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Picard-Fuchs Equations, Hauptmoduls and Integrable Systems

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Abstract

The Schwarzian equations satisfied by certain Hauptmoduls (i.e., uniformizing functions for Riemann surfaces of genus zero) are derived from the Picard–Fuchs equations for families of elliptic curves and associated surfaces. The inhomogeneous Picard–Fuchs equations associated to elliptic integrals with varying endpoints are derived and used to determine solutions of equations that are algebraically related to a class of Painlevé VI equations.

1. Differential equations for modular functions

There are a number of differential systems whose general solutions may be expressed in terms of modular functions. An example is the Darboux-Halphen system [Ha],

$$\begin{aligned} w_1' &= w_1(w_2+w_3) - w_2w_3 \\ w_2' &= w_2(w_1+w_3) - w_1w_3 \\ w_3' &= w_3(w_1+w_2) - w_1w_2, \end{aligned} \tag{1.1}$$

which was already thoroughly studied in the last century, but recently has recurred in several applications in mathematical physics [GP, AH, CAC, T, Hi, Du]. Its symmetrization under the symmetric group in three variables S_3 gives the Chazy equation [C]

$$W''' = 2WW'' - 3W'^2, \tag{1.2}$$

whose solutions are related to those of (1.1) by

$$W = 2(w_1+w_2+w_3). \tag{1.3}$$

The physical contexts in which these equations have appeared include:

1. The dynamics of magnetic monopoles pairs [AH].
2. Homogeneous, $SO(3)$ invariant solutions of self–dual Einstein equations [GP, T, Hi].
3. Solutions of the WDVV equations in topological field theory [Du].

The general solution to (1.1) was determined in 1881 by Halphen [Ha] and Brioschi [Br] in terms of the *elliptic modular function*

$$\lambda(\tau) = k^2(\tau), \tag{1.4}$$

where $k(\tau)$ is the elliptic modulus, viewed as a function of the ratio τ of the elliptic periods. A particular solution to (1.1) is given by

$$\begin{aligned} w_1 &:= \frac{1}{2}\frac{d}{d\tau}\ln\frac{\lambda'}{\lambda}, \quad w_2 := \frac{1}{2}\frac{d}{d\tau}\ln\frac{\lambda'}{(\lambda-1)}, \\ w_3 &:= \frac{1}{2}\frac{d}{d\tau}\ln\frac{\lambda'}{\lambda(\lambda-1)}. \end{aligned} \tag{1.5}$$

The general solution is obtained by composing this with a general Möbius transformation

$$T : \tau \to \frac{a\tau+b}{c\tau+d} \equiv T(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{C}). \tag{1.6}$$

Similarly, a particular solution to (1.2) is given by

$$W := 2\sigma_1 = \frac{1}{2}\frac{d}{d\tau}\ln\frac{J'^6}{J^4(J-1)^3}, \tag{1.7}$$

where J denotes Klein's J–function

$$J = \frac{4(\lambda^2-\lambda+1)^3}{27\lambda^2(\lambda-1)^2}, \tag{1.8}$$

and the general solution is again obtained by composing J with a Möbius transformation (1.6).

The key element in deriving these solutions is to first note that the period integrals, when viewed as functions of λ, are hypergeometric functions

$$\begin{aligned} K_1 &= \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-\lambda t^2)}} = \frac{\pi}{2}F(\tfrac{1}{2},\tfrac{1}{2};1;\lambda) \\ iK_2 &= \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-(1-\lambda)t^2)}} = \frac{i\pi}{2}F(\tfrac{1}{2},\tfrac{1}{2};1;1-\lambda), \end{aligned} \tag{1.9}$$

and hence satisfy the hypergeometric equation of Legendre type

$$\lambda(1-\lambda)\frac{d^2y}{d\lambda^2} + (1-2\lambda)\frac{dy}{d\lambda} - \frac{1}{4}y = 0. \tag{1.10}$$

Since $\lambda(\tau)$ is the inverse of the function $\tau(\lambda)$ given by the ratio of the periods, it follows that it satisfies the Schwarzian equation

$$\{\lambda, \tau\} + \frac{\lambda^2 - \lambda + 1}{2\lambda^2(1-\lambda)^2}\lambda'^2 = 0, \tag{1.11}$$

where

$$\{f, \tau\} := \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2, \qquad (f' := \frac{df}{d\tau}). \tag{1.12}$$

is the Schwarzian derivative [GS]. This implies that (1.5) defines a solution of (1.1) and, by the $SL(2, \mathbf{C})$ invariance of the system, composition with the Möbius transformations (1.6) gives the general solution. Similarly, the fact that (1.7) determines the general solution to (1.2) follows from the Schwarzian equation satisfied by J

$$\{J, \tau\} + \frac{36J^2 - 41J + 32}{72J^2(J-1)^2}J'^2 = 0, \tag{1.13}$$

which is obtained from (1.11) by composing with (1.8). This latter Schwarzian equation equation is analogously related to the hypergeometric equation

$$J(1-J)\frac{d^2y}{dJ^2} + \left(\frac{2}{3} - \frac{7}{6}J\right)\frac{dy}{dJ} - \frac{1}{144}y = 0. \tag{1.14}$$

Thus J is the inverse of the function obtained taking the ratio of two linearly independent solutions of (1.14).

