Sample/practice exam Fall 2016, questions

Intermediate Statistics (Carnegie Mellon University)

Practice Test for Test II

1. Let $X_1, \ldots, X_n \sim \text{Uniform}(-\theta, \theta)$ where $\theta > 0$.
   (a) Find the maximum likelihood estimator $\widehat{\theta}_n$.
   (b) Find a minimal sufficient statistic.
   (c) Show that $\widehat{\theta}_n \xrightarrow{P} \theta$.
   (d) Find the limiting distribution of $n(\theta - \widehat{\theta}_n)$.

2. Let $X \sim \text{Bernoulli}(\theta)$ be a single coin flip.
   Suppose that $\theta \in \Theta = \{1/3,\ 2/3\}$. Hence θ can only take 2 possible values.

   (a) Find the maximum likelihood estimator.
   (b) Let the loss function be

$$L(\theta, \widehat{\theta}) = \begin{cases} 1 & \text{if } \theta \neq \widehat{\theta} \\ 0 & \text{if } \theta = \widehat{\theta}. \end{cases}$$

   Find the risk function of the maximum likelihood estimator. Since θ only takes the values 1/3 and 2/3, you only need to find $R(1/3, \widehat{\theta})$ and $R(2/3, \widehat{\theta})$.
   (c) Show the the maximum likelihood estimator is minimax.

3. Let $X_1, \ldots, X_n$ be iid from a Binomial (k, θ).

   (a) Find a minimal sufficient statistic S for θ.
   (b) For each of the following say, whether it is sufficient, minimal sufficient, or not sufficient.

$$T = X_1, \quad T = \sum_i X_i, \quad T = \left(X_1, \sum_i X_i\right), \quad T = \left(X_1, \sum_{i=2}^{n} X_i\right).$$

   (c) Let $\tau(\theta) = P(X = 1) = k\theta(1-\theta)^{k-1}$. Define $U = 1$ if $X_1 = 1$ and 0 otherwise. Show that U an unbiased estimator of τ.
   (d) Find the mle $\widehat{\tau}$ of τ.
   (e) Show that $\widehat{\tau} \xrightarrow{P} \tau$.
   (f) Find the limiting distribution of $\widehat{\tau}$.

4. Construct an example where $X_n \rightsquigarrow X$ and $Y_n \rightsquigarrow Y$, but $X_n + Y_n$ does not converge in distribution to $X + Y$.

5. Let $X_i \sim N(\theta_i, 1)$ for $i = 1, \ldots, n$. Let $\gamma = \sum_{i=1}^n \theta_i^2$. Find the mle $\widehat{\gamma}$. Let $L(\gamma, \widehat{\gamma}) = (\gamma - \widehat{\gamma})^2$. Find the risk of the mle.

6. Let $X_1, \ldots, X_n \sim \text{Uniform}(0, 2)$. Let

$$W_n = \frac{1}{n} \sum_{i=1}^n X_i^2.$$

   (a) Show that there is a number μ such that W_n converges in quadratic mean to μ.

   (b) Show that W_n converges in probability μ.

   (c) What is the limiting distribution of $\sqrt{n}(W_n^2 - \mu^2)$?

7. Let $X_1, \ldots, X_n \sim \text{Bernoulli}(p)$.

   (a) Let $T = X_3$. Show that T is not sufficient.

   (b) Show that $U = \sum_{i=1}^n X_i^2$ is minimal sufficient.

8. Let

$$X_1, \ldots, X_n \sim \frac{1}{2} N(0, 1) + \frac{1}{2} N(\theta, 1).$$

   In other words, with probability 1/2, X_i is drawn from $N(0, 1)$ and with probability 1/2, X_i is drawn from $N(\theta, 1)$.

   (a) Find the method of moments estimator $\widehat{\theta}$ of θ.

   (b) Find the mean squared error of $\widehat{\theta}$.

9. Let $X_1, \ldots, X_n \sim N(\theta, 1)$. Let $\tau = e^\theta + 1$.

   (b) Consider some loss function $L(\tau, \widehat{\tau})$. Define what it means for an estimator to be a minimax estimator for τ.

   (c) Let π be a prior for θ. Find the Bayes estimator for τ under the loss $L(\tau, \widehat{\tau}) = (\widehat{\tau} - \tau)^2 / \tau$.

10. Let $X_1, \ldots, X_n \sim \text{Normal}(\theta, 1)$. Suppose that $\theta \in \{-1, 1\}$. In other words, θ can only take two possible values.

   (a) Find a minimal sufficient statistic.

   (b) Find the maximum likelihood estimator. Is the maximum likelihood estimator a sufficient statistic?

   (c) Find the risk function using the loss function

$$L(\theta, \widehat{\theta}) = \begin{cases} 1 & \text{if } \theta \neq \widehat{\theta} \\ 0 & \text{if } \theta = \widehat{\theta}. \end{cases}$$

11. Let $X_i \sim \text{Bernoulli}(p_i)$ for $i = 1, \ldots, n$. The observations are independent but each observation has a different mean. The unknown parameter is $p = (p_1, \ldots, p_n)$.

(a) Let $\psi = \sum_{i=1}^n p_i$. Find the maximum likelihood estimator of ψ.

(b) Find the mean squared error (MSE) of the maximum likelihood estimator of ψ.

(c) Suppose we use the following prior distribution

$$\pi(p_1, \ldots, p_n) = 1.$$

Find the Bayes estimator of ψ.

Hint: Recall that the Beta (α, β) density is

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}.$$

If $W \sim \text{Beta}(\alpha, \beta)$ then $\mathbb{E}[W] = \alpha/(\alpha+\beta)$ and
$\text{Var}(W) = \alpha\beta/((\alpha+\beta)^2(\alpha+\beta+1))$.

12. Let $X_1, \ldots, X_n \sim N(\mu, 1)$.

(a) Let $T = \max\{X_1, \ldots, X_n\}$. Show that T is not sufficient.

(b) Let us use the improper prior $\pi(\mu) \propto 1$. Find the Bayes estimator of $\psi = \mu^2$.