Linear Algebra Done Right (4th ed.) by Sheldon Axler.

Write-up for exercises in *Chapter 1: Vector Spaces.*

Date commenced: 20th November 2025.

Date updated: 21st November 2025.

The following are my self-study write ups for the exercises in the above textbook. The purpose of writing up my attempts at the exercises from pen and paper is to practise writing up and presenting formal proofs.

The purpose of these write-ups is so that I can:

- Practice writing up the exercises and formal mathematical proofs I completed in my scrapbook by committing my answers to the public domain.
- In doing so, learn more about logical proof structure, and to improve mathematical communication.
- Document any exercises that I attempted and struggled with, and for which I needed assistance to proceed from the instructor's solutions manual.
- In doing so, reveal any patterns of deficiencies in my own thinking when assimilating this material, which may be useful when studying fields that build on these concepts.

I have organised the exercises into problem sets corresponding to each section in the chapter. For exercises that revealed a point of broader significance after correction, I have listed this as a **Key idea** in bold. For proofs that needed more thought, I have listed the proof strategy, or a one line description of the high-level idea at play.

In cases where I got stuck with a problem, or needed instantaneous feedback on the right direction to proceed, I would use a hint from the solutions manual. This is indicated as **HINT** in bold the write-up, and will be useful as a marker on where I got stuck. These markers serve as indicators of where it is that I learnt something new. This is particularly important for later problems which use results established in previous questions.

Corrections made using the solutions manual and Googling are listed as *Corrections* in italics. Only problems which illustrated significant conceptual difficulties for me are corrections included.

Exercises 1B Vector Spaces.

Here is an example (which was my first attempt to **1B.1**), of what is not appropriate justification in these exercises:

$$-(-v) = -1(-1(v)) = v.$$

René Descartes explaining his work to Queen Christina of Sweden. Vector spaces are a generalization of the description of a plane using two coordinates, as published by Descartes in 1637.

Figure 1: image

For all of the questions here, we can only assume the axioms of vector spaces, and anything else we've proved in the chapter. And we need to be sure that every symbolic manipulation is rigorously justified with reference to these axioms and properties.

Pedantic and seemingly pointless? Here, we are being given a formal system of axioms to manipulate objects which we already have developed intuition for at secondary school. So the rigorous formality to justify these intuitions may seem like a step backwards, particularly if it feels abstract and difficult.

But the importance of this is that if we ever wish to go on to work with more complicated, or even novel mathematical structures, for which we have no priori intuitions to lean on, and no idea of their properties, then the rigorous formality becomes our *only* guiding light to develop those intuitions, one which needs to be trained in an easier domain!

1B.1.

In words, we want to prove that the additive inverse of $(-v)$ is v.

The additive inverse axiom in (1.20) states that for every $v \in V$, there exists $w \in V$ such that $v + w = 0$. And in (1.27), we establish that the additive inverse w is unique. So we have that for every vector $v \in V$, there will exist a unique additive inverse $(-v)$ such that,

$$v + (-v) = 0. \tag{*}$$

Because $(-v) \in V$, then it must also be the case that,

$$(-v) + (-(-v)) = 0. \tag{**}$$

Adding the additive inverse of $(-v)$, that is, v, to both sides yields,

$$v + ((-v) + (-(-v))) = v + 0.$$

Using associativity of addition on V on the left hand side, and the fact that $0 \in V$ is the additive identity of $v \in V$ on the right hand side yields,

$$(v + (-v)) + (-(-v)) = v.$$

Again because $(-v)$ is the additive inverse of v, we have,

$$0 + (-(-v)) = v.$$

Hence we have that the additive inverse of $(-v)$ is v,

$$(-(-v)) = v.$$

1B.2.

Hint: Break down into cases and use properties of the field $\mathbb{F}$.

Assume that $a \neq 0$. Now $a \in \mathbb{F}$ and $\mathbb{F}$ is a field.

Let $av = 0$ with $a \in \mathbb{F}$ and $v \in V$.

By properties of a field (1.3), for every $a \in \mathbb{F}$ with $a \neq 0$, there will exist a unique $a^{-1} \in \mathbb{F}$ such that $aa^{-1} = 1$.

Multiplying by a^{-1} on both sides of $av = 0$ gives,

$$a^{-1}av = a^{-1}0.$$

By definition of the multiplicative inverse and commutativity of multiplication in the field, $a^{-1}a = aa^{-1} = 1$. And also, property (1.31) that $a0 = 0$ for every $a \in \mathbb{F}$ yields,

$$1v = 0.$$

Because V is a vector space, it must also have a multiplicative identity, so $1v = v$, this implies that

$$v = 0.$$

Now assume that $v \neq 0$. Using this together with (1.30) that $0v = 0$ for every $v \in V$, and also (1.31), this implies that $a = 0$.

Corrections: The last paragraph involving as you've written is circular. To clean it up, you should assume for contradiction that $a \neq 0$, then use existence of a^{-1} and derive a contradiction.

In any case, this last part is redundant, because you've already shown in the first part that $a \neq 0$ and $av = 0$ forces $v = 0$. If $a = 0$, then the condition either $a = 0$ or $v = 0$ is already satisfied.

1B.3.

We have,

$$v + 3x = w.$$

Adding the additive inverse of w, $-w$ to both sides, we have,

$$(v + (-w)) + 3x = w + (-w).$$

Using additive inverse properties and associativity of addition yields,

$$(v - w) + 3x = 0.$$

Multiplying by the scalar $1/3 \in \mathbb{F}$ yields,

$$\frac{1}{3}(v - w) + \frac{1}{3}(3x) = \frac{1}{3} \cdot 0.$$

Using (1.31), and existence of the multiplicative inverse $1x = x$ for all $x \in V$ yields,

$$\frac{1}{3}(v - w) + x = 0.$$

As V is a vector space, $(1/3)(v - w) \in V$ must possess a unique additive inverse. This unique additive inverse is x.

