

Understanding Analysis (2nd ed.) by Stephen Abbott.

My write-up for exercises in *Chapter 2: Sequences and Series*.

17th July 2021.

The following are my write-ups for the exercises in the above textbook. The purpose of writing up my attempts at the exercises from pen and paper is to practise writing up and presenting formal proofs.

This is not intended to be a solutions manual, nor any kind of authoritative reference. There is already an official instructor's solution manual for all the exercises in the 1st edition of the book, and numerous online solutions for the additional exercises added to the 2nd edition.

Rather, committing this document to the public domain, with the possibility that it may be read, pushes me to enforce a higher standard of proof presentation than if I had confined myself to pen and paper attempts in my notebook.

I have organised the exercises into problem sets corresponding to each section in the chapter. For proofs that needed more thought, I have listed the proof strategy. For exercises that revealed a point of broader significance, I have listed this as a **Key idea** in bold.

Problem set 2.2.

2.2.1.

A sequence (x_n) converges to x if

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)(|x_n - x| < \epsilon).$$

Reversing the order of the first two quantifiers, a sequence (x_n) verconges if

$$(\exists \epsilon > 0)(\forall N \in \mathbb{N})(\forall n \geq N)(|x_n - x| < \epsilon).$$

Simplifying the definition, (x_n) verconges to x if we can find an $\epsilon > 0$ such that for all $n \geq 1$,

$$|x_n - x| < \epsilon \implies x - \epsilon < x_n < x + \epsilon \implies x_n \in V_\epsilon(x).$$

That is, a sequence (x_n) verconges if an ϵ -neighbourhood $V_\epsilon(x)$ of x can be found that contains *every term of the sequence*. In more terms of more conventional mathematical notions, a vercongent sequence (x_n) is one which is *bounded* in some open interval $(x - \epsilon, x + \epsilon)$.

An example of a vercongent sequence (x_n) would be

$$(x_n) = \left(1 - \frac{1}{n}\right) = \left(0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6} \dots\right).$$

To see how this sequence satisfies the definition note that (x_n) lies, as an example, in an open interval $(-1, 1)$. Taking $x = 0$ we have that (x_n) verconges to 0 with $\epsilon = 1$, or more generally, any $\epsilon > 1$.

An example of a vercongent sequence that is divergent would be

$$(y_n) = (0.99, 0.01, 0.99, 0.01, 0.99, 0.01, \dots).$$

This sequence satisfies the definition by noting that (y_n) lies, as an example, in an open interval $(0, 1)$. Taking $x = 1/2$ and $\epsilon \geq 1/2$ means that (x_n) verconges to $1/2$.

In order to answer whether a sequence can verconge to more than one value, it is illustrative to look at an example that diverges but does *not* verconge. Consider

$$(z_n) = ((-2)^n) = (2, -4, 8, -16, 32, -64, \dots).$$

This does not verconge because the sequence (z_n) is neither bounded above nor below.

On whether a vercongent sequence (x_n) can verconge to more than one value, this is true, but going further, a sequence (x_n) can verconge to every $x \in \mathbb{R}$.

Proof.

2.2.2.

(a)

Observe that

$$\lim_{n \rightarrow \infty} \frac{2n+1}{5n+4} = \lim_{n \rightarrow \infty} \frac{2 + (1/n)}{5 + (4/n)} = \frac{2}{5},$$

because heuristically, $(1/n) \rightarrow 0$ and $(4/n) \rightarrow 0$ at the same rate.

Proof. Consider an arbitrary $\epsilon > 0$. Because $n > 0$, we have that

$$\left| \frac{2n+1}{5n+4} - \frac{2}{5} \right| = \left| \frac{-3}{5(5n+4)} \right| = \frac{3}{5(5n+4)}.$$

Setting $N > (3 - 20\epsilon)/25\epsilon$, our proof is completed by observing that for all $n \geq N$,

$$\frac{3}{5(5n+4)} < \epsilon.$$

(b).

