Closed Form Variable Fractional Time Delay Using FFT

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***Abstract*—In this letter, a novel tunable frequency response is proposed for variable fractional delay filter with careful consideration on conjugate symmetry. This proposed fractional delay method can be efficiently realized by FFT using the tunable fractional delay parameter. Its equivalent closed form to windowing method is derived and presented. Several comparisons to other windowing methods are also demonstrated, such as the magnitude response, group delay, and approximation error between ideal frequency response.**

***Index Terms*—Fractional delay (FD) filter, windowing method.**

I. Introduction

FRACTIONAL delay (FD) is important in many signal processing applications such as communication, speech coding and sample-rate conversion, etc. [1]–[3]. However, making an arbitrary shift of time on discrete sampled signals cannot be realized directly [4]. This problem leads to the design art of fractional delay filters which have been investigated by several studies [5]–[12]. Laakso *et al.* [5] provide a comprehensive review of several standard FD filter design techniques, such as windowing method, Farrow structure, and maximally flat approximation. In Farrow structure [6], the finite-length impulse response (FIR) transfer function is treated as a polynomial of FD and solved by the mean-square-error criteria. The Farrow structure is designed offline with complicated optimization method, but the FD value can be controlled by a single parameter. Our proposed method has a closed form design solution, and is equivalent and compatible to the windowing method. Also, it can be realized by fast Fourier transform (FFT) efficiently, which has lower complexity than the Farrow structure and the traditional windowing method. Simple closed form for FD design and lower complexity using FFT are our advantages.

Here, in this letter, a novel tunable frequency response is proposed for variable FD filter, which possesses the conjugate symmetry property. Its equivalent closed form to windowing method is also derived. Meanwhile, the analytic closed form of tunable filter's frequency response is attractive and useful for variable fractional time delay in signal processing applications. This letter is organized as follows. Section II presents the frequency response and impulse response of the proposed FD filter, and reveals its relation to windowing method. Section III illustrates several simulation examples and comparisons with the traditional windowing methods. Finally, in Section IV, conclusions are made.

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II. Fractional Time Delay and its Closed Form in Windowing Method

The ideal continuous frequency response of fractional delay filter is given as

$$H_{id}(e^{j\omega}) = e^{-j\omega D},\ D = I + d \tag{1}$$

where ω is the normalized angular frequency and D is the delay amount that can be split into integer part I and fractional part d. By utilizing (1) and Fourier transforms, the delayed version of continuous-time signal $x(t)$ can be implemented as

$$x(t-D) = IFT\left\{H_{id}(e^{j\omega}) \cdot FT\left\{x(t)\right\}\right\} \tag{2}$$

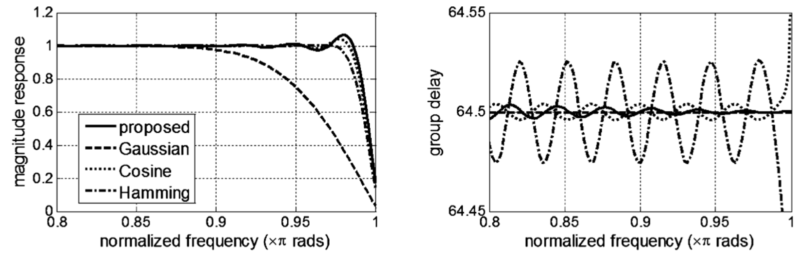
in which $FT\{\cdot\}$ and $IFT\{\cdot\}$ denote the forward and inverse Fourier transform, respectively. Equation (2) reveals that a displacement D in time corresponds to a $-\omega D$ linear phase shift in frequency. However, delaying discrete-time signals must be treated with care. A discrete periodic signal $x[n]$ delayed with an integer amount I can be realized by the following frequency response

$$G[k] = e^{-j(2\pi/N)kI},\ for\ 0 \le k \le N-1. \tag{3}$$

However, this frequency response (3) is not proper for fractional delay systems with $D = I + d$ because (3) does not possess conjugate symmetry property for fractional delay. Fractionally delaying a real discrete signal by (3) may result in unnatural complex output. In consideration of this, we propose a novel conjugate symmetric frequency response $H[k]$ for FD filter

