Understanding Analysis (2nd ed.) by Stephen Abbott.

Write-up for exercises in *Chapter 2: Sequences and Series.*
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The following are my write-ups for the exercises in the above textbook. The purpose of writing up my attempts at the exercises from pen and paper is to practise writing up and presenting formal proofs.

This is not intended to be a solutions manual, nor any kind of authoritative reference. There is already an official instructor's solution manual for all the exercises in the 1st edition of the book, and numerous online solutions for the additional exercises added to the 2nd edition.

Rather, committing this document to the public domain, with the possibility that it may be read, pushes me to enforce a higher standard of proof presentation than if I had confined myself to pen and paper attempts in my notebook.

I have organised the exercises into problem sets corresponding to each section in the chapter. For proofs that needed more thought, I have listed the proof strategy. For exercises that revealed a point of broader significance, I have listed this as a **Key idea** in bold.

Problem set 2.2.

2.2.1.

A sequence (x_n) converges to x if

$$(\forall \epsilon > 0)(\exists N \in \mathbb{N})(\forall n \geq N)(|x_n - x| < \epsilon).$$

Reversing the order of the first two quantifiers, a sequence (x_n) verconges if

$$(\exists \epsilon > 0)(\forall N \in \mathbb{N})(\forall n \geq N)(|x_n - x| < \epsilon).$$

Simplifying the definition, (x_n) verconges to x if we can find an $\epsilon > 0$ such that for all $n \geq 1$,

$$|x_n - x| < \epsilon \implies x - \epsilon < x_n < x + \epsilon \implies x_n \in V_\epsilon(x).$$

That is, a sequence (x_n) verconges if an ϵ-neighbourhood $V_\epsilon(x)$ of x can be found that contains *every term of the sequence.* In more terms of more conventional mathematical notions, a vercongent sequence (x_n) is one which is *bounded* in some open interval $(x - \epsilon, x + \epsilon)$.

An example of a vercongent sequence (x_n) would be

$$(x_n) = \left(1 - \frac{1}{n}\right) = \left(0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6} \cdots\right).$$

To see how this sequence satisfies the definition note that (x_n) lies, as an example, in an open interval $(-1, 1)$. Taking $x = 0$ we have that (x_n) verconges to 0 with $\epsilon = 1$, or more generally, any $\epsilon > 1$.

An example of a vercongent sequence that is divergent would be

$$(y_n) = (0.99, 0.01.0.99, 0.01, 0.99, 0.01, \dots).$$

This sequence satisfies the definition by noting that (y_n) lies, as an example, in an open interval $(0, 1)$. Taking $x = 1/2$ and $\epsilon \geq 1/2$ means that (x_n) verconges to $1/2$.

In order to answer whether a sequence can vercongе to more than one value, it is illustrative to look at an example that diverges but does *not* verconge. Consider

$$(z_n) = ((-2)^n) = (2, -4, 8, -16, 32, -64, \dots).$$

This does not verconge because the sequence (z_n) is neither bounded above nor below.

On whether a vercongent sequence (x_n) can verconge to more than one value, this is true, but going further, a sequence (x_n) can verconge to every $x \in \mathbb{R}$.

Proof.

2.2.2.

(a)

Observe that

$$\lim_{n\to\infty} \frac{2n+1}{5n+4} = \lim_{n\to\infty} \frac{2+(1/n)}{5+(4/n)} = \frac{2}{5},$$

because heuristically, $(1/n) \to 0$ and $(4/n) \to 0$ at the same rate.

Proof. Consider an arbitrary $\epsilon > 0$. Because $n > 0$, we have that

$$\left| \frac{2n+1}{5n+4} - \frac{2}{5} \right| = \left| \frac{-3}{5(5n+4)} \right| = \frac{3}{5(5n+4)}.$$

Setting $N > (3 - 20\epsilon)/25\epsilon$, our proof is completed by observing that for all $n \geq N$,

$$\frac{3}{5(5n+4)} < \epsilon.$$

(b).

Observe that

$$\lim_{n\to\infty} \frac{2n^2}{n^3+3} = \lim_{n\to\infty} \frac{2}{n+(3/n^2)} = 0$$

because $(3/n^2) \to 0$ and therefore $(2/n) \to 0$.

Proof. Consider an arbitrary $\epsilon > 0$. Because $n > 0$, we have that

$$\left| \frac{2n^2}{n^3+3} \right| = \frac{2n^2}{n^3+3} < \frac{2n^2}{n^3} = \frac{2}{n}$$

Setting $N > 2/\epsilon$, our proof is completed by observing that for all $n \geq N$,

$$\frac{2n^2}{n^3+3} < \epsilon.$$

Key idea: For a value $\epsilon = \epsilon_0$, we do not necessarily need to analytically solve for the exact value $N_0 = N(\epsilon_0)$. In the above, that would amount to solving a cubic equation. Instead, we can supply an $N_1 > N_0$ such that the statement we wish to prove holds for all $n \geq N_1$.

Do we require the argument that $(2/n) \to 0$ here?

(c)

Proof. Because $f(n) = \sin(n)$ is such that $f : \mathbb{N} \to [-1, 1]$, and because $n > 0$, we have that

$$\left|\frac{\sin(n^2)}{n^{1/3}}\right| \leq \frac{1}{n^{1/3}}.$$

Setting $N > \epsilon^{-3}$, we conclude that for all $n \geq N$,

$$\left|\frac{\sin(n^2)}{n^{1/3}}\right| < \epsilon.$$

Do we similarly require that $(n^{-1/3}) \to 0$ here?

2.2.3

2.2.4.

(a)

Consider the following sequence

$$(x_n) = (1, 0, 1, 0, 1, 0, 1, 0, 1, 0, \dots),$$

where each individual term x_n is given by

$$x_n = \begin{cases} 1 & \text{if} \quad n \text{ is odd,} \\ 0 & \text{if} \quad n \text{ is even.} \end{cases}$$

This sequence consists of a countably infinite number of 1s, and also a countably infinite number of 0s. It diverges because for $0 < \epsilon < 1$ it will not be possible to find an N such that $|x_n - 1| < \epsilon$ for all $n \geq N$.

