Homework 2 - September 15, 2016. Questions.

Intermediate Statistics (Carnegie Mellon University)

Homework 2
36-705
Due: Thursday Sept 15 by 3:00

1. Let X have mean 0. We say that X is sub-Gaussian if there exists $\sigma > 0$ such that
$$\log\left(\mathbb{E}[e^{tX}]\right) \leq \frac{t^2\sigma^2}{2}$$
for all t.

   (i) Show that X is sub-Gaussian if and only if $-X$ is sub-Gaussian.

   (ii) Let X have mean μ. Suppose that $X - \mu$ is sub-Gaussian. Show that
$$\mathbb{P}(|X - \mu| \geq t) \leq 2e^{-t^2/(2\sigma^2)}.$$

   Remark: When people say "X is sub-Gaussian" they often mean that "$X - \mu$ is sub-Gaussian."

   (iii) Suppose that X is sub-Gaussian. Show that, for any $p > 0$,
$$\mathbb{E}[|X|^p] \leq p2^{p/2}\sigma^p\Gamma(p/2).$$

2. Let $X_1, \ldots, X_n$ be iid, with mean μ, $\mathrm{Var}(X_i) = \sigma^2$ and $|X_i| \leq c$. Bernstein's inequality says that
$$\mathbb{P}\left(|\overline{X}_n - \mu| > t\right) \leq 2\exp\left\{-\frac{nt^2}{2\sigma^2 + 2ct/3}\right\}.$$
Suppose that $\sigma^2 = O(1/n)$. Use Bernstein's inequality to show that $\overline{X}_n - \mu = O_P(1/n)$.

3. Prove or disprove the following:
(i) If $X_n = O_P(a_n)$ and $Y_n = O(b_n)$ then $X_n + Y_n = O_P(a_n b_n)$.
(ii) If $X_n = o_P(a_n)$ and $Y_n = o_P(b_n)$ then $X_n + Y_n = o_P(\min\{a_n, b_n\})$.
(iii) If $X_n = o_P(a_n)$ and $Y_n = O_P(b_n)$ then $X_n/Y_n = o_P(a_n/b_n)$.
(iv) If $X_n = O_P(a_n)$ and $Y_n = O_P(b_n)$ then $X_n Y_n = o_P(a_n b_n)$.

4. Let $U \sim \mathrm{Unif}(0, 1)$. Let $Y = F^{-1}(U)$ where F is a continuous cdf on the real line. Show that the distribution of Y is F.