

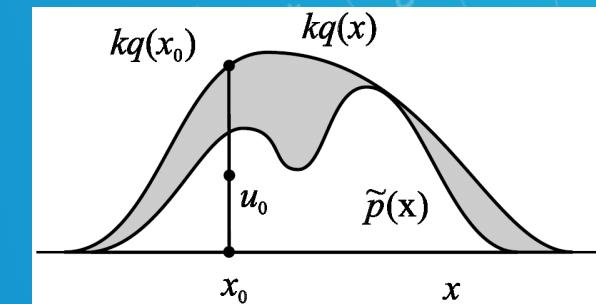
# Probabilistic Graphical Models

Approximate Inference: Monte Carlo  
and Sequential Monte Carlo methods

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Reading: see class homepage





# Approaches to inference

- ❑ Exact inference algorithms
  - ❑ The elimination algorithm
  - ❑ Message-passing algorithm (sum-product, belief propagation)
  - ❑ The junction tree algorithms
- ❑ Approximate inference techniques
  - ❑ Variational algorithms
    - ❑ Loopy belief propagation
    - ❑ Mean field approximation
  - ❑ Stochastic simulation / sampling methods
  - ❑ Markov chain Monte Carlo methods





# How to represent a joint, or a marginal distribution?

- Closed-form representation

- E.g., 
$$(x_1, \dots, x_p)^T \sim \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x - \mu)^T \Sigma^{-1} (x - \mu)\right)$$

$$\mathbb{E}_p(f(x)) = \int f(x)p(x)dx$$

- Sample-based representation:

.





# Monte Carlo methods

- Draw random samples from the desired distribution
- Yield a stochastic representation of a complex distribution
  - marginals and other expectations can be approximated using sample-based averages

$$E[f(x)] = \frac{1}{N} \sum_{t=1}^N f(x^{(t)})$$

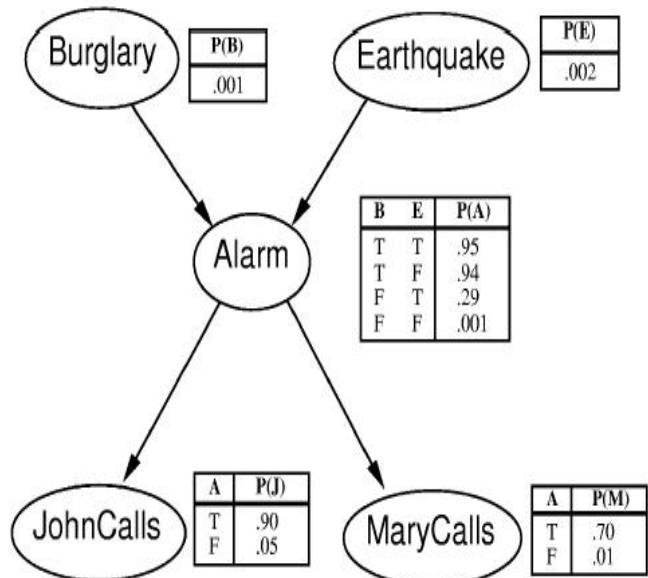
- Asymptotically exact and easy to apply to arbitrary models
- Challenges:
  - how to draw samples from a given dist. (not all distributions can be trivially sampled)?
  - how to make better use of the samples (not all samples are useful, or equally useful, see an example later)?
  - how to know we've sampled enough?





# Example: naive sampling

- Construct samples according to probabilities given in a BN.



**Alarm example:** (Choose the right sampling sequence)  
1) Sampling:  $P(B) = <0.001, 0.999>$  suppose it is false,  $B_0$ . Same for  $E_0$ .  $P(A|B_0, E_0) = <0.001, 0.999>$  suppose it is false...

2) Frequency counting: In the samples right,  
 $P(J|A_0) = P(J, A_0) / P(A_0) = <1/9, 8/9>$ .