The hypergeometric equations (1.10) and (1.14) may both be viewed as examples of *Picard–Fuchs* equations for elliptic pencils; that is, as Fuchsian differential equations determining the variation of elliptic integrals over an affine parametric family of elliptic curves. In each of these cases, the relevant inverse function is a modular function. This follows from the fact that the projectivized monodromy groups for the associated hypergeometric equations (1.10) and (1.14) are both commensurable with the modular group $\Gamma := PSL(2, \mathbf{Z})$. For the case (1.10), this gives the principal congruence subgroup $\Gamma(2)$, which is the automorphism group of the modular functions $\lambda(\tau)$, while for (1.14), it is the full modular group, the automorphism group of $J(\tau)$.

There is another sense in which Picard–Fuchs equations may be related to nonlinear equations of interest in mathematical physics; namely, the class of isomonodromic deformation equations, such as the family of Painlevé equations $P_{VI}(\alpha, \beta, \gamma, \delta)$

$$\begin{aligned} X'' =& \frac{1}{2}\left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-t}\right)(X')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{X-t}\right)X' \\ &+ \frac{X(X-1)(X-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t^2}{X^2} + \frac{\gamma(t-1)}{(X-1)^2} + \frac{\delta t(t-1)}{(X-t)^2}\right). \end{aligned} \tag{1.15}$$

It was shown in the work of Hitchin [Hi], Dubrovin [Du] and Mazzocco [M] that solutions to particular cases of (1.15) with special values of the parameters $(\alpha, \beta, \gamma, \delta)$ could be given in terms of solutions to (1.1). For example, for the case $P_{VI}(\alpha = 2, \beta = 0, \gamma = 0, \delta = \frac{1}{2})$, we have the one–parameter family of Chazy solutions given by

$$X = \frac{(w_2 w_3 - w_1 w_2 - w_1 w_3)^2}{4 w_1 w_2 w_3 (w_1 - w_3)}, \tag{1.16}$$

where the independent parameter is taken as

$$t := \frac{w_1 - w_3}{w_2 - w_3} = \lambda, \tag{1.17}$$

and (w_1, w_2, w_3) is a general solution to (1.1). (Only one of the three $SL(2, \mathbf{C})$ parameters introduced by (1.6) is effective.) More generally, a sequence of Chazy–type solutions for parameter values $\alpha = \frac{1}{2}(2\mu - 1)^2, \mu + \frac{1}{2} \in \mathbf{Z} \backslash 1$ are derived in [M] by application of discrete symmetry transformations.

Another class of solutions to (1.15) for the case $(\alpha = 0, \beta = 0, \gamma = 0, \delta = \frac{1}{2})$ was already known to Picard [Pi], who expressed them in terms of elliptic integrals with variable end-points. This allows us to relate this case to an *inhomogeneous* Picard–Fuchs equation. Namely, consider the 1–parameter family of elliptic curves

$$y^2 = 4x(x-1)(x-\lambda), \tag{1.18}$$

and corresponding period integrals:

$$K_1 = \oint_\infty^1 \frac{dx}{y}, \qquad K_2 = \oint_\infty^0 \frac{dx}{y}. \tag{1.19}$$

If the elliptic integral with *varying* (λ–dependent) endpoints

$$K := \int_\infty^{(X(\lambda), Y(\lambda))} \frac{dx}{y} \tag{1.20}$$

is required to satisfy the same hypergeometric equation as do the period integrals K_1, K_2, i.e., if it is set equal to a linear combination

$$K = AK_1 + BK_2, \tag{1.21}$$

it follows that $X = X(\lambda)$ satisfies $P_{VI}(\alpha = 0, \beta = 0, \gamma = 0, \delta = \frac{1}{2})$, providing a 2–parameter family of solutions. More generally, R. Fuchs (1907) in [Fu] showed that $P_{VI}(\alpha, \beta, \gamma, \delta)$ is equivalent to the inhomogeneous Picard–Fuchs equation

$$\begin{aligned} \lambda(1-\lambda)\frac{d^2K}{d\lambda^2} + (1-2\lambda)\frac{dK}{d\lambda} - \frac{1}{16}K \\ = \frac{Y}{\lambda(\lambda-1)}\left(\alpha + \frac{\beta\lambda^2}{X^2} + \frac{\gamma(\lambda-1)}{(X-1)^2} + \left(\delta - \frac{1}{2}\right)\frac{\lambda(1-\lambda)}{(X-\lambda)^2}\right). \end{aligned} \tag{1.22}$$

(A more recent perspective on such equations and their algebro–geometric meaning may be found in [Ma].)

In the following sections, a number of further examples of modular functions having similar properties will be considered. These all provide solutions to certain associated systems of nonlinear differential equations whose origins may be traced to Picard–Fuchs equations for families of elliptic curves. In each case, an associated inhomogeneous Picard–Fuchs equation may also be determined, and shown equivalent to an equation that is algebraically related to the Picard case of P_{VI}.

