Sample/practice exam Fall 2016, questions

Intermediate Statistics (Carnegie Mellon University)

Practice Questons for Test 3

1. Let $X_1, \ldots, X_n \sim p(x;\theta)$ where
$$p(x;\theta) = \theta e^{-x\theta}$$
for $x > 0$ and $\theta > 0$. Find the mle $\widehat{\theta}$. Find the Fisher information. Find the limiting distribution of $\widehat{\theta}$. Find the Wald test for
$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta \neq \theta_0.$$
Find the likelihood ratio test. Find a $1-\alpha$ asymptotic confidence interval for $\psi = \log\theta$.

2. Let $X_1, \ldots, X_n \sim N(\theta, 1)$. Consider testing
$$H_0: \ \theta = \theta_0 \qquad H_1: \ \theta \neq \theta_0.$$
Suppose we reject H_0 when $|W| > z_{\alpha/2}$ where $W = \sqrt{n}(\overline{X} - \theta_0)$ Suppose that the true value of the parameter is $\theta > \theta_0$. Show that $\beta(\theta) \to 1$ as $n \to \infty$ where $\beta(\theta)$ is the power function.

3. Let
$$\mathrm{KL}(p_\theta, p_\nu) = \int p_\theta(x) \log\left(\frac{p_\theta(x)}{p_\nu(x)}\right) dx.$$
Assume that $\theta \in \mathbb{R}$. Let $\nu = \theta + \epsilon$ where ϵ is small. Show that
$$\mathrm{KL}(p_\theta, p_\nu) = \frac{\epsilon^2}{2} I(\theta) + o(\epsilon^2)$$
where I is the Fisher information.

4. The asymptotic standard error of the mle is
$$\mathrm{se} = (I_n(\theta))^{-1/2} = (nI(\theta))^{-1/2}$$
where $I(\theta)$ is the Fisher information function. Suppose that $I(\theta) > 0$ and that $I(\theta)$ is a continuous function. Show that
$$\frac{\widehat{\mathrm{se}}}{\mathrm{se}} \xrightarrow{P} 1$$
where $\widehat{\mathrm{se}} = (I_n(\widehat{\theta}))^{-1/2}$.

5. Let $X_1, \ldots, X_n \sim N(\theta, 1)$. Consider the following estimator:
$$\widehat{\theta} = \begin{cases} \overline{X}_n & = \text{if } |\overline{X}_n| > \frac{1}{n^{1/4}} \\ 0 & = \text{if } |\overline{X}_n| < \frac{1}{n^{1/4}}. \end{cases}$$
Find the limiting distribution of $\widehat{\theta}$ when $\theta \neq 0$. Find the limiting distribution of $\widehat{\theta}$ when $\theta = 0$.

6. Let $X \sim \text{Binomial}(n, p)$. Consider testing

$$H_0: \ p = p_0 \qquad \text{versus} \qquad H_1: \ p \neq p_0.$$

(a) Construct the Wald test. Show that the power tends to 1 as $n \to \infty$ when $p > p_0$.

(b) Construct the (asymptotic) likelihood ratio test.

7. Let $X_1, \ldots, X_n \sim \text{Bernoulli}(p)$. Let $\psi = \log(p/(1-p))$.

(a) Find the mle $\widehat{\psi}$ of ψ and find the limiting distribution of $\widehat{\psi}$.

(b) Find the Wald test for testing

$$H_0: \ \psi = 0 \qquad \text{versus} \qquad H_1: \ \psi \neq 0.$$

(c) Construct asymptotic confidence sets for ψ by (i) inverting the Wald test; (ii) by inverting the likelihood ratio test.