1B.4.

Hint: Are there any axioms in (1.20) which mandate the existence of something that is not in the empty set?

Reasoning about this feels difficult, because many of the vector space axioms in (1.20) assume that the candidate vector space in question is a set with some $u, v \in V$, so it is difficult to see how can we engage in any reasoning whatsoever.

Here are some facts about empty sets. An empty set $= \{\}$ has no elements, and has cardinality zero. The sum of elements of the empty set is zero, the additive identity. And the product of elements of the empty set is one, the multiplicative identity.

On this basis, the only candidates in (1.20) not satisfied by the empty set are the additive inverse and additive identity.

However, the additive identity appears to require that $0 \in \{\}$, which cannot be true.

Hence I would submit that it is the additive identity axiom that isn't fulfilled by the empty set.

[C] But isn't clear to me why it also cannot be the additive inverse, which also requires the existence of some $w \in V$ such that $v + w = 0$.

Corrections: The reasoning difficulty is because the vector space axioms that are satisfied by the empty set are satisfied vacuously. For these axioms, which are conditional statements, the antecedents (ifs) cannot be satisfied, and so their consequents (thens) are vacuously true.

So all the vector space axioms whose antecedents cannot be satisfied are those that contain a universal quantifier at the beginning. In these cases, as the empty set contains no elements, the antecedent doesn't apply, and the consequents are vacuously true.

The additive identity axiom is the only one with an existential qualifier at the beginning, and so is false for the empty set.

Exercises 1C Subspaces.

To determine whether a subset U of a vector space V is a subspace, we check the three subspace conditions in (1.34). That is, the additive identity $0 \in V$ is in U, and that U is closed under the scalar multiplication and addition operations defined on V.

1C.1a.

Define the subset $U \subseteq \mathbb{F}^3$ as follows,

$$U := \{(x_1, x_2, x_3) \in \mathbb{F}^3 \mid x_1 + 2x_2 + 3x_3 = 0\}.$$

Then U is a subspace of $\mathbb{F}^3$, as shown by the following.

Additive identity. The additive identity $0 \in \mathbb{F}^3$ is $(0, 0, 0)$. So we check whether $0 \in U$. We have,

$$x_1 + 2x_3 + 3x_3\Big|_{x_1=0, x_2=0, x_3=0} = 0 + 2(0) + 3(0) = 0.$$

Hence $0 \in U$.

Closure under addition operation on $\mathbb{F}^3$. Let $u, u' \in U$, so that

$$u = (u_1, u_2, u_3), u' = (u'_1, u'_2, u'_3)$$

$$u_1 + 2u_2 + 3u_3 = 0, u'_1 + 2u'_2 + 3u'_3 = 0. \tag{*}$$

We check whether $u + u' \in U$. So we have,

$$u + u' = (u_1 + u'_1, u_2 + u'_2, u_3 + u'_3).$$

Using (*), we have,

$$(u_1 + u'_1) + 2(u_2 + u'_2) + 3(u_3 + u'_3) = (u_1 + 2u_2 + 3u_3) + (u'_1 + 2u'_2 + 3u'_3) = 0.$$

Hence $u + u' \in U$ for all $u, u' \in U$.

Closure under scalar multiplication operation on $\mathbb{F}^3$. Let $\lambda \in \mathbb{F}$ and $u \in U$ so that

$$u = (u_1, u_2, u_3).$$

$$u_1 + 2u_2 + 3u_3 = 0. \tag{**}$$

We check whether $\lambda u \in U$, and so we have,

$$\lambda u = \lambda(u_1, u_2, u_3) = (\lambda u_1, \lambda u_2, \lambda u_3).$$

Using (**), we have that

$$(\lambda u_1) + 2(\lambda u_2) + 3(\lambda u_3) = \lambda(u_1 + 2u_2 + 3u_3) = 0.$$

Hence $\lambda u \in U$ for all $\lambda \in \mathbb{F}$ and $u \in U$.

Putting all these verified conditions together shows that U is a subspace of $\mathbb{F}^3$.

1C.1b.

Define the subset $U \subseteq \mathbb{F}^3$ as follows,

$$U := \{(x_1, x_2, x_3) \in \mathbb{F} \mid x_1 + 2x_2 + 3x_3 = 4\}.$$

Then U is not a subspace of $\mathbb{F}^3$. All that is required for this is that U fails to satisfy at least one of the subspace conditions. In this particular case, all three conditions are not satisfied.

Additive identity. The additive identity $0 \in \mathbb{F}^3$ is $(0, 0, 0)$, but we conclude that $0 \notin U$ because

$$x_1 + 2x_3 + 3x_3\Big|_{x_1=0, x_2=0, x_3=0} = 0 + 2(0) + 3(0) = 0 \neq 4.$$

Closure under addition on $\mathbb{F}^3$. For $u, u' \in U$ we conclude $u + u' \notin U$ for all $u, u' \in U$ because

$$(u_1 + u_1') + 2(u_2 + u_2') + 3(u_3 + u_3') = (u_1 + 2u_2 + 3u_3) + (u_1' + 2u_2' + 3u_3') = 8 \neq 4.$$

Closure under scalar multiplication on $\mathbb{F}^3$. For $u \in U$, we conclude $\lambda u \in U$ only for $\lambda = 1$ because

$$\lambda u_1 + 2(\lambda u_2) + 3(\lambda u_3) = \lambda(u_1 + u_2 + u_3) = 4\lambda.$$

So it is not the case that $\lambda u \in U$ for all $\lambda \in \mathbb{F}$.

