Linear Algebra Done Right (4th ed.) by Sheldon Axler.

Write-up for exercises in *Chapter 2: Finite Dimensional Vector Spaces*.
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The main building of the Institute for Advanced Study, in Princeton, New Jersey. Paul Halmos (1916–2006) wrote the first modern linear algebra book in this building. Halmos's linear algebra book was published in 1942 (second edition published in 1958). The title of Halmos's book was the same as the title of this chapter.

Figure 1: image

The following are my self-study write ups for the exercises in the above textbook. The purpose of writing up my attempts at the exercises from pen and paper is to practise writing up and presenting formal proofs.

The purpose of these write-ups is so that I can:

- Practise writing up the exercises and formal mathematical proofs I completed in my scrapbook by committing my answers to the public domain.
- In doing so, learn more about logical proof structure, and to improve mathematical communication.
- Document any exercises that I attempted and struggled with, and for which I needed assistance to proceed from the instructor's solutions manual.
- In doing so, reveal any patterns of deficiencies in my own thinking when assimilating this material, which may be useful when studying fields that build on these concepts.

I have organised the exercises into problem sets corresponding to each section in the chapter. For exercises that revealed a point of broader significance after correction, I have listed this as a **Key idea** in bold. For proofs that needed more thought, I have listed the proof strategy, or a one line description of the high-level idea at play.

In cases where I got stuck with a problem, or needed instantaneous feedback on the right direction to proceed, I would use a hint from the solutions manual. This is indicated as **HINT** in bold the write-up, and will be useful as a marker on where I got stuck and on where it is that I learnt something new. This is particularly important for later problems which use results established in previous questions.

Corrections made using the solutions manual and Googling are listed as *Corrections* in italics. Only problems which illustrated significant conceptual difficulties for me are corrections included.

Exercises 2A Span and Linear Independence.

2A.1.

Define the subset U as follows,

$$U := \{(x, y, z) \in \mathbb{F}^3 \mid x + y + z = 0\}.$$

The Cartesian description of a 2-dimensional plane in $\mathbb{F}^3$ is,

$$ax + by + cz = d.$$

This passes through the origin when $d = 0$. We also have that $x+y+z = 0$ implies that $z = -(x+y)$ and our subset U becomes,

$$U = \{(x, y, -(x + y)) \in \mathbb{F}^3 \mid x, y \in \mathbb{F}\}.$$

Let $u \in U$, and so we have,

$$u = (x, y, -(x + y)) = (x, 0, -x) + (0, y, -y) = x(1, 0, -1) + y(0, 1, -1).$$

With $u_1 := (1, 0, -1)$ and $u_2 := (0, 1, -1)$, this shows both that $\operatorname{span}(u_1, u_2) \subseteq U$ and $U \subseteq \operatorname{span}(u_1, u_2)$, so that $U = \operatorname{span}(u_1, u_2)$.

Note that $\mathbb{F}^3$ is spanned by three vectors $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. Because we require *four* distinct vectors that span $\mathbb{F}^3$, and the length of every linearly independent list cannot exceed the length of the spanning list, the list we seek must be linearly dependent.

With u_1, u_2, we take a vector u_3 so that the list u_1, u_2, u_3 is linearly independent. We append some vector $u_4 = \lambda u_i$ for $i = 1, 2, 3$.

So one solution would be,

$$u_1 = (1, 0, -1), u_2 = (0, 1, -1), u_3 = (0, 0, 1), u_4 = 2(0, 0, 1) = (0, 0, 2).$$

REDO - misread the question.

2A.2.

Let $v \in \operatorname{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$, so that there exist scalars $b_1, \dots, b_4 \in \mathbb{F}$ such that

$$\begin{aligned} v &= b_1(v_1 - v_2) + b_2(v_2 - v_3) + b_3(v_3 - v_4) + b_4v_4 \\ &= b_1v_1 + (b_2 - b_1)v_2 + (b_3 - b_2)v_3 + (b_4 - b_3)v_4 \end{aligned}$$

Because $b_1, (b_2 - b_1), (b_3 - b_2), (b_4 - b_3) \in \mathbb{F}$, we have $v \in \operatorname{span}(v_1, \dots, v_4)$.

To show the reverse inclusion, let $v \in \operatorname{span}(v_1, v_2, v_3, v_4)$, and so there exist scalars $a_1, \ldots, a_4 \in \mathbb{F}$ such that

$$v = a_1 v_1 + \cdots + a_4 v_4.$$

If we now set our scalars a_i like so,

$$a_1 := b_1,\ a_2 := b_2 - b_1,\ a_3 := b_3 - b_2, a_4 := b_4 - b_3,$$

and use the previous identity in reverse, we have $v \in \operatorname{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$.

HINT: This is concise, but incomplete at your level, you need to spell out the form that the coefficients a_i take when we write $v \in \operatorname{span}(v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4)$.

Continuing more explicitly, solving for the scalars b_i we have,

$$b_1 = a_1,\ b_2 = a_1 + a_2,\ b_3 = a_1 + a_2 + a_3,\ b_4 = a_1 + a_2 + a_3 + a_4.$$

And so we have,

$$\begin{aligned} v &= a_1 v_1 + \cdots + a_4 v_4 \\ &= a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + (a_1 + a_2 + a_3 + a_4)v_4. \end{aligned}$$

Corrections: As you are still learning to write rigorous proofs, everything you've written after the ***HINT*** *is necessary to demonstrate the claim in question with absolute clarity. It will help for polynomials when we deal with changes of basis.*

2A.3.

We first show that $\operatorname{span}(v_1, \ldots, v_m) \subseteq \operatorname{span}(w_1, \ldots, w_m)$.

Let $v \in \operatorname{span}(v_1, \ldots, v_m)$, and by definition, we have for some scalars $a_i \in \mathbb{F}$,

$$v = a_1 v_1 + \cdots + a_m v_m.$$

Noting that $v_1 = w_1$ and that $v_k = w_k - w_{k-1}$ for $k \in \{2, \ldots, m\}$, we have,

$$\begin{aligned} v &= a_1 w_1 + a_2(w_2 - w_1) + \cdots + a_m(w_m - w_{m-1}) \\ &= (a_1 - a_2)w_1 + \cdots + (a_{m-1} - a_m)w_{m-1} + a_m w_m. \end{aligned}$$

As $(a_1 - a_2), \ldots, (a_{m-1} - a_m), a_m \in \mathbb{F}$, we have that $v \in \operatorname{span}(w_1, \ldots, w_m)$.

To show $\operatorname{span}(w_1, \ldots, w_m) \subseteq \operatorname{span}(v_1, \ldots, v_m)$, let $v \in \operatorname{span}(w_1, \ldots, w_m)$, so that there exist scalars $b_i \in \mathbb{F}$ such that

$$v = b_1 w_1 + \cdots + b_m w_m.$$

Now setting our scalars $b_m := a_m$ and $b_k := (a_k - a_{k+1})$ for $k = 1, \dots, m-1$ and unravelling the recursion forwards from index 1 to m to solve for the scalars a_i, we have $a_i = \sum_{k=i}^{m} b_i$, and so we have

$$\begin{aligned} v &= b_1 w_1 + \cdots + b_m w_m \\ &= (b_1 + \cdots + b_m)v_1 + (b_2 + \cdots + b_m)v_2 + \dots b_m v_m \end{aligned}$$

As $(b_1 + \cdots + b_m), (b_2 + \cdots + b_m), \dots, b_m \in \mathbb{F}$, we have that $v \in \operatorname{span}(v_1, \dots, v_m)$.

Corrections: You've opted to show that $\operatorname{span}(w_1, \dots, w_m) = \operatorname{span}(v_1, \dots, v_m)$ *using substitution, and the primary results are* $(*)$ *and* $(**)$*, whereas SM does so by grouping.*

For the second inclusion, it is faster to note that because $w_k = v_1 + \cdots + v_k$*, then* $S_1 = \{w_1, \dots, w_m\} \subseteq \operatorname{span}(S_2) = \operatorname{span}(v_1, \dots, v_m) \implies \operatorname{span}(S_1) \subseteq \operatorname{span}(S_2)$.

2A.4a.

$\implies$. Assume that the list v_1 of length 1 is linearly independent list, and also assume for contradiction that $v_1 = 0$. By definition of linear independence, the only choice of $a_1 \in \mathbb{F}$ that renders $a_1 v_1 = 0$ is when $a_1 = 0$.

But we have also assumed that $v_1 = 0$, which implies that $a_1 v_1 = 0$ for any $a_1 \in \mathbb{F}$, which contradicts linear independence of v_1.

Hence if the list v_1 of length 1 is linearly independent, then it must be the case that $v_1 \neq 0$.

$\impliedby$. Assume that $v_1 \neq 0$. Consider $a_1 v_1 = 0$. Because $k0 = 0$ for every $k \in \mathbb{F}$ and $v_1 \neq 0$, then it must be the case that $a_1 = 0$, and so $v_1 \neq 0$ implies that it is a linearly independent list of length 1.

Corrections: For the $\impliedby$ *part, you can communicate more clearly by using the fact that if* $a_1 v_1 = 0$*, then either* $a_1 = 0$ *or* $v = 0$*, and because* $v \neq 0$ *then* $a_1 = 0$.

SM proof logic. We want to show that $P \iff Q$*, where* P *is "*$\{v_1\}$ *is linearly independent", and* Q *is "*$v_1 \neq 0$*. For* $Q \implies P$*, he proves this directly by showing* $v_1 \neq 0 \implies \{v_1\}$ *is linearly independent. For* $P \implies Q$ *he proves using contrapositive, by proving* $\neg Q \implies \neg P$*, i.e.* $v_1 = 0 \implies \{v_1\}$ *is a linearly dependent list.*

2A.4b

$\implies$. Assume that the list v_1, v_2 is linearly independent. Assume for contradiction that at least one of the vectors in the list is a scalar multiple of the other. Then by definition, the only choice of $a_1, a_2 \in \mathbb{F}$ such that

$$a_1 v_1 + a_2 v_2 = 0,$$

is $a_1 = a_2 = 0$.

By rearranging, we have that

$$v_1 \neq \frac{-a_2}{a_1} v_1, \quad v_2 \neq \frac{-a_1}{a_2} v_1,$$

because $a_1 = a_2 = 0$. Which contradicts our assumption that at least one of the vectors in the list a scalar multiple of the other.

$\Longleftarrow$. Assume that neither of the two vectors in the list is a scalar multiple of the other. That is, assume that $v_1 \neq \lambda v_2$ and $v_2 \neq \mu v_1$ for $\lambda, \mu \in \mathbb{F}$. Consider,

$$a_1 v_1 + a_2 v_2 = 0.$$

Because neither of the two vectors is a scalar multiple of the other, this implies that the only way to write 0 as a linear combination of v_1 and v_2 is with $a_1 = a_2 = 0$.

Corrections: Intuition is correct, but there are some serious logical issues in both directions.

For the $\Longrightarrow$ part, you cannot use the definition of linear independence $a_1 = a_2 = 0$ to prove the non-existence of a scalar by dividing by zero!

It is better to opt for a contrapositive proof $\neg Q \implies \neg P$. So assume that one vector is a scalar multiple of the other $v_1 = \lambda v_2$. Then we have that $v_1 - \lambda v_2 = 0$, and we can represent 0 as a linear combination with (at least) one of the $a_i \neq 0$. Hence the list is linearly dependent. Contrapositively, if the list is linearly independent, then one must be a scalar multiple of the other.

