Problem Set #2: Quantitative Models of Behavior

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Introduction

In this report we will work on several classic behavior models. The topics that we will cover include conditional learning (classical conditioning and operant conditioning), decision making and reinforcement learning.

1 Rescola-Wagner model

1.1 Model description

Let's start with the Rescola-Wagner model. It's a model of classical conditioning in which learning signifies building associations between conditioned (CS) and unconditioned (UCS) stimuli. UCS are often represented in form of rewards like food (for an animal) or money (for a person) while CS are some kinds of neutral stimuli that may or may not allow us to predict the occurrence of this reward.

In a single trial, the presence or absence of the reward is respectively denoted by $r = 0$ or $r = 1$. More than one CS can be taken into account, then the presence of the ith stimulus is denoted by $u^{(i)} = 1$ and its absence by $u^{(i)} = 0$. The animal's prediction v is given by the formula

$$v = \sum_{i=1}^{m} w^{(i)} u^{(i)}$$

where m is the number of CS, and for each i, $w^{(i)}$ is the prediction parameter associated with $u^{(i)}$. If we note $u = (u_1, ..., u_m)$ and $w = (w_1, ..., w_m)$, it can also be written in the form

$$v = w \cdot u$$

where $\cdot$ denotes the scalar product. We can now calculate the predition error $\delta = r - v$ which, with the learning rate ϵ, allows us to write down the update rule for every $w^{(i)}$ after the trial at time t

$$w_{t+1}^{(i)} = w_t^{(i)} + \epsilon \delta_t u_t^{(i)}.$$

1.2   A simple test

To examine the reliability of the model, we'll first look at a very simple experiment. Only one stimulus is considered, so we identify for the moment u with $u^{(1)}$ and w with $w^{(1)}$. Let's assume that in the first 25 trials, both stimulus and reward are present, and during the next 25 trials, only the stimulus is present. The plot of r and u is shown in the figure below.
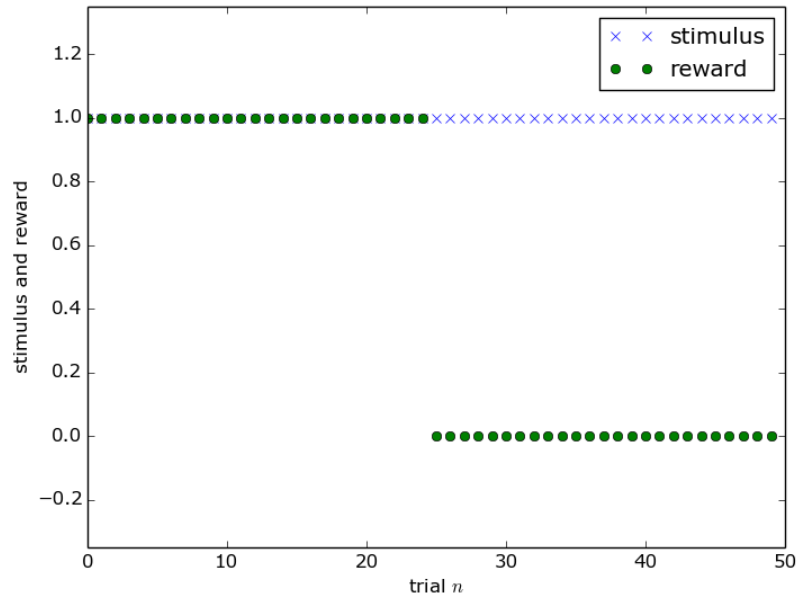

FIGURE 1: A very simple experiment

Now if we do the simulation with the learning rate $\epsilon = 0.1$, the value of w evolves as shown (since u equals always 1 here, we have also all the time $v = w$).
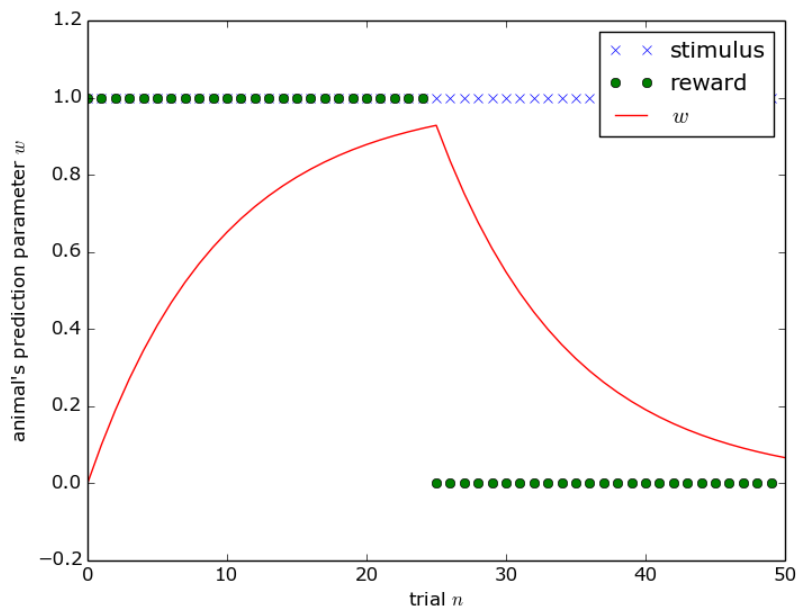

FIGURE 2: Evolution of w in this simple experiment ($\epsilon = 0.1$)

We see an exponential rise of w to value 1 during the first 25 trials and its value decays exponentially to 0 for the rest of the experiment. It's rather reasonable intuitively, and from

a mathematical point of view, we write simply $w_{t+1} - w_t = \epsilon \delta_t u_t$. We know that $\delta_t = r_t - w_t$ (remember that $v_t = w_t$ here) and $u_t = 1$ for all t. For the first 25 trials, we get $w_{t+1} - w_t = \epsilon(1 - w_t)$, so if we put it in a continuous form, it becomes

$$\frac{dw}{dt} = \epsilon(1 - w).$$

We now just need to solve this differential equation to see that $w_t = 1 - e^{-\epsilon t}$. In the same way, for the trials 26 to 50, we can get $w_t = Ae^{-\epsilon(t-25)}$ where A is a constant that can be decided given a particular value of w_t.

