FROBENIUS CONSTANTS FOR FAMILIES OF ELLIPTIC CURVES

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1. Introduction

In this paper we will consider complex analytic functions which arise in the following way. Take a family of elliptic curves with one parameter, e.g. the Legendre family $y^2 = x(x-1)(x-t)$, $t \neq 0, 1$. For this family there is a distinguished period integral which is analytic near $t = 1$, namely

$$\delta(t) = \int_t^1 \frac{dx}{\sqrt{x(x-1)(x-t)}}.$$

We now choose a path from $t = 1$ to another singular point $t = 0$, consider the analytic continuation of $\delta(t)$ along this path and take its Mellin transform. Up to a simple factor of s^2, this yields the function of our interest:

$$\kappa(s) = s^2 \int_0^1 t^{s-1} \delta(t) dt = \sum_{n \geq 0} \kappa_n s^n. \tag{1}$$

The coefficients $\kappa_0, \kappa_1, \dots$ of its power series expansion are called *Frobenius constants*. Their study was initiated in [3], where the authors consider families of algebraic surfaces arising in the context of mirror symmetry. The general definition of Frobenius constants and their basic properties can be found in [2]. These numbers are defined in terms of the differential equations satisfied by period integrals, so-called Picard–Fuchs differential equations of families of algebraic varieties. Period integrals of the Legendre family are annihilated by the hypergeometric differential operator

$$t(1-t)\frac{d^2}{dt^2} + (1-2t)\frac{d}{dt} - \frac{1}{4}. \tag{2}$$

Frobenius constants describe the monodromy of Frobenius solutions determined near one singularity along a path connecting it to another singularity, going around it and coming back. Formula (1) is a special case of the main result of [2], which connects Frobenius constants and expansion coefficients of generalized gamma functions. We will recall the definition of generalized gamma functions in Section 3. By [2, Corollary 31] Frobenius constants of families of algebraic varieties are periods in the sense of Kontsevich–Zagier, [5]. Frobenius constants are related to periods of limiting Hodge structures, see [2, 4], but their true nature is yet mysterious. For the moment we only know how to compute Frobenius constants for hypergeometric differential equations, see [2, Prop. 26]. In other cases one may hope to evaluate a few constants κ_n for small n.

Our paper deals with Frobenius constants for *stable* families of elliptic curves. A family is called stable if its singular fibres have only double points. Such families are known to have at least four singular fibres. Stable families with exactly four singular points were

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classified by Beauville, [1]. The respective Picard–Fuchs differential equations were found in [7], they are of the second order with four singular points. In Table 2 we will demonstrate results of numerical experiments in which some Frobenius constants κ_n appear to be $\mathbb{Q}$-linear combinations of products of zeta values of total weight n. Our data resembles the numerical evaluation of Frobenius constants for the Apéry family of K3 surfaces in [3]. In this paper we wanted to capture the simplest possible situation in which this curious phenomenon seems to take place.

Families of elliptic curves admit a natural parametrization by modular functions. Their period integrals then correspond to modular forms of weigth 1. In Sections 2 and 3 we will use this modular parametrization to express Frobenius constants as iterated integrals of modular forms in two different ways. The first expression given in Section 2 is rather straightforward, it uses the functional equation satisfied by generalized gamma functions. With this we will be able to evaluate Frobenius constants numerically. In Section 3 we will use a more subtle, homological expression for generalized gamma functions , which allows us to represent Frobenius constants κ_n as regularized iterated integrals and to evaluate them for small n. The main results of our paper are Theorems 3 and 4.

In the end of this introduction let us compute Frobenius constants for the Legendre family. This turns out to be a simple task. We first change the order of integration in the integral expression in (1) and then substitute $t = xu$:

$$\begin{aligned}\int_0^1\int_t^1 \frac{t^{s-1}dxdt}{\sqrt{x(1-x)(x-t)}} &= \int_0^1 x^{-1/2}(1-x)^{-1/2}\int_0^x t^{s-1}(x-t)^{-1/2}dtdx\\ &= \int_0^1 x^{-1/2}(1-x)^{-1/2}x^{s-1/2}\int_0^1 u^{s-1}(1-u)^{-1/2}dudx\\ &= \int_0^1 x^{s-1}(1-x)^{-1/2}dx\int_0^1 u^{s-1}(1-u)^{-1/2}du\\ &= \left(\frac{\Gamma(s)\Gamma(1/2)}{\Gamma(s+1/2)}\right)^2 = \pi\,\frac{\Gamma(s)^2}{\Gamma(s+\frac{1}{2})^2}.\end{aligned}$$

This yields

$$\kappa(s) = \pi\frac{s^2\,\Gamma(s)^2}{\Gamma(s+\frac{1}{2})^2} = 16^s\frac{\Gamma(1+s)^4}{\Gamma(1+2s)^2} = \exp\left(4\log(2)s + 2\sum_{k=2}^{\infty}\frac{\zeta(k)}{k}(2-2^k)(-s)^k\right),$$

where we first used the Legendre duplication formula for the classical gamma function and then the known power series expansion of the logarithm of the gamma function at 1. Therefore we have

$$\kappa_0 = 1,\ \kappa_1 = 4\log(2),\ \dots \tag{3}$$

As we mentioned above, these constants are related to the monodromy of Frobenius solutions. The classical Frobenius method gives the following standard solutions of the hypergeometric differential operator (2) near $t = 0$:

$$\phi_0(t) = \sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n^2}{n!^2}t^n,\quad \phi_1(t) = \log(t)\phi_0(t) + \sum_{n=1}^{\infty}\frac{(\frac{1}{2})_n^2}{n!^2}\left(\sum_{k=1}^{n}\frac{1}{k(k-\frac{1}{2})}\right)t^n.$$

The value $\kappa_1 = 4\log(2)$ in (3) means that the linear combination $\phi_1(t) - 4\log(2)\phi_0(t)$ is analytic at $t = 1$. This linear combination actually equals $-i\delta(t)$, see [8, (9)]. There are also higher Frobenius functions $\phi_2(t), \phi_3(t), \ldots$ which come as a natural extension of the Frobenius method, see [3] and [2, §3]. For every n the constant κ_n is determined by the fact that the linear combination $\phi_n(t) - \kappa_n\phi_0(t)$ extends analytically through $t = 1$.

2. Frobenius constants of Beauville's families

The objective of this paper is the computation of Frobenius constants for the six second order differential operators of the shape

$$L = \theta^2 - t\big(A\,\theta(\theta+1) + \lambda\big) + Bt^2\,(\theta+1)^2, \tag{4}$$

where $\theta = t\frac{d}{dt}$ and A, B, λ are certain integer numbers whose values are listed in Table 1. Denote by $\phi_0(t) = \sum_{m\geq 0} u_m t^m$ the unique power series solution to L normalized so that $u_0 = 1$. Observe that L has four singular points whenever

$$B \neq 0 \text{ and } A^2 \neq 4B. \tag{5}$$

Under assumption (5), the only known values (A, B, λ) for which $\phi_0(t) \in \mathbb{Z}[\![t]\!]$ are those listed in Table 1 along with the case **B** with $(A, B, \lambda) = (9, 27, 3)$ and the triples $(-A, B, -\lambda)$ in the cases **A**-**F**, see [10]. Each of these operators is a Picard–Fuchs operator of one of the Beauville's stable families of elliptic curves; the correspondence between them and Beauville's classification can be found in [10, §7] and references therein. We will not need the equations of those families. What we will use is the modular parametrization, which we will also borrow form [10]. Namely, for each case there exists a modular function $t(z)$ such that $f(z) = \phi_0(t(z))$ is a modular form of weight 1; here z is the variable in the upper halfplane. In all cases except **B** the value $c = t(0)$ is a singularity of the differential operator L. For brevity, we have excluded case **B** from our considerations.

The singularities of L are located at $t = 0, \infty$ and the roots of $1 - At + Bt^2 = 0$. One can check that each of them is a regular singular point with maximally unipotent local monodromy transformation. The local exponents at 0 and the roots of $1 - At + Bt^2 = 0$ are equal to 0, the local exponent at ∞ equals 1. In particular, near each singularity $t = c$ there is a unique up to constant multiple solution $\delta^{(c)}(t)$ to L which is analytic at c, and every other solution adds a constant multiple of $\delta^{(c)}(t)$ when going around this singular point. For example, one can take the above mentioned power series solution $\phi_0(t)$ $\delta^{(0)}(t)$.

