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# Criteria for complex multiplication and transcendence properties of automorphic functions

*Dedicated to Wolfgang M. Schmidt on the occasion of his 60th birthday*

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**Acknowledgment.** The results of this paper are joint work with Paula Beazley Cohen. Indeed, this paper is closely based on a larger preprint [CSW] written with her. To our regret, she prefers not to join us for this publication, but instead to give elsewhere a different account of this joint work, which will include other material (see also her prepublication [Co1], and [Co2] for a special case). With her kind permission, we make use of a lot of her ideas, remarks and numerous corrections of previous versions of the present article, without mentioning them at each item. The two authors of the present article accept full responsability for any errors in it that may have been overlooked.

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## § 1. Introduction and statement of the Main Theorem

A theorem of Kronecker-Weber and its generalisation by Shimura and Taniyama [ST] says that suitably normalised automorphic functions, whose values occur as moduli of abelian varieties, take algebraic values at complex multiplication (or "CM", for short) points. The normalisation in question determines in what sense the automorphic function is defined over $\bar{\mathbb{Q}}$. The present paper deals with the converse problem: Do suitably normalised automorphic functions take algebraic values only at those algebraic points which are also CM points? The first result in this direction is due to Th. Schneider [Sch] who in 1937 showed that the elliptic modular function $j = j(\tau)$ takes transcendental values at all algebraic arguments $\tau$ in the upper half plane, with the exception of the imaginary quadratic ones, that is the CM points. Schneider's theorem was generalised in 1972 by Morita [Mo] to automorphic functions for some quaternionic norm unit groups, and by the first author to Siegel and Picard modular functions [Sh1] and the Matsumoto theta map [Sh2]. For the Picard modular functions, see also recent work of Holzapfel [Ho]. In all these cases, the results can be stated in terms of a certain family $\{A_z | z \in D\}$ of polarised abelian varieties $A_z$ parametrised by the points $z$ of a complex bounded symmetric domain $D$. Namely, if $z$ has algebraic coordinates without being a CM point, the field of moduli of $A_z$ has positive transcendence degree.

In the present article, we give a new and unified proof embracing the above results, and extend them to all symmetric domains parametrising families of complex polarised abelian varieties as classified by Shimura and Siegel. In what follows, we shall usually denote a polarised abelian variety defined over a subfield of $\mathbb{C}$ by a single capital letter such as $A$, leaving off a separate notation specifically for the polarisation which will be clear from the context. To state our main result more precisely, recall that every algebra $L$ occurring as the endomorphism algebra of a simple polarised abelian variety is a division algebra over $\mathbb{Q}$ on which the polarisation induces a positive involution $\varrho$; for the precise connection between polarisation and involution see § 3. Such algebras have been classified (see Albert [A]) and fall into one of four types (see Proposition 1 of [Shi1]). Indeed, if $\mathbb{F}$ is the totally real number field consisting of the fixed points of $\varrho$ in the center $\mathbb{K}$ of $L$, then we have one of the following:

(Type I) $L$ is the field $\mathbb{F}$,

(Type II) $L$ is a totally indefinite quaternion algebra over $\mathbb{F}$,

(Type III) $L$ is a totally definite quaternion algebra over $\mathbb{F}$,

(Type IV) $L$ is a central simple algebra over $\mathbb{K}$, this field being a totally imaginary quadratic extension of $\mathbb{F}$.

In 1963, Siegel [Sie] and Shimura [Shi1] independently clarified the nature of the role played by certain symmetric domains for the theory of abelian varieties. Namely, fix a division algebra $L$ over $\mathbb{Q}$ and a positive involution $\varrho$ on $L$, together with a complex representation $\Phi$ of $L$ of degree $n$. Let $S = S(L, \Phi, \varrho)$ be the set of all (not necessarily simple) polarised abelian varieties $A$ of dimension $n$, defined over $\mathbb{C}$, with

$$L \subset \mathrm{End}_0(A) := \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q} ,$$

such that $\Phi$ is equivalent to a complex representation of $L$ induced by an isomorphism of the complex tangent space $T_A(\mathbb{C})$ to the origin of $A$ with $\mathbb{C}^n$, and whose polarisation is compatible with $\varrho$. Notice that $\Phi$ cannot be arbitrary in order that $S$ be not empty. We call a representation $\Phi$ "*admissible*" if $S(L, \Phi, \varrho)$ is not empty. In fact, it is not difficult to see that for each $L$ of type I, II, III, there is only one possible equivalence class for an admissible $\Phi$ ([Shi1], p. 156). For admissible $\Phi$, there is a symmetric domain $D = D_\Phi$, depending only on $L$ and the equivalence class of $\Phi$, parametrising families $\Sigma$ of elements $A_z$ in $S = S(L, \Phi, \varrho)$ where $z$ runs over $D$. The introduction to § 3 will explain in some more detail how these "*analytic families of abelian varieties*" [Shi1] are parametrised by the points $z$ of $D$.

We mention the following two extreme possibilities for the dimension of $D = D_\Phi$ as a function of $n$. If $L = \mathbb{Q}$, we obtain the Siegel upper half space

$$D = \mathfrak{H}_n^1 = \{z \in M_n(\mathbb{C}) \mid z = {}^t z, \frac{1}{2i}(z - \bar{z}) \text{ positive definite}\}$$

of dimension $\frac{1}{2}n(n+1)$. On the other hand, $D$ reduces to one point (that is $\dim D = 0$) if and only if $[L : \mathbb{Q}] = 2n$ and $L$ is a CM field, that is a purely imaginary quadratic extension of a totally real number field. In this case, $L$ and $\Phi$ determine up to isogeny a unique abelian variety of dimension $n$. When this abelian variety is simple it is said to have "*complex multiplication*" ("CM"). An abelian variety which is not necessarily simple is said to be of "CM *type*" if it is isogenous to a direct product of simple abelian varieties with CM [ST]. We call a point $z \in D$ a "CM *point*" if for each analytic family $\Sigma$ of elements of $S$ parametrized by $D$ the corresponding abelian variety $A_z \in \Sigma$ is of CM type. In fact, this property depends only on $z$, not on the choice of the analytic family $\Sigma \subset S$.

The *algebraic points* of $D_\Phi$, denoted $D_\Phi(\bar{\mathbb{Q}})$, are by definition those with all their components algebraic numbers. Here, as throughout the present article, we assume $D_\Phi$ to be normalized as in [Shi1], p. 162. In particular, the domain $D_\Phi$ is a product of $g = [\mathbb{F} : \mathbb{Q}]$ irreducible symmetric domains whose definition depends only on the type of $L$ and the equivalence class of $\Phi$. Namely, the irreducible factors are the following (notice that $m := 2n/[L : \mathbb{Q}]$ and $q$ are integers where $q^2 = [L : \mathbb{K}]$):

(Type I) the Siegel upper half space $\mathfrak{H}_{m/2}^1$, the integer $m$ being even in this case,

(Type II) the Siegel upper half space $\mathfrak{H}_m^1$,

(Type III) the space $\mathfrak{H}_m^2 := \{z \in M_m(\mathbb{C}) \mid {}^t z = -z,\ 1 - z\,{}^t\bar{z} \text{ positive hermitian}\}$,

(Type IV) the space $\mathfrak{H}_{r,s}^3 := \{z \in M_{r,s}(\mathbb{C}) \mid 1 - z\,{}^t\bar{z} \text{ positive hermitian}\}$, $(r, s) = (r_\nu, s_\nu)$, where for $\nu = 1, \dots, g$ the $r_\nu, s_\nu$ are certain positive integers satisfying $r_\nu + s_\nu = mq$ and determined by the multiplicity $r_\nu$ in $\Phi$ of a given absolutely irreducible representation of $L$ and the multiplicity $s_\nu$ in $\Phi$ of the complex conjugate representation.

With this normalisation, the CM points of $D_\Phi$ are algebraic points. Further, we call the complex abelian variety $A_z$ *defined over* $\bar{\mathbb{Q}}$ if it may be holomorphically embedded in a complex projective space as the complex points of an algebraic group variety defined over $\bar{\mathbb{Q}}$. We can now state our main result:

**Main Theorem.** *Let $D_\Phi$ be a complex symmetric domain parametrizing an analytic family $\Sigma$ of polarized abelian varieties $A_z$, $z \in D_\Phi$, which are in $S = S(L, \Phi, \varrho)$. Then the following properties are equivalent:*

(i) *$z$ is in $D_\Phi(\bar{\mathbb{Q}})$ and the corresponding polarized abelian variety $A_z \in \Sigma$ is defined over $\bar{\mathbb{Q}}$.*

(ii) *$z$ is a complex multiplication point, that is $A_z$ is of* CM *type.*

That (ii) implies (i) is well known, so in the sequel we prove only that if $A_z$ is defined over $\bar{\mathbb{Q}}$ and is not of CM type, the point $z$ cannot be algebraic. We remark in passing that by Prop. 5 of [Shi1], $A_z$ is defined over $\bar{\mathbb{Q}}$ if and only if its field of moduli is a number field.

The plan of the present article is as follows. In § 2 we concentrate on the parts of the proof of the Main Theorem coming from the theory of transcendental numbers, in particular from Wüstholz' analytic subgroup theorem. Proposition 1, § 2, shows that for abelian varieties defined over $\bar{\mathbb{Q}}$, all $\bar{\mathbb{Q}}$-linear relations between periods of differentials of the first kind defined over $\bar{\mathbb{Q}}$ are induced by endomorphisms of the abelian variety. Moreover, it is possible to calculate in Proposition 2, § 2, the precise dimension of the vector space generated over $\bar{\mathbb{Q}}$ by all these periods. There is even a more general result for periods of the first and the second kind together with 1 and $\pi$ (see § 6 of [CSW] and a forthcoming paper by P. B. Cohen). These results seem to be already known. One can compare them, for example, with Wüstholz' announcement (without proof) of Theorem 5 in [Wü2] or with Bertrand's Remarque 1 in [Be2]. Nonetheless, we prefer to include proofs for the convenience of the reader. The referee of this paper kindly informed us about a simpler version of the proof of Proposition 1 using ideas from recent work of Masser and Wüstholz [MW1], [MW2]. Nevertheless, we preferred to correct our original proof because its ideas are used again in Proposition 3. This result is an improvement of theorem M′ of [Sh1] and turns out to provide a useful criterion for complex multiplication and to give sharper versions of the Main Theorem in special cases. Indeed, sometimes one does not need to know that all the coefficients of $z \in D_\Phi$ are algebraic. In § 5, we have an example where it is already sufficient for $z$ to have only one algebraic coefficient.

In § 3, we treat that part of the proof of the Main Theorem involving moduli spaces. To begin with, this requires an application of Proposition 2, § 2. If $A_z$ is defined over $\bar{\mathbb{Q}}$ and belongs to a family $\Sigma$ of elements of $S(L, \Phi, \varrho)$ as above with $\dim D_\Phi > 0$ and if $L = \mathrm{End}_0(A_z)$, then by inspection of the definition of $z$ in terms of the periods of $A_z$, one shows that $z$ cannot be an algebraic point. If, on the other hand, we have $\mathrm{End}_0(A_z)$ strictly larger than $L$, the abelian variety $A_z$ belongs to a subfamily $\Sigma'$ (of elements of $S(L', \Phi', \varrho)$) of $\Sigma$ with $L' = \mathrm{End}_0(A_z)$. Therefore, $A_z$ can be identified with an $A_\tau$ for some $\tau \in D_{\Phi'}$. If $\dim D_{\Phi'} = 0$ then $A_z$ is of CM type. If $\dim D_{\Phi'} > 0$ we can argue as above that $\tau$ cannot be an algebraic point. To obtain the transcendence of $z$ from that of $\tau$, we have to use rationality properties of certain modular embeddings of $D_{\Phi'}$ into $D_\Phi$. In this way we avoid, given the algebraicity of $z$, having to construct in one step many elements of $\mathrm{End}_0(A_z)$ and then having to show that they mutually commute as described in [Mo] and which presents difficulties in [Sh1]. (Added in proof, February 1995: Bernhard Runge shows in a recent preprint ("On complex Shimura varieties") that all complex symmetric domains parametrizing analytic families of abelian varieties can be replaced by nicely defined subvarieties of the corresponding Siegel upper half space. Using these subvarieties instead of Shimura's symmetric domains one can even avoid the construction of modular embeddings and get a new version of the Main Theorem.)

