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Sporadic sequences, modular forms and new series for $1/\pi$

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Dedicated to the memory of Srinivasa Ramanujan

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Abstract Two new sequences, which are analogues of six sporadic examples of D. Zagier, are presented. The connection with modular forms is established and some new series for $1/\pi$ are deduced. The experimental procedure that led to the discovery of these results is recounted. Proofs of the main identities will be given, and some congruence properties that appear to be satisfied by the sequences will be stated as conjectures.

Keywords Congruence · Modular form · Pi · Sporadic sequence

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1 Introduction

The sequences $\{s_5(k)\}$, $\{s_6(k)\}$ and $\{s_{10}(k)\}$ defined by

$$s_5(k) = \binom{2k}{k} \sum_{j=0}^{k} \binom{k}{j}^2 \binom{k+j}{j}, \tag{1}$$

$$s_6(k) = \binom{2k}{k} \sum_{j=0}^{k} \binom{k}{j}^3 \tag{2}$$

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and

$$s_{10}(k) = \sum_{j=0}^{k} \binom{k}{j}^4 \tag{3}$$

have some remarkable properties. They appear as coefficients in various series for $1/\pi$, for example,

$$\frac{1}{\pi} = \frac{5\sqrt{47}}{7614} \sum_{k=0}^{\infty} (-1)^k s_5(k) \frac{(682k+71)}{15228^k}, \tag{4}$$

$$\frac{1}{\pi} = \frac{2}{25} \sum_{k=0}^{\infty} s_6(k) \frac{(9k+2)}{50^k} \tag{5}$$

and

$$\frac{1}{\pi} = \frac{\sqrt{15}}{18} \sum_{k=0}^{\infty} s_{10}(k) \frac{(4k+1)}{36^k}. \tag{6}$$

The identities (4)–(6) are analogues of formulas such as

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \left\{ \frac{(4k)!}{k!^4} \right\} \frac{(1103 + 26390k)}{396^{4k}}$$

found by Ramanujan [23]. The series (6) was discovered by Y. Yang in 2005. Yang's series led the authors of [12] to discover three families of series for $1/\pi$, of which (5) is a particular example. Three more families of series for $1/\pi$ were subsequently given in [8], and the series (4) is one of the examples. Yang's series (6) has been investigated further in [16], where yet more families of series for $1/\pi$ were given. For more detailed and recent bibliographical information about these series for $1/\pi$, the reader is referred to the survey article [4] and to Sect. 3 of [8].

Fundamental in establishing (4)–(6) are the modular identities

$$\frac{1}{4}(5P_5 - P_1) = \sum_{k=0}^{\infty} s_5(k) \left(\frac{x_5}{x_5^2 + 22x_5 + 125} \right)^k, \tag{7}$$

$$\frac{1}{6}(6P_6 + 3P_3 - 2P_2 - P_1) = \sum_{k=0}^{\infty} s_6(k) \left(\frac{x_6}{x_6^2 + 14x_6 + 81} \right)^k \tag{8}$$

and

$$\frac{1}{12}(10P_{10} + 5P_5 - 2P_2 - P_1) = \sum_{k=0}^{\infty} s_{10}(k) \frac{x_{10}^k}{(x_{10}-1)^{2k}}, \tag{9}$$

where

$$P_m = 1 - 24 \sum_{j=0}^{\infty} \frac{jq^{mj}}{1-q^{mj}}, \qquad \eta_m = q^{m/24} \prod_{j=1}^{\infty} \left(1 - q^{mj}\right)$$

and

$$x_5 = \frac{\eta_1^6}{\eta_5^6}, \qquad x_6 = \frac{\eta_1^4\eta_2^4}{\eta_3^4\eta_6^4} \quad \text{and} \quad x_{10} = \frac{\eta_2^6\eta_5^6}{\eta_1^6\eta_{10}^6}.$$

This work is concerned with finding new sequences that have properties analogous to (4)–(6) and (7)–(9). In the next section we will describe Zagier's six sporadic sequences and the connections of two of them with the $s_5(k)$ and $s_6(k)$ defined above in (1) and (2). We then outline an experimental search that produced two new sequences that are analogues of the sequence $s_{10}(k)$.

In Sect. 3 we will describe how modular analogues of (7)–(9) were obtained for the two new sequences. In Sect. 4 we outline how the corresponding series for $1/\pi$ were determined, and ten new rational series for $1/\pi$ will be exhibited. In Sect. 5 some conjectures will be given for some congruences satisfied by the two new sequences. Sections 6 and 7 contain proofs of the results in Sect. 3.

2 Sequences

Let $t_5(k)$ and $t_6(k)$ be defined by

$$t_5(k) = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{k+j}{j} \quad \text{and} \quad t_6(k) = \sum_{j=0}^{k} \binom{k}{j}^3. \tag{10}$$

Apart from a multiple of $\binom{2k}{k}$, the numbers $t_5(k)$ and $t_6(k)$ are the same as $s_5(k)$ and $s_6(k)$ defined by (1) and (2), respectively. R. Apéry found (e.g., see [28]) that $t_5(k)$ satisfies the recurrence relation

$$(k+1)^2 t_5(k+1) = \left(11k^2 + 11k + 3\right) t_5(k) + k^2 t_5(k-1), \tag{11}$$

and it is a classical result (e.g., see [3, 19, 20], [27, pp. 245, 278]) that $t_6(k)$ satisfies the three-term recurrence relation

$$(k+1)^2 t_6(k+1) = \left(7k^2 + 7k + 2\right) t_6(k) + 8k^2 t_6(k-1). \tag{12}$$

Apéry's example motivated D. Zagier [30] to use a computer to search for sequences defined by the recurrence relation

$$(k+1)^2 t(k+1) = \left(ak^2 + ak + b\right) t(k) + ck^2 t(k-1) \tag{13}$$

and initial conditions

$$t(-1) = 0, \qquad t(0) = 1, \tag{14}$$

that produce only integer values. The search yielded six sequences that are not either terminating (two successive terms that are zero), polynomial or the product of a polynomial with a multinomial coefficient. These are now called the sporadic examples.