E0	B0	A0	M0	J0
E0	B0	A0	M0	J0
E0	B0	A0	M0	J1
E0	B0	A0	M0	J0
E0	B0	A0	M0	J0
E0	B0	A0	M0	J0
E0	B0	A0	M0	J0
E1	B0	A1	M1	J1
E0	B0	A0	M0	J0
E0	B0	A0	M0	J0
E0	B0	A0	M0	J0





# Example: naive sampling

- Construct samples according to probabilities given in a BN.

**Alarm example:** (Choose the right sampling sequence)

3) what if we want to compute  $P(J|A1)$  ?

we have only one sample ...

$P(J|A1)=P(J,A1)/P(A1)=<0, 1>$ .

4) what if we want to compute  $P(J|B1)$  ?

No such sample available!

$P(J|A1)=P(J,B1)/P(B1)$  can not be defined.

For a model with hundreds or more variables, rare events will be very hard to garner enough samples even after a long time or sampling ...

E0	B0	A0	M0	J0
E0	B0	A0	M0	J0
E0	B0	A0	M0	J1
E0	B0	A0	M0	J0
E0	B0	A0	M0	J0
E0	B0	A0	M0	J0
E0	B0	A0	M0	J0
E1	B0	A1	M1	J1
E0	B0	A0	M0	J0
E0	B0	A0	M0	J0
E0	B0	A0	M0	J0





# Monte Carlo methods (cond.)

- ❑ Direct Sampling
  - ❑ We have seen it.
  - ❑ Very difficult to populate a high-dimensional state space
- ❑ Rejection Sampling
  - ❑ Create samples like direct sampling, only count samples which is consistent with given evidences.
- ❑ Likelihood weighting, ...
  - ❑ Sample variables and calculate evidence weight. Only create the samples which support the evidences.
- ❑ Markov chain Monte Carlo (MCMC)
  - ❑ Metropolis-Hastings
  - ❑ Gibbs





# Rejection sampling

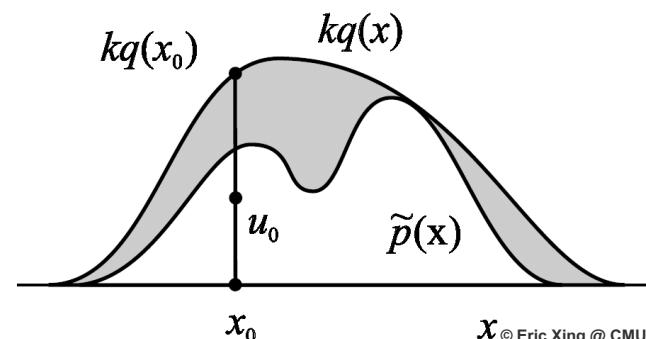
- Suppose we wish to sample from dist.  $\Pi(\mathbf{X}) = \Pi'(\mathbf{X})/Z$ .
  - $\Pi(\mathbf{X})$  is difficult to sample, but  $\Pi'(\mathbf{X})$  is easy to **evaluate**
  - Sample from a simpler dist  $Q(\mathbf{X})$
  - Rejection sampling

$$x^* \sim Q(\mathbf{X}), \quad \text{accept } x^* \text{ w.p. } \Pi'(x^*) / kQ(x^*)$$

- Correctness:

$$\begin{aligned} p(x) &= \frac{[\Pi'(x)/kQ(x)]Q(x)}{\int [\Pi'(x)/kQ(x)]Q(x)dx} \\ &= \frac{\Pi'(x)}{\int \Pi'(x)dx} = \Pi(x) \end{aligned}$$