$$H[k] = \begin{cases} 1, & k = 0 \\ e^{-jD(2\pi/N)k}, & k = 1, 2, \ldots, \frac{N}{2}-1 \\ \cos(D\pi), & k = \frac{N}{2} \\ e^{jD(2\pi/N)(N-k)}, & k = \frac{N}{2}+1, \ldots, N-1 \end{cases} \tag{4}$$

where N is an even number. This novel frequency response (4) is designed to be conjugate symmetric with respect to $N/2$; that is $H[k] = H^*[N-k]$ where the star denotes the complex conjugate. In this case, the elements $H[0]$ and $H[N/2]$ are purely



(a) (b)

Fig. 2. (a) Magnitude responses and (b) group delays for $N = 128$, $I = 64$, $d = 0.5$ of proposed method, Gaussian-window method, Cosine-window method, and Hamming-window method.

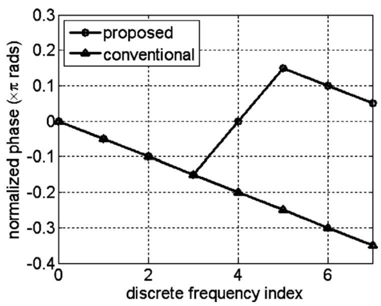


Fig. 1. Phase responses of conventional method (3) and proposed method (4) at fractional delay $d = 0.2$ and $N = 8$. Our proposed method (i.e., the curve with circle mark) is conjugate symmetric and has a phase jump around $N = 4$.

real for conjugate symmetry, and $H[N/2]$ is chosen as the real part of $\exp[-jD \times (2\pi/N) \times (N/2)] = \exp(-jD\pi)$. To discuss the major difference between (3) and (4), Fig. 1 shows the phase responses of conventional method (3) at $I = 0.2$ and proposed method (4) at fractional delay $d = 0.2$ and $N = 8$. As shown, the phase response of our proposed method (i.e., the curve with circle mark) is conjugate symmetric and has a phase jump around $N = 4$. Based on the proposed frequency response (4), the discrete delayed signal $x[n-D]$ can be implemented as

$$x[n-D] = IFFT_N\left\{H[k] \cdot FFT_N\left\{x[n]\right\}\right\}, \; D = I + d \quad (5)$$

in which $FFT_N\{\cdot\}$ and $IFFT_N\{\cdot\}$ denote N-point forward and inverse fast Fourier transform, respectively. By using (5), discrete signal $x[n]$ can be delayed with a delay amount D composed of an integer delay I and fractional delay d. It is noted that the FD method (5) has some remarkable unique features: 1) a real input can result in a real delayed output as nature as continuous signal; 2) an N-point input results in an N-point delayed output without interpolation or resampling; 3) the fractional delay amount is a tunable parameter which can be varied and the proposed method for variable fractional delay can be efficiently realized by FFT.

According to the conjugate symmetric frequency response $H[k]$ in (4) for FD filter, we derive its closed form impulse response expression $h[n]$ as

$$h[n] = \frac{\cos\left(\frac{\pi(n-D)}{N}\right)}{\mathrm{sinc}\left(\frac{n-D}{N}\right)} \cdot \mathrm{sinc}(n-D), \text{ for } 0 \le n \le N-1. \quad (6)$$

That is, (6) is the inverse DFT of frequency response $H[k]$ in (4) (see Appendix for the details). It is noteworthy that the impulse response (6) is a special kind of window for FD filter design. As discussed in [5], $\mathrm{sinc}(n-D)$ with infinite duration is the ideal solution to FD problem and, by truncation into finite length, it results in Gibbs phenomenon and ripple in magnitude response. Since a bell-shaped window can be used for time-domain weighting to reduce the Gibbs phenomenon, the windowing method for FD filter is given as

$$h_w[n] = w(n-D) \cdot \mathrm{sinc}(n-D), \text{ for } 0 \le n \le N-1 \quad (7)$$