For the $\Longleftarrow$ part, you've just assumed that neither vector is a scalar multiple of the other, then stated linear independence definition. As it stands, it asserts the truth of the statement in question, and isn't proof.

Use a contradiction proof. Assume for contradiction that the list is linearly dependent. Then $a_1 v_1 + a_2 v_2 = 0$ where at least one of the a_i is non-zero. If $a_1 \neq 0$, then $v_2 = -(a_2/a_1)v_1$, and so v_1 is a scalar multiple of v_2, which contradicts our assumption that neither is a scalar multiple of the other. Hence the list v_1, v_2 is linearly independent.

2A.5.

By definition, a list of vectors is linearly dependent if some vector in the list can be expressed as a linear combination of the other vectors, i.e. there is some $k \in \{1, \ldots, m\}$ such that $v_k \in \operatorname{span}(v_1, \ldots, v_{k-1})$.

We need to find some value of t such that the following equality holds for some choice of $\lambda, \mu \in \mathbb{F}$,

$$(5, 9, t) = \lambda(3, 1, 4) + \mu(-2, 3, 5).$$

This yields the following system of linear equations,

$$\begin{aligned} 5 &= 3\lambda - 2\mu, \\ 9 &= \lambda + 3\mu \\ t &= 4\lambda + 5\mu. \end{aligned}$$

Solving the first two equations yields $\mu = 2$ and $\lambda = 3$. Now setting $t = 22$ renders the list of three vectors linearly dependent.

2A.6.

$\Longrightarrow$. Assume that the list $(2,3,1),(1,-1,2),(7,3,c)$ is linearly dependent in $\mathbb{F}^3$. That is, there exists some c such that the following equality holds for $\lambda, \mu \in \mathbb{F}$,

$$(7,3,c) = \lambda(2,3,1) + \mu(1,-1,2).$$

Solving the implied linear equations across the first and second coordinates, we have $\lambda = 2$ and $\mu = 3$. Using the linear equation across the third coordinate gives $c = 8$.

$\Longleftarrow$. Assume that the $c = 8$. Then the list is linearly dependent because $(7,3,8) = 2(2,3,1) + 3(1,-1,2)$.

Corrections. On the $\Longrightarrow$ part, instead of "that is, there exists some…", clearer to state use of the linear dependence lemma.

2A.7a.

Key idea: Linear independence/dependence is contingent on the underlying field $\mathbb{F}$ of the vector space V. That's because the primary definition of linear independence given in Axler examines the following system for solutions a_i which reside *in the field* $\mathbb{F}$. In this example we have a situation where the same two vectors $v_1, v_2 \in \mathbb{C}$ are linearly independent when $\mathbb{F} = \mathbb{R}$, but linearly dependent when $\mathbb{F} = \mathbb{C}$.

$$a_1 v_1 + \cdots + a_n v_n = 0.$$

Let $\mathbb{C}$ be a vector space over the field of reals $\mathbb{R}$. Now consider

$$a_1(1+i) + a_2(1-i) = 0.$$

The vectors $1+i$ and $1-i$ are linearly independent if the only choice of scalars $a_1, a_2 \in \mathbb{R}$ that makes the above true is $a_1 = a_2 = 0$.

We now have have that

$$(a_1 + a_2) + (a_1 - a_2)i = 0.$$

Equating coefficients we have that $a_1 + a_2 = 0$ and $a_1 - a_2 = 0$, whose only solution is $a_1 = a_2 = 0$.

2A.7b.

Let $\mathbb{C}$ be a vector space of the field of complex numbers $\mathbb{C}$. We now consider whether the equality below has any solutions $a_1, a_2 \in \mathbb{C}$ other than $a_1 = a_2 = 0$,

$$a_1(1+i) + a_2(1-i) = 0.$$

Note that with $a_1 = 1$ and $a_2 = -(1+i)/(1-i)$ then the equality holds, meaning that the list of vectors is linearly dependent.

2A.8.

Assume that the list of vectors $v_1, \ldots, v_4$ is linearly independent in V. Let $v \in \operatorname{span}(v_1 - v_2, v_2 - v_3, v_3 - v_3, v_4)$. Then there exist $a_1, \ldots, a_4 \in \mathbb{F}$ such that

$$\begin{aligned} v &= a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4 v_4 \\ &= a_1 v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + (a_4 - a_3)v_4. \end{aligned}$$

Set $v := 0$, and consider,

$$a_1 v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + (a_4 - a_3)v_4 = 0.$$

By linear independence of $v_1, \ldots, v_4$, it must be the case that

$$a_1 = 0, (a_2 - a_1) = 0, (a_3 - a_2) = 0, (a_4 - a_3) = 0.$$

This implies that $a_1 = \cdots = a_4 = 0$. Hence the list in question is linearly independent.

Corrections: Even though you covered the idea of the span of a list of vectors in the chapter, for concision, you can leave out the part of $v \in \operatorname{span}(\ldots)$ *and* $v := 0$ *altogether, and just work with equations here.*

2A.9.

Assume that $v_1, \ldots, v_m$ is linearly independent in V. Let $v \in \operatorname{span}(5v_1 - 4v_2, \ldots, v_m)$. Expressing v as a linear combination, we have

$$\begin{aligned} v &= b_1(5v_1 - 4v_2) + b_2 v_2 + \cdots + b_m v_m \\ &= 5b_1 v_1 + (b_2 - 4b_1)v_2 + \cdots + b_m v_m. \end{aligned}$$

Set $v := 0$ and consider,

$$5b_1 v_1 + (b_2 - 4b_1)v_2 + \cdots + b_m v_m = 0.$$

By linear independence of $v_1, \ldots, v_m$, it must be the case that the only solution to the above equation is with,

$$5b_1 = 0, (b_2 - 4b_1) = 0, b_3 = 0, \ldots, b_m = 0.$$

Hence $b_1 = \cdots = b_m = 0$ and the list in question is linearly independent.

2A.10.

Assume that $v_1, \ldots, v_m$ is linearly independent in V and $\lambda \in \mathbb{F}$ with $\lambda \neq 0$.

Consider the following for scalars $a_i \in \mathbb{F}$,

$$a_1(\lambda v_1) + \cdots + a_m(\lambda v_m) = 0.$$

By associativity of scalar multiplication on V, this is equivalent to the statement that,

$$(a_1\lambda)v_1 + \cdots + (a_m\lambda)v_m = 0.$$

Linear independence of the list $v_1, \dots, v_m$ implies that the only solution to this equation is with $(a_1\lambda) = \cdots = (a_m\lambda) = 0$. Because $\lambda \neq 0$, this can only occur when $a_1 = \dots a_m = 0$.

Hence the list $\lambda v_1, \dots, \lambda v_m$ is linearly independent.

2A.11.

Not sure.

Corrections: No need for a complex counterexample. Simple is fine. Here's one from SM. Consider the vector space $\mathbb{R}$ over itself. Then you have two length 1 linearly indepedent lists, 1, and -1, but the list $1 + (-1) = 0$ is linearly dependent.

2A.12.

It doesn't seem possible to reason about this using coefficients alone.

HINT: Use the linear dependence lemma.

As the list $v_1 + w, \dots, v_m + w$ is linearly dependent, by the linear dependence lemma, there exists some $k \in \{1, \dots m\}$ for which

$$v_k + w \in \operatorname{span}(v_1 + w, \dots, v_{k-1} + w).$$

We now need to show that $w \in \operatorname{span}(v_1, \dots, v_k)$ for every possible value of $k = 1, \dots, m$ identified by the linear dependence lemma.

Consider the following,

$$a_1(v_1 + w) + \cdots + a_m(v_m + w) = 0. \tag{*}$$

As the $v_1 + w, \dots, v_m + w$ is linearly dependent, by definition there exists at least one $a_i \neq 0$ such that the above holds.

Case 1: Edge case where linear dependence lemma identifies $k = 1$, this only occurs when $v_1 + w = 0$. This cannot be because of v_1 and w both being 0 because it would contradict linear independence of $v_1, \dots, v_m$. So it can only occur when the w is the additive inverse of v_1, i.e. $w = (-v_1)$, in which case,

$$w \in \operatorname{span}(v_1).$$

Case 2: Linear dependence lemma identifies some $k \geq 2$. This k will be the largest element of $\{2, \dots, m\}$ such that $a_k \neq 0$ in (*). Hence $a_{k+1} = \cdots = a_m = 0$, and so we have,

$$a_1(v_1 + w) + \cdots + a_k(v_k + w) = 0,$$

We now have,

$$\begin{aligned} a_k v_k + a_k w &= -a_1 v_1 - \cdots - a_{k-1} v_{k-1} - (a_1 + \cdots + a_{k-1}) w \\ a_k v_k + (a_1 + \cdots + a_k) w &= -a_1 v_1 - \cdots - a_{k-1} v_{k-1} \end{aligned}$$

Defining $c_k := a_1 + \cdots + a_k$, we have that,

$$w = -\sum_{i=1}^{k} \frac{a_i}{c_k} v_i. \tag{***}$$

This shows that $w \in \operatorname{span}(v_1, \ldots, v_k)$ if and only if $c_k \neq 0$.

HINT: Use $(**)$ to show that $c_k \neq 0$, as it is not so clear from $(***)$.

We have that $c_k \neq 0$ because otherwise, $v_k \in \operatorname{span}(v_1, \ldots, v_{k-1})$, which would contradict our assumption of linear independence of $v_1, \ldots, v_m$. Hence,

$$w \in \operatorname{span}(v_1, \ldots, v_m).$$

Corrections: You needed a few hints, but it's well-written, save perhaps for the edge case part. SM is concise, by leaving out algebra in the presentation. Something to note for new proofs. Not every algebraic step needs to be included in proof presentation. Saves typing time too.

2A.13.

$\implies$. Assume that $v_1, \ldots, v_m, w$ is linearly independent. Consider

$$a_1 v_1 + \cdots + a_n v_n + a_{m+1} w = 0.$$

This is equivalent to the condition that,

$$a_{m+1} w = -\sum_{i=1}^{n} a_i v_i.$$

But by linear independence of $v_1, \ldots, v_m, w$, we have that all $a_i = 0$, including a_{m+1}, and so

$$w \neq -\sum_{i=1}^{n} \frac{a_i}{a_{m+1}} v_i \implies w \notin \operatorname{span}(v_1, \ldots, v_m).$$

$\impliedby$. Assume that $w \notin \operatorname{span}(v_1, \ldots, v_m)$. Noting that $v_1, \ldots, v_m$ is linearly independent, we use the linear dependence lemma contrapositively, and so we have that the list $v_1, \ldots, v_m, w$ must also be linearly independent.

Corrections: The $\implies$ part is problematic.

*Your erroneous reasoning is that because $a_{m+1} = 0$, we can "block" the possibility of the equality holding true, i.e. that w is a linear combination of the v_is. Similar to the error you've made in **2A.4b**, in order to make a statement $A = B$ or $A \neq B$, you need both A and B to be well-defined. Anything involving an undefined quantity, such as a quantity involving division by zero, cannot fall under the scope of your equality/inequality statement.*

Use contradiction instead. Assume for contradiction that $w \in \operatorname{span}(v_1, \ldots, v_m)$. So $w = a_1v_1 + \cdots + a_mv_m$, and so $a_1v_1 + \cdots + a_mv_m - w = 0$, and we can represent 0 as a linear combination of the v_is and w with at least one non-zero scalar coefficient, these vectors must be linearly dependent, contradiction.