Check the details of countably infinite.

(b)

I present an example in an attempt to develop a heuristic argument for why this is likely impossible. This needs development. The goal is to try and construct a sequence (x_n) that contains an infinite number of 1s, and which converges to a limit $x \neq 1$, and make arguments for why this cannot be done.

Following (a), to ensure that (x_n) contains an infinite number of 1s, we set $x_n = 1$ when n is even. We then set every $x_n = x + (1/n)$ when n is odd. That is, consider the sequence

$$(x_n) = \left(x + 1, 1, x + \frac{1}{2}, 1, x + \frac{1}{3}, 1, x + \frac{1}{4}, \dots\right).$$

This sequence diverges because for $0 < \epsilon < |1 - x|$, we cannot find an N such that $|x_n - x| < \epsilon$ for all $n \geq N$. That is, no matter how many terms we enumerate in the sequence (x_n), there is no N after which the absolute distance between x_n and x can be controlled to be less than the absolute distance between 1 and x, the proposed limit point.

The question then becomes whether it is possible to distribute the infinite number of 1s in such a way so as to avoid the divergence. One possible candidate might be a sequence that involves an infinite number of 1s, after which all numbers are x. However, it is likely that such a construction is not possible, as the number of 1s would then really be finite.

Based on these heuristic arguments, I would conclude that such a construction is not possible.

(c)

Consider the sequence

$$(x_n) = (0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, \dots, \underbrace{0, 1, \dots, 1}_{k\text{th block}}, \dots),$$

where $k \in \mathbb{N}$ and each kth block consists of $k+1$ terms, with first term set to 0 and remaining k terms set to 1.

This sequence diverges, because given any proposed limit x and $0 < \epsilon < 1$, it will not be possible to find an N such that $|x_n - x| < \epsilon$ for all $n \geq N$.

Some uncertainties about divergence. In particular:

- *What kinds of divergence are there? It looks like we can have divergence to $\pm\infty$, and a case where it does not exist.*
- *The discussion in Abbott shows means of argumentation against convergence to a specific limit x. It is difficult to see how this matches up with divergence as being negation of convergence. In particular, would we need to negate the defintion for all $x \in \mathbb{R}$? Scrutinise the definition again.*

2.2.5.

(a)

Enumerating terms in the sequence $(5/n)$, we have that

$$\left(\frac{5}{n}\right) = \left(5, \frac{5}{2}, \frac{5}{3}, \frac{5}{4}, 1, \frac{5}{6}, \frac{5}{7}, \dots\right).$$

This yields the sequence

$$(a_n) = \left[\left[\frac{5}{n}\right]\right] = (5, 2, 1, 1, 1, 0, 0, \dots).$$

This sequence converges to 0.

Proof. Let $\epsilon > 0$ be arbitrary. Regardless of choice of ϵ, as long as $\epsilon > 0$, when $N = 6$, we have that for all $n \geq N$,

$$|a_n| = \left|\left[\left[\frac{5}{n}\right]\right]\right| = 0 < \epsilon.$$

(b)

Enumerating terms in the sequence $((12 + 4n)/3n)$, we have that

$$\left(\frac{12 + 4n}{3n}\right) = \left(\frac{16}{3}, \frac{10}{3}, \frac{8}{3}, \frac{7}{3}, \frac{32}{15}, 2, \frac{40}{21}, \dots\right).$$

Observe that

$$\lim_{n\to\infty} \frac{12+4n}{3n} = \lim_{n\to\infty} \frac{(12/n)+4}{3} = \frac{4}{3}.$$

This yields the sequence

$$(a_n) = (5, 3, 2, 2, 2, 2, 1, \dots)$$

This converges to 1.

Proof. Let $\epsilon > 0$ be arbitrary. Regardless of choice of ϵ, as long as $\epsilon > 0$, when $N = 7$, we have that for all $n \geq N$,

$$|a_n - 1| = \left| \left[\left[\frac{12+4n}{3n} \right]\right] - 1 \right| = 0 < \epsilon.$$

Concerning the comment that "the smaller the ϵ-neighbourhood, the larger N may have to be," the above sequences, which not only converge to, but also reach their respective limit points, are examples where it is not necessarily the case that N must increase as the ϵ-neighbourhood getss smaller.

Key idea:. While the choice of N in response to an $\epsilon > 0$ will often depend on ϵ, so that we have $N = N(\epsilon)$, this exercise shows that is not always the case.

2.2.6.

2.2.7.

(a)

Enumerating the sequence, we find that it is divergent.

$$(-1)^n = (-1, 1, -1, 1, -1, \dots).$$

The sequence is *not eventually* in $A = \{1\}$. That is because no N exists such that $a_n \in A$ for all $n \geq N$.

The sequence is *frequently* in $A = \{1\}$. That is because for every N, there exists not only one $a_n \in A$ for all $n \geq N$, but a countably infinite number of $a_n \in A$, occurring when n is even.

(b)

The condition of a sequence being *eventually* in A is stronger than that of a sequence being *frequently* in A in the sense that

$$(x_n) \text{ is eventually in } A \implies (x_n) \text{ is frequently in } A,$$

but

$$\neg \left((x_n) \text{ is frequently in } A \implies (x_n) \text{ is eventually in } A \right)$$

(c)

Definition 2.2.3B.a (Convergence of a Sequence: Alternate Topological Version).

A sequence (a_n) converges to a, if for all $\epsilon > 0$, the sequence (a_n) is *eventually* in the ϵ-neighbourhood $V_\epsilon(a)$ of a.

(d)

Consider the following divergent sequence, which contains a countably infinite number of 2s:

$$(x_n) = (0, 2, 0, 2, 0, 2, 0, 2, \dots).$$

Using the previous definition, this sequence is *not eventually* in $A = (1.9, 2.1)$ because it diverges. Because any sequence with a countably infinite number of 2s can be constructed in such a way so as to diverge, the crucial determinant of whether a sequence is eventually in a set A is whether it converges, due to equivalence of definition.

However, the seqeuence (x_n) is *frequently* in $A = (1.9, 2.1)$. That is because for every N, the corresponding subsequence (a_n) for all $n \geq N$ contains not just one, but a countably infinite number of terms $a_n = 2 \in (1.9, 2.1)$, which occur when n is even.