An evident consequence of our results for the values of automorphic functions at algebraic points is treated in § 4: For any analytic family $\Sigma$ of abelian varieties parametrized by $D = D_\Phi$ there is a *modular group* $\Gamma = \Gamma_\Sigma$ acting discontinuously on $D$ and having orbits corresponding to isomorphic elements of $\Sigma$. More precisely, there is a one-to-one correspondence between $\Gamma \backslash D$ and the isomorphism classes of elements of $\Sigma$. As a complex analytic space, the quotient $\Gamma \backslash D$ may be embedded into $V(\mathbb{C}) \subset \mathbb{P}_N(\mathbb{C})$ where $V$ is a projective algebraic variety defined over a number field, the *Shimura variety* for the family $\Sigma$ (some more details are given in § 4). The composite map

$$J : D \to \Gamma \backslash D \hookrightarrow V(\mathbb{C})$$

can be normalized such that $A_z$ is defined over $\bar{\mathbb{Q}}$ if and only if $J(z) \in V(\bar{\mathbb{Q}})$. In particular, CM points $z \in D$ give algebraic $J$-values on $V$. Under this normalization the Main Theorem implies the

**Corollary.** *One has both $z \in D(\bar{\mathbb{Q}})$ and $J(z) \in V(\bar{\mathbb{Q}})$ if and only if $z$ is a* CM *point.*

In this sense our Main Result provides a generalisation of Schneider's 1937 result on the elliptic $j$-function. Some open problems are also presented in § 4. In § 5, a striking application to Hilbert modular functions and hypergeometric functions of one variable is made through their connection with families of abelian varieties derived from the Type IV case with $L = \mathbb{K}$. Such abelian varieties are sometimes said to have "*generalized complex multiplication*". The result on hypergeometric functions given in Corollary 5, § 5, can be summarized in a way not involving abelian varieties as follows. If a suitably normalized Schwarz hyperbolic triangle function, for a triangle with angles quotients of $\pi$ by a positive integer, takes an algebraic value at an algebraic argument $x$, then some other such functions, for triangles with angles rational multiples of the original ones, also take algebraic values at $x$. A different proof of these results may be found in Theorem A.1 of [Sh1], and for the Hilbert modular functions see also [Co2].

## § 2. Linear independence of periods

Given a complex abelian variety $A$, choose a basis $\omega_i$, $i = 1, \ldots, n$, of the complex vector space $H^0(A, \Omega)$ of holomorphic differentials on $A$ and generators $\gamma_j$, $j = 1, \ldots, 2n$, of $H_1(A, \mathbb{Z})$. The period vectors

$$\int_\gamma \underset{\sim}{\omega} := \begin{pmatrix} \int_\gamma \omega_1 \\ \vdots \\ \int_\gamma \omega_n \end{pmatrix}, \quad \gamma \in H_1(A, \mathbb{Z}),$$

form a lattice $\Lambda \subset \mathbb{C}^n$ of rank $2n$ with $\mathbb{Z}$-basis $\int_{\gamma_j} \underset{\sim}{\omega}$, $j = 1, \ldots, 2n$. Suppose now that $A$ is defined over $\bar{\mathbb{Q}}$ and that $\omega_1, \ldots, \omega_n \in H^0(A, \Omega_{\bar{\mathbb{Q}}})$, that is, they are defined over $\bar{\mathbb{Q}}$. Let $T_A$ be the tangent space at the origin of $A$. Choosing a basis of $T_A$ we identify the complex tangent space $T_A(\mathbb{C}) := T_A \otimes_{\bar{\mathbb{Q}}} \mathbb{C}$ of $A(\mathbb{C})$ with $\mathbb{C}^n$. The exponential map

$$\exp : \mathbb{C}^n \to A(\mathbb{C})$$

with $\Lambda = \operatorname{Ker} \exp$ is then defined over $\bar{\mathbb{Q}}$. The endomorphism algebra $\operatorname{End}_0 A = \operatorname{End} A \otimes_{\mathbb{Z}} \mathbb{Q}$ can be lifted by exp to a complex representation $\Phi$ of $\operatorname{End}_0 A$ on $\mathbb{C}^n$ given therefore by matrices in $M_n(\bar{\mathbb{Q}})$, i.e. with algebraic coefficients (see e.g. [Ma], Appendix). Let $\Lambda_{\mathbb{Q}}$ be the $\mathbb{Q}$-vector space $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. For any $\alpha \in \operatorname{End}_0 A$ we have $\Phi(\alpha) \Lambda_{\mathbb{Q}} \subset \Lambda_{\mathbb{Q}}$, so on comparing the action of the complex and the rational representations of $\alpha$, we obtain

$\bar{\mathbb{Q}}$-linear relations (which may of course be trivial) induced by $\alpha$ between the periods $\int_{\gamma_j} \omega_i$ on $A$. To see this, take for simplicity an $\alpha \in \operatorname{End} A$. Its rational representation on $H_1(A, \mathbb{Z})$ gives

$$\alpha\gamma_j = \sum_{\nu=1}^{2n} m_{j\nu}\gamma_\nu \quad \text{with some} \quad m_{j\nu} \in \mathbb{Z}\,.$$

On the other hand, $\underset{\sim}{\omega} \circ \alpha = \Phi(\alpha)\underset{\sim}{\omega}$, so we have algebraic $\beta_{i\mu}$ for which

$$\omega_i \circ \alpha = \sum_{\mu=1}^{n} \beta_{i\mu}\omega_\mu\,, \quad \text{hence} \quad \sum_{\nu=1}^{2n} m_{j\nu} \int_{\gamma_\nu} \omega_i = \sum_{\mu=1}^{n} \beta_{i\mu} \int_{\gamma_j} \omega_\mu\,.$$

We will show that these relations generate all $\bar{\mathbb{Q}}$-linear relations between the periods on $A$. Namely, we have

**Proposition 1.** *Let $A$ and $B$ be abelian varieties defined over $\bar{\mathbb{Q}}$, and denote by $V_A$ the $\bar{\mathbb{Q}}$-vector subspace of $\mathbb{C}$ generated by all periods $\int_\gamma \omega$ on $A$ with cycles $\gamma \in H_1(A, \mathbb{Z})$ and differentials of the first kind $\omega \in H^0(A, \Omega_{\bar{\mathbb{Q}}})$. Then*:

1) *$V_A \cap V_B \neq \{0\}$ if and only if there are simple abelian subvarieties $A'$ of $A$ and $B'$ of $B$, with $A'$ isogenous to $B'$.*

2) *Suppose in addition that $A$ is simple with endomorphism algebra $L = \operatorname{End}_0 A$. For a basis $\{\omega_i \mid i = 1, \ldots, n\}$ of $H^0(A, \Omega_{\bar{\mathbb{Q}}})$ let $\Lambda$ be the associated period lattice in $\mathbb{C}^n$. Choose a basis $\gamma_1, \ldots, \gamma_{2n}$ of $H_1(A, \mathbb{Z})$ such that the first $m$ basis period vectors $\int_{\gamma_1} \underset{\sim}{\omega}, \ldots, \int_{\gamma_m} \underset{\sim}{\omega}$ of $\Lambda$ form a basis of $\Lambda_{\mathbb{Q}}$ as a $\Phi(L)$-module where $\Phi$ denotes the complex representation of $L$ determined by $\Lambda$ (see above). Then the components $\int_{\gamma_j} \omega_i$ $(i = 1, \ldots, n;\ j = 1, \ldots, m)$ of these period vectors form a basis of $V_A$ over $\bar{\mathbb{Q}}$.*

In Part 1) it is evident that $V_A \cap V_B \neq \{0\}$ if $A$ and $B$ have some isogenous subvarieties $A' \triangleq B'$. And in part 2) it follows from the above-mentioned inclusion $\Phi(L) \subset M_n(\bar{\mathbb{Q}})$ that the $\int_{\gamma_j} \omega_i$ in question generate $V_A$. On the other hand, if in part 1) $V_A \cap V_B \neq \{0\}$ we have a nontrivial $\bar{\mathbb{Q}}$-linear relation between periods of $A \times B$. Therefore it is sufficient to show that such relations only come from isogenies between simple components of one abelian variety. In other words, we have to show only linear independence properties for non-isogenous simple abelian varieties. Without yet making the special assumptions of part 2) we consider on $A$ a nontrivial linear dependence relation between periods:

$$(1) \qquad \sum_{j=1}^{m} \sum_{i=1}^{n} a_{ji} \int_{\gamma_j} \omega_i = 0\,, \quad \text{all } a_{ji} \in \bar{\mathbb{Q}}\,, \quad \text{not all } a_{ji} = 0\,.$$

This means that in the complex tangent space $T_G(\mathbb{C}) = (\mathbb{C}^n)^m$ of the algebraic group $G := A^m$ there is a hyperplane $H$ defined over $\bar{\mathbb{Q}}$ and given by

$$H := \left\{ (z_1, \ldots, z_m) \in (\mathbb{C}^n)^m \;\middle|\; \sum_{j=1}^{m} \sum_{i=1}^{n} a_{ji}(z_j)_i = 0 \right\}$$

where $(\underset{\sim}{z}_j)_i$ denotes the $i$-th component of $\underset{\sim}{z}_j \in \mathbb{C}^n$. The hyperplane $H$ contains the nonzero period vector $v := \left( \int_{\gamma_1} \underset{\sim}{\omega}, \ldots, \int_{\gamma_m} \underset{\sim}{\omega} \right)$ of $G = A^m$. Note that this point lies in the kernel of $\exp \times \ldots \times \exp$ ($m$ times) which is the exponential map Exp for $G$. We can therefore apply the following special case of the Wüstholz analytic subgroup theorem to $G = A^m$ (see [Wü1], [Wü3] and a more general statement in [Wa]).

**Lemma 1.** *Let $G$ be a commutative algebraic group defined over $\bar{\mathbb{Q}}$, and $H$ a proper complex linear subspace, defined over $\bar{\mathbb{Q}}$, of the complex tangent space to $G$ at the origin. Suppose that $H$ contains a point $v \neq 0$ in the kernel of the exponential map of $G(\mathbb{C})$. Then there is a proper connected algebraic subgroup $W \subset G$ defined over $\bar{\mathbb{Q}}$ with tangent space $T_W$ satisfying $v \in T_W(\mathbb{C}) \subset H$.*

The subspace $H$ can be defined therefore by $\bar{\mathbb{Q}}$-linear combinations of the defining equations for $T_W$. On the other hand, there are not too many possibilities for the algebraic subgroups $W$ of $G = A^m$ thanks to the reduction theorem of Poincaré combined with a variant of Schur's Lemma:

**Lemma 2.** *The abelian variety $A$, defined over $\bar{\mathbb{Q}}$, is isogenous over $\bar{\mathbb{Q}}$ to a direct product $A_1^{k_1} \times \ldots \times A_N^{k_N}$ of simple abelian varieties $A_\nu$, $\nu = 1, \ldots, N$, all defined over $\bar{\mathbb{Q}}$ and pairwise non-isogenous. Any abelian subvariety $W$, defined over $\bar{\mathbb{Q}}$ and contained in $A^m$ for an integer $m \geqq 1$, is also isogenous to some product of powers of the simple abelian subvarieties $A_1, \ldots, A_N$.*

To simplify the proof of Proposition 1 we note that

1) neither $\Lambda_{\mathbb{Q}}$ nor $V_A$ nor $\operatorname{End}_0 A$ change if we replace $A$ by an isogenous abelian variety, so we may assume $A$ to be a direct product $A_1^{k_1} \times \ldots \times A_N^{k_N}$ of simple pairwise non-isogenous abelian subvarieties defined over $\bar{\mathbb{Q}}$,

2) the $\bar{\mathbb{Q}}$-vector space $V_A$ does not change if we replace $A$ by $A_1 \times \ldots \times A_N$, so we may assume that all simple components of $A$ occur with multiplicity 1,

3) by a decomposition argument we may assume the simplicity of $W$:

If $W \triangleq W_1 \times \ldots \times W_k$ for simple abelian subvarieties $W_\varkappa \subset W$, $\varkappa = 1, \ldots, k$, then $T_W = T_{W_1} \oplus \ldots \oplus T_{W_k}$, there is an $s \in \mathbb{N}$ and a decomposition $sv = v_1 + \ldots + v_k$ of $sv$ into period vectors $v_\varkappa \in T_{W_\varkappa}(\mathbb{C})$ such that the period relation (1) will be a $\bar{\mathbb{Q}}$-linear combination of the linear equations defining the $T_{W_\varkappa}$, after substituting for $v_\varkappa$. All $v_\varkappa$, $\varkappa = 1, \ldots, k$, belong to Ker Exp, hence are of the form

$$v_\varkappa = \left( \int_{\gamma_{\varkappa 1}} \underset{\sim}{\omega}, \ldots, \int_{\gamma_{\varkappa m}} \underset{\sim}{\omega} \right), \quad \text{all} \quad \gamma_{\varkappa j} \in H_1(A, \mathbb{Z}),$$

but their components $\int_{\gamma_{\varkappa j}} \underset{\sim}{\omega}$, $j = 1, \ldots, m$, do not necessarily form a basis of $\Lambda_{\mathbb{Q}}$ as a $\Phi(L)$-module.