They are the sequences that correspond to

$$(a,b,c) = (11,3,1), (-17,-6,-72), (10,3,-9),$$
$$(7,2,8), (12,4,-32) \text{ and } (-9,-3,-27); \tag{15}$$

two of these have already been discussed above in (11) and (12). It is believed that there are no other such examples of integer-valued sequences defined by (13) and (14) [30, conjecture on p. 354].

The modular analogues of (7)–(9) for all of the sporadic sequences were listed by Zagier [30, Sect. 5].

Let us now consider the sequences defined by (1), (2) and (3). By another classical result (e.g., see [3, 19, 21], [27, pp. 245, 278]), the sequence $\{s_{10}(k)\}$ satisfies the recurrence relation

$$(k+1)^3 s_{10}(k+1) = 2(2k+1)\big(3k^2+3k+1\big)s_{10}(k) + 4k\big(16k^2-1\big)s_{10}(k-1). \tag{16}$$

Furthermore, if

$$s(k) = \binom{2k}{k} t(k)$$

where $t(k)$ is given by (13) and (14), then it is easy to check that $s(k)$ satisfies the recurrence relation

$$(k+1)^3 s(k+1) = 2(2k+1)\big(ak^2+ak+b\big)s(k) + 4ck\big(4k^2-1\big)s(k-1). \tag{17}$$

These examples suggest there might be other integral sequences that satisfy a recurrence relation of the form

$$(k+1)^3 s(k+1) = (2k+1)\big(ak^2+ak+b\big)s(k) + k\big(ck^2+d\big)s(k-1) \tag{18}$$

and initial conditions

$$s(-1) = 0, \qquad s(0) = 1. \tag{19}$$

Both of the recurrences (13) and (18) are linear, three-term relations. The coefficients in (13) are polynomials of degree 2, while the coefficients in (18) are polynomials of degree 3.

A search over the domain

$$1 \le a \le 50, \qquad -20 \le b \le 20, \qquad -200 \le c \le 200, \qquad -30 \le d \le 30$$

was performed to look for integer-valued sequences. It is sufficient to consider $a > 0$ because replacing (a,b,c,d) with $(-a,-b,c,d)$ simply has the effect of replacing $s(k)$ with $(-1)^k s(k)$.

Some of the sequences obtained from the search are trivial, that is, terminating, polynomial or polynomial times a multinomial coefficient. The search also produced the six sequences of the form $\binom{2k}{k} \times$ one of Zagier's sporadic examples. Another set of sequences that can be anticipated are those given by the recurrence relation

$$(k+1)^3 s(k+1) = -(2k+1)\big(ak^2+ak+a-2b\big)s(k) - \big(4c+a^2\big)k^3 s(k-1)$$

where the parameters (a,b,c) take any of the six sets of values given by (15). The sequences in the cases $(a,b,c)=(-17,-6,-72)$, $(10,3,-9)$ and $(7,2,8)$ are called the Apéry numbers, Domb numbers and Almkvist–Zudilin numbers, respectively; the other three sequences are unnamed. Various of these six sequences, together with the corresponding series for $1/\pi$, have been studied by H.H. Chan, S.H. Chan and Z.-G. Liu [6], H.H. Chan and H. Verrill [13], M.D. Rogers [25] and T. Sato [26]. All six examples are discussed in the works by G. Almkvist, D. van Straten and W. Zudilin [2] and H.H. Chan and S. Cooper [8].

The only other three examples produced by the search are given by:

$$(a,b,c,d)=(6,2,64,-4),\ (13,4,27,-3)\ \text{and}\ (14,6,-192,12).$$

In these examples, $-c/d$ is a square. This suggests that a larger search over the parameter space for a, b and d, with the condition that $c=-r^2d$ for (say) $1\le r\le 10$, should be performed. No further integer sequences were produced.

When $(a,b,c,d)=(6,2,64,-4)$ the recurrence relation (18) becomes simply (16), so the sequence in this case is already known and is just $s_{10}(k)$. For more information, see [12, 16, 31].

The sequences obtained from (18) and (19) in the other two cases appear to be new, and we define $s_7(k)$ and $s_{18}(k)$ by the recurrence relations

$$\begin{aligned}(k+1)^3 s_7(k+1) &= (2k+1)\big(13k^2+13k+4\big)s_7(k)\\ &\quad + 3k\big(9k^2-1\big)s_7(k-1), \end{aligned}\tag{20}$$

$$\begin{aligned}(k+1)^3 s_{18}(k+1) &= 2(2k+1)\big(7k^2+7k+3\big)s_{18}(k)\\ &\quad - 12k\big(16k^2-1\big)s_{18}(k-1), \end{aligned}\tag{21}$$

and initial conditions

$$s_7(-1)=s_{18}(-1)=0 \quad \text{and} \quad s_7(0)=s_{18}(0)=1. \tag{22}$$

In the next section we will use modular forms to deduce that the numbers $s_7(k)$ and $s_{18}(k)$ are always integers, and the reasons for the subscripts 7 and 18 will become clear then. Explicit formulas as sums of binomial coefficients that exhibit integrality will also be given. The characteristic equations for the recurrence relations (20) and (21) are

$$m^2-26m-27=(m-27)(m+1)=0$$

and

$$m^2-28m+192=(m-16)(m-12)=0.$$

Therefore, by Poincaré's theorem on difference equations, e.g., see [18, p. 343]

$$\lim_{k\to\infty}\big|s_7(k)\big|^{1/k}\le \max\big\{27,|-1|\big\}=27$$

and

$$\lim_{k\to\infty}\big|s_{18}(k)\big|^{1/k}\le \max\{16,12\}=16.$$

It follows that the series $\sum_{k=0}^{\infty} s_7(k) y^k$ and $\sum_{k=0}^{\infty} s_{18}(k) y^k$ converge for $|y| < 1/27$ and $|y| < 1/16$, respectively.