2.2.8.

(a)

The sequence of interest is

$$(x_n) = (0, 1, 0, 1, 0, 1, 0, 1, \dots).$$

Symbolically, a sequence (x_n) is *zero-heavy* if

$$(\exists M \in \mathbb{N})(\forall N \in \mathbb{N})(\exists n \in \mathbb{N}, N \leq n \leq N + M)(x_n = 0)$$

Parsing this in words, a sequence (x_n) is zero-heavy if every periodic interval consisting of $M + 1$ terms contains at least one zero.

The above sequence is zero-heavy in a way that is stronger than the definition. Not only can we find a single value of M such that every periodic interval of $(M + 1)$ terms contains at least one 0. Rather, it is the case that *for every* value $M \in \mathbb{N}$, every periodic interval of $(M + 1)$ terms contains *at least an M number* of 0s.

(b)

We will prove that the following is true:

$$(x_n) \text{ is zero-heavy} \implies (x_n) \text{ contains a countably infinite number of 0s.}$$

Proof. For every natural number $N \in \mathbb{N}$, consider the corresponding subsequence I_N consisting of $(M + 1)$ terms of the sequence (x_n),

$$I_N = (x_N, x_{N+1}, \dots, x_{N+M}).$$

By definition of (x_n) being zero-heavy, we know that for at least one $M \in \mathbb{N}$, every subsequence $I_1, I_2, I_3, \dots$ will contain at least one 0. By virtue of each subsequence I_N being indexed to a natural $N \in \mathbb{N}$, there are a countably infinite number of subsequences I_N containing at least one 0. Hence we conclude (x_n) contains a countably infinite number of 0s.

(c)

By constructing a counter-example, we will show that the following is true:

$$\neg\left((x_n) \text{ contains a countably infinite number of 0s} \implies (x_n) \text{ is zero-heavy.}\right)$$

Our counter-example will consist of specifying a sequence (x_n) containing a countably infinite number of 0s, such that for every value of $M \in \mathbb{N}$, we can specify an algorithm for finding a periodic subsequence of $(M+1)$ terms that *does not contain a single* 0.

Counter-example. Consider the sequence

$$(x_n) = (0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, \dots, \underbrace{0, 1, \dots, 1}_{k\text{th block}}, \dots),$$

where $k \in \mathbb{N}$ and each kth block consists of $k+1$ terms, with first term set to 0 and remaining k terms set to 1.

This sequence contains a countably infinite number of 0s because we can view the sequence (x_n) as a concatenation of a countably infinite number of blocks I_k, each containing a single 0 (and also a k number of 1s). That is,

$$(x_n) = (\underbrace{0, 1}_{I_1}, \underbrace{0, 1, 1}_{I_2}, \underbrace{0, 1, 1, 1}_{I_3}, \underbrace{0, 1, 1, 1, 1}_{I_4}, 0, \dots, \underbrace{0, 1, \dots, 1}_{I_k}, \dots),$$

Now for any given $M \in \mathbb{N}$, we now consider block I_{M+1}, which has first term set to 0 and remaining $M+1$ terms set to 1:

$$I_{M+1} = (0, \underbrace{1, 1, 1, \dots, 1}_{(M+1)\text{ terms}})$$

We can therefore see that, for any M, the existence of $M+1$ terms in block I_{M+1}, consisting of all 1s and not a single 0, means that (x_n) cannot be zero-heavy.

In terms of the elements of the sequence (x_n), for any given $M \in \mathbb{N}$, we can construct a subsequence consisting of all terms x_l where $N_0 \leq l \leq N_0 + M$ as follows,

$$(x_l) = (x_{N_0}, \dots, x_{N_0+M}), \quad N_0 = 1 + \sum_{i=1}^{M+1} i.$$

Because this subsequence (x_l) will not contain a single 0, only an $M+1$ number of 1s, our conclusion follows.

(d)

There seem to be multiple ways of stating that a sequence (x_n) is *not zero-heavy* by negating the definition. Some of these negations will be more literal in order to get a better feel for how negating nested sequences of qualifiers works.

i) A sequence (x_n) is not zero-heavy if there does not exist an $M \in \mathbb{N}$ such that for all $N \in \mathbb{N}$, there exists an n satisfying $N \leq n \leq N + M$ where $x_n = 0$. Symbolically, we have that

$$(\nexists M \in \mathbb{N})(\forall N \in \mathbb{N})(\exists n \in \mathbb{N}, N \leq n \leq N + M)(x_n = 0).$$

ii) A sequence (x_n) is not zero-heavy if for all $M \in \mathbb{N}$, there exists an $N \in \mathbb{N}$ such that no n exists satisfying $N \leq n \leq N + M$ where $x_n = 0$. Symbolically,

$$(\forall M \in \mathbb{N})(\exists N \in \mathbb{N})(\nexists n \in \mathbb{N}, N \leq n \leq N + M)(x_n = 0).$$

iii) A sequence (x_n) is not zero-heavy if for all $M \in \mathbb{N}$, there exists an $N \in \mathbb{N}$ such that satisfying $N \leq n \leq N + M$ where $x_n \neq 0$. Symbolically,

$$(\forall M \in \mathbb{N})(\exists N \in \mathbb{N})(\forall n \in \mathbb{N}, N \leq n \leq N+M)(x_n \neq 0).$$

Problem set 2.3.

2.3.1.

(a)

We will prove that when $x_n \geq 0$ for all $n \geq N$, then

$$(x_n) \longrightarrow 0 \implies (\sqrt{x_n}) \longrightarrow 0.$$

Proof. Let $\epsilon > 0$ be arbitrary. Because $(x_n) \to 0$, choose N so that for all $n \geq N$

$$|x_n| = |\sqrt{x_n}| \cdot |\sqrt{x_n}| < \epsilon^2.$$

Hence, for all $n \geq N$, we have that

$$|\sqrt{x_n}| < \epsilon.$$

(b)

We will prove that when $x_n \geq 0$ for all $n \geq N$, then

$$(x_n) \longrightarrow x \implies (\sqrt{x_n}) \longrightarrow \sqrt{x}.$$

Hint. Use the following difference of two squares identity to turn the expression into a fraction.

$$a - b = \frac{a^2 - b^2}{a+b}.$$

Proof. Let $\epsilon > 0$ be arbitrary. Using the above identity, we have that

$$|\sqrt{x_n} - \sqrt{x}| = \left| \frac{x_n - x}{\sqrt{x}_n + \sqrt{x}} \right| = |x_n - x| \cdot \frac{1}{|\sqrt{x_n} + \sqrt{x}|}.$$

We want to ensure that the product of terms on the right hand side is arbitrarily small. We can do this by making $|x_n - x|$ arbitrarily small and upper bounding $1/|\sqrt{x_n} + \sqrt{x}|$.

