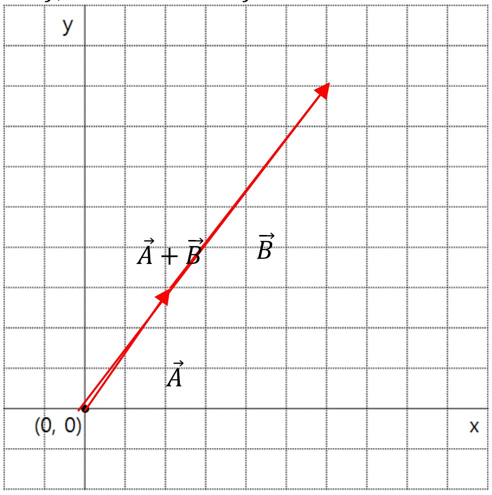
2023-1 Solid Mechanics Midterm Exam

(2023.04.25)

『Please write all the answers on this test sheets. You can also use the back side for the answers.』

1. There are two vectors: $\vec{A} = 2\vec{i} + 3\vec{j}$, and $\vec{B} = 4\vec{i} + 5\vec{j}$. Draw $\vec{A} + \vec{B}$ on the 1-unit graph paper. [5 Points]



2. Calculate the inner product of $\vec{A}\cdot\vec{B}$ (of Problem 1). [5 Points]

$$\vec{A}\cdot\vec{B} = (2\vec{i}+3\vec{j})\cdot(4\vec{i}+5\vec{j}) = (2)(4)\vec{i}\cdot\vec{i} + (2)(5)\overset{0}{\cancel{\vec{i}\cdot\vec{j}}} + (3)(4)\overset{0}{\cancel{\vec{j}\cdot\vec{i}}} + (3)(5)\vec{j}\cdot\vec{j}$$

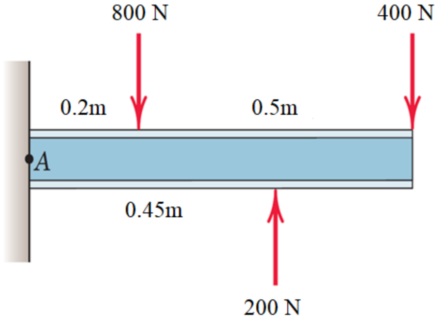
$$= 8 + 15 = \underline{23}$$

3. Calculate the cross product of $\vec{A}\times\vec{B}$ (of Problem 1). [5 Points]

$$\vec{A}\times\vec{B} = (2\vec{i}+3\vec{j})\times(4\vec{i}+5\vec{j}) = (2)(4)\overset{0}{\cancel{\vec{i}\times\vec{i}}} + (2)(5)\overset{\vec{k}}{\underline{\vec{i}\times\vec{j}}} + (3)(4)\overset{-\vec{k}}{\underline{\vec{j}\times\vec{i}}} + (3)(5)\cancel{\vec{j}\times\vec{j}}$$

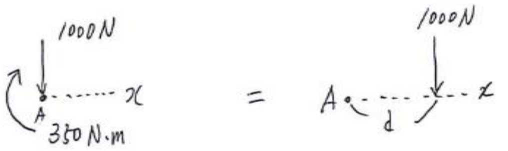
$$= (10-12)\vec{k} = \underline{-2\vec{k}}$$

4. Express the force-couple system at point A, and calculate the distance x to express the force-couple system as only one resultant force. [Total 12 Points (Answer: 3 Points, Equations: 9 Points)]



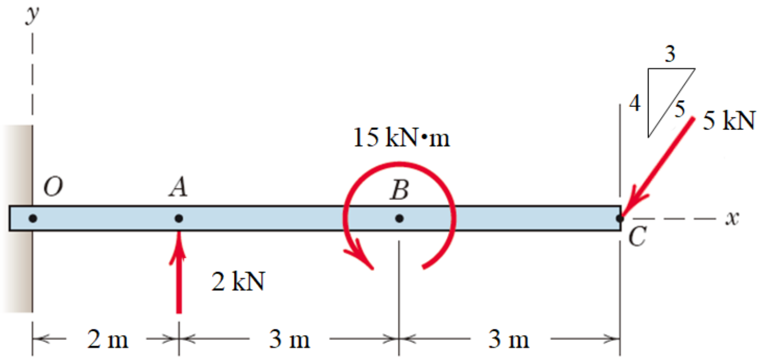
At A: $R = \Sigma F = 800 + 400 - 200 = \underline{1000\ N\ (\downarrow)}$