Observe that

$$\lim_{n \rightarrow \infty} \frac{2n^2}{n^3+3} = \lim_{n \rightarrow \infty} \frac{2}{n + (3/n^2)} = 0$$

because $(3/n^2) \rightarrow 0$ and therefore $(2/n) \rightarrow 0$.

Proof. Consider an arbitrary $\epsilon > 0$. Because $n > 0$, we have that

$$\left| \frac{2n^2}{n^3+3} \right| = \frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = \frac{2}{n}$$

Setting $N > 2/\epsilon$, our proof is completed by observing that for all $n \geq N$,

$$\frac{2n^2}{n^3 + 3} < \epsilon.$$

Key idea: For a value $\epsilon = \epsilon_0$, we do not necessarily need to analytically solve for the exact value $N_0 = N(\epsilon_0)$. In the above, that would amount to solving a cubic equation. Instead, we can supply an $N_1 > N_0$ such that the statement we wish to prove holds for all $n \geq N_1$.

Do we require the argument that $(2/n) \rightarrow 0$ here?

(c)

Proof. Because $f(n) = \sin(n)$ is such that $f : \mathbb{N} \rightarrow [-1, 1]$, and because $n > 0$, we have that

$$\left| \frac{\sin(n^2)}{n^{1/3}} \right| \leq \frac{1}{n^{1/3}}.$$

Setting $N > \epsilon^{-3}$, we conclude that for all $n \geq N$,

$$\left| \frac{\sin(n^2)}{n^{1/3}} \right| < \epsilon.$$

Do we similarly require that $(n^{-1/3}) \rightarrow 0$ here?

2.2.3

2.2.4.

(a)

Consider the following sequence

$$(x_n) = (1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \dots),$$

where each individual term x_n is given by

$$x_n = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

This sequence consists of a countably infinite number of 1s, and also a countably infinite number of 0s. It diverges because for $0 < \epsilon < 1$ it will not be possible to find an N such that $|x_n - 1| < \epsilon$ for all $n \geq N$.

Check the details of countably infinite.

(b)

I present an example in an attempt to develop a heuristic argument for why this is likely impossible. This needs development. The goal is to try and construct a sequence (x_n) that contains an infinite number of 1s, and which converges to a limit $x \neq 1$, and make arguments for why this cannot be done.

Following (a), to ensure that (x_n) contains an infinite number of 1s, we set $x_n = 1$ when n is even. We then set every $x_n = x + (1/n)$ when n is odd. That is, consider the sequence

$$(x_n) = \left(x + 1, 1, x + \frac{1}{2}, 1, x + \frac{1}{3}, 1, x + \frac{1}{4}, \dots \right).$$

This sequence diverges because for $0 < \epsilon < |1 - x|$, we cannot find an N such that $|x_n - x| < \epsilon$ for all $n \geq N$. That is, no matter how many terms we enumerate in the sequence (x_n) , there is no N after which the absolute

distance between x_n and x can be controlled to be less than the absolute distance between 1 and x , the proposed limit point.

The question then becomes whether it is possible to distribute the infinite number of 1s in such a way so as to avoid the divergence. One possible candidate might be a sequence that involves an infinite number of 1s, after which all numbers are x . However, it is likely that such a construction is not possible, as the number of 1s would then really be finite.

Based on these heuristic arguments, I would conclude that such a construction is not possible.

(c)

Consider the sequence

$$(x_n) = (0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, \dots, \underbrace{0, 1, \dots, 1}_{k\text{th block}}, \dots),$$

where $k \in \mathbb{N}$ and each k th block consists of $k + 1$ terms, with first term set to 0 and remaining k terms set to 1.

This sequence diverges, because given any proposed limit x and $0 < \epsilon < 1$, it will not be possible to find an N such that $|x_n - x| < \epsilon$ for all $n \geq N$.

Some uncertainties about divergence. In particular:

- *What kinds of divergence are there? It looks like we can have divergence to $\pm\infty$, and a case where it does not exist.*
- *The discussion in Abbott shows means of argumentation against convergence to a specific limit x . It is difficult to see how this matches up with divergence as being negation of convergence. In particular, would we need to negate the definition for all $x \in \mathbb{R}$? Scrutinise the definition again.*

2.2.5.