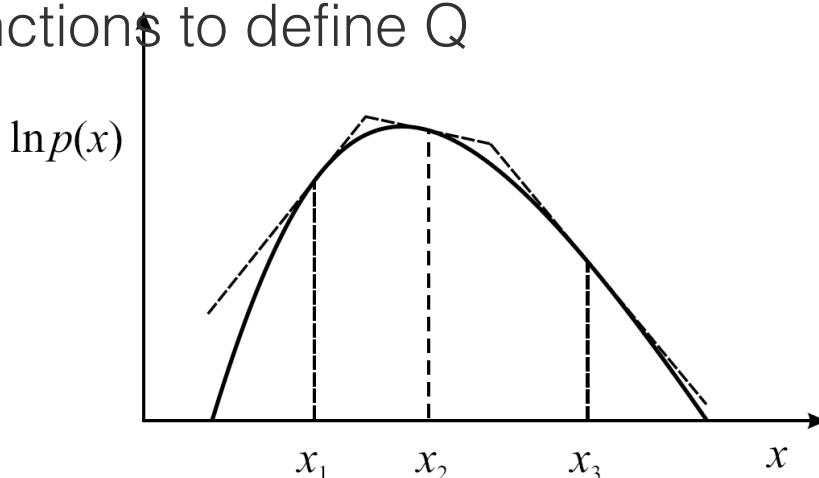
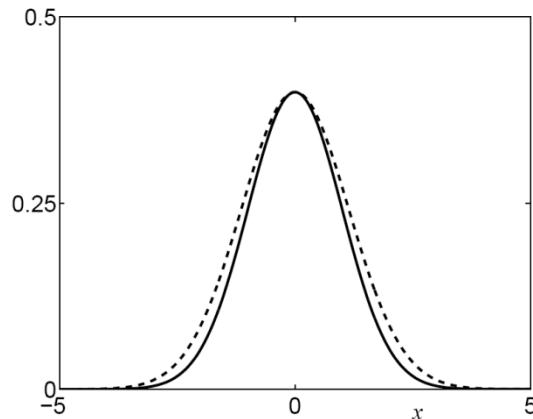
- Pitfall ...





# Rejection sampling

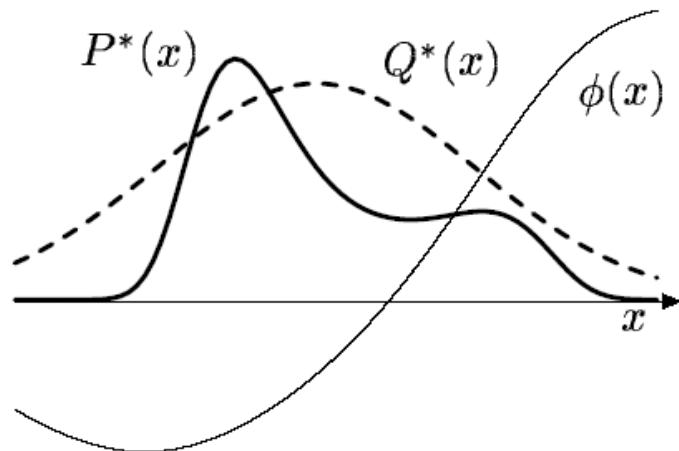
- ❑ Pitfall:
  - ❑ Using  $Q = \mathcal{M}(\mu, \sigma_q^{2/d})$  to sample  $P = \mathcal{M}(\mu, \sigma_p^{2/d})$
  - ❑ If  $\sigma_q$  exceeds  $\sigma_p$  by 1%, and  $\text{dimensional}=1000$ ,
  - ❑ The optimal acceptance rate  $k = (\sigma_q/\sigma_p)^d \approx 1/20,000$
  - ❑ Big waste of samples!
- ❑ Adaptive rejection sampling
  - ❑ Using envelope functions to define Q





# Unnormalized importance sampling

- ❑ Suppose sampling from  $P(\cdot)$  is hard.
- ❑ Suppose we can sample from a "simpler" proposal distribution  $Q(\cdot)$  instead.
- ❑ If  $Q$  dominates  $P$  (i.e.,  $Q(x) > 0$  whenever  $P(x) > 0$ ), we can sample from  $Q$  and reweight:



$$\begin{aligned}\langle f(X) \rangle &= \int f(x)P(x)dx \\ &= \int f(x) \frac{P(x)}{Q(x)} Q(x)dx \\ &\approx \frac{1}{M} \sum_m f(x^m) \frac{P(x^m)}{Q(x^m)} \quad \text{where } x^m \sim Q(X) \\ &= \frac{1}{M} \sum_m f(x^m) w^m\end{aligned}$$

- ❑ What is the problem here?