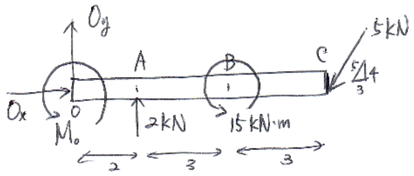
$\circlearrowleft + \ M_A = 800(0.2) + 400(0.7) - 200(0.45) = \underline{350\ N\cdot m}$



$1000(d) = 350 \quad \therefore \underline{d = 0.35\ m}$

5. Calculate the reaction forces at point O. (Neglect the mass of the beam.) [Total 12 Points (FBD: 4 Points, EoM: 6 Points, Answer: 2 Points)]





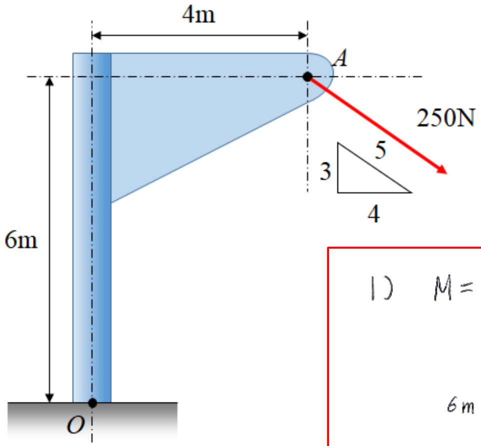
$\Sigma F_x = 0: \quad O_x - 3 = 0 \qquad \therefore \underline{O_x = 3kN}$

$\Sigma F_y = 0: \quad O_y + 2 - 4 = 0 \qquad \therefore \underline{O_y = 2kN}$

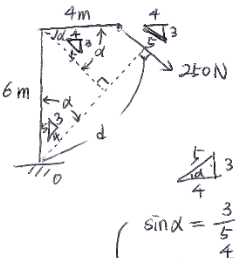
$\Sigma M_o = 0: \quad M_o + 2(2) + 15 - 8(4) = 0$

$\therefore \underline{M_o = 13\ kN\cdot m}$

6. Calculate the magnitude of the moment about the base point O of the 600-N force in five different ways. [Total 15 Points (3 Points each)]



1) $M = Fd$ 이용.



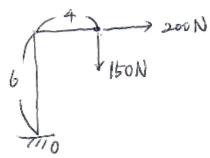
$\sin\alpha = \frac{3}{5}$

$\cos\alpha = \frac{4}{5}$

$$d = 4\sin\alpha + 6\cos\alpha = 4\left(\frac{3}{5}\right) + 6\left(\frac{4}{5}\right) = \frac{36}{5}\,(m)$$

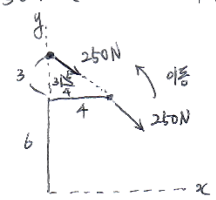
$$\therefore M_o = Fd = 250\left(\frac{36}{5}\right) = \underline{1800\ (N\cdot m)\ CW}$$

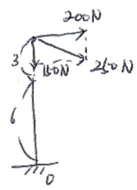
2) 250N을 F_x와 F_y로 나눔.



$$M_o = 200(6) + 150(4) = \underline{1800\ (N\cdot m)\ CW}$$

3) 250 N을 $x=0$ 위치로 이동.

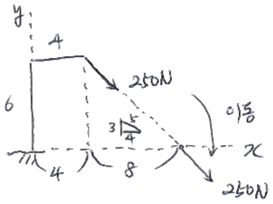


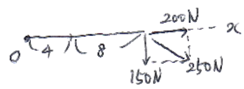


F_x 성분만 모멘트를 생성.

$$M_o = 200(9) = \underline{1800\ N\cdot m)CW}$$

4) 250 N을 $y=0$ 위치로 이동.





