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# Dynamic modeling and rest-to-rest motion for a one-link flexible arm with flexible joint

Artificial Intelligence and Robotics

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# Contents

# 1 Introduction

The mechanical flexibility is one of the major problem in control of robots. For example, to reach high-speed manipulation, the elasticity at the joints and the intrinsic flexibility of the structure (for example in lightweight robots) cause vibration effects that are not negligible. Obviously, the simplest practical way to reduce vibrations is to reduce motion speed, but from a productivity point of view this way should be avoided. Another possibility is to increase the weight of the mechanical structure, but this means that the actuators should be very powerful. Taking into account the flexibility effects of the links and the elasticity of the joints surely complicates the model, but it's mandatory to find an accurate dynamic model that is usable in the application of very lightweight robot with high motion speed. One example is for space application, in which the lightness of the robot and the use of less powerful actuators are very important.
As we'll see later, these type of system is described by both spatial and temporal quantities, and the analytic model is formed by partial differential equations.
For this reason is crucial to elaborate a robot model not so much complicated, but that can describe in an accurate way the behaviour of the system, both for analysis and synthesis.
The modal analysis is very important to derive the fundamental modes of deformation for the robotic arm. In particular the accuracy reached by the analytical modeling of the robot is very related to the adopted mode shapes, that for the elastic joint flexible link robot are surely different with respect to the rigid joint flexible arm case. In fact, the elasticity of the joints surely affects the mode shapes. Some researchers used the pinned-free or clamped-free mode shapes, or for example polynomial interpolations.
The problem to move a manipulator from one equilibrium configuration to another is simple in the case of a rigid robot, but can become a difficult problem in the case of elasticity and flexibility at the joint-link. The flexibility of the link cause residual oscillations when the robot should reach the final configuration, augmenting the time to reach the final equilibrium. Different approaches were developed: the first is the input shaping method, while the second is the inverse dynamics trajectory design. The input shaping method use a suitable input (step input convolved with impulses) such that to "kill" the modal frequencies of the vibrations. This method is good in presence of one or few flexible modes, but increasing the modes becomes more complex. In the second method, it's possible to design a suitable smooth interpolating trajectory for the end effector that automatically bound the link deformation and after it is possible to compute by inverse dynamics the feedforward torque command.

M. Burli, F. Nicolò and A. De Luca [3] describe the difference between the two methods for modeling a robotic arm of this type: analitycal method and finite elements method. The first method is very complicated by a mathematical point of view, but it's very convenient for the accuracy that the dynamical equations can reach. The second method derives simpler dynamical equations, but the accuracy is not very high because the finite elements method is based on approximations.

D. Li, J.W. Zu, and A.A. Goldenberg [1] use the dynamic modeling of an elastic joint and flexible link arm with a single joint to do a very accurate modal analysis: this simple system take into account all possible vibration effects due to joint and link. In particular they find that also a very small joint elasticity can affect the system frequency of vibrations, and also that the fundamental frequency is insensitive to the hub inertia or payload inertia, but it is affected by the payload mass. They derive the exact mode shapes for the system.

F. Bellezza, L. Lanari, G. Ulivi [4] used the assumed mode technique to found the exact eigenfunctions for the flexible slewing beam. They analyze the problem in the two formulations called "pseudo-clamped" and "pseudo-pinned" changing the rotating, non inertial frame, resulting in a change of the boundary conditions of the slewing variable. They found the two related linear dynamic model and also a change of coordinates to pass from pseudo-clamped and pseudo-pinned models.

A. De Luca and G. Di Giovanni [2] present a solution to find the correct input torque command for a rest to rest motion for the rigid joint flexible link robot arm. They design an auxiliary output such that the transfer function has no zeroes to ensure maximum relative degree. Then they plan a smooth trajectory for this designed output, that also bound link deformation. The nominal torque is obtained solving an inverse dynamic problem in time domain, and finally they present a control feedback law added to this nominal command.

In this paper we use the analytical dynamical model proposed by D. Li *et al.*[1], using also some equations derived by M. Burli *et al.*[3]. We will develop both the pseudo-clamped and pseudo-pinned case of F. Bellezza *et al.*[4], and for the rest to rest control of motion we use the inverse dynamics trajectory design by A. De Luca and G. Di Giovanni [2], for which has been necessary to develop the model also in a new environment, that we called "Frame on Rotor" case.

# 2 Model Description

In this section we want to determine the equations of motions for the robot in Fig. 4a: the robot is formed by an hub at the base where the motor is positioned, and the robot can rotate around the z-axis moving a payload mounted at the tip.
The motor gives to the structure the torque $\tau$ required for the motion, and between the hub and the rotor there's a spring to model the elasticity of the mechanical joint (it's always present, even if the joint is very stiff).
The beam has the following parameters: $E$ (Young Modulus), $I$ (Moment of Inertia of the cross section), $l$ (lenght of the beam) and $\rho$ (linear density of the link).
Regarding the payload: $I_P$ (Inertia of the payload) and $m_P$ (mass of the payload).
Finally for the hub and rotor: $I_H$ (Inertia of the hub) and $I_R$ (Inertia of the rotor).
Regarding the spring we have $k$ that models the joint stiffness.

To simplify the modeling of the system, we introduce the following assumption:

- Horizontal plane motion: we will consider only motion (both rigid and elastic) in the horizontal plane, this means that we neglect torsional and shear deflections
- Only small elastic deflection are possible
- The flexible joint is modeled as a simple spring
- No gravity effects are considered
- The Euler-Beam assumption is used

The flexible beam is generally formed of many fibers aligned longitudinally.
Under the action of downward transverse loads, the fibers near the top of the beam contract, whereas the fibers near the bottom of the beam extend.
Along the neutral axis of the beam, the fibers do not change their length.

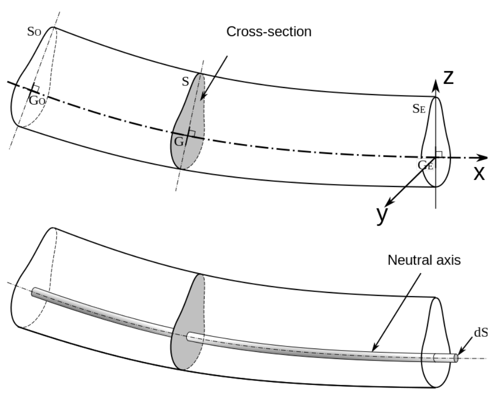



Figure 1: Beam cross section

For the Euler-Beam assumption:

- The beam section is infinitely rigid in its own plane: according to this assumption, there is no deformation in the plane of the cross-section (as we can see in Fig.2 in the bottom right part).

- The cross-section of the beam remains plane to the deformed axis of the beam: this assumption means that the cross-section of the beam remains perpendicular to the neutral axis (as we can see in Fig.2 we have a lot of cross sections all perpendicular to the neutral axis)

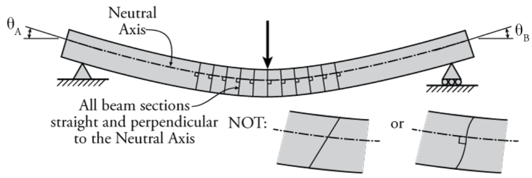
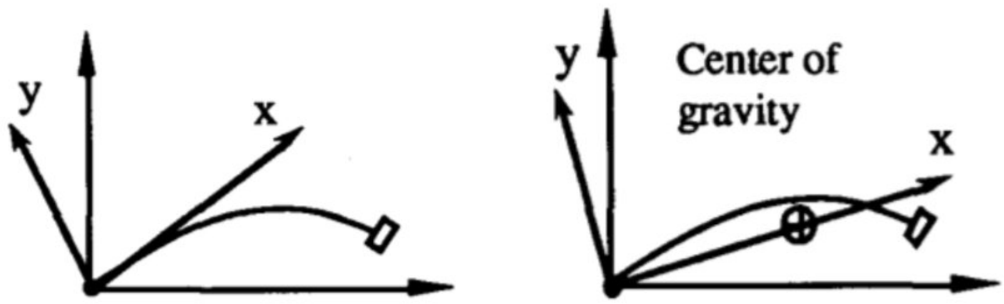



Figure 2: Euler-Beam-Assumption

In structural engineering, clamped and pinned are terms used to describe different types of connections between structural elements. These connections determine how the structure behaves.

1. Clamped: is a type of fixed connection that can resist vertical and lateral loads, it is design to hold firmly two or more parts togheter (no relative rotation)

2. Pinned: is another type of fixed connection that allows relative rotation between the connected members.




(a) Pseudo-clamped

(b) Pseudo-pinned

Figure 3: Moving reference frame

So there are two trivial different choices to describe the reference frames of this structure (Fig. 3):

1. Pseudo-clamped: the x-axis of the moving frame is tangent at the beam base, so with respect to this frame the base doesn't rotate.

2. Pseudo-pinned: The x-axis of the moving frame pass through the center of mass of the elastic beam, so its base rotates with respect to this frame.

We developed the model both in pseudo-clamped and in pseudo-pinned case; in both cases we have:

1. $Oxy$: fixed reference frame with origin at the rotor of the motor

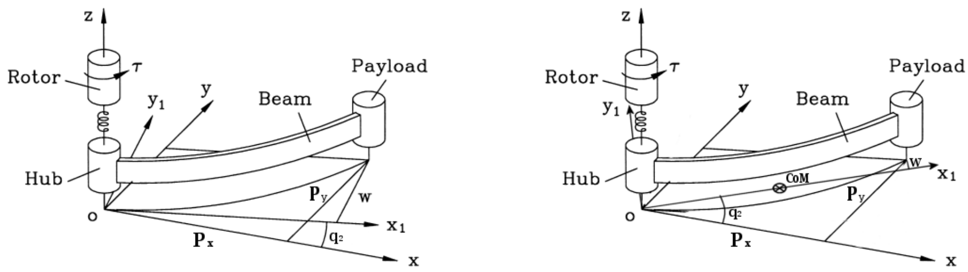
2. $Ox_ry_r$: reference frame of the rotor describing the angle $q_1(t)$ with respect to $Oxy$

then in pseudo-clamped case we can define the last reference frame:

3a. $Ox_1y_1$: reference frame with origin at the hub of the robot with the $x_1$ axis tangent to the base of the robot, that describes the angle $q_2(t)$ with respect to $Oxy$

that differs from the one used in the pseudo-pinned case:

3b. $Ox_1y_1$: reference frame that passes through the center of mass of the beam, that describes the angle $q_2(t)$ with respect to $Oxy$



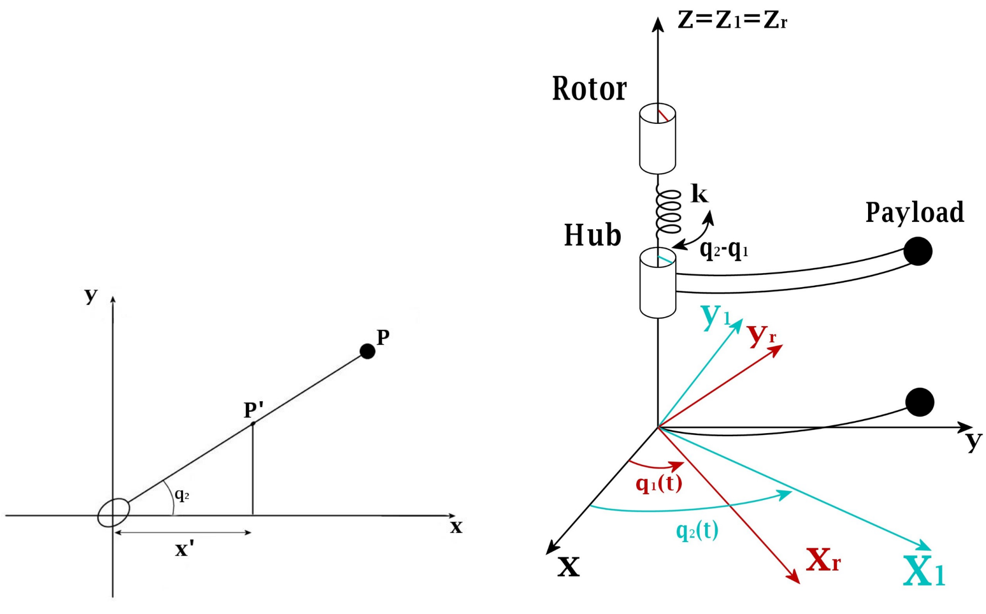

(a) Pseudo-clamped case

(b) Pseudo-pinned case

Figure 4: Description of the manipulator

As in Fig. 2, we can see two different generalized coordinates $q_1(t)$ and $q_2(t)$:

- $q_1(t)$: instantaneous angular position of the rotor with respect to the fixed frame
- $q_2(t)$: instantaneous angular position of $Ox_1y_1$ with respect to the fixed frame




(a) Rigid joint and rigid link robot

(b) Description of reference frames, pseudo-clamped case

Figure 5: Defined angles and variables

In the case of rigid joint $q_1$ coincides with $q_2$: this is because we have no angular displacement of the hub with respect to the rotor.

As we have seen with the assumptions, we have only torsional deflection around z-axis in the $Oxy$ plane, and we will use two different variables to describe this behaviour:

- $w(x_1, t)$: it is the instantaneous distance of a generic point P of the link with respect to the $x_1$ axis.

- $y(x_1, t)$: it is the instantaneous distance of a generic point P of the link with respect to the x-axis.

To simplify the notation, we will use $x$ instead of $x_1$ in all the document: this means that $w(x_1, t)$ will be $w(x, t)$ and $y(x_1, t)$ will be $y(x, t)$.

As we can see in Fig. 5b, in the case of a rigid link and rigid joint, we need just $q_1(t)$ to describe the configuration of the robot because it has only 1 degree of freedom: this means that, for example, to describe the position of a generic point P' of the structure we only need to know the $q_1(t)$ angle and use trigonometry to determine the two coordinates $P'_x$ and $P'_y$.

In the case of an elastic link robot, the information of $q_1(t)$ angle is not enough due to the deformation: in fact we should use $w(x, t)$ or $y(x, t)$ variables that depend on space and time. For example, to determine the position of P' and P" of the structure with respect to $Ox_1y_1$ reference frame at instant $t_0$ we can use the $w(x', t_0)$ and $w(x'', t_0)$ values that are the distances with respect to the $x_1$ axis. We can observe that at the following instant $t_1$ the $w(x', t_1)$ and $w(x'', t_1)$ variables are different because the link is oscillating. In the same way, for example, to determine the P' position with respect to the $Oxy$ reference frame at the initial instant $t_0$ we should use the $y(x', t_0)$ variable.

This type of structure, as we can see later, has an infinite number of degree of freedom, and this means that we should refer to an infinitesimal piece of the link: for this reason we'll obtain a dynamical system with partial differential equation (PDE).

Moreover, we can express the position of a beam point in this way:

$$P(x, t) = \begin{pmatrix} P_x \\ P_y \end{pmatrix} = \begin{pmatrix} x\cos(q_2(t)) - w(x, t)\sin(q_2(t)) \\ x\sin(q_2(t)) + w(x, t)\cos(q_2(t)) \end{pmatrix}$$

And clearly the payload is at:

$$P(l, t) = \begin{pmatrix} l\cos(q_2(t)) - w(l, t)\sin(q_2(t)) \\ l\sin(q_2(t)) + w(l, t)\cos(q_2(t)) \end{pmatrix}$$

The differential kinematics is:

$$\dot{P}(x, t) = \begin{pmatrix} \dot{P}_x \\ \dot{P}_y \end{pmatrix} = \begin{pmatrix} -x\dot{q}_2(t)\sin(q_2(t)) - \dot{w}(x, t)\sin(q_2(t)) - w(x, t)\dot{q}_2(t)\cos(q_2(t)) \\ x\dot{q}_2(t)\cos(q_2(t)) + \dot{w}(x, t)\cos(q_2(t)) - w(x, t)\dot{q}_2(t)\sin(q_2(t)) \end{pmatrix}$$

To simplify the notation from Section 3 we will drop the dependencies in the generic case $w := w(x, t)$ or $q_2 := q_2(t)$, instead we maintain the dependencies in the specific cases $w(l, t)$ and $w(0, t)$.