# Normalized importance sampling

- Suppose we can only evaluate  $P'(x) = \alpha P(x)$  (e.g. for an MRF).
- We can get around the nasty normalization constant  $\alpha$  as follows:

$$\text{Let } r(X) = \frac{P'(X)}{Q(X)} \quad \Rightarrow \quad \langle r(X) \rangle_Q = \int \frac{P'(X)}{Q(X)} Q(X) dX = \int P'(X) dX = \alpha$$

- Now

$$\begin{aligned} \langle f(X) \rangle_P &= \int f(x) P(x) dx = \frac{1}{\alpha} \int f(x) \frac{P'(x)}{Q(x)} Q(x) dx \\ &= \frac{\int f(x) r(x) Q(x) dx}{\int r(x) Q(x) dx} \\ &\approx \frac{\sum_m f(x^m) r^m}{\sum_m r^m} \quad \text{where } x^m \sim Q(X) \\ &= \sum_m f(x^m) w^m \quad \text{where } w^m = \frac{r^m}{\sum_m r^m} \end{aligned}$$





# Normalized vs unnormalized importance sampling

- Unnormalized importance sampling is unbiased:

$$E_Q[f(X)w(X)] =$$

- Normalized importance sampling is biased, e.g., for  $M = 1$ :

$$E_Q\left[\frac{f(x^1)r(x^1)}{r(x^1)}\right] =$$

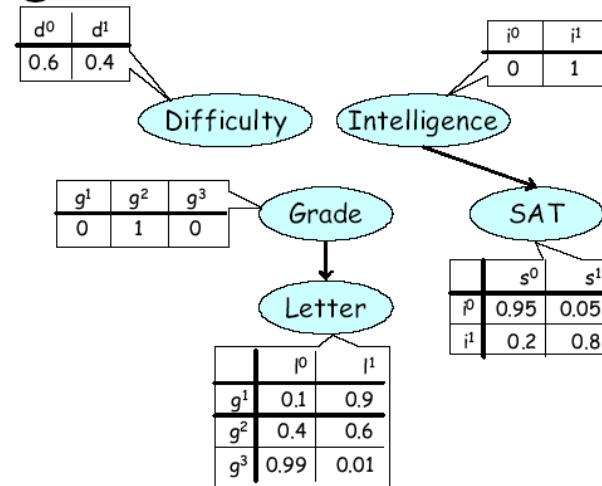
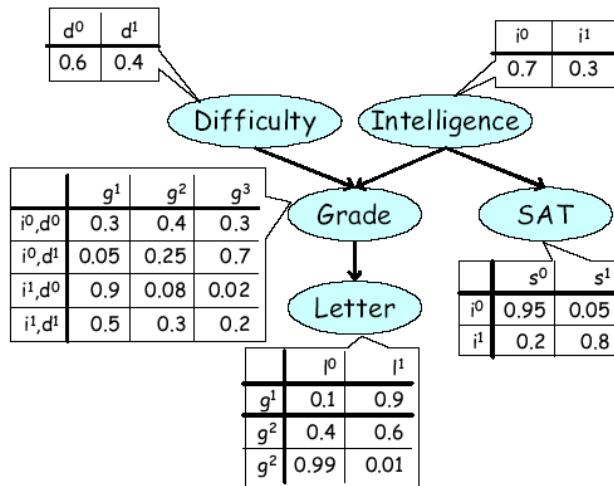
- However, the **variance** of the normalized importance sampler is usually lower in practice.
- Also, it is common that we can evaluate  $P'(x)$  but not  $P(x)$ , e.g.  $P(x|e) = P'(x, e)/P(e)$  for Bayes net, or  $P(x) = P'(x)/Z$  for MRF.





# Likelihood weighting

- We now apply normalized importance sampling to a Bayes net.
- The proposal  $Q$  is gotten from the mutilated BN where we **clamp evidence nodes**, and cut their incoming arcs. Call this  $P_M$ .



