Chapter 3 – Rigid-Body Kinetics

3.1 Newton–Euler Equations of Motion about the CG
3.2 Newton–Euler Equations of Motion about the CO
3.3 Rigid-Body Equations of Motion

To derive the marine craft equations of motion, it is necessary to study of the motion of rigid bodies, hydrodynamics, and hydrostatics.

The overall goal of Chapter 3 is to show that the rigid-body equations of motion can be expressed in matrix-vector form about the CO according to the following:

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}_{RB}$$

$\boldsymbol{M}_{\mathrm{RB}}$ Rigid-body mass matrix
$\boldsymbol{C}_{\mathrm{RB}}$ Rigid-body Coriolis and centripetal matrix due to the rotation of {b} about {n}
$\boldsymbol{\nu} = [u, v, w, p, q, r]^{\top}$ generalized velocity expressed in {b}
$\boldsymbol{\tau}_{\mathrm{RB}} = [X, Y, Z, K, M, N]^{\top}$ generalized force expressed in {b}

CO - coordinate origin o_b of $\{b\}$

CG - center of gravity relative to the CO, located at $\boldsymbol{r}^b_{bg} = [x_g, y_g, z_g]^{\top}$

CO is the body-fixed coordinate origin of the GNC system, and it is defined to meet specific control objectives. As such, the craft's CO can be stabilized or programmed to follow a time-varying trajectory in 2-D or 3-D space.

Chapter Goals

- Understand that Newton's 2nd law and its generalization to the Newton-Euler equations of motion is formulated in an "approximative" inertial frame usually chosen as the tangent plane NED.
- Understand why we get a $\boldsymbol{C}_{\mathrm{RB}}$ matrix when transforming the equations of motion to a BODY-fixed rotating reference frame instead of NED.
- Be able to write down and simulate the rigid-body equation of motion about the CG for a vehicle moving in 6 DOFs.
- Know how to transform the rigid-body equation of motion to other reference points such as the CO. This involves using the $\boldsymbol{H}$-matrix (system transformation matrix) defined in Appendix C.
- Understand the:
  - 6 x 6 rigid-body matrix $\boldsymbol{M}_{\mathrm{RB}}$
  - 6 x 6 Coriolis and centripetal matrix $\boldsymbol{C}_{\mathrm{RB}}$ and how it is computed from $\boldsymbol{M}_{\mathrm{RB}}$
  - Parallel-axes theorem and its application to moments of inertia

Chapter 3 – Rigid-Body Kinetics

The equations of motion will be represented in two body-fixed reference points

1) Center of gravity (CG), subscript *g*
2) Origin CO of {b}, subscript *b*

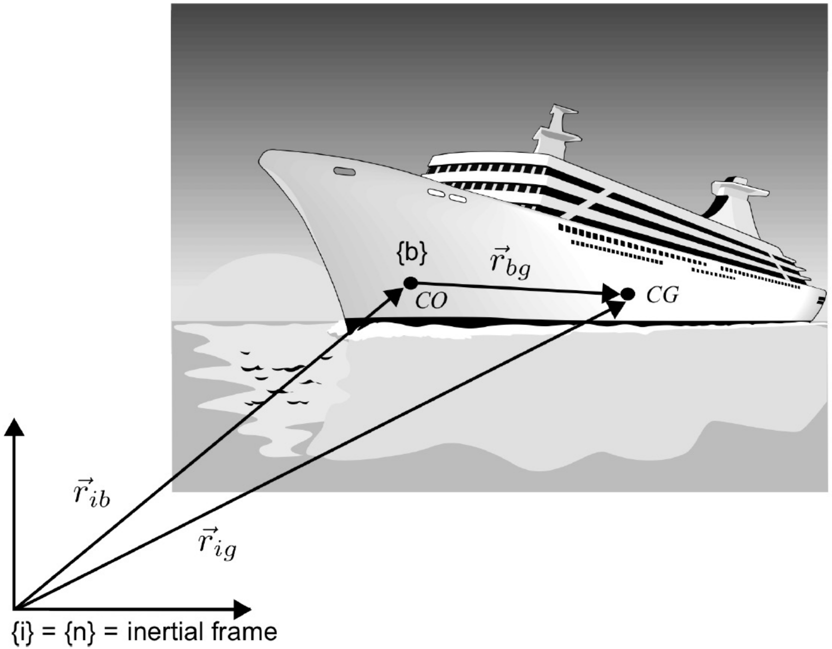
These points coincides if the vector $\boldsymbol{r}_{bg}^{b} = \mathbf{0}$

Time differentiation of a vector in a moving reference frame {b} satisfies

$$\frac{{}^{i}\mathrm{d}}{\mathrm{d}t}\vec{a} = \frac{{}^{b}\mathrm{d}}{\mathrm{d}t}\vec{a} + \vec{\omega}_{ib} \times \vec{a}$$

Time differentiation in {b} is denoted as

$$\dot{\vec{a}} := \frac{{}^{b}\mathrm{d}}{\mathrm{d}t}\vec{a}$$



3.1 Newton-Euler Equations of Motion about the CG

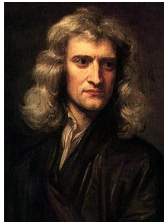
Coordinate-free vector: A vector $\vec{v}_{nb}$, velocity of {b} with respect to {n}, is defined by its magnitude and direction but without reference to a coordinate frame.
Coordinate vector: A vector $\vec{v}_{nb}$ decomposed in the inertial reference frame is denoted by

Newton-Euler Formulation

Newton's Second Law relates mass *m*, acceleration $\dot{\vec{v}}_{ig}$ and force $\vec{f}_g$ according to

$$\boxed{m\dot{\vec{v}}_{ig} = \vec{f}_g}$$

where the subscript *g* denotes the center of gravity (CG).



Isaac Newton (1642-1726)
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Euler's First and Second Axioms
Euler suggested to express Newton's Second Law in terms of conservation of both linear momentum $\vec{p}_g$ and angular momentum $\vec{h}_g$ according to:

$$\boxed{\begin{array}{ll} \dfrac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{p}_g = \vec{f}_g & \vec{p}_g = m\vec{v}_{ig} \\ \dfrac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{h}_g = \vec{m}_g & \vec{h}_g = I_g\vec{\omega}_{ig} \end{array}}$$

$\vec{f}_g$ and $\vec{m}_g$ are forces/moments about the body's CG
$\vec{\omega}_{ib}$ is the angular velocity of frame *b* relative frame *i*
I_g is the inertia dyadic about the body's CG

Leonhard Euler (1707-1783)
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3.1 Translational Motion about the CG

When deriving the equations of motion, it will be assumed that:

(1) The vessel is rigid
(2) The NED frame is inertial—that is, $\{n\} \approx \{i\}$

The first assumption eliminates the consideration of forces acting between individual elements of mass while the second eliminates forces due to the Earth's motion relative to a star-fixed inertial reference system such that

$$\vec{v}_{ig} \approx \vec{v}_{ng}$$
$$\vec{\omega}_{ig} = \vec{\omega}_{ib} \approx \vec{\omega}_{nb}$$

For guidance and navigation applications in space, it is usual to use a star-fixed reference frame or a reference frame rotating with the Earth. Marine craft are, on the other hand, usually related to the NED reference frame. This is a reasonable assumption since forces on a marine craft due to the Earth's rotation

$$\omega_{ie} = 7.2921 \times 10^{-5} \text{ rad/s}$$

are quite small compared to the hydrodynamic forces.