(a)

Enumerating terms in the sequence $(5/n)$, we have that

$$\left(\frac{5}{n}\right) = \left(5, \frac{5}{2}, \frac{5}{3}, \frac{5}{4}, 1, \frac{5}{6}, \frac{5}{7}, \dots\right).$$

This yields the sequence

$$(a_n) = \left[\left[\frac{5}{n}\right]\right] = (5, 2, 1, 1, 1, 0, 0, \dots).$$

This sequence converges to 0.

Proof. Let $\epsilon > 0$ be arbitrary. Regardless of choice of ϵ , as long as $\epsilon > 0$, when $N = 6$, we have that for all $n \geq N$,

$$|a_n| = \left|\left[\left[\frac{5}{n}\right]\right]\right| = 0 < \epsilon.$$

(b)

Enumerating terms in the sequence $((12 + 4n)/3n)$, we have that

$$\left(\frac{12 + 4n}{3n}\right) = \left(\frac{16}{3}, \frac{10}{3}, \frac{8}{3}, \frac{7}{3}, \frac{32}{15}, 2, \frac{40}{21}, \dots\right).$$

Observe that

$$\lim_{n \rightarrow \infty} \frac{12 + 4n}{3n} = \lim_{n \rightarrow \infty} \frac{(12/n) + 4}{3} = \frac{4}{3}.$$

This yields the sequence

$$(a_n) = (5, 3, 2, 2, 2, 2, 1, \dots)$$

This converges to 1.

Proof. Let $\epsilon > 0$ be arbitrary. Regardless of choice of ϵ , as long as $\epsilon > 0$, when $N = 7$, we have that for all $n \geq N$,

$$|a_n - 1| = \left| \left\lfloor \left\lceil \frac{12 + 4n}{3n} \right\rceil \right\rfloor - 1 \right| = 0 < \epsilon.$$

Concerning the comment that “the smaller the ϵ -neighbourhood, the larger N may have to be,” the above sequences, which not only converge to, but also reach their respective limit points, are examples where it is not necessarily the case that N must increase as the ϵ -neighbourhood gets smaller.

Key idea: While the choice of N in response to an $\epsilon > 0$ will often depend on ϵ , so that we have $N = N(\epsilon)$, this exercise shows that is not always the case.

2.2.6.

2.2.7.

(a)

Enumerating the sequence, we find that it is divergent.

$$(-1)^n = (-1, 1, -1, 1, -1, \dots).$$

The sequence is *not eventually* in $A = \{1\}$. That is because no N exists such that $a_n \in A$ for all $n \geq N$.

The sequence is *frequently* in $A = \{1\}$. That is because for every N , there exists not only one $a_n \in A$ for all $n \geq N$, but a countably infinite number of $a_n \in A$, occurring when n is even.

(b)

The condition of a sequence being *eventually* in A is stronger than that of a sequence being *frequently* in A in the sense that

$$(x_n) \text{ is eventually in } A \implies (x_n) \text{ is frequently in } A,$$

but

$$\neg((x_n) \text{ is frequently in } A \implies (x_n) \text{ is eventually in } A)$$

(c)

Definition 2.2.3B.a (Convergence of a Sequence: Alternate Topological Version).

A sequence (a_n) converges to a , if for all $\epsilon > 0$, the sequence (a_n) is *eventually* in the ϵ -neighbourhood $V_\epsilon(a)$ of a .

(d)

Consider the following divergent sequence, which contains a countably infinite number of 2s:

$$(x_n) = (0, 2, 0, 2, 0, 2, 0, 2, \dots).$$

Using the previous definition, this sequence is *not eventually* in $A = (1.9, 2.1)$ because it diverges. Because any sequence with a countably infinite number of 2s can be constructed in such a way so as to diverge, the crucial determinant of whether a sequence is eventually in a set A is whether it converges, due to equivalence of definition.