In other words this last step replaces (1) by minimal period relations. For the proof of part 1) of Proposition 1 we can by Lemma 1 and 2 assume that $W$ is isogenous to a simple component of $A$, say $A_1$, and that all projections

$$p : W \to S \subset A^m$$

onto other simple subvarieties $S \triangleq A_\nu$, $\nu = 2, \ldots, N$, must be zero, hence

$$T_W \cap T_{(A_2 \times \ldots \times A_N)^m} = \{0\}\,.$$

Consequently, (1) is only a period relation in $V_{A_1}$. Since all period relations can be decomposed in this way, part 1) of Proposition 1 is true.

Now we assume the hypothesis of part 2). We suppose there is a nontrivial linear period relation leading by the Lemmas 1 and 2 and the reduction process to a product of simple abelian subvarieties (describing minimal period relations)

$$W_\varkappa \subset W \subset G = A^m\,, \quad \text{all} \quad W_\varkappa \triangleq A \quad \text{for } \varkappa = 1, \ldots, k\,.$$

Since $W$ is a proper subvariety of $G$ we have $k < m$. For the moment fix one $\varkappa$ and omit in the definition of $v_\varkappa \in T_{W_\varkappa}(\mathbb{C})$ all cycles $\gamma_{\varkappa j} = 0$, that is, replace $m$ by some smaller exponent $m_\varkappa$ if necessary. Then we can assume that in

$$v_\varkappa = \left( \int_{\gamma_{\varkappa 1}} \underset{\sim}{\omega}, \ldots, \int_{\gamma_{\varkappa m_\varkappa}} \underset{\sim}{\omega} \right)$$

all $\gamma_{\varkappa j} \neq 0$ for $j = 1, \ldots, m_\varkappa$. As $A$ is simple and defined over $\bar{\mathbb{Q}}$ and as $0 \neq \omega_i \in H^0(A, \Omega_{\bar{\mathbb{Q}}})$, by an easy (and known, see [WW]) application of Lemma 1 we have $\int_{\gamma_{\varkappa j}} \omega_i \neq 0$ for all $i = 1, \ldots, n$, $j = 1, \ldots, m_\varkappa$. Since $v_\varkappa \in T_{W_\varkappa}(\mathbb{C})$, all projections $p_\mu$, $\mu = 1, \ldots, m_\varkappa$, of $W_\varkappa$ onto the factors $A$ of $A^{m_\varkappa}$ are nonzero, hence surjective. Before giving the next lemma we recall that nonzero isogenies $p_\nu, p_\mu : W_\varkappa \to A$ of simple abelian varieties have finite kernel. Hence $\operatorname{Ker} p_\nu$ consists of torsion points of an order dividing $t$, say, therefore $t p_\mu \circ p_\nu^{-1}$ will be a well-defined endomorphism of $A$, hence $p_\mu \circ p_\nu^{-1} \in \operatorname{End}_0 A$. So we obtain

**Lemma 3.** *Suppose $A$ is a simple abelian variety and $A \triangleq W_\varkappa \subset A^{m_\varkappa}$. Let the $m_\varkappa$ projections $p_\mu$, $\mu = 1, \ldots, m_\varkappa$ of $W_\varkappa$ onto the factors $A$ be $\neq 0$. Then for all $\mu, \nu = 1, \ldots, m_\varkappa$, the mapping $p_\mu \circ p_\nu^{-1}$ defines an element of* $\operatorname{End}_0 A$. *The lift of the elements $p_\mu \circ p_\nu^{-1}$ $(\nu \neq \mu)$ by* exp *gives matrix equations $\underset{\sim}{z}_\mu = C_{\mu\nu} \underset{\sim}{z}_\nu$ with $C_{\mu\nu} = \Phi(p_\mu \circ p_\nu^{-1}) \in M_n(\mathbb{C})$, defining the subspace $T_{W_\varkappa}$ of $T_{A^{m_\varkappa}}$ whose complexification is identified with*

$$(\mathbb{C}^n)^{m_\varkappa} = \{(\underset{\sim}{z}_1, \ldots, \underset{\sim}{z}_{m_\varkappa}) \mid \underset{\sim}{z}_\mu \in \mathbb{C}^n, \mu = 1, \ldots, m_\varkappa\}\,.$$

This means in particular that for all $\varkappa = 1, \ldots, k$, the components $\int_{\gamma_{\varkappa j}} \underset{\sim}{\omega}$ of $v_\varkappa$ generate a $\Phi(L)$-module of rank 1, hence the components $\int_{\gamma_j} \underset{\sim}{\omega}$ of $v = s^{-1} \sum_{\varkappa=1}^{k} v_\varkappa$ generate a $\Phi(L)$-module of rank at most $k < m$, in contradiction to the choice of $\int_{\gamma_1} \underset{\sim}{\omega}, \ldots, \int_{\gamma_m} \underset{\sim}{\omega}$ as a basis of the $\Phi(L)$-module $\Lambda_{\mathbb{Q}}$.

A simple counting argument enables one to deduce from Proposition 1 (see [Wü2])

**Proposition 2.** *Suppose that the abelian variety $A$ is isogenous to the direct product $A_1^{k_1} \times \ldots \times A_N^{k_N}$ of simple, pairwise non-isogenous abelian varieties $A_\nu$, $\nu = 1, \ldots, N$, defined over $\bar{\mathbb{Q}}$ and of dimension $n_\nu$. Then the $\bar{\mathbb{Q}}$-vector space $V_A$ generated by all periods of differentials of the first kind on $A$, defined over $\bar{\mathbb{Q}}$, has the dimension*

$$\dim_{\bar{\mathbb{Q}}} V_A = \sum_{\nu=1}^{N} \frac{2n_\nu^2}{\dim_{\mathbb{Q}} \mathrm{End}_0 A_\nu}.$$

For a more general result see the appendix §6 of [CSW]. The following can be considered as a special consequence of Proposition 1 if one uses the classification of endomorphism algebras and their action on period lattices. But we prefer to give a direct proof.

**Proposition 3.** *Let $A$ be a simple abelian variety defined over $\bar{\mathbb{Q}}$ and $\omega \neq 0$ a differential of the first kind on $A$ defined over $\bar{\mathbb{Q}}$. Suppose that all periods of $\omega$ are algebraic multiples of each other. Then $A$ has complex multiplication.*

*Proof.* 1. Note again that for all nonzero $\gamma \in H_1(A, \mathbb{Z})$ the period $\int_\gamma \omega$ is nonzero. Choose a basis $\omega = \omega_1, \omega_2, \ldots, \omega_n$ of $H^0(A, \Omega_{\bar{\mathbb{Q}}})$ and a basis $\gamma_1, \ldots, \gamma_{2n}$ of $H_1(A, \mathbb{Z})$. By our assumptions,

$$\int_{\gamma_j} \omega = \alpha_j \int_{\gamma_1} \omega \quad \text{with} \quad \alpha_j \in \bar{\mathbb{Q}} \quad \text{for all} \quad j = 1, \ldots, 2n.$$

These $\alpha_j$ are linearly independent over $\mathbb{Q}$, otherwise a nontrivial relation

$$r_1\alpha_1 + \ldots + r_{2n}\alpha_{2n} = 0$$

would hold with $r_j \in \mathbb{Z}$ for all $j$, and the cycle $\gamma := r_1\gamma_1 + \ldots + r_{2n}\gamma_{2n} \neq 0$ would give a zero period of $\omega$. Therefore, the number field $\mathbb{K} := \mathbb{Q}(\alpha_1, \ldots, \alpha_{2n})$ must have degree $[\mathbb{K} : \mathbb{Q}] \geqq 2n$.

2. Let $\mathbb{C}^{2n} = \{(\underset{\sim}{z}, \underset{\sim}{w}) \mid \underset{\sim}{z}, \underset{\sim}{w} \in \mathbb{C}^n\} = \left\{\left(\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}, \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}\right) \middle| z_i, w_i \in \mathbb{C}, i = 1, \ldots, n\right\} = T_{A \times A}(\mathbb{C})$.
The kernel of $\exp : \mathbb{C}^{2n} \to (A \times A)(\mathbb{C})$ is the period lattice

$$\Lambda = \left\{\left(\int_\gamma \underset{\sim}{\omega}, \int_\delta \underset{\sim}{\omega}\right) \middle| \gamma \text{ and } \delta \in H_1(A, \mathbb{Z})\right\}$$

and contains the points $v_j = \left(\int_{\gamma_1} \underset{\sim}{\omega}, \int_{\gamma_j} \underset{\sim}{\omega}\right)$, $j = 1, \ldots, 2n$. These are in turn contained in the hyperplanes

$$H_j := \{(\underset{\sim}{z}, \underset{\sim}{w}) \in \mathbb{C}^{2n} \mid w_1 = \alpha_j z_1\}$$

defined over $\bar{\mathbb{Q}}$. The Wüstholz analytic subgroup theorem as stated in Lemma 1 shows the existence of $2n$ connected proper abelian subvarieties $W_j \subset A \times A$ with $v_j \in T_{W_j}(\mathbb{C}) \subset H_j$. By Lemma 2, each $W_j$ is isogenous to $A$, and by Lemma 3, each $T_{W_j}$ is given by some nonsingular system of $n$ linear equations $\underset{\sim}{w} = C_j \underset{\sim}{z}$ with $C_j = \begin{pmatrix} \alpha_j\, 0 \dots 0 \\ * \end{pmatrix}$. Each $C_j$ represents an element of $\mathrm{End}_0 A$ in its complex representation $\Phi$ on the tangent space $T_A(\mathbb{C})$.

3. The representation $\Phi$ is faithful, so the $C_j$ generate over $\mathbb{Q}$ a subalgebra $C$ of the division algebra $\Phi(\mathrm{End}_0 A) \cong \mathrm{End}_0 A$. It is well known that $\dim_{\mathbb{Q}} \mathrm{End}_0 A$ divides $2n$ for $n = \dim_{\mathbb{C}} A$, hence we have in particular

$$\dim_{\mathbb{Q}} C \leqq 2n .$$

Equality is possible only for $C = \Phi(\mathrm{End}_0 A)$. The matrices of $C$ are of the form $\begin{pmatrix} \alpha\, 0 \dots 0 \\ * \end{pmatrix}$ with $\alpha \in \mathbb{K}$, so the map $h : \begin{pmatrix} \alpha\, 0 \dots 0 \\ * \end{pmatrix} \mapsto \alpha$ defines an algebra homomorphism $h$ of $C$ onto $\mathbb{K}$. Now $[\mathbb{K} : \mathbb{Q}] \geqq 2n$ implies that $\dim_{\mathbb{Q}} C \geqq 2n$. Therefore we have equality everywhere, the map $h$ must be an isomorphism and $\mathrm{End}_0 A \cong C \cong \mathbb{K}$ must be a CM field.