F_y 성분만 모멘트를 생성.

$$M_o = 150(12) = \underline{1800\ (N\cdot m)\ CW}$$

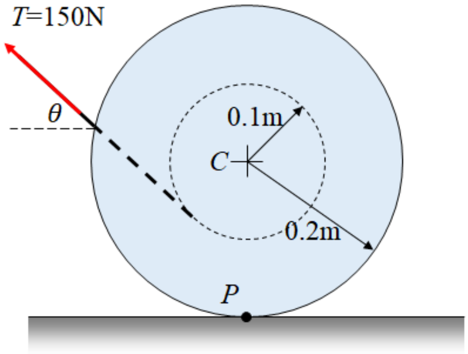
5) $\vec{M} = \vec{r} \times \vec{F}$ 이용.

$$\vec{M}_o = (4\vec{i} + 6\vec{j}) \times (200\vec{i} - 150\vec{j})$$

$$= -600\vec{k} - 1200\vec{k} = -1800\vec{k}$$

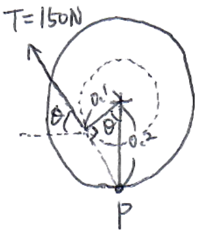
$$\therefore \underline{M_o = 1800\ (N\cdot m)\ CW}$$

7. Pull the string tightly wound around the inner hub of the drum with a force T of 150N. (1) Find the moment of T with respect to the center C of the drum. (2) Find the angle θ that makes the moment about the contact point P zero. [Total 10 Points (5 Points each)]



(1) $\circlearrowright + M_c = 150(0.1) = \underline{15\ (N \cdot m)\ CW}$

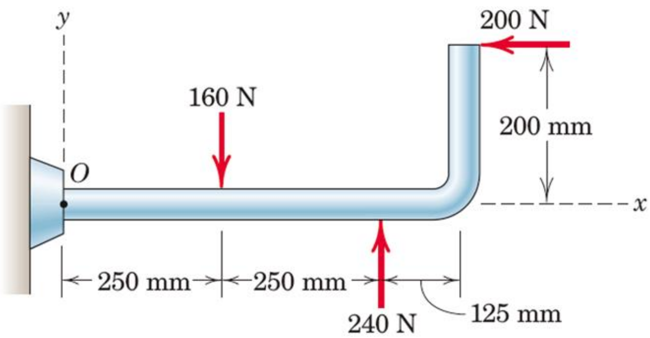
(2)



$$\cos\theta = \frac{0.1}{0.2} = \frac{1}{2}$$

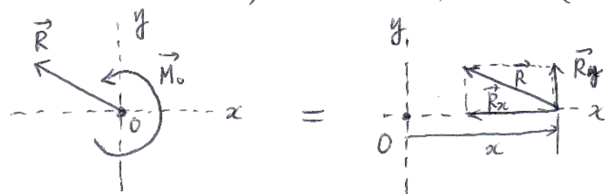
$$\therefore \theta = \cos^{-1}\left(\frac{1}{2}\right) = \underline{60^\circ}$$

8. Replace the three forces acting on the bent pipe with one equivalent force R. Find the distance x between the point O and the point on the x-axis through which the line of action of the resultant force R passes. [Total 12 Points (Answer: 4 Points, Equations: 8 Points)]



$$\vec{R} = -200\,\vec{i} + 80\,\vec{j}\ (N) \quad \cdots (*)$$

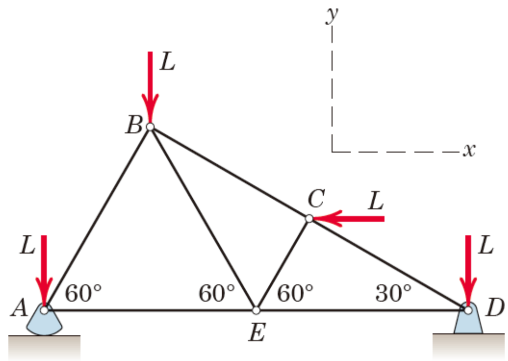
$$\circlearrowleft + \vec{M}_o = -160(0.25) + 240(0.5) + 200(0.2) = 120\ (N \cdot m)$$



$$R_y x = M_o$$

$$x = \frac{M_o}{R_y} = \frac{120}{80} = \underline{1.5\ (m)}$$

9. A simple asymmetric simple truss is loaded as shown. Determine the reactions at A and D. Neglect the weight of the structure. [Total 12 Points (FBD: 4 Points, EoM: 6 Points, Answer: 2 Points)]