where $w(n-D)$ is a shifted window sequence that can be chosen as Gaussian window, Cosine window, Hamming window, etc. By comparing (6) with (7), we can prove that the proposed FD filter (6) is a special case of windowing method (7) with window function $w[n] = \cos(\pi n/N)/\mathrm{sinc}(n/N)$. However, unlike the traditional fractional delay filter design by windowing methods, in which filtering is implemented by convolution in time domain, our proposed method can be efficiently realized by FFT (5) for its closed form frequency response in frequency domain. The required operations of proposed method (5) are two FFT operations and one N complex multiplication. But, traditional fractional delay by windowing method (7) needs N^2 multiplications by convolution in time domain. Therefore, our method requires much less complexity with $O(N\log_2 N)$ compared to other windowing methods with $O(N^2)$. Fig. 2 shows the magnitude responses and group delays of proposed method (solid curve), Gaussian-window method (dashed curve), Cosine-window method (dotted curve), and Hamming-window method (dash-dotted curve), respectively. The four normalized frequency responses are compared and shown in high-frequency band between $[0.8, 1]$. As is illustrated in Fig. 2(a), the magnitude response of our method has more sharp transition band, and in Fig. 2(b), the group delay of our method can approximate to actual delay amount.

III. Experimental Results

Since the proposed FD filter (6) is a special case of windowing method (7), in this section, we compare the proposed FD filter with traditional windowing methods in Gaussian window, Cosine window and Hamming window. To evaluate the performance of FD windowing methods, the normalized root mean square difference (RMSD) is defined as

$$RMSD = \left(\frac{1}{N}\sum_{n=0}^{N-1} \left| x_{id}[n-D] - x[n-D]\right|^2\right)^{1/2} \quad (8)$$

where $x_{id}[n-D]$ is the ideal delayed solution produced by sampling the continuous delayed signal, and $x[n-D]$ is the discrete

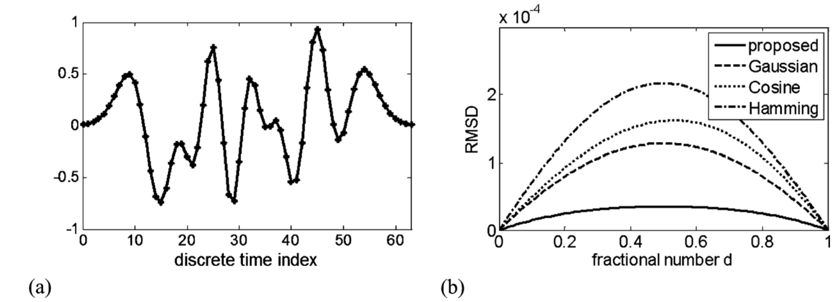


Fig. 3. (a) Input Hermite Gaussian function with $N = 64$ and (b) RMSD curves for different delays by using proposed method, Gaussian-window method, Cosine-window method, and Hamming-window method.