2. Generalized Halphen Equations

2a. Triangular cases.

Halphen [Ha] also considered generalizations of the system (1.1) related to the general hypergeometric equation

$$f(1-f)\frac{d^2y}{df^2} + (c-(a+b+1)f)\frac{dy}{df} - aby = 0. \tag{2.1}$$

Assuming the inverse function of the ratio of two linearly independent solutions of (2.1) to exist (which by no means is always the case in a global sense, in view of the infinite-valued multiplicity of the solutions of (2.1) due to monodromy), it also satisfies a Schwarzian equation of the form

$$\{f,\tau\} + 2R(f)f'^2 = 0, \tag{2.2}$$

where

$$R(f) = \frac{1}{4}\left(\frac{1-\lambda^2}{f^2} + \frac{1-\mu^2}{(f-1)^2} + \frac{\lambda^2+\mu^2-\nu^2-1}{f(f-1)}\right), \tag{2.3}$$

with the parameters (λ,μ,ν) defined by

$$\lambda := 1-c, \quad \mu := c-a-b, \quad \nu := b-a. \tag{2.4}$$

The generalized Halphen-Brioschi variables

$$\begin{aligned} W_1 &:= \frac{1}{2}\frac{d}{d\tau}\ln\frac{f'}{f}, \quad W_2 := \frac{1}{2}\frac{d}{d\tau}\ln\frac{f'}{(f-1)}, \\ W_3 &:= \frac{1}{2}\frac{d}{d\tau}\ln\frac{f'}{f(f-1)} \end{aligned} \tag{2.5}$$

then satisfy the *general* Halphen system:

$$\begin{aligned} W_1' &= W_1(W_2+W_3) - W_2W_3 + X \\ W_2' &= W_2(W_1+W_3) - W_1W_3 + X \\ W_3' &= W_3(W_1+W_2) - W_1W_2 + X, \\ X &:= \mu^2W_1^2 + \lambda^2W_2^2 + \nu^2W_3^2 + (\nu^2-\lambda^2-\mu^2)W_1W_2 \\ &\quad + (\lambda^2-\mu^2-\nu^2)W_3W_1 + (\mu^2-\lambda^2-\nu^2)W_2W_3. \end{aligned} \tag{2.6}$$

A sufficient condition for having a well-defined inverse function $f(\tau)$ is that the projectivized monodromy group of the hypergeometric equation (2.1) be a Fuchsian group of the first kind. This essentially means that it acts properly discontinuously and there is a tesselation of the image space by fundamental domains with a finite number of vertices. In the present case, the domains are necessarily triangular (with circular arcs as sides), since the vertices must map to the three regular singular points $(0, 1, \infty)$. (The more general case, with n singular points is considered in the next subsection.) The projectivized image of the monodromy representation is just the automorphism group of the function. The parameters (λ, μ, ν) are the fractions of π giving the angles at the vertices of the fundamental domain.

In general, the automorphism group $\mathfrak{G}_f$ of a function $f(\tau)$ (under Möbius transformations (1.6)) is said to be *commensurable* with the modular group Γ if it is a subgroup of $PGL(2, \mathbf{Q})$ whose intersection with Γ is of finite index in both Γ and $\mathfrak{G}_f$. Such a function and its automorphism group will be referred to as *modular*. By a *Hauptmodul*, we understand a uniformizing function for a genus zero Riemann surface; that is, the quotient $\mathbf{H}/\mathfrak{G}_f$ of the upper half τ–plane by the automorphism group defines a genus zero Riemann surface. The fact that such a function is the sole generator of the field of meromorphic functions on the Riemann surface implies that f must satisfy a Schwarzian equation of the type (2.2) for *some* rational function $R(f)$. A special class of such Hauptmoduls, referred to as *replicable functions* (due to their replication properties under the action of generalized Hecke operators [CN]), have the additional property of containing a finite index subgroup of the type

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}),\ c \equiv 0 \bmod N \right\}. \tag{2.7}$$

(Such functions arise in connection with “modular moonshine”, either as character generators for the Monster sporadic group, or in relation to these through the generalized Hecke averaging procedure [CN, FMN]). Each such function, of which there are only a finite number, is analytic in the upper half–plane, has a finite number of vertices in its fundamental domains, a cusp at ∞ and admits, up to an affine transformation, an expansion as a normalized McKay–Thompson q–series

$$F(q) = \alpha f(\tau) + \beta = \frac{1}{q} + \sum_{n=1}^{\infty} a_n q^n, \qquad q := e^{2i\pi\tau}, \tag{2.8}$$

convergent in the upper half–plane. (For the cases considered here, we also have $a_n \in \mathbf{Z}$.)

The table below (which is taken from [HM]) contains a complete list, up to equivalence under affine transformations in the τ variable, of the replicable functions of triangular type (i.e., those for which the fundamental domain has three vertices). These coincide with the arithmetic triangular functions of noncompact type [Ta]. Through formulae (2.5), they provide solutions to the general Halphen systems (2.6).

