Problem Set #4: Networks

Hsieh Yu-Guan

May 22, 2017

Introduction

Besides studying the strucutre and function of a single neuron, it's also important to understand what may happen when neurons communicate between them. In this report we'll thus look at some simple models of neural networks. What will be their dynamics and expressive power? (P.S. we'll ignore all physical units for the whole report.)

1 Some simple networks

1.1 Neuron with autapse

Let's start by working on the simplest model that one can ever imagine: there's only one neuron in the network, and its output feeds back onto itself via a synapse (such a synapse is called an "autapse"). We note x the neuron's firing rate, and it obeys the equation

$$\dot{x}(t) = -x(t) + f(wx(t) + I)$$

where $w = 0.04$ is the strength of the synaptic connection and $I = -2$ is the external (and inhibitory) background input which is constant. Finally, f is the input-output (or activation) function of the neuron having a sigmoidal form and is given by

$$f(s) = 50(1 + tanh(s))$$

where s is the total input of the neuron.

To see that f is indeed a sigmoidal function, we plot it for the range $s \in [-10, 10]$ as shown in Figure 1. Next, we plot the derivative of the firing rate $\dot{x}$ as a function of x (Figure 2). The form should be easily predictable. The function f is first stretched out and then shifted to the right, before we finally add the linear function $x \mapsto -x$ to it.

We observe three zero-crossings in this graph. Let's call them respectively x_1, x_2 and x_3 with $x_1 < x_2 < x_3$. In fact we get $x_1 \sim 2$, $x_2 = 50$ and $x_3 \sim 98$. They are the fixed points of the dynamics. However, x_1 and x_3 are stable while x_2 is unstable. We can see that if x lies between x_1 and x_2, $\dot{x}$ is negative so x will be "attract" to x_1, and if x is smaller than x_1, $\dot{x}$ is positive and x will converge to x_1. The same analysis works for x_3.