- The unnormalized posterior is  $P'(x) = P(x, e)$ .
- So for  $f(X_i) = \delta(X_i = x_i)$ , we get  $\hat{P}(X_i = x_i | e) = \frac{\sum_m w_m \delta(x_i^m = x_i)}{\sum_m w_m}$
- where  $w_m = P'(x^m, e) / P_M(x^m)$





# Likelihood weighting algorithm

$[x_{1:n}, w] = \text{function LW(CPDs, } G, E)$

let  $X_1, \dots, X_n$  be a topological ordering of  $G$

$w = 1$

$x = (0, \dots, 0)$

for  $i = 1 : n$

    let  $u_i = x(Pa_i)$

    if  $X_i \notin E$

        then sample  $x_i$  from  $P(X_i|u_i)$

    else

$x_i = e(X_i)$

$w = w * P(x_i|u_i)$





# Efficiency of likelihood weighting

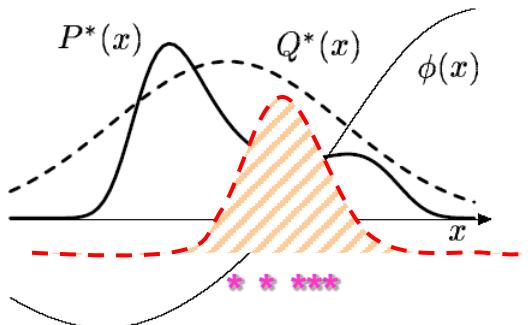
- The efficiency of importance sampling depends on how close the proposal Q is to the target P.
- Suppose all the evidence is at the roots. Then  $Q = P(X|e)$ , and all samples have weight 1.
- Suppose all the evidence is at the leaves. Then Q is the prior, so many samples might get small weight if the evidence is unlikely.
- We can use arc reversal to make some of the evidence nodes be roots instead of leaves, but the resulting network can be much more densely connected.





# Weighted resampling

- Problem of importance sampling: depends on how well  $\mathcal{Q}$  matches  $\mathcal{P}$ 
  - If  $P(x)f(x)$  is strongly varying and has a significant proportion of its mass concentrated in a small region,  $r_m$  will be dominated by a few samples



- Note that if the high-prob mass region of  $\mathcal{Q}$  falls into the low-prob mass region of  $\mathcal{P}$ , the variance of  $r^m = P(x^m)/Q(x^m)$  can be small even if the samples come from low-prob region of  $\mathcal{P}$  and potentially erroneous .
- Solution
  - Use heavy tail  $\mathcal{Q}$ .
  - Weighted resampling

$$w^m = \frac{P(x^m)/Q(x^m)}{\sum_i P(x^i)/Q(x^i)} = \frac{r^m}{\sum_m r^m}$$





# Weighted resampling

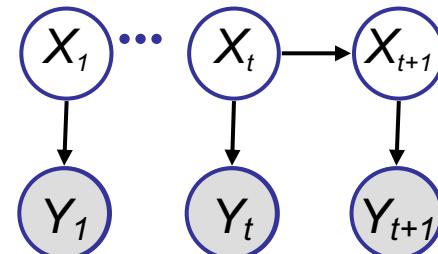
- Sampling importance resampling (SIR):

- Draw  $N$  samples from  $Q: X_1 \dots X_N$
- Constructing weights:  $w_1 \dots w_N$ ,
- Sub-sample  $x$  from  $\{X_1 \dots X_N\}$  w.p.  $(w_1 \dots w_N)$