By a decomposition into simple parts we obtain the following consequence (Cor. 2 is Theorem M′ of [Sh1]).

**Corollary 1.** *Let $A$ be an abelian variety defined over $\bar{\mathbb{Q}}$, and let $\omega_1 = \omega, \omega_2, \dots, \omega_n$ be a basis of $H^0(A, \Omega_{\bar{\mathbb{Q}}})$ and $\gamma_1, \dots, \gamma_{2n}$ a basis of $H_1(A, \mathbb{Z})$. If all periods of $\omega$ are algebraic multiples of each other, $A$ has a factor with complex multiplication.*

**Corollary 2.** *If in addition the quotients $\int_{\gamma_j} \omega_i / \int_{\gamma_k} \omega_i$ are algebraic numbers for all $i = 1, \dots, n$ and $j, k = 1, \dots, 2n$ (whenever $\int_{\gamma_k} \omega_i \neq 0$), the abelian variety $A$ is of* CM *type.*

## § 3. Proof of the Main Theorem

In this section we give the proof of the Main Theorem as outlined in § 1.

1. Let $(L, \Phi, \varrho)$ be as in the statement of that theorem and let $\Sigma$ be any analytic family of elements of $S$ parametrized by $D = D_\Phi$. Recall from § 1 that we have to prove that if $A \in \Sigma$ is defined over $\bar{\mathbb{Q}}$ and is not of CM type and if $A = A_z \in \Sigma$, $z \in D$, then $z$ is not algebraic. To understand how the analytic families $\Sigma$ are defined and how $A_z \in \Sigma$ depends on $z$, let $\mathbb{F}$ be as in § 1 the totally real number field consisting of the fixed points of $\varrho$ in the center $\mathbb{K}$ of $L$, let $g := [\mathbb{F} : \mathbb{Q}]$ and $m := 2n/[L : \mathbb{Q}]$ where as always $n$ denotes the dimension of the abelian varieties under consideration. Let $T$ be a $\varrho$-skew-symmetric matrix in $GL_m(L)$ satisfying some additional signature condition in the type IV case ([Shi1], Thm. 1 and (25), p. 160), and let $\mathcal{M}$ be any free $\mathbb{Z}$-submodule of rank $2n = m[L : \mathbb{Q}]$ in $L^m$. Then, every $z \in D$ together with $T$ and $\mathcal{M}$ determine an element $A_z$ of $S$. In fact, this construction of [Shi1] yields the complex polarized abelian variety $A_z$ as a pair $(\mathbb{C}^n/\Lambda_z, E_z)$ where $\Lambda_z$ is a lattice in $\mathbb{C}^n$ and $E_z$ is a non-degenerate Riemann form on $\mathbb{C}^n/\Lambda_z$. The involution $\varrho$ and the polarization of $A_z$ given by $E_z$ are connected by

$$E_z(\Phi(a)x, y) = E_z(x, \Phi(a^\varrho)y)$$

for all $x, y \in \mathbb{C}^n/\Lambda_z$ and all $a \in L$ ([Shi1], p. 154). We recall that $A_z$ is defined over $\bar{\mathbb{Q}}$ if, as a complex manifold, $\mathbb{C}^n/\Lambda_z$ may be embedded in a complex projective space as the complex points of an abelian algebraic group variety defined over $\bar{\mathbb{Q}}$. Hence, if $A_z$ is defined over $\bar{\mathbb{Q}}$ then so is any complex polarized abelian variety $A$ given by an isomorphic pair $(\mathbb{C}^n/\Lambda, E)$ obtained from $(\mathbb{C}^n/\Lambda_z, E_z)$ via a $\mathbb{C}$-linear isomorphism of $\mathbb{C}^n$, that is by a change of analytic coordinates. As already remarked $A_z$ is defined over $\bar{\mathbb{Q}}$ if and only if its field of moduli is a number field. Conversely, up to a change of analytic coordinates commuting with $\Phi$ and defining an isomorphism of complex polarized abelian varieties, every element of $S$ can be obtained in this way, since the existence of an appropriate $T$ is assured via its polarization and of an appropriate $\mathscr{M}$ by the $\Phi(L)$-module generated over $\mathbb{Q}$ by its lattice in $\mathbb{C}^n$. Hence on fixing $T$ and $\mathscr{M}$ we obtain a family $\Sigma = \Sigma(T, \mathscr{M})$ of elements of $S$ parametrized by the points of $D$. When $L = \mathbb{K}$ and $[L : \mathbb{Q}] = 2n$ then $L$ is a CM field and every $A_z$, $z \in D$, is of CM type, so we exclude this case and suppose therefore that $\dim D > 0$. If $\Sigma = \Sigma(T, \mathscr{M})$, the fact of whether or not $A_z$, $z \in D$, is defined over $\bar{\mathbb{Q}}$ does not change when we vary $T$ (see for example [Shi1], Proposition 6). Also, the property of $A_z$ being of CM type depends only on $z$, not on $T$ and $\mathscr{M}$. When we vary $\mathscr{M}$, the abelian variety $A_z$ changes by an isogeny only. Therefore in order to prove the Main Theorem we have to prove:

(*) *Let $z \in D$. If for some $T$ and $\mathscr{M}$ satisfying the conditions of* [Shi1], *Theorem* 1, *$A = A_z \in \Sigma(T, \mathscr{M}) \subset S(L, \Phi, \varrho)$ is defined over $\bar{\mathbb{Q}}$ and is not of* CM *type, then $z$ is not algebraic.*

Now, the domain $D$ decomposes into a product $D_1 \times \ldots \times D_g$ of irreducible symmetric domains $D_\nu$, $\nu = 1, \ldots, g$. Here $g = [\mathbb{F} : \mathbb{Q}]$ with the notations of § 1. We denote accordingly by $z = (z_\nu)_{\nu=1}^g$ the elements of $D$. Let $A = A_z$ satisfy the conditions of (*). Then $S = S(L, \Phi, \varrho) = S(L, \Phi_0, \varrho)$ for any $\Phi_0$ in the equivalence class of $\Phi$. Let $\Phi_0$ be the representative built up from the absolutely irreducible representations of $L$ which is given in [Shi1], § 2.2, p. 156/157. In particular, $\Phi_0(L) \subset M_n(\bar{\mathbb{Q}})$ and any other complex representation $\Phi$ in its equivalence class with $\Phi(L) \subset M_n(\bar{\mathbb{Q}})$ is conjugate in $M_n(\bar{\mathbb{Q}})$ to $\Phi_0$. The components $z_\nu$ of $z$ are matrices which are obtained from the period lattice $\Lambda$ of $A$ and from a polarization on $A$ in a way described by the proof of Thm. 1 in [Shi1]. We recall briefly the main steps of this construction as far as they concern our proof. Shimura starts with lattice vectors $b_1, \ldots, b_m \in \Lambda_{\mathbb{Q}}$ ($\mathfrak{r}_1, \ldots, \mathfrak{r}_m$ in his notation, p. 157) forming a basis of $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ as a $\Phi_0(L)$-module, hence $m = 2n/[L : \mathbb{Q}]$. The vectors $b_\varkappa$, the representation $\Phi_0$ and the module $\mathscr{M}$ determine

$$\Lambda = \left\{ \sum_{\varkappa=1}^{m} \Phi_0(a_\varkappa) b_\varkappa \,|\, (a_1, \ldots, a_m) \in \mathscr{M} \subset L^m \right\}.$$

Using complex conjugation and multiplication by matrices $W_\nu^{-1}$, $\nu = 1, \ldots, g$ ([Shi1], p. 160/161) he transforms

$$(b_1, \ldots, b_m) \quad \text{into} \quad \begin{cases} (U_1, V_1, \ldots, U_g, V_g) & \text{(Types I, II, III)} \\ \begin{pmatrix} \mathfrak{U}_1, \mathfrak{V}_1, \ldots, \mathfrak{U}_g, \mathfrak{V}_g \\ \mathfrak{W}_1, \mathfrak{Y}_1, \ldots, \mathfrak{W}_g, \mathfrak{Y}_g \end{pmatrix} & \text{(Type IV)} \end{cases}$$

where all $U_\nu$, $V_\nu$, $\mathfrak{U}_\nu$, $\mathfrak{V}_\nu$, $\mathfrak{W}_\nu$, $\mathfrak{Y}_\nu$ denote complex matrices. The component matrices on the right are not necessarily of the same size, but the total number of their coefficients is again $nm$. The $U_\nu$, $V_\nu$, $\mathfrak{U}_\nu$, $\mathfrak{V}_\nu$, depend linearly and $\mathfrak{W}_\nu$, $\mathfrak{Y}_\nu$ depend antilinearly on the $b_\mu$. If we replace all $\mathfrak{W}_\nu$, $\mathfrak{Y}_\nu$ by their complex conjugate matrices $\bar{\mathfrak{W}}_\nu =: \mathfrak{X}_\nu$, $\bar{\mathfrak{Y}}_\nu =: \mathfrak{Z}_\nu$ we obtain therefore a linear map (defined for arbitrary $b_\mu \in \mathbb{C}^n$, not only for bases of $\Lambda_{\mathbb{Q}}$)

$$B: \mathbb{C}^{nm} \to \mathbb{C}^{nm}: (b_1, \ldots, b_m) \mapsto \begin{cases} (U_1, V_1, \ldots, U_g, V_g) & \text{(Types I, II, III)} \\ \begin{pmatrix} \mathfrak{U}_1, \mathfrak{V}_1, \ldots, \mathfrak{U}_g, \mathfrak{V}_g \\ \mathfrak{X}_1, \mathfrak{Z}_1, \ldots, \mathfrak{X}_g, \mathfrak{Z}_g \end{pmatrix} & \text{(Type IV)}. \end{cases}$$

As Shimura's construction shows, the kernel of $B$ is $0 \in \mathbb{C}^{nm}$, hence $B$ is non-singular. Moreover, we can assume that $B$ is defined over $\bar{\mathbb{Q}}$ because the matrices $W_\nu$ can be chosen with algebraic coefficients. For the bases $b_1, \ldots, b_m$ in question, the following matrix quotients always exist and finally give the components of the point $z \in D = \prod_\nu D_\nu$ corresponding to $A$.

$$\left.\begin{array}{ll} z_\nu := V_\nu^{-1} U_\nu & \text{for Type I and II} \\ z_\nu := -V_\nu^{-1} U_\nu & \text{for Type III} \\ z_\nu := \mathfrak{U}_\nu^{-1} \mathfrak{V}_\nu = {}^t(\mathfrak{Z}_\nu^{-1} \mathfrak{X}_\nu) & \text{for Type IV} \end{array}\right\} \quad \text{for all } \nu = 1, \ldots, g.$$

For Type IV, our definition of $z_\nu$ differs from Shimura's by a complex conjugation. The equations for the $\mathfrak{U}_\nu$, $\mathfrak{V}_\nu$, $\mathfrak{X}_\nu$, $\mathfrak{Z}_\nu$ in this case are implied by Riemann's period relations. For the other types this consequence of Riemann's period relations is reflected in the symmetry properties $z_\nu = {}^t z_\nu$ (Types I and II) or $z_\nu = -{}^t z_\nu$ (Type III).

2. Now we assume in addition that $L = \mathrm{End}_0 A_z$, so that as $L$ is the endomorphism algebra of a simple abelian variety, $A_z$ itself is simple. As we supposed that $A_z$, $z \in D$, is defined over $\bar{\mathbb{Q}}$ we know from the remarks preceding Proposition 1 of §2 that there is an isomorphism of $T_{A_z}(\mathbb{C})$ with $\mathbb{C}^n$ such that the corresponding complex representation $\Phi$ satisfies $\Phi(L) \subset M_n(\bar{\mathbb{Q}})$. By a suitable choice of $\mathscr{M}$ we may assume that the $\Phi(L)$-basis $b_1, \ldots, b_m$ of $\Lambda_{\mathbb{Q}}$ in subsection 3.1 is the same as in Proposition 1, part 2), i.e. we may find $\gamma_1, \ldots, \gamma_m$ such that $b_i = \int_{\gamma_i} \underset{\sim}{\omega}$ for $\underset{\sim}{\omega} = {}^t(\omega_1, \ldots, \omega_n)$ with $\omega_1, \ldots, \omega_n$ any basis of the $\bar{\mathbb{Q}}$-vector space $H^0(A_z, \Omega_{\bar{\mathbb{Q}}})$. Then by Proposition 1, all coefficients of the matrix $(b_1, \ldots, b_m)$ are linearly independent over $\bar{\mathbb{Q}}$.