Proposition 1. *In all cases listed in Table 1 the function defined for $Re(s) > 0$ by the integral*

$$\mathbf{\Gamma}(s) = \int_0^{i\infty} t(z)^{s-1} z f(z) dt(z) \tag{6}$$

satisfies the functional equation

$$s^2\mathbf{\Gamma}(s) = (As(s+1) + \lambda)\mathbf{\Gamma}(s+1) - B(s+1)^2\mathbf{\Gamma}(s+2). \tag{7}$$

Proof. One can check that in each case $zf(z)$ has a finite limit when $z \to 0$. Consider the singularity $c = t(0)$ of the respective differential operator L. It follows that $zf(z) = \delta^{(c)}(t(z))$ for a solution $\delta^{(c)}(t)$ of L which is analytic near $t = c$. We then have

$$\mathbf{\Gamma}(s) = \int_c^0 t^{s-1}\delta^{(c)}(t)dt. \tag{8}$$

TABLE 1. Modular parametrization of some Picard–Fuchs differential operators of Beauville's families of elliptic curves, [10].

Zagier's label	A	B	λ	Modular group G	Hauptmodul $t(z)$	$f(z)$	χ	$c = t(0)$
A	7	-8	2	$\Gamma_0(6)$	$\frac{1^3 6^9}{2^3 3^9}$	$\frac{2^1 3^6}{1^2 6^3}$	$\left(\frac{3}{\cdot}\right)$	1/8
C	10	9	3	$\Gamma_0(6)$	$\frac{1^4 6^8}{2^8 3^4}$	$\frac{2^6 3^1}{1^3 6^2}$	$\left(\frac{3}{\cdot}\right)$	1/9
F	17	72	6	$h^{-1}\Gamma_0(6)h$, $h = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$	$\frac{1^5 3^1 4^5 6^2 12^1}{2^{14}}$	$\frac{2^{15} 3^2 12^2}{1^6 4^6 6^5}$		1/9
E	12	32	4	$\Gamma_0(8)$	$\frac{1^4 4^2 8^4}{2^{10}}$	$\frac{2^{10}}{1^4 4^4}$	$\left(\frac{-4}{\cdot}\right)$	1/8
G	0	-16	0	$\Gamma_0(8)$	$\frac{2^4 8^8}{4^{12}}$	$\frac{4^{10}}{2^4 8^4}$	$\left(\frac{-4}{\cdot}\right)$	1/4
D	11	-1	3	$\Gamma_1(5)$	$q\prod_{n=1}^{\infty}(1-q^n)^{5\left(\frac{n}{5}\right)}$	$\left(t^{-1}\frac{5^5}{1^1}\right)^{1/2}$	1	$-\frac{11}{2}+\frac{5}{2}\sqrt{5}$

Next we consider the identity $\displaystyle\int_0^c t^{s-1}L(\delta^{(c)}(t))dt = 0$. Integrating by parts we obtain

$$s^2 \int_0^c t^{s-1}\delta^{(c)}(t)dt - A(s(s+1)+\lambda)\int_0^c t^s\delta^{(c)}(t)dt + B(s+1)^2\int_0^c t^{s+1}\delta^{(c)}(t)dt$$
$$= sc^s\delta^{(c)}(c)(1-Ac+Bc^2) - c^{s+1}\frac{d}{dt}\delta^{(c)}(t)\mid_{t=c}(1-Ac+Bc^2).$$

Since c is a root of $1-At+Bt^2$, the right-hand side vanishes. Thus we obtain the desired functional equation of $\mathbf{\Gamma}(s)$, namely (7). □

Note that, similarly to the classical gamma function, one can extend $\mathbf{\Gamma}(s)$ to a meromorphic function in the complex plane using the the above functional equation (7). The *Frobenius constants* that we want to compute are given by the expansion coefficients at $s=0$ of the function

$$\kappa(s) = 2\pi i\, s^2\,\mathbf{\Gamma}(s) = \sum_{n\geq 0}\kappa_n s^n. \tag{9}$$

Note that $t(z)$ and $f(z)$ take real values on the imaginary half-axis. Therefore it follows from (6) and (7) that $\kappa(s)$ is real-valued for $s\in\mathbb{R}$. It follows from (7) that

$$\begin{aligned}\kappa(s) &= 2\pi i(As(s+1)+\lambda)\int_0^{i\infty} t(z)^s z f(z)t'(z)dz \\ &\quad - 2\pi i B(s+1)^2\int_0^{i\infty} t(z)^{s+1} z f(z)t'(z)dz,\end{aligned} \tag{10}$$

TABLE 2. Numerical values of some Frobenius constants κ_n.

case	κ_0	κ_1	κ_2	κ_3	κ_4	κ_5
A	1	0	$-\frac{\pi^2}{6}$	$\zeta(3)$	$\frac{\pi^4}{80}$	$-\frac{7}{16}\zeta(5)-\frac{5}{16}\pi^2\zeta(3)$
C	1	0	$-\frac{\pi^2}{6}$	$\frac{2}{3}\zeta(3)$	*	*
D	1	0	$-\frac{7}{30}\pi^2$	$2\zeta(3)$	$\frac{\pi^4}{180}$	$\zeta(5)-\frac{\pi^2}{2}\zeta(3)$
E	1	0	$-\frac{\pi^2}{12}$	$-\frac{5}{4}\zeta(3)$	*	*
F	1	0	0	$-\frac{11}{3}\zeta(3)$	*	*
G	1	0	$-\frac{\pi^2}{12}$	$\frac{1}{2}\zeta(3)$	$-\frac{\pi^4}{720}$	$\frac{3}{8}\zeta(5)-\frac{\pi^2}{24}\zeta(3)$

where the expression in the right-hand side is well-defined for $Re(s) > -1$. In particular, we see that $\kappa(s)$ is analytic at $s = 0$.

We differentiate (10) in s and subsequently evaluate it at $s = 0$, which yields an expression for κ_n involving integrals

$$\int_0^{i\infty} \log^m(t(z))zf(z)t'(z)dz, \quad \int_0^{i\infty} t(z)\log^m(t(z))zf(z)t'(z)dz$$

with $m \le n$. We computed these integrals numerically in PARI/GP with precision of about 75 digits after comma. The respective numerical values of κ_n are listed in Table 2. Remarkably, we were often able to identify these numbers as $\mathbb{Q}$-linear combinations of products of zeta values of total weight n. In case **G** this fact can be actually proved for all n, see the argument in the end of this section. However in the cases marked by * in Table 2 such identification is impossible already for $n = 4$. These Frobenius constants must be periods of more complicated nature.

In the following sections we will be able to confirm the values of κ_0, κ_1 and κ_2 using a homological interpretation of the function $\mathbf{\Gamma}(s)$. To state our main results we will need the following concept of *regularization at infinity* for functions in the upeer halfplane.

Definition 2. *We say that a function $F : \mathcal{H} \to \mathbb{C}$ is* regularizable *as $z \to \infty$ if there exists a polynomial $P \in \mathbb{C}[z]$ such that*

$$F(z) = P(z) + o(|z|^{-N}), \quad \text{for all } N \ge 1$$

when $|z| \to \infty$. We define the regularized value of $F(z)$ at infinity as

$$[F(z)]_{reg} := P(0).$$

Theorem 3. *In all our six cases one has $\kappa_0 = 1$, $\kappa_1 = 0$ and*

$$\kappa_{2+m} = \frac{4\pi^2}{b\,m!}\left[\int_{z_0/(bz_0+1)}^{z_0} \log^m(t(z))g(z)dz\right]_{reg} \tag{11}$$

for all $m \ge 0$, where $g(z) = \frac{1}{2\pi i}\frac{t'(z)}{t(z)}f(z)$ is a modular form on G of weight 3 and character χ and b is the minimal positive integer such that $\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \in G$.