For the $\Longleftarrow$ direction, SM opts for contrapositive proof, whereas you use the linear dependence lemma contrapositively. So assuming $\neg Q$, i.e. $v_1, \ldots, v_m, w$ is linearly dependent, by LDL one of the vectors must be in the span of the previous vectors, but not of one of the v_k as otherwise contradiction. Hence $\neg P$ i.e. $w \in \operatorname{span}(v_1, \ldots, v_m)$ is established, and contrapositively, the statement is proven.

2A.14.

$\Longrightarrow$. Assume that $v_1, \ldots, v_m$ is linearly independent and consider,

$$b_1w_1 + \cdots + b_mw_m = 0.$$

By definition of $w_k = v_1 + \cdots + v_m$, we have that the above is equivalent to,

$$\left(\sum_{i=1}^{m} b_i\right) v_1 + \cdots + \left(\sum_{i=m-1}^{m} b_i\right) v_{m-1} + \left(\sum_{i=m}^{m} b_i\right) v_m = 0.$$

By linear independence of the v_i, it must be the case that the only solution for this linear system is with $\sum_{i=k}^{m} b_i = 0$ for all $k = 1, \ldots, m$. Unravelling the recursion beginning with $k = m$, we have that $\sum_{i=m}^{m} b_i = b_m = 0$, and continuing, we have $b_{m-1}, \ldots b_1 = 0$.

Hence the $w_1, \ldots w_m$ is linearly independent.

$\Longleftarrow$. Assume that $w_1, \ldots, w_m$ is linearly independent, and consider,

$$a_1v_1 + \cdots + a_mv_m = 0.$$

Noting that $v_k = w_k - w_{k-1}$, we have that the above is equivalent to,

$$(a_1 - a_2)w_1 + (a_2 - a_3)w_2 + \cdots + (a_{m-1} - a_m)w_{m-1} + a_mw_m = 0.$$

By linear independence of the w_i, it must be the case that the only solution for the above system is with $a_m = 0, (a_{m-1} - a_m) = 0, \ldots, (a_1 - a_2) = 0$. Solving, we have $a_1 = \ldots a_m = 0$, and so $v_1, \ldots, v_m$ is linearly independent.

Corrections. This looks fine, but one of the themes emerging from your attempt at these exercises is that you can use the linear dependence lemma to shorten many algebraic derivations involving linear combinations and their coefficients, and so achieve greater concision.

2A.15.

The standard basis of $\mathcal{P}_4(\mathbb{F})$ is $\{1, x, x^2, x^3, x^4\}$, a list of five vectors that form a spanning list.

By (2.22), any linearly independent list in the vector space $\mathcal{P}_4(\mathbb{F})$ cannot exceed the length of its spanning list. That is, no list of vectors with length greater than five can be linearly independent in $\mathcal{P}_4(\mathbb{F})$.

And so there does not exist a list of six polynomial vectors that is linearly independent in this vector space.

2A.16

As established in **2A.15.**, the spanning list of the vector space $\mathcal{P}_4(\mathbb{F})$ is of length five. This list is also linearly independent, and so we can use (2.22)

(2.22) states that every spanning list cannot be less than the length of a linearly independent list in that vector space. That is, no list of vectors with length less than 5 can span $\mathcal{P}_4(\mathbb{F})$.

And so there does not exist a list of four polynomials that spans $\mathcal{P}_4(\mathbb{F})$.

2A.17.

Key idea: An infinite-dimensional vector space V at this stage of the text is one for which no (finite) list spans it. The result proved in this problem allows you to show that a vector space V is infinite-dimensional by supplying an infinite sequence of vectors in $v_1, v_2, \dots$ where each $v_i \in V$, and for which every finite initial segment is linearly independent.

$\Longleftarrow$. Assume for contradiction that V is finite-dimensional, that is V is spanned by a finite list of k vectors $S = v_1, \dots v_k$, with each $v_i \in V$. But then we have also assumed that there exists a sequence of vectors $v_1, v_2, \dots$ with each $v_i \in V$ such that $v_1, \dots v_m$ is linearly independent for every positive integer $m \in \mathbb{Z}$. This implies that there exists a linearly independent list of length $m = k + 1$ that is longer than the spanning list of V of length k, which is a contradiction. Hence V is infinite-dimensional.

HINT: Rather than use contradiction for the next part, try a direct, constructive approach.

$\Longrightarrow$. Assume that V is infinite-dimensional. We will construct a sequence of vectors $v_1, v_2, \dots$ such that the sub-list of the first m vectors is linearly independent, for every positive integer $m \in \mathbb{Z}$.

For step 1, select some nonzero vector $v_1 \neq 0$, and add it to the list. Then for step m, select some $v_m \notin \operatorname{span}(v_1, \dots, v_{m-1})$, and add it to the list. Using the linear dependence lemma contrapositively, at the end of every step, we have a linearly independent list.

Because V is infinite-dimensional, assuming that the Steinitz exchange lemma (2.22) holds for infinite dimensional vector spaces, the length of any linearly independent list in V is unbounded, because the length of the spanning list of V is unbounded. Hence there will also be vectors in V that will allow us to continue the construction of a linearly independent list indefinitely.

Which proves that there exists a sequence $v_1, v_2, \dots$ such that the first m vectors are linearly independent for every positive integer $m \in \mathbb{Z}$.

Corrections: The $\Longleftarrow$ direction looks good. But the $\Longrightarrow$ direction has the following deficiencies:

- *Most seriously, you need not assume that the Steinitz exchange lemma (2.22) holds for infinite dimensional vector spaces, nor use a boundedness argument to make the claim that there will always exist vectors v_m in V such that $v_m \notin \operatorname{span}(v_1, \dots, v_{m-1})$.*
- *Instead, because V is infinite-dimensional, it is not finite-dimensional. Which means that there does not exist a finite list that spans V.*
- *Now if you have chosen $v_1, \dots v_{m-1}$, we know that $\operatorname{span}(v_1, \dots, v_{m-1}) \neq V$, otherwise V would be finite-dimensional. Hence there will exist some $v_m \in V$ such that $v_m \notin \operatorname{span}(v_1, \dots, v_{m-1})$.*
- *In terms of proof presentation, at its core, it is induction. Just because you are using an Axler-type multi-step construction, doesn't mean that you can't be explicit.*

- *First, the base case should acknowledge that $V \neq \{0\}$, otherwise it would be finite-dimensional, so pick some $v_1 \neq 0$, and so this list is linearly independent.*
- *Lastly from SM, the inductive step should be, assume that you have a list of $m-1$ linearly independent vectors $v_1, \dots v_{m-1}$. Then $V \neq \operatorname{span}(v_1, \dots, v_{m-1})$ by assumption, and so pick any $v_m \notin \operatorname{span}(v_1, \dots, v_{m-1})$ so that using the LDL contrapositively, you have that the list $v_1, \dots v_{m-1}, v_m$ is linearly independent.*
- *We can continue this process for every positive integer $m \in \mathbb{Z}$. This generates an infinite sequence $v_1, v_2, \dots$ where every finite initial segment is linearly independent.*

2A.18.

We can use **2A.17** to show that $\mathbb{F}^\infty = \{(x_1, x_2, \dots) \mid x_k \in \mathbb{F},\, k = 1, 2, \dots\}$ is infinite-dimensional.

HINT: You cannot proceed with a similar induction argument as in **2A.17**, because that was an implication, and required assuming that V was infinite-dimensional. Instead, you have to leverage the fact that each vector in $\mathbb{F}^\infty$ is itself an infinite sequence - can you use this to construct a linearly independent set?

Define the infinite sequence $(e'_i)_{i=1}^\infty$ consisting of vectors $e'_k \in \mathbb{F}^\infty$. Where each vector e'_k is defined as the standard basis vector e_k, but with infinitely many coordinates 0, and with its kth coordinate defined as 1.

$$e'_k = (0, 0, 0, \dots, 1, 0, 0 \dots).$$

Let m be some positive integer. Now consider the following equation system in the list of the first m vectors of the sequence $e'_1, \dots e'_m$, where m is a positive integer,

$$a_1 {e'}_1 + \cdots + a_m {e'}_m = 0.$$

Trivially solving the system across the m coordinates implies that $a_i = 0$ for all $i = 1, \dots, m$, hence the list $e_1, \dots, e_m$ is linearly independent for all positive integers m.

As we have shown that there exists an infinite sequence of vectors in $\mathbb{F}^\infty$ with the property that any finite sub-list starting from the beginning of the sequence is linearly independent, F^∞ is infinite-dimensional.

Corrections: All good.

Exercises 2B Bases.

2B.2.

Key idea: Contextualising in terms of systems of equations, and coordinate based linear algebra, define the standard basis E. Then,

- Linear independence is about assessing whether there are any other solutions to the homogeneous system $Ax = 0$ other than $x = 0$.
- Spanning is about assessing whether there exist any b such that there are no solutions to the inhomogeneous system $Ax = b$.

Example 2.27a.

The n-dimensional vector space $\mathbb{F}^n$ over the field $\mathbb{F}$ is defined as,

$$\mathbb{F}^n = \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in \mathbb{F}\}.$$

The standard basis of $\mathbb{F}^n$ is the list of vectors $e_1, \ldots e_n$ with e_i, where each e_i consists of a 1 in the ith coordinate and a 0 at every other coordinate.

We need to show that $e_1, \ldots, e_n$ spans V, and is linearly independent.

Let $(x_1, \ldots, x_n) \in \mathbb{F}^n$, and we have $\mathbb{F}^n \subseteq \operatorname{span}(e_1, \ldots, e_n)$ because

$$(x_1, \ldots, x_n) = x_1 e_1 + \cdots + x_n e_n \in \operatorname{span}(e_1, \ldots, e_n).$$

Let $v = x_1 e_1 + \cdots + x_n e_n$ and using the above identity in reverse shows that $\operatorname{span}(e_1, \ldots, e_n) \subseteq \mathbb{F}^n$. Hence $\operatorname{span}(e_1, \ldots, e_m) = V$.

For linear independence, considering the following system of linear equations in scalars $a_i \in \mathbb{F}$ across the n coordinates implies that $a_i = 0$ for all $i = 1, \ldots, n$.

$$a_1 e_1 + \cdots + a_n e_n = 0$$

Hence $e_1, \ldots, e_m$ is linearly independent, and this shows that the standard basis of $\mathbb{F}^n$ is indeed a basis.

Example 2.27b.

Consider the system of linear equations in the scalars $a_1, a_2 \in \mathbb{F}$,

$$a_1(1, 2) + a_2(3, 5) = 0.$$

Solving this system yields only one solution, $a_1 = a_2 = 0$, and so these vectors are linearly independent.

Now if $(1, 2), (3, 5)$ spans $\mathbb{F}^2$, then the following linear system will have solutions in $a_1, a_2 \in \mathbb{F}$, for all possible vectors $(x_1, x_2) \in \mathbb{F}$,

$$a_1(1, 2) + a_2(3, 5) = (x_1, x_2).$$

For any given (x_1, x_2), we can set $a_1 = (2x_1 - x_2)$ and $a_2 = (-5x_1 + 3x_2)$, which shows that the two vectors span $\mathbb{F}^2$.

