36-705 Intermediate Statistics.

Homework 3.

Date: 18th June 2020.

Correction status: pending.

1) Let $\mathcal{C} = \mathcal{A} \bigcup \mathcal{B}$. Show that

$$s_n(\mathcal{C}) \leq s_n(\mathcal{A}) + s_n(\mathcal{B})$$

where s_n denotes the shattering number.

2) Let $\mathcal{C} = \{A \cup B; A \in \mathcal{A}, B \in \mathcal{B}\}$. Show that:

$$s_n(\mathcal{C}) \leq s_n(\mathcal{A}) s_n(\mathcal{B})$$

3) Chapter 5, problem 2.

Let $X_1, X_2, \ldots$ be a sequence of random variables. Show that $X_n \overset{\text{qm}}{\to} b$ if and only if:

$$\lim_{n\to\infty} \mathbb{E}[X_n] = b \quad \text{and} \quad \lim_{n\to\infty} \text{Var}[X_n] = 0$$

Denoting proposition (I) as:

$$X_n \overset{\text{qm}}{\to} b$$

And proposition (II) as:

$$\lim_{n\to\infty} \mathbb{E}[X_n] = b \quad \text{and} \quad \lim_{n\to\infty} \text{Var}[X_n] = 0$$

To show that (I) $\implies$ (II):

In order for convergence in quadratic mean, $X_n \overset{\text{qm}}{\to} b$, by definition, we have that

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}[(X_n - b)^2] = 0 \\
\implies &\lim_{n\to\infty} \mathbb{E}[(X_n^2 - 2bX_n + b^2] = 0 \\
\implies &\lim_{n\to\infty} \mathbb{E}[X_n^2] - 2b\mathbb{E}[X_n] + b^2 = 0 \\
\implies &\lim_{n\to\infty} \text{Var}[X_n] + \mathbb{E}[X_n]^2 - 2b\mathbb{E}[X_n] + b^2 = 0 \\
\implies &\lim_{n\to\infty} \text{Var}[X_n] + \lim_{n\to\infty} (\mathbb{E}[X_n] - b)^2 = 0
\end{aligned}$$

Where we have substituted the 2nd moment for the sum of the variance and squared mean to get from the 3rd to the 4th equality.

A property of both the variance $\mathrm{Var}[X_n]$ and the term $(\mathbb{E}[X_n]-b)^2$ is that they are both non-negative. And because the RHS is 0, in order for equality to hold, we must have that both limits on the LHS be equal to 0, which occurs when

$$\lim_{n\to\infty} \mathbb{E}[X_n] = b \quad \text{and} \quad \lim_{n\to\infty} \mathrm{Var}[X_n] = 0$$

Showing (II) $\implies$ (I):

Because it is the case that

$$\lim_{n\to\infty} \mathbb{E}[X_n] = b \quad \text{and} \quad \lim_{n\to\infty} \mathrm{Var}[X_n] = 0$$

And also because $\mathrm{Var}[X_n] = \mathbb{E}[X_n^2] - \mathbb{E}[X_n]^2$, we have the following limit on the 2nd moment

$$\lim_{n\to\infty} \mathbb{E}[X_n^2] = b^2$$

We now consider

$$\lim_{n\to\infty} \mathbb{E}[(X_n-b)^2] = \lim_{n\to\infty} \mathbb{E}[X_n^2] - 2b \lim_{n\to\infty} \mathbb{E}[X_n] + b^2 = b^2 - 2b(b) + b^2 = 0$$

Which is the required result.

4) Chapter 5, problem 5.

Let $X_1,\ldots,X_n \sim \mathrm{Bernoulli}(p)$. Prove that:

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{\text{P}} p \quad \text{and} \quad \frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{\text{qm}} p$$

Bernoulli random variables have mean $\mathbb{E}[X_i] = p$ and variance $\mathrm{Var}[X_i] = p(1-p)$.

We define a new random variable $Y_i = X_i^2$, and instead consider $Y_1, Y_2, \ldots, Y_n$.

The mean of Y_i is given by

$$\mathbb{E}[Y_i] = \mathbb{E}[X_i^2] = \mathrm{Var}[X_i] + \mathbb{E}[X_i]^2 = p(1-p) + p^2 = p$$

As each of the X_i are Bernoulli random variables taking values of either 0 or 1, the random variables $Y_i = X_i^2$ are bounded within the interval $[0,1]$.