However, the sequence (x_n) is *frequently* in $A = (1.9, 2.1)$. That is because for every N , the corresponding subsequence (a_n) for all $n \geq N$ contains not just one, but a countably infinite number of terms $a_n = 2 \in (1.9, 2.1)$, which occur when n is even.

2.2.8.

(a)

The sequence of interest is

$$(x_n) = (0, 1, 0, 1, 0, 1, 0, 1, \dots).$$

Symbolically, a sequence (x_n) is *zero-heavy* if

$$(\exists M \in \mathbb{N})(\forall N \in \mathbb{N})(\exists n \in \mathbb{N}, N \leq n \leq N + M)(x_n = 0)$$

Parsing this in words, a sequence (x_n) is zero-heavy if every periodic interval consisting of $M + 1$ terms contains at least one zero.

The above sequence is zero-heavy in a way that is stronger than the definition. Not only can we find a single value of M such that every periodic interval of $(M + 1)$ terms contains at least one 0. Rather, it is the case that *for every* value $M \in \mathbb{N}$, every periodic interval of $(M + 1)$ terms contains *at least an M number* of 0s.

(b)

We will prove that the following is true:

$$(x_n) \text{ is zero-heavy} \implies (x_n) \text{ contains a countably infinite number of 0s.}$$

Proof. For every natural number $N \in \mathbb{N}$, consider the corresponding subsequence I_N consisting of $(M + 1)$ terms of the sequence (x_n) ,

$$I_N = (x_N, x_{N+1}, \dots, x_{N+M}).$$

By definition of (x_n) being zero-heavy, we know that for at least one $M \in \mathbb{N}$, every subsequence I_1, I_2, I_3, \dots will contain at least one 0. By virtue of each subsequence I_N being indexed to a natural $N \in \mathbb{N}$, there are a countably infinite number of subsequences I_N containing at least one 0. Hence we conclude (x_n) contains a countably infinite number of 0s.

(c)

By constructing a counter-example, we will show that the following is true:

$$\neg((x_n) \text{ contains a countably infinite number of 0s}) \implies (x_n) \text{ is zero-heavy.}$$

Our counter-example will consist of specifying a sequence (x_n) containing a countably infinite number of 0s, such that for every value of $M \in \mathbb{N}$, we can specify an algorithm for finding a periodic subsequence of $(M + 1)$ terms that *does not contain a single 0*.

Counter-example. Consider the sequence

$$(x_n) = (0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 0, \dots, \underbrace{0, 1, \dots, 1}_{k\text{th block}}, \dots),$$

where $k \in \mathbb{N}$ and each k th block consists of $k + 1$ terms, with first term set to 0 and remaining k terms set to 1.

This sequence contains a countably infinite number of 0s because we can view the sequence (x_n) as a concatenation of a countably infinite number of blocks I_k , each containing a single 0 (and also a k number of 1s). That is,

$$(x_n) = (\underbrace{0, 1}_{I_1}, \underbrace{0, 1, 1}_{I_2}, \underbrace{0, 1, 1, 1}_{I_3}, \underbrace{0, 1, 1, 1, 1}_{I_4}, 0, \dots, \underbrace{0, 1, \dots, 1}_{I_k}, \dots),$$

Now for any given $M \in \mathbb{N}$, we now consider block I_{M+1} , which has first term set to 0 and remaining $M + 1$ terms set to 1:

$$I_{M+1} = (0, \underbrace{1, 1, \dots, 1}_{(M+1) \text{ terms}})$$

We can therefore see that, for any M , the existence of $M + 1$ terms in block I_{M+1} , consisting of all 1s and not a single 0, means that (x_n) cannot be zero-heavy.

In terms of the elements of the sequence (x_n) , for any given $M \in \mathbb{N}$, we can construct a subsequence consisting of all terms x_l where $N_0 \leq l \leq N_0 + M$ as follows,

$$(x_l) = (x_{N_0}, \dots, x_{N_0+M}), \quad N_0 = 1 + \sum_{i=1}^{M+1} i.$$

Because this subsequence (x_l) will not contain a single 0, only an $M + 1$ number of 1s, our conclusion follows.