3 Modular forms

The identities (7)–(9) may be phrased as

$$z_5 = \sum_{k=0}^{\infty} s_5(k) y_5^k, \qquad z_6 = \sum_{k=0}^{\infty} s_6(k) y_6^k \quad \text{and} \quad z_{10} = \sum_{k=0}^{\infty} s_{10}(k) y_{10}^k, \tag{23}$$

where y_5, y_6, y_{10}, z_5, z_6 and z_{10} are defined by

$$\begin{aligned} y_5 &= \frac{x_5}{x_5^2 + 22x_5 + 125}, \qquad y_6 = \frac{x_6}{x_6^2 + 14x_6 + 81} \quad \text{and} \quad y_{10} = \frac{x_{10}}{(x_{10}-1)^2}, \\ z_5 &= \frac{1}{4}(5P_5 - P_1), \qquad z_6 = \frac{1}{6}(6P_6 + 3P_3 - 2P_2 - P_1) \end{aligned} \tag{24}$$

and

$$z_{10} = \frac{1}{12}(10P_{10} + 5P_5 - 2P_2 - P_1).$$

In this section, we determine analogues of (23) for sequences $s_7(k)$ and $s_{18}(k)$ defined by (20)–(22) at the end of the last section. That is, we seek functions z and y with q-expansions of the form

$$z = 1 + \sum_{j=1}^{\infty} \mu(j) q^j \quad \text{and} \quad y = q + \sum_{j=2}^{\infty} \lambda(j) q^j, \tag{25}$$

and with the property that

$$z = \sum_{k=0}^{\infty} s(k) y^k, \tag{26}$$

where $s(k) = s_7(k)$ or $s(k) = s_{18}(k)$.

Equations (25) and (26) do not contain enough information to determine the coefficients $\lambda(j)$ and $\mu(j)$ uniquely. That is, there are infinitely many q-expansions given by (25) that will satisfy (26). More information will be required in order to determine z and y uniquely. To see what this additional information might look like, recall that the functions in (23) also satisfy the differential relations

$$q \frac{d}{dq} \log y_5 = z_5 \sqrt{1 - 44y_5 - 16y_5^2}, \tag{27}$$

$$q \frac{d}{dq} \log y_6 = z_6 \sqrt{1 - 28y_6 - 128y_6^2}, \tag{28}$$

$$q \frac{d}{dq} \log y_{10} = z_{10} \sqrt{1 - 12y_{10} - 64y_{10}^2}. \tag{29}$$

The identity (27) follows from [17, Theorem 2.4]; the identity (28) was given in [12]; and (29) was proved in [16]. Now recall from (16) and (17) that the sequences $s_5(k)$, $s_6(k)$ and $s_{10}(k)$ satisfy the recurrence relations

$$\begin{aligned}(k+1)^3 s_5(k+1) &= (2k+1)\big(22k^2+22k+6\big)s_5(k)\\ &\quad + k\big(16k^2-4\big)s_5(k-1), \qquad (30)\\ (k+1)^3 s_6(k+1) &= (2k+1)\big(14k^2+14k+4\big)s_6(k)\\ &\quad + k\big(128k^2-32\big)s_6(k-1) \qquad (31)\end{aligned}$$

and

$$(k+1)^3 s_{10}(k+1) = (2k+1)\big(6k^2+6k+2\big)s_{10}(k) + k\big(64k^2-4\big)s_{10}(k-1). \quad (32)$$

On comparing (30)–(32) with (27)–(29) and (23), it is not hard to make the conjecture that if $s(k)$ satisfies the recurrence relation

$$(k+1)^3 s(k+1) = (2k+1)\big(ak^2+ak+b\big)s(k) + k\big(ck^2+d\big)s(k-1) \qquad (33)$$

then perhaps the corresponding modular forms z and y will satisfy the differential relation

$$q\frac{d}{dq}\log y = z\sqrt{1-2ay-cy^2},$$

and be related by the identity

$$z = \sum_{k=0}^{\infty} s(k) y^k.$$

Let us use these ideas to determine the modular forms for the sequence $\{s_7(k)\}$ defined by (20) and (22). Let

$$y = q + \sum_{j=2}^{\infty} \lambda(j) q^j, \qquad (34)$$

where the coefficients $\lambda(j)$, $j \geq 2$, are unknowns that we seek to determine. Let z be defined in terms of y by

$$z = \frac{1}{y\sqrt{1-26y-27y^2}} q\frac{dy}{dq}. \qquad (35)$$

Now substitute (34) and (35) into the identity

$$z = \sum_{k=0}^{\infty} s_7(k) y^k$$

and equate coefficients of powers of q on each side. The coefficients $\lambda(j)$ can be determined successively, and we find that

$$\begin{aligned}y &= q - 9q^2 + 30q^3 - 15q^4 - 240q^5 + 978q^6 - 1463q^7 - 2361q^8 \\ &\quad + 18201q^9 - 42800q^{10} + 15624q^{11} + 227742q^{12} - 809028q^{13} \\ &\quad + 1088367q^{14} + 1593120q^{15} - 11383551q^{16} + 25003158q^{17} + \cdots\end{aligned}$$

and

$$\begin{aligned}z &= 1 + 4q + 12q^2 + 16q^3 + 28q^4 + 24q^5 + 48q^6 + 4q^7 + 60q^8 \\ &\quad + 52q^9 + 72q^{10} + 48q^{11} + 112q^{12} + 56q^{13} + 12q^{14} + 96q^{15} \\ &\quad + 124q^{16} + 72q^{17} + \cdots.\end{aligned}$$

Based on the examples given in (7)–(9), we may guess that z is a linear combination of P_m, for various positive integers m, and by expanding in powers of q and comparing coefficients we come up with the conjecture that

$$z = \frac{1}{6}(7P_7 - P_1).$$

This suggests that y will be a modular function of level 7. Once again, based on the examples given in (7)–(9), we let $x_7 = \eta_1^4/\eta_7^4$ and assume that y is a rational function of x_7. On comparing q-expansions we are readily led to conjecture that (cf. [7, (13)])

$$y = \frac{x_7}{x_7^2 + 13x_7 + 49}.$$

Similar calculations may be performed for the sequence $\{s_{18}(k)\}$ defined by (21) and (22): the modular form z can be identified by computing its q-expansion, then the corresponding modular function x can be guessed by referring to Table 3 in the paper by Conway and Norton [14], and finally the modular function y can be guessed to be a rational function of x and worked out by examining the q-expansions.