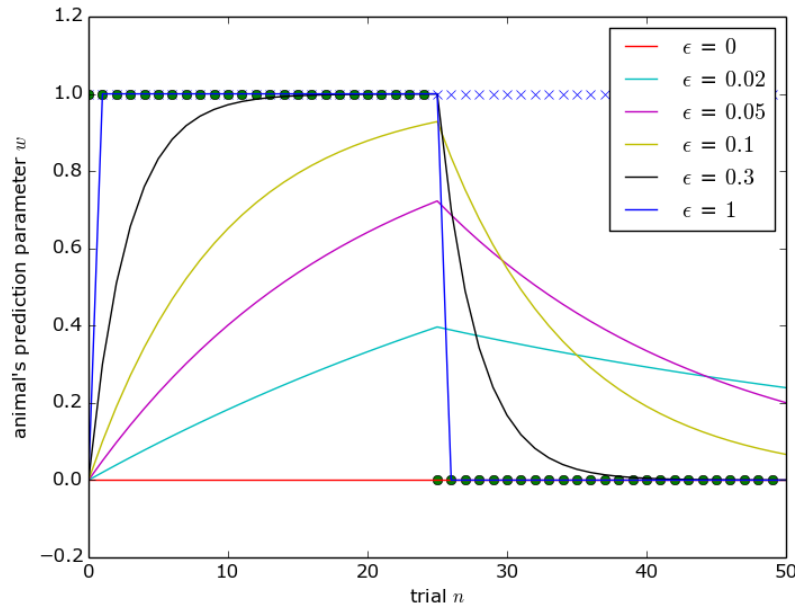The next thing to do is surely to study the impact of the learning rate, so we vary the value of ϵ.

FIGURE 3: Evolution of w in the same experiment but with different ϵ

As what our formula implies, greater the learning rate, faster the animal learns and unlearns the association. It's also interesting to notice that in most of the cases, it's more difficult to unlearn than to learn because the initial δ is smaller (in terms of the absolute value). However, even though one learns faster with a greater learning rate, it doesn't necessarily means that it's better. In fact, there are always noises in what is observed and a smaller learning rate indicates that the animal is somehow doing an average with past experiences.

Two extreme cases are shown in the figure, when $\epsilon = 0$, the animal can never learn, and when $\epsilon = 1$, the animal exploit only the information from the current trial to compute w and is hence very sensitive to noises. In cosequence, the best value of ϵ depends on the animal's environment.

1.3   Partial conditioning

In this paragraph we are interested in the partial conditioning experiment. We modify slightly the experiment of the last paragraph, there is always only one stimulus and it's all the time present, but the presence of the reward is now a random event with a fixed probability p. We simulate this experiment with $p = 0.4$ and the learning rate $\epsilon = 0.1$ over 160 trials.
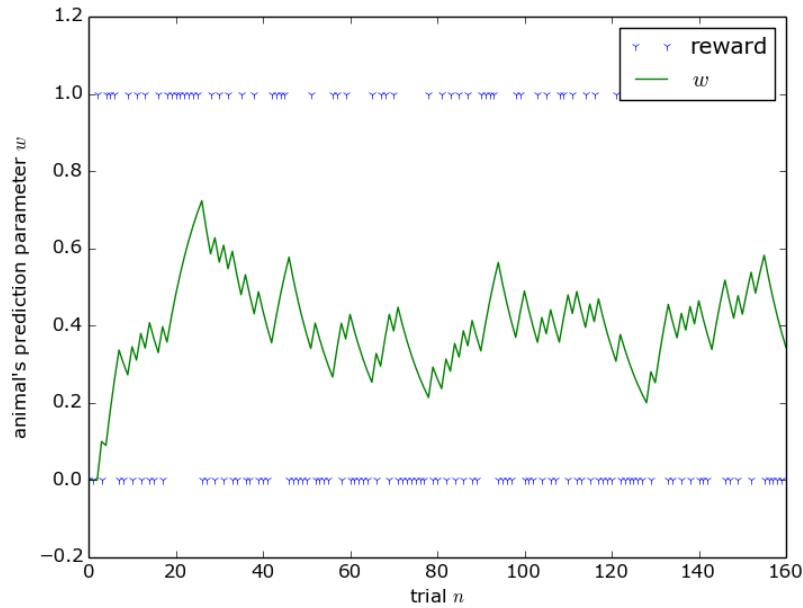
FIGURE 4: Evolution of w in the partial conditioning experiment ($p = 0.4$, $\epsilon = 0.1$)

We can see that the curve becomes quite noisy because the experiment is not deterministic anymore, but roughly speaking, w tends to increase at the beginning and then oscillates around 0.4. However, its value will never converge. As a matter of fact, the learning rate $\epsilon = 0.1$ is too high and a small number of trials can affect the animal's prediction parameter. We can redo the same simulation but with other ϵ values and over more trials to be able to see the evolution of w for smaller learning rate.
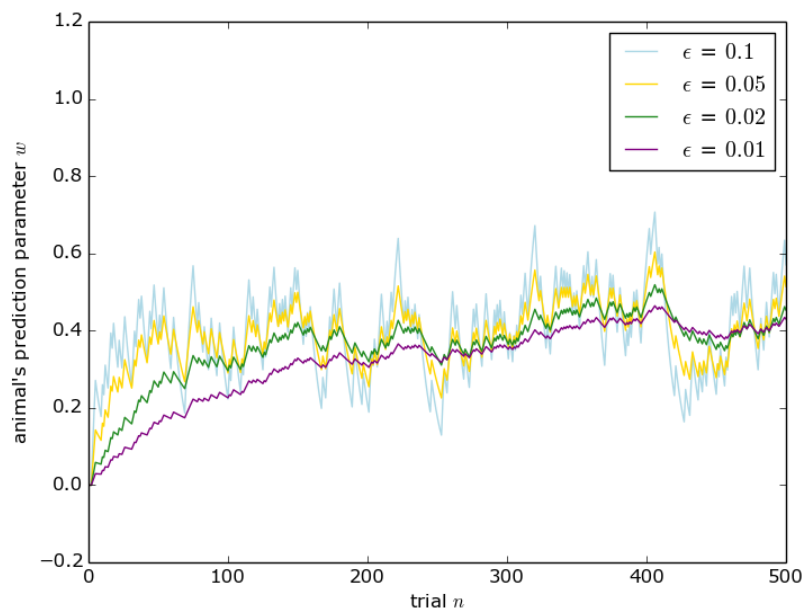

FIGURE 5: Evolution of w in the partial conditioning experiment with different ϵ ($p = 0.4$)

As predicted, when the learning rate decreases, in particular when it equals 0.01, it takes more time for the animal to learn, but the final value is also more stable and doesn't deviate very much from 0.4. On the contrary, we can imagine that if ϵ is bigger than 0.1, the curve becomes even noisier and one can never learn the probability value $p = 0.4$, which confirms what is said before.

