Understanding Analysis (2nd ed.) by Stephen Abbott.

My write-up for exercises in *Chapter 2: Sequences and Series.*

17th July 2021.

The following are my write-ups for the exercises in the above textbook. The purpose of writing up my attempts at the exercises from pen and paper is to practise writing up formal proofs.

This is not intended to be a solutions manual, nor any kind of authoritative reference. Rather, committing this document to the public domain to a virtual target audience pushes me to enforce a high standard for myself than if I had confined myself to pen and paper attempts.

There is already an official instructor's solution manual for all the exercises in the 1st edition of the book, and numerous online solutions for the additional exercises added to the 2nd edition.

I have organised the exercises into problem sets corresponding to each section in the chapter. For proofs that needed more thought, I have listed the proof strategy.

Problem set 2.2.

2.2.1.

2.2.2.

(a)

Observe that

$$\lim_{n\to\infty}\frac{2n+1}{5n+4}=\lim_{n\to\infty}\frac{2+(1/n)}{5+(4/n)}=\frac{2}{5},$$

because heuristically, $(1/n)\to 0$ and $(4/n)\to 0$ at the same rate.

Proof. Consider an arbitrary $\epsilon>0$. Because $n>0$, we have that

$$\left|\frac{2n+1}{5n+4}-\frac{2}{5}\right|=\left|\frac{-3}{5(5n+4)}\right|=\frac{3}{5(5n+4)}.$$

Setting $N>(3-20\epsilon)/25\epsilon$, our proof is completed by observing that for all $n\geq N$,

$$\frac{3}{5(5n+4)}<\epsilon.$$

(b).

Observe that

$$\lim_{n\to\infty}\frac{2n^2}{n^3+3}=\lim_{n\to\infty}\frac{2}{n+(3/n^2)}=0$$

because $(3/n^2) \to 0$ and therefore $(2/n) \to 0$.

Proof. Consider an arbitrary $\epsilon > 0$. Because $n > 0$, we have that

$$\left|\frac{2n^2}{n^3+3}\right| = \frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = \frac{2}{n}$$

Setting $N > 2/\epsilon$, our proof is completed by observing that for all $n \geq N$,

$$\frac{2n^2}{n^3+3} < \epsilon.$$

Key idea: For a value $\epsilon = \epsilon_0$, we do not necessarily need to analytically solve for the exact value $N_0 = N(\epsilon_0)$. In the above, that would amount to solving a cubic equation. Instead, we can supply an $N_1 > N_0$ such that the statement we wish to prove holds for all $n \geq N_1$.

Do we require the argument that $(2/n) \to 0$ here?

(c)

Proof. Because $f(n) = \sin(n)$ is such that $f : \mathbb{N} \to [-1, 1]$, and because $n > 0$, we have that

$$\left|\frac{\sin(n^2)}{n^{1/3}}\right| \leq \frac{1}{n^{1/3}}.$$

Setting $N > \epsilon^{-3}$, we conclude that for all $n \geq N$,

$$\left|\frac{\sin(n^2)}{n^{1/3}}\right| < \epsilon.$$

Do we similarly require that $(n^{-1/3}) \to 0$ here?

2.2.3

2.2.4.

(a)

Consider the following sequence

$$(x_n) = (1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \dots),$$

where each individual term x_n is given by

$$x_n = \begin{cases} 1 & \text{if} \quad n \text{ is odd,} \\ 0 & \text{if} \quad n \text{ is even.} \end{cases}$$

This sequence consists of a countably infinite number of 1s, and also a countably infinite number of 0s. It diverges because for $0 < \epsilon < 1$ it will not be possible to find an N such that $|x_n - 1| < \epsilon$ for all $n \geq N$.

Check the details of countably infinite.

(b)

I present an example in an attempt to develop a heuristic argument for why this is likely impossible. This needs development. The goal is to try and construct a sequence (x_n) that contains an infinite number of 1s, and which converges to a limit $x \neq 1$, and make arguments for why this cannot be done.

Following (a), to ensure that (x_n) contains an infinite number of 1s, we set $x_n = 1$ when n is even. We then set every $x_n = x + (1/n)$ when n is odd. That is, consider the sequence

$$(x_n) = \left(x + 1, 1, x + \frac{1}{2}, 1, x + \frac{1}{3}, 1, x + \frac{1}{4}, \dots\right).$$

This sequence diverges because for $0 < \epsilon < |1 - x|$, we cannot find an N such that $|x_n - x| < \epsilon$ for all $n \geq N$. That is, no matter how many terms we enumerate in the sequence (x_n), there is no N after which the absolute distance between x_n and x can be controlled to be less than the absolute distance between 1 and x, the proposed limit point.