1C.1c.

Define the subset $U \in \mathbb{F}^3$ as,

$$U := \{(x_1, x_2, x_3) \in \mathbb{F}^3 \mid x_1x_2x_3 = 0\}.$$

On inspection, it is likely that the multiplicative nature of this subset, in the sense of $x_1x_2x_3 = 0$ will mean that it will not comply with the additive structure of $\mathbb{F}^3$.

We find that U is not a subspace because it fails closure under addition. If we let $u, u' \in U$, then note that

$$u + u' = (u_1 + u_1', u_2 + u_2', u_3 + u_3').$$

Now we need to assess whether it is the case that $(u_1+u_1')(u_2+u_2')(u_3+u_3') = 0$ for all $u_1, \ldots u_3' \in \mathbb{F}$ given that $u_1u_2u_3 = 0$ and $u_1'u_2'u_3' = 0$.

We have that the vectors $u = (0, 1, 1) \in U$ and $u' = (1, 0, 0) \in U$ but $(u_1 + u_1')(u_2 + u_2')(u_3 + u_3') = 1 \neq 0$.

The next part is not necessary for the proof, but it we check anyway. U does fulfil the additive identity and closure under scalar multiplication conditions:-

Additive identity. Note that $0 \in U$ because

$$x_1x_2x_3\Big|_{x_1=0,x_2=0,x_3=0} = 0 \in U.$$

Closure under scalar multiplication on $\mathbb{F}^3$. Let $a \in \mathbb{F}$ and $u \in U$ with

$$u = (u_1, u_2, u_3),$$

$$u_1u_2u_3 = 0. \tag{*}$$

We now have that,

$$\lambda u = \lambda(u_1, u_2, u_3) = (\lambda u_1, \lambda u_2, \lambda u_3).$$

And also,

$$(\lambda u_1)(\lambda u_2)(\lambda u_3) = \lambda^3(u_1u_2u_3) = 0.$$

Using (*), this shows that $\lambda u \in U$ for all $\lambda \in \mathbb{F}$.)

1C.1d.

Define the subset $U \in \mathbb{F}^3$ as,

$$U := \{(x_1, x_2, x_3) \in \mathbb{F}^3 \mid x_1 = 5x_3\}.$$

Then U is a subspace of $\mathbb{F}^3$, as shown by the following.

Additive identity. The additive identity $0 \in \mathbb{F}$ is $(0,0,0)$. We have

$$x_1 - 5x_3\Big|_{x_1=0,x_2=0,x_3=0} = 0 - 5(0) = 0.$$

Hence $0 \in U$.

Closure under addition on $\mathbb{F}^3$. Let $u, u' \in U$ so that

$$u + u' = (u_1 + u_1', u_2 + u_2', u_3 + u_3').$$

Now because $u_1 = 5u_3$ and $u_1' = 5u_3'$, we have $u_1 + u_1' = 5(u_3 + u_3')$, and hence $u + u' \in U$.

Closure under scalar multiplication on $\mathbb{F}^3$. Let $\lambda \in \mathbb{F}$ and $u \in U$. Now because $u_1 = 5u_3$, we have $\lambda u_1 = 5\lambda u_3$, and so

$$\lambda u = \lambda(u_1, u_2, u_3) = (\lambda u_1, \lambda u_2, \lambda u_3) = (5\lambda u_3, \lambda u_2, \lambda u_3) \in U.$$

1C.5.

Key idea. If U is a subspace of a "parent" vector space V, then one way of viewing the subspace conditions is that the subspace U "inherits" the additive identity of V. And it must also exhibit closure under the "inherited" addition and scalar multiplication operations on its parent vector space V. In particular, if V has a scalar multiplication by $a \in \mathbb{F}$, then U will inherit this scalar multiplication, and be defined over the same field $\mathbb{F}$.

Key idea 2. Whether a vector space is "real" or "complex" depends on the field of scalars over which the vector space is defined.

First, we clarify the definitions of $\mathbb{C}^2$ and $\mathbb{R}^2$ as sets and as subspaces.

The definition of the *set* $\mathbb{C}^2$ is,

$$\mathbb{C}^2 = \{(z_1, z_2) \mid z_1, z_2 \in \mathbb{C}\}.$$

Equipping the set $\mathbb{C}^2$ with addition and a scalar multiplication by $\lambda \in \mathbb{R}$ yields a real vector space $\mathbb{C}^2$ over $\mathbb{R}$. Whereas if instead choose to equip $\mathbb{C}^2$ with addition and a scalar multiplication by $\lambda \in \mathbb{C}$, we get a complex vector space $\mathbb{C}^2$ over $\mathbb{C}$.

Now the definition of the set $\mathbb{R}^2$ is,

$$\mathbb{R}^2 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}.$$

Equipping $\mathbb{R}^2$ with an addition and a scalar multiplication by $\lambda \in \mathbb{R}$ yields the vector space $\mathbb{R}^2$. It is not possible for $\mathbb{R}^2$ to be defined over a field of complex scalars $\mathbb{C}$.

Now the question asks whether $\mathbb{R}^2 \subseteq \mathbb{C}^2$ is a subspace of the complex vector space $\mathbb{C}^2$, i.e. over the field of complex numbers $\mathbb{F} = \mathbb{C}$.

$\mathbb{R}^2$ is a subset of $\mathbb{C}^2$ in the following way. Any real number $a \in \mathbb{R}$ can be represented as $a + 0i \in \mathbb{C}$. And so

$$\mathbb{R}^2 = \{(x_1, x_2) \in \mathbb{C}^2 \mid x_1, x_2 \in \mathbb{R}\} \subseteq \mathbb{C}^2.$$

We now check the subspace conditions.