1.4   Blocking effect

Another advantage of the Rescola-Wagner model is that it allows us to explain the blocking effect. It means that the conditioning of an association between a CS and an US can be impaired if during the conditioning process, the CS is presented together with another CS that is already associated with the US. Therefore, in the newer experiment, we need two stimuli CS1 and CS2. In the first 25 trials, only CS1 and the reward (US) are present, and during the next 25 trials, CS1, CS2 and the reward are all present. We choose as usual $\epsilon = 0.1$.
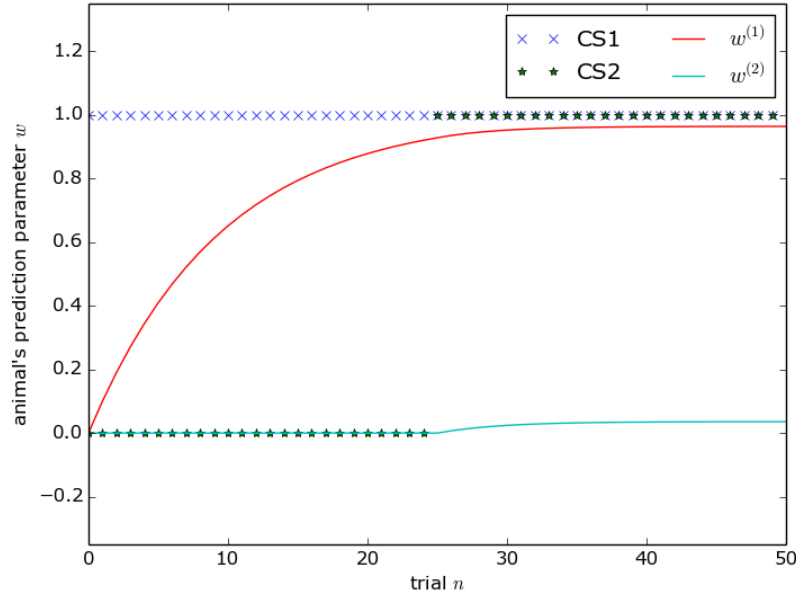

FIGURE 6: Simulation of the blocking effect ($\epsilon = 0.1$)

We get the expected result: the conditioning of CS2 is blocking by the presence of CS1. In the R-W model, the blocking effect can be explained by the fact that $\delta = r - v = r - w^{(1)} - w^{(2)}$ is small during the last 25 trials and using the update rule $w_{t+1}^{(2)} = w_t^{(2)} + \epsilon \delta_t u_t^{(2)}$ we can hardly change the value of $w^{(2)}$.

1.5   Overshadowing

In reality, there is no reason to assume that ϵ is the same for all the stimuli. We should replace the global ϵ by some individual $\epsilon^{(i)}$ for each i. Then, in order to compare different $\epsilon^{(i)}$, a simple experiment can be considered: all the stimuli and the reward are all the time present and we just need to see which particular stimulus is the most associated with the reward after a certain number of trials. For example, if we use two stimuli CS1 and CS2 with respective learning rate 0.2 and 0.1, we plot the result as below.
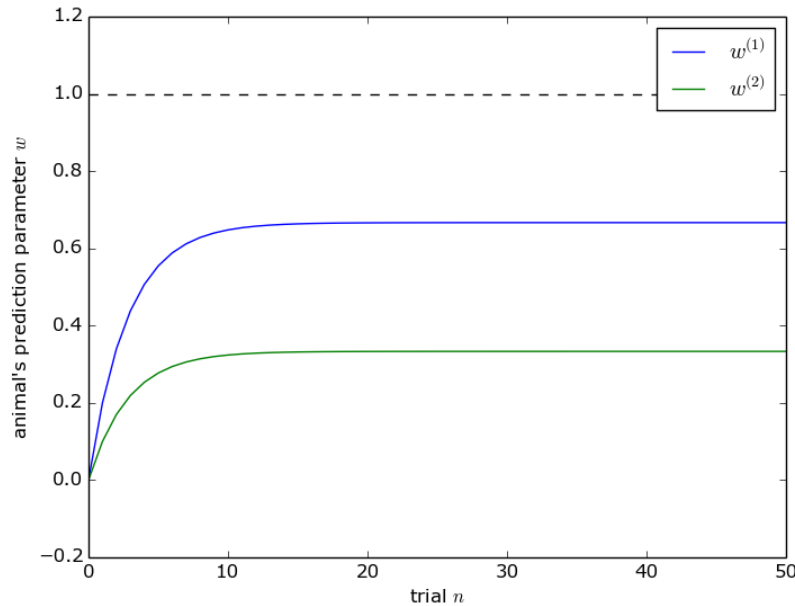
FIGURE 7: Simulation of the overshadowing effect ($\epsilon^{(1)} = 0.2$, $\epsilon^{(2)} = 0.1$)

The stimulus with a higher learning rate (in this case CS1) is more associated with the reward. And if the difference between two learning rates become even larger, it can result in a very weak conditioning of CS2. This is known as the overshadowing effect. We note that neither of the stimuli is fully associated with the reward, but if we add $w^{(1)}$ and $w^{(2)}$, we get something very close to 1.

1.6   Conclusion

In the Rescola-Wagner model conditioning means learning association between conditioned and unconditioned stimuli. At the beginning, we saw that this model could help us understand how one gets conditioned by a stimulus and how this conditioning can again disappear. The most appropriate value of the learning rate may vary from case to case. Next, it turned out that even in a non-deterministic experiment the model is able to find the key probability value. Blocking and overshadowing can also be explained.

Of course this simple model has its limit. For instance, high-order conditioning requires us to take into consideration the time factor, which is not done yet for the time being. Nonetheless, this model is not totally absurd either. In fact, studies have suggested that the activity of some dopaminergic neurons in the brain encodes effectively the prediction error δ of the model.

2   Operant conditioning

2.1   Model description

Operant conditioning is also called instrumental conditioning. It differs from classical conditioning in that the acquired reward or punishment is mainly decided by the agent's behavior, so what one needs to learn is the association between each behavior and its consequence.

We'll illustrate this idea through a small example here. A bee is collecting nectar from

yellow and blue flowers, and at every moment, each type of flower carries a specific reward, which is denoted by r_b for blue flowers and r_y for yellow ones. However, the bee is not aware of the exact values of nectar rewards. Instead, it has some internal estimates m_b and m_y. What the bee needs to do is therefore to make decisions on the basis of these two values and to do real-time updates of them using what it knows.

2.2 Softmax strategy

For the decision part, we assume that the bee adopts the softmax strategy. That is, it chooses the flower of type i with probabily

$$p_i = \frac{\exp(\beta m_i)}{\exp(\beta m_y) + \exp(\beta m_b)}$$

where $i \in \{b, y\}$ and β is the "expoitation-exploration trade-off" parameter (or the inverse temperature parameter if we refer to the Boltzmann distribution model). Writing in this form, it can be easily generalized to situations with more than two types of flowers, but we can also write, for example for p_b

$$p_b = \frac{1}{1 + \exp(\beta(m_y - m_b))}.$$

The name of β comes from the fact that it controls the bee's attitude towards explorative and exploitative behavior. To see this, we can fix $m_y - m_b$ and plot p_b as a funcion of β.
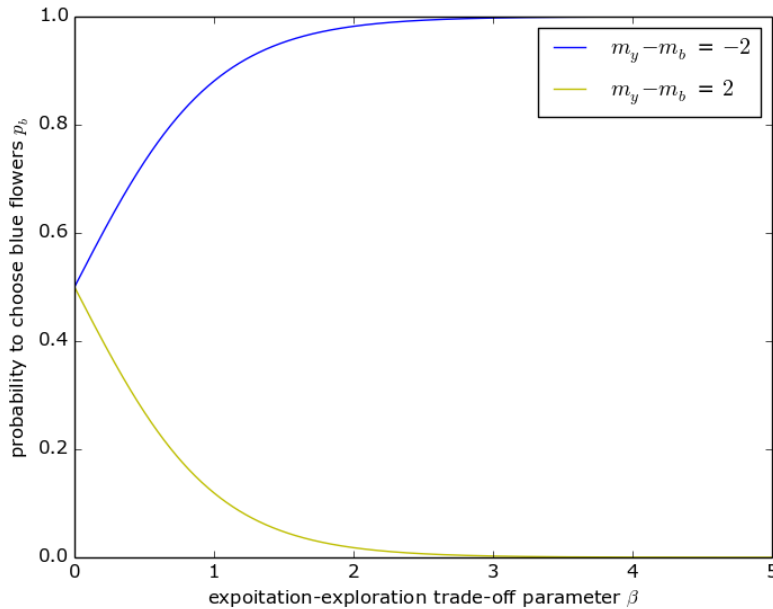