The results of the above discussion may be summarized as:

Theorem 3.1 *Let x_7, x_{18}, y_7, y_{18}, z_7 and z_{18} be defined by*

$$x_7 = \frac{\eta_1^4}{\eta_7^4}, \qquad x_{18} = \frac{\eta_3^4\eta_6^4}{\eta_1^2\eta_2^2\eta_9^2\eta_{18}^2},$$

$$y_7 = \frac{x_7}{x_7^2 + 13x_7 + 49}, \qquad y_{18} = \frac{x_{18}}{(x_{18}+3)^2},$$

$$z_7 = \frac{1}{6}(7P_7 - P_1) \quad \textit{and} \quad z_{18} = \frac{1}{4}(18P_{18} - 9P_9 - 12P_6 + 6P_3 + 2P_2 - P_1).$$

Let $s_7(k)$ and $s_{18}(k)$ be defined by (20)–(22). Then

$$\begin{aligned}&q\frac{d}{dq}\log y_7 = z_7\sqrt{1 - 26y_7 - 27y_7^2}, \qquad q\frac{d}{dq}\log y_{18} = z_{18}\sqrt{1 - 28y_{18} + 192y_{18}^2}, \\ &z_7 = \sum_{k=0}^{\infty} s_7(k)y_7^k \quad \textit{and} \quad z_{18} = \sum_{k=0}^{\infty} s_{18}(k)y_{18}^k.\end{aligned} \tag{36}$$

The discussion leading up to the statement of Theorem 3.1 by no means constitutes a proof. The experimental procedure that led to the results has been included here on account of its own interest, and perhaps similar ideas can be used to find other such identities. A proof of Theorem 3.1 will be given in Sects. 6 and 7.

A wealth of information about hauptmoduls and weight one modular forms for $\Gamma_0(N)$ in the case of genus zero (including $N=7$ and $N=18$) has been given by R.S. Maier [22]. Our recurrence relations for $s_7(k)$ and $s_{18}(k)$ in (20) and (21) are different from those for c_n in [22, Table 14]. For example, the recurrence relation for c_n in the case of level 18 given in [22, Table 14] is a 7-term recurrence relation, whereas (21) is a 3-term recurrence relation. Moreover, the functions z_7 and z_{18} in (36) are weight two modular forms, whereas weight one modular forms are studied in [22].

The following consequence of Theorem 3.1 should be noted:

Corollary 3.2 *The numbers $s_7(k)$ and $s_{18}(k)$ defined by* (20)–(22) *are all integers.*

Proof Let $z=z_j$, $y=y_j$ and $s(k)=s_j(k)$, where $j=7$ or $j=18$. From the definitions of z and y in the statement of Theorem 3.1 it follows that

$$z=1+\sum_{j=1}^{\infty}\mu(j)q^j \quad \text{and} \quad y=q+\sum_{j=2}^{\infty}\lambda(j)q^j,$$

where the coefficients $\mu(j)$ and $\lambda(j)$ are all integers. Now substitute these expansions in either of the series in (36). On equating coefficients of q^k and using induction on k it follows that the $s(k)$ is an integer. □

Explicit formulas for $s_7(k)$ and $s_{18}(k)$ as sums of binomial coefficients, which exhibit the integrality of the sequences, have been communicated to the author by W. Zudilin [32], viz.:

$$s_7(k)=\sum_{j=0}^{k}\binom{k}{j}^2\binom{2j}{k}\binom{j+k}{j}$$

and

$$s_{18}(k)=\sum_{j=0}^{\lfloor k/3\rfloor}(-1)^j\binom{k}{j}\binom{2j}{j}\binom{2k-2j}{k-j}\left\{\binom{2k-3j-1}{k}+\binom{2k-3j}{k}\right\}.$$

Zudilin found these binomial sums by looking at the formulas #26 and #183 in the database [1]. The formulas can be shown to satisfy the recurrence relations (20) and (21) by the method of creative telescoping. This has been automated and implemented in the sumtools package in Maple. For example, typing

```
> with(sumtools);
> sumrecursion(binomial(k, j)^2*binomial(2*j, k)
    *binomial(j+k, j), j, s(k));
```

produces the recurrence formula

$$\begin{aligned} &-3\,(k-1)(3\,k-4)(3\,k-2)s(-2+k) \\ &\quad -(2\,k-1)\big(13\,k^2-13\,k+4\big)s(k-1)+s(k)k^3 \end{aligned}$$

that is equal to zero.

4 Series for $1/\pi$

For $\ell \in \{5, 6, 10\}$, let $y_\ell = y_\ell(q)$ be defined by (24). The parameter ℓ is called the level. For a given level ℓ there may exist certain positive integers N for which the reciprocal of

$$y_{\ell,N} := y_\ell\big(\exp\big(-2\pi\sqrt{N/\ell}\big)\big)$$

is a positive integer. First in [12], and then subsequently in [8, 16], these values were used to construct series for $1/\pi$ of the types:

$$\sqrt{1-44y_{5,N}-16y_{5,N}^2}\sum_{k=0}^{\infty} s_5(k)(k+\lambda)y_{5,N}^k = \frac{1}{2\pi}\sqrt{\frac{5}{N}},$$

$$\sqrt{1-28y_{6,N}-128y_{6,N}^2}\sum_{k=0}^{\infty} s_6(k)(k+\lambda)y_{6,N}^k = \frac{1}{2\pi}\sqrt{\frac{6}{N}},$$