We can upper bound $1/|\sqrt{x_n} + \sqrt{x}|$ by lower bounding the denominator $|\sqrt{x}_n + \sqrt{x}|$. Because all terms x_n are non-negative, the value of the limit x is also non-negative by the order limit theorem. Hence $\sqrt{x}_n$ and $\sqrt{x}$ are well-defined in $\mathbb{R}$. Noting that $\sqrt{\cdot}$ must be the *positive* square root to be a well-defined function, we have that

$$|\sqrt{x_n} + \sqrt{x}| \geq |\sqrt{x}|.$$

We now choose $N \in \mathbb{N}$ so that for all $n \geq N$

$$|x_n - x| < \epsilon\,|\sqrt{x}|.$$

Hence, we have

$$\begin{aligned}
|\sqrt{x_n} - \sqrt{x}| &= |x_n - x| \cdot \frac{1}{|\sqrt{x_n} + \sqrt{x}|} \\
&\leq |x_n - x| \cdot \frac{1}{|\sqrt{x}|} \\
&< \epsilon\,|\sqrt{x}| \cdot \frac{1}{|\sqrt{x}|} \\
&= \epsilon.
\end{aligned}$$

This concludes the proof.

2.3.2.

(a)

$$(x_n) \longrightarrow 2 \implies \left(\frac{2x_n - 1}{3}\right) \longrightarrow 1.$$

Proof. Let $\epsilon > 0$ be arbitrary. Because $(x_n) \to 2$, choose N so that for all $n \geq N$,

$$|x_n - 2| < \frac{3}{2}\epsilon.$$

We then have that for all $n \geq N$,

$$\left|\frac{2x_n - 1}{3} - 1\right| = \left|\frac{2(x_n - 2)}{3}\right| = \frac{2}{3}\,|x_n - 2| < \epsilon.$$

(b)

$$(x_n) \longrightarrow 2 \implies \left(\frac{1}{x_n}\right) \longrightarrow \frac{1}{2}.$$

Proof. Let $\epsilon > 0$ be arbitrary. We have that

$$\left|\frac{1}{x_n} - \frac{1}{2}\right| = \left|\frac{2 - x_n}{2x_n}\right| = |x_n - 2| \cdot \frac{1}{2|x_n|}.$$

We would like to be able to make the quantity on the right hand side arbitrarily small. If we can upper bound $1/(2|x_n|)$ with some fixed constant, then we can choose to make $|x_n - 2|$ arbitrarily small in such a way as to offset the former term.

More concretely, because upper bounding $1/(2|x_n|)$ amounts to lower bounding $|x_n|$, we can use the triangle inequality to yield

$$|2| = |2 - x_n + x_n| \leq |x_n - 2| + |x_n| \implies |x_n| \geq 2 - |x_n - 2|.$$

Because $(x_n) \to 2$, we choose N so that for all $n \geq N$,

$$|x_n - 2| < \frac{4\epsilon}{1 + 2\epsilon}.$$

This means that

$$2|x_n| \geq 4 - 2\,|x_n - 2| > 4 - \frac{8\epsilon}{1+2\epsilon},$$

and so

$$\frac{1}{2|x_n|} < \frac{1+2\epsilon}{4}.$$

Putting this together, we have that

$$\begin{aligned} \left|\frac{1}{x_n} - \frac{1}{2}\right| &= |x_n - 2| \cdot \frac{1}{2|x_n|} \\ &< \frac{4\epsilon}{1+2\epsilon} \cdot \frac{1+2\epsilon}{4} \\ &= \epsilon. \end{aligned}$$

This concludes the proof.

2.3.3.

2.3.4.

(a)

By factorising the denominator, because $(a_n) \to 0$, we have

$$\lim_{n\to\infty} \frac{1+2a_n}{1+3a_n-4a_n^2} = \lim_{n\to} \frac{1+2a_n}{(1-a_n)(1+4a_n)} = 1.$$

(b)

Simplifying, and because $(a_n) \to 0$, we have

$$\lim_{n\to\infty} \frac{(a_n+2)^2-4}{a_n} = \lim_{n\to\infty} \frac{a_n(a_n+4)}{a_n} = 4.$$

(c)

Simplifying, and because $(a_n) \to 0$, we have

$$\lim_{n\to\infty} \frac{(2/a_n)+3}{(1/a_n)+5} = \lim_{n\to\infty} \frac{2+3a_n}{1+5a_n} = 2.$$

2.3.5.

2.3.6.

Using the same difference of two squares decomposition in part **(b)** of **2.3.1** to convert the expression into a fraction, we have

$$n - \sqrt{n^2+2n} = \sqrt{n^2} - (\sqrt{n^2+2n} = \frac{n^2-(n^2+2n)}{\sqrt{n^2}+\sqrt{n^2+2n}}.$$

Simplifying, we have

$$\frac{-2n}{n+\sqrt{n^2+2n}} = \frac{-2}{1+\sqrt{(n^2+2n)/n^2}} = \frac{-2}{1+\sqrt{1+2(1/n)}}.$$

Because $(1/n) \to 0$, the value of the limit is computed to be

$$\lim_{n\to\infty} n - \sqrt{n^2+2n} = \lim_{n\to\infty} \frac{-2}{1+\sqrt{1+2(1/n)}} = -1.$$

Proof. To show the existence of the limit, because $(1/n) \to 0$, the algebraic limit theorem for multiplicative constants yields

$$\lim_{n\to\infty} 2\left(\frac{1}{n}\right) = 0.$$

The result follows.