For convenience let us list the values of b in this theorem:

case	**A**	**C**	**D**	**E**	**F**	**G**
b	6	6	5	8	12	8

Theorem 3 will be proved in Section 3. In Section 4 we will evaluate the regularized integral for $m = 0$ and confirm the values of κ_2:

Theorem 4. *The numerical values of κ_2 given in Table 2 agree with formula* (11) *for $m = 0$.*

At the end let us remark that all Frobenius constants in case **G** can be evaluated using the fact that in this case $L = \theta^2 - 16t^2(\theta+1)^2$ is the change of variable $t \mapsto 16t^2$ in the hypergeometric differential equation (2). Therefore $\delta^{(c)}(t)$ in this case is a constant multiple of $\delta(16t^2)$ and by simple manipulation with integrals we find that

$$s^2 \int_0^1 t^{s-1}\delta(t)dt = 2s^2 16^s \int_0^{1/4} t^{2s-1}\delta(16t^2)dt = 16^s \kappa_{\mathbf{G}}(2s).$$

This function was computed at the end of the introductory section, and therefore we obtain

$$\kappa_{\mathbf{G}}(s) = \exp\left(2\sum_{k=2}^{\infty}\frac{\zeta(k)}{k}(2^{1-k}-1)(-s)^k\right) = 1 - \frac{\pi^2}{12}s^2 + \frac{\zeta(3)}{2}s^3 + \dots \tag{12}$$

3. Generalized gamma functions

For a differential operator L on $\mathbb{P}^1(\mathbb{C})$ with regular singularities, one can consider Mellin transforms of its solutions along closed loops, e.g.

$$\mathbf{\Gamma}_{(\sigma,\phi)}(s) = \int_\sigma t^{s-1}\phi(t)dt$$

The natural requirements here are that σ is a closed loop avoiding the singularities of L, $\phi(t)$ is a solution to $L\phi = 0$ having no monodromy along the loop σ. Moreover, we should require that t^s is single-valued along σ. This latter restriction is equivalent to the fact that σ is contractible in $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$. Such a choice of σ and ϕ may not always exist. More generally, one can attach gamma functions to homology classes with coefficients in the twisted local system of solutions of L, see [2, §1]. To describe such homology classes it is convenient to fix a regular point $p \in \mathbb{C}^*$ and consider the group $G = \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus S, p)$ of homotopy clases of loops based at p. Here S is a finite set of points consisting of $0, \infty$ and the singularities of L. Let V be the $\mathbb{C}$-vector space of solutions of L near p. Then V is a representation G by monodromy of solutions along loops. Consider the ring

$$R = \mathbb{C}[e^{\pm 2\pi i s}]$$

and fix some branch of t^s near the point $t = p$. Then R is a representation of G via the monodromy of t^s. An 1-cycle ξ for the homology of G in the twisted representation $\tilde{V} = V \otimes_{\mathbb{C}} \mathbb{C}[e^{\pm 2\pi i s}]$ is a finite sum

$$\xi = \sum_j \sigma_j \otimes \phi_j \otimes e^{2\pi i s n_j}$$

where $\sigma_j \in G$, $\phi_j \in V$, $n_j \in \mathbb{Z}$ and

$$\partial\xi = \sum_j \left(\sigma_j\left(\phi_j \otimes e^{2\pi i s n_j}\right) - \phi_j \otimes e^{2\pi i s n_j}\right) = 0.$$

The respective generalized gamma function is defined as

$$\mathbf{\Gamma}_\xi(s) = \sum_j e^{2\pi i s n_j} \int_{\sigma_j} t^{s-1}\phi_j(t)dt.$$

This is an entire function of s, which depends only on the homology class of ξ in $H_1(G, \tilde{V})$ and satisfies a functional equation, [2, Lemma 4 and Proposition 8]. For our differential operator (4) the functional equation that generalized gamma functions satisfy reads as (7).

Remark 5. *Let us mention that generalized gamma functions corresponding to a differential operator L form an R-module of finite rank,* [2, Proposition 5]. *This can be seen in the following way. Observe that $\tilde{V}$ is an R-module and the action of G is R-linear. Therefore the homology group $H_1(G, \tilde{V})$ is an R-module as well. The module of gamma functions is the quotient of $H_1(G, \tilde{V})$ by those homology classes whose gamma functions vanish.*

Though we will not use this fact, let us mention that the R-module of gamma functions of our differential operator (4) *has rank at most 2. The rank drops to 1 when $A = \lambda = 0$, in which case L comes from a hypergeometric differential operator by a change of variables. It was shown in* [2, §2] *that for hypergeometric differential operators the module of generalized gamma functions has rank 1.*

We will now consider generalized gamma functions in a situation when L is of order 2 and admits a modular parametrization. Namely, let $\mathcal{H}$ be the upper halfplane and $t : \mathcal{H} \to \mathbb{C}^*$ be a Hauptmodul for some genus zero subgroup $G \subset SL_2(\mathbb{Q})$. Let $S \subset \mathbb{P}^1(\mathbb{C})$ be the set of cuspidal values of $t(z)$. Let $f(z)$ be a modular form $f(z)$ of weight one on G with a character χ. This function satisfies

$$f\left(\frac{az+b}{cz+d}\right) = \chi(\sigma)(cz+d)f(z)$$

for any $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. In this situation there exists a differential operator L of order 2 on $\mathbb{P}^1(\mathbb{C}) \setminus S$ whose pull-back under $t(z)$ annihilates the 2-dimensional $\mathbb{C}$-vector space spanned by $f(z)$ and $zf(z)$, [11, §5.4].

We chose the base point $p \in \mathbb{P}^1(\mathbb{C}) \setminus S$ and identify the space of solutions of L near p with the space of polynomials $V = \mathbb{C}z + \mathbb{C}$. Here $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ acts on a polynomial $Q(z) \in V$ by sending it into the polynomials $\chi(\sigma)(cz+d)Q\left(\frac{az+b}{cz+d}\right)$.

To describe $\tilde{V}$, we need to twist by the monodromy of t^s. We assume that $t(z)$ has no zeroes in $\mathcal{H}$. Take any branch of $\log t(z)$, it is single-valued in $\mathcal{H}$. Then for $\sigma \in G$ we define

$$n(\sigma) = \frac{1}{2\pi i}\left(\log(t(\sigma z)) - \log t(z)\right) \in \mathbb{Z}.$$

This is an integer which is independent of z because $\mathcal{H}$ is connected. One can easily see that $n(\sigma_1\sigma_2) = n(\sigma_1) + n(\sigma_2)$. Then $\tilde{V}$ is the twist of V by the multiplicative character

$\gamma \mapsto e^{2\pi i s n(\gamma)} \in \mathbb{C}[e^{\pm 2\pi i s}]^\times$. We recall the notation $R = \mathbb{C}[e^{\pm 2\pi i s}]$ and identify $\tilde{V} \cong Rz + R$. The action of $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ on an element $Q \in \tilde{V}$ is given by formula

$$(Q|\sigma)(z) = e^{2\pi i s n(\sigma)} \chi(\sigma)(cz+d) Q\left(\frac{az+b}{cz+d}\right).$$

Our notation $Q|\sigma$ reflects the fact that this is a right action.

A homological 1-cycle with coefficients in $\tilde{V}$ is a finite sum $\xi = \sum_j \sigma_j \otimes Q_j$ satisfying the condition

$$\partial\xi = \sum_j (Q_j|\sigma_j - Q_j) = 0.$$

The respective generalized gamma function is given by

$$\mathbf{\Gamma}_\xi(s) = \sum_j \int_{z_0}^{\sigma_j z_0} t(z)^s Q_j(z) g(z) dz, \tag{13}$$

where

$$g(z) = \frac{1}{2\pi i} f(z) \frac{t'(z)}{t(z)} \tag{14}$$

is a modular form on G of weight 3 and character χ. Here $z_0 \in \mathcal{H}$ is an arbitrary point in $\mathcal{H}$, it is a preimage of the basepoint $p = t(z_0)$. Note that the above integral (13) is well-defined for any $s \in \mathbb{C}$, and therefore it defines an entire function $\mathbf{\Gamma}_\xi(s)$. The following properties are expected but worth to check.