3.1 Translational Motion about the CG

{n} is inertial

$$\vec{r}_{ng} = \vec{r}_{nb} + \vec{r}_{bg} \longleftarrow \vec{r}_{ig} = \vec{r}_{ib} + \vec{r}_{bg}$$

Time differentiation of $\vec{r}_{ng}$ in a moving reference frame {b} gives

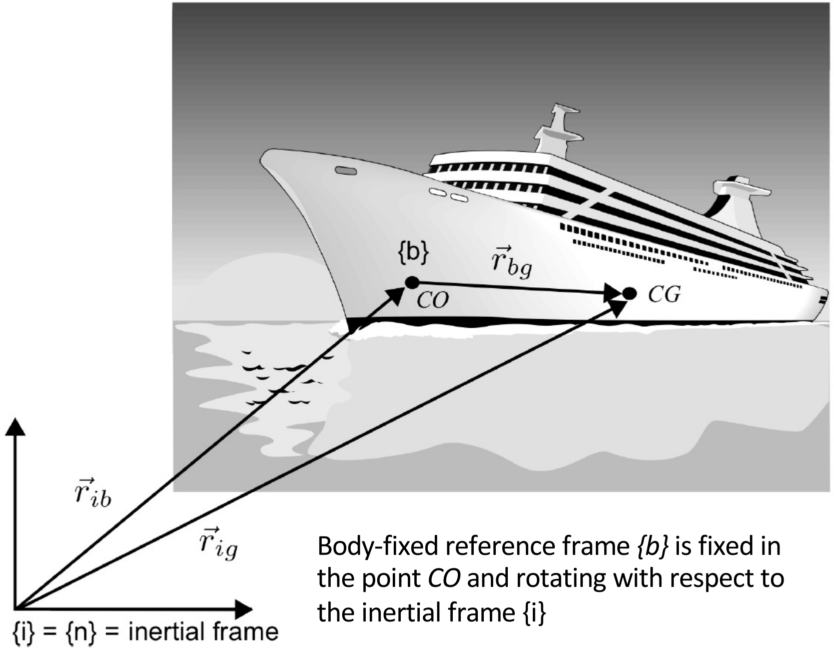
$$\vec{v}_{ng} = \vec{v}_{nb} + \left(\frac{{}^b\mathrm{d}}{\mathrm{dt}} \vec{r}_{bg} + \vec{\omega}_{nb} \times \vec{r}_{bg} \right)$$

For a rigid body, the CG satisfies

$$\frac{{}^b\mathrm{d}}{\mathrm{dt}} \vec{r}_{bg} = \vec{0}$$

$$\vec{v}_{ng} = \vec{v}_{nb} + \vec{\omega}_{nb} \times \vec{r}_{bg}$$

$$\boxed{\begin{aligned} \vec{f}_g &= \frac{{}^i\mathrm{d}}{\mathrm{dt}}(m\vec{v}_{ig}) \\ &= \frac{{}^i\mathrm{d}}{\mathrm{dt}}(m\vec{v}_{ng}) \\ &= \frac{{}^b\mathrm{d}}{\mathrm{dt}}(m\vec{v}_{ng}) + m\vec{\omega}_{nb} \times \vec{v}_{ng} \\ &= m(\dot{\vec{v}}_{ng} + \vec{\omega}_{nb} \times \vec{v}_{ng}) \end{aligned}}$$



Body-fixed reference frame *{b}* is fixed in the point *CO* and rotating with respect to the inertial frame {i}

Translational Motion about the CG Expressed in {b}

$$m[\dot{\boldsymbol{v}}^b_{ng} + \boldsymbol{S}(\boldsymbol{\omega}^b_{nb})\boldsymbol{v}^b_{ng}] = \boldsymbol{f}^b_g$$

3.1 Rotational Motion about the CG

The derivation starts with the Euler's 2nd axiom

$$
\begin{aligned}
\vec{m}_g &= \frac{{}^i\mathrm{d}}{\mathrm{d}t}(I_g\vec{\omega}_{ib}) \\
&= \frac{{}^i\mathrm{d}}{\mathrm{d}t}(I_g\vec{\omega}_{nb}) \\
&= \frac{{}^b\mathrm{d}}{\mathrm{d}t}(I_g\vec{\omega}_{nb}) + \vec{\omega}_{nb} \times (I_g\vec{\omega}_{nb}) \\
&= I_g\dot{\vec{\omega}}_{nb} - (I_g\vec{\omega}_{nb}) \times \vec{\omega}_{nb}
\end{aligned}
$$

Rotational Motion about the CG Expressed in {b}

$$\boldsymbol{I}_g^b\dot{\boldsymbol{\omega}}_{nb}^b - \boldsymbol{S}(\boldsymbol{I}_g^b\boldsymbol{\omega}_{nb}^b)\,\boldsymbol{\omega}_{nb}^b = \boldsymbol{m}_g^b$$

where $\boldsymbol{I}_g^b$ is the *inertia dyadic*

where I_x, I_y, and I_z are the *moments of inertia* about {b} and $I_{xy}=I_{yx}$, $I_{xz}=I_{zx}$ and $I_{yz}=I_{zy}$ are the *products of inertia* defined as

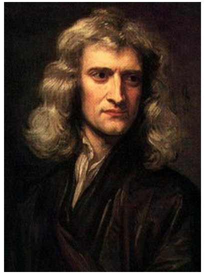
$$\boldsymbol{I}_g^b := \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{bmatrix}, \qquad \boldsymbol{I}_g^b = (\boldsymbol{I}_g^b)^\top > 0$$

$$
\begin{aligned}
I_x &= \int_V (y^2+z^2)\ \rho_m dV; & I_{xy} &= \int_V xy\ \rho_m dV = \int_V yx\ \rho_m dV = I_{yx} \\
I_y &= \int_V (x^2+z^2)\ \rho_m dV; & I_{xz} &= \int_V xz\ \rho_m dV = \int_V zx\ \rho_m dV = I_{zx} \\
I_z &= \int_V (x^2+y^2)\ \rho_m dV; & I_{yz} &= \int_V yz\ \rho_m dV = \int_V zy\ \rho_m dV = I_{zy}
\end{aligned}
$$

3.1 Equations of Motion about the CG

The Newton-Euler equations can be represented in matrix form according to

$$\boldsymbol{M}_{RB}^{CG} \begin{bmatrix} \dot{\boldsymbol{v}}_{ng}^{b} \\ \dot{\boldsymbol{\omega}}_{nb}^{b} \end{bmatrix} + \boldsymbol{C}_{RB}^{CG} \begin{bmatrix} \boldsymbol{v}_{ng}^{b} \\ \boldsymbol{\omega}_{nb}^{b} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_{g}^{b} \\ \boldsymbol{m}_{g}^{b} \end{bmatrix}$$