and

$$\sqrt{1-12y_{10,N}-64y_{10,N}^2}\sum_{k=0}^{\infty} s_{10}(k)(k+\lambda)y_{10,N}^k = \frac{1}{2\pi}\sqrt{\frac{10}{N}},$$

where $s_5(k)$, $s_6(k)$ and $s_{10}(k)$ are defined by (1)–(3), and λ is a rational number that depends on N and ℓ. By analogy, we should expect similar series of the forms:

$$\sqrt{1-26y_{7,N}-27y_{7,N}^2}\sum_{k=0}^{\infty} s_7(k)(k+\lambda)y_{7,N}^k = \frac{1}{2\pi}\sqrt{\frac{7}{N}}, \tag{37}$$

and

$$\sqrt{1-28y_{18,N}+192y_{18,N}^2}\sum_{k=0}^{\infty} s_{18}(k)(k+\lambda)y_{18,N}^k = \frac{1}{2\pi}\sqrt{\frac{18}{N}} \tag{38}$$

for the sequences $s_7(k)$ and $s_{18}(k)$ defined by (20)–(22). By analogy with further results in [8, 12, 16], we would expect identities of the forms

$$\sqrt{1-26y_{28,N}-27y_{28,N}^2}\sum_{k=0}^{\infty} s_7(k)(k+\lambda)y_{28,N}^k = \frac{1}{2\pi}\sqrt{\frac{28}{N}} \tag{39}$$

Table 1 Data for rational series for $1/\pi$ for level $\ell = 7$

q	N	$y_7(q)$	λ
$e^{-2\pi\sqrt{N/7}}$	4	$\frac{1}{5^3}$	$\frac{4}{21}$
$-e^{-\pi\sqrt{N/7}}$	13	$\frac{-1}{4^3}$	$\frac{10}{39}$
	61	$\frac{-1}{22^3}$	$\frac{1286}{11895}$

Table 2 Data for rational series for $1/\pi$ for level $\ell = 18$

q	N	$y_{18}(q)$	λ
$e^{-2\pi\sqrt{N/18}}$	2	$\frac{1}{18}$	0
	5	$\frac{1}{6^2}$	$\frac{3}{20}$
	11	$\frac{1}{12^2}$	$\frac{15}{88}$
	29	$\frac{1}{54^2}$	$\frac{789}{6380}$
$-e^{-\pi\sqrt{N/9}}$	13	$\frac{-1}{6^2}$	$\frac{9}{26}$
	25	$\frac{-1}{180}$	$\frac{3}{14}$
	37	$\frac{-1}{24^2}$	$\frac{171}{1036}$

and

$$\sqrt{1 - 28y_{36,N} + 192y_{36,N}^2}\sum_{k=0}^{\infty} s_{18}(k)(k+\lambda)y_{36,N}^k = \frac{1}{2\pi}\sqrt{\frac{36}{N}}, \tag{40}$$

where

$$y_{28,N} := y_7\big(-\exp\big(-2\pi\sqrt{N/28}\big)\big) \quad \text{and} \quad y_{36,N} := y_{18}\big(-\exp\big(-2\pi\sqrt{N/36}\big)\big).$$

A computer may be used to evaluate $y_{7,N}$, $y_{18,N}$, $y_{28,N}$ and $y_{36,N}$ to high precision, for various positive integers N. The corresponding value of λ can then be determined numerically by summing sufficiently many terms in the corresponding series for $1/\pi$. In general, the values of $y_{7,N}$, $y_{18,N}$, $y_{28,N}$, $y_{36,N}$ and λ will be algebraic numbers. The values of $y_{7,N}$, $y_{18,N}$, $y_{28,N}$ and $y_{36,N}$ that appear to be rational (and lie within the intervals of convergence determined at the end of Sect. 2) are listed in Table 1 together with the corresponding values of λ, which also appear to be rational. In this way, we determine 10 new series for $1/\pi$.

It should be noted that for several positive integers N, the value of $y_{7,N}$, $y_{18,N}$, $y_{28,N}$ and $y_{36,N}$, as well as the corresponding value of λ, are quadratic irrational numbers. We have not attempted to classify these values, and they are not included in the tables.

The series identities (37)–(40) with data given by Tables 1 and 2 may be proved using a result of H.H. Chan, S.H. Chan and Z.-G. Liu [6, Theorem 2.1]. For additional applications of this theorem, see [8, 12, 16]. As such, the method can be considered to be well understood, so it is not necessary to write out detailed proofs for each series for $1/\pi$. The main issue will be the calculation of the modular equations satisfied by $y_7(q)$ and $y_{18}(q)$.

We end this section by stating two of the new series for $1/\pi$ in full. The first one corresponds—after some slight simplification—to the data in Table 1 for $\ell = 7$, $N = 61$ and $q = -\exp(-\pi\sqrt{61/7})$:

$$\frac{1}{\pi} = \sqrt{7}\sum_{k=0}^{\infty}(-1)^k s_7(k)\frac{(11895k + 1286)}{22^{3k+3}}. \tag{41}$$

Each term adds approximately three decimal places of accuracy. The other series corresponds to the data in Table 2 for $\ell = 18$, $N = 2$ and $q = \exp(-2\pi/3)$:

$$\frac{1}{\pi} = \frac{2\sqrt{3}}{27}\sum_{k=0}^{\infty}\frac{k s_{18}(k)}{18^k}. \tag{42}$$

This formula, although compact and elegant in appearance, converges slowly and requires approximately 20 terms for each decimal place of accuracy.