Since we need also the angle that a point describes with the fixed reference frame, we introduce the variable $\alpha(x, t)$ that is equal to (see Fig. 7):

$$\alpha(x, t) = q_2(t) + \gamma(x, t)$$

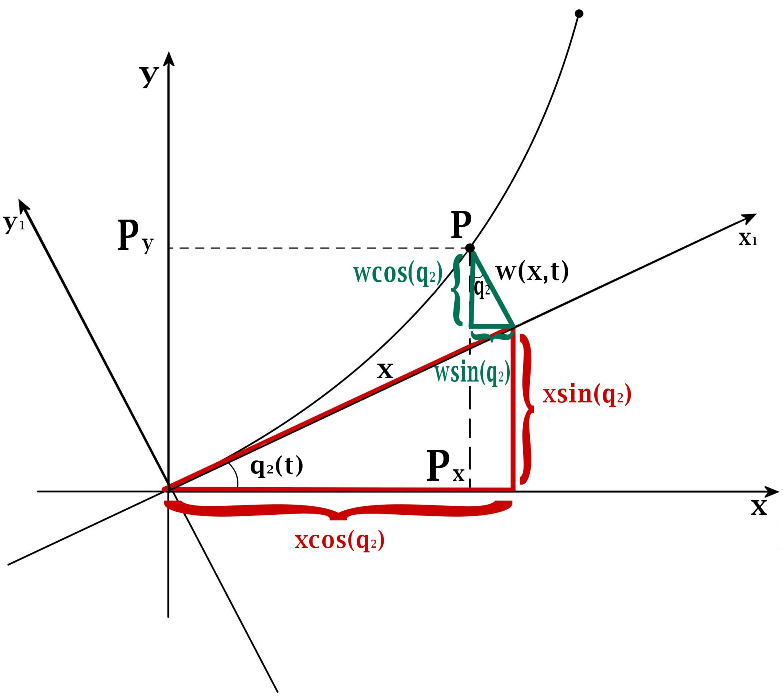



Figure 6: Geometric view of a generic beam point

for the geometric property of the derivative:

$$\gamma' = \tan^{-1}\left(\frac{\partial w(x,t)}{\partial x}\right)$$

and for small $w(x,t)$ (we remember that for the second assumption we have small deflection), we have

$$\gamma' \simeq \frac{\partial w(x,t)}{\partial x}$$
$$\gamma \simeq \gamma'$$

so we'll assume

$$\alpha(x,t) = q_2(t) + \frac{\partial w(x,t)}{\partial x}$$

Note that the relation found for the direct kinematics and angle $\gamma$ are valid for any choice of $Ox_1y_1$ (in the case of $\gamma$, the approximation is better for small $w(x,t)$).

For the rest of the paper, for a generic function $f(x,t)$, we define:

$$f'(x,t) := \frac{\partial f(x,t)}{\partial x}$$
$$\dot{f}(x,t) := \frac{\partial f(x,t)}{\partial t}$$

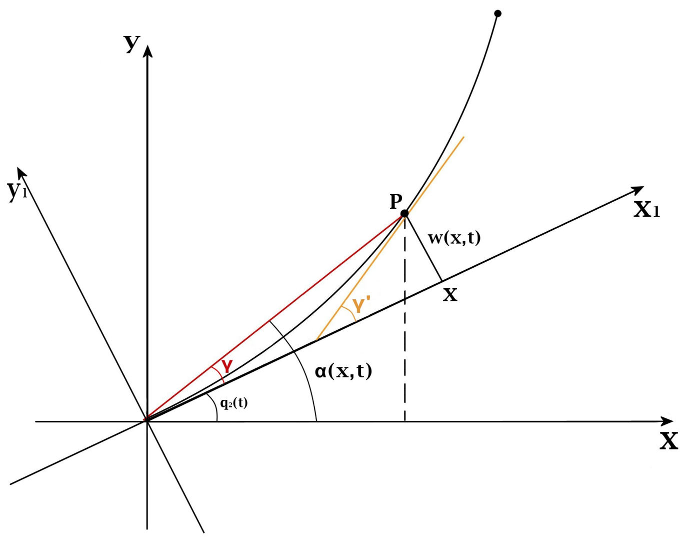



Figure 7: Approximation of the beam angle $\alpha$ in the clamped-case

# 3 Model Development

## 3.1 Hamiltonian Energy Method

To derive the equations that describe the behaviour of the beam we use a Lagrangian based approach. In particular, we search for the Lagrangian $L$ of the robot:

$$L = T - U$$

where $T$ is its kinetic energy and $U$ its potential one.

So we need the energy for each part of the structure, in particular we divide the system in several parts to obtain:

$$L = T_R + T_H + T_B + T_P - U_k - U_B$$

where:

- $T_R$ is the kinetic energy of the rotor
- $T_H$ is the kinetic energy of the hub
- $T_B$ is the kinetic energy of the beam
- $T_P$ is the kinetic energy of the payload
- $U_k$ is the potential energy of the spring between rotor and hub
- $U_B$ is the potential energy of the beam

As we can see, we don't have potential energy due to gravity effect, because from the assumption we don't consider gravity.
Following from the Hamilton principle, we should have

$$\int_{t_1}^{t_2} L(q_1, q_2, w)\, dt = \int_{t_1}^{t_2} -W(q_1)\, dt$$

where

$$W = \tau q_1$$

is the external work computed on the system. In fact that is the same of minimizing the quantity

$$\int_{t_1}^{t_2} \big(L(q_1, q_2, w) + W(q_1)\big)\ dt \tag{1}$$

Since $L$ depends on different variables, we can continue as follows: we search for the triad $(q_1^*, q_2^*, w^*)$ that minimizes (1); for doing so we separate the variation of this variables introducing an arbitrary variable $\mu$:

$$\begin{aligned} q_1(t,\mu) &= \underline{q_1}(t) + \mu\, \delta q_1(t) \\ q_2(t,\mu) &= \underline{q_2}(t) + \mu\, \delta q_2(t) \\ w(x,t,\mu) &= \underline{w}(x,t) + \mu\, \delta w(x,t) \end{aligned}$$

Since $t_1$ and $t_2$ are arbitrary, we choose "far enough" them such that:

$$\delta q_1(t_1) = \delta q_1(t_2) = \delta q_2(t_1) = \delta q_2(t_2) = \delta w(x, t_1) = \delta w(x, t_2) = 0$$

In this way the Hamilton problem is simplified as minimizing

$$I(\mu) = \int_{t_1}^{t_2} \big(L(\mu) + W(\mu)\big)\; dt = \int_{t_1}^{t_2} \big(T(\mu) - U(\mu) + W(\mu)\big)\; dt$$

so the only thing to solve is

$$\frac{dI(\mu)}{d\mu}\Big|_{\mu=0} = 0$$

This means solving the following:

$$\int_{t_1}^{t_2} \left(\frac{dT(\mu)}{d\mu} - \frac{dU(\mu)}{d\mu} + \frac{dW(\mu)}{d\mu}\right)\; dt = 0$$

For all these derivation of kinetic energy, potential energy and work with respect to $\mu$ we use the chain rule of derivation. For example, since we'll find that $T_B$ depends on $\dot{q}_2$ and $\dot{w}$, we'll have:

$$T_B(\dot{q}_2(\mu), \dot{w}(\mu)) = \frac{1}{2}\rho \int_0^l \big(x\dot{q}_2(\mu) + \dot{w}(\mu)\big)^2\, dx$$

$$\frac{dT_B}{d\mu} = \frac{dT_B}{d\dot{q}_2}\frac{d\dot{q}_2}{d\mu} + \frac{dT_B}{d\dot{w}}\frac{d\dot{w}}{d\mu}$$

We should solve all terms starting from all kinetic energy terms, where we can notice a very important simplification solving by part the following integral (since $T_R$ depends just on $\dot{q}_1(t)$):

$$\begin{aligned}
\int_{t_1}^{t_2} \frac{dT_R}{d\mu}\, dt &= \int_{t_1}^{t_2} \frac{dT_R}{d\dot{q}_1}\frac{d\dot{q}_1}{d\mu}\, dt = \frac{dT_R}{d\dot{q}_1}\frac{dq_1}{d\mu}\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\frac{dT_R}{d\dot{q}_1}\frac{dq_1}{d\mu}\, dt = \\
&= \frac{dT_R}{d\dot{q}_1}\Big|_{t_2} \cancelto{0}{\delta q_1(t_2)} - \frac{dT_R}{d\dot{q}_1}\Big|_{t_1} \cancelto{0}{\delta q_1(t_1)} - \int_{t_1}^{t_2} \frac{d}{dt}\frac{dT_R}{d\dot{q}_1}\delta q_1(t)\, dt = \\
&= -\int_{t_1}^{t_2} \frac{d}{dt}\frac{dT_R}{d\dot{q}_1}\delta q_1(t)\, dt \qquad (2)
\end{aligned}$$

### 3.1.1 Pseudo-clamped case

In the pseudo-clamped case the energy has these expressions:

$$
T_R = \frac{1}{2} I_R \dot{q}_1^2 \tag{3}
$$

$$
T_H = \frac{1}{2} I_H \dot{q}_2^2 \tag{4}
$$

since the rotor and the hub just rotate

$$
\begin{aligned}
T_B &= \frac{1}{2} \int_0^l \rho \dot{P}^T \dot{P}\, dx = \frac{1}{2} \rho \int_0^l \left( x^2 \dot{q}_2^2 + \cancelto{0}{w^2} \dot{q}_2^2 + \dot{w}^2 + 2x \dot{q}_2 \dot{w} \right) dx = \\
&= \frac{1}{2} \rho \int_0^l \left( x\dot{q}_2 + \dot{w} \right)^2 dx
\end{aligned} \tag{5}
$$

that is the translational energy of each point of the beam

$$
\begin{aligned}
T_P &= \frac{1}{2} m_P \dot{P}^T(l,t) \dot{P}(l,t) + \frac{1}{2} I_P \dot{\alpha}(l,t)^2 \\
&= \frac{1}{2} m_P \left( l^2 \dot{q}_2^2 + \cancelto{0}{w(l,t)^2} \dot{q}_2^2 + \dot{w}(l,t)^2 + 2l\dot{q}_2 \dot{w}(l,t) \right) + \frac{1}{2} I_P (\dot{q}_2 + \dot{w}'(l,t))^2 = \\
&= \frac{1}{2} m_P \left( l\dot{q}_2 + \dot{w}(l,t) \right)^2 + \frac{1}{2} I_P (\dot{q}_2 + \dot{w}'(l,t))^2
\end{aligned} \tag{6}
$$

that is the payload translational and rotational energy, the latter due to the rotation around the $z$ axis with a time variable angle of $q_2(t) + w'(l,t)$

$$
U_k = \frac{1}{2} k (q_2 - q_1)^2 \tag{7}
$$

that is the potential elastic energy of the spring

$$
U_B = \frac{1}{2} EI \int_0^l w''^2\, dx \tag{8}
$$

that is the potential energy of a flexible beam[1].

The boundary conditions in the pseudo-clamped case are given by the following two equations:

$$
w(0,t) = 0 \tag{9}
$$

$$
w'(0,t) = 0 \tag{10}
$$

from which

$$
\delta w(0,t) = 0 \tag{11}
$$

$$
\delta w'(0,t) = 0 \tag{12}
$$

Equation (9) means that the deflection of the origin O is always zero.
We know that $w'(x,t)$ is the angle between the line that pass from the point $P(x,y)$ on the link and the $x_1$ axis: the equation (10) means that $w'(0,t)$ is zero (this is true only in the pseudo-clamped case, because as we can see later, taking $x_1$ passing through the center of

[1] See Section 8.1 for details

mass, as in pseudo-pinned case, will result in $w'(0,t)$ different from zero).
Finally the two equations (11) and (12) are trivial, because the variation of an identically null function is null as well.

So we have all the derived terms expressed in term of $\delta q_1(t)$, $\delta q_2(t)$ or $\delta w(x,t)$ (with some exceptions for the boundary conditions), and we need to compare only the terms that affect the same variation; we proceed by expressing each kinetic energy term highlighting the variation that it affects, as we've done for the rotor kinetic energy.

$$\begin{aligned}
&\int_{t_1}^{t_2} \frac{dT_R}{d\mu}\,dt = -\int_{t_1}^{t_2} \frac{d}{dt}\frac{dT_R}{d\dot{q}_1}\delta q_1(t)\,dt \\
&\int_{t_1}^{t_2} \frac{dT_H}{d\mu}\,dt = \int_{t_1}^{t_2} \frac{dT_H}{d\dot{q}_2}\frac{d\dot{q}_2}{d\mu}dt = -\int_{t_1}^{t_2} \frac{d}{dt}\frac{dT_H}{d\dot{q}_2}\delta q_2\,dt \\
&\int_{t_1}^{t_2} \frac{dT_B}{d\mu}\,dt = \int_{t_1}^{t_2} \frac{dT_B}{d\dot{q}_2}\frac{d\dot{q}_2}{d\mu}dt + \int_{t_1}^{t_2} \frac{dT_B}{d\dot{w}}\frac{d\dot{w}}{d\mu}dt \\
&\qquad\qquad\quad = -\int_{t_1}^{t_2} \frac{d}{dt}\frac{dT_B}{d\dot{q}_2}\delta q_2\,dt - \int_{t_1}^{t_2} \frac{d}{dt}\frac{dT_B}{d\dot{w}}\delta w\,dt \\
&\int_{t_1}^{t_2} \frac{dT_P}{d\mu}\,dt = \int_{t_1}^{t_2} \frac{dT_P}{d\dot{q}_2}\frac{d\dot{q}_2}{d\mu}dt + \int_{t_1}^{t_2} \frac{dT_P}{d\dot{w}(l,t)}\frac{d\dot{w}(l,t)}{d\mu}dt + \int_{t_1}^{t_2} \frac{dT_P}{d\dot{w}'(l,t)}\frac{d\dot{w}'(l,t)}{d\mu}dt \\
&\qquad\qquad\quad = -\int_{t_1}^{t_2} \frac{d}{dt}\frac{dT_P}{d\dot{q}_2}\delta q_2\,dt - \int_{t_1}^{t_2} \frac{d}{dt}\frac{dT_P}{d\dot{w}(l,t)}\delta w(l,t)\,dt - \int_{t_1}^{t_2} \frac{d}{dt}\frac{dT_P}{d\dot{w}'(l,t)}\delta w'(l,t)\,dt
\end{aligned}$$

For the potential energy terms and the work term, we don't need the procedure that we've done for the rotor kinetic energy, but we can immediately see the dependence on $\delta q_1(t)$, $\delta q_2(t)$ or $\delta w(x,t)$ using the chain rule of derivation.

$$\begin{aligned}
\int_{t_1}^{t_2} -\frac{dU_k}{d\mu}\,dt &= -\int_{t_1}^{t_2} \frac{dU_k}{dq_1}\frac{dq_1}{d\mu}dt - \int_{t_1}^{t_2} \frac{dU_k}{dq_2}\frac{dq_2}{d\mu}dt \\
&= -\int_{t_1}^{t_2} \frac{dU_k}{dq_1}\delta q_1(t)\,dt - \int_{t_1}^{t_2} \frac{dU_k}{dq_2}\delta q_2\,dt
\end{aligned}$$

For the beam potential energy $U_B$ we can see that to highlight the dependence on $\delta w(l,t)$, $\delta w'(l,t)$ and $\delta w(x,t)$, need several calculations, first substituting the definition of $\frac{dU_B}{dw''}$ and solving the integral by part different times:

$$
\begin{aligned}
&\int_{t_1}^{t_2} -\frac{dU_B}{d\mu}\,dt = -\int_{t_1}^{t_2} \frac{dU_B}{dw''}\frac{\partial w''}{\partial \mu}\,dt = -\int_{t_1}^{t_2} EI\int_0^l w''\delta w''\,dx\,dt = \\
&= -EI\int_{t_1}^{t_2}\left(w''\delta w'|_0^l - \int_0^l w'''\delta w'\,dx\right)dt = \\
&= -EI\int_{t_1}^{t_2}\left(w''(l,t)\delta w'(l,t) - w''(0,t)\cancelto{0}{\delta w'(0,t)} - \left(w'''\delta w|_0^l - \int_0^l w''''\delta w\,dx\right)\right)dt = \\
&= -EI\int_{t_1}^{t_2}\left(w''(l,t)\delta w'(l,t) - w'''(l,t)\delta w(l,t) + w'''(0,t)\cancelto{0}{\delta w(0,t)} + \int_0^l w''''\delta w\,dx\right)dt = \\
&= -EI\int_{t_1}^{t_2}\left(w''(l,t)\delta w'(l,t) - w'''(l,t)\delta w(l,t) + \int_0^l w''''\,\delta w dx\right)dt
\end{aligned}
$$

Finally the work:

$$
\int_{t_1}^{t_2}\frac{dW}{d\mu}\,dt = \int_{t_1}^{t_2}\frac{dW}{dq_1}\frac{dq_1}{d\mu}dt = \int_{t_1}^{t_2}\frac{dW}{dq_1}\delta q_1\,dt
$$

Now I can set equal to zero the terms that affects the same variation, starting from $\delta q_1(t)$:

$$
\begin{aligned}
-\frac{d}{dt}\frac{dT_R}{d\dot{q}_1} - \frac{dU_k}{dq_1} + \frac{dW}{dq_1} &= 0 \\
-\frac{d}{dt}I_R\dot{q}_1 + k(q_2-q_1) + \tau &= 0 \\
-I_R\ddot{q}_1 + k(q_2-q_1) + \tau &= 0 \\
I_R\ddot{q}_1 = \tau + k(q_2-q_1) & \qquad (13)
\end{aligned}
$$