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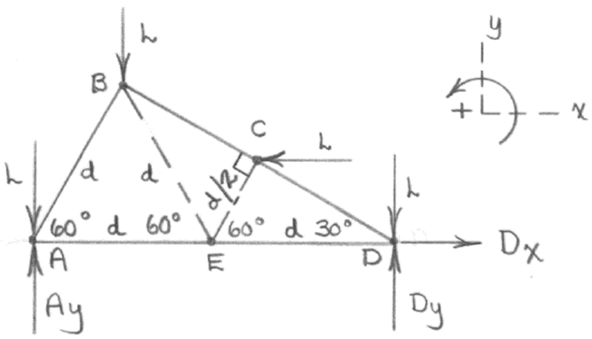


$\Sigma F_x = 0: \quad D_x - L = 0, \qquad \underline{D_x = L}$

$\Sigma F_y = 0: \quad A_y + D_y - 3L = 0$

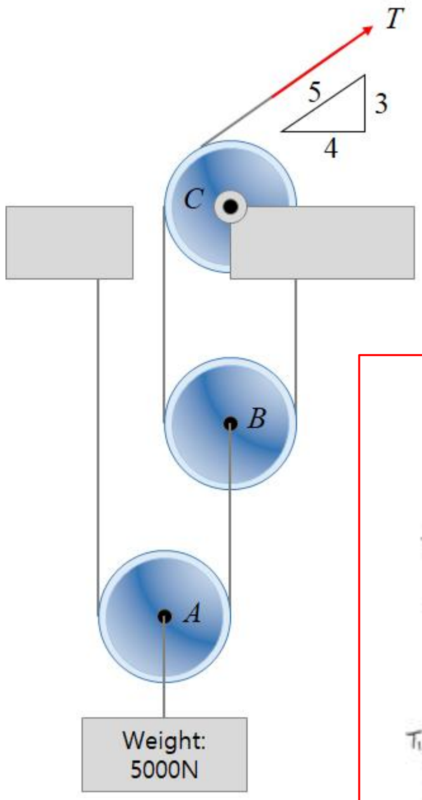
$\Sigma M_A = 0: \quad D_y(2d) + L\left(\frac{d}{2}\frac{\sqrt{3}}{2}\right) - L\left(\frac{d}{2}\right) - L(2d) = 0$

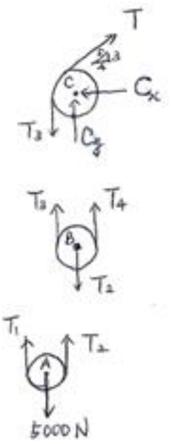
Solving the last 2 equations: $\quad A_y = \frac{L}{4}\left(7 + \frac{\sqrt{3}}{2}\right)$

$\underline{D_y = \frac{L}{4}\left(5 - \frac{\sqrt{3}}{2}\right)}$

(or $A_y = 1.967L$, $D_y = 1.033L$)

10. Calculate the tension T and the total force acting on the bearing of the pulley C in the cable supporting the pulley with a weight of 5000 N as shown in the figure. (Assume that the pulley can rotate freely for each bearing, and the weight of each part is smaller than the load of the load.) [Total 12 Points (Tension: 6 Points, Forces at C: 6 Points)]





A 도르래

도르래 반지름을 r이라 하자.