Lemma 6. *The function $\mathbf{\Gamma}_\xi(s)$ defined by* (13) *is independent of $z_0 \in \mathcal{H}$. This function depends only on the class of ξ in the homology group $H_1(G, \tilde{V})$.*

Proof. To prove independence of z_0 we transform the difference

$$\begin{aligned} &\sum_j \left(\int_{z_0}^{\sigma_j z_0} t(z)^s Q_j(z) g(z) dz - \int_{z_1}^{\sigma_j z_1} t(z)^s Q_j(z) g(z) dz \right) \\ &= \sum_j \left(\int_{z_0}^{z_1} t(z)^s Q_j(z) g(z) dz - \int_{\sigma_j z_0}^{\sigma_j z_1} t(z)^s Q_j(z) g(z) dz \right) \end{aligned}$$

and perform the change of variable $z = \sigma_j u$ in the second integral in the brackets. Since $g(z)$ is a modular form of weight 3 on G with character χ, then for any $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $Q \in \tilde{V}$ one has

$$t(z)^s Q(z) g(z) dz|_{z=\sigma u} = t(u)^s (Q|\sigma)(u) g(u) du.$$

With this observation, the above difference becomes

$$= \int_{z_0}^{z_1} t(z)^s \sum_j \left(Q_j(z) - Q_j|\sigma_j(z)\right) g(z) dz = -\int_{z_0}^{z_1} t(z)^s (\partial\xi)(z) g(z) dz = 0.$$

Here we used the fact that $\partial\xi = 0$. For the second statement we evaluate $\mathbf{\Gamma}_\xi(s)$ for an 1-boundary

$$\xi = \partial\ ((\sigma_1, \sigma_2) \otimes Q) = \sigma_2 \otimes (Q|\sigma_1) - \sigma_1\sigma_2 \otimes Q + \sigma_1 \otimes Q.$$

One has

$$\begin{aligned}\int_{z_0}^{\sigma_1\sigma_2 z_0} t(z)^s Q(z) g(z) dz &= \int_{z_0}^{\sigma_1 z_0} t(z)^s Q(z) g(z) dz + \int_{\sigma_1 z_0}^{\sigma_1\sigma_2 z_0} t(z)^s Q(z) g(z) dz \\ &= \int_{z_0}^{\sigma_1 z_0} t(z)^s Q(z) g(z) dz + \int_{z_0}^{\sigma_2 z_0} t(z)^s (Q|\sigma_1)(z) g(z) dz,\end{aligned}$$

which precisely means that $\mathbf{\Gamma}_\xi(s) = 0$. □

We now turn to the proof of Theorem 3. From now on we assume that L is of the shape (4) with (A, B, λ), $t(z)$ and $f(z)$ listed in Table 1. Note that in all our cases $t(i\infty) = 0$. We consider the singularity of L given by the cuspidal value $c = t(0)$ and the direct path from 0 to c along the real line. It was noticed in [2, §2] that if we restrict gamma functions to a neighbourhood of this path in $\mathbb{P}^1(\mathbb{C}) \setminus S$, there will be essentially a unique such function to consider. This is the function $\mathbf{\Gamma}(s)$ that was introduced in formula (6). More precisely, we will show that

$$\mathbf{\Gamma}_\xi(s) = \frac{(e^{2\pi i s} - 1)^2}{2\pi i}\, \mathbf{\Gamma}(s), \tag{15}$$

where ξ is a homology 1-cycle whose class generates the quotient

$$H_1(\langle \sigma_0, \sigma_c \rangle, \tilde{V}) / H_1(\langle \sigma_c \rangle, \tilde{V}) \tag{16}$$

as a $\mathbb{C}[e^{\pm 2\pi i s}]$-module. Here by $\langle .. \rangle$ we denote the subgroup of G generated by the respective elements and σ_0, σ_c are elements of G which in t-plane correspond to the loops around 0 and c respectively. We take for σ_0 and σ_c the generators of stabilizers of the respective cusps $z = i\infty$ and $z = 0$ in G. In all our cases these generators are of the form

$$\sigma_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_c = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix},$$

where b is precisely the integer from Theorem 3. It is easy to see that gamma functions corresponding to homology classes in $H_1(\langle \sigma_c \rangle, \tilde{V})$ vanish.

Lemma 7. *The quotient* (16) *is a* $\mathbb{C}[e^{\pm 2\pi i s}]$*-module of rank 1 generated by the class of*

$$\xi = \sigma_0 \otimes Q_0 + \sigma_c \otimes Q_c.$$

with $Q_0 = (1 - e^{2\pi i s}) z + e^{2\pi i s}$, $Q_c = \frac{1}{b}(1 - e^{2\pi i s})^2$.

Proof. Let $Q_0(z) = \kappa z + \mu$ and $Q_c(z) = uz + v$, for some $\kappa, \mu, u, v \in \mathbb{C}[e^{\pm 2\pi i s}]$ such that $\sigma_0 \otimes Q_0 + \sigma_c \otimes Q_c$ is an arbitary element of $H_1(\langle \sigma_0, \sigma_c \rangle, \tilde{V})$. We can assume $u = 0$ because $\partial(\sigma_c \otimes z) = z - z = 0$ and thus $\sigma_c \otimes z \in H_1(\langle \sigma_c \rangle, \tilde{V})$. Therefore, we need $\kappa, t, v \in \mathbb{Q}[e^{\pm 2\pi i s}]$ such that

$$\begin{aligned} 0 = \partial(\sigma_0 \otimes Q_0 + \sigma_c \otimes Q_c) &= e^{2\pi i s}(\kappa(z+1) + \mu) - (\kappa z + \mu) + (bz+1)v - v \\ &= z(e^{2\pi i s}\kappa - \kappa + bv) + (e^{2\pi i s}(\kappa + \mu) - \mu). \end{aligned}$$

It gives $e^{2\pi i s}\kappa = \mu(1 - e^{2\pi i s})$ and $bv = \kappa(1 - e^{2\pi i s})$. Here $\mu \in \mathbb{C}[e^{\pm 2\pi i s}]$ can be chosen arbitrarily, which shows that the quotient module is of rank one. Taking the unit value $\mu = e^{2\pi i s}$ we obtain the polynomials Q_0 and Q_c given in the statement of the lemma. □

The gamma function corresponding to the homological cycle ξ in Lemma 7 is given by

$$\mathbf{\Gamma}_\xi(s) = \int_{z_0}^{z_0+1} t(z)^s Q_0(z) g(z) dz + \int_{z_0}^{z_0/(bz_0+1)} t(z)^s Q_c(z) g(z) dz. \tag{17}$$

To prove relation (15) we break each of the integrals into two parts as follows:

$$\begin{aligned}
\mathbf{\Gamma}_\xi(s) &= \int_{z_0}^{i\infty} Q_0(z) t(z)^s g(z) dz + \int_{i\infty}^{z_0+1} Q_0(z) t(z)^s g(z) dz \\
&\quad + \int_{z_0}^{0} Q_c(z) t(z)^s g(z) dz + \int_{0}^{z_0/(bz_0+1)} Q_c(z) t(z)^s g(z) dz \\
&= \int_{i\infty}^{z_0} \left(e^{2\pi i s} Q_0(z+1) - Q_0(z)\right) t(z)^s g(z) dz \\
&\qquad + \int_0^{z_0} \left((bz+1) Q_c(\frac{z}{bz+1}) - Q_c(z)\right) t(z)^s g(z) dz \\
&= -(1-e^{2\pi i s})^2 \int_{i\infty}^{z_0} z t(z)^s g(z) dz \\
&\qquad + \frac{1}{b}(1-e^{2\pi i s})^2 \int_0^{z_0} ((bz+1)-1) t(z)^s g(z) dz \\
&= (1-e^{2\pi i s})^2 \int_0^{i\infty} z t(z)^s g(z) dz = \frac{(1-e^{2\pi i s})^2}{2\pi i} \mathbf{\Gamma}(s).
\end{aligned}$$

To prove our theorem we need to compute the Taylor expansion at $s = 0$ of the function

$$\kappa(s) = \left(\frac{2\pi i s}{1 - e^{2\pi i s}}\right)^2 \Gamma_\xi(s).$$

Note that each of the two integrals in the right-hand side of (17) may depend on z_0 while their sum is independent of z_0. Consider the Taylor expansion of the first integral at $s = 0$. It turns out that that the coefficients of this expansion have polynomial asymptopics when $z_0 \to \infty$.

Lemma 8. *In each of our six cases one has*

$$\left[\int_{z_0}^{z_0+1} t(z)^s g(z) Q_0(z) dz\right]_{reg} = \left(\frac{1 - e^{2\pi i s}}{2\pi i s}\right)^2.$$

Here the integral is first expanded as a formal power series in s and then each term is regularized as a function of z_0.