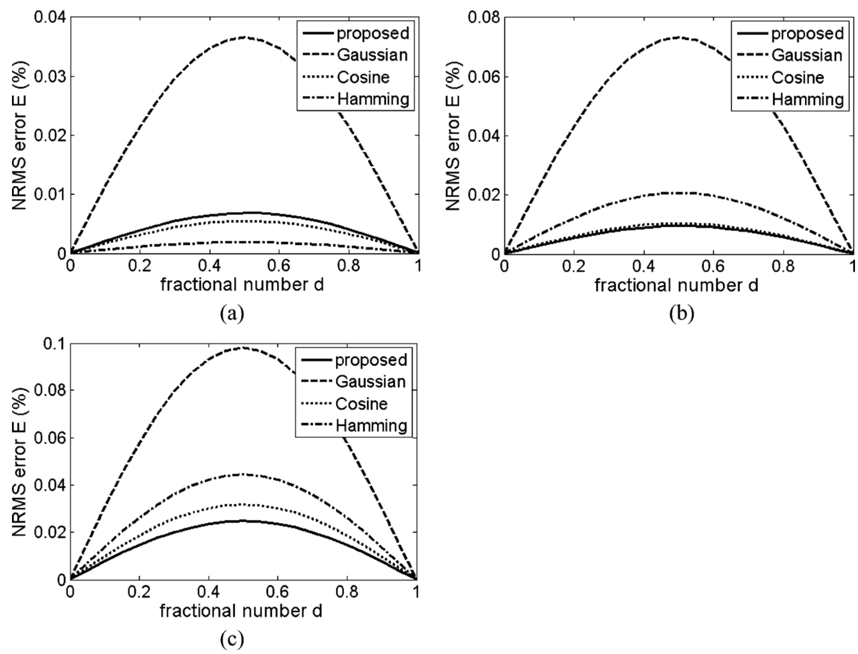


Fig. 4. NRMS error curves in different fractional delay filter design for $d \in [0, 1)$ using proposed method, Gaussian-window method, Cosine-window method, and Hamming-window method. $N = 64$, and bandwidth parameter (a) $\lambda = 0.95$, (b) $\lambda = 0.97$, (c) $\lambda = 0.98$.

delayed output generated from windowing method. Fig. 3(a) shows a tested input function $f(t)$ with $N = 64$, which is a combination of the 5th and 12th order Hermite Gaussian functions:

$$f(t) = \Phi_5(t) + \Phi_{12}(t)$$
$$\text{where } \Phi_p(t) = \frac{1}{\sqrt{2^p p! \sqrt{\pi}}} \exp\left(\frac{-t^2}{2}\right) H_p(t). \tag{9}$$

These Hermite Gaussian functions $\Phi_p(t)$ are the functions generated by the product of Gaussian function and Hermite polynomial $H_p(t)$. The experimental RMSD curves (8) for different delay amounts d between $[0, 1)$ by using proposed method (solid curve), Gaussian-window method (dashed curve) with $\sigma = 1.6$, Cosine-window method (dotted curve), and Hamming-window method (dash-dotted curve) are demonstrated in Fig. 3(b), respectively. As shown in this example, the delayed result from our method has much smaller RMSD than the others, and the largest RMSD for all methods occurs around $d = 0.5$. In addition, we compare the transfer function of FD windowing methods with the ideal frequency response (1). The transfer function $H(z)$ of length-N windowing method filter (7) can be calculated by

$$H(z) = \sum_{n=0}^{N-1} h_w[n] z^{-n} \tag{10}$$

and the normalized root mean square (NRMS) error for judging the quality of windowing method filters is defined as

$$NRMS = \left(\frac{\int_0^{\lambda\pi} \left| H_{id}(e^{j\omega}) - H(e^{j\omega}) \right|^2 d\omega}{\int_0^{\lambda\pi} \left| H_{id}(e^{j\omega}) \right|^2 d\omega} \right)^{1/2} \times 100\% \tag{11}$$

where the parameter λ is the "bandwidth" parameter. Obviously, the smaller the NRMS error (11) is, the better the performance of the FD method is. Fig. 4 shows the NRMS error curves (11) for different delay amounts d between $[0, 1)$ from proposed method (solid curve), Gaussian-window method (dashed curve) with $\sigma = 1.6$, Cosine-window method (dotted curve), and Hamming-window method (dash-dotted curve). The experimental bandwidth parameter λ is chosen as 0.95, 0.97, and 0.98 where their corresponding NRMS error curves are plotted in

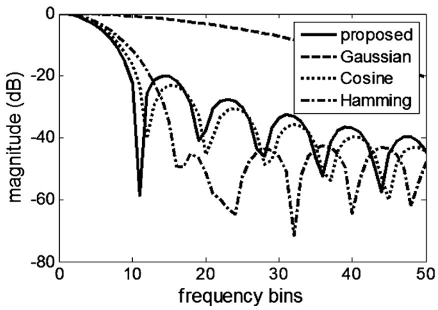


Fig. 5. Comparison of frequency magnitude responses of proposed window, Gaussian window, Cosine window, and Hamming window, $N = 128$.