$$w^m = \frac{P(x^m)/Q(x^m)}{\sum_i P(x^i)/Q(x^i)} = \frac{r^m}{\sum_m r^m}$$

- Particular Filtering

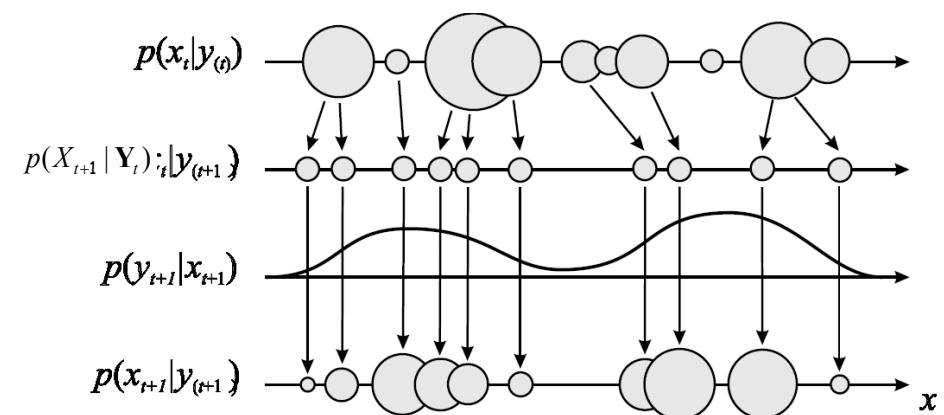
- A special weighted resampler
- Yield samples from posterior  $p(X_t|Y_{1:t})$
- Also known as sequential Monte Carlo





# Sketch of Particle Filters

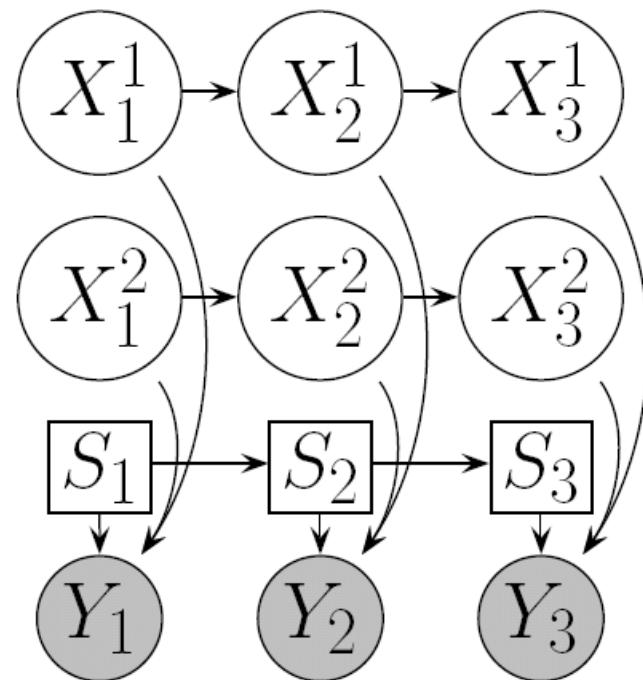
- The starting point  $p(X_t | Y_{1:t}) = p(X_t | Y_t, Y_{1:t-1}) = \frac{p(X_t | Y_{1:t-1})p(Y_t | X_t)}{\int p(X_t | Y_{1:t-1})p(Y_t | X_t) dX_t}$ 
  - Thus  $p(X_t | Y_{1:t})$  is represented by  $\left\{ X_t^m \sim p(X_t | Y_{1:t-1}), w_t^m = \frac{p(Y_t | X_t^m)}{\sum_{m=1}^M p(Y_t | X_t^m)} \right\}$
- A sequential weighted resampler
  - Time update
$$p(X_{t+1} | Y_{1:t}) = \int p(X_{t+1} | X_t)p(X_t | Y_{1:t}) dX_t$$
$$= \sum_m w_t^m p(X_{t+1} | X_t^{(m)}) \quad (\text{sample from a mixture model})$$
  - Measurement update
$$p(X_{t+1} | Y_{1:t+1}) = \frac{p(X_{t+1} | Y_{1:t})p(Y_{t+1} | X_{t+1})}{\int p(X_{t+1} | Y_{1:t})p(Y_{t+1} | X_{t+1}) dX_{t+1}}$$
$$\Rightarrow \left\{ X_{t+1}^m \sim p(X_{t+1} | Y_{1:t}), w_{t+1}^m = \frac{p(Y_{t+1} | X_{t+1}^m)}{\sum_{m=1}^M p(Y_{t+1} | X_{t+1}^m)} \right\} \quad (\text{reweight})$$