Additive identity. The additive identity of the complex vector space $\mathbb{C}^2$ is $(0, 0)$. Because $(0, 0)$ is also in $\mathbb{R}^2$, this condition is satisfied regardless of whether $\mathbb{C}^2$ is defined over the field $\mathbb{R}$ or over $\mathbb{C}$.

Closure under addition. Let $u, v \in \mathbb{R}^2$ so that $u = (u_1, u_2)$ and $v = (v_1, v_2)$, with $u_1, u_2, v_1, v_2 \in \mathbb{R}$. Because the field of real numbers is closed under addition, and vector addition on the complex vector space $\mathbb{C}^2$ is defined coordinate-wise, we have that for all $u, v \in \mathbb{R}^2$,

$$u + v = (u_1 + v_1, u_2 + v_2) \in \mathbb{R}^2.$$

Hence $\mathbb{R}^2$ is closed under addition regardless of whether $\mathbb{C}^2$ is defined over the field $\mathbb{R}$ or over $\mathbb{C}$.

Closure under scalar multiplication. Let $u \in \mathbb{R}^2$. As $\mathbb{C}^2$ is a complex vector space, i.e. defined over the field $\mathbb{C}$, then if $\lambda \in \mathbb{C}$, then

$$\lambda u = (\lambda u_1, \lambda u_2).$$

But because field of real numbers is not closed under scalar multiplication by a complex number, we have that there exists $u \in \mathbb{R}^2$ and $\lambda \in \mathbb{C}$ such that $\lambda u \notin \mathbb{R}^2$. Take for example $u = (1, 0) \in \mathbb{C}$ and $\lambda = i$, so that $\lambda u = (i, 0) \notin \mathbb{R}^2$.

So when $\mathbb{C}^2$ is a complex vector space defined over the field $\mathbb{C}$, then $\mathbb{R}^2$ is not a subspace, because $\mathbb{R}^2$ is not closed under scalar multiplication by a complex number.

However, if $\mathbb{C}^2$ is a real vector space defined over the field $\mathbb{R}$, then $\mathbb{R}^2$ is a subspace, as it is closed under scalar multiplication by a real number (as well as satisfying other subspace conditions).

1C.6a.

Define the subset U as,

$$U := \{(a, b, c) \in \mathbb{R}^3 \mid a^3 = b^3\}.$$

Then U is a subspace of $\mathbb{R}^3$ as shown by the following.

First note that $a^3 = b^3$ implies that $a^3 - b^3 = 0$, which can be factorised as,

$$(a - b)(a^2 + ab + b^2) = 0.$$

And so $a^3 = b^3$ when either $a = b$, or $(a^2 + ab + b^2) = 0$, or when both are true.

We can view the second condition as a quadratic equation in the variable a, and restrict attention to the case where a and b are real.

Now the quadratic equation has real roots in a if and only if $(b\sqrt{3})i \geq 0$. So there are only real roots in a when $b = 0$, and this implies that $a^2 = 0$, and hence $a = 0$. Hence $(a^2 + ab + b^2) = 0$ has real roots if and only if $a = 0$ and $b = 0$.

We can rewrite our set U as,

$$U = \{(x_1, x_1, x_2) \in \mathbb{R}^3 \mid x_1, x_2 \in \mathbb{R}\}.$$

We now show that U is indeed a subspace.

Additive identity. The additive identity $0 \in \mathbb{R}^3$ is $(0,0,0)$. Now $0 \in U$ because its first two coordinates are both the same, and also 0.

Closure under addition. Let $u = (u_1, u_1, u_2) \in U$ and $u' = (u_1', u_1', u_2') \in U$. We now have,

$$u + u' = (u_1 + u_1', u_1 + u_1', u_2 + u_2') \in U,$$

because the first two coordinates of $u + u'$ are the same, and also 0 when both u_1 and u_1' are 0.

Closure under scalar multiplication. Let $\lambda \in \mathbb{R}$ and $u = (u_1, u_1, u_2) \in U$. Then

$$\lambda u = (\lambda u_1, \lambda u_1, \lambda u_2) \in U,$$

because the first two coordinates of λu are the same, and also 0 when $u_1 = 0$.

1C.6b.

Define the subset U as,

$$U := \{(a, b, c) \in \mathbb{C}^3 \mid a^3 = b^3\}.$$

We no longer restrict attention to cases where $a, b \in \mathbb{R}$, as in **1C.6a.**. Note that $a^3 = b^3$ when either $a = b$ or exclusively when $a = -b(1 \pm i\sqrt{3})/2$, the subset U can be written as,

$$U = U_1 \cup U_2 \cup U_3,$$

where $U_1 = \{(a, a, c) \in \mathbb{C}^3 \mid a, c \in \mathbb{C}\}$, $U_2 = \{(-a(1 + i\sqrt{3})/2, a, c) \in \mathbb{C}^3 \mid a, c \in \mathbb{C}\}$, $U_3 = \{(-a(1 - i\sqrt{3})/2, a, c) \in \mathbb{C}^3 \mid a, c \in \mathbb{C}\}$.

The additive identity of $\mathbb{C}^3$ is $(0,0,0) \in U_1 \cap U_2 \cap U_3$, and in fact $U_1 \cap U_2 \cap U_3 = \{0\}$, because $a = -a(1 \pm i\sqrt{3})/2$ when $a = 0$.

(?) However, for addition and scalar multiplication, it is not clear to me how we would go about representing an "arbitrary" element of $u \in U$ to check the conditions. Because when $a \neq 0$, then $a \neq -a(1 \pm i\sqrt{3})/2$, and we also have that $U_1 \cap U_2 = \{0\}$, $U_1 \cap U_3 = \{0\}$, $U_2 \cap U_3 = \{0\}$.