Proof. We denote $q = e^{2\pi i z}$. After writing $\log t(z) = 2\pi i z + \sum_{k\ge1} c_k q^k$ and $g(z) = \sum_{n\ge0} a_n q^n$, we have

$$t(z)^s \cdot g(z) = \sum_{m\ge0} \frac{s^m}{m!} \left(2\pi i z + \sum_{k\ge1} c_k q^k\right)^m \left(\sum_{n\ge0} a_n q^n\right).$$

The term near each s^m a polynomial in z whose coefficients are convergent q-series. Obeserve that

$$\int_{z_0}^{z_0+1} z^k (\sum_{n\ge0} b_n q^n) dz = b_0 \int_{z_0}^{z_0+1} z^k dz + O(|z_0|^{-N}), \quad \text{for all } N \ge 1$$

and therefore

$$\left[\int_{z_0}^{z_0+1} z^k (\sum_{n\geq 0} b_n q^n) dz\right]_{reg} = b_0 \int_0^1 z^k dz = b_0/(k+1).$$

Hence, as equality of formal power series in s, we have

$$\left[\int_{z_0}^{z_0+1} t(z)^s g(z) Q_0(z) dz\right]_{reg} = a_0 \int_0^1 e^{2\pi i z s} Q_0(z) dz = a_0 \left(\frac{1-e^{2\pi i s}}{2\pi i s}\right)^2.$$

Observe that $g(z)$ in (14) has $a_0 = 1$ for all cases under consideration, which concludes the proof of the lemma. □

Proof of Theorem 3. By Lemma 6 the sum of integrals in (17) is independent of z_0. Using Lemma 8 for the first integral, we conclude that each coeffcient in the expansion of the second integral considered as a power series in s is regularizable. With the expression for Q_c from Lemma 7 we conclude that

$$\begin{aligned}\kappa(s) = \left(\frac{2\pi i s}{1-e^{2\pi i s}}\right)^2 \mathbf{\Gamma}_\xi(s) &= 1 - \frac{4\pi^2 s^2}{b}\left[\int_{z_0}^{z_0/(bz_0+1)} t(z)^s g(z) dz\right]_{reg} \\ &= 1 + \frac{4\pi^2}{b}\sum_{m\geq 0}\frac{s^{m+2}}{m!}\left[\int_{z_0/(bz_0+1)}^{z_0} \log^m(t(z)) g(z) dz\right]_{reg}.\end{aligned}$$

□

4. Evaluating regularized integrals

In this section we prove Theorem 4. What we need for this is to evaluate the regularized integral

$$\left[\int_{z_0}^{z_0/(bz_0+1)} g(z) dz\right]_{reg} \tag{18}$$

in each of the cases in Table 1. Here $g(z) = \frac{1}{2\pi i}\frac{t'(z)}{t(z)} f(z)$ is the weight 3 modular form on a modular group $G \subset SL_2(\mathbb{Z})$ with character χ and b is the minimal positive integer such that $\begin{pmatrix} 1 & 0 \\ b & 1\end{pmatrix} \in G$. In all our cases we can restrict G to the respective subgroup $\Gamma_1(N)$. Therefore we assume that $\chi = 1$, which will simplify the considerations in this section.

We first review some properties of integrals (18) and their relation to periods of modular forms. Let $g \in M_k(G)$ be a modular form of weight k. For any z_0 in the upper halfplane $\mathcal{H}$, consider the map $\psi_{z_0} : G \to V_{k-2}$ from G into the set $V_{k-2} \subset \mathbb{C}[X]$ of polynomials of degree $\leq k-2$ defined as

$$\psi_{z_0}(\gamma) = \int_{z_0}^{\gamma z_0} (X-z)^{k-2} g(z) dz. \tag{19}$$

The space V_{k-2} is a representation of G under the usual right action in weight $2-k$ which is given for $Q(X) \in V_{k-2}$ and $\gamma = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in G$ by $(Q|\gamma)(X) = (cX+d)^{k-2} Q(\frac{aX+b}{cX+d}) \in V_{k-2}$. We denote $\gamma Q = Q|\gamma^{-1}$ to turn it into the left action. The map (19) then satisfies the relation $\psi_{z_0}(\gamma_1\gamma_2) = \gamma_1\psi_{z_0}(\gamma_2) + \psi_{z_0}(\gamma_1)$, which precisely means that ψ_{z_0} is an 1-cocyle for G with coefficients in the representation V_{k-2}. Note that

$$\begin{aligned}\psi_{z_0}(\gamma) - \psi_{z_1}(\gamma) &= \int_{z_0}^{z_1} (X-z)^{k-2} g(z)dz - \int_{\gamma z_0}^{\gamma z_1} (X-z)^{k-2} g(z)dz \\ &= (1-\gamma)\int_{z_0}^{z_1} (X-z)^{k-2} g(z)dz\end{aligned}$$

is a coboundary, and therefore we conclude that the class of ψ_{z_0} in $H^1(G, V_{k-2})$ is independent of z_0. We denote this class by $[g] \in H^1(G, V_{k-2})$, it depends only on the modular form g.

Proposition 9. *When $Im(z_0) \to \infty$ we have $\psi_{z_0}(\gamma) = \sum_{i=0}^{k-1} P_{i,\gamma}(X) z_0^i + o(1)$ for some $P_{i,\gamma} \in V_{k-2}$ independent of z_0.*

Moreover, the maps $\gamma \mapsto P_{i,\gamma}(X)$ are coboundaries for $i > 0$ and $\gamma \mapsto P_{0,\gamma}(X)$ is an 1-cocycle whose class in $H^1(G, V_{k-2})$ is equal to $[g]$.

Proof. One can express the above integral $\psi_{z_0}(\gamma)$ like

$$\sum_{i=0}^{k-2}\left[(-1)^i \binom{k-2}{i} X^i\right] \int_{z_0}^{\gamma z_0} g(z) z^{k-2-i} dz.$$

For each i, we take a specific $\left(\frac{d}{dz}\right)^{k-1-i}$-th anti-derivative of $g(z)$. After considering those anti-derivatives in the above integral, we get $\psi_{z_0}(\gamma) = \sum_{i=0}^{k-1} P_{i,\gamma}(X) z_0^i + \sum_{i=0}^{k-1} P_{i,\gamma}(X) g(z_0)$. When $Im(z_0) \to \infty$, the first fact follows.

As we mentioned above, if we consider a $(k-1)$-th anti-derivative of $g(w)$ as $f(w) = \int_{z_0}^{w} (w-z)^{k-2} g(z)dz$. We denote the map $\varphi_g : G \to V_{k-2}$ as $\gamma \mapsto f(z) - (f\mid_{2-k}\gamma)(z)$. Since $\left(\frac{1}{2\pi i}\frac{d}{dz}\right)^{k-1}(f(z)|_{2-k}\gamma - f(z)) = g(z)|_k\gamma - g(z) = 0$, $\varphi_g(\gamma)$ is a polynomial of degree $\leq k-2$. It is possible to observe that $\varphi_g(\gamma) = \psi_{z_0}(\gamma^{-1})$ which leads to $[\varphi_g] = [g]$.

Moreover, if we consider a map $G \to H^1(G, V_{k-2})$ like $\gamma \mapsto P_{0,\gamma}(X)$. To see this as a 1 -cocycle in $H^1(G, V_{k-2})$, we observe $\varphi(\gamma_1\gamma_2) = P_{0,\gamma_1\gamma_2}(X)$ which is the constant term of $\psi_{z_0}(\gamma_1\gamma_2)$. Similar to above, we have the relation $\psi_{z_0}(\gamma_1\gamma_2) = \gamma_1\psi_{z_0}(\gamma_2) + \psi_{z_0}(\gamma_1)$. Hence $P_{0,\gamma_1\gamma_2}(X)$ is the constant term of $(\gamma_1\psi_{z_0}(\gamma_2) + \psi_{z_0}(\gamma_1))$ which is nothing but $\gamma_1 P_{0,\gamma_2}(X) + P_{0,\gamma_1}(X)$ which proves that $\gamma \mapsto P_{0,\gamma}(X)$ is a 1-cocycle. Along with the above observations of identification between two classes, it is clear that $[g] = [\gamma \mapsto P_{0,\gamma}(X)]$. Moreover, $\gamma \mapsto P_{i,\gamma}(X)$ are coboundaries for all $i > 0$ due to alternative definition of the class $[g]$. □

The above proposition shows that the coefficient in $\psi_{z_0}(\gamma)$ at every power of X is regularizable as $z_0 \to \infty$ in the sense of Definition 2. We denote

$$\psi_\infty(\gamma) := P_{0,\gamma}(X) = \sum_{i=0}^{k-2} (-1)^i \binom{k-2}{i} X^{k-2-i} \Big[\int_{z_0}^{\gamma z_0} z^i g(z)dz\Big]_{reg}.$$

Since $\psi_\infty : G \to V_{k-2}$ is an 1-cocycle, we have the following relations among regularized integrals:

$$\psi_\infty(\gamma_1\gamma_2) = \gamma_1\,\psi_\infty(\gamma_2) + \psi_\infty(\gamma_1), \qquad \gamma_1, \gamma_2 \in G. \tag{20}$$

TABLE 3. Expressions for $g(z)$ via the Eisenstein series.