$\circlearrowleft + \Sigma M_A = 0 : \quad T_2 r - T_1 r = 0$

$\therefore T_1 = T_2$

$\Sigma F_y = 0 : \quad T_1 + T_2 - 5000 = 0$

$\therefore T_1 = T_2 = \underline{2500\ (N)}$

B 도르래

$\circlearrowleft + \Sigma M_B = 0 : \quad T_4 r - T_3 r = 0$

$\therefore T_3 = T_4$

$\Sigma F_y = 0 : \quad T_3 + T_4 - T_2 = 0$

$\therefore T_3 + T_4 = 2500$

$\therefore T_3 = T_4 = \underline{1250\ (N)}$

C 도르래

$\circlearrowleft + \Sigma M_C = 0 : \quad T_3 r - T r = 0$

$\therefore T = T_3 = \underline{1250\ (N)}$ ...⊛

$\Sigma F_x = 0 : \quad 1250 \left(\frac{4}{5}\right) - C_x = 0$

$\therefore C_x = \underline{1000\ (N)}$

$\Sigma F_y = 0 : \quad C_y + 1250 \left(\frac{3}{5}\right) - 1250 = 0$

$\therefore C_y = \underline{500\ (N)}$

$\therefore C = \sqrt{C_x^2 + C_y^2} = \sqrt{1000^2 + 500^2}$

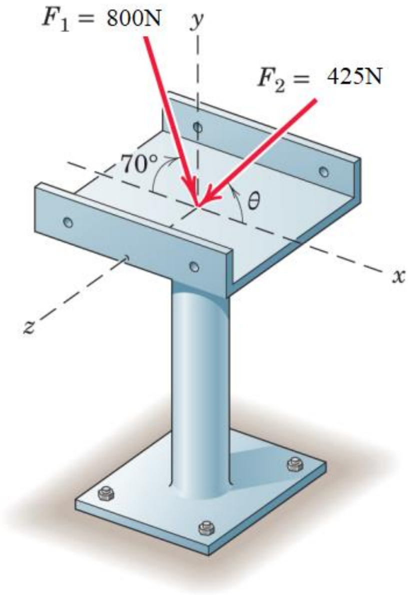
$= \underline{500\sqrt{5}\ (N)}$

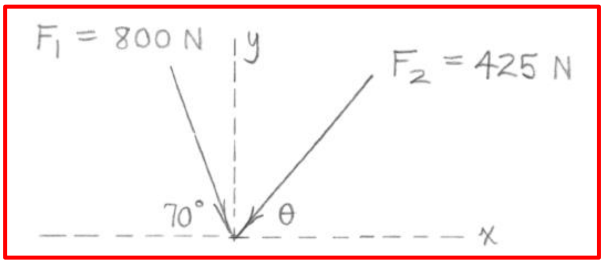
2023-1 Solid Mechanics Quiz #1

(2023.04.11.)

『Please write all the answers on this test sheets. You can also use the back side for the answers.』

1. Two forces are applied to the construction bracket as shown. (a) Determine the angle which makes the resultant of the two forces vertical. (b) Determine the magnitude R of the resultant. [20 Points]





$R_x = \sum F_x = 800 \cos 70° - 425 \cos\theta = 0$

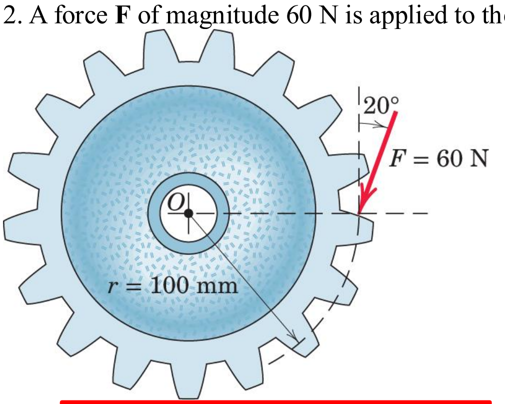
$\underline{\theta = 49.9°}$

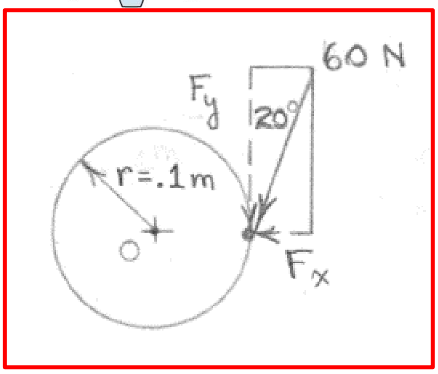
$R_y = \sum F_y = -800 \sin 70° - 425 \sin 49.9°$