Hint: It may *not* be a subspace! If not, then show how.

Let $u_1 \in U_1$ and $u_2 \in U_2$, so that $u_1, u_2 \in U$. So we have,

$$u_1 + u_2 = (a, a, c) + \left(\frac{-b + (1 + i\sqrt{3})}{2}, b, d\right) = \left(\frac{2a - b(1 + i\sqrt{3})}{2}, a + b, c + d\right).$$

Now $u_1 + u_2 \notin U_1$, because it does not have first and second coordinate equal (unless u_1 and u_2 have first and second coordinate equal). And also $u_1 + u_2 \notin U_2, U_3$ because for nonzero a and b, $2a - b(1 + i\sqrt{3})/2 \neq (a + b)$.

Because $u_1 + u_2 \notin U$, this is a counterexample, and so U is not closed under addition.

Hence U is not a subspace.

Corrections: Note that the arbitrary element of U here can either be in U_1, U_2 or U_3, but not in any combination of the U_is, and this is probably why it felt strange picking a "representative element" of U. Unless the arbitrary element is the additive identity, because the pairwise and 3-tuplewise intersections only contain 0, as you've established.

1C.9.

Key idea:

$\mathbb{F}^S$ is the vector space of all functions $f : S \to \mathbb{F}$, so $\mathbb{R}^\mathbb{R}$ is the vector space of all functions $f : \mathbb{R} \to \mathbb{R}$.

Define the subset U of all periodic functions,

$$U := \{f : \mathbb{R} \to \mathbb{R} \mid \exists\, p \in \mathbb{R}^+ \text{ s.t. } \forall x\, f(x) = f(x + p)\}.$$

As an example, $\sin(x), \tan(x) \in U$ because $\sin(x) = \sin(x + 2\pi)$ and $\tan(x) = \tan(x + \pi)$.

Additive identity. The additive identity $0 \in \mathbb{R}^\mathbb{R}$ is the zero function $0 : \mathbb{R} \to \mathbb{R}$, $0(x) = 0$ for all $x \in \mathbb{R}$. So by definition, we have that for all $x \in \mathbb{R}$, and for all $p \in \mathbb{R}$ (not just $p > 0$),

$$0(x) = 0(x + p) = 0.$$

Hence $0 \in U$.

Closure under scalar multiplication. Let $f \in U$ and $\lambda \in \mathbb{R}$ (remember that for the vector space of functions $\mathbb{F}^S, \lambda \in \mathbb{F}$). Using the fact that f is periodic, there exists some $p > 0$ such that for all $x \in \mathbb{R}$,

$$(\lambda f)(x) = \lambda f(x) = \lambda f(x + p) = (\lambda f)(x + p).$$

Hence $f \in U$ implies that $\lambda f \in U$.

Closure under addition. Let $f_1, f_2 \in U$. Assume that they have period $p_1, p_2 > 0$ respectively, so that for all $x \in \mathbb{R}$, $f_1(x) = f_1(x + p_1)$ and $f_2(x) = f_2(x + p_2)$. We now consider,

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = f_1(x + p_1) + f_2(x + p_2).$$

And it's not so clear from this whether there is any reason why we would expect the sum of two periodic functions with different periods to be itself, periodic.

Now if $p_1 = p_2$, so that f_1 and f_2 have the same period, then we have that there exists some $p > 0$ such that

$$f_1(x+p) + f_2(x+p) = (f_1 + f_2)(x+p).$$

That is, if we have some subspace W_p,

$$W_p = \{f : \mathbb{R} \to \mathbb{R} \mid \forall x\, f(x) = f(x+p)\}.$$

consisting only of functions with the same period p then it will meet all the subspace conditions, including closure under addition, and will be a subspace of $\mathbb{R}^{\mathbb{R}}$.

However, U is a set of functions which may not necessarily all have the same period p and so closure under addition will likely not be satisfied.

And complicating things further is the following example. The functions $\sin(3x), \sin(4x) \in U$ as both have period $2\pi/3$ and $\pi/2$ respectively. However, a graph of the sum of functions $\sin(3x)+\sin(4x) \in U$ with period 2π.

Not clear whether closure under addition holds. Needs further investigation.

Corrections: Your solution is partially correct in that the subset U of periodic functions from $\mathbb{R}$ to $\mathbb{R}$ satisifies the additive identity and closure under scalar multiplication conditions, and that closure under addition may be problematic. And you've also correctly established that the set of all functions with the same period W_p is a subspace of $\mathbb{R}^{\mathbb{R}}$.

Here is the key fact you couldn't quite get, and needed more experimenting plotting graphs on Wolfram. For two periodic functions f_1, f_2 in U with periods p_1 and p_2, if the ratio p_1/p_2 is rational, then $f_1 + f_2$ will itself be periodic. If the ratio p_1/p_2 is irrational, then $f_1 + f_2$ will not be periodic.

Lastly, note that the set of all periodic functions U is a non-disjoint union over all sets of periodic functions W_p with fixed period p.

$$U = \bigcup_{p \in \mathbb{R},\, p>0} W_p.$$

In SM proof, why aren't we using a counterexample to prove that $f(x) = \sin(x) + \sin(\sqrt{2}x)$ is not periodic, rather assuming p exists then showing contradiction? Because the statement we are trying to prove is that there does not exist some value of p such that $f(x) = f(x+p)$ for all x. If we use counterexample, then picking some value of p so that f is not periodic only shows that specific p is not the period. It doesn't rule out possibility that some other p which is the period doesn't exist. Analogous to issue of black swans and induction, but in context of non-existence.

1C.10

Key idea: The intersection of two subspaces is itself a subspace.

Additive identity. The additive identity is $0 \in V$. Because V_1 and V_2 are subspaces, $0 \in V_1$ and $0 \in V_2$. So $0 \in V_1 \cap V_2$.

