,-1,1,-2,1]$
$f_{24,5d}$	$\eta_{24}[0,0,-4,6,2,-3,-2,1]$	$12\eta_{24}[2,-1,-2,-2,1,1,4,1]$
$f_{24,6a}$	$\eta_{24}[-1,1,3,1,-3,-4,3,0]$	$\eta_{24}[-1,7,-1,0,1,-2,-2,2]$
$f_{24,6b}$	$\eta_{24}[-4,1,0,1,3,-1,-3,3]$	$2\eta_{24}[-2,0,2,7,-2,-1,1,-1]$
$f_{24,6c}$	$\eta_{24}[-3,3,1,-3,-1,0,-1,4]$	$3\eta_{24}[-1,1,-1,-2,7,2,0,-2]$
$f_{24,6d}$	$\eta_{24}[0,-3,4,3,-1,-3,-1,1]$	$6\eta_{24}[2,-2,-2,1,0,-1,7,-1]$
$f_{24,7a}$	$\eta_{24}[-2,5,6,-10,-3,4,0,0]$	$2\eta_{24}[2,4,-2,-3,2,1,-1,1]$
$f_{24,7b}$	$\eta_{24}[4,-10,0,5,0,-2,-3,6]$	$4\eta_{24}[1,-3,1,4,-1,2,2,-2]$
$f_{24,7c}$	$\eta_{24}[-6,3,2,0,-5,0,10,-4]$	$6\eta_{24}[-2,2,2,-1,4,1,-3,1]$
$f_{24,7d}$	$\eta_{24}[0,0,-4,3,10,-6,-5,2]$	$12\eta_{24}[1,-1,1,2,-3,-2,4,2]$
$f_{24,8a}$	$\eta_{24}[-2,1,6,8,-15,-4,6,0]$	$2\eta_{24}[-2,10,2,-5,-4,1,1,1]$
$f_{24,8b}$	$\eta_{24}[-4,8,0,1,6,-2,-15,6]$	$4\eta_{24}[1,-5,1,10,1,-2,-4,2]$
$f_{24,8c}$	$\eta_{24}[-6,15,2,-6,-1,0,-8,4]$	$6\eta_{24}[2,-4,-2,1,10,1,-5,1]$
$f_{24,8d}$	$\eta_{24}[0,-6,4,15,-8,-6,-1,2]$	$12\eta_{24}[1,1,1,-4,-5,2,10,-2]$
$f_{36,1}$	$\eta_{36}[-1,1,0,-1,0,1,0,-1,1]$	$\eta_{36}[-1,1,-2,-1,10,-1,-2,1,-1]$
$f_{36,2a}$	$\eta_{36}[-3,3,2,-3,-4,-3,2,9,-3]$	$\eta_{36}[-3,6,-2,-3,10,1,-2,-4,1]$
$f_{36,2b}$	$\eta_{36}[3,-9,-2,3,4,3,-2,-3,3]$	$3\eta_{36}[1,-4,-2,1,10,-3,-2,6,-3]$
$f_{36,3a}$	$\eta_{36}[-1,1,2,-1,0,3,2,-9,3]$	$\eta_{36}[-1,4,-6,-1,14,3,-6,-6,3]$
$f_{36,3b}$	$\eta_{36}[-3,9,-2,-3,0,1,-2,-1,1]$	$3\eta_{36}[3,-6,-6,3,14,-1,-6,4,-1]$
$f_{36,4}$	$\eta_{36}[-2,5,0,-2,0,2,0,-5,2]$	$2\eta_{36}[2,-5,-4,2,14,2,-4,-5,2]$