5 Congruences

For any prime $p > 3$ and any positive integer k, the numbers $s_{10}(k)$ are known to satisfy the congruence relation

$$s_{10}(pk) \equiv s_{10}(k) \pmod{p^3}.$$

See [9] for a proof and a discussion of other congruences, as well as some conjectures. The analogous congruences for $s_7(k)$ and $s_{18}(k)$ are given by

Conjecture 5.1 *Let k be a positive integer and let $s_7(k)$ and $s_{18}(k)$ be defined by* (20)–(22). *The following congruences for $s_7(k)$ hold*:

$$s_7(2k) \equiv s_7(k) \pmod{32}, \quad \textit{provided } k \geq 2$$

and if p is any odd prime then

$$s_7(pk) \equiv s_7(k) \pmod{p^3}.$$

The following congruences for $s_{18}(k)$ hold:

$$s_{18}(2k) \equiv s_{18}(k) \pmod{32}, \quad \textit{provided } k \geq 2,$$
$$s_{18}(3k) \equiv s_{18}(k) \pmod{27}, \quad \textit{provided } k \geq 3,$$

and if $p > 3$ is prime then

$$s_{18}(pk) \equiv s_{18}(k) \pmod{p^2}.$$

6 Proofs of the results for level 7

In this section we will prove the results in Theorem 3.1 that involve y_7 and z_7. We begin with

Lemma 6.1 *Suppose $\sigma = \sigma(y)$ is a solution of the second order, linear, homogeneous differential equation*

$$\frac{d^2\sigma}{dy^2} + f_1(y)\frac{d\sigma}{dy} + f_2(y)\sigma = 0.$$

Then the function z defined by $z = z(y) = (\sigma(y))^2$ satisfies the third order linear differential equation

$$\frac{d^3 z}{dy^3} + 3f_1\frac{d^2 z}{dy^2} + \left(f_1' + 4f_2 + 2f_1^2\right)\frac{dz}{dy} + \left(2f_2' + 4f_1 f_2\right)z = 0.$$

Proof This is a classical result. The idea is to note that z, z', z'' and z''' can each be expressed as linear combinations of σ^2, $\sigma\sigma'$ and $(\sigma')^2$, and hence they are linearly related. The claimed differential equation is just an explicit statement of the linear dependence. □

We are now ready for:

Proof of Theorem 3.1 for the results that involve x_7, y_7 and z_7 Let us write x, y and z for x_7, y_7 and z_7, respectively, and let us define u and σ by $u = x^{-1}$ and $\sigma = z^{1/2}$; that is,

$$x = \frac{\eta_1^4}{\eta_7^4}, \qquad z = \sigma^2 = \frac{1}{6}(7P_7 - P_1),$$
$$u = \frac{1}{x} \quad \text{and} \quad y = \frac{x}{x^2 + 13x + 49} = \frac{u}{1 + 13u + 49u^2}.$$

Some simple calculations give

$$1 - 26y - 27y^2 = \frac{(1 - 49u^2)^2}{(1 + 13u + 49u^2)^2} \tag{43}$$

and

$$\frac{dy}{du} = \frac{1 - 49u^2}{(1 + 13u + 49u^2)^2}. \tag{44}$$

Then

$$q\frac{dy}{dq} = \frac{dy}{du} \times q\frac{du}{dq}$$
$$= \frac{u(1 - 49u^2)}{(1 + 13u + 49u^2)^2}\, q\frac{d}{dq}\log u$$

$$
\begin{aligned}
&= y\sqrt{1-26y-27y^2}\, q\frac{d}{dq}\log\left(\frac{\eta_7^4}{\eta_1^4}\right)\\
&= y\sqrt{1-26y-27y^2}\, \times \frac{1}{6}(7P_7-P_1)\\
&= zy\sqrt{1-26y-27y^2}.
\end{aligned}
$$

This proves the first result in Theorem 3.1 for the functions y_7 and z_7. Next, recall the differential equation from [17, Corollary 3.7] (cf., [11, Theorem 2.4]):

$$u\frac{d}{du}\left(u\frac{d\sigma}{du}\right) = 2u\frac{(1+16u+49u^2)}{(1+13u+49u^2)^2}\sigma. \tag{45}$$

By (43) and (44) we find that

$$u\frac{d}{du} = y\sqrt{1-26y-27y^2}\frac{d}{dy},$$

so the differential equation (45) becomes

$$y\sqrt{1-26y-27y^2}\frac{d}{dy}\left(y\sqrt{1-26y-27y^2}\frac{d\sigma}{dy}\right) = 2y(1+3y)\sigma,$$

or simply

$$y\big(1-26y-27y^2\big)\frac{d^2\sigma}{dy^2} + \big(1-39y-54y^2\big)\frac{d\sigma}{dy} - 2(1+3y)\sigma = 0.$$

Since $z=\sigma^2$, we may apply Lemma 6.1 to deduce that z satisfies the third order differential equation

$$
\begin{aligned}
y^2\big(1-26y-27y^2\big)\frac{d^3z}{dy^3} &+ 3y\big(1-39y-54y^2\big)\frac{d^2z}{dy^2}\\
&+ \big(1-86y-186y^2\big)\frac{dz}{dy} - 4(1+6y)z = 0.
\end{aligned}
$$

We seek a power series solution of the form

$$z = \sum_{k=0}^{\infty} s(k)y^k, \tag{46}$$

and deduce from the differential equation that the coefficients must satisfy the recurrence relation

$$(k+1)^3 s_7(k+1) = (2k+1)\big(13k^2+13k+4\big)s_7(k) + 3k\big(9k^2-1\big)s_7(k-1),$$

the same recurrence as (20) satisfied by the sequence $s_7(k)$. If we compare the first few terms in the q-expansion of (46), we find that $s(0)=1$ and $s(1)=4$ and hence, from the recurrence relation, we obtain $s(-1)=0$. It follows that $s(k)=s_7(k)$, and this completes the proof of Theorem 3.1 for the functions y_7 and z_7. □

7 Proofs of the results for level 18

In this section we will prove the results in Theorem 3.1 that involve x_{18}, y_{18} and z_{18}. Let u, v, w, x, y, z and ϕ be defined by

$$u=\frac{\eta_3\eta_{18}^3}{\eta_6\eta_9^3},\qquad v=\frac{\eta_2^6\eta_3^{12}\eta_{18}^6}{\eta_1^6\eta_6^{12}\eta_9^6},\qquad w=u^3,$$

$$x=\frac{\eta_3^4\eta_6^4}{\eta_1^2\eta_2^2\eta_9^2\eta_{18}^2},\qquad y=\frac{x}{(x+3)^2}$$

$$z=q\frac{d}{dq}\log v=\frac{1}{4}(18P_{18}-9P_9-12P_6+6P_3+2P_2-P_1)$$

and

$$\phi=\frac{\eta_6\eta_9^6}{\eta_3^2\eta_{18}^3}.$$

The function u is Ramanujan's cubic continued fraction but with q^3 in place of q. The functions x, y and z correspond to x_{18}, y_{18} and z_{18} from Theorem 3.1, respectively; the subscript "18" has been dropped for convenience. The function v has been introduced because of its connection with the function z. The function ϕ is a weight one modular form; see [10, 15, 29, 30] for more information.