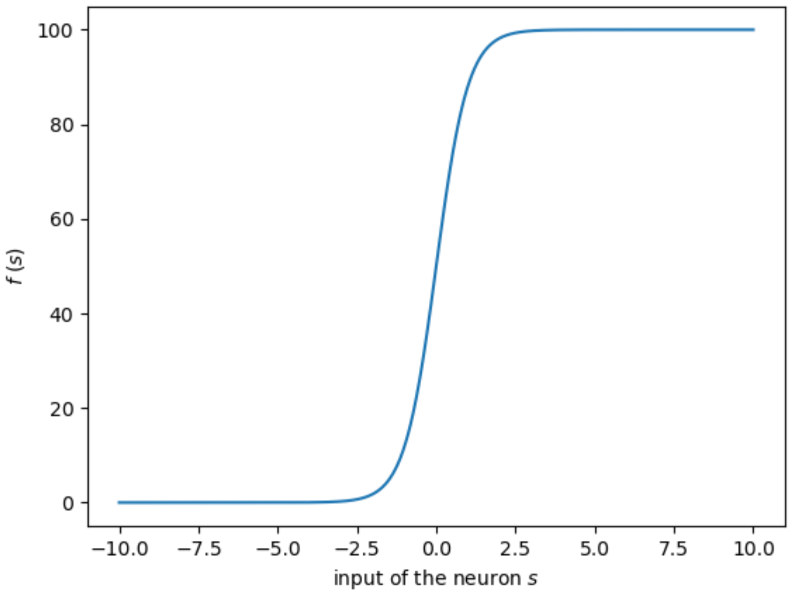

FIGURE 1: The activtion function f of the neuron

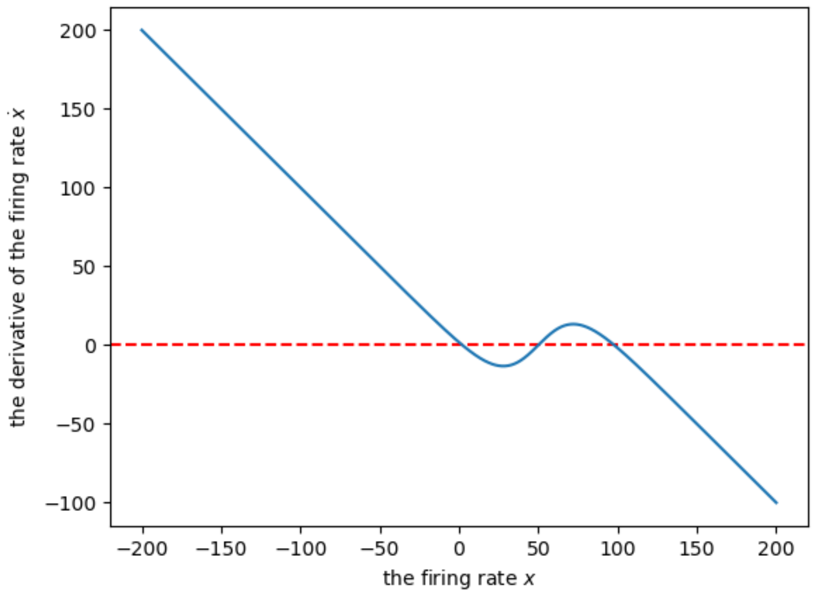

FIGURE 2: $\dot{(x)}$ as a function of x

Now we'll simulate the dynamics of the system by taking a time step $\Delta t = 0.1$ and a total time period $T = 100$. First consider $x(0) = 49$.

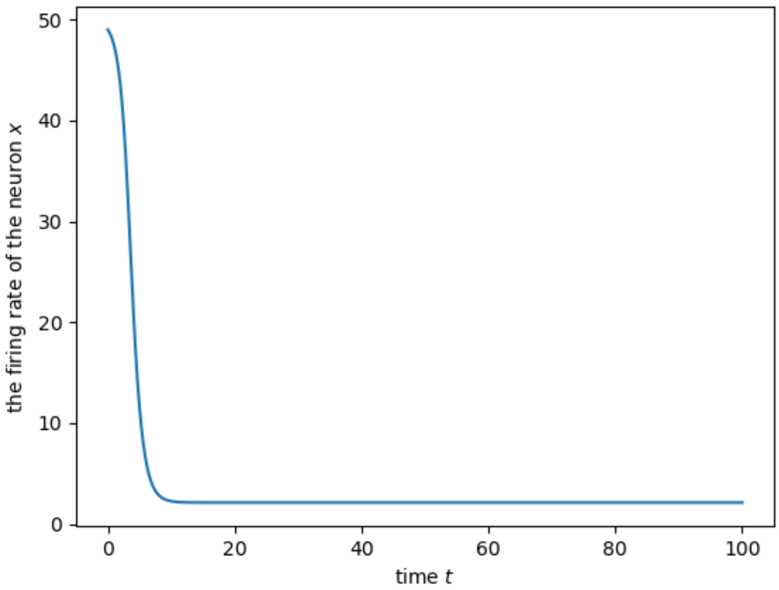

FIGURE 3: The evolution of x for $x(0) = 49$

As predicted before x is attracted to the dynamics attractor x_1. We redo the simulation for $x(0) = 50$.