$= -1077$ N

So $\underline{R = 1077 \text{ N}}$

2. A force **F** of magnitude 60 N is applied to the gear. Determine the moment of **F** about point O. [10 Points]



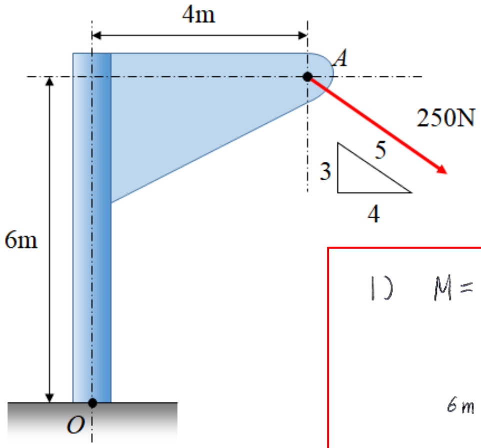


$+\circlearrowleft M_o = r F_y$

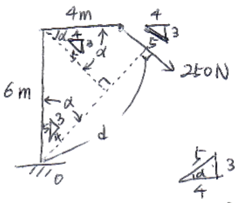
$= (0.1)(60 \cos 20°)$

$= \underline{5.64 \text{ N}\cdot\text{m}}$

3. Calculate the magnitude of the moment about the base point O of the 600-N force in five different ways. [20 Points]



1) $M = Fd$ 이용.



$d = 4\sin\alpha + 6\cos\alpha$

$= 4\left(\frac{3}{5}\right) + 6\left(\frac{4}{5}\right)$

$= \frac{36}{5}$ (m)

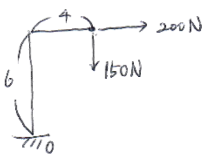
$\sin\alpha = \frac{3}{5}$

$\cos\alpha = \frac{4}{5}$

$\therefore M_o = Fd = 250\left(\frac{36}{5}\right)$

$= 1800$ (N·m) CW

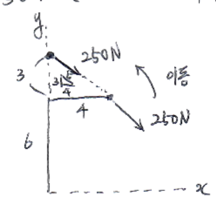
2) 250N을 F_x 와 F_y로 나눔.

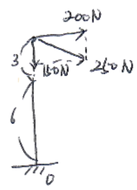


$M_o = 200(6) + 150(4)$

$= 1800$ (N·m) CW

3) 250N을 $x=0$ 위치로 이동.

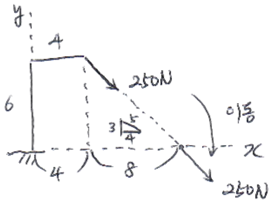


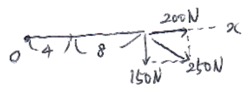


F_x 성분만 모멘트를 생성.

$M_o = 200(9) = 1800$ (N·m) CW

4) 250N을 $y=0$ 위치로 이동.





F_y 성분만 모멘트를 생성.

$M_o = 150(12) = 1800$ (N·m) CW

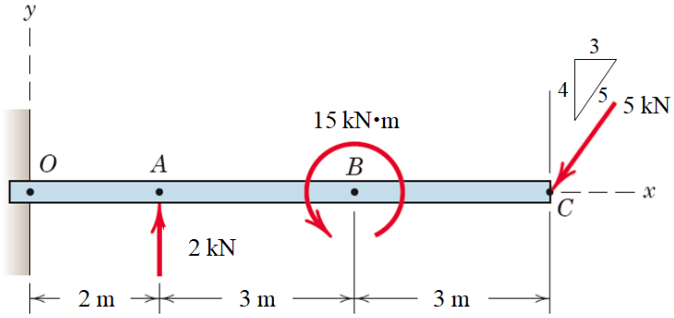
5) $\vec{M} = \vec{r} \times \vec{F}$ 이용.

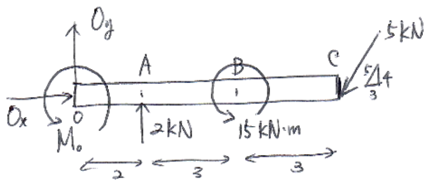
$\vec{M}_o = (4\vec{i} + 6\vec{j}) \times (200\vec{i} - 150\vec{j})$

$= -600\vec{k} - 1200\vec{k} = -1800\vec{k}$

$\therefore M_o = 1800$ (N·m) CW

4. Calculate the reaction forces at point O. (Neglect the weight of the beam.) [30 Points]





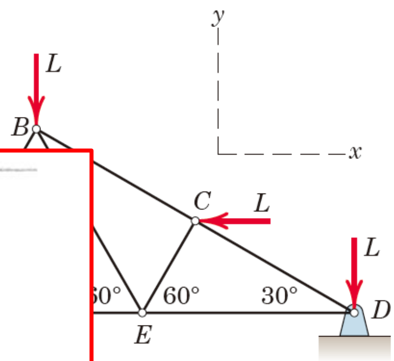
$\Sigma F_x = 0$: $O_x - 3 = 0$ $\therefore O_x = 3\text{kN}$