Since $B$ is nonsingular and defined over $\bar{\mathbb{Q}}$, the coefficients of $B((b_1, \ldots, b_m))$ are also linearly independent over $\bar{\mathbb{Q}}$. Hence no matrix quotient $z_\nu$ can have algebraic components only. (Note that in spite of $\dim D_\Phi > 0$, in the case of Type IV some individual $D_\nu$ may reduce to one point.)

3. Next we consider a simple abelian variety $A_z$ in $\Sigma(T, \mathscr{M}) \subset S(L, \Phi, \varrho)$ defined over $\bar{\mathbb{Q}}$ but with

$$L' = \mathrm{End}_0 A_z \supsetneqq L$$

and complex representation $\Phi'$ with $\Phi'|_L = \Phi$. Hence $A_z$ belongs as well to a subfamily $\Sigma'(T', \mathscr{M}') \subset \Sigma(T, \mathscr{M}) \subset S(L, \Phi, \varrho)$, and this subfamily $\Sigma' \subset S(L', \Phi', \varrho)$ is again parametrized by a symmetric domain $D_{\Phi'}$ of dimension $\leqq \dim D_\Phi$, hence $A_z = A_\tau$ for some $\tau \in D_{\Phi'}$. If $\dim D_{\Phi'} = 0$, we have an $A_z$ with complex multiplication, so we have nothing to show. If $\dim D_{\Phi'} > 0$ we know by the preceding part that there is a non-algebraic $\tau \in D_{\Phi'}$

corresponding to $A_z$. Here we show that the transcendence of $\tau$ implies the transcendence of $z$. This will follow by some useful properties of the natural modular embedding

$$\mu : D_{\Phi'} \to D_{\Phi} \quad \text{with} \quad \mu(\tau) = z$$

lifting to the parametrizing symmetric domains the natural inclusion of the isomorphism classes of $\Sigma(T', \mathcal{M}')$ into those of $\Sigma(T, \mathcal{M})$. The explicit construction will be given in 4. a) below.

4. Such modular embeddings have been studied long before in various special cases, mostly for purposes connected with modular forms and functions (see e.g. Shimura's embedding of the products of $\mathfrak{H}^3_{r,s}$-spaces into a Siegel upper half space $\mathfrak{H}^1_n$ [Shi3], (4.7)). Generalizing this approach we need three facts about $\mu$:

a) The modular embedding $\mu$ is a rational map. By Thm. 1 of [Shi1], $\tau \in D_{\Phi'}$ is an $h$-tuple $(\tau_1, \dots, \tau_h)$ of matrices. As illustrated by the example given below (the most complicated case which can occur), first one has to go back the way described in part 1 from $\tau \in D_{\Phi'}$ to a basis of a period lattice $\Lambda = \Lambda_\tau = \Lambda_z$ of $A_z$. The matrices $\tau_\nu$, together with unit matrices of appropriate size, can be arranged as blocks of a matrix $(c_1, \dots, c_k)$, $k = 2n/[L' : \mathbb{Q}]$, whose column vectors $c_\varkappa$ generate the period lattice $\Lambda_\tau$ of $A_\tau$ as described by [Shi1], (10), p. 157,

$$\Lambda_\tau = \left\{ \sum_{\varkappa=1}^{k} \Phi'(d_\varkappa) c_\varkappa \,|\, (d_1, \dots, d_k) \in \mathcal{M}' \subset (L')^k \right\}.$$

All coefficients of all vectors $b \in \Lambda_\tau$ are linear polynomials in the coefficients of $\tau$. Now $z$ depends on $\Lambda$ as described in part 1 of this paragraph: One has to choose $b_1, \dots, b_m \in \Lambda_{\mathbb{Q}}$ with $m = 2n/[L : \mathbb{Q}]$ such that

$$\Lambda = \left\{ \sum_{\lambda=1}^{m} \Phi(a_\lambda) b_\lambda \,|\, (a_1, \dots, a_m) \in \mathcal{M} \subset L^m \right\},$$

apply the linear map $B$ on $(b_1, \dots, b_m)$ and take matrix quotients. Since the coefficients of the $b_\lambda$ are linear polynomials in the coefficients of $\tau$, the map $\tau \mapsto z = \mu(\tau)$ is rational. The choice of $(b_1, \dots, b_m) = (b_1(\tau), \dots, b_m(\tau))$ is not unique, but we fixed $A_\tau$, $\mathcal{M}$, $T$ and $B$, hence the point $z$ is uniquely determined up to a birational transformation (in fact, up to an element of the modular group $\Gamma_\Sigma$ for the family $\Sigma(T, \mathcal{M})$). Therefore, $\mu(\tau)$ is uniquely determined up to elements of $\Gamma_\Sigma$ only. To see that a fixed choice of the functions $b_\lambda(\tau)$ give a well-defined rational map $\mu$ everywhere on $D_{\Phi'}$ we observe the following: Assume that for some $\tau = \tau_0$ the $b_1(\tau_0), \dots, b_m(\tau_0)$ fail to be a basis for $\Lambda_{\mathbb{Q}}$ over $L$. This can happen in a proper analytic subset of $D_{\Phi'}$ only. In the point $\tau_0$ there would exist another choice $b'_1(\tau), \dots, b'_m(\tau)$ giving a differnt rational map $\mu'(\tau)$ well-defined locally in $\tau_0$. In an open dense subset of $D_{\Phi'}$, the maps $\mu$ and $\mu'$ are regular and differ only by the composition with an element $\gamma \in \Sigma(T, \mathcal{M})$. Therefore, $\mu = \gamma \circ \mu'$ everywhere on $D_{\Phi'}$, and $\gamma$ is regular everywhere on $D_{\Phi}$, hence $\mu$ is regular in $\tau_0$ also.

**Example.** To illustrate the construction we consider the most general Type IV case of an endomorphism algebra $L' = \mathrm{End}_0 A_z$ where we identify $A_z \in \Sigma(T, \mathcal{M}) \subset S(L, \Phi, \varrho)$ with $A_\tau \in \Sigma(T', \mathcal{M}') \subset S(L', \Phi', \varrho)$, $\tau \in D_{\Phi'}$. In this case, $\tau$ is an $h$-tuple $(\tau_1, \dots, \tau_h)$ of matrices

$$\tau_\nu \in \mathfrak{H}^3_{r_\nu, s_\nu} \subset M_{r_\nu, s_\nu}(\mathbb{C})$$

where $h$ is the degree $[\mathbb{F}' : \mathbb{Q}]$ of the totally real field of fixed points of $\varrho$ in the center $\mathbb{K}'$ of $L'$ and where

$$r_\nu + s_\nu = kq \quad \text{for any} \quad \nu = 1, \ldots, h\,, \quad k = 2n/[L' : \mathbb{Q}]\,,$$

and

$$q^2 = [L' : \mathbb{K}'] \,.$$

The pairs $(r_\nu, s_\nu)$ are determined by the decomposition of $\Phi'$ into absolutely irreducible representations as explained in §1. Now define

$$Y_\nu := \begin{bmatrix} 1_{r_\nu} & \tau_\nu \\ {}^t\bar{\tau}_\nu & 1_{s_\nu} \end{bmatrix} \in M_{kq}(\mathbb{C})$$

(see [Shi1], (35), p. 162) where $1_t$ denotes the $t \times t$ identity matrix. In the case $r_\nu$ or $s_\nu = 0$ we get $Y_\nu = 1_{kq}$ this occurring when $\mathfrak{H}_{r_\nu, s_\nu}$ consists of one point. To reconstruct $A_\tau$ from the matrices $Y_\nu$, $\nu = 1, \ldots, h$, we can follow the proof of [Shi1], Thm. 1. The procedure is considerably simplified by assuming that $T'$ is normalized in the form given in [Shi1], 2.5, that is Shimura's matrices $W_\nu$ can assumed to be $= 1_{kq}$. The matrices occurring in [Shi1], (32), p. 161, can as well assumed to be $= 1_n$; both normalizations mean a (linear) change of analytic coordinates of $A_\tau$ only and do not change $\mathcal{M}'$ or the polarization. Therefore, (20) of [Shi1], p. 159, reads as

$$X_\nu = \begin{bmatrix} u^\nu_{11} \ldots u^\nu_{k1} & u^\nu_{12} \ldots u^\nu_{k2} & \ldots & u^\nu_{1q} \ldots u^\nu_{kq} \\ \bar{v}^\nu_{11} \ldots \bar{v}^\nu_{k1} & \bar{v}^\nu_{12} \ldots \bar{v}^\nu_{k2} & \ldots & \bar{v}^\nu_{1q} \ldots \bar{v}^\nu_{kq} \end{bmatrix} = \begin{bmatrix} 1_{r_\nu} & \tau_\nu \\ {}^t\bar{\tau}_\nu & 1_{s_\nu} \end{bmatrix}$$

where all $u^\nu_{ij} \in \mathbb{C}^{r_\nu}$ and all $v^\nu_{ij} \in \mathbb{C}^{s_\nu}$, $\nu = 1, \ldots, h$. We rearrange the coefficients of all these $X_\nu$ into column vectors $c_\varkappa \in \mathbb{C}^n$, $\varkappa = 1, \ldots, k = 2n/[L' : \mathbb{Q}]$ defining

$${}^tc^\nu_\varkappa := ({}^tu^\nu_{\varkappa 1} \ldots {}^tu^\nu_{\varkappa q}\, {}^tv^\nu_{\varkappa 1} \ldots {}^tv^\nu_{\varkappa q}) \quad \text{where } q \cdot (r_\nu + s_\nu) = kq^2\,,$$

$${}^tc_\varkappa := ({}^tc^1_\varkappa \ldots {}^tc^h_\varkappa) \quad \text{where } kq^2 h = n$$

(observe $kq^2h = \dfrac{2n}{[L' : \mathbb{Q}]} \cdot [L' : \mathbb{K}'] \cdot [\mathbb{F}' : \mathbb{Q}]$). By construction, all coefficients of all $c_\varkappa$, $\varkappa = 1, \ldots, k$, are constant or coefficients of $\tau$, hence all coefficients of all period vectors

$$b = \sum_{\varkappa = 1}^{k} \Phi'(d_\varkappa) c_\varkappa \in \Lambda_\tau$$

are linear polynomials in the coefficients of $\tau$. The same is true for the coefficients of the vectors in $\Lambda_\mathbb{Q}$ for $\Lambda = \Lambda_\tau = \Lambda_z$.

b) The modular embedding $\mu$ is a rational map defined over $\bar{\mathbb{Q}}$. Probably this property can be seen by a careful case-by-case analysis making explicit the idea of a), but we give a straightforward general argument as follows. In $D_{\Phi'}$ there is a dense subset of CM points all of which are algebraic. These points are applied by $\mu$ to CM points of $D_\Phi$ which are again algebraic. So we obtain sufficiently many linear equations over $\bar{\mathbb{Q}}$ determining the coefficients of $\mu$ by evaluating $\mu$ at special algebraic points.

c) The modular embedding $\mu$ preserves the dimension, i.e.

$$\dim \mu(D_{\Phi'}) = \dim D_{\Phi'} .$$

This is easily seen by considering the corresponding abelian varieties in the analytic family $\Sigma'$ parametrized by $D_{\Phi'}$. These are the same for images and preimages of $\mu$, and two points of $D_{\Phi'}$ (and $D_{\Phi}$ respectively) correspond to isomorphic abelian varieties if and only if the points belong to the same (discrete) orbit for the action of the respective modular group $\Gamma_{\Sigma'}$ and $\Gamma_{\Sigma}$ ([Shi1], Thm. 2).