The next three lemmas establish the connections of v, x and y with u.

Lemma 7.1 *The following identities hold*:

$$1+u=\frac{\eta_2^2\eta_3\eta_{18}}{\eta_1\eta_6\eta_9^2},\qquad 1-2u=\frac{\eta_1^2\eta_{18}}{\eta_2\eta_9^2},\qquad 1+u^3=\frac{\eta_6^5\eta_{18}}{\eta_3\eta_9^5}$$

$$\textit{and}\quad 1-8u^3=\frac{\eta_3^8\eta_{18}^4}{\eta_6^4\eta_9^8}.$$

Proof Most of these identities were given by Ramanujan in his second notebook [24, Chap. 20, Entry 1]. See [5, pp. 345–347] for proofs. It should be pointed out that the results for $1+u$ and $1-2u$ may be proved by beginning with the series rearrangements

$$\sum_{j=0}^{\infty}q^{j(j+1)/2}=q\sum_{j=0}^{\infty}q^{9j(j+1)/2}+\sum_{j=-\infty}^{\infty}q^{3j(3j+1)/2}$$

and

$$\sum_{j=-\infty}^{\infty}(-1)^jq^{j^2}=\sum_{j=-\infty}^{\infty}(-1)^jq^{9j^2}-2\sum_{j=-\infty}^{\infty}(-1)^jq^{(3j+1)^2},$$

respectively, and converting each series to a product by using Jacobi's triple product identity. The identity for $1+u^3$ can then be obtained from the result for $1+u$ by replacing q with $q\exp(2\pi ij/3)$, $j \in \{0,1,2\}$, and multiplying the three identities together. The identity for $1-8u^3$ may be obtained from the result for $1-2u$ by a similar procedure. □

Lemma 7.2 *We have*

$$v = \frac{u(1+u)^2(1-8u^3)}{(1-2u)^2(1+u^3)},$$

$$x = \frac{(1+u^3)(1-8u^3)}{u(1+u)^2(1-2u)^2},$$

and

$$y = \frac{u(1+u^3)(1-8u^3)}{(1+2u-2u^2)^4}.$$

Proof These are immediate from the definitions of v, x and y, and Lemma 7.1. □

Lemma 7.3 *We have*

$$1-28y+192y^2 = \frac{(1+2u^2)^2(1-4u-2u^2)^2(1-2u+6u^2+4u^3+4u^2)^2}{(1+2u-2u^2)^8}$$

and

$$\frac{dy}{du} = \frac{(1+2u^2)(1-4u-2u^2)(1-2u+6u^2+4u^3+4u^4)}{(1+2u-2u^2)^5}.$$

Hence,

$$\left(1+2u-2u^2\right)\frac{d}{du} = \sqrt{1-28y+192y^2}\,\frac{d}{dy}.$$

Proof These are immediate from Lemma 7.2. □

The next result summarizes some known properties of the functions ϕ and w.

Lemma 7.4 *We have*

$$q\frac{d}{dq}\log w = 3\phi^2\,(1+w)(1-8w)$$

and

$$\frac{d}{dw}\left(w(1+w)(1-8w)\frac{d\phi}{dw}\right) = 2(1+4w)\phi. \tag{47}$$

Proof Proofs of both of these results have been given in [10]. The factor 3 on the right hand side of the first result occurs because our definitions of w and ϕ involve q^3 in place of the variable q used in [10]. □

The first result in the next lemma establishes part of Theorem 3.1. The second result gives a connection between the weight 2 modular form z and the weight one modular form ϕ.

Lemma 7.5 *We have*

$$q\frac{d}{dq}\log y = z\sqrt{1-28y+192y^2}$$

and

$$z = \left(1+2u-2u^2\right)^3\phi^2.$$

Proof By the chain rule and the definition of z, we have

$$\begin{aligned} q\frac{d}{dq}\log y &= \frac{1}{y}q\frac{dy}{dq} \\ &= \frac{1}{y}\left(\frac{dy}{du}\bigg/\frac{dv}{du}\right)q\frac{dv}{dq} \\ &= \frac{v}{y}\left(\frac{dy}{du}\bigg/\frac{dv}{du}\right)q\frac{d}{dq}\log v \\ &= \frac{v}{y}\left(\frac{dy}{du}\bigg/\frac{dv}{du}\right)z. \end{aligned}$$

By Lemma 7.2, each of v and y, and hence dv/du and dy/du, can be expressed as rational functions of u, and we find that

$$q\frac{d}{dq}\log y = \frac{(1+2u^2)(1-4u-2u^2)(1-2u+6u^2+4u^3+4u^4)}{(1+2u-2u^2)^4}z.$$

If we now apply Lemma 7.3, we complete the proof of the first identity.

The second identity may be proved in a similar way: by the chain rule and the definitions of u, v, w and z, we have

$$z = \frac{1}{v}q\frac{dv}{dq} = \frac{u}{v}\frac{dv}{du}q\frac{d}{dq}\log u = \frac{u}{3v}\frac{dv}{du}q\frac{d}{dq}\log w.$$

Now apply Lemmas 7.2 and 7.4 and simplify to obtain the required result. □

The next goal is to determine a third order linear differential equation for z with respect to y. In order to accomplish this, we consider the differential operator D defined by

$$D = \sqrt{1-28y+192y^2}\,\frac{d}{dy}. \tag{48}$$

By Lemma 7.3, an equivalent expression for D in terms of the variable u is given by

$$D = \left(1+2u-2u^2\right)\frac{d}{du}. \tag{49}$$

It will be useful to establish the following intermediate result.