$\Sigma F_y = 0$: $O_y + 2 - 4 = 0$ $\therefore O_y = 2\text{kN}$

$\Sigma M_o = 0$: $M_o + 2(2) + 15 - 8(4) = 0$

$\therefore M_o = 13 \text{ kN}\cdot\text{m}$

5. A simple asymmetric simple truss is loaded as shown. Determine the reactions at A and D. Neglect the weight of the structure compared with the applied loads. [20 Points]



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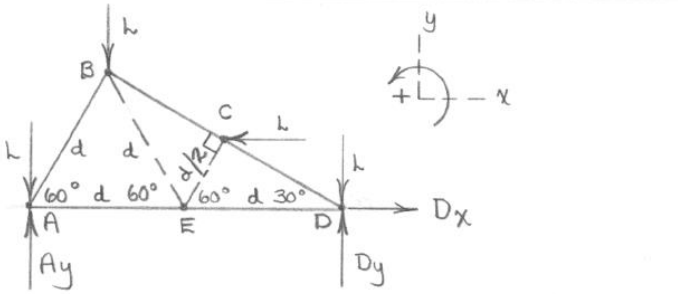


$\Sigma F_x = 0$: $D_x - L = 0$, $D_x = L$

$\Sigma F_y = 0$: $A_y + D_y - 3L = 0$

$\Sigma M_A = 0$: $D_y(2d) + L\left(\frac{d}{2}\frac{\sqrt{3}}{2}\right) - L\left(\frac{d}{2}\right) - L(2d) = 0$

Solving the last 2 equations : $A_y = \frac{L}{4}\left(7 + \frac{\sqrt{3}}{2}\right)$

$D_y = \frac{L}{4}\left(5 - \frac{\sqrt{3}}{2}\right)$

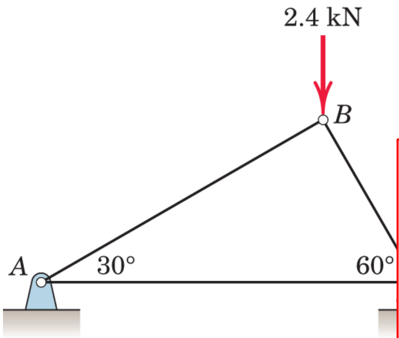
(or $A_y = 1.967L$, $D_y = 1.033L$)

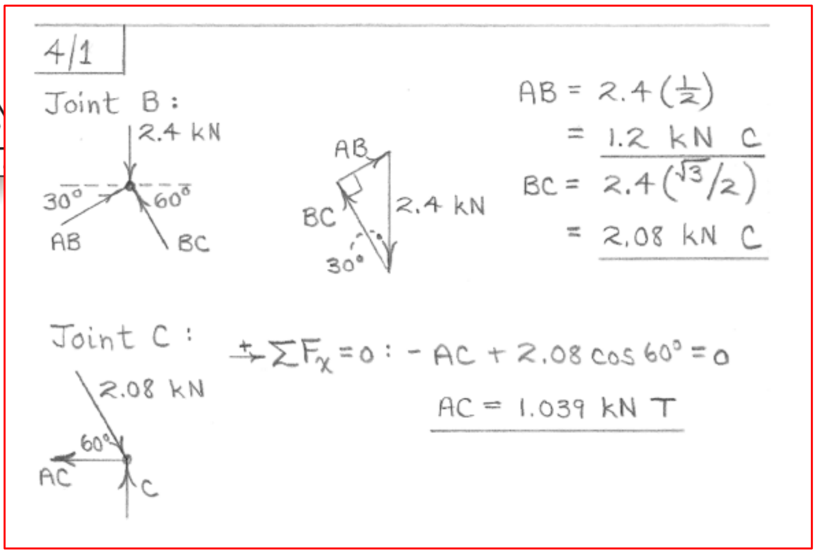
2023-1 Solid Mechanics Quiz #2

(2023.06.01.)