$f_{36,5}$	$\eta_{36}[3,-9,0,3,0,-3,0,9,-3]$	$\eta_{36}[3,-9,-10,3,30,3,-10,-9,3]$
$f_{36,6a}$	$\eta_{36}[-3,6,10,0,-24,-3,8,6,0]$	$3\eta_{36}[-1,3,6,-2,-6,-1,4,3,-2]$
$f_{36,6b}$	$\eta_{36}[0,-6,-8,3,24,0,-10,-6,3]$	$12\eta_{36}[-2,3,4,-1,-6,-2,6,3,-1]$
$f_{36,7a}$	$\eta_{36}[2,-5,-1,2,2,1,-1,-1,1]$	$\eta_{36}[2,-5,-3,2,12,-1,-3,1,-1]$
$f_{36,7b}$	$\eta_{36}[-1,1,1,-1,-2,-2,1,5,-2]$	$\eta_{36}[-1,1,-3,-1,12,2,-3,-5,2]$
$f_{36,8a}$	$\eta_{36}[3,-3,-10,3,6,3,-2,-3,3]$	$3\eta_{36}[1,0,-6,-1,12,1,-2,0,-1]$
$f_{36,8b}$	$\eta_{36}[-3,3,2,-3,-6,-3,10,3,-3]$	$3\eta_{36}[-1,0,-2,1,12,-1,-6,0,1]$
$f_{36,9a}$	$\eta_{36}[-6,3,2,0,0,0,-2,-3,6]$	$6\eta_{36}[-1,0,0,1,4,1,0,0,-1]$
$f_{36,9b}$	$\eta_{36}[0,-3,2,6,0,-6,-2,3,0]$	$6\eta_{36}[1,0,0,-1,4,-1,0,0,1]$
$f_{36,10a}$	$\eta_{36}[-3,6,2,0,0,-3,-8,6,0]$	$3\eta_{36}[1,-3,2,2,6,1,-4,-3,2]$
$f_{36,10b}$	$\eta_{36}[0,-6,8,3,0,0,-2,-6,3]$	$12\eta_{36}[2,-3,-4,1,6,2,2,-3,1]$
$f_{36,11a}$	$\eta_{36}[-2,5,6,-2,-16,0,6,3,0]$	$2\eta_{36}[-1,4,4,-1,-6,-3,4,6,-3]$
$f_{36,11b}$	$\eta_{36}[0,-3,-6,0,16,2,-6,-5,2]$	$6\eta_{36}[-3,6,4,-3,-6,-1,4,4,-1]$
$f_{36,12a}$	$\eta_{36}[-6,3,4,0,-6,-6,2,15,-6]$	$6\eta_{36}[-2,3,0,-1,4,2,0,-3,1]$
$f_{36,12b}$	$\eta_{36}[0,-3,-2,6,6,6,-4,-15,6]$	$6\eta_{36}[-1,3,0,-2,4,1,0,-3,2]$
$f_{36,12c}$	$\eta_{36}[-6,15,4,-6,-6,-6,2,3,0]$	$6\eta_{36}[2,-3,0,1,4,-2,0,3,-1]$
$f_{36,12d}$	$\eta_{36}[6,-15,-2,6,6,0,-4,-3,6]$	$6\eta_{36}[1,-3,0,2,4,-1,0,3,-2]$
$f_{36,13a}$	$\eta_{36}[6,-15,-2,6,4,0,-2,3,0]$	$2\eta_{36}[3,-6,-4,3,14,1,-4,-4,1]$
$f_{36,13b}$	$\eta_{36}[0,-3,2,0,-4,-6,2,15,-6]$	$6\eta_{36}[1,-4,-4,1,14,3,-4,-6,3]$