Closure under addition. Let $u_1, u_2 \in V_1 \cap V_2$. Now as $u_1, u_2 \in V_1$, and V_1 is a subspace that is closed under addition, we have that $u_1 + u_2 \in V_1$. Similarly, as $u_1, u_2 \in V_2$ and V_2 is a subspace, we have that $u_1 + u_2 \in V_2$. Because the sum $u_1 + u_2$ is in both V_1 and V_2, $u_1 + u_2 \in V_1 \cap V_2$.

Closure under scalar multiplication. Let $u \in V_1 \cap V_2$, and let $\lambda \in \mathbb{F}$. Then $\lambda u \in V_1$ and $\lambda u \in V_2$ as both V_1 and V_2 are subspaces which are closed under scalar multiplication. Hence $\lambda u \in V_1 \cap V_2$.

1C.11

Key idea: The intersection of a collection of subspaces is itself a subspace.

Let $a \in A$ be an arbitrary member of an index set A, where we first assume that A is finite. And also let V_a be an arbitrary subspace of the V.

Define the subset U as,

$$U := \bigcap_{a \in A} V_a.$$

Additive identity. Let $0 \in V$ be the additive identity of V. Then $0 \in V_a$ for all $a \in A$, because each V_a is a subspace, and every subspace must contain the additive identity of V. Hence $0 \in U$.

Closure under addition. Let u_1 and u_2 be arbitrary elements of U, that is, u_1 and u_2 is a member of every subspace V_a. Because every subspace V_a is closed under addition, we have that $u_1 + u_2 \in V_a$ for all $a \in A$. Hence $u_1 + u_2 \in U$.

Closure under scalar multiplication. Let u be an arbitrary element of U, so that u is a member of every subspace V_a. Because each subspace V_a is closed under scalar multiplication, $\lambda u \in V_a$ for all $a \in A$, and hence $\lambda u \in U$ for all $a \in A$.

As we assumed A was a finite index set, this shows that the intersection of a finite collection of subspaces is itself a subspace. And there appears to be nothing we know of suggesting that this cannot be extended to countable and uncountable A.

Corrections: In SM, Axler uses a general $\mathcal{U}$ to refer to a set of subspaces, whereas in your case, you have indexed these subspaces, which is not wrong. And yes, it is a general fact that this results holds regardless of whether we have a finite, countable, or uncountable collection of subspaces.

1C.12.

Key idea: In general, unions of subspaces are not subspaces. This question illustrates an exception.

Define the subset $U := V_1 \cup V_2$.

$\Longleftarrow$. Without loss of generality, let $V_1 \subseteq V_2$. We now check that U is a subspace.

Additive identity. As V_1 and V_2 are subspaces, the additive identity of V, $0 \in V$, is in V_1 and V_2. So we have that,

$$0 \in V_1 \cap V_2 \implies 0 \in V_1 \cup V_2.$$

Closure under addition. Let $u_1 \in V_1 \cup V_2$, which means that either $u_1 \in V_1$, or $u_1 \in V_2$ or $u_1 \in V_1 \cap V_2$. But because $V_1 \subseteq V_2$, this simplifies to $u_1 \in V_2$. By the same argument, $u_2 \in V_2$. Hence we have that,

$$u_1 + u_2 \in V_2 \implies u_1 + u_2 \in V_1 \cup V_2.$$

Closure under scalar multiplication. Let $u \in V_1 \cup V_2$, and by a similar argument as before, $u \in V_2$. As V_2 is a subspace, we have,

$$\lambda u \in V_2 \implies \lambda u \in V_1 \cup V_2.$$

$\implies$. We will prove this by contradiction. Assume that $V_1 \cup V_2$ is a subspace. Assume for contradiction that V_1 is not contained in V_2 and V_2 is not contained in V_1.

Let $v_1, v_2 \in V_1 \cup V_2$, but also, $v_1 \in V_1 \backslash V_2$ and $v_2 \in V_2 \backslash V_1$. Because $V_1 \cup V_2$ is a subspace, it must be closed under addition, meaning that

$$v_1 + v_2 \in V_1 \cup V_2.$$

This implies that either $v_1 + v_2 \in V_1$ or $v_1 + v_2 \in V_2$ or $v_1 + v_2 \in V_1 \cap V_2$. But each of these conditions in turn would require that either $v_2 \in V_1$, or $v_1 \in V_2$ or lastly, $v_2 \in V_1$ and $v_1 \in V_2$ respectively.

However, our assumption for contradiction means that none of these hold, and so we have the contradiction that $V_1 \cup V_2$ is a subspace that is not closed under addition.

Which means that either V_1 is contained in V_2 or V_2 is contained in V_1, or both.

Corrections: Comparing your $\impliedby$ proof with SM reveals that you just went for routine checking, but instead, a more sophisticated approach would shortcut it by reasoning that if $V_1 \subseteq V_2$ or $V_2 \subseteq V_1$ then $V_1 \cup V_2 = V_2$ or $V_1 \cup V_2 = V_1$, respectively.

For your $\implies$ proof, you opt for contradiction by assuming $V_1 \cup V_2$ is a subspace (P) and neither of V_1, V_2 are contained in each other ($\neg Q$), then show contradiction. You could be more explicit in showing for example that $v_1 + v_2 \in V_1 \implies v_2 \in V_1$ because $(v_1 + v_2) - v_1 = v_2 \in V_1$ and $v_1 \in V_1 \backslash V_2$. No need to be hesitant in using subtraction in your arguments just because you've formalised it in this course as an additive inverse.

SM opts for a direct proof with assuming $V_1 \cup V_2$ is a subspace, and that V_1 is not contained in V_2, to then show that V_2 must be contained in V_1. Which is cleaner and slicker.