The question then becomes whether it is possible to distribute the infinite number of 1s in such a way so as to avoid the divergence. One possible candidate might be a sequence that involves an infinite number of 1s, after which all numbers are x. However, it is likely that such a construction is not possible, as the number of 1s would then really be finite.

Based on these heuristic arguments, I would conclude that such a construction is not possible.

(c)

Consider the sequence

$$(x_n) = (0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, \dots, \underbrace{0, 1, \dots, 1}_{k\text{th block}}, \dots),$$

where $k \in \mathbb{N}$ and each kth block consists of $k + 1$ terms, with first term set to 0 and remaining k terms set to 1.

This sequence diverges, because given any proposed limit x and $0 < \epsilon < 1$, it will not be possible to find an N such that $|x_n - x| < \epsilon$ for all $n \geq N$.

Some uncertainties about divergence. In particular:

- What kinds of divergence are there? It looks like we can have divergence to $\pm\infty$, and a case where it does not exist.
- The discussion in Abbott shows means of argumentation against convergence to a specific limit x. It is difficult to see how this matches up with divergence as being negation of convergence. In particular, would we need to negate the defintion for all $x \in \mathbb{R}$? Scrutinise the definition again.

2.2.5.

(a)

Enumerating terms in the sequence $(5/n)$, we have that

$$\left(\frac{5}{n}\right) = \left(5, \frac{5}{2}, \frac{5}{3}, \frac{5}{4}, 1, \frac{5}{6}, \frac{5}{7}, \dots\right).$$

This yields the sequence

$$(a_n) = \left[\left[\frac{5}{n}\right]\right] = (5, 2, 1, 1, 1, 0, 0, \dots).$$

This sequence converges to 0.

Proof. Let $\epsilon > 0$ be arbitrary. Regardless of choice of ϵ, as long as $\epsilon > 0$, when $N = 6$, we have that for all $n \geq N$,

$$|a_n| = \left| \left[\left[\frac{5}{n} \right]\right] \right| = 0 < \epsilon.$$

(b)

Enumerating terms in the sequence $((12 + 4n)/3n)$, we have that

$$\left(\frac{12 + 4n}{3n} \right) = \left(\frac{16}{3}, \frac{10}{3}, \frac{8}{3}, \frac{7}{3}, \frac{32}{15}, 2, \frac{40}{21}, \dots \right).$$

Observe that

$$\lim_{n \to \infty} \frac{12 + 4n}{3n} = \lim_{n \to \infty} \frac{(12/n) + 4}{3} = \frac{4}{3}.$$

This yields the sequence

$$(a_n) = (5, 3, 2, 2, 2, 2, 1, \dots)$$

This converges to 1.

Proof. Let $\epsilon > 0$ be arbitrary. Regardless of choice of ϵ, as long as $\epsilon > 0$, when $N = 7$, we have that for all $n \geq N$,

$$|a_n - 1| = \left| \left[\left[\frac{12 + 4n}{3n} \right]\right] - 1 \right| = 0 < \epsilon.$$

Key idea:. While the choice of N in response to an $\epsilon > 0$ will often depend on ϵ, so that we have $N = N(\epsilon)$, this exercise shows that is not always the case.

2.2.6.

2.2.7.

(a)

Enumerating the sequence, we find that it is divergent.

$$(-1)^n = (-1, 1, -1, 1, -1, \dots).$$

The sequence is *not eventually* in $A = \{1\}$. That is because no N exists such that $a_n \in A$ for all $n \geq N$.

The sequence is *frequently* in $A = \{1\}$. That is because for every N, there exists not only one $a_n \in A$ for all $n \geq N$, but a countably infinite number of $a_n \in A$, occurring when n is even.

(b)

The condition of a sequence being *eventually* in A is stronger than that of a sequence being *frequently* in A in the sense that

$$(x_n) \text{ is eventually in } A \implies (x_n) \text{ is frequently in } A,$$

but

$$\neg\left((x_n) \text{ is frequently in } A \implies (x_n) \text{ is eventually in } A\right)$$

(c)

Definition 2.2.3B.a (Convergence of a Sequence: Alternate Topological Version).

A sequence (a_n) converges to a, if for all $\epsilon > 0$, the sequence (a_n) is *eventually* in the ϵ-neighbourhood $V_\epsilon(a)$ of a.