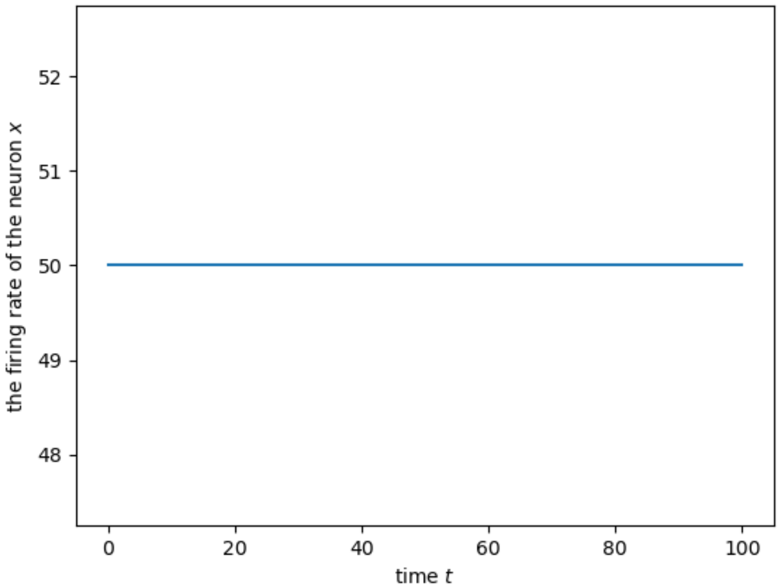

FIGURE 4: The evolution of x for $x(0) = 50$

This time since 50 is itself a fixed point of the dynamics, the system is at equilibrium and the solution doesn't change with time (though 50 is a repeller). Finally let $x(0) = 51$.

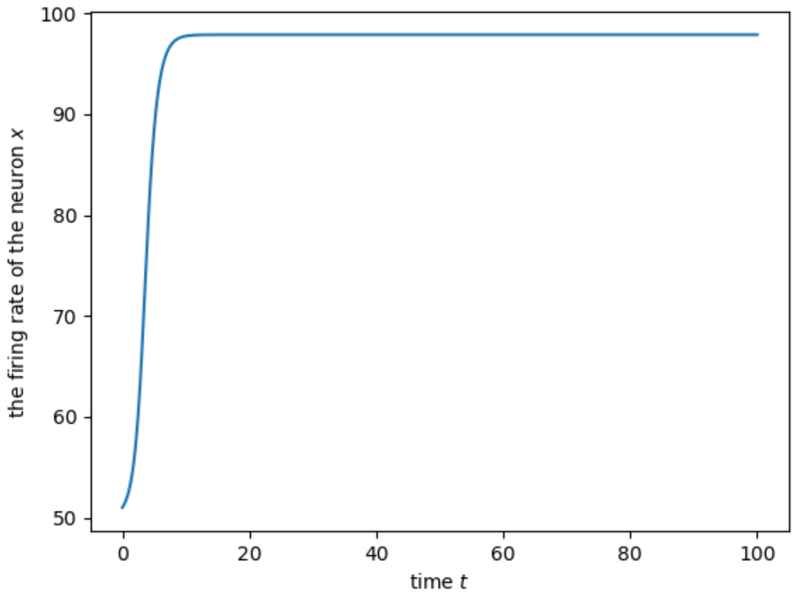

FIGURE 5: The evolution of x for $x(0) = 51$

It's the symmetry of the case $x(0) = 49$. Between the repeller x_2 and the attractor x_3, x evolves towards x_3.

We then add noise to the system, so the differential equation becomes

$$\dot{x}(t) = -x(t) + f(wx(t) + I) + \sigma\eta(t)$$

where $\sigma(t)$ is Gaussian white noise with varaiance 1. First we suppose $\sigma = 5$ and we simulate for $x(0) = 49$.