Applying Hoeffding's inequality to the sequence of sample means $\bar{Y}_n$ and considering when $n\to\infty$ we have that $\forall\ \epsilon > 0$,

$$P(|\bar{Y}_n - p| \geq \epsilon) \leq 2e^{-2n\epsilon^2} \to 0$$

As $Y_i = X_i^2$, we have that $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^n X_i^2$, and hence we have the required result concerning convergence in probability of the 2nd sample moment:

$$\bar{Y}_n \xrightarrow{\mathrm{P}} p \implies \frac{1}{n}\sum_{i=1}^n X_i^2 \xrightarrow{\mathrm{P}} p$$

To show convergence in quadratic mean, we consider the expression $\mathbb{E}[(\bar{Y}_n - p)^2]$:

$$\mathbb{E}[(\bar{Y}_n - p)^2] = \mathbb{E}[(\bar{Y}_n^2 - 2p\bar{Y}_n + p^2)] = \mathbb{E}[\bar{Y}_n^2] - 2p\mathbb{E}[Y_n] + p^2$$

Substituting the variance and mean of the sample mean in place of its 2nd moment we have that

$$\mathbb{E}[(\bar{Y}_n - p)^2] = \mathrm{Var}[\bar{Y}_n] + \mathbb{E}[\bar{Y}_n]^2 - 2p\mathbb{E}[\bar{Y}_n] + p^2$$

As $\bar{Y}_n$ is a sample mean, we have that $\mathrm{Var}[\bar{Y}_n] = \mathrm{Var}[Y_i]/n$, and evaluating the expression for the variance of Y_i, we have that

$$\mathrm{Var}[Y_i] = \mathbb{E}[{Y_i}^2] - \mathbb{E}[Y_i]^2 = \mathbb{E}[X_i^4] - p^2$$

We now need to evaluate the fourth moment of the Bernoulli random variable X_i. We find the moment generating function $\psi_{X_i}(t)$ as follows:

$$\begin{aligned}
\psi_{X_i}(t) = \mathbb{E}[e^{tX}] &= \int e^{tX} dF(x) \\
&= \sum_{x_i} e^{tx_i} P(X_i = x_i) \\
&= e^t P(X_i = 1) + e^0 P(X_i = 0) \\
&= pe^t + (1-p)
\end{aligned}$$

As $\frac{d}{dt}e^t = e^t$ for all t, we have that the kth derivative of the moment generating function has the form

$$\psi_{X_i}^{(k)}(t) = pe^t \quad \forall\, k \geq 1$$

Evaluating at $t = 0$, we find that for Bernoulli random variables, all kth moments have the form $\psi_{X_i}^{(k)}(0) = \mathbb{E}[X_i^k] = p$ for $k \geq 1$.

We then have that

$$\mathrm{Var}[\bar{Y}_n] = \frac{\mathrm{Var}[Y_i]}{n} = \frac{p - p^2}{n} = \frac{p(1-p)}{n}$$

As $\mathbb{E}[\bar{Y}_n] = \mathbb{E}[Y_i] = p$, we have that

$$\mathbb{E}[(\bar{Y}_n - p)^2] = \mathrm{Var}[\bar{Y}_n] + \mathbb{E}[\bar{Y}_n]^2 - 2p\mathbb{E}[\bar{Y}_n] + p^2 = \frac{p(1-p)}{n} + p^2 - 2p(p) + p^2 = \frac{p(1-p)}{n}$$

As $n \to \infty$, we have that $\mathbb{E}[(\bar{Y}_n - p)^2] = \frac{p(1-p)}{n} \to 0$, which is the required result.

It was only observed after completing the problem that for Bernoulli random variables, $X_i^2 = X_i$, thereby rendering the need for computations using moment generating functions redundant in the calculation of $\mathrm{Var}[Y_i] = \mathrm{Var}[X_i^2]$, as it is the case that $\mathbb{E}[X_i^2] = \mathbb{E}[X_i] = p$.

5) Chapter 5, problem 12.

Let $X_1, X_2, \ldots$ be random variables that are positive and integer valued.

Show that $X_n \xrightarrow{D} X$ if and only if:

$$\lim_{n\to\infty} P(X_n = k) = P(X = k)$$

6) Chapter 5, problem 15.

Let

$$\begin{pmatrix} X_{11} \\ X_{21} \end{pmatrix}, \begin{pmatrix} X_{12} \\ X_{22} \end{pmatrix}, \cdots \begin{pmatrix} X_{1n} \\ X_{2n} \end{pmatrix}$$

be IID random vectors with mean $\boldsymbol{\mu} = (\mu_1, \mu_2)$ and variance Σ. Assume that $\mu_2 \neq 0$. Then let

$$\bar{X}_1 = \frac{1}{n}\sum_{i=1}^{n} X_{1i}, \quad \bar{X}_2 = \frac{1}{n}\sum_{i=1}^{n} X_{2i}$$

and define $Y_n = \bar{X}_1 / \bar{X}_2$. Find the limiting distribution of Y_n.