Then, for $\delta q_2(t)$:

$$
-\frac{d}{dt}\frac{dT_H}{d\dot{q}_2} - \frac{d}{dt}\frac{dT_B}{d\dot{q}_2} - \frac{d}{dt}\frac{dT_P}{d\dot{q}_2} - \frac{dU_k}{dq_2} = 0
$$

$$
-\frac{d}{dt}I_H\dot{q}_2 - \rho\frac{d}{dt}\int_0^l\left(x^2\dot{q}_2 + x\dot{w}\right)dx - \frac{d}{dt}m_p(l^2\dot{q}_2 + l\dot{w}(l,t)) - \frac{d}{dt}I_P(\dot{q}_2 + \dot{w}'(l,t)) - k(q_2-q_1) = 0
$$

$$
I_H\ddot{q}_2 + \rho\int_0^l(x^2\ddot{q}_2 + x\ddot{w})\,dx + m_P(l^2\ddot{q}_2 + l\ddot{w}(l,t)) + I_P(\ddot{q}_2 + \ddot{w}'(l,t)) + k(q_2-q_1) = 0
$$

$$
I_S\ddot{q}_2 + \mu(t) + k(q_2-q_1) = 0 \qquad (14)
$$

being

$$
I_S = I_H + \rho\frac{l^3}{3} + m_P l^2 + I_P
$$

$$
\mu(t) = \rho\int_0^l x\ddot{w}\,dx + m_P l\ddot{w}(l,t) + I_P\ddot{w}'(l,t)
$$

Now I can proceed for $\delta w(x,t)$:

$$\begin{aligned}
-\frac{d}{dt}\frac{dT_B}{d\dot{w}} - EI\int_0^l w'''' \, dx = 0 \\
-\frac{d}{dt}\rho\int_0^l (\dot{w} + x\dot{q}_2)\, dx - EI\int_0^l w'''' \, dx = 0 \\
-\int_0^l \Big(\rho(\ddot{w} + x\ddot{q}_2) + EIw''''\Big)\, dx = 0 \\
\rho(\ddot{w} + x\ddot{q}_2) + EIw'''' = 0 \qquad (15)
\end{aligned}$$

and the evaluation on $\delta w(l,t)$ and $\delta w'(l,t)$ give boundary conditions:

$$\begin{aligned}
-\frac{d}{dt}\frac{dT_P}{d\dot{w}(l,t)} + EIw'''(l,t) = 0 \\
-\frac{d}{dt}m_P(\dot{w}(l,t) + l\dot{q}_2) + EIw'''(l,t) = 0 \\
EIw'''(l,t) = m_P(\ddot{w}(l,t) + l\ddot{q}_2) \qquad (16) \\
-\frac{d}{dt}\frac{dT_P}{d\dot{w}'(l,t)} - EIw''(l,t) = 0 \\
-\frac{d}{dt}I_P(\dot{q}_2 + \dot{w}'(l,t)) - EIw''(l,t) = 0 \\
EIw''(l,t) = -I_P(\ddot{q}_2 + \ddot{w}'(l,t)) \qquad (17)
\end{aligned}$$

Now I report all the equations got with the four boundary conditions:

$$\begin{aligned}
I_R\ddot{q}_1 = \tau + k(q_2 - q_1) \qquad (13) \\
I_S\ddot{q}_2 + \mu(t) = -k(q_2 - q_1) \qquad (14) \\
EIw'''' + \rho(\ddot{w} + x\ddot{q}_2) = 0 \qquad (15) \\
w(0,t) = 0 \qquad (9) \\
w'(0,t) = 0 \qquad (10) \\
EIw'''(l,t) = m_P(\ddot{w}(l,t) + l\ddot{q}_2) \qquad (16) \\
EIw''(l,t) = -I_P(\ddot{w}'(l,t) + \ddot{q}_2) \qquad (17)
\end{aligned}$$

that are equal to the equations found in [1] (pagg. 48 and 49)
One could merge (13) and (14) obtaining

$$I_R\ddot{q}_1 + I_S\ddot{q}_2 + \mu(t) = \tau \qquad (18)$$

That is the equivalent of equation (1b) found in [2]

### 3.1.2 Frame-on-rotor case

The Frame-on-rotor case (FOR) is a particular choice for the reference frames in which $Ox_1y_1 \equiv Ox_ry_r$

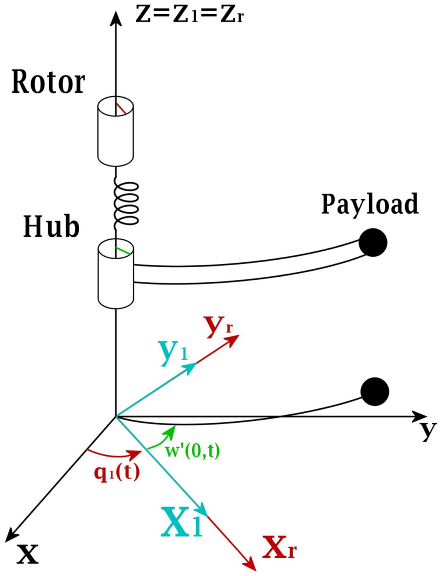



Figure 8: FOR References Frames

Now the angle $q_2(t) \equiv q_1(t)$ because the rotor frame $Ox_ry_r$ and moving frame $Ox_1y_1$ coincide.

With these change of reference frames, the boundary conditions in the FOR case change also:

$$w(0,t) = 0 \tag{9}$$
$$w'(0,t) \neq 0$$

from which their variation:

$$\delta w(0,t) = 0 \tag{11}$$
$$\delta w'(0,t) \neq 0$$

Equations (9) and (11) remain the same of the pseudo-clamped case meaning that the deflection and its variation at the origin O is always zero. Recalling that $w'(x,t)$ is the angle between the line passing from a generic point $P(x,y)$ on the beam and the $x_1$ axis, then in this case $w'(0,t)$ at the origin should be not null (the same for its variation).
The angle of the beam at origin is $\alpha(0,t) = q_1(t) + w'(0,t)$ (Fig. 8).
This choice is crucial for the control since, as we'll see later, we need to compute the mode analysis of this system, and the FOR system simplifies it a lot.

We report all energy terms, both changed (Hub KE, Spring PE) and unchanged (Rotor KE, Beam KE and PE, Payload KE, Work):

$$
\begin{aligned}
T_R^* &= T_R = \frac{1}{2} I_R \dot{q}_1^2 \\
T_H^* &= \frac{1}{2} I_H (\dot{q}_1 + \dot{w}'(0,t))^2 \\
T_B^* &= T_B = \frac{1}{2} \rho \int_0^l \left( x\dot{q}_1 + \dot{w}(x,t) \right)^2 dx \\
T_P^* &= T_P = \frac{1}{2} m_P \left( l\dot{q}_1 + \dot{w}(l,t) \right)^2 + \frac{1}{2} I_P (\dot{q}_1 + \dot{w}'(l,t))^2 \\
U_k^* &= \frac{1}{2} k w'(0,t)^2 \\
U_B^* &= U_B = \frac{1}{2} EI \int_0^l w''(x,t)^2 \, dx \\
W^* &= W = \tau q_1
\end{aligned}
$$

Knowing all energy terms, we can proceed as in the pseudo-clamped case, expressing all integrals as variation of $\delta q_1(t)$, $\delta w(x,t)$, $\delta w(l,t)$, $\delta w'(0,t)$ and $\delta w'(l,t)$:

$$
\begin{aligned}
\int_{t_1}^{t_2} \frac{dT_R^*}{d\mu} \, dt &= -\int_{t_1}^{t_2} \frac{d}{dt} \frac{dT_R^*}{d\dot{q}_1} \delta q_1(t) \, dt \\
\int_{t_1}^{t_2} \frac{dT_H^*}{d\mu} \, dt &= -\int_{t_1}^{t_2} \frac{d}{dt} \frac{dT_H^*}{d\dot{q}_1} \delta q_1(t) \, dt - \int_{t_1}^{t_2} \frac{d}{dt} \frac{dT_H^*}{d\dot{w}'(0,t)} \delta w'(0,t) \, dt \\
\int_{t_1}^{t_2} \frac{dT_B^*}{d\mu} \, dt &= -\int_{t_1}^{t_2} \frac{d}{dt} \frac{dT_B^*}{d\dot{q}_1} \delta q_1(t) \, dt - \int_{t_1}^{t_2} \frac{d}{dt} \frac{dT_B^*}{d\dot{w}} \delta w(x,t) \, dt \\
\int_{t_1}^{t_2} \frac{dT_P^*}{d\mu} \, dt &= -\int_{t_1}^{t_2} \frac{d}{dt} \frac{dT_P^*}{d\dot{q}_1} \delta q_1(t) \, dt - \int_{t_1}^{t_2} \frac{d}{dt} \frac{dT_P^*}{d\dot{w}(l,t)} \delta w(l,t) \, dt - \int_{t_1}^{t_2} \frac{d}{dt} \frac{dT_P^*}{d\dot{w}'(l,t)} \delta w'(l,t) \, dt \\
\int_{t_1}^{t_2} -\frac{dU_k^*}{d\mu} \, dt &= -\int_{t_1}^{t_2} \frac{dU_k^*}{dw'(0,t)} \delta w'(0,t) \, dt \\
\int_{t_1}^{t_2} -\frac{dU_B^*}{d\mu} \, dt &= -\int_{t_1}^{t_2} \frac{dU_B^*}{dw''} \frac{\partial w''}{\partial \mu} \, dt \\
\int_{t_1}^{t_2} \frac{dW^*}{d\mu} \, dt &= \int_{t_1}^{t_2} \frac{dW^*}{dq_1} \delta q_1(t) \, dt
\end{aligned}
$$

Now I can evaluate the terms that affects the same variation, starting from $\delta q_1(t)$:

$$
\begin{aligned}
-\frac{d}{dt}\frac{dT_R^*}{d\dot{q}_1} - \frac{d}{dt}\frac{dT_H^*}{dq_1} - \frac{d}{dt}\frac{dT_B^*}{dq_1} - \frac{d}{dt}\frac{dT_P^*}{dq_1} + \frac{dW^*}{dq_1} &= 0 \\
-\frac{d}{dt} I_R \dot{q}_1 - \frac{d}{dt} I_H (\dot{q}_1 + \dot{w}'(0,t)) - \frac{d}{dt} \rho \int_0^l (x\dot{q}_1 + \dot{w}(x,t)) x \, dx + & \\
-\frac{d}{dt} m_P (l\dot{q}_1 + \dot{w}(l,t)) l - \frac{d}{dt} I_P (\dot{q}_1 + \dot{w}'(l,t)) + \tau &= 0 \\
-I_R \ddot{q}_1 - I_H (\ddot{q}_1 + \ddot{w}'(0,t)) - \rho \int_0^l (x^2 \ddot{q}_1 + x\ddot{w}(x,t)) \, dx + & \\
-m_P (l^2 \ddot{q}_1 + l\ddot{w}(l,t)) - I_P (\ddot{q}_1 + \ddot{w}'(l,t)) + \tau &= 0 \\
I_T \ddot{q}_1 + \nu(t) &= \tau \qquad (19)
\end{aligned}
$$

being

$$I_T = I_R + I_H + \rho\frac{l^3}{3} + m_P l^2 + I_P = I_R + I_S$$

$$\nu(t) = I_H\ddot{w}'(0,t) + \rho\int_0^l x\ddot{w}(x,t)\,dx + m_P l\ddot{w}(l,t) + I_P\ddot{w}'(l,t) = I_H\ddot{w}'(0,t) + \mu(t)$$

Then, for $\delta w(x,t)$:

$$\begin{aligned}
-\frac{d}{dt}\frac{dT_B^*}{d\dot{w}} - EI\int_0^l w'''' \,dx &= 0\\
-\frac{d}{dt}\rho\int_0^l (\dot{w} + x\dot{q}_1)\,dx - EI\int_0^l w''''\,dx &= 0\\
-\rho\int_0^l (\ddot{w} + x\ddot{q}_1) + EIw'''')\,dx &= 0\\
\rho(\ddot{w} + x\ddot{q}_1) + EIw'''' &= 0 \qquad (20)
\end{aligned}$$

and the boundary conditions, obtained from $\delta w(l,t)$, $\delta w'(0,t)$ and $\delta w'(l,t)$:

$$\begin{aligned}
-\frac{d}{dt}\frac{dT_P^*}{dw(\dot{l},t)} + EIw'''(l,t) &= 0\\
-\frac{d}{dt}m_P(\dot{w}(l,t) + l\dot{q}_1) + EIw'''(l,t) &= 0\\
EIw'''(l,t) &= m_P(\ddot{w}(l,t) + l\ddot{q}_1) \qquad (21)
\end{aligned}$$

$$\begin{aligned}
-\frac{d}{dt}\frac{dT_H^*}{d\dot{w}'(0,t)} - \frac{dU_k^*}{dw'(0,t)} + EIw''(0,t) &= 0\\
-\frac{d}{dt}I_H(\dot{q}_1 + \dot{w}'(0,t)) - kw'(0,t) + EIw''(0,t) &= 0\\
-I_H(\ddot{q}_1 + \ddot{w}'(0,t)) - kw'(0,t) + EIw''(0,t) &= 0\\
EIw''(0,t) &= I_H(\ddot{q}_1 + \ddot{w}'(0,t)) + kw'(0,t) \qquad (22)
\end{aligned}$$

$$\begin{aligned}
-\frac{d}{dt}\frac{dT_P^*}{d\dot{w}'(l,t)} - EIw''(l,t) &=\\
-\frac{d}{dt}I_P(\dot{q}_1 + \dot{w}'(l,t)) - EIw''(l,t) &= 0\\
EIw''(l,t) &= -I_P(\ddot{q}_1 + \ddot{w}'(l,t)) \qquad (23)
\end{aligned}$$

All the obtained equations are

$$\begin{aligned}
I_T\ddot{q}_1 + \nu(t) &= \tau && (19)\\
\rho(\ddot{w} + x\ddot{q}_1) + EIw'''' &= 0 && (20)\\
w(0,t) &= 0 && (9)\\
EIw'''(l,t) &= m_P(\ddot{w}(l,t) + l\ddot{q}_1) && (21)\\
EIw''(0,t) &= I_H(\ddot{q}_1 + \ddot{w}'(0,t)) + kw'(0,t) && (22)\\
EIw''(l,t) &= -I_P(\ddot{q}_1 + \ddot{w}'(l,t)) && (23)
\end{aligned}$$

We can also use the boundary condition (22) to explicit the influence of $k$ on $q_1$; the (19) is

$$(I_R + \rho\frac{l^3}{3} + m_P l^2 + I_P)\ddot{q}_1 + I_H(\ddot{q}_1 + \ddot{w}'(0,t)) + \mu(t) = \tau$$

so

$$(I_T - I_H)\ddot{q}_1 + \mu(t) + EIw''(0,t) = \tau + kw'(0,t)$$

### 3.1.3 Pseudo-pinned case

In this section we develop the pseudo-pinned model of the robot.
The boundary conditions in the pseudo-pinned case change with respect to the pseudo-clamped case:

$$\begin{aligned} w(0,t) &= 0 \qquad (9) \\ w'(0,t) &\neq 0 \end{aligned}$$

from which

$$\begin{aligned} \delta w(0,t) &= 0 \qquad (11) \\ \delta w'(0,t) &\neq 0 \qquad (24) \end{aligned}$$

Equations (9) and (11) remain the same of the pseudo-clamped case meaning that the deflection and its variation at the origin O is always zero.
In the pseudo-pinned case, since $w'(x,t)$ is the angle between the line passing from a generic point $P(x,y)$ on the link and the $x_1$ axis, the angle $w'(0,t)$ at the base is not null anymore, neither its variation.
Since $q_2(t)$ is the angle between the fixed reference frame and the rotating frame, now the angle of the beam at the origin is $q_2(t) + w'(0,t)$, so the energy terms that contains this angle has to be changed.
We report all energy terms, both changed and unchanged:

$$\begin{aligned}
T_R^* &= T_R = \frac{1}{2} I_R \dot{q}_1^2 \\
T_H^* &= \frac{1}{2} I_H (\dot{q}_2 + \dot{w}'(0,t))^2 \\
T_B^* &= T_B = \frac{1}{2}\rho \int_0^l (x\dot{q}_2 + \dot{w})^2 \, dx \\
T_P^* &= T_P = \frac{1}{2} m_P \left(l\dot{q}_2 + \dot{w}(l,t)\right)^2 + \frac{1}{2} I_P (\dot{q}_2 + \dot{w}'(l,t))^2 \\
U_k^* &= \frac{1}{2} k (q_2 + w'(0,t) - q_1)^2 \\
U_B^* &= U_B = \frac{1}{2} EI \int_0^l w''^2 \, dx \\
W^* &= W = \tau q_1
\end{aligned}$$

it's easy to notice that these expressions are the same as before if we consider $w'(0,t) = 0$.