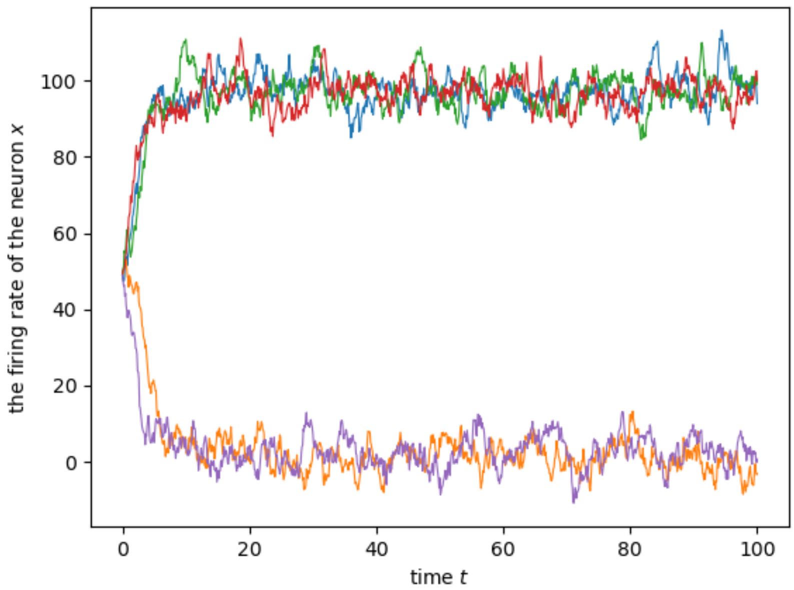

FIGURE 6: The evolution of x with noise $\sigma = 5$ for $x(0) = 49$

We see that there are two different scinarios. With noise we can no longer ensure that the system will converge towards x_1. Since the evolutions of the system are very different for

$x < 50$ and $x > 50$, and 49 is close to 50, slight noise in the model may lead to totaly distinct results. This can be again shown for $x(0) = 50$ and $x(0) = 51$.

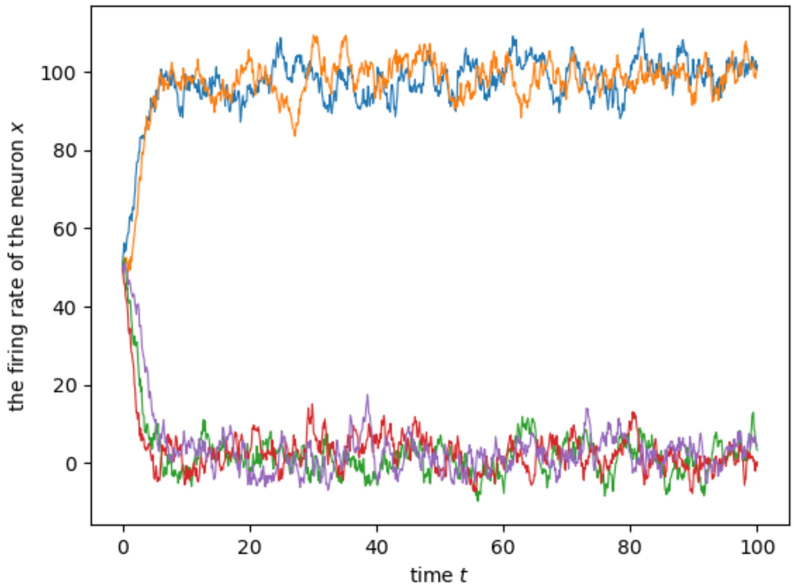

FIGURE 7: The evolution of x with noise $\sigma = 5$ for $x(0) = 50$

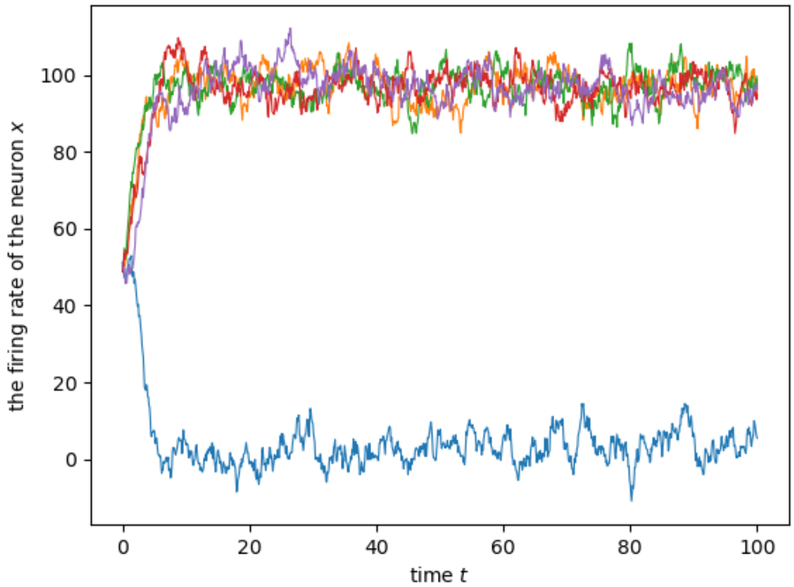

FIGURE 8: The evolution of x with noise $\sigma = 5$ for $x(0) = 51$

Also notice that for $x(0) = 50$, x will not stay anymore at the value 50 because as mentioned before, $x_2 = 50$ is a repeller. In a model with noise, the probability that x is always 50 becomes null.