Isaac Newton (1642-1726)
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Leonhard Euler (1707-1783)
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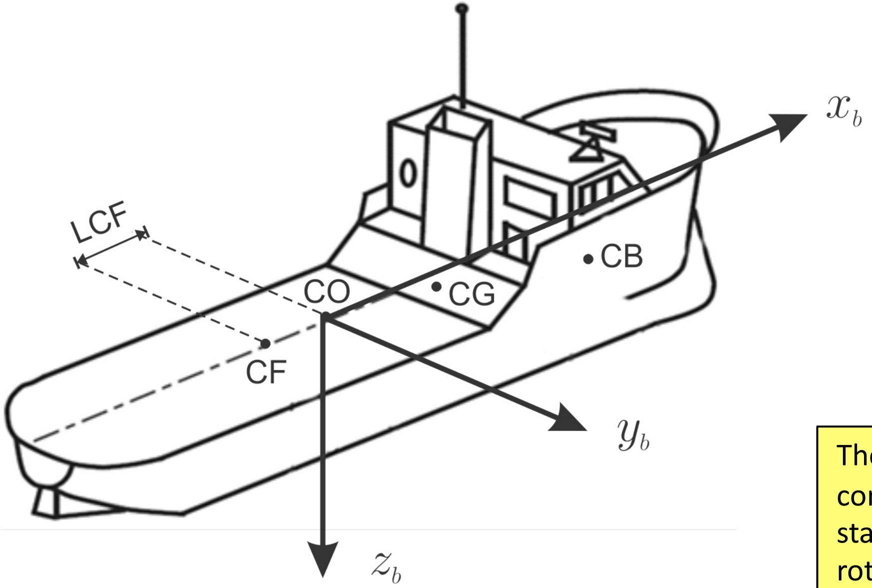
Expanding the matrices give

$$\underbrace{\begin{bmatrix} m\boldsymbol{I}_3 & \boldsymbol{0}_{3\times 3} \\ \boldsymbol{0}_{3\times 3} & \boldsymbol{I}_g^b \end{bmatrix}}_{\boldsymbol{M}_{RB}^{CG}} \begin{bmatrix} \dot{\boldsymbol{v}}_{ng}^{b} \\ \dot{\boldsymbol{\omega}}_{nb}^{b} \end{bmatrix} + \underbrace{\begin{bmatrix} m\boldsymbol{S}(\boldsymbol{\omega}_{nb}^{b}) & \boldsymbol{0}_{3\times 3} \\ \boldsymbol{0}_{3\times 3} & -\boldsymbol{S}(\boldsymbol{I}_g^b \boldsymbol{\omega}_{nb}^{b}) \end{bmatrix}}_{\boldsymbol{C}_{RB}^{CG}} \begin{bmatrix} \boldsymbol{v}_{ng}^{b} \\ \boldsymbol{\omega}_{nb}^{b} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_{g}^{b} \\ \boldsymbol{m}_{g}^{b} \end{bmatrix}$$

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}_{RB}$$

Which Coordinate Origin Should I Use?



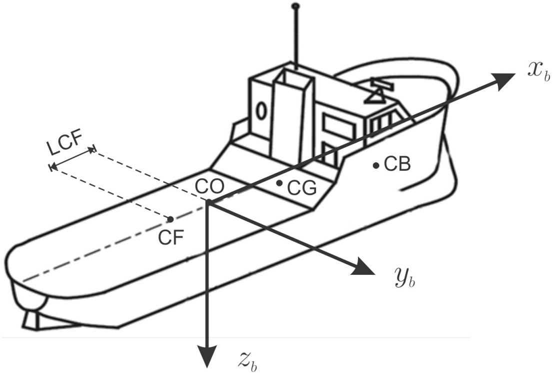
It is recommended to represent the equations of motion in the origin of an arbitrary body-fixed Coordinate Origin (CO) instead of the Centre of Gravity (CG), which can be time-varying due to varying payload and fuel consumption.

The CO of the GNC system is chosen to satisfy the control objective. For instance, the CO can be either stabilized, acting as the point about which the craft rotates during stationkeeping, or directed to track a time-varying path in 2-D or 3-D space, serving as the point that follows the designated trajectory.

3.2 Newton-Euler Equations of Motion about the CO

For marine craft, it is desirable to derive the equations of motion for an arbitrary Coordinate Origin (CO) usually chosen as coordinate origin of the GNC system.

The equations of motion can be transformed from the CG to the CO using the following coordinate transformation (see Appendix C)



$$
\begin{aligned}
\boldsymbol{v}_{ng}^{b} &= \boldsymbol{v}_{nb}^{b} + \boldsymbol{\omega}_{nb}^{b} \times \boldsymbol{r}_{bg}^{b} \\
&= \boldsymbol{v}_{nb}^{b} - \boldsymbol{r}_{bg}^{b} \times \boldsymbol{\omega}_{nb}^{b} \\
&= \boldsymbol{v}_{nb}^{b} + \boldsymbol{S}^{\top}(\boldsymbol{r}_{bg}^{b})\boldsymbol{\omega}_{nb}^{b}
\end{aligned}
\quad \Rightarrow \quad
\boxed{\begin{bmatrix} \boldsymbol{v}_{ng}^{b} \\ \boldsymbol{\omega}_{nb}^{b} \end{bmatrix} = \boldsymbol{H}(\boldsymbol{r}_{bg}^{b}) \begin{bmatrix} \boldsymbol{v}_{nb}^{b} \\ \boldsymbol{\omega}_{nb}^{b} \end{bmatrix}}
$$

The transformation matrix is

$$
\boldsymbol{H}(\boldsymbol{r}_{bg}^{b}) := \begin{bmatrix} \boldsymbol{I}_3 & \boldsymbol{S}^{\top}(\boldsymbol{r}_{bg}^{b}) \\ \boldsymbol{0}_{3\times3} & \boldsymbol{I}_3 \end{bmatrix}, \qquad \boldsymbol{H}^{\top}(\boldsymbol{r}_{bg}^{b}) = \begin{bmatrix} \boldsymbol{I}_3 & \boldsymbol{0}_{3\times3} \\ \boldsymbol{S}(\boldsymbol{r}_{bg}^{b}) & \boldsymbol{I}_3 \end{bmatrix}
$$