(d)

There seem to be multiple ways of stating that a sequence (x_n) is *not zero-heavy* by negating the definition. Some of these negations will be more literal in order to get a better feel for how negating nested sequences of qualifiers works.

- i) A sequence (x_n) is not zero-heavy if there does not exist an $M \in \mathbb{N}$ such that for all $N \in \mathbb{N}$, there exists an n satisfying $N \leq n \leq N + M$ where $x_n = 0$. Symbolically, we have that

$$(\nexists M \in \mathbb{N})(\forall N \in \mathbb{N})(\exists n \in \mathbb{N}, N \leq n \leq N + M)(x_n = 0).$$

- ii) A sequence (x_n) is not zero-heavy if for all $M \in \mathbb{N}$, there exists an $N \in \mathbb{N}$ such that no n exists satisfying $N \leq n \leq N + M$ where $x_n = 0$. Symbolically,

$$(\forall M \in \mathbb{N})(\exists N \in \mathbb{N})(\nexists n \in \mathbb{N}, N \leq n \leq N + M)(x_n = 0).$$

- iii) A sequence (x_n) is not zero-heavy if for all $M \in \mathbb{N}$, there exists an $N \in \mathbb{N}$ such that satisfying $N \leq n \leq N + M$ where $x_n \neq 0$. Symbolically,

$$(\forall M \in \mathbb{N})(\exists N \in \mathbb{N})(\forall n \in \mathbb{N}, N \leq n \leq N + M)(x_n \neq 0).$$

Problem set 2.3.

2.3.1.

(a)

We will prove that when $x_n \geq 0$ for all $n \geq N$, then

$$(x_n) \longrightarrow 0 \implies (\sqrt{x_n}) \longrightarrow 0.$$

Proof. Let $\epsilon > 0$ be arbitrary. Because $(x_n) \rightarrow 0$, choose N so that for all $n \geq N$

$$|x_n| = |\sqrt{x_n}| \cdot |\sqrt{x_n}| < \epsilon^2.$$

Hence, for all $n \geq N$, we have that

$$|\sqrt{x_n}| < \epsilon.$$

(b)

We will prove that when $x_n \geq 0$ for all $n \geq N$, then

$$(x_n) \longrightarrow x \implies (\sqrt{x_n}) \longrightarrow \sqrt{x}.$$

Hint. Use the following difference of two squares identity to turn the expression into a fraction.

$$a - b = \frac{a^2 - b^2}{a + b}.$$

Proof. Let $\epsilon > 0$ be arbitrary.

2.3.2.

(a)

$$(x_n) \longrightarrow 2 \implies \left(\frac{2x_n - 1}{3} \right) \longrightarrow 1.$$

Proof. Let $\epsilon > 0$ be arbitrary. Because $(x_n) \rightarrow 2$, choose N so that for all $n \geq N$,

$$|x_n - 2| < \frac{3}{2}\epsilon.$$

We then have that for all $n \geq N$,

$$\left| \frac{2x_n - 1}{3} - 1 \right| = \left| \frac{2(x_n - 2)}{3} \right| = \frac{2}{3} |x_n - 2| < \epsilon.$$

(b)

$$(x_n) \rightarrow 2 \implies \left(\frac{1}{x_n}\right) \rightarrow \frac{1}{2}.$$

Proof. Let $\epsilon > 0$ be arbitrary. We have that

$$\left|\frac{1}{x_n} - \frac{1}{2}\right| = \left|\frac{2 - x_n}{2x_n}\right| = |x_n - 2| \cdot \frac{1}{2|x_n|}.$$

We would like to be able to make the quantity on the right hand side arbitrarily small. If we can upper bound $1/(2|x_n|)$ with some fixed constant, then we can choose to make $|x_n - 2|$ arbitrarily small in such a way as to offset the former term.