1C.15.

Key idea: The addition of subspaces is idempotent, and this arises from subspaces being closed under addition.s

With $V = \mathbb{R}^n$ or $V = \mathbb{C}^n$, we can show that $U + U = U$.

(?) Not sure how to show this for the general case where we cannot construct a set-theoretic, coordinate description of U and V.

If instead we use vector space axioms algebraically, let $u \in U \subseteq V$. Then by the distributive and multiplicative inverse properties of V,

$$(1u + 1u) = (1 + 1)u = 2u = \lambda u \in U.$$

This shows that $u + u \in U$ with $\lambda = 2$.

(?) But not sure how to proceed in the other direction.

Corrections: Main stumbling block here was that you assumed that some element of $U+U$ needed to be of the form $u+u$, i.e. the sum of the same element. Here is solution from SM. To show that $U+U \subseteq U$, first let $u+v \in U+U$, then $u+v \in U$ as U is a subspace. To show that $U \subseteq U+U$, let $u \in U$, then $u+0 \in U+U$.

1C.16.

Key idea: The operation of addition on subspaces of V is commutative. Remember to show equivalence of sets, need to show via bidirectional set inclusion.

Let v be an arbitrary element of $U+W$, so that for $u \in U$ and $w \in W$,

$$v = u + w.$$

As u and w are vectors in V, and vector addition in V is commutative, we have,

$$v = u + w = w + u \in W + U.$$

Because $v \in U+W \implies v \in W+U$, we have $U+W \subseteq W+U$.

Now let v' be an arbitrary element of $W+U$, so that for $w \in W$ and $u \in U$,

$$v' = w + u$$

As w and u are vectors in V, and vector addition on V is commutative,

$$v' = w + u = u + w \in U + W.$$

Because $v' \in W+U \implies v' \in U+W$ we have $W+U \subseteq U+W$. Putting this all together, we have that $U+W = W+U$.

1C.17.

Key idea: The operation of addition on arbitrary subspaces of V is associative.

After completion of **1C.16.**, the operation of addition on arbitrary subspaces should be associative. Because each individual element of a subspace that is being summed over is also an element of the parent vector space, and so should obey the associativity vector space axiom. We will again prove by bidirectional set inclusion.

Let v be an arbitrary element of $(V_1+V_2)+V_3$ so that for $v_i \in V_i$ for $i = 1,2,3$, we have,

$$v = (v_1 + v_2) + v_3 = v_1 + (v_2 + v_3) \in V_1 + (V_2 + V_3).$$

Where we have used the fact that because $V_i \subseteq V$, each $v_i \in V$, and also the associativity of addition property of V. And so $(V_1+V_2)+V_3 \subseteq V_1+(V_2+V_3)$.

Similarly, let v' be an arbitrary element of $V_1+(V_2+V_3)$. Similar arguments show that

$$v' = v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3 \in (V_1 + V_2) + V_3.$$

And so $V_1 + (V_2 + V_3) \subseteq (V_1 + V_2) + V_3$.

Putting this altogether, we have $(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)$.

1C.18

Key idea: The addition of subspaces is an accumulative/monotonic operation. Hence for arbitrary subspaces $U \subseteq V$, the additive identity is the subspace $\{0\}$. And for subspaces $U \neq \{0\}$, no additive inverse exists.

Not really sure what the question means here.

Corrections: This correction is lengthy as I didn't even understand the scope of the problem. Main stumbling block is that the abstraction in the question has ratcheted up a notch. Note that the vector space axioms (1.20) are couched in terms of elements of sets being commutative, having additive identities etc. Whereas what has thrown you here is the notion that a set itself, i.e. a subspace, can have an additive inverse, and as solution shows, this additive inverse being itself subspace $\{0\}$.

The difficulty you are having appears to be moving between statements about operations on subsets/subspaces, and operations on their elements and vice versa - because there are some subtleties in moving from element-wise summation to subspace set-summation.

What follows is the notational overloading you need to be clear about. At the level of elements, we can have vector addition, $u + w$, *which is adding two points in space. At the level of subspaces, we have subspace addition* $U + W$, *which is adding two sets of vectors to produce a set containing all possible element-wise sums,*

$$U + W = \{u + w \mid u \in U, w \in W\}.$$

*From SM, for (the operation of addition on) an arbitrary subspace U to have an additive identity, we require a subspace $E \subseteq V$ such that for all subspaces $U \subseteq V$,

$$U + E = U.$$

Moving back to element-wise notation, let U *be an arbitrary subspace of* V, *and let* $u \in U$. *We require a subspace* E *such that for every* $e \in E$, *and every* $u \in U$,

$$u + e = u.$$

That is, every possible summation of $u \in U$ *and* $e \in E$ *yields* $u \in U$.*In order for this to hold, we require that* $e = 0$. *And this suggests that* $E = \{0\}$.

We now need to show that $U + \{0\} = U$. *First, letting* $u \in U$, *note that* $u + 0 = u \in U$. *Hence* $U + \{0\} \subseteq U$. *Also note that if* $u \in U$, *then we can write* $u = u + 0 \in U + \{0\}$, *where* $0 \in \{0\}$.

This concludes proof that $U + \{0\} = U$ *for all subspaces* $U \subseteq V$, *i.e. that the additive identity of addition on an arbitrary subspace of* V *is* $\{0\}$.