3.2 Newton-Euler Equations of Motion about the CO

Newton-Euler equations in matrix-vector form about the CG (see Section 3.1)

$$\boldsymbol{M}_{RB}^{CG}\begin{bmatrix}\dot{\boldsymbol{v}}_{ng}^{b}\\ \dot{\boldsymbol{\omega}}_{nb}^{b}\end{bmatrix}+\boldsymbol{C}_{RB}^{CG}\begin{bmatrix}\boldsymbol{v}_{ng}^{b}\\ \boldsymbol{\omega}_{nb}^{b}\end{bmatrix}=\begin{bmatrix}\boldsymbol{f}_{g}^{b}\\ \boldsymbol{m}_{g}^{b}\end{bmatrix}$$

$$\boxed{\begin{bmatrix}\boldsymbol{v}_{ng}^{b}\\ \boldsymbol{\omega}_{nb}^{b}\end{bmatrix}=\boldsymbol{H}(\boldsymbol{r}_{bg}^{b})\begin{bmatrix}\boldsymbol{v}_{nb}^{b}\\ \boldsymbol{\omega}_{nb}^{b}\end{bmatrix}}$$

See App. C for more details

Newton-Euler equations in matrix-vector form about the CO

$$\boldsymbol{H}^{\top}(\boldsymbol{r}_{bg}^{b})\boldsymbol{M}_{RB}^{CG}\boldsymbol{H}(\boldsymbol{r}_{bg}^{b})\begin{bmatrix}\dot{\boldsymbol{v}}_{nb}^{b}\\ \dot{\boldsymbol{\omega}}_{nb}^{b}\end{bmatrix}+\boldsymbol{H}^{\top}(\boldsymbol{r}_{bg}^{b})\boldsymbol{C}_{RB}^{CG}\boldsymbol{H}(\boldsymbol{r}_{bg}^{b})\begin{bmatrix}\boldsymbol{v}_{nb}^{b}\\ \boldsymbol{\omega}_{nb}^{b}\end{bmatrix}=\boldsymbol{H}^{\top}(\boldsymbol{r}_{bg}^{b})\begin{bmatrix}\boldsymbol{f}_{g}^{b}\\ \boldsymbol{m}_{g}^{b}\end{bmatrix}$$

$$\boldsymbol{M}_{RB}:=\boldsymbol{H}^{\top}(\boldsymbol{r}_{bg}^{b})\boldsymbol{M}_{RB}^{CG}\boldsymbol{H}(\boldsymbol{r}_{bg}^{b})$$

$$\boldsymbol{C}_{RB}:=\boldsymbol{H}^{\top}(\boldsymbol{r}_{bg}^{b})\boldsymbol{C}_{RB}^{CG}\boldsymbol{H}(\boldsymbol{r}_{bg}^{b})$$

Expanding the matrices

$$\boldsymbol{M}_{RB}=\begin{bmatrix}m\boldsymbol{I}_3 & -m\boldsymbol{S}(\boldsymbol{r}_{bg}^{b})\\ m\boldsymbol{S}(\boldsymbol{r}_{bg}^{b}) & \boldsymbol{I}_{g}^{b}-m\boldsymbol{S}^{2}(\boldsymbol{r}_{bg}^{b})\end{bmatrix}$$

$$\boldsymbol{C}_{RB}=\begin{bmatrix}m\boldsymbol{S}(\boldsymbol{\omega}_{nb}^{b}) & -m\boldsymbol{S}(\boldsymbol{\omega}_{nb}^{b})\boldsymbol{S}(\boldsymbol{r}_{bg}^{b})\\ m\boldsymbol{S}(\boldsymbol{r}_{bg}^{b})\boldsymbol{S}(\boldsymbol{\omega}_{nb}^{b}) & -m\boldsymbol{S}(\boldsymbol{r}_{bg}^{b})\boldsymbol{S}(\boldsymbol{\omega}_{nb}^{b})\boldsymbol{S}(\boldsymbol{r}_{bg}^{b})-\boldsymbol{S}(\boldsymbol{I}_{g}^{b}\boldsymbol{\omega}_{nb}^{b})\end{bmatrix}$$

It is possible to rewrite $M_{RB}^{\{22\}}$ and $C_{RB}^{\{22\}}$ using the Parallel-Axis Theorem. The motivation for this is to replace I_g with I_b (that is the inertia tensor in the CO instead of the CG).

3.2 Newton-Euler Equations of Motion about the CO

Theorem 3.1 (Huygens–Steiner's Parallel-Axis Theorem)
The inertia dyadic $\boldsymbol{I}_b^b = (\boldsymbol{I}_b^b)^\top \in \mathbb{R}^{3\times 3}$ *about the origin CO is*

$$\begin{aligned} \boldsymbol{I}_b^b &= \boldsymbol{I}_g^b - m\boldsymbol{S}^2(\boldsymbol{r}_{bg}^b) \\ &= \boldsymbol{I}_g^b + m\left((\boldsymbol{r}_{bg}^b)^\top \boldsymbol{r}_{bg}^b \boldsymbol{I}_3 - \boldsymbol{r}_{bg}^b(\boldsymbol{r}_{bg}^b)^\top\right) \end{aligned} \tag{3.36}$$

where $\boldsymbol{I}_g^b = (\boldsymbol{I}_g^b)^\top \in \mathbb{R}^{3\times 3}$ *is the inertia dyadic about the CG. Expanding (3.36) gives*

$$\boldsymbol{I}_b^b = \boldsymbol{I}_g^b + m\begin{bmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{bmatrix} \tag{3.37}$$

$$\boldsymbol{M}_{RB} = \begin{bmatrix} m\boldsymbol{I}_3 & -m\boldsymbol{S}(\boldsymbol{r}_{bg}^b) \\ m\boldsymbol{S}(\boldsymbol{r}_{bg}^b) & \underbrace{\boldsymbol{I}_g^b - m\boldsymbol{S}^2(\boldsymbol{r}_{bg}^b)}_{\boldsymbol{I}_b^b} \end{bmatrix}$$

Christian Huygens (1629-1695)
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3.2 Newton-Euler Equations of Motion about the CO

$$\boldsymbol{C}_{RB} = \begin{bmatrix} m\boldsymbol{S}(\boldsymbol{\omega}^b_{nb}) & -m\boldsymbol{S}(\boldsymbol{\omega}^b_{nb})\boldsymbol{S}(\boldsymbol{r}^b_{bg}) \\ m\boldsymbol{S}(\boldsymbol{r}^b_{bg})\boldsymbol{S}(\boldsymbol{\omega}^b_{nb}) & \boxed{-m\boldsymbol{S}(\boldsymbol{r}^b_{bg})\boldsymbol{S}(\boldsymbol{\omega}^b_{nb})\boldsymbol{S}(\boldsymbol{r}^b_{bg}) - \boldsymbol{S}(\boldsymbol{I}^b_g\boldsymbol{\omega}^b_{nb})} \end{bmatrix}$$

$$\boldsymbol{S}^\top(\boldsymbol{a})\boldsymbol{b} = -\boldsymbol{S}(\boldsymbol{a})\boldsymbol{b} = \boldsymbol{S}(\boldsymbol{b})\boldsymbol{a}.$$