Because these two vectors span $\mathbb{F}^2$ and are linearly independent, they form a basis of $\mathbb{F}^2$. Meaning that any vector $v = (x_1, x_2) \in \mathbb{F}^2$ can be uniquely represented as a linear combination of these two vectors by a (unique) choice of scalars a_1 and a_2.

Example 2.27c.

Consider the system of linear equations in scalars $a_1, a_2 \in \mathbb{F}$,

$$a_1(1, 2, -4) + a_2(7, 5 - 6) = 0.$$

Solving the system yields only one solution, $a_1 = a_2 = 0$, and so these vectors are linearly independent in $\mathbb{F}^3$.

The standard basis of $\mathbb{F}^3$ is a linearly independent list of length 3. Using (2.22), every spanning list in $\mathbb{F}^3$ cannot be less than the length of a linearly independent list. So no list of less than length 3 can span $\mathbb{F}^3$.

As the length of these two vectors is 2, we know they cannot span $\mathbb{F}^3$.

Example 2.27d.

In example 2.27b, we established that the list $(1, 2)$ and $(3, 5)$ of length 2 is linearly independent and spans $\mathbb{F}^2$, i.e. is a basis of $\mathbb{F}^2$.

Using (2.22), we know that the length of any linearly independent list in $\mathbb{F}^2$ cannot exceed the length of the spanning list of $\mathbb{F}^2$, which is of length 2.

Hence the new list $(1, 2), (3, 5), (4, 13)$ of length 3 is not linearly independent, and also because $(4, 13) = \lambda(1, 2) - \mu(3, 5)$ with $\lambda = 19$ and $\mu = -5$.

Because $(4, 13) \in \text{span}((1, 2), (3, 5))$, the second part of the linear dependence lemma implies that we can remove this vector and the span of the new list (Which we know to span $\mathbb{F}^2$) will be the same as the original list, i.e.

$$\mathbb{F}^2 = \text{span}((1, 2), (3, 5)) = \text{span}((1, 2), (3, 5), (4, 13)).$$

Example 2.27e.

Key idea: If a set of vectors form a basis of a subspace $U \subseteq V$, then they span U, and we say that they are linearly independent in V rather than U. Saying that they are linearly independent in U is technically correct, but not conventional, as redundant - U inherits its vector space structure from V.

Define the subset $U \subset \mathbb{F}^3$ as follows,

$$U = \{(x, x, y) \in \mathbb{F}^3 \mid x, y \in \mathbb{F}\}.$$

By inspection, U is a subspace of $\mathbb{F}^3$.

Consider the system of linear equations in scalars $a_1, a_2 \in \mathbb{F}$,

$$a_1(1, 1, 0) + a_2(0, 0, 1) = (x_1, x_1, x_2).$$

Solving the system with $(x_1, x_1, x_2) = (0, 0, 0)$ yields only one solution, $a_1 = a_2 = 0$, and so these vectors are linearly independent in $\mathbb{F}^3$.

Let $(x_1, x_2, x_3) \in U$, and $U \subseteq \text{span}((1, 1, 0), (0, 0, 1))$ because

$$(x_1, x_2, x_3) = x_1(1, 1, 0) + x_2(0, 0, 1) \in \text{span}((1, 1, 0), (0, 0, 1)).$$

Letting $u \in \text{span}((1,1,0),(0,0,1))$ and using the above identity in reverse also shows $\text{span}((1,1,0),(0,0,1)) \subseteq U$ and hence $U = \text{span}((1,1,0),(0,0,1))$.

This can also be shown by solving the linear system above, yielding $a_1 = x_1$ and $a_2 = x_3$.

Example 2.27f.

Define the subspace $U \in \mathbb{F}^3$ as,

$$U = \{(x,y,z) \in \mathbb{F}^3 \mid x+y+z=0\} = \{(x_1, x_2, -x_2 - x_1) \in \mathbb{F}^3 \mid x_1, x_2 \in \mathbb{F}\}.$$

Considering the following linear system, we have

$$a_1(1,-1,0) + a_2(1,0,-1) = (x_1, x_2, -x_2 - x_1).$$

Setting the RHS to the zero vector and solving yields only one solution $a_1 = a_2 = 0$ which shows that these vectors are linearly independent in $\mathbb{F}^3$.

Solving the linear system above without setting the RHS to the zero vector yields $a_1 = -x_2$ and $a_2 = x_1 + x_2$, and hence $U = \text{span}((1,-1,0),(1,0,-1))$.

Example 2.27g.

HINT: You have to be a bit careful about turning the questions of linear independence and span when it comes to polynomials into a linear system. Because the coordinate isomorphism is not an identity between $\mathcal{P}_m(\mathbb{F})$ and $\mathbb{F}^{m+1}$, you cannot rely on the fact that n-tuples are implicitly defined in the standard basis, and this is felt when you attempt to write down a polynomial as a coordinate representation - you cannot do so without declaring a basis, or rely on the standard basis as implicit. Instead just work algebraically.

To see that $\{1, z, \ldots, z^m\}$ is linearly independent, consider the linear system,

$$a_0(1) + a_1 z + \ldots a_m z^m = 0.$$

We can see that the only solution is with all the $a_i = 0$, and so linear independence is shown.

That $\{1, z, \ldots, z^m\}$ spans $\mathcal{P}_m(\mathbb{F})$ arises by definition of a polynomial of degree m, in particular that any polynomial $\mathcal{P}_m(\mathbb{F})$ is uniquely determined by its coefficients a_i.

Corrections. For many of the spanning arguments, you proceeded by bidirectional set-inclusion. This is great for learning proof writing. But in many of these cases, the equality you've specified already establishes the inclusion both ways. Something to note as a way to shorten your proofs slightly.

2B.3a.

Define the subspace $U \subseteq \mathbb{R}^5$,

$$\begin{aligned} U &= \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1 = 3x_2, x_3 = 7x_4\} \\ &= \{(3x_1, x_1, 7x_2, x_2, x_3) \in \mathbb{R}^5 \mid x_1, \ldots, x_3 \in \mathbb{F}\}. \end{aligned}$$

Let $u \in U$, and note the following linear decomposition.

$$
\begin{aligned}
u &= (3x_1, x_1, 7x_2, x_2, x_3) \\
&= x_1(3,1,0,0,0) + x_2(0,0,7,1,0) + x_3(0,0,0,0,1).
\end{aligned}
$$

This suggests a natural choice of basis is $u_1 := (3,1,0,0,0), u_2 := (0,0,7,1,0), u_3 = (0,0,0,0,1)$.

To show that $\operatorname{span}(u_1, u_2, u_3) = U$, and noting that the above decomposition already implicitly shows that $U \subseteq \operatorname{span}(u_1, u_2, u_3)$, we only need to show $\operatorname{span}(u_1, u_2, u_3) \subseteq U$.

Let $u' \in \operatorname{span}(u_1, u_2, u_3)$, and the above shows that $u' = (3x_1, x_1, 7x_2, x_2, x_3) \in U$, so $\operatorname{span}(u_1, u_2, u_3) \subseteq U$.

For linear independence, consider the linear system,

$$
a_1 u_1 + \cdots + a_3 u_3 = 0,
$$

which has as its only solution that $a_1 = a_2 = a_3 = 0$, showing that u_1, u_2, u_3 are linearly independent, and hence form a basis of U.

Corrections: Recalling earlier correction, your linear decomposition shows the bidirectional set-inclusion both ways, you can state this next time.

2B.3b.

Let the basis of subspace U be $\beta_U = \{u_1, u_2, u_3\}$ from **2B.3a.** and also define the standard basis of $\mathbb{R}^5$ to be $\beta_V = \{e_1, \ldots, e_5\}$.

We can use (2.32) by taking the linearly independent list β_U and appending the spanning list β_V of $\mathbb{R}^5$ to get a new list $\beta_U \cup \beta_V$, which will also span $\mathbb{R}^5$, so that

$$
\operatorname{span}(\beta_U \cup \beta_V) = \mathbb{R}^5.
$$

However, this new list of length 8 will not be linearly independent, and will not form a basis. But because every spanning list contains a basis, we can apply (2.30) to reduce the spanning list $\beta_U \cup \beta_V$ to a new basis β'_V of $\mathbb{R}^5$.

And so (2.30) will do this by sequentially pruning vectors from $\beta_U \cup \beta_V$ one by one until we get β'_V that is linearly independent, and also spans $\mathbb{R}^5$.

Using the algorithm in (2.30), we first define the list B,

$$
B = \beta_U \cup \beta_V = u_1, u_2, u_3, e_1, \ldots, e_5.
$$

Step 1 of the algorithm is to check if the first vector in the list B is zero, and if so, delete it, otherwise leaves the list unchanged. Now because β_U is linearly independent, none of the $u_i = 0$, and so $u_1 \neq 0$.

Step k of the algorithm applies the linear dependence lemma to check whether the kth vector in the initial list B is in the span of previous vectors, and if so deletes it (which will maintain the span of the new list as equal to that of the original list). Otherwise it leaves the list unchanged.

Because the list B is of length 8, the algorithm will run for 8 steps.

Now because β_U is linearly independent, we only need to consider step 4 onwards.

Step 4: $e_1 = (1,0,0,0,0) \notin \text{span}(u_1, u_2, u_3)$. To see this, note that if we try and find scalars $a_i \in \mathbb{F}$ to express e_1 as a linear combination,

$$e_1(1,0,0,0,0) = a_1(3,1,0,0,0) + a_2(0,0,7,1,0) + a_3(0,0,0,0,1),$$

then we find that even with $a_2 = a_3 = 0$, we cannot find a value of a_1 that can simultaneously map the first coordinate of u_1 from 3 to 1, and also map the second coordinate of u_1 from 1 to 0. Hence we leave e_1 in the list.

Step 5: $e_2 = (0,1,0,0,0) \in \text{span}(u_1, u_2, u_3, e_1)$. Now considering,

$$e_2 = (0,1,0,0,0) = a_1(3,1,0,0,0) + a_2(0,0,7,1,0) + a_3(0,0,0,0,1) + a_4(1,0,0,0,0),$$

we find that $a_2 = a_3 = 0$ and $x_1 = 1, x_4 = -2$, and so we delete e_2.

Step 6: $e_3 \notin \text{span}(u_1, u_2, u_3, e_1)$. Considering the equation in step 5, but now with e_2 on the left hand side, we find that with $a_1 = a_3 = a_4 = 0$, no choice of scalar a_2 can simulatenously map the 3rd coordinate of u_2 from 7 to 1 and 4th coordinate of u_2 from 1 to 0. So we leave e_3.

At the completion of step 6, we have the list

$$B = u_1, u_2, u_3, e_1, e_3, e_4, e_5.$$

However, we need not run step 7 and 8 to consider whether e_4 and e_5 are in the span of the previous vectors, we can just remove them. That is because the first 5 vectors $u_1, \dots e_3$ span $\mathbb{R}^5$ and are linearly independent by construction.

Hence we have a new basis β'_V of $\mathbb{R}^5$,

$$\beta'_V = \{u_1, u_2, u_3, e_1, e_3\}.$$

Corrections: All good.