Case	$g(z)$	b
A	$g_A(z) = 1 - q - 5q^2 - q^3 + 11q^4 + 24q^5 - \cdots$ $= -G_{3,\chi}(z) - 8G_{3,\chi}(2z)$, where $\chi = \left(\frac{3}{\cdot}\right)$ is the non-trivial Dirichlet character modulo 3	6
C	$g_C(z) = g_A(z)$	6
F	$g_F(z) = 1 + q - 5q^2 + q^3 + \cdots = g_A(z + \frac{1}{2})$	12
E	$g_E(z) = 1 - 4q^2 - 4q^4 + \cdots = -4G_{3,\chi}(2z)$ where $\chi = \left(\frac{-4}{\cdot}\right)$ is the non-trivial Dirichlet character modulo 4	8
G	$g_G(z) = g_E(z)$	8
D	$g_D(z) = 1 - 2q - 6q^2 + 7q^3 + 26q^4 + \ldots$ $= (-1 + \frac{i}{2})G_{3,\chi}(z) + (-1 - \frac{i}{2})G_{3,\overline{\chi}}(z)$, where χ is the odd Dirichlet character modulo 5 with $\chi(2) = i$	5

We now return to the computation of regularized integrals (18). For this purpose it will be convenient to write $g(z)$ using the standard Eisenstein series

$$G_{3,\chi}(z) = \frac{1}{2}L(\chi, -2) + \sum_{n \geq 1} \left(\sum_{m|n} \chi(m)m^2 \right) q^n \ \in M_3(\Gamma_0(N), \chi)$$

defined for an odd Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. These expressions are given in Table 3, where in all cases except **D** we use the Dirichlet character χ given in Table 1.

We denote

$$\tilde{g}(z) = \sum_{n \geq 0} \tilde{a}_n q^n := g(z + 1/b) = 1 + \sum_{n \geq 1} a_n \exp(2\pi i/b)^n q^n.$$

Note that from the above representation in terms of Eisenstein series it follows that $|\tilde{a}_n| = |a_n| = O(n^2)$ as $n \to \infty$ and therefore the twisted Dirichlet series $L(\tilde{g}, s) = \sum_{n \geq 1} \frac{\tilde{a}_n}{n^s}$ is convergent for $Re(s) > 3$. Below we will evaluate it explicitly in some of our cases, and from this expressions it will follow that $L(\tilde{g}, s)$ is holomorphic for $s \in \mathbb{C}$ except for $s = 3$ where it has a simple pole.

Lemma 10. *As* $|z_0| \to \infty$ *one has*

$$\int_{z_0}^{\frac{z_0}{bz_0+1}} g(z)dz = \frac{1}{b} - z_0 + \frac{L(\tilde{g},1)}{2\pi i} - \frac{b^2(bz_0+1)^2}{8\pi^3 i} Res_{s=3} L(\tilde{g},s) + o(1). \tag{21}$$

Proof. We will do the change of variables $z = w + \frac{1}{b}$. Denote $\tilde{g}^*(z) = \tilde{g}(z) - a_0 = \tilde{g}(z) - 1$. We then have

$$\begin{aligned}\int_{z_0}^{\frac{z_0}{bz_0+1}} g(z)dz = \int_{w_0}^{-\frac{1}{b(bw_0+2)}} \tilde{g}(w)dw &= -\frac{1}{b(bw_0+2)} - w_0 + \int_{w_0}^{-\frac{1}{b(bw_0+2)}} \tilde{g}^*(w)dw \\ &= -w_0 + \frac{1}{2\pi i} F\left(-\frac{1}{b(bw_0+2)}\right) + o(1), \text{ for } |w_0| \to \infty, \end{aligned} \tag{22}$$

where

$$F(z) = \sum_{n \geq 1} \frac{\tilde{a}_n}{n} q^n$$

is the function satisfying $\frac{1}{2\pi i}\frac{d}{dz}F = q\frac{d}{dq}F = \tilde{g}^*$. We will now show that

$$F(it) = L(\tilde{g},1) + \frac{1}{(2\pi t)^2} Res_{s=3} L(\tilde{g},s) + o(1), \ t \to 0+\,. \tag{23}$$

Formula (21) then follows from this and (22). Recall that the Dirichlet series for $L(\tilde{g},s)$ is convergent for $Re(s) > 3$. Therefore for $Re(s) > 2$ we have

$$\frac{\Gamma(s)}{(2\pi)^s} L(\tilde{g}, 1+s) = \frac{\Gamma(s)}{(2\pi)^s} \sum_{n\geq 1} \frac{\tilde{a}_n}{n^{s+1}} = \sum_{n \geq 1} \frac{\tilde{a}_n}{n} \int_0^\infty t^{s-1} e^{-2\pi n t} dt = \int_0^\infty t^{s-1} F(it) dt.$$

Since $|\Gamma(s)|$ is small when $|Im(s)|$ is large and $L(\tilde{g}, 1+s)$ is uniformly bounded when $Re(s) \geq 2 + \varepsilon$ with any fixed $\varepsilon > 0$, we can apply the Mellin inversion theorem. One recovers

$$F(it) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)}{(2\pi t)^s} L(\tilde{g}, 1+s) ds$$

with any real $c > 2$. If we integrated over the vertical line with $c < 0$, then the integral would be $o(1)$ when $t \to 0_+$. This is because $|t^{-s}| = t^{-Re(s)}$. We can move the line of integration from $c > 2$ to $-1 < c < 0$. The integrand has two poles between those lines: $s = 0$ is a pole for $\Gamma(s)$ and $s = 2$ is a pole for $L(\tilde{g}, 1+s)$. Therefore one needs to add the residues of the integrand at those points, which yields formula (23). □

Lemma 11. *In case* **A** *one has*

$$\begin{aligned} L(\tilde{g},s) = &-\frac{1}{2}(1-3^{1-s})(1-2^{2-s})(1+2^{1-s})\zeta(s)L(\chi,s-2) \\ &- i\frac{\sqrt{3}}{2}(1-3^{2-s})(1-2^{1-s})(1+2^{2-s})\zeta(s-2)L(\chi,s). \end{aligned} \tag{24}$$

For this L-function we have

$$L(\tilde{g},1) = 0 \quad \textit{and} \quad Res_{s=3} L(\tilde{g},3) = -i\frac{\pi^3}{54}.$$

Proof. From the expression for $g(x)$ in Table 3 we find that its Fourier coefficients are given by $a_n = \sum_{d|n}(-1)^d\chi(d)d^2$. Denote $\alpha = \exp(2\pi i/6)$,

$$A_k(s) = \sum_{m\geq 1}\frac{\alpha^{km}}{m^s} \qquad \textit{and} \qquad B_k(s) = \sum_{m\geq 0}\frac{1}{(6m+k)^s}$$

for $k \geq 1$. For the twisted L-function we then have

$$\begin{aligned} L(\tilde{g}, s) &= \sum_{d\geq 1}\sum_{m\geq 1}\frac{(-1)^d\chi(d)d^2\alpha^{md}}{(md)^s} = \sum_{d\geq 1}\frac{(-1)^d\chi(d)}{d^{s-2}}A_d(s) \\ &= -B_1(s-2)A_1(s) - B_2(s-2)A_2(s) + B_4(s-2)A_4(s) + B_5(s-2)A_5(s). \end{aligned} \tag{25}$$

We now list some observations about the A's:

$$A_1(s) + A_4(s) = \sum_{m=1}^{\infty}\frac{\alpha^m}{m^s} + \sum_{m=1}^{\infty}(-1)^m\frac{\alpha^m}{m^s} = 2\sum_{2|m}\frac{\alpha^m}{m^s} = 2^{1-s}A_2(s),$$

$$A_2(s) + A_5(s) = \sum_{m=1}^{\infty}\frac{\alpha^{2m}}{m^s} + \sum_{m=1}^{\infty}(-1)^m\frac{\alpha^{2m}}{m^s} = 2\sum_{2|m}\frac{\alpha^{2m}}{m^s} = 2^{1-s}A_4(s).$$