Table 3
Equivalent eta quotients under $q \mapsto -q$.

$f_{4,1a} \sim f^-_{4,2}$	$f_{4,1b} \sim f^-_{4,1b}$		
$f_{6,1a} \sim f^-_{12,10a}$	$f_{6,1b} \sim f^-_{12,23a}$	$f_{6,1c} \sim f^-_{12,10b}$	$f_{6,1d} \sim f^-_{12,23b}$
$f_{6,2a} \sim f^-_{12,17a}$	$f_{6,2b} \sim f^-_{12,17b}$	$f_{6,3a} \sim f^-_{12,27}$	$f_{6,3b} \sim f^-_{12,30}$
$f_{6,4a} \sim f^-_{12,19a}$	$f_{6,4b} \sim f^-_{12,19b}$		
$f_{8,1a} \sim f^-_{8,2}$	$f_{8,1b} \sim f^-_{8,3}$		
$f_{9,1} \sim f^-_{36,5}$			
$f_{12,1a} \sim f^-_{12,8b}$	$f_{12,1b} \sim f^-_{12,8c}$	$f_{12,1c} \sim f^-_{12,11c}$	$f_{12,1d} \sim f^-_{12,11d}$
$f_{12,2a} \sim f^-_{12,25b}$	$f_{12,2b} \sim f^-_{12,25d}$	$f_{12,2c} \sim f^-_{12,13d}$	$f_{12,2d} \sim f^-_{12,13c}$
$f_{12,3a} \sim f^-_{12,26a}$	$f_{12,3b} \sim f^-_{12,26d}$	$f_{12,3c} \sim f^-_{12,14c}$	$f_{12,3d} \sim f^-_{12,14d}$
$f_{12,4a} \sim f^-_{12,9a}$	$f_{12,4b} \sim f^-_{12,9d}$	$f_{12,4c} \sim f^-_{12,5c}$	$f_{12,4d} \sim f^-_{12,5d}$
$f_{12,5a} \sim f^-_{12,9b}$	$f_{12,5b} \sim f^-_{12,9c}$		
$f_{12,6a} \sim f^-_{12,6b}$	$f_{12,6c} \sim f^-_{12,20a}$	$f_{12,6d} \sim f^-_{12,20b}$	
$f_{12,7a} \sim f^-_{12,22a}$	$f_{12,7b} \sim f^-_{12,21b}$	$f_{12,7c} \sim f^-_{12,22c}$	$f_{12,7d} \sim f^-_{12,21c}$
$f_{12,8a} \sim f^-_{12,11a}$	$f_{12,8d} \sim f^-_{12,11b}$		
$f_{12,12a} \sim f^-_{12,15a}$	$f_{12,12b} \sim f^-_{12,16b}$	$f_{12,12c} \sim f^-_{12,15d}$	$f_{12,12d} \sim f^-_{12,16d}$
$f_{12,13a} \sim f^-_{12,25c}$	$f_{12,13b} \sim f^-_{12,25a}$	$f_{12,14a} \sim f^-_{12,26b}$	$f_{12,14b} \sim f^-_{12,26c}$
$f_{12,15b} \sim f^-_{12,16a}$	$f_{12,15c} \sim f^-_{12,16c}$		
$f_{12,18a} \sim f^-_{12,24a}$	$f_{12,18b} \sim f^-_{12,18b}$	$f_{12,18c} \sim f^-_{12,24b}$	$f_{12,18d} \sim f^-_{12,18d}$
$f_{12,21a} \sim f^-_{12,22b}$	$f_{12,21d} \sim f^-_{12,22d}$		
$f_{12,28a} \sim f^-_{12,29a}$	$f_{12,28b} \sim f^-_{12,29b}$	$f_{12,28c} \sim f^-_{12,28d}$	
$f_{16,1a} \sim f^-_{16,2}$	$f_{16,1b} \sim f^-_{16,3}$		
$f_{18,1a} \sim f^-_{36,2a}$	$f_{18,1b} \sim f^-_{36,13a}$	$f_{18,1c} \sim f^-_{36,2b}$	$f_{18,1d} \sim f^-_{36,13b}$
$f_{18,2a} \sim f^-_{36,3a}$	$f_{18,2b} \sim f^-_{36,11a}$	$f_{18,2c} \sim f^-_{36,3b}$	$f_{18,2d} \sim f^-_{36,11b}$
$f_{18,3a} \sim f^-_{36,7a}$	$f_{18,3b} \sim f^-_{36,7b}$	$f_{18,4a} \sim f^-_{36,1}$	$f_{18,4b} \sim f^-_{36,4}$
$f_{20,1a} \sim f^-_{20,3a}$	$f_{20,1b} \sim f^-_{20,2a}$	$f_{20,1c} \sim f^-_{20,2c}$	$f_{20,1d} \sim f^-_{20,3c}$
$f_{20,2b} \sim f^-_{20,3b}$	$f_{20,2d} \sim f^-_{20,3d}$		
$f_{24,1a} \sim f^-_{24,3a}$	$f_{24,1b} \sim f^-_{24,6b}$	$f_{24,1c} \sim f^-_{24,3b}$	$f_{24,1d} \sim f^-_{24,6d}$
$f_{24,2a} \sim f^-_{24,4a}$	$f_{24,2b} \sim f^-_{24,4b}$	$f_{24,2c} \sim f^-_{24,8b}$	$f_{24,2d} \sim f^-_{24,8d}$
$f_{24,3c} \sim f^-_{24,5c}$	$f_{24,3d} \sim f^-_{24,5d}$	$f_{24,4c} \sim f^-_{24,7b}$	$f_{24,4d} \sim f^-_{24,7d}$
$f_{24,5a} \sim f^-_{24,6a}$	$f_{24,5b} \sim f^-_{24,6c}$	$f_{24,7a} \sim f^-_{24,8a}$	$f_{24,7c} \sim f^-_{24,8c}$
$f_{36,6a} \sim f^-_{36,8a}$	$f_{36,6b} \sim f^-_{36,10b}$	$f_{36,8b} \sim f^-_{36,10a}$	$f_{36,9a} \sim f^-_{36,12d}$
$f_{36,9b} \sim f^-_{36,12b}$	$f_{36,12a} \sim f^-_{36,12c}$		

$$z_d = \eta_6[1, -2, -3, 6](\tau),$$

then the ten entries for level 6 in Table 2 correspond respectively to Eisenstein series representations for

$$z_a^2, z_b^2, z_c^2, z_d^2, z_a z_d, z_b z_c, z_a z_c, z_b z_d, z_a z_b \text{ and } z_c z_d.$$