2B.3c.

Key idea. If and only if U and W are subspaces of V that form a direct sum $U \oplus W = V$, and β_U and β_W are bases of U and W respectively, then $\beta_U \cup \beta_W$ is a basis of V.

From **2B.3b.**, we let $W = \text{span}(e_1, e_3)$.

To show that $U + W = \text{span}(\beta'_V) = \mathbb{R}^5$, note that β'_V is a basis of $\mathbb{R}^5$.

To show that $U \oplus W$ is a direct sum, let $z \in U \cap W$. Then $z \in W$ implies that $z = a_1 e_1 + a_2 e_3 = (a_1, 0, a_2, 0, 0)$. We also have that $z \in U$ implies that $z = b_1 u_1 + b_2 u_2 + b_3 u_3 = (3b_1, b_1, 7b_2, b_2, b_3)$.

Because $z \in U$ and $z \in W$ simultaneously, this implies that $b_1 = b_2 = b_3 = 0$, meaning that $z = 0$. Hence $U \cap W = \{0\}$, and so $U \oplus W = \mathbb{R}^5$.

2B.4a.

Define the subspace $U \in \mathbb{C}^5$,

$$\begin{aligned} U &= \{(z_1, \dots, z_5) \in \mathbb{C}^5 \mid 6z_1 = z_2, z_3 + 2z_4 + 3z_5 = 0\} \\ &= \{(z_1, 6z_1, -(2z_2 + 3z_3), 2z_2, 3z_3) \in \mathbb{C}^5 \mid x_1, x_2, x_3 \in \mathbb{C}\}. \end{aligned}$$

Noting that our subspace is defined by two linear constraints, and so has three "free parameters", we have the following decomposition on an arbitrary $u \in U$,

$$\begin{aligned} u &= (z_1, 6z_1, -(2z_2 + 3z_3), 2z_2, 3z_3) \\ &= (z_1, 6z_1, 0, 0, 0) + (0, 0, -2z_2, 2z_2, 0) + (0, 0, -3z_3, 0, 3z_3) \\ &= z_1(1, 6, 0, 0, 0) + z_2(0, 0, -2, 2, 0) + z_3(0, 0, -3, 0, 3) \end{aligned}$$

This suggests the following natural choice of basis, $u_1 := (1, 6, 0, 0, 0)$, $u_2 := (0, 0, -2, 2, 0)$, and $u_3 := (0, 0, -3, 0, 3)$ then this shows that $U \subseteq \operatorname{span}(u_1, u_2, u_3)$ and $U \supseteq \operatorname{span}(u_1, u_2, u_3)$.

For linear independence of the u_i, consider the following linear system,

$$z_1 u_1 + z_2 u_2 + z_3 u_3 = 0,$$

which has as its only solution, $z_1 = z_2 = z_3 = 0$, and so the u_i are linearly independent, and hence form a basis of U.

Corrections: There is an error in your simultaneous translation and relabelling of variables in the constraint. Correct answer here is,

$$\begin{aligned} U &= \{(z_1, \dots, z_5) \in \mathbb{C}^5 \mid 6z_1 = z_2, z_3 + 2z_4 + 3z_5 = 0\} \\ &= \{(z_1, 6z_1, -(2z_2 + 3z_3), z_2, z_3) \in \mathbb{C}^5 \mid x_1, x_2, x_3 \in \mathbb{C}\}. \end{aligned}$$

And so the natural choice of basis is,

$$u_1 := (1, 6, 0, 0, 0),\ u_2 := (0, 0, -2, 1, 0),\ u_3 := (0, 0, -3, 0, 1)$$

2B.4b.

Following the approach in **2B.3b.**, let $\beta_V = \{e_1, \dots, e_5\}$ be the standard basis of $\mathbb{C}^5$, and $\beta_U = \{u_1, u_2, u_3\}$ be the basis of U as established in **2B.4a.**.

Now $\operatorname{span}(\beta_U \cup \beta_V) = \mathbb{C}^5$, and so we can run the spanning list reduction algorithm to remove linearly dependent vectors on the following list B,

$$B = \beta_U \cup \beta_V = \{u_1, u_2, u_3, e_1, \dots, e_5\}.$$

We skip the first 3 steps of the algorithm because of the linear independence of β_U.

Step 4: $e_1 \notin \operatorname{span}(u_1, \dots, u_3)$. Considering $e_1 = z_1 u_1 + z_2 u_2 + z_3 u_3$, even with $z_2 = z_3 = 0$, no choice of $z_1 \in \mathbb{C}$ can multiply the first and second coordinates of u_1 to be the same as the coordinates of e_1. Hence we leave e_1 in the list.

Step 5: $e_2 \in \text{span}(u_1, u_2, u_3, e_1)$. Note that $e_2 = 1/6u_1 + 0u_2 + 0u_3 + (-1/6)e_1$. Hence we delete e_2.

Step 6: $e_3 \notin \text{span}(u_1, u_2, u_3, e_1)$. Considering $e_2 = z_1u_1 + z_2u_2 + z_3u_3 + z_4e_1$, even with $z_1 = z_4 = 0$, there is no combination of $z_2, z_3 \in \mathbb{C}$, such that the coordinates of e_3 can be attained.

However, we need not run step 7 and 8 to consider whether e_4 and e_5 are in the span of the previous vectors, we can just remove them. That is because the first 5 vectors $u_1, \dots e_3$ span $\mathbb{C}^5$ and are linearly independent by construction.

Hence we have a new basis β'_V of $\mathbb{R}^5$,

$$\beta'_V = \{u_1, u_2, u_3, e_1, e_3\}.$$

Corrections. Wrong starting basis from previous part, but algorithm correctly implemented.

2B.4c.

From **2B.3b.**, we let $W = \text{span}(e_1, e_3)$.

To show that $U + W = \text{span}(\beta'_V) = \mathbb{C}^5$, note that β'_V is a basis of $\mathbb{C}^5$.

To show that $U \oplus W$ is a direct sum, let $z \in U \cap W$. Then $z \in W$ implies that $z = a_1e_1 + a_2e_3 = (a_1, 0, a_2, 0, 0)$. We also have that $z \in U$ implies that $z = b_1u_1 + b_2u_2 + b_3u_3 = (b_1, 6b_1, -(2b_2 + 3b_3), 2b_2, 3b_3)$.

Because $z \in U$ and $z \in W$ simultaneously, this implies that $a_1 = b_1 = 0$ and $b_2 = b_3 = 0$, and the latter implies that $a_2 = 0$, meaning that $z = 0$. Hence $U \cap W = \{0\}$, and so $U \oplus W = \mathbb{C}^5$.

Corrections: Again, amending the error from previous parts, this answer can be corrected by noting that "$z \in U$ implies that $z = b_1u_1 + b_2u_2 + b_3u_3 = (b_1, 6b_1, -(2b_2 + 3b_3), b_2, b_3)$," instead of what you wrote.

2B.5.

HINT: You have to prove that if β is a basis of V, then $\beta \subseteq U \cup W$.

Define a basis $u_1, \dots, u_m$ of U and a basis $w_1, \dots, w_n$ of W.

(?) Strategy here is to construct a spanning list of V, using these two bases, then prune, but it's not clear to me whether we can be sure "putting" these two bases together via taking a union establishes that they span V.

Corrections. Strategy is correct. You've omitted using the given fact that $V = U + W$, that's why you couldn't proceed. From SM,

- *You are able to define the bases of the subspaces U and W as you have done due to the fact that any subspace of a finite-dimensional vector space is finite-dimensional (2.25), and also because any subspace, as a finite-dimensional vector space, has a basis.*
- *To see that $V = U + W$ implies $V = \text{span}(u_1, \dots, u_m, w_1, \dots, w_n)$, it's a simple matter of observing the following equality, which shows what we need via bidirectional set inclusion.*

$$v = a_1v_1 + \cdots + a_mv_m + b_1w_1 + \cdots + b_nw_n.$$

- *Then we just prune the spanning list $\beta = \{u_1, \dots, u_m, w_1, \dots, w_n\}$, and this may or may not remove linearly dependent vectors, so $\beta \subseteq U \cup W$.*

2B.7.

Because $v_1, \dots, v_4$ is a basis of V, it must be that $\operatorname{span}(v_1, \dots, v_4) = V$. Let $v' \in \operatorname{span}(v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4)$. Then there exist scalars $b_i \in \mathbb{F}$ such that

$$\begin{aligned} v' &= b_1(v_1 + v_2) + b_2(v_2 + v_3) + b_3(v_3 + v_4) + b_4 v_4 \\ &= b_1 v_1 + (b_1 + b_2)v_2 + (b_2 + b_3)v_3 + (b_3 + b_4)v_4 \\ &\in \operatorname{span}(v_1, \dots, v_4). \end{aligned}$$

This shows that $\operatorname{span}(v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4) \subseteq \operatorname{span}(v_1, \dots, v_4)$. For the reverse inclusion, let $v \in \operatorname{span}(v_1, \dots, v_4)$, so that there exist scalars $a_i \in \mathbb{F}$. Then,

$$\begin{aligned} v &= a_1 v_1 + \cdots + a_4 v_4 \\ &= a_1(v_1 + v_2) + (a_2 - a_1)(v_2 + v_3) + (a_3 - a_2 + a_1)(v_3 + v_4) + (a_4 - a_3 + a_2 - a_1)v_4 \\ &\in \operatorname{span}(v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4). \end{aligned}$$

Hence, $\operatorname{span}(v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4) = \operatorname{span}(v_1, \dots, v_4) = V$.

Setting $v = 0$ in $(*)$, linear independence of $v_1, \dots, v_4$ implies that the only solution in scalars $a_i \in \mathbb{F}$ is with all the $a_i = 0$. This implies that all the scalar coefficients of the vectors in the second equality must be 0.

Therefore, $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ is also linearly independent, and so forms a basis of V.

Corrections. All good.

2B.10.

Key idea:

Assume that U, W are subspaces of V such that $V = U \oplus W$. Define the basis of U as $\beta_U = \{u_1, \dots, u_m\}$ and $\beta_W = \{w_1, \dots, w_n\}$ are bases of U and W respectively.

Define the following,

$$\beta_U \cup \beta_W = \{u_1, \dots, u_m, w_1, \dots, w_n\}.$$

We first show that $\operatorname{span}(\beta_U \cup \beta_W) = V$. Let $v \in V$. Because $V = U + W$, we can write

$$v = u + w, \quad u \in U, w \in W.$$

Expressing u and w as a linear combination of the basis vectors β_U and β_W respectively, we have that there exist scalars $a_i, b_i \in \mathbb{F}$ such that,

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 w_1 + \cdots + b_n w_n \in \operatorname{span}(\beta_U \cup \beta_W).$$

This shows that $V \subseteq \operatorname{span}(\beta_U \cup \beta_W)$.

To show the reverse inclusion, let $v' \in \operatorname{span}(\beta_U \cup \beta_W)$. Then expressing v' as a linear combination of the vectors in $\beta_U \cup \beta_W$, we have,

$$\begin{aligned}v' &= a_1u_1 + \cdots + a_mu_m + b_1w_1 + \cdots + b_nw_n \\ &= u' + w'.\end{aligned}$$

Because β_U and β_W are bases of U and V respectively, we have that $\text{span}(u_1, \ldots, u_m) = U$ and $\text{span}(w_1, \ldots, w_n) = W$, which implies that $u' \in U$ and $w' \in W$. Because we have written v' as a sum $u' + w'$, this implies that $v' \in V$.