2.3.7.

2.3.8.

(a)

We want to prove that for some polynomial $p(x)$,

$$(x_n) \longrightarrow x \implies p(x_n) \to p(x)$$

We will first prove the following lemma:

Lemma. For all natural numbers $m \in \mathbb{N}$,

$$(x_n) \longrightarrow x \implies (x_n^m) \longrightarrow x^m.$$

Proof.

Proof. Define an arbitrary polynomial $p(x)$ of degree k

$$p(x) = \sum_{i=0}^{k} c_i x^{k-i} = c_0 x^k + c_1 x^{k-1} + c_2 x^{k-2} + \cdots + c_k.$$

Using the triangle inequality, we have

$$\begin{aligned}
|p(x_n) - p(x)| &= \left| \sum_{i=0}^{k} c_i x_n^{k-i} - \sum_{i=0}^{k} c_i x^{k-i} \right| \\
&= \left| \sum_{i=0}^{k-1} c_i \left(x_n^k - x^k \right) \right| \\
&\leq \sum_{i=0}^{k-1} |c_i|\, |x_n^{k-i} - x^{k-i}|.
\end{aligned}$$

Using the lemma, we have that $(x_n^{k-i}) \to x^{k-i}$ for all $i = 0, \ldots k-1$. For every $i = 0, \ldots, k-1$, we select a corresponding N_{k-i} such that for all $n \geq N_{k-i}$,

$$|x_n^{k-i} - x^{k-i}| < \frac{\epsilon}{k\,|c_i|}.$$

Setting $N = \max\{N_1, \ldots, N_k\}$, we have that for all $n \geq N$,

$$\begin{aligned} |p(x_n) - p(x)| &< \sum_{i=0}^{k-i} \left(|c_i| \cdot \frac{\epsilon}{k\,|c_i|} \right) \\ &= \epsilon. \end{aligned}$$

This concludes the proof.

2.3.9.

2.3.10.

2.3.11.

2.3.12.

(a)

This statement is true. Assuming that $(a_n) \to a$, then we have that

$$(\forall n \in \mathbb{N})(\forall b \in B)(a_n \geq b) \implies (\forall b \in B)(a \geq b).$$

Proof. Let $s = \sup B$ and consider the following two cases:

Case 1. When $a \geq \sup B$ (or $a \geq \max B$ if B is a closed interval).

In this case, because $s = \sup B$ is the least upper bound, we have that for all $b \in B$,

$$b \leq s \leq a.$$

Hence in this case, a is an upper bound for B.

Case 2. When $a < \sup B$.

In this case, we prove that if every a_n is an upper bound for B, then a must also an upper bound, and therefore it cannot be the case that $a < \sup B$.

Assume that $a < \sup B$. Because $s = \sup B$ is the least upper bound, it must be the case that a is not an upper bound and therefore we know that there exists some $b \in B$ such that

$$a < b.$$

Because $(a_n) \to a$, we now choose N such that for all $n \geq N$,

$$|a_n - a| < b - a \implies$$

(b)

This statement is true. Assuming that $(a_n) \to a$, then we have that

$$(\forall n \in \mathbb{N})(a_n \in (0,1)^c) \implies a \in (0,1)^c.$$

Note that the complement of $(0,1)$ is $(0,1)^c = (-\infty, 0] \cup [1, \infty)$. We prove the above by contradiction.

Proof. Assume that $a \in (0,1)$. Construct an ϵ-neighbourhood $V_\epsilon(a)$ by setting $\epsilon < \max\{1-a, a\}$ so that

$$V_\epsilon(a) \subset (0,1).$$

Because $(a_n) \to a$, there exists an $N \in \mathbb{N}$ such that $a_n \in V_\epsilon(a)$ for all $n \geq N$. Together with the fact that $V_\epsilon(a) \subset (0,1)$, it must therefore be the case that $a_n \in (0,1)$ for all $n \geq N$. But this contradicts our assumption that no term a_n in the sequence lies in $(0,1)$. Hence we conclude that $a \in (0,1)^c$.

(c)

This is false. Assuming that $(a_n) \to a$, then we have that

$$\neg \left((\forall n \in \mathbb{N})(a_n \in \mathbb{Q}) \implies a \in \mathbb{Q} \right).$$

Counter-example. Define the nth term in the sequence (a_n) with the following partial sum

$$a_n = \sum_{m=0}^{n} \frac{1}{m!}.$$

As every a_n is the sum of fractions consisting of only integers in the numerator and denominator, we have that a_n is rational for all $n \in \mathbb{N}$. Computing the limit, we have that

$$a = \lim_{n\to\infty} a_n = \sum_{m=0}^{\infty} \frac{1}{m!} = e.$$

But Euler's number e is irrational, and our conclusion follows.

Problem set 2.4.

2.4.1.

(a)

We want to prove convergence of the sequence defined by the recursive relation

$$x_1 = 3, \quad x_{n+1} = \frac{1}{4 - x_n}.$$

Proof. We first prove by induction that (x_n) is decreasing. The base case is clearly satisfied with $x_1 \geq x_2$. Assuming that $x_k \geq x_{k+1}$ for some $k \in \mathbb{N}$, we have that

$$x_k \geq x_{k+1} \implies \frac{1}{4 - x_k} \geq \frac{1}{4 - x_{k+1}} \implies x_{k+1} \geq x_{k+2}.$$

Hence (x_n) is decreasing, and together with the fact that $x_1 \leq 3$, we have that the sequence is bounded with $|x_n| \leq 3$ for all $n \in \mathbb{N}$. Applying the monotone convergence theorem, our conclusion follows.

(b)

For clarity, the sequence (x_{n+1}) for $n \in \mathbb{N}$ is a proper subsequence of (x_n). That is

$$(x_{n+1}) = (x_2, x_3, x_4, \dots).$$

If we know that the limit of the original sequence $\lim_{n\to\infty} x_n$ exists, then the limit of the proper subsequence $\lim_{n\to\infty} x_{n+1}$ exists. As an informal justification, this is because (x_{n+1}) is a proper subsequence of (x_n), and more specifically, the former is the latter with its first term x_1 deleted. Limits of a convergent sequence are governed by asymptotic tail behaviour i.e. when n is sufficiently large, rather than behaviour when n is small.