Table 1. Triangular Replicable Functions

Name	(a,b,c)	(λ,μ,ν)	ρ_0	$f(\tau)$
$1A$ $\sim\Gamma$	$(\frac{1}{12},\frac{1}{12},\frac{2}{3})$	$(\frac{1}{3},\frac{1}{2},0)$	$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$	$J=\frac{(\vartheta_2^8+\vartheta_3^8+\vartheta_4^8)^3}{54\vartheta_2^8\vartheta_3^8\vartheta_4^8}$
$2A$	$(\frac{1}{8},\frac{1}{8},\frac{3}{4})$	$(\frac{1}{4},\frac{1}{2},0)$	$\begin{pmatrix} 0 & -1 \\ 2 & -2 \end{pmatrix}$	$\frac{\left(\vartheta_3^4+\vartheta_4^4\right)^4}{16\vartheta_2^8\vartheta_3^4\vartheta_4^4}$
$3A$	$(\frac{1}{6},\frac{1}{6},\frac{5}{6})$	$(\frac{1}{6},\frac{1}{2},0)$	$\begin{pmatrix} 0 & -1 \\ 3 & -3 \end{pmatrix}$	$\frac{\left(\eta^{12}(\tau)+27\eta^{12}(3\tau)\right)^2}{108\eta^{12}(\tau)\eta^{12}(3\tau)}$
$2B$ $\sim\Gamma_0(2)$	$(\frac{1}{4},\frac{1}{4},\frac{1}{2})$	$(\frac{1}{2},0,0)$	$\begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$	$1+\frac{1}{64}\left(\frac{\eta(\tau)}{\eta(2\tau)}\right)^{24}$ $=\frac{\left(\vartheta_3^4(\tau)+\vartheta_4^4(\tau)\right)^2}{\vartheta_2^8(\tau)}$
$3B$ $\sim\Gamma_0(3)$	$(\frac{1}{3},\frac{1}{3},\frac{2}{3})$	$(\frac{1}{3},0,0)$	$\begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}$	$1+\frac{1}{27}\left(\frac{\eta(\tau)}{\eta(3\tau)}\right)^{12}$
$4C^*$ $\sim\Gamma_0(4)$	$(\frac{1}{2},\frac{1}{2},1)$	$(0,0,0)$	$\begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}$	$\frac{1}{\lambda(2\tau)}=\frac{\vartheta_3^4(2\tau)}{\vartheta_2^4(2\tau)}$ $=1+\frac{1}{16}\left(\frac{\eta(\tau)}{\eta(4\tau)}\right)^8$
$2a$	$(\frac{1}{6},\frac{1}{6},\frac{2}{3})$	$(\frac{1}{3},\frac{1}{3},0)$	$\begin{pmatrix} 2 & -3 \\ 4 & -4 \end{pmatrix}$	$\frac{\sqrt{3}i\left(e^{\pi i/3}\vartheta_3^4(2\tau)-\vartheta_2^4(2\tau)\right)^3}{9\vartheta_2^4(2\tau)\vartheta_3^4(2\tau)\vartheta_4^4(2\tau)}$
$4a$	$(\frac{1}{4},\frac{1}{4},\frac{3}{4})$	$(\frac{1}{4},\frac{1}{4},0)$	$\begin{pmatrix} 4 & -5 \\ 8 & -8 \end{pmatrix}$	$-\frac{i\left(\vartheta_3^2(2\tau)+i\vartheta_4^2(2\tau)\right)^4}{8\vartheta_2^4(2\tau)\vartheta_3^2(2\tau)\vartheta_4^2(2\tau)}$
$6a$	$(\frac{1}{3},\frac{1}{3},\frac{5}{6})$	$(\frac{1}{6},\frac{1}{6},0)$	$\begin{pmatrix} 6 & -7 \\ 12 & -12 \end{pmatrix}$	$-\frac{\sqrt{3}i\left(\eta^6(2\tau)+3\sqrt{3}i\eta^6(6\tau)\right)^2}{36\eta^6(2\tau)\eta^6(6\tau)}$

In this table, the first column gives the labelling according to the notation of refs. [CN, FMN], consistent with the finite group atlas. The second and third columns give the hypergeometric parameters (a,b,c) and (λ,μ,ν). The fourth column contains the generator of the automorphism group which stabilizes a vertex mapping to 0. The corresponding generator stabilizing the cusp at ∞ is

$$\rho_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{2.9}$$

and the third generator ρ_1, stabilizing a vertex mapping to 1 is determined by the relation

$$\rho_\infty\rho_1\rho_0 = \mathbf{I}. \tag{2.10}$$

The last column in Table 1 gives explicit expressions for the Hauptmoduls in terms of null theta functions $\vartheta_2(\tau),\vartheta_3(\tau),\vartheta_4(\tau)$ or the Dedekind eta–function $\eta(\tau)$.

In section 3, it will be shown how the corresponding hypergeometric equations, which imply the Schwarzian equation (2.2), and hence provide solutions to the generalized Halphen equations for the triangular cases, may be derived as Picard–Fuchs equations for families of elliptic curves. But first we consider some further generalizations of the Halphen system (1.1) corresponding to second order Fuchsian equations with more than three regular singular points. Such generalized systems were introduced by Ohyama [Oh].