The series representations can be found in Theorems 6.17 and 6.19 of [9]. (Remark: there is a minor misprint in the identity for z_c^2.) One should consult Chapter 6 of the excellent treatise by Cooper [9] to understand the rich interplay between these ten weight 2 eta quotients of level 6. Equivalent statements for $f_{6,1a}$, $f_{6,3a}$, $f_{6,3b}$, $f_{6,4a}$ and $f_{6,4b}$ can also be found in [11].

Under the transformation $q \mapsto -q$, identities of level N are mapped to level $4N/\gcd(4, N)$. Thus, all the level 6 identities are equivalent to some level 12 identities.

The four identities of level 8 in Table 2 are all equivalent up to Atkin-Lehner involutions or the transformation $q \mapsto -q$. Equivalent statements can be found in Theorem 8.16 of [9]. In particular, $f_{8,1a}$, $f_{8,1b}$, $f_{8,2}$, and $f_{8,3}$ correspond to Equations (8.38), (8.41), (8.39) and (8.42) respectively. As mentioned previously, $f_{8,2}$ can also be found in [11].

There are a total of 100 *distinct* identities of level 12 and the relationships between these identities are worth investigating. From Table 3, we can see that the twelve identities labelled $f_{12,1*}$, $f_{12,8*}$ and $f_{12,11*}$ are all equivalent. In total, there are six groups of 12 equivalent identities accounting for a total of 72. There are another ten identities equivalent to the ten from level 6. The remaining 18 out of the 100 identities can be classified into three groups of six equivalent identities each. It is interesting to note that identities $f_{12,18b}$ and $f_{12,18d}$ are invariant under $q \mapsto -q$.

As mentioned in Section 2, $f_{12,27}$ can be found in [11]. It is equivalent to the representations of $\varphi(q)^2\varphi(q^3)^2$ and the result was known to Ramanujan [4, p. 223]. (See also [9, Theorem 3.27].) Equivalent forms of identities $f_{12,24a}$, $f_{12,29a}$, $f_{12,18a}$ and $f_{12,28a}$ can be found in [9], where they correspond respectively to the second, fourth, fifth and sixth equations in the statement of Theorem 12.8.

In [23], Williams studied the problem of finding eta quotients in $M_2(\Gamma_0(12))$ whose Fourier coefficients can be represented by divisor sums. In other words, he attempted to determine eta quotients in E_{12}, which by Lemma 2.1 is equivalent to the problem of finding eta quotients whose derivatives are eta quotients. Williams found 126 eta quotients, each of which can be easily mapped to the identities in Table 2. The number 126 came about because he did not restrict his search to primitive identities (see Section 2). With reference to Table 1, the 3 identities from level 4 and the 10 identities from level 6, are each counted twice at level 12. Together with the 100 primitive identities at level 12 totals the 126 eta quotients listed in [23].

The four identities of level 16 in Table 2 are all equivalent up to Atkin-Lehner involutions or the transformation $q \mapsto -q$. Identity $f_{16,2}$ appeared in [6, p. 608]. An application of this identity to the theory of partitions is described in the next section.

Each of the 12 identities of level 18 are equivalent to some level 36 identities under the transformation $q \mapsto -q$. We are not aware of any previous appearances of these identities.

The 12 identities of level 20 in Table 2 are all equivalent up to Atkin-Lehner involutions or the transformation $q \mapsto -q$. We are not aware of any previous appearances of these identities.

The 32 identities of level 24 can be divided into two groups, each of which consisting of 16 equivalent identities. In [1], Alaca, Alaca and Aygin attempted to determine eta quotients in E_{24}. Among the eta quotients that they found, they listed 32 which do not arise directly from level 12 [1, Theorem 5.1]. These 32 can be easily mapped to the 32 level 24 identities in Table 2. For the more general problem of determining which modular forms space is spanned by eta quotients, see [20].