In all the above examples, though noise can have great influence on the evolution of the system, once x gets far enough from x_2, the evolution is still mainly dominated by the drift term $-x(t) + f(wx(t) + I)$. Nonetheless, this is not the case for a greater noise level, for example, when $\sigma = 80$, as shown in the top of the next page.

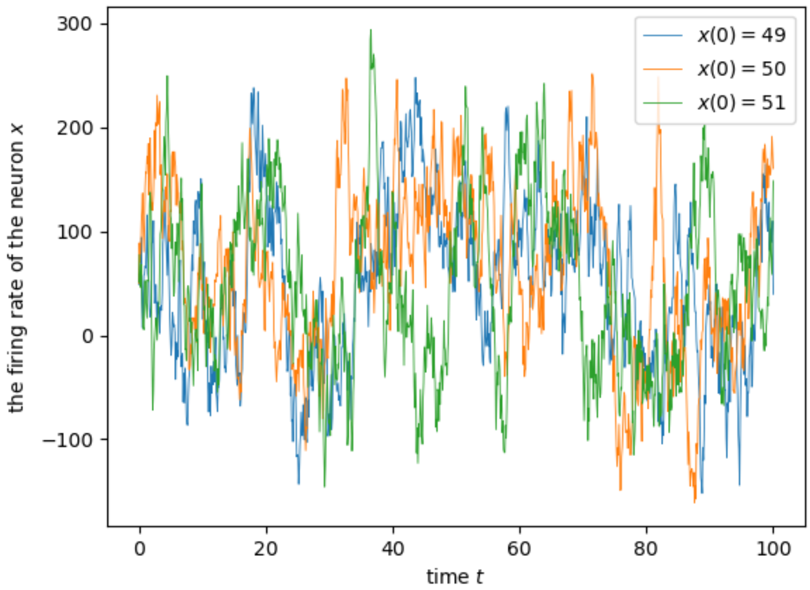

FIGURE 9: The evolution of x with noise $\sigma = 80$ for $x(0) = 49, 50, 51$

As the resulting curves follow almost the same pattern for different initial values of x, I plot all of them on the same graph. It's not worth it to plot several simulations for a same $x(0)$ because we would not be able to see great differences. In short, the evolution of x is donimated by the noise term and becomes just noisy.

1.2 Circuit with mutual inhibition

The second model we'll discuss here is made up of two neurons that are coupled by mutual inhibition. We note the firing rate of the two neurons respectively x_1 and x_2, then the whole system is governed by the differential equations

$$\dot{x}_1 = -x_1(t) + f(wx_2(t) + I)$$
$$\dot{x}_2 = -x_2(t) + f(wx_1(t) + I)$$

where f is defined as before and the inhibitory synaptic weights are given by $w = 0.1$. The external inputs are now excitatory, $I = 5$.

We may also want to use the vector notation that would turn out to be quite useful when the population of neurons gets larger, then the system of differential equations can be put in the form:

$$\dot{\mathbf{x}} = -\mathbf{x}(t) + f(\mathbf{W}\mathbf{x}(t) + \mathbf{I})$$

where $\mathbf{x}$ is the vector of firing rates, $\mathbf{W}$ is the synaptic weight matrix and $\mathbf{I}$ is the vector of input currents. In this particular case, we have

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \qquad \mathbf{W} = \begin{pmatrix} 0 & w \\ w & 0 \end{pmatrix} \qquad \mathbf{I} = \begin{pmatrix} I \\ I \end{pmatrix}.$$

To study the dynamics of this system, we first plot its nullclines, that is, the line for which

$\dot{x}_1(t) = 0$ and the line for which $\dot{x}_2(t) = 0$.