FIGURE 8: Plot of p_b as a funcion of β with fixed $m_y - m_b$

When the difference $m_y - m_b$ is positive, yellow flowers are more rewarding to the bee than blue flowers, so generally speaking it tends to go to yellow flowers to collect nectar. This tendancy is however not very clear when β is small, it suggests that the bee doesn't trust very much its own estimates and puts emphasis on the exploration side.

On the other hand, when β gets larger (in this case typically when it's greater than 2), the bee goes to yellow flowers almost all the time. This implies the bee exploits a lot what it

has learned and doesn't explore much. Now if $m_y - m_b$ is negative, it means that blue flowers are more attractive to bees so the two curves are horizontally symmetrical but the effect of β is the same. We can also plot p_b as a function of $m_y - m_b$ by fixing β.
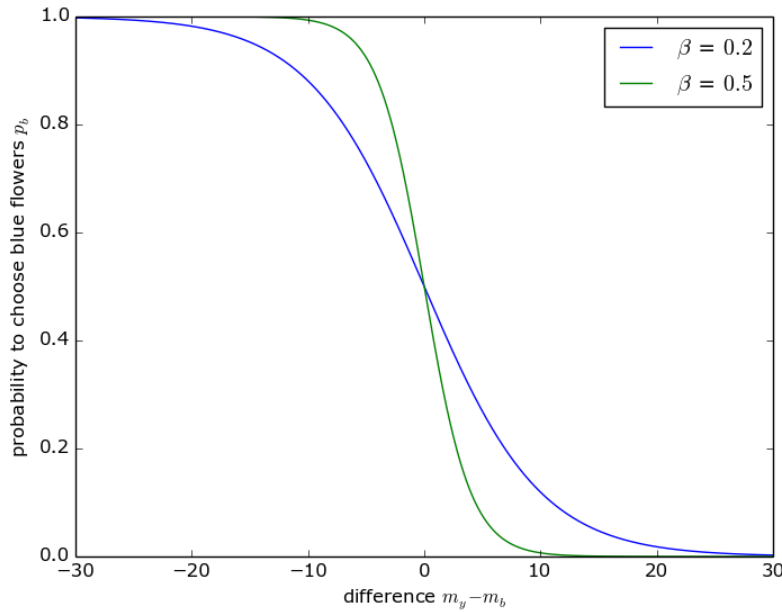

FIGURE 9: Plot of p_b as a funcion of $m_y - m_b$ with fixed β

It's not surprising to get some sigmoid curves whose steepness depends on parameter β. When $m_y - m_b$ tends to minus infinity, p_b tends to 1 because the bee believes that blue flowers are much better than their yellow counterparts, and when $m_y - m_b$ tends to plus infinity, p_b tends to 0 (here we're supposing implicitly $\beta > 0$).

2.3   Dumb bee

Before discussing how to update estimated rewards, we'll first do some simulations with the policy given in the last paragraph. The simulation is for two days. During the fisrt day we have $r_b = 8$ and $r_y = 2$. During the second day, they're set to $r_b = 2$ and $r_y = 8$. The bee is able to sample 100 flowers during one day. Throughout the simulation, the bee can never learn and believes that $m_y = 5$ and $m_b = 0$. First let's assume $\beta = 0$.
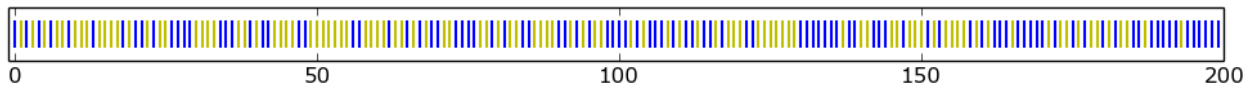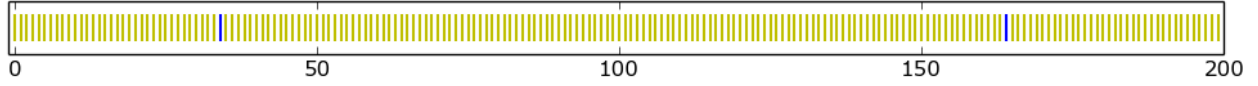

FIGURE 10: The choice of the dumb bee for 200 samplings during two days, $\beta = 0$

It's known that when $\beta = 0$ the bee chooses blue and yellow flowers equiprobably regarless of its internal estimates. This is indeed what is observed here, it's not obvious to say the bee goes to blue or yellow flowers more often (naturally, blue bars for blue flowers and yellow bars for yellow flowers). We do one more simulation with this time $\beta = 0.8$.

FIGURE 11: The choice of the dumb bee for 200 samplings during two days, $\beta = 0.8$

In this case using the given formula we have $p_b = 1/(1 + e^4) \sim 1.7\%$. Our result of the simulation is not very far from this value. The bee visits blues flowers only twice which stands for a probability of 1%. Anyway, in the two cases, we can say that the bee's performance is quite poor because it doesn't learn from experiences, in average the reward that the bee gets is 5 but it could have done better.
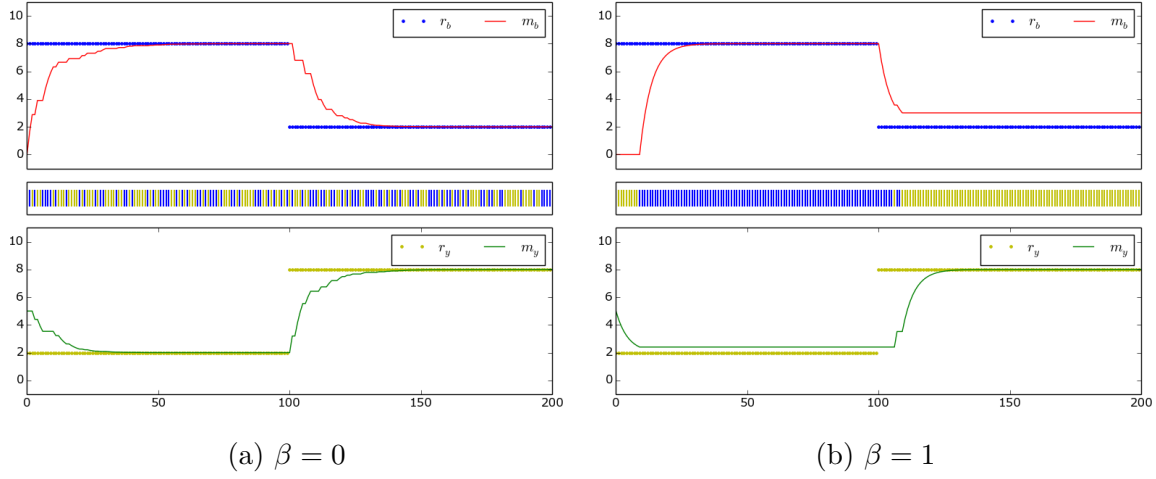
2.4 Smart bee