Finally, there are 25 identities of level 36. One is equivalent to the unique level 9 identity, while another 12 are each equivalent to an identity from level 18. The remaining 12 identities can be divided into two groups, each of which consisting of 6 equivalent identities.

6. Applications

In his lost notebook [19], Ramanujan recorded eight evaluations of integrals of theta functions. Four of these are equivalent to identities in Table 2. For example, Entry 14.3.1 [2, p. 314] is as follows: For $0 < q < 1$,

$$\frac{\varphi(-q^3)}{\varphi(-q)} = \exp\left(2\int_0^q \psi(t)^2\psi(t^3)^2 dt\right). \tag{6.1}$$

Identity (6.1) is equivalent to $f_{6,3b}$ from Table 2, namely

$$q\frac{d}{dq}\log\left(\eta_6[-2,1,2,-1](\tau)\right) = 2\eta_6[-2,4,-2,4](\tau). \tag{6.2}$$

Similarly, Entries 14.3.2, 14.3.3 and 14.3.4 of [2] correspond respectively to $f_{12,27}$, $f_{4,1b}$ and $f_{4,2}$. The last of which is simply Jacobi's identity (1.4). Other analogues of (6.1) may be deduced from Table 2. For example, the identity labelled $f_{24,2c}$ is equivalent to the following.

Theorem 6.1. *For* $0 < q < 1$,

$$\frac{\varphi(q^6)^3}{\varphi(-q)^2\varphi(q^2)} = \exp\left(4\int_0^q \frac{\psi(t^2)^3\psi(t^3)\varphi(-t^4)^2}{\psi(-t)\varphi(t^6)}dt\right). \tag{6.3}$$

A second application is to the evaluation of certain integrals in terms of algebraic constants. Fine evaluated the following three integrals [11, Eq. (33.43), (33.53), (33.63)]:

$$\int_0^{e^{-\pi}} \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{20}}{(1-q^n)^{16}}dq = \frac{1}{16}, \tag{6.4}$$

$$\int_0^{e^{-\pi/\sqrt{2}}} \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{8}(1-q^{4n})^4}{(1-q^n)^{8}}dq = \frac{\sqrt{2}}{8}, \tag{6.5}$$

$$\int_0^{e^{-\pi/\sqrt{3}}} \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{14}(1-q^{6n})^6}{(1-q^n)^{8}(1-q^{4n})^8}dq = \frac{1}{3}. \tag{6.6}$$

The proof of each boils down to integrating the identities $f_{4,2}$, $f_{8,2}$ and $f_{12,27}$ respectively, and subsequently evaluating the resulting eta quotients. Cooper [8] employed a similar technique to the identity associated with $f_{9,1}$, i.e. the identity discovered by Borwein and Garvan (1.5), to obtain

$$\int_0^{e^{-2\pi/3}} \prod_{n=1}^{\infty} \frac{(1-q^{3n})^{10}}{(1-q^n)^{6}}dq = \frac{1}{3\sqrt{3}}. \tag{6.7}$$

By combing through our list of identities, we are able to obtain the following 15 evaluations. We organize the results according to level.

Theorem 6.2. *Let* $j \in \mathbb{Z}^+$. *The following evaluations hold:*

$$\int_0^{e^{-2\pi/\sqrt{6}}} q^{j-1}\prod_{n=1}^{\infty} \frac{(1-q^{2n})^{7+j}(1-q^{3n})^{7-j}}{(1-q^n)^{5+5j}(1-q^{6n})^{5-5j}}dq = \frac{1}{j}\left(\frac{1}{6\sqrt{2}}\right)^j, \tag{6.8}$$

$$\int_0^{e^{-2\pi/\sqrt{6}}} q^{j-1}\prod_{n=1}^{\infty} \frac{(1-q^{n})^{3+3j}(1-q^{2n})^{3-3j}}{(1-q^{3n})^{1+9j}(1-q^{6n})^{1-9j}}dq = \frac{1}{j}\left(\frac{3}{2\sqrt{2}}-1\right)^j. \tag{6.9}$$

Proof. From the identity associated with $f_{6,2b}$ in Table 2, we can deduce that

$$q\frac{d}{dq}\left(\eta_6[-5,1,-1,5](\tau)\right)^j = j\left(\eta_6[-5,1,-1,5](\tau)\right)^j \cdot \eta_6[-5,7,7,-5](\tau).$$