2b. Generalized Halphen Systems with n singular points.

Consider second order Fuchsian equations of the form

$$\frac{d^2y}{df^2} + R(f)y = 0, \tag{2.11}$$

where $R(f)$ is a rational function of the form

$$R(f) = \frac{N(f)}{(D(f))^2}, \qquad D(f) = \prod_{i=1}^{n}(f - a_i), \tag{2.12}$$

and $N(f)$ is a polynomial of degree $\leq 2n-2$. (Any second order Fuchsian equation is projectively equivalent to one of this form; i.e., it may be transformed to this form, with no first derivative term, by multiplication of the solutions by a suitably chosen function.) Let $\tau(f)$ again denote the ratio

$$\tau(f) := \frac{y_1}{y_2} \tag{2.13}$$

of two linearly independent solutions of (2.11), and suppose again that the inverse function $f = f(\tau)$ is well-defined. It then satisfies the Schwarzian differential equation

$$\{f, \tau\} + 2R(f)f'^2 = 0, \tag{2.14}$$

and conversely, at least locally, all solutions of the Fuchsian equation (2.11) are expressible as:

$$y = \frac{(A + B\tau(f))}{(\tau')^{\frac{1}{2}}}. \tag{2.15}$$

The image of the monodromy representation

$$\begin{aligned} M &: \pi_1(\mathbf{P} - \{a_1, \dots a_n, \infty\}) \to GL(2, \mathbf{C}) \\ M &: \gamma \mapsto M_\gamma =: \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \end{aligned} \tag{2.16}$$

defined up to global conjugation by

$$\gamma : (y_1, y_2)|_{f_0} = (y_1, y_2)|_{f_0} M_\gamma, \tag{2.17}$$

determines a subgroup $\mathfrak{G}_f \subset GL(2, \mathbf{C})$ that acts on τ by Möbius transformations (1.6), leaving $f(\tau)$ invariant. Introducing the new variables [Oh, HM]

$$u := X_0 = \frac{1}{2}\frac{f''}{f'}, \quad v_i := \frac{1}{2}(X_0 - X_i) = \frac{1}{2}\frac{f'}{f - a_i}, \tag{2.18}$$

these satisfy the set of quadratic constraints

$$(a_i - a_j)v_i v_j + (a_j - a_k)v_j v_k + (a_k - a_i)v_k v_i = 0, \tag{2.19}$$

and the differential equations:

$$v_i' = -2v_i^2 + 2uv_i, \qquad i = 1, \dots n \tag{2.20a}$$

$$u' = u^2 - \sum_{i,j=1}^{n} r_{ij} v_i v_j, \tag{2.20b}$$

where the quadratic form $\sum_{i,j=1}^{n} r_{ij} v_i v_j$ appearing (2.20b) is determined by expressing of the rational function $R(f)$ in the form

$$R(f) = \frac{1}{4} \sum_{i,j=1}^{n} \frac{r_{ij}}{(f - a_i)(f - a_j)}. \tag{2.21}$$

(There is a nonuniqueness in such expressions for $R(f)$, but this just corresponds to the freedom of adding any linear combination of the vanishing quadratic forms (2.19) to the right hand side of eq. (2.20b).) In this case, if the automorphism group of f is again Fuchsian, the angles $\{\alpha_i \pi\}_{i=1,n}$ at the finite vertices of the fundamental polygon are related to the diagonal part of the quadratic form by

$$r_{ii} = 1 - \alpha_i^2. \tag{2.22}$$

Table 2 below, which is a shortened version of one appearing in [HM], again contains a list of Hauptmoduls that are replicable functions. But in these cases, their fundamental domains have four vertices, and hence there are four generalized Halphen variables (u, v_1, v_2, v_3) appearing in eqs. (2.20a), (2.20b).

The first column again identifies the functions and their groups according to the notation of refs. [CN, FMN], the second gives the values of $f(\tau)$ at the finite vertices (i.e., the location of the poles of $R(f)$), and the third lists the generators of the automorphism group of f corresponding to these vertices (i.e., the projectivized monodromy group generators). The fourth column gives the quadratic form appearing in (2.20b), and serves to define the rational function $R(f)$ through (2.21). The last column lists explicit formulae for $f(\tau)$ in terms of the Dedekind η–function. This table contains all the geometrical and group theoretical data characterizing the Hauptmoduls listed. A similar set of data may be determined for all the replicable functions appearing in [CN, FMN], and from this the corresponding Schwarzian equations and generalized Halphen equations may be deduced. (It should be noted that the number of vertices for the corresponding fundamental domains never exceeds 26, and that there are algebraic relations interlinking all these various cases.)