To learn the estimated reward, we inspire from the first part of the report, the R-W model. Consequently, the online update rules are given by

$$m_b \rightarrow m_b + \epsilon(r_b - m_b)$$
$$m_y \rightarrow m_y + \epsilon(r_y - m_y)$$

where as usual ϵ is the learning rate. The first rule is only used when the bee visits a blue flower and similarly the second rule is considered only when it visits a yellow flower. We start again from $m_y = 5$, $m_b = 0$ and we give $\epsilon = \beta = 0.2$.
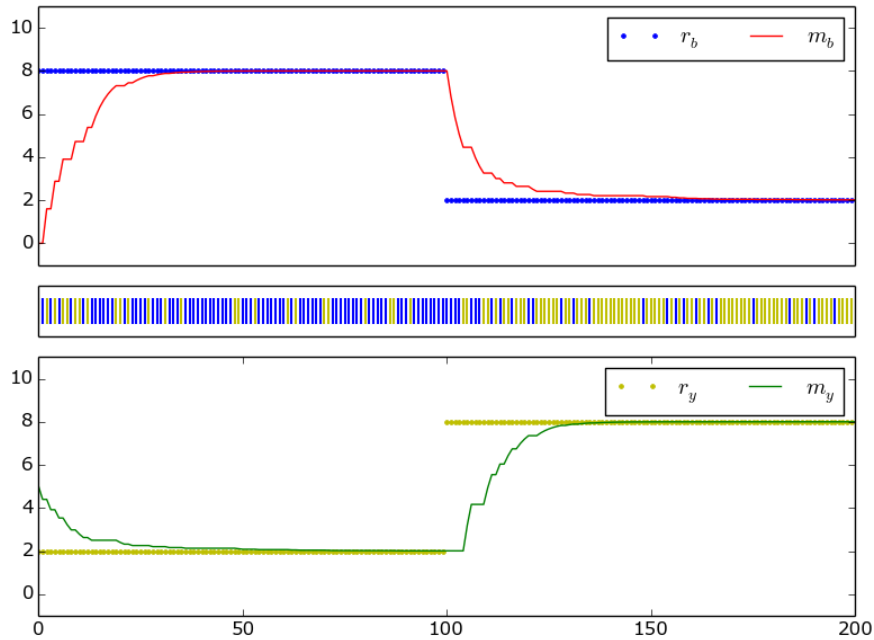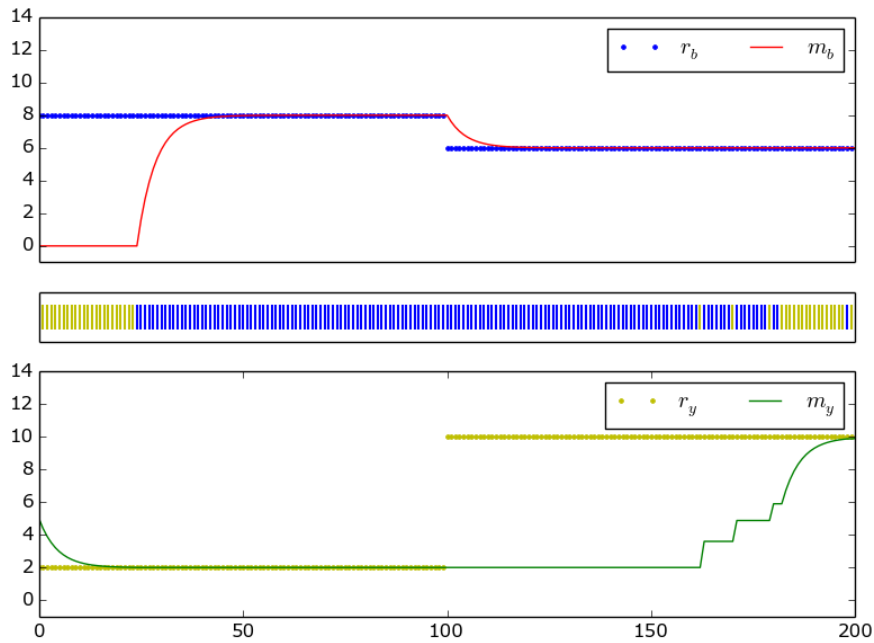

FIGURE 12: Smart bee's choices and internal estimates over two days with $\epsilon = \beta = 0.2$

For the bee this is rather a satisfying result. It's able to learn correctly the reward values and it can also take advantage of what it learns. On average it gets almost 8 units of reward per trial, which is optimal. We will then look at some extreme cases, such as the purely explorative behavior with $\beta = 0$ and the case of strongly exploitative behavior with $\beta = 1$.

(a) $\beta = 0$ (b) $\beta = 1$

FIGURE 13: Smart bee's choices and internal estimates over two days with $\epsilon = 0.2$

For $\beta = 0$, in terms of estimated reward the real value can be learned so it works correctly. Nonetheless, if we take a look at the bee's choices, it's completely random and it doesn't benefit at all from what it has learned.

On the other hand, when $\beta = 1$, surprisingly the outcome is quite ideal. In reality, at the beginning of the two days, the bee learns that the real reward that yellow/blue flowers carry is only 2 so it would also give a try to the other kind of flowers and it discovers that it's pretty appealing. However, we can also imagine a scenario where in the second day the blue flowers have always almost the same amount of nectar but the nectar that one yellow flower carries gets even higher.