Jacobi identity

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{0}$$

Choosing $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$ gives

$$\begin{aligned} \boldsymbol{a} \times (\boldsymbol{b} \times (\boldsymbol{a} \times \boldsymbol{b})) &= -\boldsymbol{b} \times ((\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{a}) \\ &= ((\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{a}) \times \boldsymbol{b} \\ &= -(\boldsymbol{a} \times (\boldsymbol{a} \times \boldsymbol{b})) \times \boldsymbol{b} \end{aligned}$$

which is equivalent to

$$\boldsymbol{S}(\boldsymbol{a})\boldsymbol{S}(\boldsymbol{b})\boldsymbol{S}(\boldsymbol{a})\boldsymbol{b} = -\boldsymbol{S}(\boldsymbol{S}^2(\boldsymbol{a})\boldsymbol{b})\boldsymbol{b}$$

Let $\boldsymbol{a} = \boldsymbol{r}^b_{bg}$ and $\boldsymbol{b} = \boldsymbol{\omega}^b_{nb}$ such that

$$-\boldsymbol{S}(\boldsymbol{r}^b_{bg})\boldsymbol{S}(\boldsymbol{\omega}^b_{nb})\boldsymbol{S}(\boldsymbol{r}^b_{bg})\boldsymbol{\omega}^b_{nb} = \boldsymbol{S}(\boldsymbol{S}^2(\boldsymbol{r}^b_{bg})\boldsymbol{\omega}^b_{nb})\boldsymbol{\omega}^b_{nb}$$

The $C_{RB}^{\{22\}}$ element in (3.32) multiplied with $\boldsymbol{\omega}^b_{nb}$ is equivalent to

$$\begin{aligned} -\left[m\boldsymbol{S}(\boldsymbol{r}^b_{bg})\boldsymbol{S}(\boldsymbol{\omega}^b_{nb})\boldsymbol{S}(\boldsymbol{r}^b_{bg}) + \boldsymbol{S}(\boldsymbol{I}^b_g\boldsymbol{\omega}^b_{nb})\right]\boldsymbol{\omega}^b_{nb} \\ &= m\boldsymbol{S}(\boldsymbol{S}^2(\boldsymbol{r}^b_{bg})\boldsymbol{\omega}^b_{nb})\boldsymbol{\omega}^b_{nb} - \boldsymbol{S}(\boldsymbol{I}^b_g\boldsymbol{\omega}^b_{nb})\boldsymbol{\omega}^b_{nb} \\ &= m\boldsymbol{S}(\boldsymbol{S}^2(\boldsymbol{r}^b_{bg})\boldsymbol{\omega}^b_{nb})\boldsymbol{\omega}^b_{nb} - \boldsymbol{S}([\boldsymbol{I}^b_b + m\boldsymbol{S}^2(\boldsymbol{r}^b_{bg})]\boldsymbol{\omega}^b_{nb})\boldsymbol{\omega}^b_{nb} \\ &= -\boldsymbol{S}(\boldsymbol{I}^b_b\boldsymbol{\omega}^b_{nb})\boldsymbol{\omega}^b_{nb} \end{aligned}$$

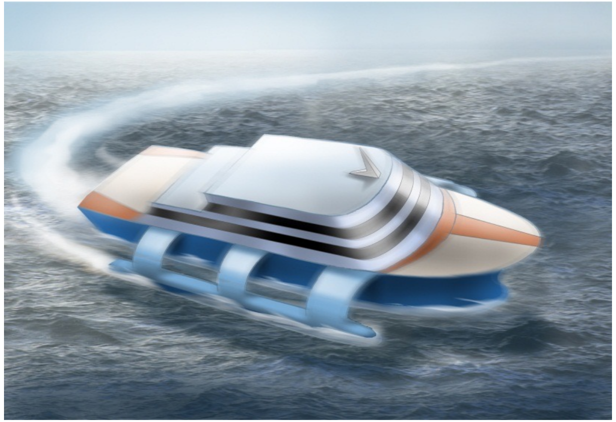
3.2 Translational Motion about the CO

Translational Motion about the CO Expressed in {b}

$$m[\dot{\boldsymbol{v}}_{nb}^b + \boldsymbol{S}(\dot{\boldsymbol{\omega}}_{nb}^b)\boldsymbol{r}_{bg}^b + \boldsymbol{S}(\boldsymbol{\omega}_{nb}^b)\boldsymbol{v}_{nb}^b + \boldsymbol{S}^2(\boldsymbol{\omega}_{nb}^b)\boldsymbol{r}_{bg}^b] = \boldsymbol{f}_b^b$$

An alternative representation using vector cross products is

$$m[\dot{\boldsymbol{v}}_{nb}^b + \dot{\boldsymbol{\omega}}_{nb}^b \times \boldsymbol{r}_{bg}^b + \boldsymbol{\omega}_{nb}^b \times \boldsymbol{v}_{nb}^b + \boldsymbol{\omega}_{nb}^b \times (\boldsymbol{\omega}_{nb}^b \times \boldsymbol{r}_{bg}^b)] = \boldsymbol{f}_b^b$$



3.2 Rotational Motion about the CO

Rotational Motion about the CO Expressed in {b}

$$\boldsymbol{I}_b^b \dot{\boldsymbol{\omega}}_{nb}^b + \boldsymbol{S}(\boldsymbol{\omega}_{nb}^b)\boldsymbol{I}_b^b \boldsymbol{\omega}_{nb}^b + m\boldsymbol{S}(\boldsymbol{r}_{bg}^b)\dot{\boldsymbol{v}}_{nb}^b + m\boldsymbol{S}(\boldsymbol{r}_{bg}^b)\boldsymbol{S}(\boldsymbol{\omega}_{nb}^b)\boldsymbol{v}_{nb}^b = \boldsymbol{m}_b^b$$

An alternative representation using vector cross products is

$$\boldsymbol{I}_b^b \dot{\boldsymbol{\omega}}_{nb}^b + \boldsymbol{\omega}_{nb}^b \times \boldsymbol{I}_b^b \boldsymbol{\omega}_{nb}^b + m\boldsymbol{r}_{bg}^b \times (\dot{\boldsymbol{v}}_{nb}^b + \boldsymbol{\omega}_{nb}^b \times \boldsymbol{v}_{nb}^b) = \boldsymbol{m}_b^b$$

Theorem 3.1 (Huygens–Steiner's Parallel-Axis Theorem)
The inertia dyadic $\boldsymbol{I}_b^b = (\boldsymbol{I}_b^b)^\top \in \mathbb{R}^{3\times 3}$ about the origin CO is

$$\begin{aligned} \boldsymbol{I}_b^b &= \boldsymbol{I}_g^b - m\boldsymbol{S}^2(\boldsymbol{r}_{bg}^b) \\ &= \boldsymbol{I}_g^b + m\left((\boldsymbol{r}_{bg}^b)^\top \boldsymbol{r}_{bg}^b \boldsymbol{I}_3 - \boldsymbol{r}_{bg}^b (\boldsymbol{r}_{bg}^b)^\top\right) \end{aligned} \tag{3.36}$$

where $\boldsymbol{I}_g^b = (\boldsymbol{I}_g^b)^\top \in \mathbb{R}^{3\times 3}$ is the inertia dyadic about the CG. Expanding (3.36) gives

$$\boldsymbol{I}_b^b = \boldsymbol{I}_g^b + m\begin{bmatrix} y^2+z^2 & -xy & -xz \\ -yx & x^2+z^2 & -yz \\ -zx & -zy & x^2+y^2 \end{bmatrix} \tag{3.37}$$