Thus

$$\int_0^{e^{-2\pi/\sqrt{6}}} \frac{1}{q}\cdot\left(\eta_6[-5,1,-1,5](\tau)\right)^j \cdot \eta_6[-5,7,7,-5](\tau)dq$$

$$= \frac{1}{j}\left(\eta_6[-5,1,-1,5](\tau)\right)^j\Big|_{q=e^{-2\pi/\sqrt{6}}}$$

$$= \frac{1}{j}\left(\frac{\eta(6\tau)^5}{\eta(\tau)^5}\cdot\frac{\eta(2\tau)}{\eta(3\tau)}\right)^j\Bigg|_{\tau=i/\sqrt{6}}$$

$$= \frac{1}{j}\left(\frac{\eta(i\sqrt{6})^5}{\eta(i/\sqrt{6})^5}\cdot\frac{\eta(i\sqrt{2/3})}{\eta(i\sqrt{3/2})}\right)^j. \tag{6.10}$$

Using the well known inversion formula

$$\eta(-1/\tau) = \sqrt{\tau/i}\cdot\eta(\tau) \tag{6.11}$$

with $\tau = i/\sqrt{6}$ and $\tau = i\sqrt{3/2}$, the right side of (6.10) becomes $\frac{1}{j}\left(\frac{1}{6\sqrt{2}}\right)^j$ which proves (6.8). The proof of (6.9) follows in a similar manner from the identity $f_{6,4a}$ in Table 2. □

Theorem 6.3. *Let $s \in \mathbb{Z}, s \neq 0$. The following evaluation holds:*

$$\int_0^{e^{-\pi/\sqrt{2}}} 2\prod_{n=1}^{\infty}\frac{(1-q^n)^{4-2s}(1-q^{8n})^{4+2s}}{(1-q^{2n})^{2-7s}(1-q^{4n})^{2+7s}}dq = \frac{1}{s}\left(2^{s/4}-1\right). \tag{6.12}$$

Proof. Use the identity $f_{8,3}$ in Table 2 and the inversion formula (6.11). □

Theorem 6.4. *Let $j \in \mathbb{Z}^+$ and $s \in \mathbb{Z}, s \neq 0$. The following evaluations hold:*

$$\int_0^{e^{-\pi/\sqrt{3}}} 2q^{2j-1}\prod_{n=1}^{\infty}\frac{(1-q^{3n})^6(1-q^{4n})^6}{(1-q^n)^{2+4j}(1-q^{2n})^{2+j}(1-q^{6n})^{2-j}(1-q^{12n})^{2-4j}}dq = \frac{1}{j}\left(\frac{1}{12\cdot 3^{1/4}}\right)^j, \tag{6.13}$$

$$\int_0^{e^{-\pi/\sqrt{3}}} 2q\prod_{n=1}^{\infty}\frac{(1-q^n)^6(1-q^{12n})^6}{(1-q^{2n})^{2+s}(1-q^{3n})^{2-4s}(1-q^{4n})^{2+4s}(1-q^{6n})^{2-s}}dq = \frac{1}{s}\left(\frac{4}{3\cdot 3^{1/4}}\right)^s - \frac{1}{s}, \tag{6.14}$$

$$\int_0^{e^{-\pi/\sqrt{3}}} q^{j-1}\prod_{n=1}^{\infty}\frac{(1-q^{2n})^{4+2j}(1-q^{3n})^{1+j}(1-q^{4n})^{1-j}(1-q^{6n})^{4-2j}}{(1-q^n)^{3+3j}(1-q^{12n})^{3-3j}}dq = \frac{1}{j}\left(\frac{1}{2\sqrt{3}}\right)^j, \tag{6.15}$$

$$\int_0^{e^{-\pi/\sqrt{3}}} \prod_{n=1}^{\infty}\frac{(1-q^n)^{1-s}(1-q^{2n})^{4+2s}(1-q^{6n})^{4-2s}(1-q^{12n})^{1+s}}{(1-q^{3n})^{3-3s}(1-q^{4n})^{3+3s}}dq = \frac{1}{s}\left(\frac{2}{\sqrt{3}}\right)^s - \frac{1}{s}, \tag{6.16}$$

$$\int_0^{e^{-\pi/\sqrt{3}}} 2\prod_{n=1}^{\infty}\frac{(1-q^n)^{2-2s}(1-q^{3n})^{2+2s}(1-q^{4n})^{2-2s}(1-q^{12n})^{2+2s}}{(1-q^{2n})^{2-5s}(1-q^{6n})^{2+5s}}dq = \frac{1}{s}\left(3^{s/4}-1\right). \tag{6.17}$$