In (25) we will substitute $A_1(s) = 2^{1-s}A_2(s) - A_4(s)$ and $A_5(s) = 2^{1-s}A_4(s) - A_2(s)$. Since $\alpha^2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $\alpha^4 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$ and $\alpha^6 = 1$, we obtain that

$$\begin{aligned} A_2(s) &= -\frac{1}{2}\sum_{3\nmid m}\frac{1}{m^s} + i\frac{\sqrt{3}}{2}L(\chi, s) + \sum_{3|m}\frac{1}{m^s} \\ &= -\frac{1}{2}(1-3^{-s})\zeta(s) + i\frac{\sqrt{3}}{2}L(\chi, s) + 3^{-s}\zeta(s) \\ &= -\frac{1}{2}(1-3^{1-s})\zeta(s) + i\frac{\sqrt{3}}{2}L(\chi, s) \end{aligned}$$

and similarly

$$A_4(s) = -\frac{1}{2}(1-3^{1-s})\zeta(s) - i\frac{\sqrt{3}}{2}L(\chi, s).$$

Substituting these expressions into (25) we obtain

$$\begin{aligned} L(\tilde{g}, s) &= -B_1(s-2)\left(2^{1-s}A_2(s) - A_4(s)\right) - B_2(s-2)A_2(s) \\ &\quad + B_4(s-2)A_4(s) + B_5(s-2)\left(2^{1-s}A_4(s) - A_2(s)\right) \\ &= \left(-2^{-1-s}B_1 - B_2 - B_5\right)(s-2)\,A_2(s) \\ &\quad + \left(B_1 + B_4 + 2^{-1-s}B_5\right)(s-2)\,A_4(s) \\ &= \left(-(1-2^{-1-s})(B_1 - B_5) + (B_2 - B_4)\right)(s-2) \times \frac{1}{2}(1-3^{1-s})\zeta(s) \\ &= \left(-(1+2^{-1-s})(B_1 + B_5) - (B_2 + B_4)\right)(s-2) \times i\frac{\sqrt{3}}{2}L(\chi, s). \end{aligned} \tag{26}$$

It remains to do a few observations about B's. One can immediately see that

$$B_2(s) - B_4(s) = 2^{1-s}L(\chi, s), \qquad B_1(s) - B_5(s) = (1+2^{-s})L(\chi, s).$$

Substituting these formulas into the first row of our latest expression for $L(\tilde{g}, s)$ yields the first row in formula (24). Now observe that $B_1(s)+B_3(s)+B_5(s) = \sum_{m\geq 1, 2\nmid m} m^{-s} = (1-2^{-s})\zeta(s)$ and $B_3(s) = 3^{-s}\sum_{m\geq 1,2\nmid m} m^{-s} = 3^{-s}(1-2^{-s})\zeta(s)$. This implies

$$B_1(s) + B_5(s) = (1-2^{-s})(1-3^{-s})\zeta(s). \tag{27}$$

Subtracting this from $B_1(s) + B_2(s) + B_4(s) + B_5(s) = \sum_{m\geq 1,3\nmid m} m^{-s} = (1-3^{-s})\zeta(s)$ we obtain

$$B_2(s) + B_4(s) = 2^{-s}(1-3^{-s})\zeta(s). \tag{28}$$

Substituting (27) and (28) into the last row in (26) we then obtain the second row in formula (24). This completes the proof of (24).

Since $L(\chi, s)$ is entire and $\zeta(s)$ is holomorphic everywhere except for a simple pole at $s = 1$, we find that the first row in (24) is entire and the second row has a unique simple pole at $s = 3$. To evaluate the twisted L-function at $s = 1$ we note that $L(\chi, -1) = 0$, see [9, Theorem 4.2]. Therefore there is no contribution from the first row of (24), and the term $(1-2^{1-s})$ in the second row causes it to vanish at $s = 1$. Therefore $L(\tilde{g}, 1) = 0$. Since $Res_{s=1}\zeta(s) = 1$, we compute that

$$\begin{aligned} Res_{s=3}L(\tilde{g}, s) &= -i\frac{\sqrt{3}}{2}(1-3^{-1})(1-2^{-2})(1+2^{-1})L(\chi, 3) \\ &= -i\frac{\sqrt{3}}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{3}{2} \times \frac{4\pi^3\sqrt{3}}{243} = -i\frac{\pi^3}{54}. \end{aligned}$$

Here we substituted the value $L(\chi, 3) = \frac{4\pi^3\sqrt{3}}{243}$ which follows from $L(\chi, -2) = -\frac{2}{9}$ and the functional equation of $L(\chi, s)$ with respect to $s \leftrightarrow 1 - s$, see [9, Chapter 4]. □

Lemma 12. *For case* **D**, *with* $\alpha = \exp(2\pi i/5)$ *and character* χ *mentioned in Table 3, one has*

$$\begin{aligned} L(\tilde{g}, s) = {} & L(2Re(\chi) + Im(\chi), s-2)\, 5^{-s}\zeta(s) \\ & + \frac{1}{2}Re(\alpha)\Big[- L(2Re(\chi) + Im(\chi), s-2)\zeta(s)(1-5^{-s}) \\ & \qquad - L(2Re(\chi) - Im(\chi), s-2)\, L(\chi^2, s)\Big] \\ & + \frac{1}{2}Re(\alpha^2)\Big[- L(2Re(\chi) + Im(\chi), s-2)\zeta(s)(1-5^{-s}) \\ & \qquad + L(2Re(\chi) - Im(\chi), s-2)\, L(\chi^2, s)\Big] \\ & + \frac{i}{2}Im(\alpha)\Big[- L(2Re(\chi) - Im(\chi), s)\, \zeta(s-2)(1-5^{2-s}) \\ & \qquad - L(2Re(\chi) + Im(\chi), s)\, L(\chi^2, s-2)\Big] \\ & + \frac{i}{2}Im(\alpha^2)\Big[- L(Re(\chi) + 2Im(\chi), s)\, \zeta(s-2)(1-5^{2-s}) \\ & \qquad + L(Re(\chi) - 2Im(\chi), s)\, L(\chi^2, s-2)\Big]. \end{aligned} \tag{29}$$

For this L-function, we have

$$L(\tilde{g},1) = -\frac{\pi}{60}i \text{ and } Res_{s=3}L(\tilde{g},s) = -\frac{4\pi^3}{125}i$$

where χ is the character modulo 5 mentioned in Table 3.

Proof. From the expression for $g(z)$ in Table 3 we find that its Fourier coefficients are given by $a_n = \sum_{d|n}\tau(d)d^2$, where $\tau = -2Re(\chi) - Im(\chi)$. Denote

$$A_k(s) = \sum_{m\geq 1}\frac{\alpha^{km}}{m^s} \qquad \text{and} \qquad B_k(s) = \sum_{m\geq 0}\frac{1}{(5m+k)^s}$$

for $k \geq 1$. For the twisted L-function, we then have

$$\begin{aligned} L(\tilde{g},s) &= \sum_{d\geq 1}\sum_{m\geq 1}\frac{\tau(d)d^2\alpha^{md}}{(md)^s} = \sum_{d\geq 1}\frac{\tau(d)}{d^{s-2}}A_d(s) \\ &= -2B_1(s-2)A_1(s) + 2B_4(s-2)A_4(s) - B_2(s-2)A_2(s) + B_3(s-2)A_3(s). \end{aligned} \tag{30}$$

We express each of $B_1(s), B_2(s), B_3(s)$ and $B_4(s)$ can be expressed as a linear combination of $L(\chi^j, s)$ for $j = 1,2,3,4$:

$$\begin{aligned} B_1(s) &= \frac{1}{4}L(\chi,s) + \frac{1}{4}L(\chi^2,s) + \frac{1}{4}L(\chi^3,s) + \frac{1}{4}L(\chi^4,s), \\ B_2(s) &= -\frac{i}{4}L(\chi,s) - \frac{1}{4}L(\chi^2,s) + \frac{i}{4}L(\chi^3,s) + \frac{1}{4}L(\chi^4,s), \\ B_3(s) &= \frac{i}{4}L(\chi,s) - \frac{1}{4}L(\chi^2,s) - \frac{i}{4}L(\chi^3,s) + \frac{1}{4}L(\chi^4,s), \\ B_4(s) &= -\frac{1}{4}L(\chi,s) + \frac{1}{4}L(\chi^2,s) - \frac{1}{4}L(\chi^3,s) + \frac{1}{4}L(\chi^4,s). \end{aligned}$$

Note that $\chi^3 = \overline{\chi}$ and $L(\chi^4,s) = (1-5^{-s})\zeta(s)$. Functions $A_k(s)$ can be also written as such linear combinations using the above formulas for B's and the fact that $A_k = \sum_{j=1}^{4}\alpha^{kj}B_j + 5^{-s}\zeta(s)$. With the help of computer, we substituted all these expressions into (30) and found that the result equals (29).

