HW 2A – ENGR 4399: Machine Learning

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Problem 1 (Linear Regression with One Coefficient and an Intercept) *Solve the following optimization problem:*

$$\min_{a,b} \sum_{i=1}^{n} (y_i - ax_i - b)^2.$$

Hint*: Set the derivatives with respect to a and b to zero to get two equations that you have to solve. This is similar to linear regression with two variables.*

Problem 2 (Orthogonality) *Two vectors v_1 and v_2 are defined to be orthogonal if*

$$\langle v_1, v_2 \rangle = v_1^T v_2 = 0.$$

1. *Show that the unit vectors in the Euclidean space $\mathbb{R}^3$, $[1, 0, 0]$, $[0, 1, 0]$, and $[0, 0, 1]$ are orthogonal.*
2. *Recall that a set of vectors $v_1, v_2, \cdots, v_n$ are called linearly independent if for constants $c_1, \cdots, c_n$*

   $$c_1 v_1 + \cdots + c_n v_n = 0 \quad \textit{implies } c_i = 0, \ \textit{for all } i.$$

   Show that orthogonal vectors are linearly independent.

Hint*: Start with the linear combination $c_1 v_1 + \cdots + c_n v_n$ and take its inner product with v_1 (then v_2, etc.).*

Problem 3 (Projection) *The projection of a vector v_1 over another vector v_2 is defined by the inner product $\langle v_1, v_2 \rangle = v_1^T v_2$.*

1. *Verify that for the orthogonal vectors provided in the previous problem, the projection of one over the other is zero.*
2. *Consider the 3-dimensional vector $x = [2, 5, 3]$. Find the projection of x on the unit vectors $e_1 = [1, 0, 0]$, $e_2 = [0, 1, 0]$, and $e_3 = [0, 0, 1]$. Verify that every such vector x can be written as*

   $$x = \langle x, e_1 \rangle e_1 \ + \ \langle x, e_2 \rangle e_2 \ + \ \langle x, e_3 \rangle e_3.$$

   In vector space terminology, we say that the vectors e_1, e_2, and e_3 span the vector space $\mathbb{R}^3$.

Problem 4 (Gradient Calculations) *Obtain the gradient ∇_β of the following function.*

$$f(\beta) = \beta^T ABC \, \beta + b^T \beta + \beta^T a + (c + d + e)^T \beta + \lambda \beta^T \beta.$$

Here, A, B, C are appropriate matrices, a, b, c, d, e are column vectors, and λ is a constant.
Hint*: Use the gradient identities studied in the class.*

Problem 5 (Convex Functions (30 Points)) *Let $f_1(x)$ and $f_2(x)$ be two convex functions. Let $\lambda_1 > 0$ and $\lambda_2 > 0$ are two positive real numbers. Then show that $\lambda_1 f_1(x) + \lambda_2 f_2(x)$ is convex.*

Hint*: Use the definition of convex functions.*