This shows that $\text{span}(\beta_U \cup \beta_W) \subseteq V$, and so $\text{span}(\beta_U \cup \beta_W) = V$.

Assume, for contradiction, that $\beta_U \cup \beta_W$ is a linearly dependent list.

Let $v \in \text{span}(\beta_U \cup \beta_W)$. Consider the following,

$$a_1u_1 + \cdots + a_mu_m + b_1w_1 + \cdots + b_nw_n = 0. \tag{*}$$

Because $\beta_U \cup \beta_W$ is linearly dependent, this implies that there exist scalars $a_i, b_i \in \mathbb{F}$, not all zero, such that $(*)$ holds true.

Because $V = U \oplus W$, this must mean that the only way to write $0 \in V$ as a sum of subspace elements $u + w$, with $u \in U$ and $w \in W$, is by taking $u = 0$ and $w = 0$. But because β_U and β_W are bases of U and W respectively, this implies that all the scalars $a_i, b_i \in \mathbb{F}$ in $(*)$ must be zero.

This implies that $0 \in V$ has two representations, rather than a unique representation. This contradicts our assumption that $V = U \oplus W$. Hence we conclude that $\beta_U \cup \beta_W$ must be linearly independent, and so $\beta_U \cup \beta_W$ is a basis of V.

Corrections: Proof is logically sound, but verbose. Note following,

- *You can remove the section that* $\text{span}(\beta_U \cup \beta_W) \subseteq V$ *entirely. Instead, you can just say that vector space closure properties imply that* $\beta_U \cup \beta_W \subseteq V \implies \text{span}(\beta_U \cup \beta_W) \subseteq V$.
- *From the SM, it is much cleaner and faster to rely on "uniqueness of representation". First note that because* $V = U \oplus W$, *every* v *can be uniquely expressed as* $v = u + w$, *and then use the uniqueness of representation implied by the individual bases* β_U *and* β_W *to note that* v *can be uniquely represented as the following, implying that the vectors are a basis of* V.

$$v = a_1u_1 + \cdots + a_mu_m + b_1w_1 + \cdots + b_nw_n.$$

Exercises 2C Dimension.

For these problems, especially **2C.4. - 2C.7.**, the following genreal example here on shifted bases served as a useful source of truth.

A polynomial $p \in \mathcal{P}_n(\mathbb{F})$ is a function $p : \mathbb{F} \to \mathbb{F}$ such that there exist $a_0 \ldots a_n \in \mathbb{F}$, with

$$p(z) = a_0 + a_1z + \cdots + a_nz^n.$$

Whilst p may be viewed as a vector in the vector space of polynomials $\mathcal{P}_n(\mathbb{F})$ without ever defining a basis, we can also view the above as specified in the standard basis $\{1, z, \ldots, z^n\}$.

Consider the following *shifted basis* (by constant c), $\{1, (z-c), \ldots, (z-c)^n\}$. This is a basis as the list is linearly independent, and it is of length $(n+1)$, hence spans $\mathcal{P}_n(\mathbb{F})$. If we wish to represent an arbitrary polynomial q in this new basis, then we have for a choice of $b_0, \ldots, b_n \in \mathbb{F}$,

$$q(z) = b_0 + b_1(z-c) + \ldots b_n(z-c)^n.$$

Algebraically, we can ask, given p, what values of $b_0 \ldots, b_n$ make the polynomials p and q equal, i.e. for $p(z) = q(z)$ for all $z \in \mathbb{F}$. In linear algebra terms, this amounts to requesting the coordinate vector of p when it is expressed in the shifted basis β'.

A cumbersome way of getting to the solution might be to expand $q(z)$ by doing binomial expansions, collecting terms in z^j, then equating coefficients of p and q, then solving for the scalars b_i.

A cleaner way to do this that avoids binomial expansions is to note that two identical polynomials have identical kth derivatives,

$$p(z) = q(z)\,\forall z \in \mathbb{F} \iff p^{(k)}(z) = q^{(k)}(z)\,\forall z \in \mathbb{F}.$$

This arises because polynomials are infinitely differentiable, i.e. belong to the class smooth of functions C^∞, and because a polynomial of degree n is uniquely determined by its value and first n derivatives at any single point $z \in \mathbb{F}$.

Now considering the kth derivative of $q(z)$, we have,

$$q^{(k)}(z) = \frac{d^k}{dz^k}\left[b_0 + b_1(z-c) + \ldots b_n(z-c)^n\right].$$

Noting that,

$$\frac{d^k}{dz^k} b_j(z-c)^j = \begin{cases} 0 & \text{when } j < k \\ k! \cdot b_k & \text{when } j = k \\ \frac{j!}{(j-k)!} \cdot b_j(z-c)^{j-k} & \text{when } j > k \end{cases}$$

So we have,

$$q^{(k)}(z) = k! \cdot b_k + \sum_{j>k} \frac{j!}{(j-k)!} \cdot b_j(z-c)^{j-k}.$$

We can exploit this and isolate the coefficients b_k by considering $p^{(k)}(z) = q^{(k)}(z)$ at $z = c$, and so we have for all $k \in \{0, \ldots, n\}$,

$$p^{(k)}(c) = q^{(k)}(c) \implies b_k = \frac{p^{(k)}(c)}{k!}.$$

2C.1.

Key idea: Dimension is a measure of the size of vector spaces in linear algebra in an analogous way to how cardinality is a measure of the size of sets in set theory.

To strengthen this further, the dimension of a vector space is the cardinality of its basis. In set theory, two sets are of the same cardinality if we can find a bijection between them. In later chapters, you will find that in linear algebra, two vector spaces (over the same field) have the same dimension i.e. are isomorphic, if an isomorphism exists between them i.e. a linear bijection.

An isomorphism is a structure-preserving map, and specifically in linear algebra, the vector space structure that is preserved are the operations of addition and scalar multiplication. This means that from a linear algebra perspective, two isomorphic vector spaces behave identically, abstracting away any differences not captured by the header of vector space structure. Phrased differently, we cannot distinguish between two isomorphic vector spaces using only tools from linear algebra.

This means we can study vector spaces, such as polynomial spaces of degree $n-1$ (which has dimension n), by instead studying $\mathbb{R}^n$, and because they both are linear structures, we can use the same algebraic and geometric lenses.

Dimension allows us to classify vector spaces (and therefore subspaces) up to isomorphism, and this is because it is a vector space *invariant* (up to isomorphism). That is, no matter how you transform the space under an isomorphism i.e. a linear bijective map, the number of basis vectors needed to describe both spaces remains the same.

Coordinate free linear algebra is the study of transformations that are intrinsic to the space, rather than artefacts of particular coordinate systems. Dimension is also invariant under change of basis. Invariants under change of basis are part of what allows coordinate free linear algebra to work. Other linear map/linear operator invariants under change of basis/similarity include the *determinant*, *trace*, *eigenvalues*, *eigenvectors*, *rank*, *characteristic polynomial*, *spectrum* and *kernel*.

If U is a subspace of V, then we know from (2.37) that $\dim(U) \leq \dim(V)$. So if U is a subspace of $\mathbb{R}^2$, then $\dim(U) \leq 2$, with equality when $U = \mathbb{R}^2$. As the dimension of any vector space is defined to be the length of the list of the basis vectors needed to represent it, it is a non-negative integer. So $\dim(U) \in \{0, 1, 2\}$.

Classifying each subspace of $\mathbb{R}^2$ according to its dimension, we have that,

$$\begin{aligned}
\dim(U) = 0 &\implies U = \{0\} \\
\dim(U) = 1 &\implies U = \operatorname{span}(v_1) = \left\{u \in \mathbb{R}^2 \mid u = a_1 v_1\right\} \\
\dim(U) = 2 &\implies = \operatorname{span}(v_1, v_2) = \left\{u \in \mathbb{R}^2 \mid u = a_1 v_1 + a_2 v_2\right\}.
\end{aligned}$$

This shows that all subspaces of $\mathbb{R}^2$ are $\{0\}$, the set of lines going through the origin, and $\mathbb{R}^2$ itself.

Corrections: Small detail you need to mention for U with $\dim U = 1$ is that it is a line through the origin with direction vector $v_1 \neq 0$. Can also make the link between dimension and existence of $\dim U$ number of basis vectors clearer.

2C.2

Following **2C.2.**, we know that $\dim(U) \in \{0, 1, 2, 3\}$, and we can classify each subspace of $\mathbb{R}^3$ according to its dimension,

$$\dim(U) = 0 \implies U = \{0\}$$
$$\dim(U) = 1 \implies U = \text{span}(v_1) = \left\{u \in \mathbb{R}^3 \mid u = a_1 v_1\right\}$$
$$\dim(U) = 2 \implies = \text{span}(v_1, v_2) = \left\{u \in \mathbb{R}^3 \mid u = a_1 v_1 + a_2 v_2\right\}.$$
$$\dim(U) = 3 \implies = \text{span}(v_1, v_2, v_3) = \left\{u \in \mathbb{R}^3 \mid u = a_1 v_1 + a_2 v_2 + a_3 v_3\right\}.$$

This shows that all subspaces of $\mathbb{R}^3$ are $\{0\}$, the set of all lines going through the origin, the set of all planes through the origin, and $\mathbb{R}^3$ itself.

Corrections: Same omission as before. For this one, for the $\dim U = 2$ *case, note that linear independence of* v_1 *and* v_2 *implies that neither is a scalar multiple the other. Geometrically, they are not collinear, and hence span a plane through the origin in* $\mathbb{R}^2$.

2C.3a.

This answer is slightly longer, as it gives a template to deal with problems **2C.4 - 2C.7**.

Define the following subset U of $\mathcal{P}_4(\mathbb{F})$,

$$U = \{p \in \mathcal{P}_4(\mathbb{F}) \mid p(6) = 0\}.$$

U is a subspace. The additive identity $0_{\mathcal{P}_4(\mathbb{F})} \in U$ because $p(6) = 0$ when $p = 0$. And also for $p, q \in U$, we have $(p+q)(6) = p(6) + q(6) = 0$ and $(\lambda p)(6) = \lambda p(6) = 0$ for all $\lambda \in \mathbb{F}$, showing closure under addition and scalar multiplication respectively.

Construction of a basis of U contains a problematic circularity that links dimension and number of basis vectors. To construct a basis, we need to know how many basis vectors comprise the basis, i.e. the dimension of U. But to establish the dimension of U, we first need to construct a basis of U.

Rather than guess a basis, establish that it is a basis with dimensions checks, as in example (2.41), we use some extra results not explicitly established in the textbook to help guess the basis.

By definition, $p \in \mathcal{P}_4(\mathbb{F})$ implies that there exist $a_0, \ldots, a_4 \in \mathbb{F}$ such that $p(z) = a_0 + \cdots + a_4 z^4$. If $p \in U$, then $p(6) = 0$ implies that, $a_0 + 6a_1 + \cdots + 1296a_4 = 0$, which is a linearly independent constraint. Hence there are four degrees of freedom in U, measured by the number of freely varying coefficients a_i and so $\dim(U) = 4$.