(c)

Define $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = x$. Beginning with computing the limits of both sides of the recursive relation, we have

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{4 - x_n} \implies x = \frac{1}{4 - x}.$$

Using the algebraic limit theorem on the right hand side and rearranging yields the quadratic equation

$$x^2 - 4x + 1 = 0.$$

Solving the above, the sequence converges to

$$x = 2 - \sqrt{3}.$$

We discard the root $x = 2 + \sqrt{3} > 3$ because we proved earlier that the sequence is bounded, that is $x_n \leq 3$ for all $n \in \mathbb{N}$.

2.4.2.

(a)

We enumerate the recursive sequence defined by

$$y_1, \quad y_{n+1} = 3 - y_n.$$

This gives

$$(y_n) = (1, 2, 1, 2 \dots).$$

(y_n) alternates between 1 and 2 and diverges. This means that (y_n) does not have a limit, and consequently, (y_n) and (y_{n+1}) cannot be said to tend to the same limit. For these reasons, the argument in the question is invalidated.

(b)

The strategy employed in part **(a)** of **2.4.2.** can be employed for the recursive sequence defined by

$$y_1 = 1, \quad y_{n+1} = 3 - \frac{1}{y_n}.$$

That is because this sequence (y_n) converges. We will first prove convergence then show the limit can be computed using the strategy outlined.

Proof.

2.4.3.

(a)

$$(x_n) = \left(\sqrt{2}, \sqrt{2 + \sqrt{2}}, \sqrt{2 + \sqrt{2 + \sqrt{2}}}, \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}} \right)$$

Notice that the sequence can be defined recursively as

$$x_1 = \sqrt{2}, \quad x_{n+1} = \sqrt{2 + x_n}.$$

We will prove that (x_n) converges to some limit.

Proof. We use induction to prove that the sequence is increasing. For the base case we have $x_1 = \sqrt{2}$ and $x_2 = \sqrt{2+\sqrt{2}}$ and so clearly $x_1 \leq x_2$. Assuming that $x_k \leq x_{k+1}$ for some $k \in \mathbb{N}$, we have

$$x_k \leq x_{k+1} \implies \sqrt{2 + x_k} \leq \sqrt{2 + x_{k+1}} \implies x_{k+1} \leq x_{k+2}.$$

Therefore (x_n) is increasing.

We now use induction to prove that the sequence is bounded above, specifically, that $x_n \leq 2$ for all $n \in \mathbb{N}$. The base case is satisfied because $x_1 = \sqrt{2} \leq 2$. Assuming that $x_k \leq 2$ for some $k \in \mathbb{N}$, we have that

$$x_k \leq 2 \implies x_{k+1} = \sqrt{2 + x_k} \leq 2.$$

Therefore the sequence (x_n) is bounded above. By the monotone convergence theorem we have that $(x_n) \to x$ where $x = \sup\{x_n : n \in \mathbb{N}\}$. $\square$

Because (x_n) converges, (x_n) and (x_{n+1}) converge to the same limit, so define $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = x$. Taking limits of the both sides of the recursion we have

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2 + x_n} \implies x = \sqrt{2 + x}.$$

Which yields the quadratic equation

$$x^2 - x - 2 = (x-2)(x+1) = 0,$$

with roots

$$x = 2, \quad x = -1.$$

How to discard the negative root?

(b)

$$(x_n) = \left(\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}\right).$$

Enumerating the above sequence gives

$$(x_n) = \left(2^{1/2}, 2^{3/4}, 2^{7/8}, 2^{15/16}, 2^{31/32}, \dots\right).$$

Hence (x_n) can be defined recursively as

$$x_1 = \sqrt{2}, \quad x_{n+1} = \sqrt{2x_n}.$$

Or one can do better and define it in terms of n so that

$$x_n = 2^{(2^n-1)/2^n}.$$

We will now prove that (x_n) converges.

Proof. We use induction to prove that the sequence is increasing. For the base case $x_1 = 2^{1/2}$ and $x_2 = 2^{3/4}$ and so $x_1 \leq x_2$. Assuming that $x_k \leq x_{k+1}$ for some $k \in \mathbb{N}$, we have that

$$x_k \leq x_{k+1} \implies \sqrt{2x_k} \leq \sqrt{2x_{k+1}} \implies x_{k+1} \leq x_{k+1}.$$

And so (x_n) is increasing.

We now use induction to prove that (x_n) is bounded above, so that $x_n \leq 2$ for all $n \in \mathbb{N}$. The base case is satisfied because $x_1 = \sqrt{2} \leq 2$. Assuming that $x_k \leq 2$ for some $k \in \mathbb{N}$, we have that

$$x_k \leq 2 \implies x_{k+1} = \sqrt{2x_k} \leq 2.$$

And so (x_n) is bounded above. Applying the monotone convergence theorem, we have that $(x_n) \to x$ where $x = \sup\{x_n : n \in \mathbb{N}\}$.

Due to convergence of (x_n) we define $\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = x$ and compute limits on both sides of the recursion to get

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \sqrt{2x_n} \implies x = \sqrt{2x} \implies x = 2.$$

Hence $(x_n) \to 2$.

2.4.4.

2.4.5. (Calculating square roots).

(a)

We have the recursive sequence

$$x_1 = 2, \quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

We first prove that $x_n^2 \geq 2$ by induction.

Proof. The base case is satisfied because $x_1 = 4 > 2$. Assuming that $x_k^2 \geq 2$ for some $k \in \mathbb{N}$, we have that

$$x_k^2 \geq 2 \implies (x_k^2 - 2)^2 + 8x_k^2 \geq 8x_k^2 \implies x_k^2 + 4 + \frac{4}{x_k^2} \geq 8.$$

Further simplifying, we have

$$\frac{1}{4}\left(x_k^2 + 4 + \frac{4}{x_k^2}\right) \geq 2 \implies \frac{1}{2}\left(x_k + \frac{2}{x_k}\right)^2 \geq 2 \implies x_{k+1}^2 \geq 2.$$

This concludes our proof that $x_n^2 \geq 2$ for all $n \in \mathbb{N}$.