To determine $L(\tilde{g},1)$, we first note that $L(\chi,-1) = 0$. Thus the first three rows of (29) vanish at $s = 1$. Moreover, note that $L(\chi,1) = \left(\frac{2\pi}{25} + \frac{6\pi}{25}i\right)\sin\left(\frac{4\pi}{5}\right) + \left(\frac{6\pi}{25} - \frac{2\pi}{25}i\right)\sin\left(\frac{2\pi}{5}\right)$ and $L(\chi^2,-1) = -2/5$. After putting all these values in the main statement, we get the residue first required L-value.

Similar to Lemma 11, for computing $Res_{s=3}L(\tilde{g},s)$, we note that $Res_{s=1}\zeta(s) = 1$ and $L(\chi,3) = \left(\frac{8\pi^3}{625} + \frac{16\pi^3}{625}i\right)\sin\left(\frac{4\pi}{5}\right) + \left(\frac{16\pi^3}{625} - \frac{8\pi^3}{625}i\right)\sin\left(\frac{2\pi}{5}\right)$. Substituting this value into (29), we evaluated the residue.

□

Proof of Theorem 4.

(Case **A** and **C**:) By Proposition 11, we have the values of $L(\tilde{g},1)$ and $Res_{s=3}L(\tilde{g},3)$. Putting these values in Proposition 10, we get

$$\int_{z_0}^{\frac{z_0}{6z_0+1}} g(z)dz = \frac{1}{4} + 3z_0^2 + o(1) \text{ as } z_0 \to i\infty. \tag{31}$$

Therefore, we conclude $\left[\int_{z_0}^{\frac{z_0}{6z_0+1}} g(z)dz\right]_{reg} = \frac{1}{4}$ and by Theorem 3 the value of κ_2 equals $-\frac{\pi^2}{6}$.

(Case **D**:) Following the similar line of argument as above, in this case,

$$\left[\int_{z_0}^{\frac{z_0}{5z_0+1}} g_D(z)dz\right]_{reg} = \frac{7}{24}$$

and by Theorem 3 the value of κ_2 equals $-\frac{7}{30}\pi^2$.

(Case **F**:) Performing the change of variable $z = u - \frac{1}{2}$ we obtain

$$\int_{z_0}^{\frac{z_0}{12z_0+1}} g_F(z)dz = \int_{u_0}^{\frac{7u_0-3}{12u_0-5}} g(z)dz, \tag{32}$$

where $g(z) = g_A(z)$ is the modular form on $\Gamma_1(6)$ as in the previous considered case. We will use Proposition 9 to find the asymptotics of this integral as $u_0 \to \infty$. Let $G = \Gamma_1(6)$. One can check that this group is generated by the parabolic elements

$$\sigma_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_c = \begin{pmatrix} 1 & 0 \\ 6 & 1 \end{pmatrix}, \qquad \sigma_{c'} = \begin{pmatrix} 7 & -3 \\ 12 & -5 \end{pmatrix}$$

fixing the cusps $\lambda = \infty, 0$ and $\frac{1}{2}$ respectively. Following the notation of Proposition 9 for $\gamma \in G$ we write

$$\psi_{z_0}(\gamma) = \int_{z_0}^{\gamma z_0} (X - z)g(z)dz = \sum_{i=0}^{2} P_{i,\gamma}(X) z_0^i + o(1), \quad z_0 \to \infty.$$

Since

$$\psi_{z_0}(\sigma_0) = \int_{z_0}^{z_0+1} (X - z)g(z)dz = X - \frac{(z_0+1)^2 - z_0^2}{2} + o(1), \quad z_0 \to \infty$$

we have

$$P_{0,\sigma_0}(X) = X - \frac{1}{2}, \quad P_{1,\sigma_0}(X) = -1, \quad P_{2,\sigma_0}(X) = 0. \tag{33}$$

We note that if $\sigma \in G$ is a parabolic element fixing a cusp $\lambda \in \mathbb{Q}$ and the form $(z-\lambda)g(z)dz$ is bounded near $z = \lambda$ then for any $z_0 \in \mathcal{H}$ one has

$$\int_{z_0}^{\sigma z_0} (z - \lambda)g(z)dz = 0.$$

Indeed, this integral is independent of z_0 because the form is σ-invariant and we send $z_0 \to \lambda$ to show that the integral vanishes. Therefore in the discussed case one has

$$\psi_{z_0}(\sigma) = \int_{z_0}^{\sigma z_0} (X - z)g(z)dz = (X - \lambda) \int_{z_0}^{\sigma z_0} g(z)dz. \tag{34}$$

Boundedness of $(z - \lambda)g(z)dz$ near $z = \lambda$ depends only on the G-orbit of λ. There are 4 G-orbits of cusps for our group, they are represented by $\lambda = \infty, 0, \frac{1}{2}$ and $\frac{1}{3}$, and we have boundedness for the latter three orbits. Formulas (31) and (34) for $\sigma = \sigma_c$ yield

$$P_{0,\sigma_c}(X) = \frac{1}{4}X, \quad P_{1,\sigma_c}(X) = 0, \quad P_{2,\sigma_c}(X) = 3X. \tag{35}$$

Recall that for $i = 1,2$ the maps $\gamma \mapsto P_{i,\gamma}(X)$ are 1-coboundaries, that is there exist polynomials $Q_1(X), Q_2(X)$ of degree ≤ 1 such that $P_{i,\gamma} = (1-\gamma)Q_i$ for $i = 1,2$. From (33) and (35) we recover that $Q_1(X) = -X$ and $Q_2(X) = \frac{1}{2}$. With this we obtain $P_{1,\sigma_{c'}}(X) = (1-\sigma_{c'})Q_1 = -6X+3$ and $P_{2,\sigma_{c'}}(X) = (1-\sigma_{c'})Q_2 = 6X-3$. In the view of (34) for $\sigma = \sigma_{c'}$ we have

$$P_{0,\sigma_{c'}}(X) = \beta(X-\frac{1}{2}), \quad P_{1,\sigma_{c'}}(X) = -6(X-\frac{1}{2}), \quad P_{2,\sigma_{c'}}(X) = 6(X-\frac{1}{2}), \tag{36}$$

with the unknown constant $\beta = \left[\int_{z_0}^{\sigma_{c'}z_0} g(z)dz\right]_{reg}$. We will now determine β using the 1-cocycle property of $\gamma \mapsto P_{0,\gamma}(X) = \psi_\infty(\gamma)$. To this end we note that $\sigma = \sigma_c\sigma_0^{-1}\sigma_{c'} = \begin{pmatrix} -5 & 2 \\ -18 & 7 \end{pmatrix}$ is a parabolic element fixing the cusp $\lambda = \frac{1}{3}$. Using the cocycle relation (20) we obtain

$$\begin{aligned}\psi_\infty(\sigma) &= \psi_\infty(\sigma_c) + \sigma_c\psi_\infty(\sigma_0^{-1}) + \sigma_c\sigma_0^{-1}\psi_\infty(\sigma_{c'}) \\ &= \psi_\infty(\sigma_c) + (\psi_\infty(\sigma_{c'}) - \psi_\infty(\sigma_0))\,|\sigma_0\sigma_c^{-1} \\ &= \frac{X}{4} + (\beta-1)(X-\frac{1}{2})\Big|\begin{pmatrix} -5 & 1 \\ -6 & 1 \end{pmatrix} = (-2\beta+\frac{9}{4})X + \frac{1}{2}(\beta-1).\end{aligned}$$

In the view of (34) this polynomial should be a multiple of $(X-\frac{1}{3})$, from which we find that $\beta = \frac{3}{2}$. With this value of β, formula (36) yields

$$\int_{z_0}^{\sigma_{c'}z_0} g(z)dz = \frac{3}{2} - 6z_0 + 6z_0^2 + o(1), \quad z_0 \to \infty.$$

Using this asymptotics in the right-hand side of (32) we find that $\left[\int_{z_0}^{\frac{z_0}{12z_0+1}} g_F(z)dz\right]_{reg} = 0.$ It follows that $\kappa_2 = 0$.

(Cases **E** and **G**:) Since $g_E = g_G$ and $b = 8$ in both cases, the values of κ_2 should be also equal. For the case **G** we already computed in (12) that $\kappa_2 = -\frac{\pi^2}{12}$. ∎

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