We have to evaluate again the lagrangian terms:

$$\begin{aligned}
\int_{t_1}^{t_2} \frac{dT_R^*}{d\mu} \, dt &= -\int_{t_1}^{t_2} \frac{d}{dt}\frac{dT_R^*}{d\dot{q}_1} \delta q_1 \, dt \\
\int_{t_1}^{t_2} \frac{dT_H^*}{d\mu} \, dt &= -\int_{t_1}^{t_2} \frac{d}{dt}\frac{dT_H^*}{d\dot{q}_2} \delta q_2 \, dt - \int_{t_1}^{t_2} \frac{d}{dt}\frac{dT_H^*}{d\dot{w}'(0,t)} \delta w'(0,t) \, dt \\
\int_{t_1}^{t_2} \frac{dT_B^*}{d\mu} \, dt &= -\int_{t_1}^{t_2} \frac{d}{dt}\frac{dT_B^*}{d\dot{q}_2} \delta q_2 \, dt - \int_{t_1}^{t_2} \frac{d}{dt}\frac{dT_B^*}{d\dot{w}} \delta w \, dt \\
\int_{t_1}^{t_2} \frac{dT_P^*}{d\mu} \, dt &= -\int_{t_1}^{t_2} \frac{d}{dt}\frac{dT_P^*}{d\dot{q}_2} \delta q_2(t) \, dt - \int_{t_1}^{t_2} \frac{d}{dt}\frac{dT_P^*}{d\dot{w}(l,t)} \delta w(l,t) \, dt - \int_{t_1}^{t_2} \frac{d}{dt}\frac{dT_P^*}{d\dot{w}'(l,t)} \delta w'(l,t) \, dt
\end{aligned}$$

The term relative to the beam potential energy in this case is different, because in the pseudo-clamped case was evaluated with the assumption of $w'(0,t)=0$.

$$\int_{t_1}^{t_2} -\frac{dU_k^*}{d\mu}\,dt = -\int_{t_1}^{t_2} \frac{dU_k^*}{dq_1}\delta q_1\,dt - \int_{t_1}^{t_2} \frac{dU_k^*}{dq_2}\delta q_2\,dt - \int_{t_1}^{t_2} \frac{dU_k^*}{dw'(0,t)}\delta w'(0,t)\,dt$$
$$\begin{aligned}\int_{t_1}^{t_2} -\frac{dU_B^*}{d\mu}\,dt &= -\int_{t_1}^{t_2} \frac{dU_B^*}{dw''}\frac{\partial w''}{\partial \mu}\,dt \\ &= -EI\int_{t_1}^{t_2}\left( w''(l,t)\delta w'(l,t) - w''(0,t)\delta w'(0,t) - w'''(l,t)\delta w(l,t) + \int_0^l w''''\delta w\,dx \right) dt\end{aligned}$$
$$\int_{t_1}^{t_2} \frac{dW^*}{d\mu}\,dt = \int_{t_1}^{t_2} \frac{dW^*}{dq_1}\delta q_1\,dt$$

Now, as in the pseudo-clamped case, we can evaluate the terms that affects the same variation, starting from $\delta q_1(t)$:

$$\begin{aligned}-\frac{d}{dt}\frac{dT_R^*}{d\dot q_1} - \frac{dU_k^*}{dq_1} + \frac{dW^*}{dq_1} &= 0 \\ -\frac{d}{dt}I_R\dot q_1 + k(q_2 + w'(0,t) - q_1) + \tau &= 0 \\ -I_R\ddot q_1 + k(q_2 + w'(0,t) - q_1) + \tau &= 0 \\ I_R\ddot q_1 &= \tau + k(q_2 + w'(0,t) - q_1) \qquad (25)\end{aligned}$$

Then, for $\delta q_2(t)$:

$$-\frac{d}{dt}\frac{dT_H^*}{d\dot q_2} - \frac{d}{dt}\frac{dT_B^*}{d\dot q_2} - \frac{d}{dt}\frac{dT_P^*}{d\dot q_2} - \frac{dU_k^*}{dq_2} = 0$$
$$-\frac{d}{dt}I_H(\dot q_2 + \dot w'(0,t)) - \rho\frac{d}{dt}\int_0^l \left(x^2\dot q_2 + x\dot w\right)dx +$$
$$-\frac{d}{dt}m_p(l^2\dot q_2 + l\dot w(l,t)) - \frac{d}{dt}I_P(\dot q_2 + \dot w'(l,t)) - k(q_2 + w'(0,t) - q_1) = 0$$
$$I_H(\ddot q_2 + \ddot w'(0,t)) + \rho\int_0^l (x^2\ddot q_2 + x\ddot w)\,dx + m_P(l^2\ddot q_2 + l\ddot w(l,t)) + I_P(\ddot q_2 + \ddot w'(l,t)) = -k(q_2 + w'(0,t) - q_1)$$
$$I\ddot q_2 + I_H\ddot w'(0,t) + \rho\int_0^l x\ddot w\,dx + m_P l\ddot w(l,t) + I_P\ddot w'(l,t) = -k(q_2 + w'(0,t) - q_1)$$
$$I_S\ddot q_2 + \nu(t) = -k(q_2 + w'(0,t) - q_1) \qquad (26)$$

being

$$I_S = I_H + \rho\frac{l^3}{3} + m_P l^2 + I_P$$
$$\nu(t) = I_H\ddot w'(0,t) + \rho\int_0^l x\ddot w\,dx + m_P l\ddot w(l,t) + I_P\ddot w'(l,t) = I_H\ddot w'(0,t) + \mu(t)$$

Now I can proceed for $\delta w(x,t)$:

$$\begin{aligned}-\frac{d}{dt}\frac{dT_B^*}{d\dot w} - EI\int_0^l w''''\,dx &= 0 \\ -\frac{d}{dt}\rho\int_0^l (\dot w + x\dot q_2)\,dx - EI\int_0^l w''''\,dx &= 0 \\ -\int_0^l \rho(\ddot w + x\ddot q_2) + EIw''''\,dx &= 0 \\ \rho(\ddot w + x\ddot q_2) + EIw'''' &= 0 \qquad (15)\end{aligned}$$

and the evaluation on $\delta w(l,t)$, $\delta w'(0,t)$ and $\delta w'(l,t)$ gives boundary conditions:

$$\begin{aligned}
-\frac{d}{dt}\frac{dT_P^*}{d\dot{w}(l,t)} + EIw'''(l,t) = 0 \\
-\frac{d}{dt}m_P(\dot{w}(l,t) + l\dot{q}_2) + EIw'''(l,t) = 0 \\
EIw'''(l,t) = m_P(\ddot{w}(l,t) + l\ddot{q}_2)
\end{aligned} \tag{27}$$

$$\begin{aligned}
-\frac{d}{dt}\frac{dT_H^*}{d\dot{w}'(0,t)} - \frac{dU_k^*}{dw'(0,t)} + EIw''(0,t) = 0 \\
-\frac{d}{dt}I_H(\dot{q}_2 + \dot{w}'(0,t)) - k(q_2 + w'(0,t) - q_1) + EIw''(0,t) = 0 \\
-I_H(\ddot{q}_2 + \ddot{w}'(0,t)) - k(q_2 + w'(0,t) - q_1) + EIw''(0,t) = 0 \\
EIw''(0,t) = I_H(\ddot{q}_2 + \ddot{w}'(0,t)) + k(q_2 + w'(0,t) - q_1)
\end{aligned} \tag{28}$$

$$\begin{aligned}
-\frac{d}{dt}\frac{dT_P^*}{d\dot{w}'(l,t)} - EIw''(l,t) = \\
-\frac{d}{dt}I_P(\dot{q}_2 + \dot{w}'(l,t)) - EIw''(l,t) = 0 \\
EIw''(l,t) = -I_P(\ddot{q}_2 + \ddot{w}'(l,t))
\end{aligned} \tag{29}$$

All the obtained equations are

$$I_R\ddot{q}_1 = \tau + k(q_2 + w'(0,t) - q_1) \tag{25}$$

$$I_S\ddot{q}_2 + \nu(t) = -k(q_2 + w'(0,t) - q_1) \tag{26}$$

$$\rho(\ddot{w} + x\ddot{q}_2) + EIw'''' = 0 \tag{15}$$

$$w(0,t) = 0 \tag{9}$$

$$EIw'''(l,t) = m_P(\ddot{w}(l,t) + l\ddot{q}_2) \tag{27}$$

$$EIw''(l,t) = -I_P(\ddot{q}_2 + \ddot{w}'(l,t)) \tag{29}$$

$$EIw''(0,t) = I_H(\ddot{q}_2 + \ddot{w}'(0,t)) + k(q_2 + w'(0,t) - q_1) \tag{28}$$

Merging (25) and (26) leads to

$$I_R\ddot{q}_1 + I_S\ddot{q}_2 + \nu(t) = \tau \tag{30}$$

while merging (25) and (28) leads to

$$EIw''(0,t) = I_R\ddot{q}_1 + I_H(\ddot{q}_2 + \ddot{w}'(0,t)) - \tau \tag{31}$$

# 4 Dynamic Model

Since the system is continuous and not discrete, the oscillation of the beam is represented as the superposition of infinite oscillation modes. A generic oscillation mode has the form

$$w_i(x,t) = \phi_i(x)\delta_i(t)$$

so the continuous system has an oscillation that is

$$w(x,t) = \sum_{i=1}^{\infty} w_i(x,t) = \sum_{i=1}^{\infty} \phi_i(x)\delta_i(t) \tag{32}$$

With this space-time separation and $\ddot{q}_2$ set to zero, we can solve equation (15) to find the free behaviour of the $\phi_i$ and $\delta_i$ functions; we get:

$$EI \sum_{i=1}^{\infty} \phi_i''''(x)\delta_i(t) = -\rho \sum_{i=1}^{\infty} \phi_i(x)\ddot{\delta}_i(t)$$

Since this relation holds for each vibration mode (i.e. for each $i$), we can write this relation for a generic mode:

$$EI\phi_i''''(x)\delta_i(t) = -\rho\phi_i(x)\ddot{\delta}_i(t)$$

From which we get one equation for $\phi(x)$ and another for $\delta(t)$:

$$\frac{EI}{\rho}\frac{\phi_i''''(x)}{\phi_i} = -\frac{\ddot{\delta}_i(t)}{\delta_i(t)} = \omega^2$$

$$\phi_i'''' = \frac{\rho\omega^2}{EI}\phi_i = \beta^4\phi_i \tag{33}$$

$$\ddot{\delta}_i = -\omega^2\delta_i \tag{34}$$

We can solve the (33) as an eigenvalue problem, in fact we can write it as

$$\mathcal{L}(\phi_i) = \omega^2\phi_i(x)$$

with $\mathcal{L}$ similar to a Sturm-Liouville operator:

$$\mathcal{L} = \frac{EI}{\rho}\frac{d^4}{dx^4}$$

Introduced the $L^2$ inner product

$$< f, g >= \int_0^l f(x)g(x)\,dx$$

We have that $\mathcal{L}$ is self-adjoint[2], so

$$< \mathcal{L}(f), g >=< f, \mathcal{L}(g) >$$

in particular

$$\int_0^l \mathcal{L}(\phi_i)\phi_j\,dx = EI\int_0^l \phi_i''''(x)\phi_j(x)\,dx = EI\int_0^l \phi_i(x)\phi_j''''(x)\,dx = \int_0^l \phi_i\mathcal{L}(\phi_j)\,dx$$

[2]Proof in Section 8.2

moreover, with the self-adjointness of $\mathcal{L}$ we know that the solutions are orthogonal[3]; in particular, we impose

$$< \phi_i, \phi_j >= \int_0^l \phi_i(x)\phi_j(x)\, dx = \frac{\Delta_{ij}}{\rho} \tag{35}$$

from which

$$\int_0^l \mathcal{L}(\phi_i)\phi_j\, dx = \omega^2 \int_0^l \phi_i(x)\phi_j(x)\, dx = \frac{\omega^2}{\rho}\Delta_{ij}$$

We can use these results only in environments in which the Sturm-Liouville operator is self-adjoint and this is the case for the pseudo-clamped and frame-on-rotor scenarios.

## 4.1 Pseudo-clamped case

If we neglect the payload the equations are, separating $w(x,t)$ with (32) and considering a finite mode number $n$:

$$I_R\ddot{q}_1 = \tau + k(q_2 - q_1) \tag{13a}$$

$$I_S\ddot{q}_2 + \sum_{i=1}^{n} N_i\ddot{\delta}_i = -k(q_2 - q_1) \tag{14a}$$

$$EI\sum_{i=1}^{n}\phi_i''''\delta_i + \rho\left(\sum_{i=1}^{n}\phi_i\ddot{\delta}_i + x\ddot{q}_2\right) = 0 \tag{15a}$$

$$\phi(0) = 0 \tag{9a}$$

$$\phi'(0) = 0 \tag{10a}$$

$$\phi'''(l) = 0 \tag{11a}$$

$$\phi''(l) = 0 \tag{12a}$$

with

$$N_i = \int_0^l \rho x\phi_i\, dx$$

Taking (15a) and dividing by $\rho$ we have:

$$\sum_{i=0}^{n}\mathcal{L}(\phi_i)\delta_i + \sum_{i=0}^{n}\phi_i\ddot{\delta}_i = -x\ddot{q}_2$$

and if we multiply by $\phi_j$ and integrate in the spatial domain we get:

$$\sum_{i=0}^{n}\int_0^l \mathcal{L}(\phi_i)\phi_j\, dx\delta_i + \sum_{i=0}^{n}\int_0^l \phi_i\phi_j\, dx\ddot{\delta}_i = -\int_0^l x\phi_j(x)\, dx\ddot{q}_2$$

that results in, for the only non-null term when $i = j$,

$$\frac{\omega_i^2}{\rho}\delta_i + \frac{1}{\rho}\ddot{\delta}_i = -\int_0^l x\phi_j(x)\, dx\ddot{q}_2$$

$$\ddot{\delta}_i + \omega_i^2\delta_i = -N_i\ddot{q}_2 \tag{36}$$

[3]Proof in Section 8.2

Hence we got all the $n+2$ dynamic equations, here reported:

$$I_R\ddot{q}_1 - k(q_2 - q_1) = \tau \tag{13a}$$

$$I_S\ddot{q}_2 + \sum_{i=1}^{N} N_i\ddot{\delta}_i + k(q_2 - q_1) = 0 \tag{14a}$$

$$N_i\ddot{q}_2 + \ddot{\delta}_i + \omega_i^2\delta_i = 0 \quad i = 1\dots n \tag{36}$$

If we introduce the coordinate vector

$$q = \begin{pmatrix} q_1 \\ q_2 \\ \delta_1 \\ \vdots \\ \delta_n \end{pmatrix}$$

We can write the model in vectorial form:

$$M\ddot{q} + Kq = U\tau$$

with

$$M = \begin{pmatrix} I_R & 0 & 0 & \dots & 0 \\ 0 & I_S & N_1 & \dots & N_n \\ 0 & N_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & N_n & 0 & \dots & 1 \end{pmatrix}, K = \begin{pmatrix} k & -k & 0 & \dots & 0 \\ -k & k & 0 & \dots & 0 \\ 0 & 0 & \omega_1^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \omega_n^2 \end{pmatrix}, U = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

## 4.2 Frame-on-rotor case

For an easy read, we report the model equations withuot payload and $I_H = 0$, with (32) and a finite mode number $n$:

$$I_T\ddot{q}_1 + \sum_{i=1}^{n} N_i\ddot{\delta}_i(t) = \tau \tag{19}$$

$$EI\sum_{i=1}^{n} \phi_i''''\delta_i + \rho\left(\sum_{i=1}^{n} \phi_i\ddot{\delta}_i + x\ddot{q}_1\right) = 0 \tag{20a}$$

$$\phi(0) = 0 \tag{9a}$$

$$\phi'''(l) = 0 \tag{21a}$$

$$\phi''(0) = \frac{k}{EI}\phi'(0) \tag{22a}$$

$$\phi''(l) = 0 \tag{23a}$$

With these boundary conditions the Stourm-Liouvlle operator is still self-adjoint[4], so the analogous of (36) is

$$\ddot{\delta}_i + \omega_i^2\delta_i = -N_i\ddot{q}_1 \quad i = 1\dots n \tag{37}$$