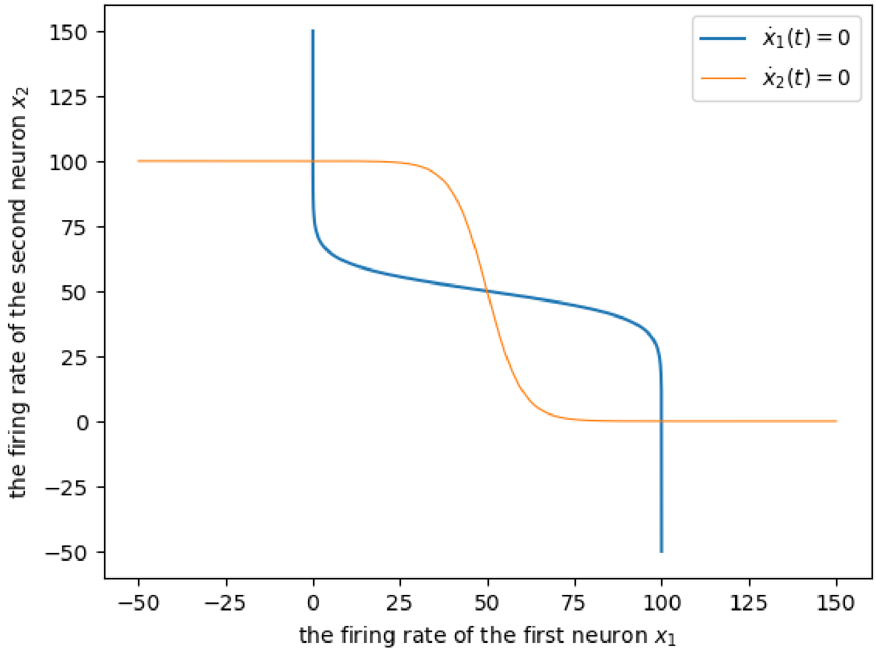

FIGURE 10: The nullclines of the system of two mutual inhibitory neurons

We observe three crossing points of the the two lines. From left to right, we name them respectively z_1, z_2 and z_3 (so $z_2 = (50, 50)$). These are the points such that $\dot{x}_1 = \dot{x}_2 = 0$. In other words, they're the fixed points of the dynamics. The plane is then divided into six zones and in each zone the system evolves in a specific direction. To put it simply, on the left of the blue line we have $\dot{x}_1 > 0$ while on the right $\dot{x}_1 < 0$. Similarly, below the orange line we get $\dot{x}_2 > 0$ whereas above it $\dot{x}_2 < 0$.

To better undestand what this implies, we simulate the system and plot the evolution of the firing rates for different initial conditions in the figure below.