Christian Huygens (1629-1695)
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Jakob Steiner (1796-1863)
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3.3 Rigid-Body Equations of Motion

$$
\begin{array}{lll}
\boldsymbol{f}_b^b &= [X,\ Y,\ Z]^\top & \text{force through the CO expressed in } \{b\} \\
\boldsymbol{m}_b^b &= [K,\ M,\ N]^\top & \text{moment about the CO expressed in } \{b\} \\
\boldsymbol{v}_{nb}^b &= [u,\ v,\ w]^\top & \text{linear velocity of the CO relative to } o_n \text{ expressed in } \{b\} \\
\boldsymbol{\omega}_{nb}^b &= [p,\ q,\ r]^\top & \text{angular velocity of } \{b\} \text{ relative to } \{n\} \text{ expressed in } \{b\} \\
\boldsymbol{r}_{bg}^b &= [x_g,\ y_g,\ z_g]^\top & \text{vector from the CO to the CG expressed in } \{b\}
\end{array}
$$

Component Form (SNAME 1950)

$$
\begin{aligned}
m\left[\dot{u} - vr + wq - x_g(q^2 + r^2) + y_g(pq - \dot{r}) + z_g(pr + \dot{q})\right] &= X \\
m\left[\dot{v} - wp + ur - y_g(r^2 + p^2) + z_g(qr - \dot{p}) + x_g(qp + \dot{r})\right] &= Y \\
m\left[\dot{w} - uq + vp - z_g(p^2 + q^2) + x_g(rp - \dot{q}) + y_g(rq + \dot{p})\right] &= Z \\
I_x\dot{p} + (I_z - I_y)qr - (\dot{r} + pq)I_{xz} + (r^2 - q^2)I_{yz} + (pr - \dot{q})I_{xy} & \\
+m\left[y_g(\dot{w} - uq + vp) - z_g(\dot{v} - wp + ur)\right] &= K \\
I_y\dot{q} + (I_x - I_z)rp - (\dot{p} + qr)I_{xy} + (p^2 - r^2)I_{zx} + (qp - \dot{r})I_{yz} & \\
+m\left[z_g(\dot{u} - vr + wq) - x_g(\dot{w} - uq + vp)\right] &= M \\
I_z\dot{r} + (I_y - I_x)pq - (\dot{q} + rp)I_{yz} + (q^2 - p^2)I_{xy} + (rq - \dot{p})I_{zx} & \\
+m\left[x_g(\dot{v} - wp + ur) - y_g(\dot{u} - vr + wq)\right] &= N
\end{aligned}
$$

These are the 6-DOF rigid-body equations of motion commonly found in textbooks.

The upcoming formulas will provide an alternative *matrix-vector* representation.

3.3 Rigid-Body Equations of Motion

Matrix-Vector Form

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}_{RB}$$

$\boldsymbol{\nu} = [u, v, w, p, q, r]^\top$ Generalized velocity

$\boldsymbol{\tau}_{RB} = [X, Y, Z, K, M, N]^\top$ Generalized force

Property 3.1 (Rigid-Body System Inertia Matrix)

$$\boldsymbol{M}_{RB} = \begin{bmatrix} m\boldsymbol{I}_3 & -m\boldsymbol{S}(\boldsymbol{r}_{bg}^b) \\ m\boldsymbol{S}(\boldsymbol{r}_{bg}^b) & \boldsymbol{I}_b^b \end{bmatrix}$$
$$= \begin{bmatrix} m & 0 & 0 & 0 & mz_g & -my_g \\ 0 & m & 0 & -mz_g & 0 & mx_g \\ 0 & 0 & m & my_g & -mx_g & 0 \\ 0 & -mz_g & my_g & I_x & -I_{xy} & -I_{xz} \\ mz_g & 0 & -mx_g & -I_{yx} & I_y & -I_{yz} \\ -my_g & mx_g & 0 & -I_{zx} & -I_{zy} & I_z \end{bmatrix}$$

$$\boldsymbol{M}_{RB} = \boldsymbol{M}_{RB}^\top > 0, \qquad \dot{\boldsymbol{M}}_{RB} = \boldsymbol{0}_{6\times 6}$$

3.3 Rigid-Body Equations of Motion

Matlab:

The rigid-body system inertia matrix M_{RB} can be computed in Matlab as

```
r_g = [10 0 1]';      % location of the CG with respect to the CO
R44 = 10;             % radius of gyration in roll
R55 = 20;             % radius of gyration in pitch
R66 = 5;              % radius of gyration in yaw
m = 1000;             % mass
I_g = m * diag([R44^2 R55^2 R66^2]);     % inertia dyadic (CG)

% rigid-body system inertia matrix
S = Smtrx(r_g);
MRB = [ m * eye(3)   -m * S
        m * S        I_g - m * S^2 ]

MRB =

        1000           0           0           0        1000           0
           0        1000           0       -1000           0       10000
           0           0        1000           0      -10000           0
           0       -1000           0      101000           0      -10000
        1000           0      -10000           0      501000           0
           0       10000           0      -10000           0      125000
```

The rigid-body system inertia matrix can also be computed using the command

```
MRB = rbody(m, R44, R55, R66, zeros(3,1), r_g)
```

3.3 Rigid-Body Equations of Motion

Theorem 3.2 (Coriolis-Centripetal Matrix from System Inertia Matrix)

Let $\mathbf{M}$ be a 6×6 *system inertia matrix* defined as:

$$\mathbf{M} = \mathbf{M}^\top = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} > 0$$

where $\mathbf{M}_{21} = \mathbf{M}_{12}^\top$. Then the *Coriolis-centripetal matrix* can always be parameterized such that

$$\mathbf{C}(\boldsymbol{\nu}) = -\mathbf{C}^\top(\boldsymbol{\nu})$$

$$\mathbf{C}(\boldsymbol{\nu}) = \begin{bmatrix} \mathbf{0}_{3\times3} & -\mathbf{S}(\mathbf{M}_{11}\boldsymbol{\nu}_1 + \mathbf{M}_{12}\boldsymbol{\nu}_2) \\ -\mathbf{S}(\mathbf{M}_{11}\boldsymbol{\nu}_1 + \mathbf{M}_{12}\boldsymbol{\nu}_2) & -\mathbf{S}(\mathbf{M}_{21}\boldsymbol{\nu}_1 + \mathbf{M}_{22}\boldsymbol{\nu}_2) \end{bmatrix}$$

where $\boldsymbol{\nu}_1 = [u, v, w]^\top$, $\boldsymbol{\nu}_2 = [p, q, r]^\top$

Proof: Sagatun and Fossen (1991).

Sagatun, S. I. and T. I. Fossen (1991). Lagrangian Formulation of Underwater Vehicles' Dynamics. *IEEE International Conference on Systems, Man, and Cybernetics*, Charlottesville, VA, USA, IEEE Xplore, pp. 1029-1034.