(d)

Consider the following divergent sequence, which contains a countably infinite number of 2s:

$$(x_n) = (0, 2, 0, 2, 0, 2, 0, 2, \dots).$$

Using the previous definition, this sequence is *not eventually* in $A = (1.9, 2.1)$ because it diverges. Because any sequence with a countably infinite number of 2s can be constructed in such a way so as to diverge, the crucial determinant of whether a sequence is eventually in a set A is whether it converges, due to equivalence of definition.

However, the seqeuence (x_n) is *frequently* in $A = (1.9, 2.1)$. That is because for every N, the corresponding subsequence (a_n) for all $n \geq N$ contains not just one, but a countably infinite number of terms $a_n = 2 \in (1.9, 2.1)$, which occur when n is even.

2.2.8

(a)

The sequence of interest is

$$(x_n) = (0, 1, 0, 1, 0, 1, 0, 1, \dots).$$

Symbolically, a sequence (x_n) is *zero-heavy* if

$$(\exists M \in \mathbb{N})(\forall N \in \mathbb{N})(\exists n \in \mathbb{N}, N \leq n \leq N + M)(x_n = 0)$$

Parsing this in words, a sequence (x_n) is zero-heavy if every periodic interval consisting of $M + 1$ terms contains at least one zero.

The above sequence is zero-heavy in a way that is stronger than the definition. Not only can we find a single value of M such that every periodic interval of $(M + 1)$ terms contains at least one 0. Rather, it is the case that *for every* value $M \in \mathbb{N}$, every periodic interval of $(M + 1)$ terms contains *at least an M number* of 0s.

(b)

We will prove that the following is true:

$$(x_n) \text{ is zero-heavy} \implies (x_n) \text{ contains a countably infinite number of 0s.}$$

Proof. For every natural number $N \in \mathbb{N}$, consider the corresponding subsequence I_N consisting of $(M + 1)$ terms of the sequence (x_n),

$$I_N = (x_N, x_{N+1}, \dots, x_{N+M}).$$

By definition of (x_n) being zero-heavy, we know that for at least one $M \in \mathbb{N}$, every subsequence $I_1, I_2, I_3, \dots$ will contain at least one 0. By virtue of each subsequence I_N being indexed to a natural $N \in \mathbb{N}$, there are a countably infinite number of subsequences I_N containing at least one 0. Hence we conclude (x_n) contains a countably infinite number of 0s.

(c)

By constructing a counter-example, we will show that the following is true:

$$\neg\,((x_n) \text{ contains a countably infinite number of 0s } \implies (x_n) \text{ is zero-heavy.})$$

Our counter-example will consist of specifying a sequence (x_n) containing a countably infinite number of 0s, such that for every value of $M \in \mathbb{N}$, we can specify an algorithm for finding a periodic subsequence of $(M+1)$ terms that *does not contain a single* 0.

Counter-example. Consider the sequence

$$(x_n) = (0,1,0,1,1,0,1,1,1,0,1,1,1,1,0,\ldots,\underbrace{0,1,\ \ldots,1}_{k\text{th block}},\ldots),$$

where $k \in \mathbb{N}$ and each kth block consists of $k+1$ terms, with first term set to 0 and remaining k terms set to 1.

This sequence contains a countably infinite number of 0s because we can view the sequence (x_n) as a concatenation of a countably infinite number of blocks I_k, each containing a single 0 (and also a k number of 1s). That is,

$$(x_n) = (\underbrace{0,1}_{I_1},\underbrace{0,1,1}_{I_2},\underbrace{0,1,1,1}_{I_3},\underbrace{0,1,1,1,1}_{I_4},0,\ldots,\underbrace{0,1,\ \ldots,1}_{I_k},\ldots),$$

Now for any given $M \in \mathbb{N}$, we now consider block I_{M+1}, which has first term set to 0 and remaining $M+1$ terms set to 1:

$$I_{M+1} = (0,\underbrace{1,1,1,\ldots,1}_{(M+1)\text{ terms}})$$

We can therefore see that, for any M, the existence of $M+1$ terms in block I_{M+1}, consisting of all 1s and not a single 0, means that (x_n) cannot be zero-heavy.

In terms of the elements of the sequence (x_n), for any given $M \in \mathbb{N}$, we can construct a subsequence consisting of all terms x_l where $N_0 \le l \le N_0 + M$ as follows,

$$(x_l) = (x_{N_0},\ldots,x_{N_0+M}), \quad N_0 = \sum_{i=1}^{M+1} i.$$

Because this subsequence (x_l) will not contain a single 0, only an $M+1$ number of 1s, our conclusion follows.

Problem set 2.3.

Problem set 2.4.

Problem set 2.5.

Problem set 2.6

Problem set 2.7

Project section 2.8