[4]Proof in Section 8.2

and the model is

$$M\ddot{q} + Kq = U\tau$$

with

$$M = \begin{pmatrix} I_T & N_1 & N_2 & \dots & N_n \\ N_1 & 1 & 0 & \dots & 0 \\ N_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ N_n & 0 & 0 & \dots & 1 \end{pmatrix}, K = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \omega_1^2 & 0 & \dots & 0 \\ 0 & 0 & \omega_2^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \omega_N^2 \end{pmatrix}, U = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

## 4.3 Pseudo-pinned case

Since with this case boundary conditions the operator $\mathcal{L}$ is not self-adjoint, the solutions are not orthogonal and we need to find the dynamic model using a change of coordinates from the pseudo-clamped system.
We need now to distinguish between the deflection variable $w_c$ used in the pseudo-clamped case and the $w_p$ used in the pseudo-pinned case; from geometric considerations we have

$$q_{c1} = q_{p1}$$

$$q_{c2} = q_{p2} + w_p'(0,t) = q_{p2} + \sum_{i=1}^{N} \phi_{pi}'(0)\delta_{pi}(t)$$

since the frequencies of the modes must be the same we have also

$$\delta_{ci} = \delta_{pi} \quad \forall i \in [1, N]$$

and so, the linear change of coordinate can be expressed as

$$T = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & \phi_{p1}'(0) & \dots & \phi_{pn}'(0) \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

such that

$$q_c = Tq_p$$

for the principle of virtual work

$$U_c = T^{-T}U_p$$

so the model in the pseudo-clamped case is

$$\begin{aligned} M_c\ddot{q}_c + K_c q_c &= U_c\tau \\ M_c T\ddot{q}_p + K_c T q_p &= T^{-T}U_p\tau \\ T^T M_c T\ddot{q}_p + T^T K_c T q_p &= U_p\tau \\ M_p\ddot{q}_p + K_p q_p &= U_p\tau \end{aligned}$$

hence

$$M_p = T^T M_c T$$
$$K_p = T^T K_c T$$
$$U_p = T^T U_c$$

Then, seen that:

$$\phi_{ci}(x) = \phi_{pi}(x) - x\phi'_{pi}(0)$$

$$N_{ci} = N_{pi} - I_B \phi'_{pi}(0) \Longrightarrow N_{ci} + I_S \phi'_{pi}(0) = N_{pi} + I_H \phi'_{pi}(0) = \nu_i$$

Up to matrix moltiplications, we get

$$M_p = \begin{pmatrix} I_R & 0 & 0 & \dots & 0 \\ 0 & I_S & \nu_1 & \dots & \nu_n \\ 0 & \nu_1 & 1 + N_{c1}\phi'_{p1}(0) + \nu_1\phi'_{p1}(0) & \dots & N_{c1}\phi'_{pn}(0) + \nu_n\phi'_{p1}(0) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \nu_n & N_{c1}\phi'_{pn}(0) + \nu_n\phi'_{p1}(0) & \dots & 1 + N_{cn}\phi'_{pn}(0) + \nu_n\phi'_{pn}(0) \end{pmatrix},$$

$$K_p = \begin{pmatrix} k & -k & -k\phi'_{p1}(0) & \dots & -k\phi'_{pn}(0) \\ -k & k & k\phi'_{p1}(0) & \dots & k\phi'_{pn}(0) \\ -k\phi'_{p1}(0) & k\phi'_{p1}(0) & \omega_1^2 + k\phi'_{p1}(0)^2 & \dots & k\phi'_{pn}(0)\phi'_{p1}(0) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k\phi'_{pn}(0) & k\phi'_{pn}(0) & k\phi'_{p1}(0)\phi'_{pn}(0) & \dots & \omega_n^2 + k\phi'_{pn}(0)^2 \end{pmatrix},$$

$$U_p = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The matrices right-bottom corner block can be generalized as

$$m_{ij} = \Delta_{ij} + N_{cj}\phi'_{pi}(0) + \nu_i\phi'_{pj}(0) \quad i, j \in [1, n]$$

$$k_{ij} = \Delta_{ij}\omega_i^2 + k\phi'_{pi}(0)\phi'_{pj}(0) \quad i, j \in [1, n]$$

### 4.4 State Space Form

Starting from the dynamic model:

$$M\ddot{q} + Kq = U\tau$$

Knowing that:

$$\ddot{q} = M^{-1}(U\tau - Kq)$$

We can always write that model in a State Space Form representation of the type:

$$\dot{x} = Ax + Bu$$

Where:

$$x = \begin{pmatrix} q_1 \\ q_2 \\ \delta_1 \\ \vdots \\ \delta_n \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{\delta}_1 \\ \vdots \\ \dot{\delta}_n \end{pmatrix} = \begin{pmatrix} q \\ \dot{q} \end{pmatrix}$$

Then:

$$\dot{x} = \begin{pmatrix} \dot{q} \\ \ddot{q} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -M^{-1}K & 0 \end{pmatrix} \begin{pmatrix} q \\ \dot{q} \end{pmatrix} + \begin{pmatrix} 0 \\ M^{-1}U \end{pmatrix} \tau$$

This is important because we will use this form in the Simulink part for the implementation of the rest-to-rest motion.

# 5 Mode Analysis

## 5.1 Pseudo-pinned case

For an easy read, we report the pseudo-pinned model equations:

$$I_R\ddot{q}_1 = \tau + k(q_2 + w'(0,t) - q_1) \tag{25}$$
$$I\ddot{q}_2 + \nu(t) = -k(q_2 + w'(0,t) - q_1) \tag{26}$$
$$I_R\ddot{q}_1 + I\ddot{q}_2 + \nu(t) = \tau \tag{30}$$
$$\rho(\ddot{w} + x\ddot{q}_2) + EIw'''' = 0 \tag{15}$$
$$w(0,t) = 0 \tag{9}$$
$$EIw'''(l,t) = m_P(\ddot{w}(l,t) + l\ddot{q}_2) \tag{27}$$
$$EIw''(l,t) = -I_P(\ddot{q}_2 + \ddot{w}'(l,t)) \tag{29}$$
$$EIw''(0,t) = I_H(\ddot{q}_2 + \ddot{w}'(0,t)) + k(q_2 + w'(0,t) - q_1) \tag{28}$$
$$EIw''(0,t) = I_R\ddot{q}_1 + I_H(\ddot{q}_2 + \ddot{w}'(0,t)) - \tau$$

being

$$I_S = I_H + \rho\frac{l^3}{3} + m_P l^2 + I_P$$

$$\nu(t) = I_H\ddot{w}'(0,t) + \rho\int_0^l x\ddot{w}\,dx + m_P l\ddot{w}(l,t) + I_P\ddot{w}'(l,t) = I_H\ddot{w}'(0,t) + \mu(t)$$

To analyze the evolution of the system we need to introduce the variable[5]

$$z(x,t) = w(x,t) + xq_2(t)$$

with this choice we have that

$$\begin{aligned} w(0,t) &= z(0,t) & w''(x,t) &= z''(x,t) \\ w'''(x,t) &= z'''(x,t) & w''''(x,t) &= z''''(x,t) \\ \ddot{w}(x,t) + x\ddot{q}_2(t) &= \ddot{z}(x,t) & \ddot{w}'(x,t) + \ddot{q}_2 &= \ddot{z}'(x,t) \end{aligned}$$

in particular we study the term $I\ddot{q}_2 + \nu(t)$:

$$\begin{aligned} I\ddot{q}_2 + \nu(t) &= I_H(\ddot{q}_2 + \ddot{w}'(0,t)) + \rho\int_0^l (x^2\ddot{q}_2 + x\ddot{w})\,dx + m_P(l^2\ddot{q}_2 + l\ddot{w}(l,t)) + I_P(\ddot{q}_2 + \ddot{w}'(l,t)) = \\ &= I_H\ddot{z}'(0,t) + \rho\int_0^l x\ddot{z}(x,t)\,dx + m_P l\ddot{z}(l,t) + I_P\ddot{z}'(l,t) = \nu_z(t) \end{aligned}$$

[5]It's important to notice that this variable doesn't have a geometric meaning. In fact, it could be thought as an approximation of $y(x,t)$, the vertical coordinate with respect to the fixed reference frame $Oxy$, valid just for small $q_2(t)$.

So we can rewrite the equations with the new variable:

$$I_R\ddot{q}_1 = \tau + k(z'(0,t) - q_1) \tag{38}$$

$$\nu_z(t) = -k(z'(0,t) - q_1) \tag{39}$$

$$I_R\ddot{q}_1 + \nu_z(t) = \tau \tag{40}$$

$$\rho\ddot{z}(x,t) + EIz''''(x,t) = 0 \tag{41}$$

$$z(0,t) = 0 \tag{42}$$

$$EIz'''(l,t) = m_P\ddot{z}(l,t) \tag{43}$$

$$EIz''(l,t) = -I_P\ddot{z}'(l,t) \tag{44}$$

$$EIz''(0,t) = I_H\ddot{z}'(0,t) + k(z'(0,t) - q_1) \tag{45}$$

$$EIz''(0,t) = I_R\ddot{q}_1 + I_H\ddot{z}'(0,t) - \tau \tag{46}$$

from the (45) we can extract a relation for $q_1(t)$:

$$q_1(t) = z'(0,t) + \frac{I_H}{k}\ddot{z}'(0,t) - \frac{EI}{k}z''(0,t)$$

to substitute in (38):

$$I_R\left(\ddot{z}'(0,t) + \frac{I_H}{k}\ddddot{z}'(0,t) - \frac{EI}{k}\ddot{z}''(0,t)\right) = \tau + EIz''(0,t) - I_H\ddot{z}'(0,t) \tag{47}$$

Now I separate $z(x,t)$ in space and time:

$$z(x,t) = \psi(x)\delta(t) \tag{48}$$

From (41) we can get $z(x,t)$ behaviour:

$$\begin{aligned}
\rho\psi(x)\ddot{\delta}(t) + EI\psi''''(x)\delta(t) &= 0 \\
\frac{EI}{\rho}\frac{\psi''''(x)}{\psi(x)} = -\frac{\ddot{\delta}(t)}{\delta(t)} &= \omega^2 \\
\psi''''(x) = \frac{\rho\omega^2}{EI}\psi(x) &= \beta^4\psi(x) \\
\ddot{\delta}(t) &= -\omega^2\delta(t)
\end{aligned}$$

the solutions to these partial differential equations are

$$\psi(x) = A\sin(\beta x) + B\cos(\beta x) + C\sinh(\beta x) + D\cosh(\beta x) \tag{49}$$

$$\delta(t) = \dot{\delta}(0)\sin(\omega t) + \delta(0)\cos(\omega t) \tag{50}$$

where coefficients $A$, $B$, $C$ and $D$ are chosen as to satisfy the four boundary conditions (42) - (44) and (47) written using (48) in free motion ($\tau = 0$):

$$\begin{aligned}
\psi(0) &= 0 \\
EI\psi'''(l)\delta(t) &= m_P\psi(l)\ddot{\delta}(t) \\
EI\psi''(l)\delta(t) &= -I_P\psi'(l)\ddot{\delta}(t) \\
I_R\left(\psi'(0)\ddot{\delta}(t) + \frac{I_H}{k}\psi'(0)\ddddot{\delta}(t) - \frac{EI}{k}\psi''(0)\ddot{\delta}(t)\right) &= EI\psi''(0)\delta(t) - I_H\psi'(0)\ddot{\delta}(t)
\end{aligned}$$

that can be expanded further using (50):

$$\begin{aligned}
\psi(0) &= 0 \\
EI\psi'''(l) &= -\omega^2 m_P \psi(l) \\
EI\psi''(l) &= \omega^2 I_P \psi'(l) \\
I_R\left(-\omega^2\psi'(0) + \omega^4\frac{I_H}{k}\psi'(0) + \omega^2\frac{EI}{k}\psi''(0)\right)\delta(t) &= EI\psi''(0)\delta(t) + \omega^2 I_H\psi'(0)\delta(t) \\
\Rightarrow \omega^2 I_R(\omega^2\frac{I_H}{k} - \frac{I_H}{I_R} - 1)\psi'(0) &= -EI(\omega^2\frac{I_R}{k} - 1)\psi''(0)
\end{aligned}$$

where we used

$$\ddddot{\delta}(t) = -\omega^2\ddot{\delta}(t) = \omega^4\delta(t)$$

that, written using $\beta^4 = \frac{\rho\omega^2}{EI}$ becomes:

$$\psi(0) = 0 \tag{51}$$

$$\psi'''(l) = -\beta^4\frac{m_P}{\rho}\psi(l) \tag{52}$$

$$\psi''(l) = \beta^4\frac{I_P}{\rho}\psi'(l) \tag{53}$$

$$\beta^4\frac{I_R}{\rho}\left(\beta^4\frac{EI}{\rho}\frac{I_H}{k} - \frac{I_H}{I_R} - 1\right)\psi'(0) = \left(1 - \beta^4\frac{EI}{\rho}\frac{I_R}{k}\right)\psi''(0) \tag{54}$$

Now we can start evaluating the coefficients $A$, $B$, $C$ and $D$. From (51) we get

$$D = -B$$

so $\psi(x)$ becomes

$$\psi(x) = A\sin(\beta x) + B(\cos(\beta x) - \cosh(\beta x)) + C\sinh(\beta x)$$

and we have

$$\begin{aligned}
\psi'(0) &= \beta(A + C) \\
\psi''(0) &= -2\beta^2 B
\end{aligned}$$

so (54) becomes

$$Q\beta(A + C) = 2P\beta^2 B \Longrightarrow B = -\frac{Q}{2P\beta}(A + C) = V(A + C)$$

with

$$P = 1 - \beta^4\frac{EI}{\rho}\frac{I_R}{k} \qquad Q = \beta^4\frac{I_R}{\rho}\left(1 + \frac{I_H}{I_R} - \beta^4\frac{EI}{\rho}\frac{I_H}{k}\right)$$

then if we define

$$\begin{aligned}
c &:= \cos(\beta l) \quad & ch &:= \cosh(\beta l) \\
s &:= \sin(\beta l) \quad & sh &:= \sinh(\beta l)
\end{aligned}$$

we get

$$\begin{aligned}
\psi(l) &= As + V(A + C)(c - ch) + Csh \\
\psi'(l) &= \beta(Ac - V(A + C)(s + sh) + Cch) \\
\psi''(l) &= \beta^2(-As - V(A + C)(c + ch) + Csh) \\
\psi'''(l) &= \beta^3(-Ac + V(A + C)(s - sh) + Cch)
\end{aligned}$$

and using the last two boundary equations we get

$$\begin{aligned}(-Ac+V(A+C)(s-sh)+Cch)+R(As+V(A+C)(c-ch)+Csh)=0\\(As+V(A+C)(c+ch)-Csh)+T(Ac-V(A+C)(s+sh)+Cch)=0\end{aligned}$$

with

$$R=\beta\frac{m_P}{\rho}\qquad T=\beta^3\frac{I_P}{\rho}$$

to find $A$ and $C$ we have now a linear system:

$$\begin{pmatrix}Rs-c+V(s-sh)+VR(c-ch) & Rsh+ch+V(s-sh)+VR(c-ch)\\Tc+s+V(c+ch)-VT(s+sh) & Tch-sh+V(c+ch)-VT(s+sh)\end{pmatrix}\begin{pmatrix}A\\C\end{pmatrix}=H\begin{pmatrix}A\\C\end{pmatrix}=0$$

Clearly we need

$$\det(H)=0$$

to avoid a trivial solution, and this condition gives the different mode frequencies $\beta$ (or $\omega$). The equation to solve is

$$\begin{aligned}&[Rs-c+V(s-sh)+VR(c-ch)][Tch-sh+V(c+ch)-VT(s+sh)]+\\&-[Rsh+ch+V(s-sh)+VR(c-ch)][Tc+s+V(c+ch)-VT(s+sh)]=0\end{aligned}$$

that brings to[6]

$$\begin{aligned}&2V(1+RT)+2R\sin(\beta l)\sinh(\beta l)+(1-2VR-2VT-RT)\sin(\beta l)\cosh(\beta l)+\\&-(1-2VR+2VT-RT)\cos(\beta l)\sinh(\beta l)-2(VRT-V-T)\cos(\beta l)\cosh(\beta l)=0\end{aligned}\tag{55}$$

it's evident that this equation has infinite solutions, so infinite modes evolve from the system. In particular, we could say that

$$z(x,t)=\sum_{i=1}^{\infty}\psi_i(x)\delta_i(t)$$

where each $\psi_i(x)$ and $\delta_i(t)$ is associated to a solution $\beta_i$ to the above characteristic equation. It's interesting to notice that this modes has been evaluated in the pseudo-pinned reference frame, and the characteristic equation coincides with the (1.48) found in [1] (pag. 55) that operates in a pseudo-clamped environment. This shouldn't be a surprise, since the modes frequencies can't depend on the chosen reference frame.