Proof. These five evaluations can be proved by respectively utilizing the identities $f_{12,20a}$, $f_{12,20b}$, $f_{12,29a}$, $f_{12,29b}$ and $f_{12,30}$ in Table 2 and the inversion formula (6.11). □

Theorem 6.5. *Let $s \in \mathbb{Z}, s \neq 0$. The following evaluation holds:*

$$\int_0^{e^{-\pi/2}} 2\prod_{n=1}^{\infty} \frac{(1-q^n)^{2-2s}(1-q^{4n})^{10}(1-q^{16n})^{2+2s}}{(1-q^{2n})^{5-5s}(1-q^{8n})^{5+5s}} dq = \frac{1}{s}\left(2^{s/2}-1\right). \tag{6.18}$$

Proof. Use identity $f_{16,3}$ in Table 2 and the inversion formula (6.11). □

Theorem 6.6. *Let $j \in \mathbb{Z}^+$. The following evaluations hold:*

$$\int_0^{e^{-\sqrt{2}\pi/3}} q^{j-1}\prod_{n=1}^{\infty} \frac{(1-q^{2n})^{1+j}(1-q^{3n})^{3+j}(1-q^{6n})^{3-j}(1-q^{9n})^{1-j}}{(1-q^n)^{2+2j}(1-q^{18n})^{2-2j}} dq = \frac{1}{j}\left(\frac{1}{\sqrt{6}}\right)^j, \tag{6.19}$$

$$\int_0^{e^{-2\sqrt{2}\pi/3}} q^{j-1}\prod_{n=1}^{\infty} \frac{(1-q^n)^{1+j}(1-q^{3n})^{2}(1-q^{6n})^{4}(1-q^{9n})^{1-j}}{(1-q^{2n})^{2+2j}(1-q^{18n})^{2-2j}} dq = \frac{1}{j}\left(1-\sqrt{\frac{2}{3}}\right)^j. \tag{6.20}$$

Theorem 6.7. *Let $j \in \mathbb{Z}^+$ and $s \in \mathbb{Z}, s \neq 0$. The following evaluations hold:*

$$\int_0^{e^{-\pi/3}} q^{j-1}\prod_{n=1}^{\infty} \frac{F_1(q^n)(1-q^{2n})^{1+j}(1-q^{18n})^{1-j}}{(1-q^n)^{1+j}(1-q^{4n})^{1+j}(1-q^{9n})^{1-j}(1-q^{36n})^{1-j}} dq = \frac{1}{j}\left(\frac{1}{\sqrt{3}}\right)^j \tag{6.21}$$

where

$$F_1(q^n) = \frac{(1-q^{6n})^{10}}{(1-q^{3n})^2(1-q^{12n})^2};$$

$$\int_0^{e^{-\pi/3}} 2\prod_{n=1}^{\infty} \frac{F_2(q^n)(1-q^n)^{2-2s}(1-q^{4n})^{2-2s}(1-q^{9n})^{2+2s}(1-q^{36n})^{2+2s}}{(1-q^{2n})^{5-5s}(1-q^{18n})^{5+5s}} dq = \frac{1}{s}\left(3^{s/2}-1\right) \tag{6.22}$$

where

$$F_2(q^n) = \frac{(1-q^{6n})^{14}}{(1-q^{3n})^4(1-q^{12n})^4};$$

$$\int_0^{e^{-\pi/3}} q^{j-1}\prod_{n=1}^{\infty} \frac{F_3(q^n)(1-q^n)^{3+3j}(1-q^{4n})^{3+3j}(1-q^{9n})^{3-3j}(1-q^{36n})^{3-3j}}{(1-q^{2n})^{9+9j}(1-q^{18n})^{9-9j}} dq = \frac{1}{j}\left(\frac{1}{3\sqrt{3}}\right)^j \tag{6.23}$$

where

$$F_3(q^n) = \frac{(1-q^{6n})^{30}}{(1-q^{3n})^{10}(1-q^{12n})^{10}};$$

$$\int_0^{e^{-\pi/3}} 6q^{6j-1}\prod_{n=1}^{\infty} \frac{F_4(q^n)(1-q^{2n})^{3j}(1-q^{3n})^{2j}(1-q^{12n})^{-2j}(1-q^{18n})^{-3j}}{(1-q^n)^{1+6j}(1-q^{36n})^{1-6j}} dq = \frac{1}{j}\left(\frac{1}{12\sqrt{3}}\right)^j \tag{6.24}$$

where

$$F_4(q^n) = (1-q^{4n})(1-q^{6n})^4(1-q^{9n}).$$

Proof. These four evaluations can be proved by respectively utilizing the identities $f_{36,1}$, $f_{36,4}$, $f_{36,5}$ and $f_{36,9a}$ in Table 2 and the inversion formula (6.11). $\square$