Table 2. Four Vertex Replicable Functions

Name	(a_1, a_2, a_3)	ρ_1 ρ_2 ρ_3	$\sum_{i,j=1}^{3} r_{ij} v_i v_j$	$f(\tau) - 1$
$6C$	$(-3, 0, 1)$	$\begin{pmatrix} 3 & -2 \\ 6 & -3 \end{pmatrix}$ $\begin{pmatrix} 3 & -1 \\ 12 & -3 \end{pmatrix}$ $\begin{pmatrix} -1 & 0 \\ 6 & -1 \end{pmatrix}$	$\frac{3}{4}v_1^2 + \frac{3}{4}v_2^2 + v_3^2$ $-\frac{1}{2}v_2v_3 - v_1v_3$	$\frac{1}{4}\frac{\eta^6(\tau)\eta^6(3\tau)}{\eta^6(2\tau)\eta^6(6\tau)}$
$6D$	$(\beta, \bar{\beta}, 1)$ $\beta :=$ $-\frac{3}{4} + \sqrt{2}i$	$\begin{pmatrix} 4 & -3 \\ 6 & -4 \end{pmatrix}$ $\begin{pmatrix} 2 & -1 \\ 6 & -2 \end{pmatrix}$ $\begin{pmatrix} -1 & 0 \\ 6 & -1 \end{pmatrix}$	$\frac{3}{4}v_1^2 + \frac{3}{4}v_2^2 + v_3^2$ $+\frac{131}{162}v_1v_2$ $-\frac{28-16\sqrt{2}i}{81}v_1v_3$ $-\frac{28+16\sqrt{2}i}{81}v_2v_3$	$\frac{1}{4}\frac{\eta^4(\tau)\eta^4(2\tau)}{\eta^4(3\tau)\eta^4(6\tau)}$
$6E$ $(\Gamma_0(6))$	$(-\frac{1}{8}, 0, 1)$	$\begin{pmatrix} 5 & -3 \\ 12 & -7 \end{pmatrix}$ $\begin{pmatrix} 5 & -2 \\ 18 & -7 \end{pmatrix}$ $\begin{pmatrix} -1 & 0 \\ 6 & -1 \end{pmatrix}$	$v_1^2 + v_2^2 + v_3^2$ $-\frac{10}{9}v_2v_3 - \frac{8}{9}v_1v_3$	$\frac{1}{8}\frac{\eta^5(\tau)\eta(3\tau)}{\eta(2\tau)\eta^5(6\tau)}$
$8E$ $(\Gamma_0(8))$	$(-1, 0, 1)$	$\begin{pmatrix} 3 & -2 \\ 8 & -5 \end{pmatrix}$ $\begin{pmatrix} 3 & -1 \\ 16 & -5 \end{pmatrix}$ $\begin{pmatrix} -1 & 0 \\ 8 & -1 \end{pmatrix}$	$v_1^2 + v_2^2 + v_3^2$ $-2v_1v_3$	$\frac{1}{4}\frac{\eta^4(\tau)\eta^2(4\tau)}{\eta^2(2\tau)\eta^4(8\tau)}$
$9B$ $(\Gamma_0(9))$	$(\omega, \bar{\omega}, 1)$ $\omega := e^{\frac{2\pi i}{3}}$	$\begin{pmatrix} 5 & -4 \\ 9 & -7 \end{pmatrix}$ $\begin{pmatrix} 2 & -1 \\ 9 & -4 \end{pmatrix}$ $\begin{pmatrix} -1 & 0 \\ 9 & -1 \end{pmatrix}$	$v_1^2 + v_2^2 + v_3^2$ $-v_1v_2$ $-(1-\omega)v_1v_3$ $-(1-\bar{\omega})v_2v_3$	$\frac{1}{3}\frac{\eta^3(\tau)}{\eta^3(9\tau)}$

3. Picard–Fuchs Equations on elliptic families

In this section, the parametric families of elliptic curves whose associated Picard–Fuchs equations underlie the Scwharzian equations governing these Hauptmoduls will be given for three of the examples appearing in the tables above. Only those cases are treated which are actually subgroups of the full modular group Γ, but an indication will be given at the end of this section how the other cases may be similarly derived. (A more complete version of these results will appear elsewhere [H].)

3a. Arithmetic triangular subgroups of Γ.

Four of the cases appearing in Table 1 involve automorphism groups that are contained in the full modular group; these are: $1A \sim \Gamma$, $2B \sim \Gamma_0(2)$, $3B \sim \Gamma_0(3)$ and $4C \sim \Gamma_0(4) \sim \Gamma(2)$. For each of these, it is possible to find a 1–parameter family of elliptic curves for which the associated Picard–Fuchs equation gives the required hypergeometric equation. We illustrate this below for the two cases: $1A \sim \Gamma$ and $2B \sim \Gamma_0(2)$. The first of these leads to the hypergeometric equation (1.14) associated with the J–function. (The case $4C \sim \Gamma_0(4) \sim \Gamma(2)$ gives the corresponding Legendre hypergeometric equation (1.10) associated with λ, and hence the original Halphen system.)