Now the transcendence of $\tau \in D_{\Phi'}$ implies the transcendence of $z = \mu(\tau) \in D_{\Phi}$ as follows. If $z$ were algebraic, the action of $\mathrm{Gal}(\mathbb{C}/\bar{\mathbb{Q}})$ on $\tau$ would give a nondiscrete subset of $D_{\Phi'}$, mapped by the $\bar{\mathbb{Q}}$-rational map $\mu$ to the single image point $z$, in contradiction to part c) above.

5. Finally we consider the case where $A_z$ is defined over $\bar{\mathbb{Q}}$ and not simple. Then $A_z$ is isogenous to a product $A_1^{k_1} \times \ldots \times A_N^{k_N}$ with pairwise non-isogenous simple abelian varieties $A_\nu$, all defined over $\bar{\mathbb{Q}}$ (Lemma 2). We may in fact assume that $A_z$ is equal to this product because isogenies change $z$ only by birational maps defined over $\bar{\mathbb{Q}}$. Every $A_\nu$, $\nu = 1, \ldots, N$, is a member of some set $S(L_\nu, \Phi_\nu, \varrho_\nu)$ with equality $L_\nu = \mathrm{End}_0 A_\nu$ as in part 2 of the proof. Furthermore, we have

$$L \subset M_{k_1}(L_1) \times \ldots \times M_{k_N}(L_N) .$$

If all $A_\nu$ have complex multiplication, $A_z$ is of CM type. If not, at least one complex symmetric domain $D_{\Phi_\nu}$ parametrizing analytic families of elements of $S(L_\nu, \Phi_\nu, \varrho_\nu)$ has positive dimension, hence by part 2 there is a transcendental point $\tau_\nu \in D_{\Phi_\nu}$ corresponding to $A_\nu$. By a construction completely analogous to parts 3 and 4 a modular embedding

$$\mu : D_{\Phi_1} \times \ldots \times D_{\Phi_N} \to D_{\Phi}$$

shows that $z$ is also a transcendental point of $D_{\Phi}$.

## § 4. Values of arithmetic automorphic functions at algebraic arguments

Let $\Gamma$ be an arithmetic group acting properly discontinuously on a complex bounded symmetric domain $D$ with finite covolume. It is well known that $\Gamma \backslash D$ has a natural structure as a complex space and, if not compact, can be compactified to a complex space $\overline{\Gamma \backslash D}$ (see [BB]; different compactifications are possible and useful, but we are interested in the finite part $\Gamma \backslash D$ only). On $D$ one has $\Gamma$-automorphic forms $f_0, \ldots, f_N$ of equal weight such that the map

$$J := (f_0, \ldots, f_N) : D \to \mathbb{P}_N(\mathbb{C})$$

defines an embedding of $\Gamma \backslash D$ into a complex algebraic variety $V(\mathbb{C}) \subset \mathbb{P}_N(\mathbb{C})$ analytically isomorphic to $\overline{\Gamma \backslash D}$. By the work of Baily, Shimura, Miyake, Shih, Deligne, Borovoi and finally Milne [Mi] on the existence and the properties of canonical models it is known that $V(\mathbb{C})$ are the complex points of a projective algebraic variety $V = V_\Gamma$ ("*Shimura variety*") defined over a number field $S$. That is, by a suitable choice of $J$ we may assume that for any $g$ in the function field (defined over $S$) of $V$ we obtain a $\Gamma$-automorphic function on $D$

$$g \circ J \in S\left(\frac{f_1}{f_0}, \ldots, \frac{f_N}{f_0}\right).$$

The functions $\frac{f_1}{f_0}, \ldots, \frac{f_N}{f_0}$ generate over $\mathbb{C}$ the field of all $\Gamma$-automorphic functions on $D$, of transcendence degree $\dim D$ over $\mathbb{C}$. To formulate our transcendence results we better consider the intermediate field

$$K = K(D, \Gamma) := S\left(\frac{f_1}{f_0}, \ldots, \frac{f_N}{f_0}\right) \otimes_S \bar{\mathbb{Q}}$$

of "*arithmetic automorphic functions*" (defined over $\bar{\mathbb{Q}}$) for $\Gamma$ corresponding via $J$ to the function field for the algebraic points $V(\bar{\mathbb{Q}})$ of $V$. For example, Siegel modular functions (in the terminology of §§1 and 3, they arise in the case $L = \mathbb{Q}$, $D$ the Siegel upper half space $\mathfrak{H}_n^1$ of dimension $\frac{1}{2}\, n\,(n+1)$, with $\Gamma = Sp_{2n}\mathbb{Z}$) or Hilbert modular functions ($L = \mathbb{F}$ totally real of degree $g = n = \dim A_z$, $D_\Phi = \mathfrak{H}^g$ for the upper half plane $\mathfrak{H}$, with $\Gamma = SL_2\mathcal{O}$ for the ring of integers $\mathcal{O}$ of $\mathbb{F}$) are arithmetic if they are quotients of modular forms with algebraic Fourier coefficients. The field $K$ has transcendence degree $\dim D$ over $\bar{\mathbb{Q}}$, hence over $\mathbb{Q}$.

Now consider the special situation of a modular group $\Gamma = \Gamma_\Sigma$ acting on $D = D_\Phi$ associated to the family $\Sigma = \Sigma(T, \mathcal{M})$ of elements of $S(L, \Phi, \varrho)$ for suitable $T, \mathcal{M}$ as in §1, §3. For these groups and their arithmetic automorphic functions we have the following fact from the theory of canonical models, needed in a very coarse version only: The abelian variety $A_z \in \Sigma$ is defined over $\bar{\mathbb{Q}}$ if and only if all functions $f \in K$ defined at $z \in D$ take algebraic values $f(z)$ or equivalently, if and only if $J(z) \in V(\bar{\mathbb{Q}})$. Therefore the Main Theorem immediately implies

**Corollary 3.** *Let $D$ be a bounded complex symmetric domain parametrizing a family $\Sigma$ of abelian varieties with associated modular group $\Gamma$. At an algebraic point $z \in D$, all arithmetic $\Gamma$-automorphic functions $f \in K$ defined at $z$ take algebraic values $f(z)$ if and only if $z$ is a* CM *point.*

**Corollary 4.** *If under the same hypotheses $z$ is an algebraic non-*CM *point of $D$, then the image $J(z)$ is not an algebraic point of the Shimura variety $V$.*

As mentioned in the introduction, these corollaries cover all known transcendence results on arithmetic automorphic functions. However, there are some problems left open.

**Problem 1.** Let now $\Gamma$ be an arbitrary arithmetic group acting on a complex symmetric domain $D$ with finite covolume. There are cases where $D$ does not parametrize a family $\Sigma$ of elements of an $S(L, \Phi, \varrho)$. Can one also obtain results analogous to Corollary 3 for the field $K$ of arithmetic $\Gamma$-automorphic functions on $D$? It would be interesting to know if the theory of motives (see e.g. [Mi]) could give a wide generalization of Corollary 3.

The following remark gives an obvious partial solution of Problem 1. Alice Silverberg kindly pointed out to us that there are examples of symmetric domains $D$ only parametrizing proper subfamilies of elements of the families $S(L, \Phi, \varrho)$ considered in Cor. 3 and 4 (see e.g. [Mu], [Shi2]). With this in mind we make the following observations.

**Remark.** Let $D$ be a bounded symmetric domain and $\Gamma$ an arithmetic group acting discontinuously on $D$ with finite covolume. Suppose there exists a family $\Sigma$ of elements of $S(L, \Phi, \varrho)$ parametrized by the bounded symmetric domain $D_\Phi$ with corresponding modular group $\Gamma_\Sigma$ together with a rational injection defined over $\bar{\mathbb{Q}}$,

$$\mu : D \hookrightarrow D_\Phi$$

and a compatible group inclusion

$$h : \Gamma \hookrightarrow \Gamma_\Sigma$$

that is satisfying the "*modular embedding*" property

$$\mu(\gamma\tau) = h(\gamma)\mu(\tau)$$

for all $\gamma \in \Gamma$ and $\tau \in D$, hence inducing a quotient map

$$\bar{\mu} : \Gamma \backslash D \to \Gamma_\Sigma \backslash D_\Phi$$

which we assume to extend as a rational map $V_\Gamma \to V_{\Gamma_\Sigma}$ defined over $\bar{\mathbb{Q}}$ for the associated Shimura varieties. Let $K$ be the field of $\Gamma$-automorphic functions

$$\{f \circ \mu \mid f \in K(D_\Phi, \Gamma_\Sigma) \text{ meromorphic on } \mu(D)\}\,.$$

Then at any algebraic point $\tau \in D$, all functions $g \in K$ defined at $\tau$ take algebraic values if and only if $\mu(\tau)$ is a CM point of $D_\Phi$. We can extend this conclusion in two directions:

1) It remains true with $K$ replaced by a finite field extension of $K$ in the field of all $\Gamma$-automorphic functions on $D$.

2) It remains even true for some non-arithmetic groups $\Gamma$, where $\mu$ is only a holomorphic embedding instead of being $\bar{\mathbb{Q}}$-rational. The main results of [CW1], [CW2] say that such modular embeddings $\mu : D \hookrightarrow D_\Phi$, $h : \Gamma \hookrightarrow \Gamma_\Sigma$ exist also for non-arithmetic Fuchsian triangle groups and Picard-Terada-Mostow-Deligne groups. In all these cases, the quotient space can also be compactified to a projective algebraic variety $V_\Gamma$ defined over $\bar{\mathbb{Q}}$, and the quotient morphisms $\bar{\mu}$ constructed in [CW1], [CW2] are again defined over $\bar{\mathbb{Q}}$. The space $D_\Phi$ is a power of $D$, and the first component of $\mu$ is the identity. Therefore for an algebraic point $\tau \in D$ whose image $\mu(\tau)$ is not a CM point of $D_\Phi$ one component of $\mu(\tau) \in D_\Phi$ is algebraic, and at least in the triangle group case this is sufficient for the transcendence of $f(\mu(\tau))$ for $f \in K(D_\Phi, \Gamma_\Sigma)$, $f \circ \mu$ meromorphic and nonconstant. This follows from Cor. 5 of the next paragraph. We conjecture that this result extends also to the Picard-Terada-Mostow-Deligne groups, see Problem 4 after Cor. 5.

**Problem 2.** It would be even more interesting to avoid the detour on families of abelian varieties or motives and to prove Cor. 3 directly only using properties of automorphic functions. In the simplest case ($D = \mathfrak{H}$ the upper half plane parametrising the family of elliptic curves with $L = \mathbb{Q}$, $\Gamma = SL_2\mathbb{Z}$, $K = \bar{\mathbb{Q}}(j)$ for the classical $j$-function) this is known as Schneider's second problem. Even there, it is still unsolved. (The method proposed by Holzapfel [Ho] uses Wüstholz' analytic subgroup theorem for elliptic curves. The proof of the Wüstholz analytic subgroup theorem requires the exponential map and so in this case the Weierstraß $\wp$-function.)