Clearly, from the built linear system we have that, if (55) is satisfied,

$$A_i=-\frac{R_ish+ch+V_i(s-sh)+V_iR_i(c-ch)}{R_is-c+V_i(s-sh)+V_iR_i(c-ch)}C_i=\sigma_iC_i$$

so

$$\psi_i(x)=C_i\left(\sigma_i\sin(\beta_ix)+V_i(1+\sigma_i)(\cos(\beta_ix)-\cosh(\beta_ix))+\sinh(\beta_ix)\right)$$

and $C_i$ is chosen accordingly to the normalization of the modes similar to (35).

[6]Details in Section 8.3

## 5.2 Frame-on-rotor case

For an easy read, we report the model equations:

$$I_T\ddot{q}_1 + \nu(t) = \tau \tag{19}$$
$$\rho(\ddot{w} + x\ddot{q}_1) + EIw'''' = 0 \tag{20}$$
$$w(0,t) = 0 \tag{9}$$
$$EIw'''(l,t) = m_P(\ddot{w}(l,t) + l\ddot{q}_1) \tag{21}$$
$$EIw''(0,t) = I_H(\ddot{q}_1 + \ddot{w}'(0,t)) + kw'(0,t) \tag{22}$$
$$EIw''(l,t) = -I_P(\ddot{q}_1 + \ddot{w}'(l,t)) \tag{23}$$

From (19) we have that

$$\ddot{q}_1 = \frac{1}{I_T}(\tau - \nu(t))$$

From now on we'll analyze the problem in free motion, so we put $\tau = 0$; the expression of $\ddot{q}_1$ put in (20) gives

$$EIw'''' + \rho\ddot{w} - x\frac{\rho}{I_T}\nu(t) = 0$$

If we separate

$$w(x,t) = \phi(x)\delta(t) \tag{56}$$

we get that

$$\nu(t) = I_H\phi'(0)\ddot{\delta}(t) + \rho\int_0^l x\phi(x)\,dx\ddot{\delta}(t) + m_P\phi(l)l\ddot{\delta}(t) + I_P\phi'(l)\ddot{\delta}(t) = \nu_0\ddot{\delta}(t)$$

so the (20) becomes

$$EI\phi''''(x)\delta(t) + \rho\phi(x)\ddot{\delta}(t) - x\frac{\rho}{I_T}\nu_0\ddot{\delta}(t) = 0$$

that is partitioned as

$$EI\phi''''(x) - \rho\omega^2\phi(x) + x\frac{\rho}{I_T}\omega^2\nu_0 = 0 \tag{57}$$
$$\ddot{\delta}(t) + \omega^2\delta(t) = 0 \tag{34}$$

From (57) we get that $\phi(x)$ should be

$$\phi(x) = A\sin(\beta x) + B\cos(\beta x) + C\sinh(\beta x) + D\cosh(\beta x) + Fx = \hat{\phi}(x) + Fx$$

with

$$\phi''''(x) = \beta^4\hat{\phi}(x) = \frac{\rho\omega^2}{EI}\hat{\phi}(x)$$

we can analyze further the term $\nu_0$:

$$\begin{aligned}
\nu_0 &= I_H\phi'(0) + \rho\int_0^l x\phi(x)\,dx + m_P\phi(l)l + I_P\phi'(l) = \\
&= I_H\hat{\phi}'(0) + I_HF + \rho\int_0^l x\hat{\phi}(x)\,dx + \rho\int_0^l x^2\,dxF + m_P\hat{\phi}(l)l + m_PFl^2 + I_P\hat{\phi}'(l) + I_PF = \\
&= \hat{\nu}_0 + IF
\end{aligned} \tag{58}$$

so follows from (57)

$$
\begin{aligned}
\rho\omega^2\hat{\phi}(x) - \rho\omega^2\hat{\phi}(x) - x\rho\omega^2 F + x\frac{\rho}{I_T}\omega^2\nu_0 &= 0 \\
-F + \frac{\nu_0}{I_T} &= 0 \\
\nu_0 &= I_T F \\
\hat{\nu}_0 + IF &= I_T F \\
\hat{\nu}_0 &= I_R F \qquad (59)
\end{aligned}
$$

$$
F = \frac{\hat{\nu}_0}{I_R} = \frac{1}{I_R}\left(I_H\hat{\phi}'(0) + \rho\int_0^l x\hat{\phi}(x)\,dx + m_P l\hat{\phi}(l) + I_P\hat{\phi}'(l)\right) \qquad (60)
$$

Using (56) we can further decompose $\ddot{q}_1$:

$$
\ddot{q}_1 = -\frac{\nu_0}{I_T}\ddot{\delta}(t) = -F\ddot{\delta}(t)
$$

Proceeding in the same way as for the pseudo-pinned case, we search for the $\phi(x)$ with the the found $F$ that satisfies boundary conditions (9) and (21) - (23), written using (56):

$$
\begin{aligned}
\phi(0)\delta(t) &= 0 \\
EI\phi'''(l)\delta(t) &= m_P\left(\phi(l)\ddot{\delta}(t) - lF\ddot{\delta}(t)\right) \\
EI\phi''(0)\delta(t) &= I_H\left(-F\ddot{\delta}(t) + \phi'(0)\ddot{\delta}(t)\right) + k\phi'(0)\delta(t) \\
EI\phi''(l)\delta(t) &= -I_P\left(-F\ddot{\delta}(t) + \phi'(l)\ddot{\delta}(t)\right)
\end{aligned}
$$

using (34) and dividing by $\delta(t)$ these conditions become

$$
\begin{aligned}
\phi(0) &= 0 \\
EI\phi'''(l) &= -\omega^2 m_P\left(\phi(l) - lF\right) = -\omega^2 m_P\hat{\phi}(l) \\
EI\phi''(0) &= -\omega^2 I_H\left(-F + \phi'(0)\right) + k\phi'(0) = -\omega^2 I_H\hat{\phi}'(0) + k\phi'(0) \\
EI\phi''(l) &= \omega^2 I_P\left(-F + \phi'(l)\right) = \omega^2 I_P\hat{\phi}'(l)
\end{aligned}
$$

that, written using $\beta^4 = \frac{\rho\omega^2}{EI}$ become:

$$
\begin{aligned}
\phi(0) &= 0 \qquad (61) \\
\phi'''(l) &= -\beta^4\frac{m_P}{\rho}\hat{\phi}(l) \qquad (62) \\
\phi''(0) &= -\beta^4\frac{I_H}{\rho}\hat{\phi}'(0) + \frac{k}{EI}\phi'(0) \qquad (63) \\
\phi''(l) &= \beta^4\frac{I_P}{\rho}\hat{\phi}'(l) \qquad (64)
\end{aligned}
$$

Now we can start evaluating the coefficients $A$, $B$, $C$ and $D$. From (61) we get

$$
D = -B
$$

so $\phi(x)$ becomes

$$
\phi(x) = A\sin(\beta x) + B(\cos(\beta x) - \cosh(\beta x)) + C\sinh(\beta x) + Fx
$$

and we have

$$\phi'(0) = \hat{\phi}'(0) = \beta(A + C)$$
$$\phi''(0) = \hat{\phi}''(0) = -2\beta^2 B$$

then, if we define

$$c := \cos(\beta l) \quad ch := \cosh(\beta l)$$
$$s := \sin(\beta l) \quad sh := \sinh(\beta l)$$

we can write

$$\phi(l) = \hat{\phi}(l) + Fl = As + B(c - ch) + Csh + Fl$$
$$\phi'(l) = \hat{\phi}'(l) + F = \beta(Ac - B(s + sh) + Cch) + F$$
$$\phi''(l) = \hat{\phi}''(l) = \beta^2(-As - B(c + ch) + Csh)$$
$$\phi'''(l) = \hat{\phi}'''(l) = \beta^3(-Ac + B(s - sh) + Cch)$$

$$F = \frac{1}{I_R}\left(I_H\beta(A + C) + \rho\int_0^l x\hat{\phi}(x)\,dx + m_P l\hat{\phi}(l) + I_P\hat{\phi}'(l)\right)$$

and

$$\begin{aligned}\rho\int_0^l x\phi(x)\,dx &= \frac{\rho}{\beta^2}\Big(A(s - \beta lc) + B(c + ch + \beta l(s - sh) - 2) + C(\beta lch - sh)\Big) + F\frac{l^3}{3} = \\ &= \rho\int_0^l x\hat{\phi}(x)\,dx + F\frac{l^3}{3} = \\ &= \left(\frac{l^3}{3I_R} + 1\right)\rho\int_0^l x\hat{\phi}(x)\,dx + \frac{l^3}{3I_R}(I_H\beta(A + C) + m_P l\hat{\phi}(l) + I_P\hat{\phi}'(l))\end{aligned}$$

We introduce some variable

$$R^+ = \beta\frac{m_P}{\rho} \qquad S^+ = -\frac{k}{2EI\beta} \qquad T^+ = \beta^3\frac{I_P}{\rho} \qquad U^+ = \beta^3\frac{I_H}{2\rho}$$

to rewrite the 3 unused conditions:

$$-Ac + B(s - sh) + Cch = -R^+(As + B(c - ch) + Csh)$$
$$B = U^+(A + C) + S^+(A + C)$$
$$-As - B(c + ch) + Csh = T^+(Ac - B(s + sh) + Cch)$$

From the second we get

$$B = V^+(A + C)$$

with

$$V^+ = U^+ + S^+$$

and the remaining boundary equations are:

$$-Ac + V^+(A + C)(s - sh) + Cch + R^+(As + V^+(A + C)(c - ch) + Csh) = 0$$
$$As + V^+(A + C)(c + ch) - Csh + T^+(Ac - V^+(A + C)(s + sh) + Cch) = 0$$

So $A$ and $C$ are found by solving the linear system:

$$\begin{pmatrix} R^+s - c + V^+(s-sh) + V^+R^+(c-ch) & R^+sh + ch + V^+(s-sh) + V^+R^+(c-ch) \\ T^+c + s + V^+(c+ch) - V^+T^+(s+sh) & T^+ch - sh + V^+(c+ch) - V^+T^+(s+sh) \end{pmatrix} \begin{pmatrix} A \\ C \end{pmatrix} =$$

$$= H^+ \begin{pmatrix} A \\ C \end{pmatrix} = 0$$

as before, we need

$$\det(H^+) = 0$$

since the expression of $H^+$ differs from the one of $H$ only for the definition of its coefficients, its characteristic equation has already been found in Section 5.1 and it's[7]

$$\begin{gathered} 2V^+(1+R^+T^+) + 2R^+\sin(\beta l)\sinh(\beta l) + (1 - 2V^+R^+ - 2V^+T^+ - R^+T^+)\sin(\beta l)\cosh(\beta l) + \\ -(1 - 2V^+R^+ + 2V^+T^+ - R^+T^+)\cos(\beta l)\sinh(\beta l) - 2(V^+R^+T^+ - V^+ - T^+)\cos(\beta l)\cosh(\beta l) = 0 \end{gathered} \tag{65}$$

As before, we have infinite solutions, so

$$w(x,t) = \sum_{i=1}^{\infty} \phi_i(x)\delta_i(t)$$

but now the mode frequencies are slightly different, maybe due to the different expression of the space component. However, from the built linear system we have that, if (65) is satisfied,

$$A_i = -\frac{R_i^+ sh + ch + V_i^+(s-sh) + V_i^+R_i^+(c-ch)}{R_i^+ s - c + V_i^+(s-sh) + V_i^+R_i^+(c-ch)} C_i = \sigma_i^+ C_i$$

so

$$\hat{\phi}_i(x) = C_i\left(\sigma_i^+ \sin(\beta_i x) + V_i^+(1+\sigma_i)(\cos(\beta_i x) - \cosh(\beta_i x)) + \sinh(\beta_i x)\right)$$

and finally, from (60):

$$\begin{aligned} F_i &= \frac{1}{I_R}\left(I_H\hat{\phi}_i'(0) + \rho\int_0^l x\hat{\phi}_i(x)\,dx + m_P l\hat{\phi}_i(l) + I_P\hat{\phi}_i'(l)\right) = \\ &= \frac{1}{I_R}\left(I_H C_i(\sigma_i^+ + 1) + \rho\int_0^l x\hat{\phi}_i(x)\,dx + m_P l\hat{\phi}_i(l) + I_P\hat{\phi}_i'(l)\right) \end{aligned}$$

recalling that $\int_0^l x\hat{\phi}(x)\,dx$ has a closed expression and all the terms are function of the only unknown $C_i$, we have found the complete expression for each $\phi_i(x)$ and so for $w(x,t)$. As before, $C_i$ is chosen accordingly to the normalization of the modes given by the self-adjoint problem conditions chosen in (35).

We can analyze further the behaviour of $F$: the value of $I_R$ appears only in $F$, so if we take a very soft rotor, with $I_R \to 0$ we have $F \to \infty$ and so

$$\phi_i(x) \simeq F_i x \Longrightarrow w(x,t) = \sum_{i=1}^{N}\phi_i(x)\delta_i(t) \simeq \sum_{i=1}^{N} F_i\delta_i(t)x = \Delta(t)x$$

with

$$\Delta(t) = \sum_{i=1}^{N} F_i\delta_i(t) \qquad \ddot{\Delta}(t) = \sum_{i=1}^{N} F_i\ddot{\delta}_i(t)$$

[7]See details and the payload-free equation in Section 8.4

in particular

$$\ddot{w}(x,t) = \ddot{\Delta}(t)x \qquad \ddot{w}'(x,t) = \ddot{\Delta}(t)$$

this means that, with a very soft rotor the beam behaves as it's rigid, and oscillates only at the hub, as if the oscillations of the beam are absorbed by the spring between hub and rotor.

# 6 Rest-to-Rest Motion Design

In general, given a dynamic linear system of the form

$$M\ddot{q} + Kq = U\tau$$

we have that

$$\ddot{q} = M^{-1}(U\tau - Kq)$$

then, if we choose an output function as

$$y = Cq$$

we can make its even derivative independent from $\tau$ and evaluate the $c_i$ coefficients.

In particular,

$$\begin{aligned}\ddot{y} = C\ddot{q} = CM^{-1}(U\tau - Kq) = \\ = CM^{-1}U\tau - CM^{-1}Kq\end{aligned}$$

and we impose

$$CM^{-1}U = 0$$

We can derive furhter, searching for a generalization:

$$\frac{d^4y}{dt^4} = -CM^{-1}K\ddot{q} = -CM^{-1}KM^{-1}U\tau + C(M^{-1}K)^2q$$

giving the condition

$$CM^{-1}KM^{-1}U = 0$$

So, for a generic even derivative (if we eliminate each time the $\tau$ coefficient):

$$\begin{aligned}\frac{d^{2m}y}{dt^{2m}} &= CM^{-1}K\frac{d^{2(m-1)}q}{dt^{2(m-1)}} = \\ &= (-1)^{m-1}C(M^{-1}K)^{m-1}M^{-1}U\tau + (-1)^m C(M^{-1}K)^m q\end{aligned}$$

that give the conditions

$$C(M^{-1}K)^{m-1}M^{-1}U = 0$$

So the problem is to solve the system of $n$ equations to find $n$ coefficients:

$$C(M^{-1}K)^{m-1}M^{-1}U = 0 \quad m \in [1, n]$$

## 6.1 Pseudo-clamped case

The model without the payload in the pseudo-clamped case is

$$M\ddot{q} + Kq = U\tau$$

with

$$M = \begin{pmatrix} I_R & 0 & 0 & \dots & 0 \\ 0 & I_S & N_1 & \dots & N_n \\ 0 & N_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & N_n & 0 & \dots & 1 \end{pmatrix}, K = \begin{pmatrix} k & -k & 0 & \dots & 0 \\ -k & k & 0 & \dots & 0 \\ 0 & 0 & \omega_1^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \omega_n^2 \end{pmatrix}, U = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

If the output function is defined as

$$y = c_0 q_1 + q_2 + \sum_{i=1}^{n} c_i \delta_i = (c_0 1 c^T) q = Cq$$

from its second derivative we get $c_0 = 0$, so we need to solve these $n-1$ equations system:

$$(0\ 1\ c^T)(M^{-1}K)^m M^{-1}U = 0 \quad m \in [1, n-1]$$

however, since we have access to the complete model only with the frame-on-rotor choice, we analyze the conditions in that reference frame, in the following section.