1. Hauptmodul $1A$. The associated family of elliptic curves is given by the elliptic pencil

$$y^2 = 4(a-1)x^3 - 3ax - a \qquad (a = J). \tag{3.1}$$

Denote by K and L the elliptic integrals of the first and second kind,

$$K := \int_\infty^{(X,Y)} \frac{dx}{y}, \qquad L := \int_\infty^{(X,Y)} \frac{x\,dx}{y}, \tag{3.2}$$

where we allow the endpoints to depend upon the parameter a. Differentiating these with respect to a, taking the endpoint contributions into account, we deduce the *inhomogeneous* Gauss-Manin system

$$K' + \frac{(5a-2)K}{12a(a-1)} - \frac{L}{6a} = \frac{a + (2+a)X + 2(1-a)X^2}{6a(a-1)Y} + \frac{X'}{Y} \tag{3.3a}$$

$$L' + \frac{K + (14a+4)L}{24a(a-1)} = \frac{-a - aX + (4+2a)X^2}{12a(a-1)Y} + \frac{XX'}{Y}. \tag{3.3b}$$

(Usually, the term "Gauss–Manin system" refers to the equations satisfied by the corresponding differential cohomology classes, but here we consider integrals along a suitable path, with varying endpoints.) If instead of taking variable endpoints, we integrate around a cycle, the right hand sides of eqs. (3.3a), (3.3b) vanishes, and we have the more usual homogeneous Gauss–Manin system. Eliminating the L integral from this system gives the inhomogeneous Picard–Fuchs equation

$$K'' + \frac{(2a-1)}{a(a-1)}K' + \frac{(36a^2 - 41a - 4)}{144a^2(a-1)^2}K = \frac{1}{Y}\left(X'' - \mathcal{A}_2(X)X'^2 - \mathcal{A}_1(X)X' - \mathcal{A}_0(X)\right), \tag{3.4}$$

where

$$\mathcal{A}_2(X) := \frac{12(a-1)X^2 - 3a}{2Y^2} \tag{3.5a}$$

$$\mathcal{A}_1(X) := \frac{a^2 + 3a^2X - 4(a-1)^2X^3}{a(a-1)Y^2} \tag{3.5b}$$

$$\begin{aligned}\mathcal{A}_0(X) := \frac{1}{72a^2(a-1)^2Y^2}\Big(& -3a^2 + (16a - 34a^2)X \\ & + (60a - 87a^2)X^2 + (16 - 140a + 124a^2)X^4\Big)\end{aligned} \tag{3.5c}$$

$$Y^2 := 4(a-1)X^3 - 3aX - a. \tag{3.5d}$$

Taking the path defining K to be a cycle, the right hand side of (3.4) disappears and we obtain the homogeneous Picard–Fuchs equation, which is projectively equivalent to the hypergeometric equation (1.14) under the identification $a = J$. (More precisely, the function $a^{\frac{1}{6}}(a-1)^{\frac{1}{4}}K$, taken over a generating pair of cycles gives a basis of solutions of (1.14).) Setting the integral K with variable endpoints equal to a linear combination

$$K = AK_1 + BK_2, \tag{3.6}$$

where K_1 and K_2 denote the values of the integral taken over a basis of cycles, amounts to choosing $X(a)$ as an elliptic function, defined as the inverse of the elliptic integral K, with its argument taken as $AK_1 + BK_2$. This implies that the right hand side of of (3.4) disappears, giving an equation for $X(a)$ of the same type as Picard's case of P_{VI}. (In fact, the two are algebraically related through through (1.8).)

2. Hauptmodul 2B. The associated family of elliptic curves in this case is given by the elliptic pencil

$$y^2 = 4x^3 - 3(1+3a)x + 9a - 1. \tag{3.7}$$

(To be precise, the Hauptmodul f_{2B} normalized as in Table 1 corresponds to the inverse $\frac{1}{a}$ of the parameter appearing in (3.7).) Denoting again by K and L the elliptic integrals of the first and second kind, respectively, defined as in (3.2), and differentiating with respect to the parameter, we obtain the corresponding inhomogeneous Gauss-Manin system:

$$K' + \frac{(3a-1)K - 2L}{12a(a-1)} = \frac{1 + 3a + (1-3a)X - 2X^2}{6a(a-1)Y} + \frac{X'}{Y} \tag{3.8a}$$

$$L' + \frac{(1+3a)K + (2-6a)L}{24a(a-1)} = -\frac{1 - 9a + (1+3a)X - (2-6a)X^2}{12a(a-1)Y} + \frac{XX'}{Y} \tag{3.8b}$$

and the resulting inhomogeneous Picard–Fuchs equation of type $2B$

$$K'' + \frac{(2a-1)}{a(a-1)}K' + \frac{3}{16a(a-1)}K = \frac{1}{Y}\left(X'' - \mathcal{A}_2(X)X'^2 - \mathcal{A}_1(X)X' - \mathcal{A}_0(X)\right), \tag{3.9}$$

where

$$\mathcal{A}_2(X) := \frac{1}{2(X-1)} + \frac{2+4X}{1 - 9a + 4X + 4X^2} \tag{3.10a}$$

$$\mathcal{A}_1(X) := -\frac{1}{a-1} - \frac{1}{a} - \frac{9}{1 - 9a + 4X + 4X^2} \tag{3.10b}$$

$$\mathcal{A}_0(X) := \frac{3(X-1)(3 - 3a + 8X + 4X^2)}{8a(a-1)(1 - 9a + 4X + 4X^2)}. \tag{3.10c}$$

Again, if the integrals are taken over cycles, the right hand side of (3.9) vanishes, and the independent variable transformation $a \to \frac{1}{a}$ gives an equation that is projectively equivalent to the hypergeometric equation satisfied by $F(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}; \frac{1}{a})$. Choosing K once again to equal a linear combination of the period integrals as in (3.6); i.e., expressing $X(a)$ again as an elliptic function of this argument, this defines a 2–parameter family of solutions to the equation obtained by equating the right hand side of (3.9) to zero. (This again is algebraically related to the Picard type solutions of P_{VI}).