Defining the sample mean vector $\bar{\mathbf{X}}_n \in \mathbb{R}^2$ as follows:

$$\bar{\mathbf{X}}_n = \begin{pmatrix} \bar{X}_1 \\ \bar{X}_2 \end{pmatrix}$$

Then the multivariate CLT states that

$$\sqrt{n}(\bar{\mathbf{X}}_n - \boldsymbol{\mu}) \xrightarrow{\mathrm{d}} N(\mathbf{0}, \Sigma)$$

We now define the following scalar function of a vector, $g : \mathbb{R}^2 \mapsto \mathbb{R}$:

$$g\left[\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right] := \frac{y_1}{y_2}$$

Which has gradient given by

$$\nabla_{\mathbf{y}} g(\mathbf{y}) = \begin{pmatrix} \frac{\partial g}{\partial y_1} \\ \frac{\partial g}{\partial y_2} \end{pmatrix}$$

Denoting the gradient evaluated at the mean, $\nabla_{\mathbf{y}} g(\mathbf{y})\big|_{\mathbf{y}=\boldsymbol{\mu}}$ as $\nabla_{\boldsymbol{\mu}}$, the multivariate Delta Method states that

$$\sqrt{n}\left(g(\bar{\mathbf{X}}_n) - g(\boldsymbol{\mu})\right) \xrightarrow{\mathrm{d}} N\left(\mathbf{0}, \nabla_{\boldsymbol{\mu}}^T \Sigma \nabla_{\boldsymbol{\mu}}\right)$$

if $g(\cdot)$ is smooth and differentiable.

The partial derivatives of the scalar function $g(\cdot)$ with respect to y_1 and y_2 are

$$\frac{\partial g}{\partial x_1} = \frac{1}{x_2} \quad , \quad \frac{\partial g}{\partial x_2} = \frac{-x_1}{x_2^2}$$

Evaluating this at $\mathbf{y} = \boldsymbol{\mu}$ we have that

$$\nabla_{\boldsymbol{\mu}} = \begin{pmatrix} \frac{1}{\mu_2} \\ \frac{-\mu_1}{\mu_2^2} \end{pmatrix}$$

In order to compute the asymptotic variance of $\sqrt{n}\left(g(\bar{\mathbf{X}}_n) - g(\boldsymbol{\mu})\right)$, we have to compute

$$\begin{aligned}
\nabla_{\boldsymbol{\mu}}^T \Sigma \nabla_{\boldsymbol{\mu}} &= \left(\tfrac{1}{\mu_2}, \tfrac{-\mu_1}{\mu_2^2} \right) \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \frac{1}{\mu_2} \\ \frac{-\mu_1}{\mu_2^2} \end{pmatrix} \\
&= \left(\tfrac{\sigma_{11}}{\mu_2} - \tfrac{\sigma_{21}\mu_1}{\mu_2^2}, \tfrac{\sigma_{12}}{\mu_2} - \tfrac{\sigma_{22}\mu_1}{\mu_2^2} \right) \begin{pmatrix} \frac{1}{\mu_2} \\ \frac{-\mu_1}{\mu_2^2} \end{pmatrix} \\
&= \frac{1}{\mu_2}\left(\sigma_{11}\mu_2 - \frac{\sigma_{21}\mu_1}{\mu_2^2} \right) - \frac{\mu_1}{\mu_2^2}\left(\frac{\sigma_{12}}{\mu_2} - \frac{\sigma_{22}\mu_1}{\mu_2^2} \right) \\
&= \left(\frac{\sigma_{11}}{\mu_2^2} - \frac{\sigma_{21}\mu_1}{\mu_2^3} \right) - \left(\frac{\mu_1\sigma_{12}}{\mu_2^3} - \frac{\sigma_{22}\mu_1^2}{\mu_2^4} \right) \\
&= \frac{\sigma_{11}\mu_2^2 - \sigma_{21}\mu_1\mu_2 - \mu_1\mu_2\sigma_{12} + \sigma_{22}\mu_1^2}{\mu_2^4} \\
&= \frac{1}{\mu_2^4}\left(\sigma_{11}\mu_2^2 - (\sigma_{12} + \sigma_{21})\mu_1\mu_2 + \sigma_{22}\mu_1^2 \right)
\end{aligned}$$

And we have the limiting distribution

$$\sqrt{n}\left(\bar{X}_1/\bar{X}_2 - \mu_1/\mu_2\right) \xrightarrow{\mathrm{d}} N\left(0, \frac{1}{\mu_2^4}(\sigma_{11}\mu_2^2 - (\sigma_{12} + \sigma_{21})\mu_1\mu_2 + \sigma_{22}\mu_1^2\right)$$

Which yields the following result on the limiting distribution of Y_n:

$$Y_n = \frac{\bar{X}_1}{\bar{X}_2} \xrightarrow{\mathrm{d}} N\left(\frac{\mu_1}{\mu_2}, \frac{\sigma^2}{n}\right)$$

where $\sigma^2 = \frac{1}{\mu_2^4}(\sigma_{11}\mu_2^2 - (\sigma_{12} + \sigma_{21})\mu_1\mu_2 + \sigma_{22}\mu_1^2$