FIGURE 14: Smart bee's choices and internal estimates over another two days with $\epsilon = 0.2$, $\beta = 1$

We see that if the bee is lucky it may be able to learn correctly r_b and r_y, but it takes quite a long time because the bee goes very rarely to the type of flowers that it estimates not rewarding. As a consequence, it can hardly benefits from this change.
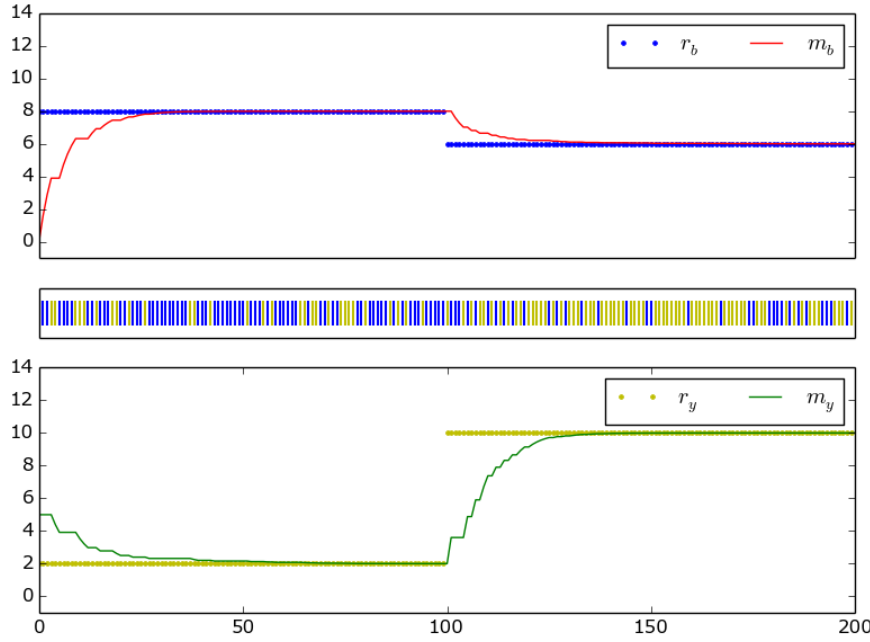
FIGURE 15: Smart bee's choices and internal estimates over another two days with $\epsilon = \beta = 0.2$

On the contrary, if we come back to the initial value of β the bee becomes quickly aware of the modification and can therefore take advantage of it.

2.5 Conclusion

In operant conditioning the agent needs to make choices in order to receive the corresponding award or punishment. So the reward value is now associated with each action. In addiction to learn these values, one also needs to carry out its actions according to its internal estimates. The softmax strategy is often considered for the policy part. It seems rational, but the agent must adapt its β parameter to the environment and finds a trade off between explorative and exploitative behavior. The delta rule is again used for the learning part. By combining these two sides with some adequate parameter values, the agent is able to make nice reponses to environmental changes.

3 Drift diffusion model

3.1 Model description

In a two-alternative forced task (2AFC-task), subjects are asked to choose between two alternative actions. For example, we can consider that the subject is looking at a set of dots on a screen that are moving in different directions and needs to indicate whether they're moving upwards or downwards. A stimulus can be very noisy. In such case, only a small part of the dots are moving in the right direction and the other points move randomly.

Then one way to describe the decision-making process is the drift-diffusion model. We assume that the subject is accumulating evidence for one or other of the alternatives at each time step. In our example, the subject compares the firing rate m_A of an upward-motion sensitive neuron with the firing rate m_B of an downward-motion sensitive neuron. From this,

it computes an integration variable x that obeys

$$\dot{x} = m_A - m_B + \sigma \eta(t)$$

where $\eta(t)$ is a noise term (Guassian white noise with unit standard deviation) simulating the noisiness of real neurons and σ represents the noise level. If x surpasses a threshold μ, the subject decides for outcome A; on the contrary, if x gets lower than $-\mu$, the subject decides for outcome B. To compute x in our program, we simply use the Euler method

$$x(t + \Delta t) = x(t) + (m_A - m_B)\Delta t + \sigma \tilde{\eta}(t)\sqrt{\Delta t}$$

where $\tilde{\eta}(t)$ is selected randomly from a standard Gussian distribution.

3.2   Evolution of x

We'll thus simulate the decision-making process by computing the evolution of x. We choose $m_A = 1$, $m_B = 0.95$, $\Delta t = 0.1$ms, $\sigma = 1/2$, $\mu = 0.4$ and finally $x(0) = 0$ (the subject is neutral at the beginning of the experiment). The simulation is run until $t = 1$s.
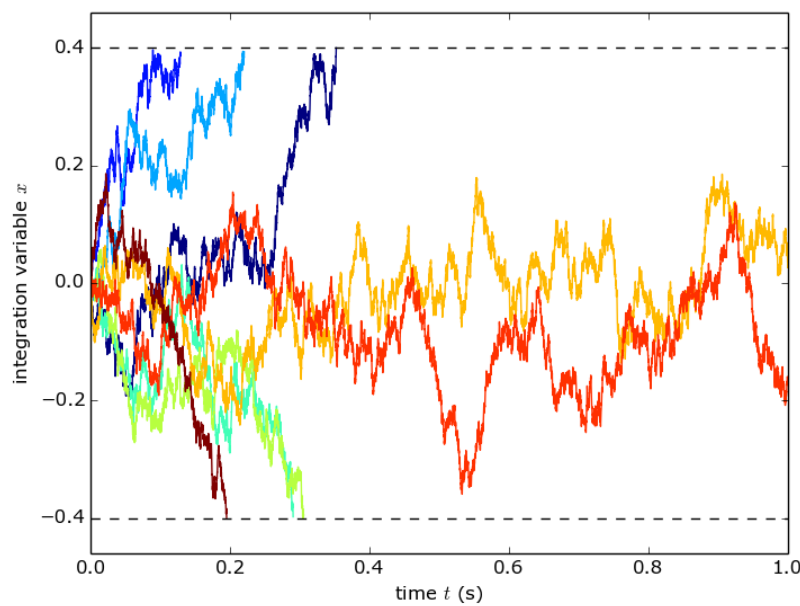

FIGURE 16: Eight runs of the drift diffusion model

The curves are very noisy and even though $m_A > m_B$, the subject chooses B as the final outcome quite often; it's not clear that the probability of choosing A is really higher here. It's simply because m_A is very close to m_B in this case and more simulations are needed to do a deeper analysis. We also see that the subject may not be able to make the decision by the time $t = 1$s.

3.3   Reaction time

We are now intrested in the reaction time of the subject. If x cross the threshold at time t_i, we claim that the reaction time of this run is $\text{RT}_i = 100 + t_i$ (take into consideration the