## 6.2 Frame-on-rotor case

Starting from the matrices

$$M = \begin{pmatrix} I_T & N_1 & N_2 & \dots & N_n \\ N_1 & 1 & 0 & \dots & 0 \\ N_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ N_n & 0 & 0 & \dots & 1 \end{pmatrix}, K = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & \omega_1^2 & 0 & \dots & 0 \\ 0 & 0 & \omega_2^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \omega_N^2 \end{pmatrix}, U = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Since in this case we choice

$$y = (1\ c^T) q$$

we only need $n$ conditions, that are always of the form

$$C(M^{-1}K)^{m-1} M^{-1}U = 0 \quad m \in [1, n]$$

For compactness and speed up numerical calculations, we could write the conditions as

$$(1\ c^T) B^m M^{-1}U = 0 \quad m \in [1, n-1]$$

with

$$B = M^{-1}K$$

To find the conditions on $c$ we need to inspect the above for some value of $m$, starting from $m = 1$; the conditions is

$$\sum_{i=1}^{n} c_i N_i = 1$$

starting from $m = 2$ the conditions are much complex and it's convenient to use each condition to express a $c_i$ as a function of the others $c_j$, with $j > i$. For example, from the above condition we get

$$c_1 = \frac{1 - \sum_{i=2}^{n} c_i N_i}{N_1}$$

with this assumption we have, for $m = 2$ the condition

$$\sum_{i=2}^{n} c_i N_i \left( -\omega_i^2 + \omega_1^2 \right) = \omega_1^2$$

that gives

$$c_2 = \frac{\omega_1^2 - \sum_{i=3}^n c_i N_i(-\omega_i^2 + \omega_1^2)}{N_2(-\omega_2^2 + \omega_1^2)}$$

then for $m = 3$ the condition is

$$\sum_{i=3}^{n} c_i N_i(\omega_i^4 - \omega_i^2(\omega_1^2 + \omega_2^2) + \omega_1^2\omega_2^2) = \omega_1^2\omega_2^2$$

that gives

$$c_3 = \frac{\omega_1^2\omega_2^2 - \sum_{i=4}^n c_i N_i(\omega_i^4 - \omega_i^2(\omega_1^2 + \omega_2^2) + \omega_1^2\omega_2^2)}{N_3(\omega_3^4 - \omega_3^2(\omega_1^2 + \omega_2^2) + \omega_1^2\omega_2^2)}$$

in general, it's not so easy to find a closed form for the conditions, but it could be (in fact we haven't proved it), for a generic $m$:

$$\sum_{i=m}^{n} c_i N_i \left( (-\omega_i^2)^{m-1} + (-\omega_i^2)^{m-2} \left( \sum_{j=1}^{m-1} \omega_j^2 \right) + (-\omega_i)^{m-3} \left( \sum_{\substack{j_1=1 \\ j_2 \neq j_1}}^{\binom{m-1}{2}} \omega_{j_1}^2 \omega_{j_2}^2 \right) + \cdots + \prod_{j=1}^{m-1} \omega_j^2 \right) = \prod_{j=1}^{m-1} \omega_j^2$$

that could be written compactly as

$$\sum_{i=m}^{n} c_i N_i \left( (-\omega_i^2)^m + \sum_{j=2}^{m} (-\omega_i^2)^{m-j} \sum_{\substack{k_1=1 \\ k_2 \neq k_1 \\ k_3 \neq k_2, k_1 \\ \vdots \\ k_{j-1} \neq k_{j-2}, \ldots, k_1}}^{\binom{m-1}{j-1}} \omega_{k_1}^2 \omega_{k_2}^2 \ldots \omega_{k_{j-1}}^2 \right) = \prod_{j=1}^{m} \omega_j^2$$

for example, the next condition, for $m = 4$, is[8]

$$\sum_{i=4}^{n} c_i N_i \left( -\omega_i^6 + \omega_i^4(\omega_1^2 + \omega_2^2 + \omega_3^2) - \omega_i^2(\omega_1^2\omega_2^2 + \omega_1^2\omega_3^2 + \omega_2^2\omega_3^2) + \omega_1^2\omega_2^2\omega_3^2 \right) = \omega_1^2\omega_2^2\omega_3^2$$

the underlying pattern should be visible.

With these assumptions we can also conjecture the value for a generic $c_m$ by inverting the general formulation. Then, since each $c$ is defined with respect to the next coefficients, in practice when the number of modes is defined as $n$, all the $c_i$ with $i > n$ are set to zero, then $c_n$ is found as

$$c_n = \frac{1}{N_n} \prod_{j=1}^{n-1} \omega_j^2 \left( (-\omega_i^2)^n + \sum_{j=2}^{n} (-\omega_i^2)^{n-j} \sum_{\substack{k_1=1 \\ k_2 \neq k_1 \\ k_3 \neq k_2, k_1 \\ \vdots \\ k_{j-1} \neq k_{j-2}, \ldots, k_1}}^{\binom{n-1}{j-1}} \omega_{k_1}^2 \omega_{k_2}^2 \ldots \omega_{k_{j-1}}^2 \right)^{-1}$$

[8]This has been verified with MatLab

and then back up to $c_1$.
In general with $n \geq 3$ become hard to write in a closed form all the coefficients. For this reason we can use numerically a "null-space" method, in which on MatLab we solve an homogeneous system of equations. In fact rewriting the system of n equations:

$$\begin{pmatrix} CM^{-1}U \\ CM^{-1}KM^{-1}U \\ C(M^{-1}K)^2M^{-1}U \\ \vdots \\ C(M^{-1}K)^{n-1}M^{-1}U \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

We can rewrite the system as a linear system:

$$\begin{pmatrix} U^TM^{-T} \\ U^TM^{-T}K^TM^{-T} \\ U^TM^{-T}(K^TM^{-T})^2 \\ \vdots \\ U^TM^{-T}(K^TM^{-T})^{n-1} \end{pmatrix} C^T = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

This is an homogenoeus linear system of the type $Ax = 0$, and calling:

$$A = \begin{pmatrix} U^TM^{-T} \\ U^TM^{-T}K^TM^{-T} \\ U^TM^{-T}(K^TM^{-T})^2 \\ \vdots \\ U^TM^{-T}(K^TM^{-T})^{n-1} \end{pmatrix}$$

The solution $C^T \in \mathcal{N}(A)$, this means that we can compute the null space of that matrix, and normalizing (dividing by $c_0$) the resultant vector

Following from [4] we have now that the chosen output can be linked to the state with the transformation

$$Y = \begin{pmatrix} y \\ y^{[2]} \\ y^{[4]} \\ \vdots \\ y^{[2n]} \end{pmatrix}(t) = Q \begin{pmatrix} q_1 \\ \delta_1 \\ \delta_2 \\ \vdots \\ \delta_n \end{pmatrix}(t), \qquad \begin{pmatrix} \dot{y} \\ y^{[3]} \\ y^{[5]} \\ \vdots \\ y^{[2n+1]} \end{pmatrix}(t) = Q \begin{pmatrix} \dot{q}_1 \\ \dot{\delta}_1 \\ \dot{\delta}_2 \\ \vdots \\ \dot{\delta}_n \end{pmatrix}(t)$$

where

$$Q = \begin{pmatrix} C \\ -CB \\ CB^2 \\ \vdots \\ (-1)^n CB^n \end{pmatrix}$$

Then, since we want a rest-to-rest motion from time $t = 0$ to $t = T$, we have

$$\begin{pmatrix} q_1 \\ \delta_1 \\ \vdots \\ \delta_n \\ \dot{q}_1 \\ \dot{\delta}_1 \\ \vdots \\ \dot{\delta}_n \end{pmatrix}(0) = \begin{pmatrix} \theta_i \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \implies \begin{pmatrix} y \\ y^{[2]} \\ \vdots \\ y^{[2n]} \\ \dot{y} \\ y^{[3]} \\ \vdots \\ y^{[2n+1]} \end{pmatrix}(0) = \begin{pmatrix} \theta_i \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

since the first column of $Q$ is 1 followed by $n$ zeroes. The same is true for the state at $t = T$ by changing $\theta_i$ with $\theta_f$, so the problem is just to define a trajectory $y(t) = y_d(t)$ that goes from $\theta_i$ to $\theta_f$ and choose $\tau$ by controlling its $(2n+2)^{\text{th}}$ derivative and imposing all the other derivatives up to $y^{[2n+1]}$ to zero at the boundaries. For doing so, a polynomial of degree $4n + 3$ is sufficient.
Then, the desired $\tau_d$ is found by inverting the expression of $y^{[2n+2]}$:

$$y^{[2n+2]}(t) = (-1)^n CB^n M^{-1} U \tau_d(t) + (-1)^{n+1} CB^{n+1} q(t)$$

since $Y(t) = Qq(t)$, it results in:

$$\tau_d(t) = \Big((-1)^n CB^n M^{-1} U\Big)^{-1} \Big(y_d^{[2n+2]}(t) + (-1)^n CB^{n+1} Q^{-1} Y(t)\Big) \tag{66}$$

### 6.3 Regulation

Clearly, in addition to the feedforward command $\tau_d$ one should put a regulation with the feedback given by the encoders, for example a PD control can be achieved, and the final control law would be

$$\tau = \tau_d + K_d(\dot{q}_{1d} - \dot{q}_1) + K_p(q_{1d} - q_1)$$

Note that not the whole $q$ coordinate vector is present in the law, since only the first component can be got from the rotor's encoder. One could also add the $w'(0, t)$ that is

got from the hub encoder, but clearly the whole coordinate vector can't be obtained from encoders only. In fact this control is effective only for really small trajectory errors, since it causes oscillations of the beam for higher error values.
A future work would be to develop a suitable regulation control.

# 7 Simulation and results

## 7.1 Frame-on-rotor case

Since we managed to find the behaviour of $\phi(x)$ only in the frame-on-rotor case, we are able to simulate the system only with this choice. We choose to model the system by considering 3 modes, in this way we can evaluate coefficients $c_1$, $c_2$ and $c_3$, starting from $c_3$ and then back, as seen in Section 6.2. For the following parameters[9]:

- $EI = 2.4507$ N m$^2$
- $l = 0.7$ m
- $I_h = 1.95$ x $10^{-3}$ Kg m$^2$
- $I_r = 1$ Kg m$^2$
- $\rho = 2.975$ Kg / m$^3$
- $k = 1000$ N/m [(10)]
- $m_P = 0.117$ Kg
- $I_p = 0$ Kg m$^2$

We have

$$N = \begin{pmatrix} -0.5771 \\ 0.1006 \\ 0.0551 \end{pmatrix}, \quad \omega^2 = \begin{pmatrix} 34.14 \\ 1383.24 \\ 11084.49 \end{pmatrix} (\text{rad/s})^2$$

that give

$$c_3 = \frac{\omega_1^2 \omega_2^2}{N_3(\omega_3^4 - \omega_3^2(\omega_1^2 + \omega_2^2) + \omega_1^2 \omega_2^2)} = 0.0079994681$$

$$c_2 = \frac{\omega_1^2 - c_3 N_3(-\omega_3^2 + \omega_1^2)}{N_2(-\omega_2^2 + \omega_1^2)} = -0.28731398$$

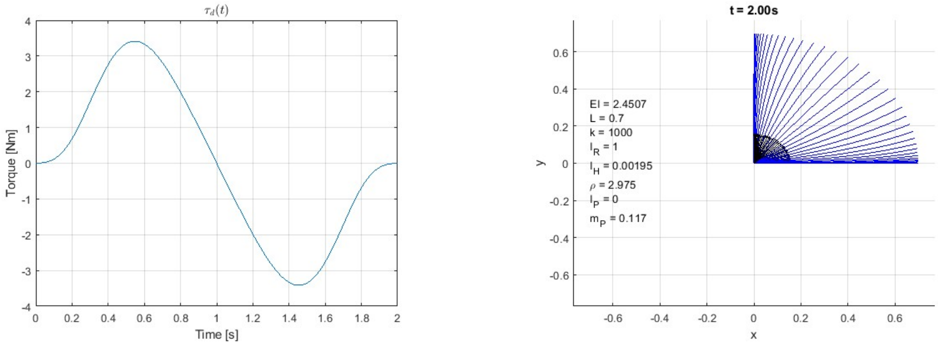
$$c_1 = \frac{1 - c_2 N_2 - c_3 N_3}{N_1} = -1.7820124$$

We chose a polynomial trajectory of degree $4 \cdot 3 + 3 = 15$ to let the beam tip going from an angle $\theta_i = 0$ to $\theta_f = \frac{\pi}{2}$. As first attempt, we set a motion time of $T = 2s$; the desired $\tau_d$ for the trajectory is plotted in Fig.9a, while in Fig.9b we can see a pretty exact motion. Then, in Fig. 10a we can see that the beam tip follows the designed trajectory with an error in the interval $[-0.0828, 0.0828]$ rad $\simeq [-4.74, 4.74]°$.

Then we repeated the simulations with an halved motion time of $T = 1s$; the first thing to notice is the different profile of $\tau_d$, shown in Fig.11a; in fact, the presence of the spring between the rotor and the hub causes a change in the shape of $\tau_d$ for fast motion that is dependent on the spring stiffness.
Now the error on the tip displacement lies in the interval of $[-0.3151, 0.3151]$ rad $\simeq [-18.05, 18.05]°$.

[9]The non-null values for the hub inertia and payload mass are given to emphasize the flexibility of the beam, although the model is not accurate

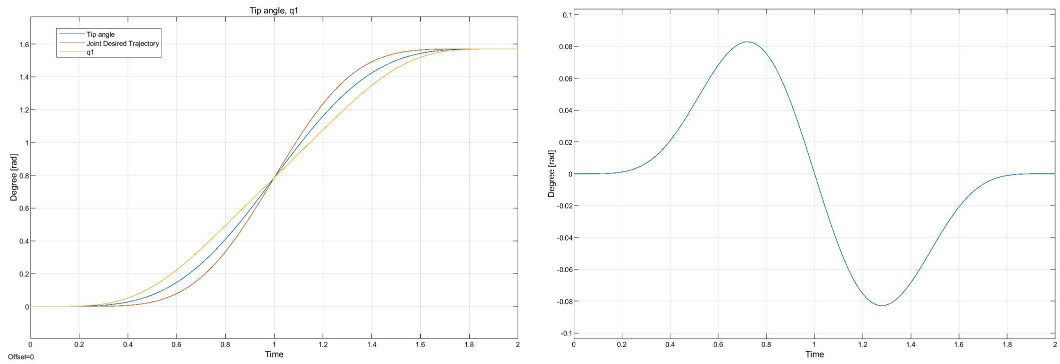
[10]The choice of a so high stiffness constant will be clarified later




(a) $\tau_d(t)$

(b) Stroboscopic view of the beam motion, the black arrow at the origin represent the angle of the rotor

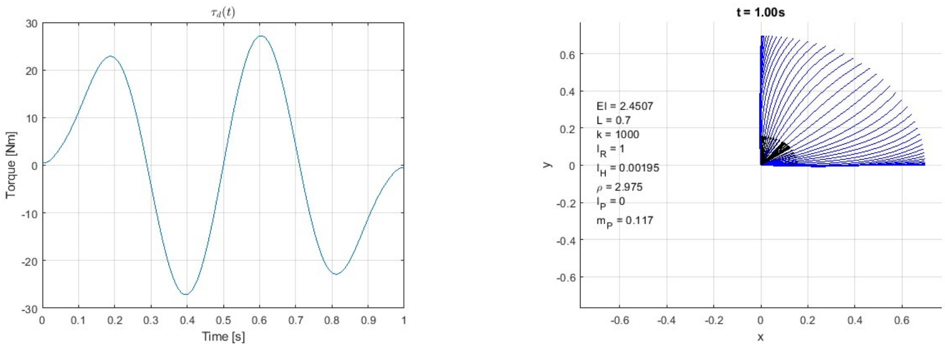
Figure 9: Motion time of $T = 2s$




(a) Tip angle, rotor angle and desired trajectory for a motion time of $T = 2s$

(b) Tip displacement error

Figure 10: Tip displacement

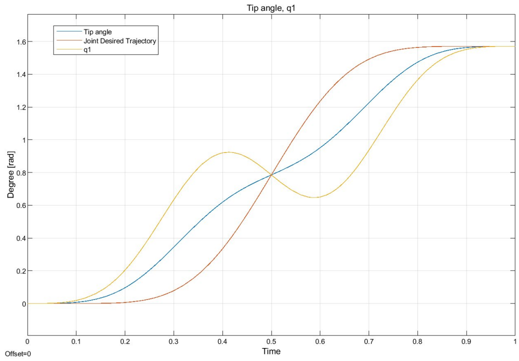



(a) $\tau_d(t)$

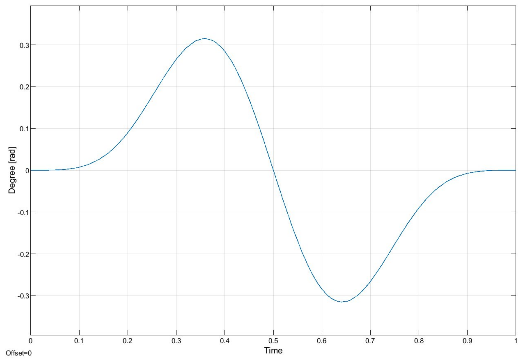
(b) Stroboscopic view of the beam motion, the black arrow at the origin represent the angle of the rotor