More concretely, because upper bounding $1/(2|x_n|)$ amounts to lower bounding $|x_n|$, we can use the triangle inequality to yield

$$|2| = |2 - x_n + x_n| \leq |x_n - 2| + |x_n| \implies |x_n| \geq 2 - |x_n - 2|.$$

Because $(x_n) \rightarrow 2$, we choose N so that for all $n \geq N$,

$$|x_n - 2| < \frac{4\epsilon}{1 + 2\epsilon}.$$

This means that

$$2|x_n| \geq 4 - 2|x_n - 2| > 4 - \frac{8\epsilon}{1 + 2\epsilon},$$

and so

$$\frac{1}{2|x_n|} < \frac{1 + 2\epsilon}{4}.$$

Putting this altogether, we have that

$$\begin{aligned} \left|\frac{1}{x_n} - \frac{1}{2}\right| &= |x_n - 2| \cdot \frac{1}{2|x_n|} \\ &< \frac{4\epsilon}{1 + 2\epsilon} \cdot \frac{1 + 2\epsilon}{4} \\ &= \epsilon. \end{aligned}$$

This concludes the proof.

2.3.3.

2.3.4.

(a)

By factorising the denominator, because $(a_n) \rightarrow 0$, we have

$$\lim_{n \rightarrow \infty} \frac{1 + 2a_n}{1 + 3a_n - 4a_n^2} = \lim_{n \rightarrow \infty} \frac{1 + 2a_n}{(1 - a_n)(1 + 4a_n)} = 1.$$

(b)

Simplifying, and because $(a_n) \rightarrow 0$, we have

$$\lim_{n \rightarrow \infty} \frac{(a_n + 2)^2 - 4}{a_n} = \lim_{n \rightarrow \infty} \frac{a_n(a_n + 4)}{a_n} = 4.$$

(c)

Simplifying, and because $(a_n) \rightarrow 0$, we have

$$\lim_{n \rightarrow \infty} \frac{(2/a_n) + 3}{(1/a_n) + 5} = \lim_{n \rightarrow \infty} \frac{2 + 3a_n}{1 + 5a_n} = 2.$$

2.3.5.

2.3.6.

Using the same difference of two squares decomposition in part (b) of **2.3.1** to convert the expression into a fraction, we have

$$n - \sqrt{n^2 + 2n} = \sqrt{n^2} - (\sqrt{n^2 + 2n}) = \frac{n^2 - (n^2 + 2n)}{\sqrt{n^2} + \sqrt{n^2 + 2n}}.$$

Simplifying, we have

$$\frac{-2n}{n + \sqrt{n^2 + 2n}} = \frac{-2}{1 + \sqrt{(n^2 + 2n)/n^2}} = \frac{-2}{1 + \sqrt{1 + 2(1/n)}}.$$

Because $(1/n) \rightarrow 0$, the value of the limit is computed to be

$$\lim_{n \rightarrow \infty} n - \sqrt{n^2 + 2n} = \lim_{n \rightarrow \infty} \frac{-2}{1 + \sqrt{1 + 2(1/n)}} = -1.$$

Proof. To show the existence of the limit, because $(1/n) \rightarrow 0$, the algebraic limit theorem for multiplicative constants yields

$$\lim_{n \rightarrow \infty} 2 \left(\frac{1}{n} \right) = 0.$$

The result follows.

2.3.7.

2.3.8.

(a)

We want to prove that for some polynomial $p(x)$,

$$(x_n) \longrightarrow x \implies p(x_n) \rightarrow p(x)$$

We will first prove the following lemma:

Lemma. For all natural numbers $m \in \mathbb{N}$,

$$(x_n) \longrightarrow x \implies (x_n^m) \longrightarrow x^m.$$

Proof.