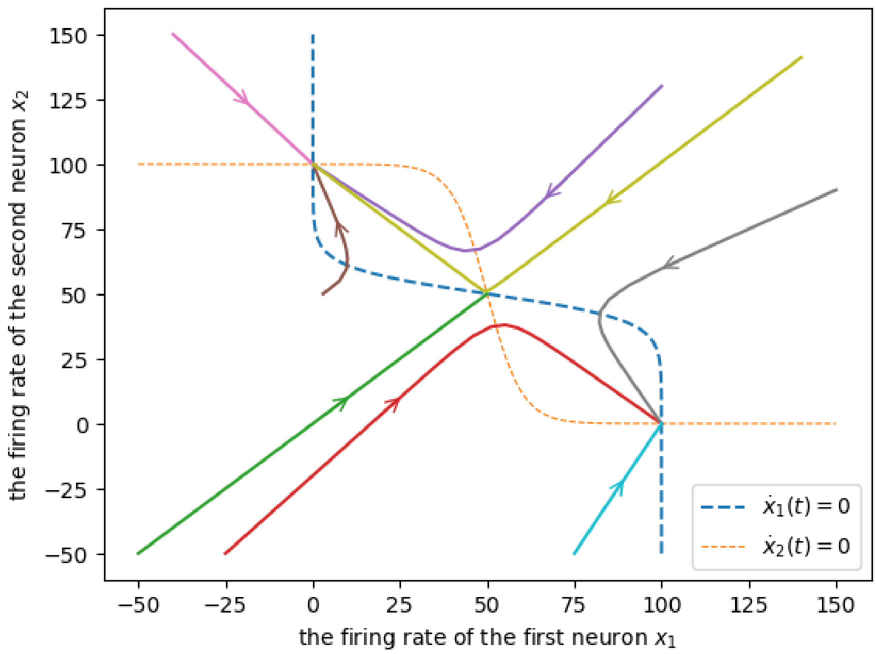

FIGURE 11: The evolution of the firing rates for different initial conditions

The initial conditions that are considered here are $(-50,-50)$, $(-40,150)$, $(-25,50)$, $(3,50)$, $(75,-50)$, $(100,130)$, $(140,141)$ and $(150,90)$. First we notice that all of the simulations end up in some fixed point and the directions of the evolutions follow roughly what is described above. Further, it seems that (but without rigorous mathematical proof here) given the initial condition $(x(0), y(0))$, if $x(0) < y(0)$ the system evolves to z_1; if $x(0) = y(0)$ the system converges to z_2; finally if $x(0) > y(0)$ the system moves to z_3.

The yellow arrow serves as quite a good example: the initial condition is $(150, 151)$, and the system gets once very clear to z_2 but then it again leaves away from this point and converges to z_1 at the end. As a result, being fixed point, z_1 and z_3 are stable while z_2 is unstable. Finally, a plot of the vector field of derivatives can better explain all of this.

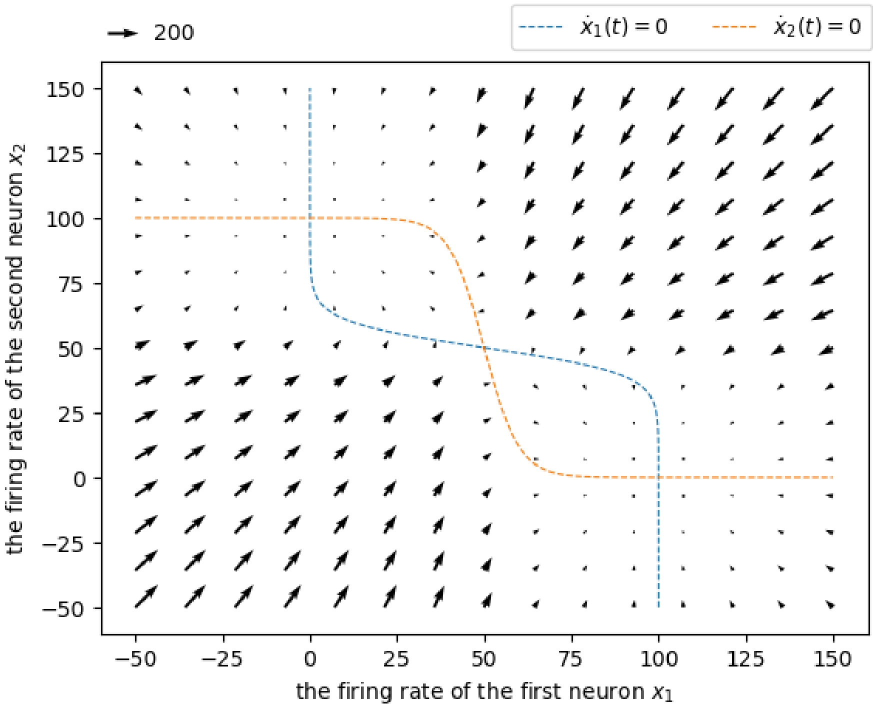

FIGURE 12: Derivatives at some points of the system

1.3 Conclusion

In this section, we have investigated two relatively simple networks and spent time studying their dynamics. Several simulations have also been carried out. We saw that there are diffrent kinds of fixed points of dynamics and if the external currents are constant, the system will evolve towards some final state (which is often an attractor). Finally, the presence of noise may more or less affect the evolution of the system.