We know that $p \in U$ implies that $p(6) = 0$, then $(z-6)$ is a factor of $p(z)$ by the factor theorem. This suggests the following basis of polynomial functions, which do not include the constant polynomial,

$$\beta_U = \{(z-6), (z-6)^2, (z-6)^3, (z-6)^4\}.$$

This implies that the polynomial $p \in U$ can be represented via scalars $b_i \in \mathbb{F}$ as

$$p(z) = b_0(z-6) + b_1(z-6)^2 + b_2(z-6)^3 + b_3(z-6)^4.$$

To check linear independence of β_U, we consider the following system in $b_i \in \mathbb{F}$,

$$b_0(z-6)+b_1(z-6)^2+b_2(z-6)^3+b_3(z-6)^4=0.$$

Following the arguments in (2.41), we have that $b_3=0$ because there are no z^4 terms on RHS; and that $b_2=0$ because all z^3 terms in $(z-6)^4$ vanish as $b_3=0$ and because there are no z^3 terms on RHS; etc., and so $b_3=\dots b_0=0$, hence β_U is linearly independent.

Because β_U is a linearly independent list of length 4 of length $\dim(U)=4$, by (2.38), β_U is a basis.

Corrections: All good. SM gets around the circularity you mentioned as follows. Propose a basis of U (implicitly using factor theorem), then show that it is linearly independent, showing that $\dim U \geq 4$. Then show that $\dim U < 5$ using a combination of $\dim U \leq \dim V$ and the fact that U does not contain constant polynomials. Then use a "squeeze argument" to establish that $\dim U = 4$, and so the proposed basis is indeed a basis.

2C.3b.

To extend the basis β_U, we append the standard basis of $\mathcal{P}_4(\mathbb{F})$, $\beta=\{1,z,\dots,z^4\}$ to form $\beta_U\cup\beta$ and then run the spanning list reduction algorithm (2.32),

$$\beta_U\cup\beta=\{(z-6),\dots,(z-6)^4,1,z,\dots,z^4\}.$$

Skipping steps 1 - 4, as β_U is a basis, we have that $1\notin\operatorname{span}((z-6),\dots,(z-6)^4)$, as it is a constant polynomial. Hence we leave this in the list, and because we have a linearly independent list of length 5, which is the same as $\dim(\mathcal{P}_4(\mathbb{F}))$, by (2.38), we can delete the remaining $z,\dots,z^4$.

Hence we have the following basis β' of $\mathcal{P}_4(\mathbb{F})$,

$$\beta'=\{1,(z-6),\dots,(z-6)^4\}.$$

2C.3c.

Using the result established in **2B.3.** connecting bases of subspaces and direct sums, if β_U is a basis of U and β' is a basis of $\mathcal{P}_4(\mathbb{F})$, then $\beta'\backslash\beta_U=\{1\}$ will be the basis of a subspace W such that $U\oplus W=\mathcal{P}_4(\mathbb{F})$,

$$W=\operatorname{span}(1)=\{p\in\mathcal{P}_4(\mathbb{F})\mid p(z)=c, c\in\mathbb{F}\}.$$

That is, W is the subspace of all constant polynomials in $\mathcal{P}_4(\mathbb{F})$.

2C.4a.

Define the following subset U of $\mathcal{P}_4(\mathbb{F})$,

$$U=\{p\in\mathcal{P}_4(\mathbb{F})\mid p''(6)=0\}.$$

Then U is a subspace. The additive identity $0_{\mathcal{P}_4(\mathbb{F})}\in U$ because $p=0$ implies that $p''(6)=0$. And also for $p,q\in U$, we have $(p''+q'')(6)=p''(6)+q''(6)=0$ and $(\lambda p'')(6)=\lambda p''(6)=0$ for all $\lambda\in\mathbb{F}$, showing closure under addition and scalar multiplication respectively.

For the dimension of U, then if $p \in \mathcal{P}_4(\mathbb{F})$ is described by $p(z) = a_0 + \ldots a_4 z^4$, then

$$p''(6) = 0 \implies a_2 + 36a_3 + 2592a_4 = 0.$$

Because this is one linearly independent constraint on the a_i defining $p(z)$, or because there are four "freely varying" a_is, then $\dim(U) = 4$.

To get a basis of U, first note that if $p \in \mathcal{P}_4(\mathbb{F})$, then $\deg p'' = 2$. If $p \in U$, then $p''(6) = 0$, and so by the factor theorem, $(z-6)$ is a factor of $p''(z)$. Both observations imply that $p''(z)$ takes the following form,

$$p''(z) = b_2 + b_3(z-6) + b_4(z-6)^2,$$

where $b_2 = 0$, which suggests that $(z-6)^2$ should not be included in our basis. The functional form of $p''(z)$ also suggests that we should include $(z-6)^3$ and $(z-6)^4$ in basis. And also that degree 0 and 1 basis vectors can be arbitrarily chosen, as long as the entire list is linearly independent.

This suggests the following basis, which is linearly independent by the incremental degree argument in **2B.3a**, and which spans U, because it is a linearly independent length of length $\dim(U) = 4$.

$$\beta_U = \{1, (z-6), (z-6)^3, (z-6)^4\}.$$

*Corrections: All good, except an arithmetic error,

$$p''(6) = 0 \implies 2a_2 + 36a_3 + 432a_4 = 0.$$

2B.4b.

Using a similar argument to **2B.3b**, we note that $z^2 \notin \operatorname{span}(1, (z-6))$, and so this suggests the following basis β' of $\mathcal{P}_4(\mathbb{F})$,

$$\beta' = \{1, (z-6), z^2, (z-6)^3, (z-6)^4\}.$$

Which is a basis because the vectors in β' are linearly independent, and because β' has length $\dim(\mathcal{P}_4(\mathbb{F})) = 5$, and so must span $\mathcal{P}_4(\mathbb{F})$.

2C.4c.

Using the result established in **2B.3**,

$$W = \operatorname{span}(z^2).$$

2C.5a.

Define the subset U of $\mathcal{P}_4(\mathbb{F})$ as,

$$U = \{p \in \mathcal{P}_4(\mathbb{F}) \mid p(5) = p(2)\}.$$

Then U is a subspace. The additive identity $0_{\mathcal{P}_4(\mathbb{F})} \in U$ because $p = 0$ implies that $p(5) = p(2) = 0$. And also for $p, q \in U$, we have $(p+q)(5) = p(5) + q(5) = p(2) + q(2) = (p+q)(2)$, and $(\lambda p)(5) = \lambda p(5) = 0 = \lambda p(2) = (\lambda p)(2)$ for all $\lambda \in \mathbb{F}$, showing closure under addition and scalar multiplication respectively.

For the dimension of U, the constraint $p(5) = p(2)$ amounts to one linear constraint on the coefficients of p, and so $\dim(U) = 4$.

To get a basis of U, first note that the constant polynomial $p(z) = 1$ is also in U, hence can be our first basis vector. And if $p(5) = p(2)$, then this is satisfied when $p(5) = p(2) = 0$, which suggests that we can use $(z-2)(z-5)$ as our second basis vector. We can multiply the second basis vector by z and z^2, to yield the following basis of U,

$$\beta_U = \{1, (z-2)(z-5), z(z-2)(z-5), z^2(z-2)(z-5)\}.$$

Which is linearly independent by the incremental degree argument in **2B.3a**, and which spans U, because it is a linearly independent length of length $\dim(U) = 4$.

2C.5b.

Comparing β_U with the standard basis β of $\mathcal{P}_4(\mathbb{F})$, β_U is missing a degree 1 basis vector, which suggests the following basis of $\mathcal{P}_4(\mathbb{F})$,

$$\beta' = \{1, z, (z-2)(z-5), z(z-2)(z-5), z^2(z-2)(z-5)\}.$$

Which is the same result as if we extended the linearly independent list β_U with the standard basis, also a spanning list, of $\mathcal{P}_4(\mathbb{F})$, and then pruned this new list to remove linearly dependent vectors with (2.32).

Corrections: All good.

2C.5c.

Using the result established in **2B.3**,

$$W = \operatorname{span}(z).$$

2C.6a.

Define the subset U of $\mathcal{P}_4(\mathbb{F})$,

$$U = \{p \in \mathcal{P}_4(\mathbb{F}) \mid p(5) = p(2) = p(6)\}.$$

Then U is a subspace. The additive identity $0_{\mathcal{P}_4(\mathbb{F})} \in U$ because $p = 0$ implies that $p(5) = p(2) = p(6) = 0$. For closure under addition, for $p, q \in U$, we have,

$$p(5) + q(5) = p(2) + q(2) = p(6) + q(6),$$

and so this implies that $(p+q)(5) = (p+q)(2) = (p+q)(6)$. For closure under scalar multiplication, if $p \in U$, then we have,

$$\lambda p(5) = \lambda p(2) = \lambda p(6),$$

which implies that $(\lambda p)(5) = (\lambda p)(6) = (\lambda p)(2)$ for all $\lambda \in \mathbb{F}$.

For the dimension of U, the constraint $p(5) = p(2) = p(6)$ amounts to one linear constraint on the coefficients of p, and so $\dim(U) = 4$.

To get a basis of U, first note that the constant polynomial $p(z) = 1$ satisfies $p(5) = p(2) = p(6) = 0$, hence is in U, to form our first basis vector. Then noting that $p(5) = p(2) = p(6)$ is satisfied when $p(5) = p(2) = p(6) = 0$, we use $(z-2)(z-5)(z-6)$ as our second basis vector. Multiplying the second basis vector by z and $1/z$ yields the basis,

$$\beta_U = \left\{1, \frac{1}{z}(z-2)(z-5)(z-6), (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)\right\}.$$

Which is linearly independent by the incremental degree argument in **2B.3a**, and which spans U, because it is a linearly independent length of length $\dim(U) = 4$.

Corrections: Some serious errors here. They are,

- *The constraint* $p(5) = p(2) = p(6) = 0$ *amounts to two linear constraints, not one linear constraint,*
$$p(5) - p(2) = 0, p(6) - p(2) = 0.$$
- *If it's not clear, expand the polynomials.*
- *This implies that* $\dim U = 3$.
- *The basis vector* $\frac{1}{z}(z-2)(z-5)(z-6)$ *is not a polynomial, as if you expand, you get a term* $\frac{-60}{z}$.
- Instead you could use the basis...

2C.6b.

Comparing β_U with the standard basis β of $\mathcal{P}_4(\mathbb{F})$, β_U is missing a degree 1 basis vector, which suggests the following basis of $\mathcal{P}_4(\mathbb{F})$,

$$\beta' = \left\{1, z, \frac{1}{z}(z-2)(z-5)(z-6), (z-2)(z-5)(z-6), z(z-2)(z-5)(z-6)\right\}.$$

Which is the same result as if we extended the linearly independent list β_U with the standard basis, also a spanning list, of $\mathcal{P}_4(\mathbb{F})$, and then pruned this new list to remove linearly dependent vectors with (2.32).

2C.6c.

Using the result established in **2B.3**,

$$W = \operatorname{span}(z).$$

2C.7a.