Proof. Define an arbitrary polynomial $p(x)$ of degree k

$$p(x) = \sum_{i=0}^k c_i x^{k-i} = c_0 x^k + c_1 x^{k-1} + c_2 x^{k-2} + \cdots + c_k.$$

Using the triangle inequality, we have

$$\begin{aligned} |p(x_n) - p(x)| &= \left| \sum_{i=0}^k c_i x_n^{k-i} - \sum_{i=0}^k c_i x^{k-i} \right| \\ &= \left| \sum_{i=0}^{k-1} c_i (x_n^{k-i} - x^{k-i}) \right| \\ &\leq \sum_{i=0}^{k-1} |c_i| |x_n^{k-i} - x^{k-i}|. \end{aligned}$$

Using the lemma, we have that $(x_n^{k-i}) \rightarrow x^{k-i}$ for all $i = 0, \dots, k-1$. For every $i = 0, \dots, k-1$, we select a corresponding N_{k-i} such that for all $n \geq N_{k-i}$,

$$|x_n^{k-i} - x^{k-i}| < \frac{\epsilon}{k |c_i|}.$$

Setting $N = \max\{N_1, \dots, N_k\}$, we have that for all $n \geq N$,

$$\begin{aligned} |p(x_n) - p(x)| &< \sum_{i=0}^{k-1} \left(|c_i| \cdot \frac{\epsilon}{k |c_i|} \right) \\ &= \epsilon. \end{aligned}$$

This concludes the proof.

2.3.9.

2.3.10.

2.3.11.

2.3.12.

(a)

This statement is true. Assuming that $(a_n) \rightarrow a$, then we have that

$$(\forall n \in \mathbb{N})(\forall b \in B)(a_n \geq b) \implies (\forall b \in B)(a \geq b).$$

Proof. Let $s = \sup B$ and consider the following two cases:

Case 1. When $a \geq \sup B$ (or $a \geq \max B$ if B is a closed interval).

In this case, because $s = \sup B$ is the least upper bound, we have that for all $b \in B$,

$$b \leq s \leq a.$$

Hence in this case, a is an upper bound for B .

Case 2. When $a < \sup B$.

In this case, we prove that if every a_n is an upper bound for B , then a must also be an upper bound, and therefore it cannot be the case that $a < \sup B$.

Assume that $a < \sup B$. Because $s = \sup B$ is the least upper bound, it must be the case that a is not an upper bound and therefore we know that there exists some $b \in B$ such that

$$a < b.$$

Because $(a_n) \rightarrow a$, we now choose N such that for all $n \geq N$,

$$|a_n - a| < b - a \implies$$

(b)

This statement is true. Assuming that $(a_n) \rightarrow a$, then we have that

$$(\forall n \in \mathbb{N})(a_n \in (0, 1)^c) \implies a \in (0, 1)^c.$$

Note that the complement of $(0, 1)$ is $(0, 1)^c = (-\infty, 0] \cup [1, \infty)$. We prove the above by contradiction.

Proof. Assume that $a \in (0, 1)$. Construct an ϵ -neighbourhood $V_\epsilon(a)$ by setting $\epsilon < \max\{1 - a, a\}$ so that

$$V_\epsilon(a) \subset (0, 1).$$

Because $(a_n) \rightarrow a$, there exists an $N \in \mathbb{N}$ such that $a_n \in V_\epsilon(a)$ for all $n \geq N$. Together with the fact that $V_\epsilon(a) \subset (0, 1)$, it must therefore be the case that $a_n \in (0, 1)$ for all $n \geq N$. But this contradicts our assumption that no term a_n in the sequence lies in $(0, 1)$. Hence we conclude that $a \in (0, 1)^c$.

(c)

This is false. Assuming that $(a_n) \rightarrow a$, then we have that

$$\neg((\forall n \in \mathbb{N})(a_n \in \mathbb{Q}) \implies a \in \mathbb{Q}).$$

Counter-example. Define the n th term in the sequence (a_n) with the following partial sum

$$a_n = \sum_{m=0}^n \frac{1}{m!}.$$

As every a_n is the sum of fractions consisting of only integers in the numerator and denominator, we have that a_n is rational for all $n \in \mathbb{N}$. Computing the limit, we have that

$$a = \lim_{n \rightarrow \infty} a_n = \sum_{m=0}^{\infty} \frac{1}{m!} = e.$$

But Euler's number e is irrational, and our conclusion follows.

Problem set 2.4.

Problem set 2.5.

Problem set 2.6

Problem set 2.7

Project section 2.8