2.4.6 (Arithmetic-geometric mean).

(a)

For positive real numbers x and y, arithmetic mean $(x+y)/2$ is always greater or equal to than the geometric mean $\sqrt{xy}$ due to non-negativity of square numbers.

Proof. For positive $x, y \in \mathbb{R}$, we have that all square numbers are non-negative, and so

$$(x-y)^2 \geq 0.$$

Hence,

$$x^2 + 2xy + y^2 \geq 4xy \implies \frac{(x+y)^2}{4} \geq xy \implies \sqrt{xy} \leq \frac{x+y}{2}.$$

2.4.7 (Limit superior).

(a)

For a bounded sequence (a_n), that is, $|a_n| \leq M$ for some $M > 0$ for all $n \in \mathbb{N}$, we want to prove that $(y_n) \to y$ where

$$y_n = \sup\{a_n : k \geq n\}.$$

Proof. The proof follows from an application of the monotone convergence theorem to the sequence (y_n). This requires that every element of (y_n) exists, that (y_n) is decreasing and therefore monotone; and also bounded.

That y_n exists for all $n \in \mathbb{N}$ follows from the fact that (a_n) is bounded, and by applying the axiom of completeness to every set $\{a_k : k \geq n\}$.

That (y_n) is decreasing and monotone relies on the fact proved in part **(a)** question **1.3.11** that for non-empty bounded sets $A, B \in \mathbb{R}$, if $A \supseteq B$ then $\sup(A) \geq \sup(B)$. Applying this with $A = \{a_k : k \geq n\}$ and $B = \{a_k : k \geq n+1\}$ for all $n \in \mathbb{N}$ yields the intermediate result.

That (y_n) is bounded begins from the fact that (a_n) is bounded; that is, for some $M > 0$, $|a_n| \leq M$ for all $n \in \mathbb{N}$. Because (a_n) is bounded, we have that $y_1 = \sup\{a_k : k \geq n\} \leq M$ by definition of the least upper bound. Because (y_n) is decreasing, it follows that $|y_n| \leq M$ for all $n \in \mathbb{N}$; that is, (y_n) is bounded.

Applying the monotone convergence theorem, yields the result $(y_n) \to y$ where $y = \inf\{y_n : n \in \mathbb{N}\}$. That is,

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \sup\{a_k : k \geq n\} = \inf\{\sup\{a_k : k \geq n\} : n \in \mathbb{N}\}.$$

Justification of existence of y seems not quite there.

(b)

Beginning with a formal definition of limit superior based on part **(a)** above, we have the following.

Definition. (Limit superior). The *limit superior* of a sequence (a_n) is defined by

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} y_n = \inf\{\sup\{a_k : k \geq n\} : n \in \mathbb{N}\}.$$

Where $(y_n) = \sup\{a_k : k \geq n\}$.

Using an analogous proof strategy applied to the sequence (x_n) defined by $x_n = \inf\{a_k : k \geq n\}$ for the bounded sequence (a_n) suggests the following definition of the limit inferior.

Definition. (Limit inferior). The *limit inferior* of a sequence (a_n) is defined by

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} x_n = \sup\{\inf\{a_k : k \geq n\} : n \in \mathbb{N}\}.$$

Where $(x_n) = \inf\{a_k : k \geq n\}$.

The $\liminf_{n\to\infty} a_n$ always exists for a bounded sequence (a_n). Because the entire sequence (a_n) is bounded, the sets $\{a_k : k \geq n\}$ for all $n \in \mathbb{N}$ are all non-empty and bounded. The axiom of completeness therefore guarantees that every term of (x_n) exists. Using similar arguments...

Justification of existence of x seems not quite there.

(c)

(d)

2.4.8.

(a)

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1.$$

Evaluating the expression for the partial sum makes use of the general identity for the difference of mth powers

$$a^m - b^m = (a-b)(a^{m-1} + a^{m-2}b + \cdots + ab^{m-2} + b^{m-1}).$$

To get the mth partial sum s_m, we have

$$\begin{aligned} s_m = \sum_{n=1}^{m} \frac{1}{2^n} &= \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{m-1}} + \frac{1}{2^m} \\ &= \frac{2^{m-1} + 2^{m-2} + \cdots + 2 + 1}{2^m} \\ &= \frac{(2-1)(2^{m-1} + \cdots 2^{m-2} + \cdots + 2^1 + 2^0)}{2^m} \\ &= \frac{2^m - 1}{2^m} \\ &= 1 - \frac{1}{2^m}. \end{aligned}$$

From above, we can see that $(s_m) \to 1$. We now prove this.

Proof. Let $\epsilon > 0$ be arbitrary. We have that because $m > 0$,

$$\left| \frac{2^m - 1}{2^m} - 1 \right| = \frac{1}{2^m}.$$

Choosing $N > \log_2(1/\epsilon)$ means that for all $m \geq N$,

$$\frac{1}{2^m} < \epsilon.$$

This concludes the proof and the result follows.

(b)

(c)

$$\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) \text{ diverges.}$$

Evaluating the mth partial sum, we have that

$$\begin{aligned} s_m = \sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right) &= \log(2) + \log\left(\frac{3}{2}\right) + \cdots + \log\left(\frac{m}{m-1}\right) + \log\left(\frac{m+1}{m}\right) \\ &= \log(2) + \log(3) - \log(2) + \cdots + \log(m) - \log(m-1) + \log(m+1) - \log(m) \\ &= \log(m+1). \end{aligned}$$

From above, we note that (s_m) diverges.

2.4.9.

We want to prove the contrapositive of **theorem 2.4.6**. That is,

$$\sum_{n=0}^{\infty} 2^n b_{2^n} \text{ diverges} \iff \sum_{n=1}^{\infty} b_n \text{ diverges .}$$

2.4.10 (Infinite products).