3.3 Rigid-Body Equations of Motion

Property 3.2 (Rigid-Body Coriolis and Centripetal Matrix)

The *rigid-body Coriolis and centripetal matrix* $\mathbf{C}_{RB}(\boldsymbol{\nu})$ can always be represented such that $\mathbf{C}_{RB}(\boldsymbol{\nu})$ is skew-symmetric. This is mathematically equivalent to

$$\mathbf{C}_{RB}(\boldsymbol{\nu}) = -\mathbf{C}_{RB}^{\top}(\boldsymbol{\nu}), \qquad \forall \boldsymbol{\nu} \in \mathbb{R}^6$$

The skew-symmetric property is very useful when designing nonlinear motion control system since the quadratic form

$$\boldsymbol{\nu}^{\top}\mathbf{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} \equiv 0$$

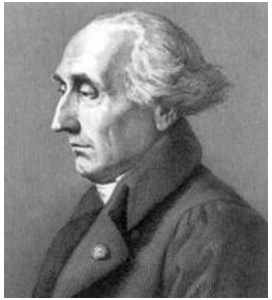
This is exploited in energy-based control designs where Lyapunov functions play a key role. The same property is also used in nonlinear observer design.

There exist several parameterizations that satisfy Property 3.2. Two important ones are the:

- Lagrangian parametrization
- Linear velocity-independent parameterization

which are presented on the forthcoming pages.

3.3 Rigid-Body Equations of Motion



Joseph-Louis Lagrange (1736-1813)
[Wikimedia Commons]

Lagrangian Parameterization

Application of the Theorem 3.2 with $\boldsymbol{M} = \boldsymbol{M}_{RB}$ yields the following expression

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} \boldsymbol{0}_{3\times 3} & -m\boldsymbol{S}(\boldsymbol{\nu}_1) - m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{r}^b_{bg}) \\ -m\boldsymbol{S}(\boldsymbol{\nu}_1) - m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{r}^b_{bg}) & m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_1)\boldsymbol{r}^b_{bg}) - \boldsymbol{S}(\boldsymbol{I}^b_b\boldsymbol{\nu}_2) \end{bmatrix}$$

which can be rewritten according to

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} \boldsymbol{0}_{3\times 3} & -m\boldsymbol{S}(\boldsymbol{\nu}_1) - m\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{S}(\boldsymbol{r}^b_{bg}) \\ -m\boldsymbol{S}(\boldsymbol{\nu}_1) + m\boldsymbol{S}(\boldsymbol{r}^b_{bg})\boldsymbol{S}(\boldsymbol{\nu}_2) & -\boldsymbol{S}(\boldsymbol{I}^b_b\boldsymbol{\nu}_2) \end{bmatrix}$$

To ensure that $\mathbf{C}_{RB}(\boldsymbol{\nu}) = -\mathbf{C}_{RB}(\boldsymbol{\nu})^{\top}$, it is necessary to use $\mathbf{S}(\boldsymbol{\nu}_1)\boldsymbol{\nu}_1 = 0$ and add $\mathbf{S}(\boldsymbol{\nu}_1)$ in C_{RB}[21]

Sagatun, S. I. and T. I. Fossen (1991). Lagrangian Formulation of Underwater Vehicles' Dynamics. *IEEE International Conference on Systems, Man, and Cybernetics*, Charlottesville, VA, USA, IEEE Xplore, pp. 1029-1034.

3.3 Rigid-Body Equations of Motion

Lagrangian Parameterization

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} \boldsymbol{0}_{3\times 3} & -m\boldsymbol{S}(\boldsymbol{\nu}_1) - m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{r}_{bg}^b) \\ -m\boldsymbol{S}(\boldsymbol{\nu}_1) - m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{r}_{bg}^b) & m\boldsymbol{S}(\boldsymbol{S}(\boldsymbol{\nu}_1)\boldsymbol{r}_{bg}^b) - \boldsymbol{S}(\boldsymbol{I}_b^b\boldsymbol{\nu}_2) \end{bmatrix}$$

Component Form

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \left[\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -m(y_g q + z_g r) & m(y_g p + w) & m(z_g p - v) \\ m(x_g q - w) & -m(z_g r + x_g p) & m(z_g q + u) \\ m(x_g r + v) & m(y_g r - u) & -m(x_g p + y_g q) \end{matrix}\right. \tag{3.59}$$
$$\left.\begin{matrix} m(y_g q + z_g r) & -m(x_g q - w) & -m(x_g r + v) \\ -m(y_g p + w) & m(z_g r + x_g p) & -m(y_g r - u) \\ -m(z_g p - v) & -m(z_g q + u) & m(x_g p + y_g q) \\ 0 & -I_{yz} q - I_{xz} p + I_z r & I_{yz} r + I_{xy} p - I_y q \\ I_{yz} q + I_{xz} p - I_z r & 0 & -I_{xz} r - I_{xy} q + I_x p \\ -I_{yz} r - I_{xy} p + I_y q & I_{xz} r + I_{xy} q - I_x p & 0 \end{matrix}\right]$$

3.3 Rigid-Body Equations of Motion

Linear Velocity-Independent Parameterization

By using the cross-product property $\mathbf{S}(\boldsymbol{\nu}_1)\boldsymbol{\nu}_2 = -\mathbf{S}(\boldsymbol{\nu}_2)\boldsymbol{\nu}_1$, it is possible to move $\mathbf{S}(\boldsymbol{\nu}_1)\boldsymbol{\nu}_2$ from $C_{RB}^{\{12\}}$ to $C_{RB}^{\{11\}}$. This gives an expression for $\mathbf{C}_{RB}(\boldsymbol{\nu})$ that is independent of linear velocity $\boldsymbol{\nu}_1$ (Fossen and Fjellstad 1995)

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} m\boldsymbol{S}(\boldsymbol{\nu}_2) & -m\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{S}(\boldsymbol{r}_{bg}^b) \\ m\boldsymbol{S}(\boldsymbol{r}_{bg}^b)\boldsymbol{S}(\boldsymbol{\nu}_2) & -\boldsymbol{S}(\boldsymbol{I}_b^b\boldsymbol{\nu}_2) \end{bmatrix}$$

This formula is the <u>preferred representation</u> when ocean currents enter the equations of motion. The main reason is that $\mathbf{C}_{RB}(\boldsymbol{\nu})$ does not depend on the linear velocity vector $\boldsymbol{\nu}_1 = [u, v, w]^\top$. For a marine craft exposed to irrotational ocean currents, it follows from Property 10.1 in Section 10.3 that

$$\boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}} + \boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} \equiv \boldsymbol{M}_{RB}\dot{\boldsymbol{\nu}}_r + \boldsymbol{C}_{RB}(\boldsymbol{\nu}_r)\boldsymbol{\nu}_r$$

where the relative velocity vector $\boldsymbol{\nu}_r = \boldsymbol{\nu} - \boldsymbol{\nu}_c$ is defined such that only linear ocean current velocities are used

$$\boldsymbol{\nu}_c = [u_c,\ v_c,\ w_c,\ 0,\ 0,\ 0]^\top$$

Fossen, T. I. and O.-E. Fjellstad (1995). Nonlinear Modelling of Marine Vehicles in 6 Degrees of Freedom. *International Journal of Mathematical Modelling of Systems* 1(1), 17-27.