time that is needed for the nerve signal to be transmitted). We'll then do the simulation 1000 times to see the distribution of reactions times for outcome A and for outcome B.
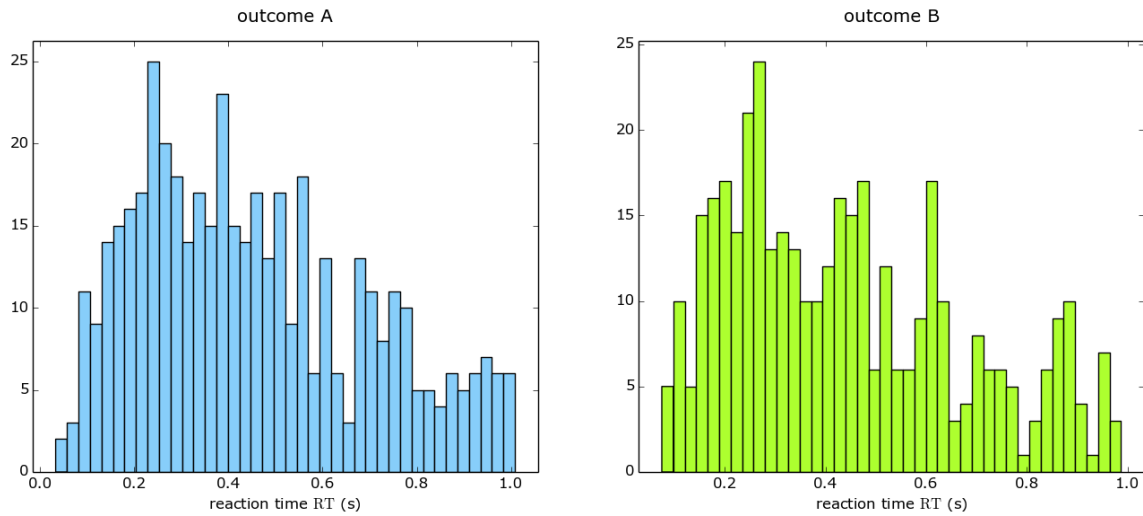

FIGURE 17: Distribution of reaction times

The two histograms have quite similar patterns. It might not be evident to find an exact function to describe the distibution, but at least we observe that decisions are the most often made at the early stage of the simulation (between 0.2 and 0.4s). In addition, with 1000 simulations, we can also see that A is more often chosen than B. In fact, in the above figure, A is chosen 453 times while B is chosen only 389 times (we notice that naturally the sum is not 1000 due to what is mentioned in the last paragraph).

3.4 Probability of choosing a specific outcome

We now denote the evidence for outcome A versus outcome B as $m_E = m_A - m_B$. It's intuitive that when m_E gets larger, the probability p_A of choosing A as outcome also increases. In reality, we have the analytical formula

$$p_A = \frac{1}{1 + \exp(\beta(m_B - m_A))}$$

where $\beta = 2\mu/\sigma^2$. To deduce the probability value by using computer simulations, for a fixed m_E, we'll carry out the simulation 1000 times to calculate the corresponding p_A. We plot the probabilities computed in this manner as a function of m_E for m_E ranging from -0.2 to 0.2. The result is shown at the top of the next page.

Even though it's not yet perfect, the simulation result follows globally the given formula. We can imagine that when the number of simulations used to caclulate the probability tends

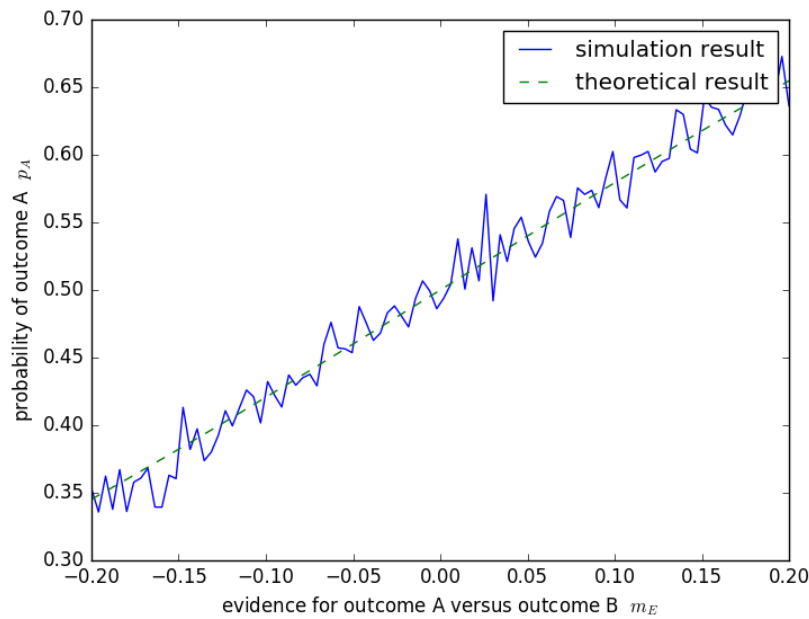to infinity, the simulation curve will converge to the theoretical one.

FIGURE 18: Probability of choosing A for different values of m_E

3.5   Conclusion

Many models are proposed to explain experimental data for 2AFC tasks. DDM is just one of them. The model suggests that the subject is accumulating evidence from the past to make decisions and simulation results seem to match real observations in terms of both external behaviors and neuron activities. As a matter of fact, in the task that is mentioned here, the lateral intraparietal area (LIP) of parietal cortex involved in saccadic movements (we ask the subject to respond by making a saccade to the location of the target) may have neurons that are somehow doing what is described in the model.

4   Temporal difference (TD) learning

4.1   Model description

In a reinforcement learning problem an agent needs to act in the world to maximize its rewards. It's a little like the operant conditioning problem that we've discussed in the second part, but now the time dimension is integrated and the reward is thus no longer immediate. In a way, the agent must be able to "predict the future" to perform the right sequence of actions.

We consider for example a simple maze navigation problem. A rat enters a maze having 7 states $\{A, B, C, D, E, F, G\}$. It starts always at state A, it can then moves on to either state B or C. At B it can go to either D or E and when it's at C, F and G are accessible. Among all of these states, only states E and F are rewarding, with respectively 5 and 2 units of reward. Finally, the rat is taken out of the maze by the experimenter, and thereby moves into the "terminal" state H.

4.2   Just be random

To make things easier, the rat can of course adopt a totally random stategy. Every time when it is at a junction, it goes to either left or right with 50% probability. Now if we do 100 simulations, theoritically, the number of times it visits each state will be: 100 for A and H, 50 for B and C, and 25 for the rest. What we actually get is:

TABLE 1: Number of visits for each state in 100 trials with a random strategy

State	A	B	C	D	E	F	G	H
Number of visits	100	51	49	28	23	22	27	100

That is not very far from the theoritical value. Next, we want to associate a value with each state, which will be evaluated as the expected sum of all possible future rewards when the rat is at this state. While the rat is carrying out the experiment, we want to update these values as well. This can be done through the temporal difference learning rule