Only in the case $\dim D = 1$ is Cor. 3/4 a result about transcendental values of an individual nonconstant arithmetic automorphic function. This special case implies also, that if $z = \mu(\tau)$ is an algebraic non-CM-point in the image of a modular embedding

$$\mu : D \to D_\Phi \quad \text{with} \quad \dim D = 1\,,$$

then all $f \in K(D_\Phi, \Gamma_\Sigma)$ with nonconstant $f \circ \mu \in K$ (see Remark) take at $z$ a transcendental value in the field $\{g(\tau) \mid g \in K\}$, whose transcendence degree is therefore 1 as $\dim D = 1$. To give an explicit example showing that the field of values $f(z)$ at such algebraic non-CM points may collapse to a field of transcendence degree 1, consider the subspace

$$\mathfrak{D} := \left\{ z = \begin{pmatrix} \tau_0 & 0 \\ 0 & \tau_2 \end{pmatrix} \middle|\, \tau_0 \text{ and } \tau_2 \in \mathfrak{H} \right\} \subset \mathfrak{H}_2^1$$

in the Siegel upper half space for $n = 2$. Using a result of Freitag ([Fr], Kap. III, § 1), there are three generating Siegel modular functions $f_1, f_2, f_3 \in K(\mathfrak{H}_2^1, Sp_4\mathbb{Z})$ whose restrictions on $\mathfrak{D}$ give

$$f_1|_\mathfrak{D} \equiv 0\,, \quad f_2|_\mathfrak{D} = g(\tau_0) + g(\tau_2)\,, \quad f_3|_\mathfrak{D} = \frac{1}{g(\tau_0)} + \frac{1}{g(\tau_2)}$$

where $g$ denotes the modular function

$$g(\tau) := G_6^2(\tau)/G_4^3(\tau)$$

and $G_k$ the normalized Eisenstein series of weight $k$ in one variable. It is easily seen that $g$ is arithmetic and for, say, $\tau_0$ and $\tau_2 \in \bar{\mathbb{Q}}$ with

$$[\mathbb{Q}(\tau_0) : \mathbb{Q}] = 2 \quad \text{and} \quad [\mathbb{Q}(\tau_2) : \mathbb{Q}] > 2$$

we obtain an algebraic non-CM point $z$ on $\mathfrak{D}$ because $g(\tau_2) \notin \bar{\mathbb{Q}}$. As $g(\tau_0) \in \bar{\mathbb{Q}}$, there are two algebraically independent functions in $K(\mathfrak{H}_2^1, Sp_4\mathbb{Z})$ with algebraic values at $z$. In the language of § 3, the corresponding $A_z$ is a direct product of an elliptic curve $E_{\tau_0}$ with CM and an elliptic curve $E_{\tau_2}$ without CM.

**Problem 3.** It would be very interesting to know if for every algebraic $z \in D_\Phi$ the transcendence degree of the moduli field of $A_z$ is precisely

$t := \mathrm{Min}\{\dim D \mid D$ a complex symmetric domain with a modular embedding
$\mu : D \to D_\Phi$ as defined in the Remark and with $z \in \mu(D)\}$.

A complementary question is related to Grothendieck's period conjecture: If the moduli field of $A_z$ is contained in $\bar{\mathbb{Q}}$, what is the connection between this minimal dimension $t$ and the transcendence degree of the field generated by the entries of $z$?

A sharpening of Corollary 3, but in a different direction, concerns Hilbert modular functions. There are several proofs, and we will give one of them in the next paragraph. We state the result already as

**Proposition 4.** *Let $\mathbb{F}$ be a totally real number field of degree $g = [\mathbb{F} : \mathbb{Q}]$, let $\mathcal{O}$ be its ring of integers, $SL_2\mathcal{O}$ the Hilbert modular group acting discontinuously on $\mathfrak{H}^g$, and $K$ the field of arithmetic Hilbert modular functions for $SL_2\mathcal{O}$. If $z = (z_1, \ldots, z_g) \in \mathfrak{H}^g$ is a non-*CM*-point with one algebraic coordinate $z_j$, then there is an arithmetic Hilbert modular function $f \in K$ with a transcendental value at $z$.*

## § 5. Abelian varieties with generalized complex multiplication

In this section we consider abelian varieties with the following property:

**Definition.** Let $\mathbb{K}$ be a CM field. We call $A$ an *abelian variety with generalized complex multiplication by* $\mathbb{K}$ if $\mathbb{K} \subset \mathrm{End}_0(A)$.

Let $m$ be the integer $2n/[\mathbb{K} : \mathbb{Q}]$ where $n = \dim A$. If $m = 1$, then $A$ is an abelian variety of CM type as defined in §1. In general, on choosing a polarisation of $A$ and an isomorphism of $T_A(\mathbb{C})$ with $\mathbb{C}^n$, the inclusion of $\mathbb{K}$ in $\mathrm{End}_0(A)$ determines a complex representation $\Phi$ of $\mathbb{K}$ of degree $n$ and hence we have an element of a family $\Sigma$ of members of $S(\mathbb{K}, \Phi, \varrho)$. (See [Shi3], §4 for full details. Notice that the set $S(\mathbb{K}, \Phi, \varrho)$ corresponds to the Type IV case with $L = \mathbb{K}$, see §1.) As usual we write simply $A \in \Sigma$. As explained in §1, the complex domain $D = D_\Phi$ parametrizing $\Sigma$ depends only on $\mathbb{K}$ and on the equivalence class of $\Phi$. To recapitulate in more detail, choose a complete set $R$ of representatives modulo complex conjugation of the field embeddings $\sigma : \mathbb{K} \to \mathbb{C}$. The space $H^0(A, \Omega)$ of differentials of the first kind splits into $[\mathbb{K} : \mathbb{Q}] = 2g$ eigenspaces under the induced action of $\mathbb{K}$, where $g$ is as in §1 the degree of the maximal totally real subfield of $\mathbb{K}$ over $\mathbb{Q}$, hence $mg = n$. For $\sigma \in R$ we write

$$V_\sigma := \{\omega \mid \omega \circ \lambda = \sigma(\lambda) \cdot \omega,\ \forall \lambda \in \mathbb{K}\}\,,$$

$$\bar{V}_\sigma := \{\omega \mid \omega \circ \lambda = \overline{\sigma(\lambda)} \cdot \omega,\ \forall \lambda \in \mathbb{K}\}\,.$$

By $\overline{\sigma(\lambda)}$ we mean the complex conjugate of $\sigma(\lambda)$, and we call $V_\sigma$ and $\bar{V}_\sigma$ eigenspaces with "*conjugated* $\mathbb{K}$*-action*". Let

$$r_\sigma := \dim V_\sigma \quad \text{and} \quad s_\sigma := \dim \bar{V}_\sigma\,,$$

then by [Shi1], 4.1, we have

$$r_\sigma + s_\sigma = m\,.$$

The equivalence class of the complex representation $\Phi$ and the parameter space $D_\Phi$ are uniquely determined by the eigenspace splitting of $H^0(A, \Omega)$. We have in fact

$$D_\Phi = \prod_{\sigma \in R} \mathfrak{H}^3_{r_\sigma, s_\sigma}$$

(compare with the definition given in § 1) of dimension $\sum_R r_\sigma s_\sigma$. Note that $r_\sigma$ or $s_\sigma$ may be 0 for some $\sigma \in R$. In this case $\mathfrak{H}^3_{r_\sigma, s_\sigma}$ consists of one point. Even $\sum_R r_\sigma s_\sigma = 0$ is possible, but then the abelian variety $A$ is automatically of CM type ([Shi1], Prop. 14). If

1) $\sum_R r_\sigma s_\sigma > 0$,

2) $r_\sigma = 0$ or $1$ for every $\sigma \in R$,

we let $R_1$ be the subset $\sigma \in R$ with $r_\sigma = \dim V_\sigma = 1$. Condition 1) excludes the CM case $m = 1$, and for $m = 2$ and 3, condition 2) is automatically satisfied for a suitably chosen $R$. Then the parametrizing domain will be a product

$$D_\Phi = \mathfrak{B}^d = (\mathfrak{H}^3_{1, m-1})^d$$

of $d$ complex balls of dimension $m - 1$ where $d$ is the number of one dimensional eigenspaces $V_\sigma$, $\sigma \in R_1$. In the case $m = 2$ we can of course replace $\mathfrak{B}^d$ by $\mathfrak{H}^d$, and among the cases where all $r_\sigma = s_\sigma = 1$ ($\Rightarrow d = g = [\mathbb{F} : \mathbb{Q}]$) occur the families whose modular groups $\Gamma_\Sigma$ are the Hilbert modular groups ([Shi1], Prop. 18, Remark 7).

Under these assumptions, we describe how $z \in \mathfrak{B}^d$ depends on the periods of $A_z \in \Sigma$ making explicit the general ideas already used in § 3. Choose a generator $\omega_\sigma$ of $V_\sigma$ for every $\sigma \in R_1$. If $A_z$ is defined over $\bar{\mathbb{Q}}$, we can even assume $\omega_\sigma$ to be in $H^0(A_z, \Omega_{\bar{\mathbb{Q}}})$, that is defined over $\bar{\mathbb{Q}}$. By the rational representation of $\mathbb{K}$, the $\mathbb{Q}$-vector space $H_1(A_z, \mathbb{Q})$ is a $\mathbb{K}$-module, hence there is a basis $\gamma_0, \gamma_1, \ldots, \gamma_{m-1} \in H_1(A_z, \mathbb{Z})$ of the $\mathbb{K}$-module $H_1(A_z, \mathbb{Q})$. This basis can be chosen such that the vector

$$z_\sigma := {}^t\!\left( \int_{\gamma_1} \omega_\sigma \Big/ \int_{\gamma_0} \omega_\sigma, \ldots, \int_{\gamma_{m-1}} \omega_\sigma \Big/ \int_{\gamma_0} \omega_\sigma \right)$$

is contained in the ball $\mathfrak{B} \subset \mathbb{C}^{m-1}$, and

$$z := (z_\sigma)_{\sigma \in R_1} \in \mathfrak{B}^d$$

is in fact the point in the parameter space $D_\Phi$ corresponding to $A_z$. By the Main Theorem we know that $A_z$ is of CM type if and only if $A_z$ is defined over $\bar{\mathbb{Q}}$ and all period quotients $\int_{\gamma_j} \omega_\sigma / \int_{\gamma_0} \omega_\sigma$, $\sigma \in R_1$ and $j = 1, \ldots, m-1$, are algebraic numbers whenever $\int_{\gamma_0} \omega_\sigma \neq 0$. But here we can give also a different and sometimes more useful condition:

**Proposition 5.** *Let $A$ be an abelian variety defined over $\bar{\mathbb{Q}}$ with generalized complex multiplication by $\mathbb{K}$. Then $A$ is of* CM *type if and only if there is a field embedding $\sigma : \mathbb{K} \to \mathbb{C}$ and a $\mathbb{K}$-basis $\omega^{(1)}, \ldots, \omega^{(m)}$ of the direct sum of eigenspaces $V_\sigma \oplus \bar{V}_\sigma \subset H^0(A, \Omega_{\bar{\mathbb{Q}}})$ such that for each $\mu = 1, \ldots, m$, the periods $\int_\gamma \omega^{(\mu)}$, $\gamma \in H_1(A, \mathbb{Z})$ generate a 1-dimensional vector space over $\bar{\mathbb{Q}}$ depending only on $\mu$.*

The proof of the "if" part is based on Corollary 1 of § 2 and on an argument of Bertrand ([Be1], § 1, Ex. 3) giving two alternative possibilities for $A$:

(i) $A$ has no proper abelian subvariety stable under the action of $\mathbb{K}$. Then $A$ is isogenous to a pure power $B^k$ of a simple abelian variety $B$. On the other hand the assumptions of the proposition guarantee that we can apply Cor. 1 of § 2, hence $A$ must contain a simple subvariety with complex multiplication. Up to isogeny, this must be $B$, so $A$ is of CM type.

(ii) $A$ has proper abelian subvarieties stable under the action of $\mathbb{K}$. Then $A$ is isogenous to a product $A_1 \times \ldots \times A_t$, each factor $A_\tau$ being defined over $\bar{\mathbb{Q}}$ with generalized complex multiplication by $\mathbb{K}$, but being without proper $\mathbb{K}$-stable abelian subvarieties. Each factor $A_\tau$ has dimension $\frac{1}{2} m_\tau[\mathbb{K} : \mathbb{Q}]$ for some positive integer $m_\tau$ with $\sum_{\tau=1}^{t} m_\tau = m$. The intersection

$$H^0(A_\tau, \Omega) \cap (V_\sigma \oplus \bar{V}_\sigma)$$

has dimension $m_\tau > 0$, so some $\omega^{(\mu)}$ has a nontrivial restriction to $A_\tau$. Therefore we can apply the first alternative to each $A_\tau$ showing again that $A$ is of CM type. We omit the "only if" part of the proof since it is very easy.