(a)

Definition (Infinite product). An *infinite product* is defined by

$$\prod_{n=1}^{\infty} b_n = b_1 b_2 b_3 \dots .$$

Definition (Partial product). A *partial product* p_m is defined by

$$p_m = \prod_{n=1}^{m} b_1 b_2 b_3 \dots b_m.$$

We consider special class of infinite products of the form

$$\prod_{n=1}^{\infty} (1 + a_n) = (1 + a_1)(1 + a_2)(1 + a_3) \dots ,$$

where $a_n \geq 0$ for all $n \in \mathbb{N}$.

Case 1. $a_n = (1/n)$.

The mth partial product is

$$\begin{aligned} p_m = \prod_{n=1}^{m} (1 + \frac{1}{n}) &= 2\left(1 + \frac{1}{2}\right) \dots \left(1 + \frac{1}{m-1}\right)\left(1 + \frac{1}{m}\right) \\ &= 2 \cdot \frac{3}{2} \cdot \dots \cdot \frac{m}{m-1} \cdot \frac{m+1}{m} \\ &= m + 1. \end{aligned}$$

Hence the sequence of partial products (p_m) diverges.

Informally reasoning by analogy with that of infinite sums, one might conjecture that convergence of an infinite product occurs when the sequence of partial products converges so that $(p_m) \to p$.

In this case, the sequence of partial products (p_m) diverges and hence this infinite product diverges.

Case 2. $a_n = (1/n^2)$.

Enumerating a few terms of the sequence of partial products (p_m) using Mathematica gives

$$\begin{aligned}(p_m) &= \left(2, \frac{5}{2}, \frac{25}{9}, \frac{425}{144}, \frac{221}{72}, \frac{8177}{2592}, \frac{204425}{63504}, \frac{13287625}{4064256}, \dots\right) \\ &\approx (2.00, 2.50, 2.78, 2.95, 3.07, 3.15, 3.22, 3.27).\end{aligned}$$

Because each term p_m is increasing by a smaller amount each time, this suggests that (p_m) might converge to some number a little greater than 3.

(b)

We will prove that for the class of infinite product being considered, then

$$(p_m) \text{ converges} \iff \sum_{n=1}^{\infty} a_n \text{ converges.}$$

Proof. Assume that $\sum_{n=1}^{\infty} a_n$ converges. We will show that the sequence of partial products (p_m) converges using the monotone convergence theorem.

Note that sequence of partial products (p_m) is increasing. That is because $a_n \geq 0$ for all $n \in \mathbb{N}$ and because for all $m \geq 1$,

$$\frac{p_{m+1}}{p_m} = (1 + a_{m+1}) \geq 1.$$

The sequence of partial products (p_m) is also bounded. To see this, first note that convergence of $\sum_{n=1}^{\infty} a_n$ means that its sequence of partial sums (s_m) converges, where

$$s_m = \sum_{n=1}^{m} = a_n.$$

Therefore (s_m) is also bounded. That is, for some $M > 0$, $|s_m| \leq M$ for all $m \in \mathbb{N}$. Using the inequality that $(1 + x) \leq 3^x$ for all $x > 0$, we have that for all $m \in \mathbb{N}$,

$$\begin{aligned}p_m = \prod_{n=1}^{m} (1 + a_n) &\leq \prod_{n=1}^{m} 3^{a_n} \\ &= 3^{(a_1 + a_2 + \cdot a_m)} \\ &< 3^M.\end{aligned}$$

The monotone convergence theorem then allows us to assert that $(p_m) \to p$ where $p = \sup\{p_m : m \in \mathbb{N}\}$.

Assume that $(p_m) \to p$ converges. We will apply the monotone convergence theorem to the sequence of partial sums (s_m).

The sequence of partial sums (s_m) is increasing because $a_n \geq 0$ for all $n \in \mathbb{N}$.

The sequence of partial sums (s_m) is also bounded. We will show that $s_m <> p_m \leq M$ for some $M > 0$. That $p_m \leq M$ arises because (p_m) is convergent and therefore bounded.

To see that $s_m < p_m$ explicitly, we define some notation. Define the set $A = \{a_1, a_2, \ldots, a_m\}$. Let $\binom{A}{n}$ denote the set of all n-length combinations on A. As an example, when $m = 3$, we have $A = \{a_1, a_2, a_3\}$ and therefore

$$\binom{A}{2} = \binom{\{a_1, a_2, a_3\}}{2} = \{\{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}\}\,.$$

Continuing our example, this allows us to write

$$\sum_{A' \in \binom{A}{2}} \left(\prod_{a \in A'} a \right) = (a_1 a_2 + a_1 a_3 + a_2 a_3).$$

Where the sum is over all sets A' of 2-length combinations on A in $\binom{A}{2}$, and where the product is over all elements in each A'.

We can now express the partial product p_m as

$$\begin{aligned}
p_m &= \prod_{n=1}^{m} (1 + a_n) \\
&= 1 + \sum_{n=1}^{m} \sum_{A' \in \binom{A}{n}} \left(\prod_{a \in A'} a \right) \\
&= 1 + \sum_{A' \in \binom{A}{1}} \left(\prod_{a \in A'} a \right) + \sum_{A' \in \binom{A}{2}} \left(\prod_{a \in A'} a \right) + \cdots + \sum_{A' \in \binom{A}{m}} \left(\prod_{a \in A'} a \right) \\
&= 1 + \sum_{n=1}^{m} a_n + \sum_{A' \in \binom{A}{2}} \left(\prod_{a \in A'} a \right) + \cdots + \sum_{A' \in \binom{A}{m}} \left(\prod_{a \in A'} a \right). \\
&= 1 + s_m + \sum_{A' \in \binom{A}{2}} \left(\prod_{a \in A'} a \right) + \cdots + \sum_{A' \in \binom{A}{m}} \left(\prod_{a \in A'} a \right).
\end{aligned}$$

Each $\sum_{A' \in \binom{A}{j}} \left(\prod_{a \in A'} a \right)$ term on the right hand side of the final equality is non-negative because it is a sum of the product of non-negative terms, and so $s_m < p_m$. Hence we have that

$$s_m < p_m \leq M.$$

Hence, this shows that (s_m) is bounded. Applying the monotone convergence theorem to (s_m) therefore allows us to conclude the convergence of $\sum_{n=1}^{\infty} a_n$.

Problem set 2.5.

Problem set 2.6

Problem set 2.7

Project section 2.8