$$V(s_t) \leftarrow V(s_t) + \epsilon[r(s_t) + V(s_{t+1}) - V(s_t)]$$

where s_t with $t = \{1, 2, 3, 4\}$ denotes the sequence of states in a trial. The idea is that the reward that the rat is meant to get at s_t is $V(s_{t+1}) - V(s_t)$ and we apply just the delta rule. The update is performed after each trial.
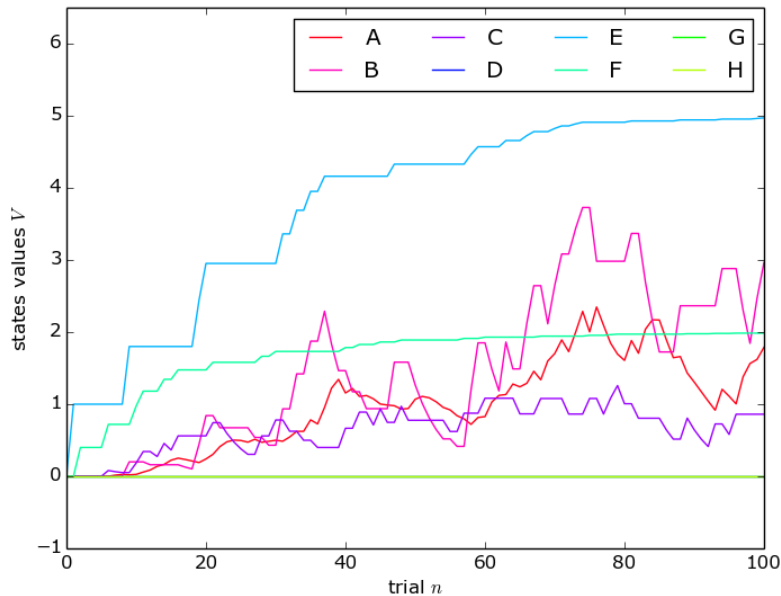

FIGURE 19: Evolution of state values using a random strategy

The rat manages to learn correctly the state values. For example we indeed obtain $V(E) \sim 5$ and $V(F) \sim 2$. Notice that B is not as valuable as E even though one can choose deterministically to go to E at B. This is simply due to the fact that the rat is using a random strategy. In a similar way, C is not as valuable as F. We'll see later that if the rat chooses almost surely to go to the side with the larger value, the result will be different.

4.3 And softmax again

Now that we're able to learn the state values, we'll study a more complicated way to make decisions. There is nothing new, at each junction, the rat chooses which state to go using the softmax strategy. We'll first pick up a quite neutral value of β, say, $\beta = 0.3$. As the purely explorative behavior is exactly the same thing as the random strategy, instead of $\beta = 0$ we'll choose $\beta = 0.05$ to have another example of highly explorative behvaior.

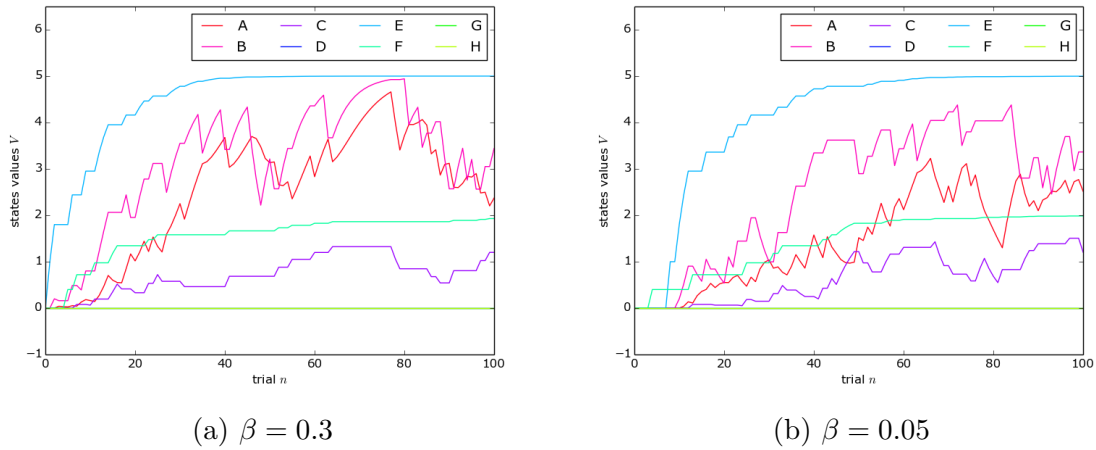(a) $\beta = 0.3$ (b) $\beta = 0.05$

FIGURE 20: Evolution of state values using the softmax strategy

Naturally, the rat succeeds in learning the right values. $V(A)$ and $V(B)$ are higher than before. Moreover, we know that by using the softmax decision rule the rat benefits from its own knowledge. It's shown in the table below, the rat reaches state E much more often.

TABLE 2: Number of visits for each state in 100 trials with a softamx strategy, $\beta = 0.3$

State	A	B	C	D	E	F	G	H
Number of visits	100	75	25	17	58	15	10	100

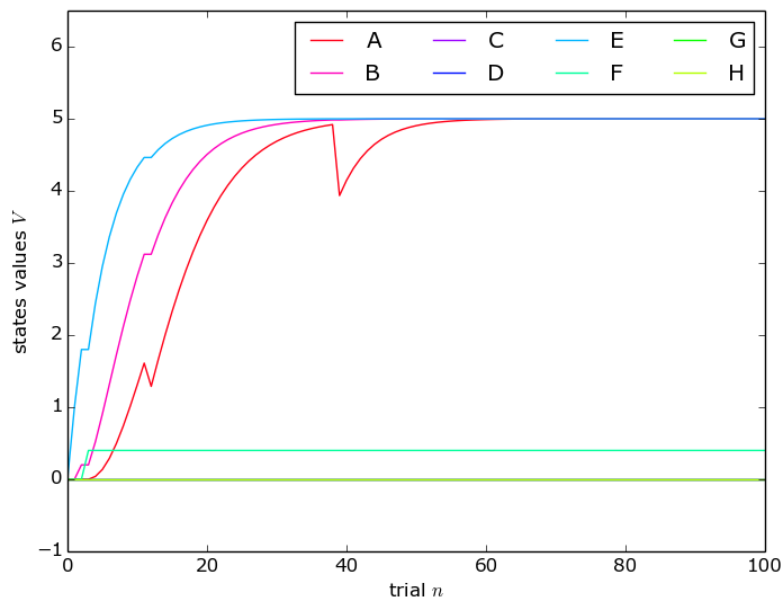What if the rat is very greedy? We assume that $\beta = 1$.

FIGURE 21: Evolution of state values using the softmax strategy, $\beta = 1$

As promised before, both state A and B are as valuable as state E because the rat will almost always chooses to go to E when it's on these two states. We might also note that the rat never learns the correct value of $V(F)$. It doesn't really matter here since E provides more reward than F, but we can also imagine that if very unluckily, at the beginning of the experiment, the rat gets to F a lot of times, it may learn $V(F) = 2$ while $V(E)$ is still zero and then decides to go to F all the time. Generally speaking, being too exploitative/greedy prevents the agent from learning new things and decreases its adaptability to environment changes.

4.4 Conclusion

In this final section, we have worked on a simple maze navigation problem and tried to solve it using TD learning. The key point is to represent all the future possible rewards as the value of each state to allow us to have a long-term consideration.

We didn't take into account the discount factor which supposes that the future doesn't have the same worth as the present. Other similar techniques include Q learning and SARSA that combines states and actions and give values to each possible couple $(state, action)$. Nevertheless, all of them share the same idea: in these values we integrate the present and the future, and in each new trial they're updated to take into consideration the newest information.