A further class of examples is provided by the 4–vertex cases listed in Table 2 that correspond to subgroups of Γ. Up to projective equivalence, these are $6E \sim \Gamma_0(6)$, $8E' \sim \Gamma_0(8)' \sim \Gamma_1(4) \cap \Gamma(2)$, and $9B' \sim \Gamma_0(9)' \sim \Gamma(3)$. (The primes $'$ in this notation just denote composition with transformations of the type $\tau \to \tau/2$ or $\tau \to \tau/3$, which do not affect the resulting Schwarzian equations.) These are all cases of elliptic pencils corresponding to Beauville's elliptic surfaces [Be]. The only case of Beauville's surfaces that does not appear in Table 2 is the one corresponding to $\Gamma_1(5)$, which does not give a replicable function since this contains no $\Gamma_0(N)$ subgroup. Nevertheless, it can be can similarly dealt with.

As an illustration, here are the details in the case of the Hauptmodul with automorphism group $\Gamma_0(8)$. The elliptic pencil is defined by

$$x^3 + 2xy + a(x^2 - y^2) + x = 0. \tag{3.11}$$

In this case, the elliptic integrals of first and second kinds are

$$K = \int_\infty^{(X,Y)} \frac{dx}{x - ay} \qquad L = \int_\infty^{(X,Y)} \frac{xdx}{x - ay}. \tag{3.12}$$

The inhomogeneous Gauss-Manin system is

$$K' + \frac{aK + L}{a^2 - 1} = \frac{(1 + a^2)X + 2aX^2}{a(a^2 - 1)(X - aY)} + \frac{X'}{X - aY} \tag{3.13a}$$

$$L' - \frac{aK + L}{a(a^2 - 1)} = -\frac{2aX + (1 + a^2)X^2}{a(a^2 - 1)(X - aY)} + \frac{XX'}{X - aY}, \tag{3.13b}$$

and the inhomogeneous Picard–Fuchs equation is

$$K'' + \frac{(3a^2 - 1)}{a(a^2 - 1)}K' + \frac{1}{(a^2 - 1)}K = \frac{1}{X - aY}\left(X'' - \mathcal{A}_2(X)X'^2 - \mathcal{A}_1(X)X' - \mathcal{A}_0(X)\right), \tag{3.14}$$

where

$$\mathcal{A}_2(X) := \frac{a + 2(1 + a^2)X + 3aX^2}{2(aX^3 + (1 + a^2)X^2 + aX)} \tag{3.15a}$$

$$\mathcal{A}_1(X) := -\frac{2a^3 - (1 - 4a^2 - a^4)X + 2a^3X^2}{a(a^2 - 1)(aX^2 + (1 + a^2)X + a)} \tag{3.15b}$$

$$\mathcal{A}_0(X) := \frac{X(X^2 - 1)}{2a(aX^2 + (1 + a^2)X + a)}. \tag{3.15a}$$

The associated Fuchsian equation for this case is the corresponding homogeneous Picard–Fuchs equation, and the result of choosing the argument of the elliptic function defining $X(a)$ as in (3.6) again defines a Picard type solution of an equation algebraically related to P_{VI}.

We conclude with the following remarks.

1. The nonlinear monodromy [Du, M] of the Picard–type solutions of these P_{VI}–like equations coincides in each case with the monodromy of the associated homogeneous Picard–Fuchs equation, since it just involves a linear transformation of the coefficients in the linear combination (3.6), giving the argument of the elliptic function that defines the solution $X(a)$.
2. The Fuchsian and associated Schwarzian equations governing the other cases of Hauptmoduls, which do not involve subgroups of Γ, may also be derived from Picard–Fuchs equations, since they are obtained by taking extensions of the automorphism groups by Atkin-Lehner involutions [CN, FMN]. Instead of considering families of elliptic curves, however, one must consider families of *surfaces* obtained essentially by taking the topological products of the pairs curves involved. (Further details will be provided in [H].)
3. The various P_{VI}–like equations that arise are all related by the same algebraic transformations that connect the corresponding homogeneous Picard–Fuchs (and Schwarzian) equations [HM].

From the viewpoint of generalizations and applications of integrable dynamical systems involving such modular functions, it seems natural to try to extend these considerations to families of higher genus curves, and also to determine whether analogs of the Chazy solutions exist, which might provide new cases of Frobenius manifolds.

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