Fig. 4(a)–(c), respectively. As shown, our method has better performance in high frequency band for $\lambda \geq 0.97$. To further discuss the window choice considerations, we plot frequency responses of proposed window $w[n] = \cos(\pi n/N)/\mathrm{sinc}(n/N)$, Gaussian window with $\sigma = 1.6$, Cosine window, and Hamming window in Fig. 5. As can be seen in this figure, the main lobe width of the proposed window is narrower with trade-off slightly higher sidelobes. That is, the proposed FD filter (6) has better frequency resolution characteristics than other three window methods.

IV. Conclusions

This letter investigates the fractional delay in time using FFT. With careful consideration on conjugate symmetry, a tunable frequency response for variable FD filter is discussed, and its equivalent time-domain closed form in windowing method is also derived. Compared with other windowing methods, our proposed FD filter can be efficiently implemented by FFT using the tunable fractional delay parameter and has better performance characteristics than the other three window methods.

Appendix

Based on the frequency response $H[k]$ in (4), we can derive its impulse response expression by inverse DFT as

$$
\begin{aligned}
&IDFT\left\{H[k]\right\}\\
&= \frac{1}{N}\sum_{k=0}^{N-1} H[k]e^{j(2\pi/N)nk}\\
&= \frac{1}{N}\Bigg\{1+\cos(D\pi)e^{j(2\pi/N)n(N/2)}\\
&\quad + \sum_{k=1}^{(N/2)-1} e^{-jD(2\pi/N)k}e^{j(2\pi/N)nk}\\
&\quad + \sum_{k=(N/2)+1}^{N-1} e^{jD(2\pi/N)(N-k)}e^{j(2\pi/N)nk}\Bigg\}\\
&= \frac{1}{N}\Bigg\{1+\cos(D\pi)\cos(n\pi)\\
&\quad + \frac{e^{j(2\pi/N)(n-D)}-e^{j\pi(n-D)}}{1-e^{j(2\pi/N)(n-D)}} + e^{j2D\pi}\\
&\quad \cdot \frac{e^{j(2\pi/N)((N/2)+1)(n-D)}-e^{j(2\pi)(n-D)}}{1-e^{j(2\pi/N)(n-D)}}\Bigg\}\\
&= \frac{1}{N}\Bigg\{1+\cos(D\pi)\cos(n\pi)-1\\
&\quad + \frac{e^{j(2\pi/N)(n-D)+j\pi(n+D)}-e^{j\pi(n-D)}}{1-e^{j(2\pi/N)(n-D)}}\Bigg\}\\
&= \frac{1}{N}\Bigg\{\cos(D\pi)\cos(n\pi)-\cos(n\pi)\\
&\quad \cdot \frac{\sin\left(\frac{\pi(n-D)}{N}\right)\cos(D\pi)+\cos\left(\frac{\pi(n-D)}{N}\right)\sin(D\pi)}{\sin\left(\frac{\pi(n-D)}{N}\right)}\Bigg\}\\
&= \frac{1}{N}\cdot\frac{\cos\left(\frac{\pi(n-D)}{N}\right)}{\sin\left(\frac{\pi(n-D)}{N}\right)}\cdot\left[-\cos(n\pi)\sin(D\pi)\right]\\
&= \frac{\pi(n-D)}{N}\cdot\frac{1}{\pi(n-D)}\cdot\frac{\cos\left(\frac{\pi(n-D)}{N}\right)}{\sin\left(\frac{\pi(n-D)}{N}\right)}\\
&\quad \cdot\left[\sin(n\pi)\cos(D\pi)-\cos(n\pi)\sin(D\pi)\right]\\
&= \frac{\pi(n-D)}{N}\cdot\frac{1}{\pi(n-D)}\\
&\quad \cdot\frac{\cos\left(\frac{\pi(n-D)}{N}\right)}{\sin\left(\frac{\pi(n-D)}{N}\right)}\cdot\sin\left(\pi(n-D)\right)\\
&= \frac{\cos\left(\frac{\pi(n-D)}{N}\right)}{\mathrm{sinc}\left(\frac{n-D}{N}\right)}\cdot\mathrm{sinc}(n-D).
\end{aligned}
$$

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