Our 15 evaluations in Theorems 6.2 to 6.7 relied on evaluating eta quotients with the inversion formula (6.11). In principle, all the 203 identities in Table 2 can be evaluated at special values. Doyle and Williams [10] gave a very well written account of how to use the Chowla-Selberg formula for fundamental discriminants and its generalizations to evaluate such integrals. In their work, they proved generalizations of (6.4), (6.5) and (6.6). In addition, they also gave two other evaluations by using the identities $f_{12,3c}$ and $f_{16,2}$.

Our third application comes from the theory of partitions. Let $p(n)$ denote the partition function, then it is known that for primes ℓ,

$$p(\ell n+\delta_\ell)\equiv 0 \pmod{\ell} \tag{6.25}$$

holds if and only if $\ell = 5$, 7 or 11. Thus it is somewhat surprising that Kim and Toh [15] showed that for every odd integer m, the function $c(n)$ satisfies

$$c(mn+\delta_m)\equiv 0 \pmod{m} \tag{6.26}$$

where $4\delta_m \equiv 1 \pmod{m}$. Here $c(n)$ counts the number of cubic partition pairs of n, weighted by the parity of the crank. The generating function of $c(n)$ is

$$\sum_{n=0}^{\infty} c(n)q^n = \prod_{n=1}^{\infty} \frac{(1-q^n)^6(1-q^{2n})^2}{(1-q^{4n})^4}. \tag{6.27}$$

The proof of (6.26) relies on the fact that the generating function of $c(n)$ is essentially an eta quotient that happens to be the derivative of some power series in q with integer coefficients. The precise identity responsible is $f_{16,2}$ in Table 2. Thus our list of identities paves the way to look for other partition functions with similar properties. We shall illustrate this with identity $f_{16,1b}$. It is equivalent to

$$q\frac{d}{dq}\eta_{16}[-2,1,-2,5,-2](\tau) = 2\eta_{16}[-4,2,6,0,0](\tau). \tag{6.28}$$

In other words, suppose $d(n)$ has the generating function,

$$\begin{aligned}\sum_{n=0}^{\infty} d(n)q^n &= \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2(1-q^{4n})^6}{(1-q^n)^4}\\ &= \prod_{n=1}^{\infty} \frac{(1-q^{4n})^4}{(1-q^n)} \cdot \prod_{n=1}^{\infty} \frac{(1-q^{2n})^2}{(1-q^n)} \cdot \prod_{n=1}^{\infty} \frac{(1-q^{4n})^2}{(1-q^n)^2}.\end{aligned} \tag{6.29}$$

We can interpret $d(n)$ as the number of four-colored partitions, where the first color consists of 4-core partitions [12], the second color consists of 2-core partitions and the third and fourth are partitions where no parts divisible by 4 appears. Now observe that

$$\eta_{16}[-2,1,-2,5,-2](\tau) = 1 + 2\sum_{n=1}^{\infty} a_n q^n \tag{6.30}$$

where $a_n \in \mathbb{Z}$ for every positive integer n. Identity (6.28) is then equivalent to

$$\sum_{n=0}^{\infty} d(n)q^{n+1} = \sum_{n=1}^{\infty} d(n-1)q^n = \sum_{n=1}^{\infty} na_n q^n. \tag{6.31}$$

In other words, for every $n, m \in \mathbb{Z}^+$ where $m > 1$, the following congruence holds

$$d(mn-1) \equiv 0 \pmod{m}. \tag{6.32}$$

We end by discussing a related work. Choi, Kim and Lim [7] also consider the problem of determining eta quotients which are mapped to other eta quotients by the differential operator. Their approach to the problem is combinatorial and quite different from our computational approach. They found that the only results for square-free levels are, up to Atkin-Lehner involutions, the ten level 6 identities in Table 2, which corroborates with our findings. It will be interesting if their methods can be used to show that there are no other identities for levels $N > 36$.

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