3.3 Rigid-Body Equations of Motion

Linear Velocity-Independent Parameterization

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \begin{bmatrix} m\boldsymbol{S}(\boldsymbol{\nu}_2) & -m\boldsymbol{S}(\boldsymbol{\nu}_2)\boldsymbol{S}(\boldsymbol{r}^b_{bg}) \\ m\boldsymbol{S}(\boldsymbol{r}^b_{bg})\boldsymbol{S}(\boldsymbol{\nu}_2) & -\boldsymbol{S}(\boldsymbol{I}^b_b\boldsymbol{\nu}_2) \end{bmatrix}$$

This formula can also be expressed in terms of the $\boldsymbol{C}_{\mathrm{RB}}$ matrix in the CG, and the transformation matrix from the CG to the CO

$$\boxed{\begin{bmatrix} \boldsymbol{v}^b_{ng} \\ \boldsymbol{\omega}^b_{nb} \end{bmatrix} = \boldsymbol{H}(\boldsymbol{r}^b_{bg}) \begin{bmatrix} \boldsymbol{v}^b_{nb} \\ \boldsymbol{\omega}^b_{nb} \end{bmatrix}}$$

See Appendix C for more details

$$\boldsymbol{C}_{RB}(\boldsymbol{\nu}) = \boldsymbol{H}^\top(\boldsymbol{r}^b_{bg}) \begin{bmatrix} 0 & -mr & mq & 0 & 0 & 0 \\ mr & 0 & -mp & 0 & 0 & 0 \\ -mq & mp & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_z r & -I_y q \\ 0 & 0 & 0 & -I_z r & 0 & I_x p \\ 0 & 0 & 0 & I_y q & -I_x p & 0 \end{bmatrix} \boldsymbol{H}(\boldsymbol{r}^b_{bg})$$

3.3 Rigid-Body Equations of Motion

Matlab:

The Lagrangian parametrization (Theorem 3.2) is implemented in the Matlab MSS toolbox in the function `m2c.m`, while the linear-velocity independent parametrization (3.63) is implemented in the more generic function `rbody.m`. The following example demonstrates how $\boldsymbol{C}_{RB}(\boldsymbol{\nu})$ can be computed numerically

```
r_g = [10 0 1]';      % location of the CG relative to the CO
R44 = 10;             % radius of gyration in roll
R55 = 20;             % radius of gyration in pitch
R66 = 5;              % radius of gyration in yaw
m = 1000;             % mass
nu = [8 0.5 0.1 0.2 -0.3 0.2]'; % velocity vector
```

```
% Method 1: Linear velocity-independent parametrization
nu2 = nu(4:6);
[MRB,CRB] = rbody(m, R44, R55, R66, nu2, r_g)

MRB =
        1000           0           0           0        1000           0
           0        1000           0       -1000           0       10000
           0           0        1000           0      -10000           0
           0       -1000           0      101000           0      -10000
        1000           0      -10000           0      501000           0
           0       10000           0      -10000           0      125000
CRB =
           0        -200        -300         200        3000       -2000
         200           0        -200           0        2200           0
         300         200           0        -200         300        2000
        -200           0         200           0        2800      120000
       -3000       -2200        -300       -2800           0       -2000
        2000           0       -2000     -120000        2000           0

% Method 2: Lagrangian parametrization
CRB = m2c(MRB, nu)

CRB =
           0           0           0           0        3100       -2300
           0           0           0       -3100           0        7700
           0           0           0        2300       -7700           0
           0        3100       -2300           0       28000      143300
       -3100           0        7700      -28000           0       17700
        2300       -7700           0     -143300      -17700           0
```

Even though the numerical values for the two $\boldsymbol{C}_{RB}(\boldsymbol{\nu})$ matrices are different, they both produce the same product $\boldsymbol{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu}$.

3.3 Linearized 6-DOF Rigid-Body Equations of Motion

The nonlinear rigid-body equations of motion

$$\mathbf{M}_{RB}\dot{\boldsymbol{\nu}} + \mathbf{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}_{RB}$$

can be linearized about $\boldsymbol{\nu}_0 = [U, 0\ , 0, 0, 0, 0]^{\top}$ for a marine craft moving at forward speed U.

$$\mathbf{M}_{RB}\dot{\boldsymbol{\nu}} + \mathbf{C}^{*}_{RB}\boldsymbol{\nu} = \boldsymbol{\tau}_{RB}$$

$$\mathbf{C}^{*}_{RB} = \mathbf{M}_{RB}\mathbf{L}U$$

$$\mathbf{L} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The linearized Coriolis and centripetal forces are recognized as

$$\mathbf{f}_c = \mathbf{C}^{*}_{RB}\boldsymbol{\nu} = \begin{bmatrix} 0 \\ mUr \\ -mUq \\ -my_g Uq - mz_g Ur \\ mx_g Uq \\ mx_g Ur \end{bmatrix}$$

3.3 Linearized 6-DOF Rigid-Body Equations of Motion

Matlab:

The linearized model (3.67) is computed using the following Matlab commands

```
U = 1;
MRB = [ 1000       0       0       0    1000       0
           0    1000       0   -1000       0   10000
           0       0    1000       0  -10000       0
           0   -1000       0  101000       0  -10000
        1000       0  -10000       0  501000       0
           0   10000       0  -10000       0  125000];

L = zeros(6,6); L(2,6) = 1; L(3,5) = -1;
CRB = U * MRB * L

CRB =
         0       0       0       0       0       0
         0       0       0       0       0    1000
         0       0       0       0   -1000       0
         0       0       0       0       0   -1000
         0       0       0       0   10000       0
         0       0       0       0       0   10000
```

Note that the skew-symmetric property is destroyed by linearization. In other words, $C^*_{\mathrm{RB}} \neq -(C^*_{\mathrm{RB}})^\top$

Chapter Goals - Revisited

- Understand that Newton's 2nd law and its generalization to the Newton-Euler equations of motion is formulated in an "approximative" inertial frame usually chosen as the tangent plane NED.
- Understand why we get a C_{RB} matrix when transforming the equations of motion to a BODY-fixed rotating reference frame instead of NED.
- Be able to write down and simulate the rigid-body equation of motion about the CG for a vehicle moving in 6 DOFs.
- Know how to transform the rigid-body equation of motion to other reference points such as the CO. This involves using the H-matrix (system transformation matrix) defined in Appendix C.
- Understand the:
  - 6 x 6 rigid-body matrix M_{RB}
  - 6 x 6 Coriolis and centripetal matrix C_{RB} and how it is computed from M_{RB}
  - Parallel-axes theorem and its application to moments of inertia