Lemma 7.6 *The function z satisfies the following second order nonlinear differential equation*:

$$D^2z-\frac{1}{2z}(Dz)^2+\frac{1}{y}\sqrt{1-28y+192y^2}\,Dz=\frac{6}{y}(1-15y)\,z.$$

Proof Recall, from Lemma 7.5, that

$$z=\bigl(1+2u-2u^2\bigr)^3\phi^2. \tag{50}$$

Apply the operator D in the form given by (49) to obtain

$$Dz=6(1-2u)\bigl(1+2u-2u^2\bigr)^3\phi^2+2\bigl(1+2u-2u^2\bigr)^4\phi\phi'. \tag{51}$$

Here, and in the remainder of the proof, primes will denote differentiation with respect to u. Apply the operator D again, to get

$$D^2z=c_1\phi^2+c_2\phi\phi'+c_3\bigl(\phi'\bigr)^2+c_4\phi\phi'' \tag{52}$$

where

$$c_1=24\bigl(1+2u-2u^2\bigr)^3\bigl(1-7u+7u^2\bigr),\qquad c_2=28\bigl(1+2u-2u^2\bigr)^4(1-2u)$$

and

$$c_3=c_4=2\bigl(1+2u-2u^2\bigr)^5.$$

The identities (50) and (51) may be used to eliminate the $(\phi')^2$ term from (52). We find that

$$D^2z-\frac{1}{2z}(Dz)^2=d_1\phi^2+d_2\phi\phi'+d_3\phi\phi'' \tag{53}$$

where

$$d_1=6\bigl(1+2u-2u^2\bigr)^3\bigl(1-16u+16u^2\bigr),$$
$$d_2=16\bigl(1+2u-2u^2\bigr)^4(1-2u)$$

and $d_3=c_3$. Next, the differential equation (47), when expressed in terms of the variable u, takes the form

$$\phi''=\frac{18u(1+4u^3)}{(1+u^3)(1-8u^3)}\phi-\frac{(1-28u^3-56u^6)}{u(1+u^3)(1-8u^3)}\phi'.$$

This may be used to eliminate the $\phi\phi''$ term from (53). The result is

$$D^2z-\frac{1}{2z}(Dz)^2=e_1\phi^2+e_2\phi\phi' \tag{54}$$

where

$$e_1 = 6\frac{(1+2u-2u^2)^3(1-10u+40u^2-7u^3+88u^4+8u^5-8u^6-64u^7-32u^8)}{(1+u^3)(1-8u^3)}$$
$$= \frac{6}{y}\big(1+2u-2u^2\big)^3\Big(1-15y-(1-2u)\sqrt{1-28y+192y^2}\Big)$$

and

$$e_2 = -2\,\frac{(1+2u-2u^2)^4(1+2u^2)(1-4u-2u^2)(1-2u+6u^2+4u^3+4u^4)}{u(1+u^3)(1-8u^3)}$$
$$= -2\big(1+2u-2u^2\big)^4\,\frac{\sqrt{1-28y+192y^2}}{y},$$

where the last step in each case is a consequence of the identities in Lemmas 7.2 and 7.3. Finally, (51) can be used to eliminate the $\phi\phi'$ term from (54) and then (50) can be used to express ϕ^2 in terms of z. This completes the proof. □

We can now deduce a third order linear differential equation for z.

Lemma 7.7 *We have*

$$y^2\big(1-28y+192y^2\big)\frac{d^3z}{dy^3}+3y\big(1-42y+384y^2\big)\frac{d^2z}{dy^2}$$
$$+\big(1-96t+1332y^2\big)\frac{dz}{dy}-6(1-30y)z=0. \tag{55}$$

Proof The differential equation in Lemma 7.6 simplifies to

$$\frac{1}{2z}\left(\frac{dz}{dy}\right)^2=\frac{d^2z}{dy^2}+r_1\frac{dz}{dy}+r_2z \tag{56}$$

where

$$r_1=\frac{1-42y+384y^2}{y(1-28y+192y^2)} \quad \text{and} \quad r_2=\frac{-6(1-15y)}{y(1-28y+192y^2)}.$$

Now take the derivative of (56). The nonlinear terms that arise on the left hand side can be eliminated using (56) (twice), and the resulting differential equation simplifies to the linear equation

$$\frac{d^3z}{dy^3}+3r_1\frac{d^2z}{dy^2}+\left(2r_1^2+2r_2+\frac{dr_1}{dy}\right)\frac{dz}{dy}+\left(2r_1r_2+\frac{dr_2}{dy}\right)z=0.$$

On further simplification, this yields (55). □

We are now ready for

Proof of Theorem 3.1 for the results that involve x_{18}, y_{18} and z_{18} The result

$$q\frac{d}{dq}\log y_{18} = z_{18}\sqrt{1-28y_{18}+192y_{18}^2}$$

was established in Lemma 7.5. Next, we seek a power series solution of the differential equation (55) of the form

$$z = \sum_{k=0}^{\infty} s(k)y^k. \tag{57}$$

The coefficients must satisfy the recurrence relation

$$(k+1)^3 s(k+1) = 2(2k+1)\big(7k^2+7k+3\big)s(k) - 12k\big(16k^2-1\big)s(k-1),$$

the same recurrence as (21) satisfied by the sequence $s_{18}(k)$. If we compare the first few terms in the q-expansions of (57), we find that $s(0)=1$ and $s(1)=6$ (and hence $s(-1)=0$). It follows that $s(k)=s_{18}(k)$, and this completes the proof of Theorem 3.1 for the functions y_{18} and z_{18}. □

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