Figure 11: Motion time of $T = 1s$

In fact, if we use a spring stiffness of $k_1 = k/100 = 10N/m$ we can see that the "bounce" of $\tau_d$ begins to emerge, as seen in Fig. 13a. We can notice that, although the motion time is $T = 2s$, the plots are similar to the one found for $T = 1s$, and the tip displacement error lies in the interval $[-0.1504, 0.1504]$ rad $\simeq [-8.62, 8.62]°$.
For smaller stiffness values the beam behaviour becomes even more similar to the one get

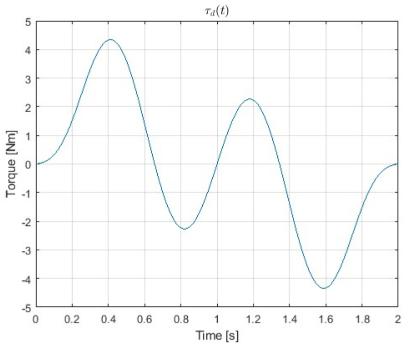



(a) Tip angle, rotor angle and desired trajectory for a motion time of $T = 1s$

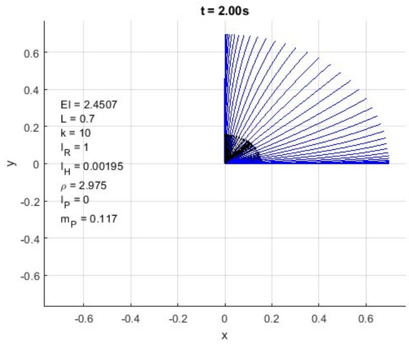



(b) Tip displacement error

Figure 12: Tip displacement

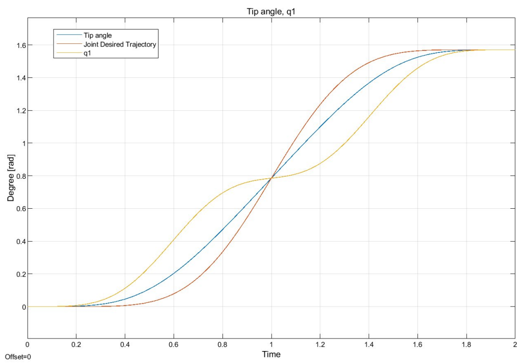



(a) $\tau_d(t)$

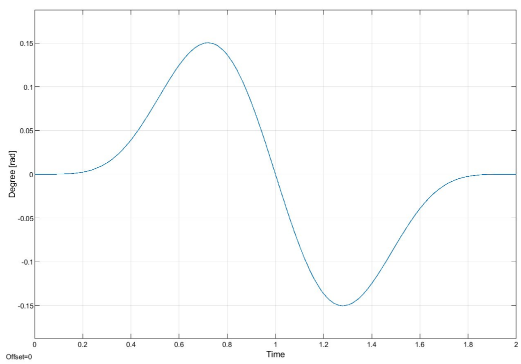



(b) Stroboscopic view of the beam motion, the black arrow at the origin represent the angle of the rotor

Figure 13: Motion time of $T = 2s$, spring stiffness of $k_1 = 10N/m$




(a) Tip angle, rotor angle and desired trajectory for a motion time of $T = 1s$




(b) Tip displacement error

Figure 14: Tip displacement

for the faster motion, but the assumption on the beam deformations doesn't hold anymore.

# 8 Appendix

## 8.1 Potential energy of an elastic beam

If we indicate the beam moment with $M$ and its curvature with $C$, we have that, by definition

$$EI := \frac{M}{C}$$

than the integral of $M$ in the curvature gives the beam's energy per unit length:

$$\frac{\partial U_B}{\partial x} = \int_0^C M \, dC = \frac{1}{2} EIC^2$$

so the total energy is

$$U_B = \int_0^l \frac{1}{2} EIC^2 \, dx = \frac{1}{2} EI \int_0^l C^2 \, dx$$

then, since the curvature of a function $w(x)$ is, with the assumption of slow variation $\left(\frac{dw}{dx} \simeq 0\right)$

$$C = \frac{\frac{d^2w}{dx^2}}{\left(1 + \left(\frac{dw}{dx}\right)^2\right)^{3/2}} \simeq \frac{d^2w}{dx^2}$$

that gives

$$U_B = \frac{1}{2} EI \int_0^l \left(\frac{d^2w}{dx^2}\right)^2 \, dx$$

that is the result we were looking for.

## 8.2 Properties of the generalized Sturm-Liouville operator

**Theorem:** The similar Sturm-Liouville operator

$$\mathcal{L} = \alpha \frac{d^4}{dx^4}$$

is self-adjoint in the spatial domain $[0, l]$ if applied to a function $f$ that satisfies

$$f(0) = 0 \quad f'(0) = 0$$
$$f''(l) = 0 \quad f'''(l) = 0$$

**Proof:** Given two functions $f$ and $g$ that satisfies the given conditions, we can proceed as for a classical Sturm-Liouville operator by integrating by parts:

$$
\begin{aligned}
< \mathcal{L}(f), g > &= \frac{EI}{\rho} \int_0^l f''''g \; dx = \frac{EI}{\rho} \left( f'''g \Big|_0^l - \int_0^l f'''g' \; dx \right) = \\
&= \frac{EI}{\rho} \left( \cancelto{0}{f'''(l)} g(l) - f'''(0) \cancelto{0}{g(0)} - f''g' \Big|_0^l + \int_0^l f''g'' \; dx \right) = \\
&= \frac{EI}{\rho} \left( - \cancelto{0}{f''(l)} g'(l) + f''(0) \cancelto{0}{g'(0)} + f'g'' \Big|_0^l - \int_0^l f'g''' \; dx \right) = \\
&= \frac{EI}{\rho} \left( f'(l) \cancelto{0}{g''(l)} - \cancelto{0}{f'(0)} g''(0) - fg''' \Big|_0^l + \int_0^l fg'''' \; dx \right) = \\
&= \frac{EI}{\rho} \left( -f(l) \cancelto{0}{g'''(l)} + \cancelto{0}{f(0)} g'''(0) + \int_0^l fg'''' \; dx \right) = \\
&= \int_0^l f\mathcal{L}(g) \; dx =< f, \mathcal{L}(g) >
\end{aligned}
$$

**Theorem:** The solutions of similar Sturm-Liouville problem

$$\alpha f''''(x) = \omega^2 f(x)$$

relative to different eigenvalues $\omega^2$ are orthogonal.
**Proof:** With the operator $\mathcal{L}$ defined above we have

$$\mathcal{L}(f_i) = \omega_i^2 f_i$$

and, since it's self-adjoint,

$$
\begin{aligned}
< \mathcal{L}(f_i), f_j > &=< f_i, \mathcal{L}(f_j) > \\
\omega_i^2 < f_i, f_j > &= \omega_j^2 < f_i, f_j > \\
(\omega_i^2 - \omega_j^2) < f_i, f_j > &= 0 \\
< f_i, f_j > &= 0
\end{aligned}
$$

since $\omega_i^2 \neq \omega_j^2$.

**Theorem:** The similar Sturm-Liouville operator

$$\mathcal{L} = \alpha \frac{d^4}{dx^4}$$

is self-adjoint in the spatial domain $[0, l]$ if applied to a function $f$ that satisfies

$$
\begin{aligned}
f(0) = 0 &\quad f''(0) = Cf'(0) \\
f''(l) = 0 &\quad f'''(l) = 0
\end{aligned}
$$

**Proof:** Given two functions $f$ and $g$ that satisfies the given conditions, we can proceed as for a classical Sturm-Liouville operator by integrating by parts:

$$
\begin{aligned}
<\mathcal{L}(f), g> &= \frac{EI}{\rho}\int_0^l f''''g\ dx = \frac{EI}{\rho}\left( f'''g\Big|_0^l - \int_0^l f'''g'\ dx\right) = \\
&= \frac{EI}{\rho}\left( \cancelto{0}{f'''(l)}\, g(l) - f'''(0)\cancelto{0}{g(0)} - f''g'\Big|_0^l + \int_0^l f''g''\ dx\right) = \\
&= \frac{EI}{\rho}\left( -\cancelto{0}{f''(l)}\, g'(l) + f''(0)g'(0) + f'g''\Big|_0^l - \int_0^l f'g'''\ dx\right) = \\
&= \frac{EI}{\rho}\left( Cf'(0)g'(0) + f'(l)\cancelto{0}{g''(l)} - f'(0)g''(0) - fg'''\Big|_0^l + \int_0^l fg''''\ dx\right) = \\
&= \frac{EI}{\rho}\left( Cf'(0)g'(0) - f'(0)Cg'(0) - f(l)\cancelto{0}{g'''(l)} + \cancelto{0}{f(0)}\, g'''(0) + \int_0^l fg''''\ dx\right) = \\
&= \int_0^l f\mathcal{L}(g)\ dx = < f, \mathcal{L}(g) >
\end{aligned}
$$

### 8.3 Evaluation of $det(H) = 0$ (Section 5.1)

Here are reported all the computations made to find eq. (55)

$$
\begin{aligned}
&[Rs - c + V(s - sh) + VR(c - ch)][Tch - sh + V(c + ch) - VT(s + sh)]+ \\
&- [Rsh + ch + V(s - sh) + VR(c - ch)][Tc + s + V(c + ch) - VT(s + sh)] = 0
\end{aligned}
$$

$$
\begin{aligned}
&RTsch - Rssh + VRs(c + ch) - VRTs(s + sh) - Tcch + csh - Vc(c + ch) + VTc(s + sh)+ \\
&+ VTch(s - sh) - Vsh(s - sh) + V^2(s - sh)(c + ch) - V^2T(s - sh)(s + sh)+ \\
&+ VRTch(c - ch) - VRsh(c - ch) + V^2R(c - ch)(c + ch) - V^2RT(c - ch)(s + sh)+ \\
&- RTcsh - Rssh - VRsh(c + ch) + VRTsh(s + sh) - Tcch - sch - Vch(c + ch) + VTch(s + sh)+ \\
&- VTc(s - sh) - Vs(s - sh) - V^2(s - sh)(c + ch) + V^2T(s - sh)(s + sh)+ \\
&- VRTc(c - ch) - VRs(c - ch) - V^2R(c - ch)(c + ch) + V^2RT(c - ch)(s + sh) = 0
\end{aligned}
$$

$$
\begin{aligned}
&RTsch - 2Rssh + VRs(c + ch) - VRTs(s + sh) - 2Tcch + csh - Vc(c + ch) + VTc(s + sh)+ \\
&+ VTch(s - sh) - Vsh(s - sh)+ \\
&+ VRTch(c - ch) - VRsh(c - ch)+ \\
&- RTcsh - VRsh(c + ch) + VRTsh(s + sh) - sch - Vch(c + ch) + VTch(s + sh)+ \\
&- VTc(s - sh) - Vs(s - sh)+ \\
&- VRTc(c - ch) - VRs(c - ch) = 0
\end{aligned}
$$

$$
\begin{aligned}
&- VRTs(s+sh) + VRTsh(s+sh) + VRTch(c-ch) - VRTc(c-ch)+\\
&+ VRs(c+ch) - VRs(c-ch) - VRsh(c+ch) - VRsh(c-ch)\\
&+ VTc(s+sh) - VTc(s-sh) + VTch(s+sh) + VTch(s-sh)+\\
&- Vc(c+ch) - Vch(c+ch) - Vsh(s-sh) - Vs(s-sh)+\\
&RTsch - RTcsh - 2Rssh - 2Tcch + csh - sch = 0
\end{aligned}
$$

$$
\begin{aligned}
&- VRTs^2 + VRTsh^2 + 2VRTcch - VRTch^2 - VRTc^2+\\
&+ 2VRsch - 2VRcsh+\\
&+ 2VTcsh + 2VTsch(s)+\\
&- Vc^2 - 2Vcch - Vch^2 + Vsh^2 - Vs^2+\\
&RTsch - RTcsh - 2Rssh - 2Tcch + csh - sch = 0
\end{aligned}
$$

$$
\begin{aligned}
&- VRT - VRT + 2VRTcch+\\
&+ 2VRsch - 2VRcsh+\\
&+ 2VTcsh + 2VTsch+\\
&- V - 2Vcch - V+\\
&RTsch - RTcsh - 2Rssh - 2Tcch + csh - sch = 0
\end{aligned}
$$

$$
\begin{aligned}
&2V(1+RT) + 2R\sin(\beta l)\sinh(\beta l)+\\
&+ (1 - 2VR - 2VT - RT)\sin(\beta l)\cosh(\beta l)+\\
&- (1 - 2VR + 2VT - RT)\cos(\beta l)\cosh(\beta l)+\\
&- 2(VRT - V - T)\cos(\beta l)\cosh(\beta l) = 0
\end{aligned}
$$

## 8.4 Evaluation of $det(H^+) = 0$ (Section 5.2)

Starting from

$$
\begin{aligned}
&2V^+(1+R^+T^+) + 2R^+\sin(\beta l)\sinh(\beta l) + (1 - 2V^+R^+ - 2V^+T^+ - R^+T^+)\sin(\beta l)\cosh(\beta l)+\\
&-(1 - 2V^+R^+ + 2V^+T^+ - R^+T^+)\cos(\beta l)\sinh(\beta l) - 2(V^+R^+T^+ - V^+ - T^+)\cos(\beta l)\cosh(\beta l) = 0
\end{aligned}
\tag{65}
$$

we can derive the model without the presence of the payload, in particular

$$m_P = 0 \Rightarrow R^+ = 0, \quad I_P = 0 \Rightarrow T^+ = 0$$

hence the equation becomes

$$2V^+ + \sin(\beta l)\cosh(\beta l) - \cos(\beta l)\sinh(\beta l) + 2V^+\cos(\beta l)\cosh(\beta l) = 0$$

Recalling

$$V^+ = U^+ + S^+ = \beta^3\frac{I_H}{2\rho} - \frac{k}{2EI\beta}$$

the equation is

$$\beta^3\frac{I_H}{\rho} - \frac{k}{EI\beta} + \sin(\beta l)\cosh(\beta l) - \cos(\beta l)\sinh(\beta l) + \left(\beta^3\frac{I_H}{\rho} - \frac{k}{EI\beta}\right)\cos(\beta l)\cosh(\beta l) = 0$$

that can be manipulated further:

$$(\cos(\beta l)\sinh(\beta l) - \sin(\beta l)\cosh(\beta l)) - \frac{I_H}{\rho}\beta^3(1+\cos(\beta l)\cosh(\beta l)) + \frac{k}{EI\beta}(1+\cos(\beta l)\cosh(\beta l)) = 0$$

and now the similarities between this equation and the one found for the non-elastic joint can be clearly seen.

## 8.5 Polynomial trajectory

Starting from a doubly-normalized polynomial $y(x)$ of odd degree $n$ of the form

$$y(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

with the following conditions:

$$\begin{gathered} y(0)=0 \quad y(1)=1 \\ y'(0) = y^{[2]}(0) = \cdots = y^{[\frac{n-1}{2}]}(0) = 0 \\ y'(1) = y^{[2]}(1) = \cdots = y^{[\frac{n-1}{2}]}(1) = 0 \end{gathered}$$

we know that the coefficents from $a_0$ up to $a_k$ are zero, where $k = \frac{n-1}{2}$. Then we can define the two matrices

$$S = \begin{pmatrix} n!^{-1} & (n-1)!^{-1} & (n-2)!^{-1} & \cdots & \left(\frac{n+1}{2}\right)!^{-1} \\ (n-1)!^{-1} & (n-2)!^{-1} & (n-3)!^{-1} & \cdots & \left(\frac{n-1}{2}\right)!^{-1} \\ (n-2)!^{-1} & (n-3)!^{-1} & (n-4)!^{-1} & \cdots & \left(\frac{n-3}{2}\right)!^{-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \left(\frac{n+1}{2}\right)!^{-1} & \left(\frac{n-1}{2}\right)!^{-1} & \left(\frac{n-3}{2}\right)!^{-1} & \cdots & 1 \end{pmatrix} = S^T$$

$$D = \begin{pmatrix} n! & 0 & 0 & \cdots & 0 \\ 0 & (n-1)! & 0 & \cdots & 0 \\ 0 & 0 & (n-2)! & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \left(\frac{n+1}{2}\right)! \end{pmatrix}$$

and the remaining coefficients are the first column of the matrix

$$(SD)^{-1}$$

or equally the component of the vector

$$a = (SD)^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

# References

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[2] F. Bellezza L. Lanari and G. Ulivi. "Exact modeling of the flexible slewing link". In: *IEEE Int. Conf. on Robotics and Automation* (1990), pp. 734–739.

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