We have to point out that even if $A$ is of CM type, in general not every basis $\omega^{(1)}, \ldots, \omega^{(m)}$ of $V_\sigma \oplus \bar{V}_\sigma$ will satisfy the last condition of Proposition 5. But in the case $m = 2$ we can give a sharper version avoiding this problem. Namely, with the notations introduced above we have

**Proposition 6.** *Let $A$ be an abelian variety defined over $\bar{\mathbb{Q}}$ with generalized complex multiplication by $\mathbb{K}$ and* $\dim A = [\mathbb{K} : \mathbb{Q}]$. *Suppose that $\sum_R r_\sigma s_\sigma > 0$ and $r_\sigma = 0$ or $1$ for every $\sigma \in R$. Let $R_1$ be the subset of those $\sigma \in R$ with $r_\sigma = \dim V_\sigma = 1$. Then $A$ is of* CM *type if and only if there is one $\sigma \in R_1$ such that $\int_{\gamma_1} \omega_\sigma / \int_{\gamma_0} \omega_\sigma$ is an algebraic number.*

Again, the "only if" direction is trivial. For the "if" part, we use Bertrand's alternative as in the preceding proof. If $A$ has no proper abelian subvariety stable under the action of $\mathbb{K}$, the same arguments as before prove that $A$ is of CM type. But if now $A$ has a proper abelian subvariety $A_1$ stable under the action of $\mathbb{K}$, we know on the one hand $\dim A_1 < n$ and on the other hand that $[\mathbb{K} : \mathbb{Q}] = n$ divides $2 \dim A_1$. Hence

$$\dim A_1 = \frac{1}{2} n = \frac{1}{2} [\mathbb{K} : \mathbb{Q}],$$

which means that $A_1$ is of CM type. Up to isogeny, $A_1$ has a complement $A_2$ in $A$ also stable under the action of $\mathbb{K}$. By the same reasons of dimension, $A_2$ is of CM type also.

As indicated in the introduction of this paragraph, the cases with $m = 2$ and $d = g = \frac{1}{2} [\mathbb{K} : \mathbb{Q}] = [\mathbb{F} : \mathbb{Q}]$ include the families of abelian varieties whose modular groups are the Hilbert modular groups. Here, Proposition 6 means that if $A$ is defined over $\bar{\mathbb{Q}}$ and $A = A_z$, for $z = (z_1, \ldots, z_g) \in \mathfrak{H}^g$ where one of the $z_i$, $i = 1, \ldots, g$ is algebraic, then $A_z$ is of CM type. Therefore, Proposition 4, § 4, follows.

Another field of application are certain hypergeometric functions. These Appell-Lauricella functions $F_1$ (or $F_D$ in some notations) satisfy systems of partial linear differential equations for which a basis of solutions can be given as period integrals over the differentials

$$\omega := u^{-\mu_0}(u-1)^{-\mu_1} \prod_{j=2}^{N+1} (u-x_j)^{-\mu_j} du\,,$$

where the $N$ complex numbers $x_2, \ldots, x_{N+1}$ are the arguments and $\mu_0, \ldots, \mu_{N+1}, \mu_{N+2}$ are the parameters of $F_1$, the last $\mu_{N+2}$ being defined by the condition

$$\sum_{j=0}^{N+2} \mu_j = 2\,. \tag{2}$$

We will further assume that:

3) All $\mu_j$ are non-integral (to avoid logarithmic singularities),

4) All $\mu_j$ are rational numbers. Then the period integrals $\int \omega$ can be viewed as periods on a smooth projective algebraic curve $X = X(x_2, \ldots, x_{N+1})$ depending on the parameters $x_2, \ldots, x_{N+1}$. An affine plane model of $X$ is given by

$$w^k = u^{k\mu_0}(u-1)^{k\mu_1} \prod_{j=2}^{N+1} (u-x_j)^{k\mu_j} \tag{3}$$

where $k$ is the least common denominator of the $\mu_j$. On this model we have $\omega = \frac{du}{w}$.

5) All $x_j$ are $\neq 0, 1$ and pairwise different (to exclude singular points of the differential equations for $F_1$).

6) All $\mu_j < 1$, $j = 0, \ldots, N+2$. Then $\omega$ gives a differential of the first kind on $X$.

To explain the application of the Propositions 5 and 6 to the functions $F_1$, we have to reconsider the program of the previous papers [CW1], [CW2]. The main purpose was to construct a modular embedding of the – in general non-arithmetic – monodromy group of $F_1$ into a suitable modular group. To this end, an abelian subvariety $T = T(x_2, \ldots, x_{N+1})$ of $\operatorname{Jac} X$ turned out to be crucial. This $T$ is the common kernel

$$T := \bigcap_{d \mid k,\, d \neq k} \operatorname{Ker} m_d$$

of all canonical morphisms $m_d$ of $\operatorname{Jac} X$ onto the Jacobians of all curves $X_d$ for which $w^k$ in equation (3) is replaced by $w^d$, $d$ a proper divisor of $k$. One computes easily

$$n := \dim T = \frac{1}{2}(N+1)\varphi(k)$$

where $\varphi$ denotes the Euler function. Induced by the automorphism of (3) given by

$$u \mapsto u, \quad w \mapsto \zeta_k^{-1} w \quad \text{with} \quad \zeta_k = e^{2\pi i/k},$$

the cyclotomic field $\mathbb{K} = \mathbb{Q}(\zeta_k)$ is contained in $\mathrm{End}_0 T$. So $T$ has generalized complex multiplication by $\mathbb{K}$ with $m = N+1$ in the previous notation.

**Convention.** In the case $\mathbb{K} = \mathbb{Q}(\zeta_k)$ we identify the embedding $\sigma : \mathbb{K} \hookrightarrow \mathbb{C}$ with the residue class $\nu \in (\mathbb{Z}/k\mathbb{Z})^*$ if $\sigma(\zeta_k) = \zeta_k^\nu$. Then, $R$ is a set of representatives of $(\mathbb{Z}/k\mathbb{Z})^*$ $\bmod\{\pm 1 \bmod k\}$. We replace the eigenspace-notation $V_\sigma$, $\bar{V}_\sigma$ by $V_\nu$ and $\bar{V}_\nu = V_{-\nu} \subset H^0(T, \Omega)$ with the dimensions $r_\nu$ and $s_\nu = r_{-\nu}$.

For $T$, we have of course

$$r_\nu + s_\nu = r_\nu + r_{-\nu} = N+1 .$$

These dimensions $r_\nu$ can in fact be determined by a result of Chevalley and Weil [Che We]. It turns out that they depend only on the parameters $\mu_j$ by the formula

$$r_\nu = -1 + \sum_{j=0}^{N+2} \langle \nu\mu_j \rangle \tag{4}$$

where $\langle\alpha\rangle = \alpha - [\alpha]$ denotes the fractional part of $\alpha \in \mathbb{R}$. So all $T(x_2, \ldots, x_{N+1})$ belong to the same family $\Sigma$ of elements of $S(\mathbb{K}, \Phi, \varrho)$. The monodromy group $\Delta$ of $F_1(x_2, \ldots, x_{N+1})$ is contained in the modular group $\Gamma_\Sigma$ of $\Sigma$ in a natural way [CW1], [CW2]. We will make the additional assumptions 1) and 2) to be sure that $\Sigma$ is parametrized by a power $\mathfrak{B}^d$, $d > 0$, of the complex ball of dimension $m-1 = N$. Among these cases, the most interesting monodromy groups $\Delta$ occur, i.e. the Picard-Terada-Mostow-Deligne groups acting discontinuously already on one factor $\mathfrak{B}$ of $\mathfrak{B}^d$. Note that $\Delta$ is arithmetic if and only if $d = 1$, or in other words if the index $[\Gamma_\Sigma : \Delta]$ is finite. Note also that by the action of $\mathbb{K}$ on $T$ and on $\omega = \dfrac{du}{w}$ (see (3)) $\omega$ corresponds to an element of $V_1$. We normalize the situation by assuming $1 \in R_1$, i.e. $r_1 = 1$. By 6), (2) and (4), this means that we assume:

7) $\mu_j > 0$ for all $j = 0, \ldots, N+2$.

Finally note that $X$, $\mathrm{Jac}\,X$ and $T$ are defined over $\bar{\mathbb{Q}}$ if $x_2, \ldots, x_{N+2} \in \bar{\mathbb{Q}}$. For $\nu \in R_1$, the generators $\omega_\nu$ of $V_\nu \subset H^0(T, \Omega)$ correspond on $X$ to the differentials

$$u^{-\langle\nu\mu_0\rangle}(u-1)^{-\langle\nu\mu_1\rangle} \prod_{j=2}^{N+1} (u-x_j)^{-\langle\nu\mu_j\rangle} du .$$

In the case of $N = 1$ variables, the periods $\int \omega_\nu$ are solutions of some Gauss' hypergeometric differential equations in the variable $x = x_2$. For the cycles $\gamma_0$ and $\gamma_1$ used in Proposition 6, these solutions are in fact linearly independent over $\mathbb{C}$. Therefore, the quotients $\int_{\gamma_1} \omega_\nu / \int_{\gamma_0} \omega_\nu$ are Schwarz triangle functions, more precisely for hyperbolic triangles with angles

$$|\pi(1 - \langle\nu\mu_0\rangle - \langle\nu\mu_2\rangle)|, \quad |\pi(1 - \langle\nu\mu_0\rangle - \langle\nu\mu_1\rangle)| \quad \text{and} \quad |\pi(1 - \langle\nu\mu_1\rangle - \langle\nu\mu_2\rangle)| .$$

In conclusion, Proposition 6 implies therefore

**Corollary 5.** *Let x be an algebraic number* $\neq 0$ *or* 1. *Then the following three properties are equivalent.*

(i) *The abelian variety* $T(x)$ *is of* CM *type.*

(ii) *For all* $v \in R_1$, *the triangle functions* $z_v = \int_{\gamma_1} \omega_v / \int_{\gamma_0} \omega_v$ *take algebraic values at* $x$.

(iii) *For one* $v \in R_1$, *the triangle function* $z_v = \int_{\gamma_1} \omega_v / \int_{\gamma_0} \omega_v$ *takes an algebraic value at* $x$.

**Problem 4.** We conjecture that an analogous result should be true for the hypergeometric functions $F_1$ in more variables $x_2, \ldots, x_{N+1}$ – at least if the monodromy groups $\Delta$ belongs to the Picard-Terada-Mostow-Deligne groups. More precisely, we expect that for $x_2, \ldots, x_{N+1} \in \bar{\mathbb{Q}}$, the abelian variety $T(x_2, \ldots, x_{N+1})$ is of CM type if and only if there is one $v \in R_1$, e.g. $v = 1$, such that all periods

$$\int_{\gamma_0} \omega_v, \int_{\gamma_1} \omega_v, \ldots, \int_{\gamma_N} \omega_v$$

are algebraic multiples of each other. This conjecture is certainly true if $\Delta$ is an arithmetic group because $R_1 = \{1\}$ in this case, hence the Main Theorem applies. In the nonarithmetic cases, this conjecture might be true for reasons of dimension: The abelian varieties $T(x_2, \ldots, x_{N+1})$ form a subfamily of $\Sigma$ of complex dimension $N$. On the other hand, the only $T$ with the property in question but which might not be of CM type are isogenous by Bertrand's arguments to a nontrivial product $A \times B$ of abelian varieties $A$ and $B$, both with generalized CM by $\mathbb{K}$. But these products form a subfamily of $\Sigma$ of codimension $> N$. Unfortunately, a proof of the conjecture along these lines would need more specific information about the subfamily of $\Sigma$ formed by the varieties $T(x_2, \ldots, x_{N+1})$. So we have only the following consequence of Proposition 5.

**Corollary 6.** *Let* $x_2, \ldots, x_{N+1}$ *be algebraic numbers. Under the notations introduced above and the assumptions* 3) *to* 7), *the abelian variety* $T(x_2, \ldots, x_{N+1})$ *is of* CM *type if and only if there are basis differentials* $\omega^{(2)}, \ldots, \omega^{(N+1)}$ *of* $V_{-1} = \overline{V_1}$, *all defined over* $\bar{\mathbb{Q}}$, *such that* (*with* $\omega^{(1)} := \omega_1 = \omega \in V_1$) *for all* $j = 1, \ldots, N+1$, *the periods*

$$\int_{\gamma_0} \omega^{(j)}, \int_{\gamma_1} \omega^{(j)}, \ldots, \int_{\gamma_N} \omega^{(j)}$$

*are algebraic multiples of each other.*

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