Similarly, for the (operation of addition on) an arbitrary subspace of U *to have an additive inverse, we require a subspace* $E \subseteq V$ *such that for all subspaces* $U \subseteq V$,

$$U + E = \{0\}.$$

Moving back to element-wise notation, let U be an arbitrary subspace of V, and let $u \in U$. We require a subspace E such that for every $e \in E$, and every $u \in U$,

$$u + e = 0.$$

*Now at purely an element-wise level as a vector equation without consideration of subsets/subspaces, setting $e = (-u)$ i.e. e to be the additive inverse of u, appears to solve this equation. Now because $(-u) \in U$, we might be tempted to use $E = U$, but then we've established that $U + U = U$ in **2C.15**. So what exactly is going on here?*

Notice that if we set $E = U$, then written set-theoretically,

$$U + U = \{u_1 + u_2 \mid u_1 \in U_1, u_2 \in U_2\}.$$

We can take $u_1 = u$ and $u_2 = (-u)$, so that $u_1 + u_2 = u + (-u) = 0$, but then we have to include all other sums e.g. $u_1 = u$ and $u_2 = u$ so that $u_1 + u_2 = 2u \in U$. Part of the issue then is that any definition of E is going to depend on $u \in U$.

Going back the to the subspace equation, we need to select a single set E such that every summation of $u \in U$ and $e \in E$, $u + e$ lands in the set $\{0\}$. But because $\{0\} \subseteq U \subseteq U + E$, this is hopeless unless $U = \{0\}$ and $E = \{0\}$.

This means we can now attempt the proof. When $U = \{0\}$, because $\{0\} + \{0\} = \{0\}$, the additive inverse of the subspace $U = \{0\}$ is the subspace$E = \{0\}$.

When U is a subspace of V and $U \neq \{0\}$, then no additive inverse exists for U, i.e. there does not exist a subspace $E \subseteq V$ such that $U + E = \{0\}$. This arises because $U \subseteq U + E$ for any subspace E, and so $U + W \neq \{0\}$.

1C.20

We have the subspace U of $\mathbb{F}^4$,

$$U := \{(x, x, y, y) \in \mathbb{F}^4 \mid x, y \in \mathbb{F}\}.$$

Note that this subspace has two "freely varying parameters", x and y. To find a subspace W of $\mathbb{F}^4$ such that $U \oplus W = \mathbb{F}^4$, we require W to satisfy the following two conditions,

$$U + W = \mathbb{F}^4, \quad U \cap W = \{0\}.$$

Let $v = (w, x, y, z) \in \mathbb{F}^4$ and $u = (a, a, b, b) \in U$, and we seek some vector w' such that

$$v = u + w \implies (w, x, y, z) = (a, a, b, b) + w' \implies w' = (w - a, x - a, y - b, z - b).$$

We now need to choose the "free parameters" a and b to ensure that $U \cap W = \{0\}$. We can do this by selecting a and b so that $w' \notin U$ for nonzero w'.

Setting $a = x$ and $b = z$ yields,

$$w' = (w - x, 0, y - z, 0) \notin U.$$

Now as w, x, y, z are also arbitrary i.e. "free parameters", we have W as the following candidate subspace, consisting of vectors whose second and fourth coordinate are 0,

$$W := \{(w, 0, z, 0) \in \mathbb{F}^4 \mid w, z \in \mathbb{F}^4\}.$$

To verify that W is indeed a subspace, the additive identity $(0, 0, 0, 0) \in \mathbb{F}^4$ is also in W by setting $w = z = 0$.

For closure under addition, note that $w' + w'' = (w_1, 0, z_1, 0) + (w_2, 0, z_2, 0) = (w_1 + w_2, 0, z_1 + z_2, 0) \in W$ because it has second and fourth coordinate 0.

For closure under scalar multiplication, we have $\lambda w' = (\lambda w, 0, \lambda z, 0) \in W$ also.

To verify that $U + W$ is indeed a direct sum, let $z' \in U \cap W$. Hence we require that $z' \in U$ and $z' \in W$, so that for suitable choices of w, x, y, z,

$$z' = (x, x, y, y) = (w, 0, z, 0).$$

This implies that $x = y = w = z = 0$, so that $z' = 0$, showing that $U \cap W = \{0\}$.

Corrections: All good. SM doesn't check W is a subspace (possibly as it's routine), and also doesn't show the reasoning for why the candidate subspace W takes the form it does, so it's good you've included here.

1C.21

We have the following subspace U of $\mathbb{F}^5$,

$$U := \{(x, y, x + y, x - y, 2x) \in \mathbb{F}^5 \mid x, y \in \mathbb{F}^5\}.$$

Again, this subspace has two "freely varying parameters". We require a subspace W such that $U \oplus W = \mathbb{F}^5$. Using the same criteria as for **1C.20**, we have for $v = (v_1, \ldots, v_5) \in \mathbb{F}^5$ and $u = (a, b, a + b, a - b, 2a) \in U$, and we seek a w' of the form,

$$w' = (v_1 - a, v_2 - b, v_3 - (a + b), v_4 - (a - b), v_5 - 2a).$$

Now we need to set a and b so that $w \notin U$ for $w \neq 0$. Setting $a = v_1$ and $b = v_2$ yields the following vector

$$w' = (0, 0, v_3 - (v_1 + v_2), v_4 - (v_1 - v_2), v_5 - 2v_1).$$

Noting that $w' \notin U$ in general for $v_3, v_4, v_5 \neq 0$, our candidate subspace is then

$$W := \{(0, 0, a, b, c) \in \mathbb{F}^5 \mid a, b, c \in \mathbb{F}\}.$$

Now W satisfies the additive identity, addition and scalar multiplication conditions, so it is indeed a subspace.

To verify that $U + W$ is a direct sum, let $z' \in U \cap W$, so that for suitable choices of x, y, a, b, c,

$$z' = (x, y, x + y, x - y, 2x) = (0, 0, a, b, c).$$

This implies that $x = y = a = b = c = 0$, and so $U + W$ is a direct sum.