# PF for switching SSM

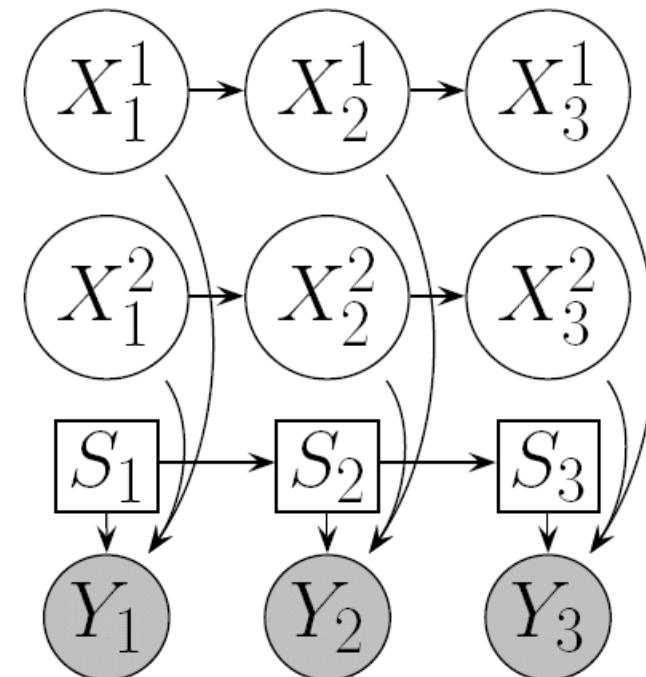
- Recall that the belief state has  $O(2^t)$  Gaussian modes





# PF for switching SSM

- ❑ Key idea: if you knew the discrete states, you can apply the right Kalman filter at each time step.
- ❑ So for each old particle  $m$ , sample  $S_t^m \sim P(S_t | S_{t-1}^m)$  from the prior, apply the KF (using parameters for  $S_t^m$ ) to the old belief state  $(\hat{x}_{t-1|t-1}^m, P_{t-1|t-1}^m)$  to get an approximation to  $P(X_t | y_{1:t}, s_{1:t}^m)$
- ❑ Useful for online tracking, fault diagnosis, etc.





# Rao-Blackwellised sampling

- Sampling in high dimensional spaces causes high variance in the estimate.
- RB idea: sample some variables  $X_p$ , and conditional on that, compute expected value of rest  $X_d$  analytically:

$$\begin{aligned} E_{p(X|e)}[f(X)] &= \int p(x_p, x_d | e) f(x_p, x_d) dx_p dx_d \\ &= \int_{x_p} p(x_p | e) \left( \int_{x_d} p(x_d | x_p, e) f(x_p, x_d) dx_d \right) dx_p \\ &= \int_{x_p} p(x_p | e) E_{p(X_d|x_p,e)}[f(x_p, X_d)] dx_p \\ &= \frac{1}{M} \sum_m E_{p(X_d|x_p^m,e)}[f(x_p^m, X_d)], \quad x_p^m \sim p(x_p | e) \end{aligned}$$

- This has lower variance, because of the identity:

$$\text{var}[\tau(X_p, X_d)] = \text{var}[E[\tau(X_p, X_d) | X_p]] + E[\text{var}[\tau(X_p, X_d) | X_p]]$$







# Rao-Blackwellised sampling

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$$\text{var}[\tau(X_p, X_d)] = \text{var}[E[\tau(X_p, X_d) | X_p]] + E[\text{var}[\tau(X_p, X_d) | X_p]]$$

- Hence  $\text{var}[E[\tau(X_p, X_d) | X_p]] \leq \text{var}[\tau(X_p, X_d)]$ , so  $\tau(X_p, X_d) = E[f(X_p, X_d) | X_p]$  is a lower variance estimator.





# Summary: Monte Carlo Methods

- ❑ Direct Sampling
  - ❑ Very difficult to populate a high-dimensional state space
- ❑ Rejection Sampling
  - ❑ Create samples like direct sampling, only count samples which is consistent with given evidences.
- ❑ Likelihood weighting, ...
  - ❑ Sample variables and calculate evidence weight. Only create the samples which support the evidences.
- ❑ Markov chain Monte Carlo (MCMC)
  - ❑ Metropolis-Hastings
  - ❑ Gibbs