Define the subset U of $\mathcal{P}_4(\mathbb{F})$,

$$U = \left\{ p \in \mathcal{P}_4(\mathbb{F}) \mid \int_{-1}^{1} p = 0 \right\}.$$

Then U is a subspace. The additive identity $0_{\mathcal{P}_4(\mathbb{F})} \in U$ because $p = 0$ implies that $\int_{-1}^{1} p = 0$. And also for $p, q \in U$, we have via linearity of definite integration, $\int_{-1}^{-1}(p + q) = \int_{-1}^{-1} p + \int_{-1}^{-1} q = 0$, and $\int_{-1}^{1}(\lambda p) = \lambda \int_{-1}^{1} p = 0$ for all $\lambda \in \mathbb{F}$, showing closure under addition and scalar multiplication respectively.

For the dimension of U, if $p \in \mathcal{P}_4(\mathbb{F})$, and $p(z) = a_0 + \ldots a_4 z^4$, and $p \in U$, then

$$\int_{-1}^{1} p = 0 \implies 2a_0 + \frac{2}{5}a_2 + \frac{2}{5}a_4 = 0,$$

which is one linearly independent constraint on the coefficients a_i, and hence $\dim(U) = 4$.

To get a basis of U, note that polynomial functions that are odd functions, i.e. p satisfying $p(-z) = -p(z)$ for all $z \in \mathbb{F}$, will be in U. As polynomial functions that are odd functions also have odd powers, this suggests using z and z^3 as our first two basis vectors.

For our other two basis vectors, note that affine transformations of x and x^3 either will generate a new linearly dependent vector, or yield a vector that is not in U. So we use odd cubic polynomials that are centred about the origin, yielding,

$$\beta_U = \{x, x^3, \}.$$

2C.9.

Let $p_0, \ldots p_m$ be polynomials in $\mathcal{P}_m(\mathbb{F})$, where each polynomial p_k is of degree k. Consider,

$$a_0 p_0(z) + a_1 p_1(z) + \ldots a_m p_m(z) = 0.$$

Now using a similar incremental degree argument as in (2.41), consider the right hand side. Because there is no z^m term on the RHS, and the polynomials $p_0, \ldots p_{m-1}$ contain no z^m terms, as they have maximum degree $m - 1$, this must mean that $a_m = 0$.

Similarly, there are no z^{m-1} terms on the RHS, and the only source of z^{m-1} terms comes from p_{m-1}. Which is because all polynomials $p_0, \ldots p_{m-2}$, having maximum degree $m - 2$, contain no x^{m-1} terms, and because all x^{m-1} terms in p_m vanish as $a_m = 0$. Hence $a_{m-1} = 0$.

Continuing this argument until z^0 shows that $a_0 = \ldots a_m = 0$, and so $p_0, \ldots, p_m$ is linearly independent.

Now because $p_0, \ldots p_m$ is a linearly independent list of length $\dim \mathcal{P}_m(\mathbb{F}) = m + 1$, it must be a basis of $\mathcal{P}_m(\mathbb{F})$.

Corrections: All good. Minor quibble is you could more be more precise and write, "since the coefficient of z^{m-1} in p_{m-1} is non-zero (by definition of degree), a_{m-1} must be zero."

2C.12.

Using the dimension of a sum of a subspaces (2.27), it must be the case that $\dim(U \cap W) = \dim(U + W) - \dim(U) - \dim(W)$. Because $U + W = \mathbb{R}^8$, we know that $\dim(U + W) = 8$, and as $\dim(U) = 3$ and $\dim(W) = 5$, this implies that $\dim(U \cap W) = 0$, and so $U \cap W = \{0\}$. Hence R^8 is a direct sum of U and W.

2C.13.

The question asks us to prove that if U and W are subspaces of $\mathbb{R}^9$, with $\dim(U) = \dim(W) = 5$, then $\mathbb{R}^9$ cannot be decomposed as a direct sum of U and W, i.e. the sum of subspaces $U + W$ do not 'cleanly jigsaw' R^9 without overlap. That is, without creating spaces where a single vector can have more than one representation.

Using (2.43) on dimension of sums of subspaces,

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W) = 10 - \dim(U \cap W).$$

HINT: Can you upper bound $\dim(U + W)$?

As the sum of subspaces $U+W$ is itself a subspace of $\mathbb{R}^9$, then we have by (2.37) that $\dim(U+W) \leq \dim(\mathbb{R}^9)$.

So we have,

$$\begin{aligned} \dim(\mathbb{R}^9) &\geq \dim(U + W) \\ &= \dim(U) + \dim(W) - \dim(U \cap W), \end{aligned}$$

which implies that,

$$\dim(U \cap W) \geq \dim(U) + \dim(W) - \dim(\mathbb{R}^9).$$

Hence $\dim(U \cap W) \geq 1$, which implies that $U \cap W \neq \{0\}$.

2C.14.

HINT: Even though you haven't established a formula for $\dim(V_1 + V_2 + V_3)$, it doesn't mean you can't extract the maximum amount of information from (2.37) for sums of two subspaces by setting one of the subspaces appropriately.

Proof strategy: Lower bound the quantity of interest $\dim(V_1 \cap V_2 \cap V_3)$. Then lower bound the intermediate quantities making up the previous lower bound. Scrutinising exactly how this works is important for the next problem.

Using the result that intersections of arbitrary collections of subspaces of V are themselves subspaces, result (2.37) that subspace dimensions are upper bounded by $\dim(V)$, and result (2.43) for the dimension of a sum of two subspaces, we have,

$$\begin{aligned} \dim(V) &\geq \dim((V_1 \cap V_2) + V_3) \\ &= \dim(V_1 \cap V_2) + \dim(V_3) + \dim(V_1 \cap V_2 \cap V_3). \end{aligned}$$

This implies the following lower bound on $\dim(V_1 \cap V_2 \cap V_3)$,

$$\dim(V_1 \cap V_2 \cap V_3) \geq \dim(V_1 \cap V_2) + \dim(V_3) - \dim(V)$$

We can lower bound $\dim(V_1 \cap V_2)$ using result (2.37) and (2.43) as follows,

$$\begin{aligned} \dim(V) &\geq \dim(V_1 + V_2) \\ &= \dim(V_2) + \dim(V_3) - \dim(V_1 \cap V_2), \end{aligned}$$

which implies that $\dim(V_1 \cap V_2) \geq \dim(V_2) + \dim(V_3) - \dim(V) = 4$.

And so we have,

$$\begin{aligned} \dim(V_1 \cap V_2 \cap V_3) &\geq \dim(V_1 \cap V_2) + \dim(V_3) - \dim(V) \\ &\geq 4 + \dim(V_3) - \dim(V) \\ &= 4 + 7 - 10 = 1. \end{aligned}$$

Now because $\dim(V_1 \cap V_2 \cap V_3) \geq 1$, we must have that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

2C.15.

Using similar reasoning to **2C.14**, we have,

$$\begin{aligned} \dim(V) &\geq \dim((V_i \cap V_j) + V_k) \\ &= \dim(V_i \cap V_j) + \dim(V_k) - \dim(V_i \cap V_j \cap V_k). \end{aligned}$$

As there are $\binom{3}{2}$ ways of selecting two indices from an index set $\{1, 2, 3\}$ to form the pairwise intersection $(V_i \cap V_j)$, the above generates three inequalities, which we can add together and rearrange for a lower bound on $\dim(V_1 \cap V_2 \cap V_3)$ to yield,

$$\begin{aligned} 3\dim(V_1 \cap V_2 \cap V_3) \geq\ &\dim(V_1) + \dim(V_2) + \dim(V_3) \\ &+ \dim(V_1 \cap V_2) + \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3) \\ &- 3\dim(V). \end{aligned}$$

Using similar reasoning to **2C.14**, we have the following lower bound on the pairwise intersections of subspaces,

$$\dim(V_i \cap V_j) \geq \dim(V_i) + \dim(V_j) - \dim(V).$$

Again, this generates three lower bounds, which we can substitute into the lower bound on $\dim(V_1 \cap V_2 \cap V_3)$, and so we have,

$$\begin{aligned} 3\dim(V_1 \cap V_2 \cap V_3) \geq\ &\dim(V_1) + \dim(V_2) + \dim(V_3) \\ &+ 2[\dim(V_1) + \dim(V_2) + \dim(V_3)] - 3\dim V \\ &- 3\dim(V) \\ =\ &3[\dim(V_1) + \dim(V_2) + \dim(V_3)] - 6\dim V \end{aligned}$$

Now using the strict lower bound $\dim(V_1) + \dim(V_2) + \dim(V_3) > 2\dim(V)$, we have,

$$\begin{aligned}
3\dim(V_1 \cap V_2 \cap V_3) &\geq 3[\dim(V_1) + \dim(V_2) + \dim(V_3)] - 6\dim V \\
&> 6\dim V - 6\dim V \\
&= 0.
\end{aligned}$$

Hence $\dim(V_1 \cap V_2 \cap V_3) > 0$, and so $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

Corrections: All good and well done. Nice little refresher on how to deal with inequalities.

2C.18

Key idea: The entire basis(!) of our ability to write any vector $v \in V$ uniquely as a linear combination of n basis vectors $v_1, \dots, v_n$ comes from a structural decomposition of the vector space V into a direct sum of 1-dimensional subspaces V_i,

$$V = V_1 \oplus \cdots \oplus V_n.$$

Geometrically, this corresponds splitting V into n distinct lines through the origin.

Property of a Basis	Property of a Direct Sum
Spanning: Every vector v is a sum of scaled basis vectors: $v = \sum a_i v_i$.	Summation: The total space is the sum of the subspaces: $V = V_1 + \cdots + V_n$.
Linear Independence: The only way to represent the zero vector is the trivial one ($a_i = 0$).	Disjointness: The subspaces are essentially "independent"; they only overlap at the origin ($\vec{0}$).
Uniqueness: Each vector v has exactly one representation as a linear combination.	Directness ($\oplus$): Each vector v can be written as a sum of subspace elements in exactly one way.

The structural decomposition into 1-d subspaces is more "fundamental" in the following sense. Any basis $v_1, \dots, v_n$ defines n one-dimensional subspaces $V_i = \text{span}(v_i)$. Conversely, the structural decomposition of V into n one-dimensional subspaces generates a whole family of bases - any set of nonzero vectors $v_1, \dots, v_n$ where each $v_i \in V_i$ is automatically linear independent and spans V.

As $\dim(V) = n \geq 1$, let $v_1, \dots, v_n$ be a basis of V. Now if v is some arbitrary vector in V, then we can express it as a linear combination of these basis vectors,

$$v = a_1 v_1 + \cdots + a_n v_n.$$

If we define one-dimensional subspaces $V_k = \text{span}(v_k)$ and let $u_k = a_k v_k$ then

$$v = u_1 + \cdots + u_n \in V_1 + \cdots + V_n.$$

And so $V \subseteq V_1 + \cdots + V_n$. Noting that $V_1 + \cdots + V_n$ is a subspace of V shows the reverse direction that $V_1 + \cdots + V_n \subseteq V$, and so $V = V_1 + \cdots + V_n$.

Now letting $v = 0$, we have,

$$a_1v_1 + \cdots + a_nv_n = u_1 + \cdots + u_m = 0.$$

By definition of $v_1, \ldots, v_n$ as a basis, which is linearly independent, the only way of writing 0 is with all scalars $a_i = 0$. Which implies that the only way to write 0 as a sum $u_1 + \cdots + u_m$ of subspace elements $u_i \in V_k$ is by setting all the $u_i = 0$. Hence V is a direct sum of one dimensional subspaces V_i.