『Please write all the answers on this test sheets. You can also use the back side for the answers.』

1. Determine the force in each member of the loaded truss. [20 Points]





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Joint B:

$AB = 2.4\left(\frac{1}{2}\right)$

$= \underline{1.2 \text{ kN } C}$

$BC = 2.4\left(\sqrt{3}/2\right)$

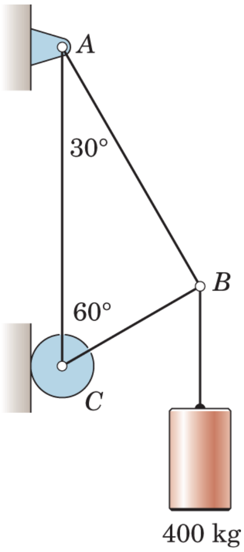
$= \underline{2.08 \text{ kN } C}$

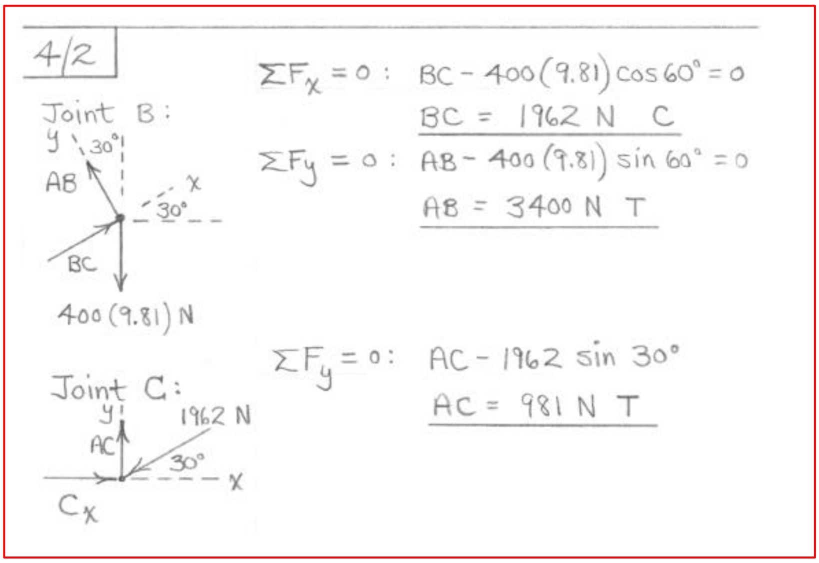
Joint C:

$\xrightarrow{+} \Sigma F_x = 0: \; -AC + 2.08 \cos 60^\circ = 0$

$\underline{AC = 1.039 \text{ kN } T}$

2. Determine the force in each member of the loaded truss. ($g=9.81 m/s^2$) [20 Points]





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Joint B:

$\Sigma F_x = 0: \; BC - 400(9.81)\cos 60^\circ = 0$

$\underline{BC = 1962 \text{ N } C}$

$\Sigma F_y = 0: \; AB - 400(9.81)\sin 60^\circ = 0$

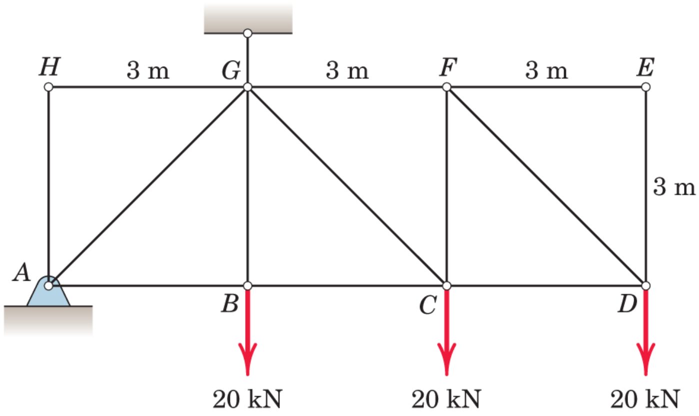
$\underline{AB = 3400 \text{ N } T}$

Joint C:

$\Sigma F_y = 0: \; AC - 1962 \sin 30^\circ$

$\underline{AC = 981 \text{ N } T}$

3. Determine the force in member CG. [20 Points]



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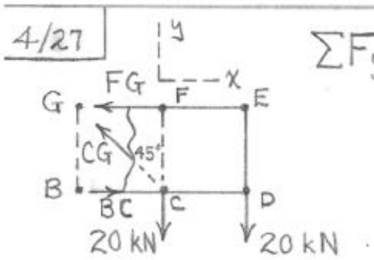
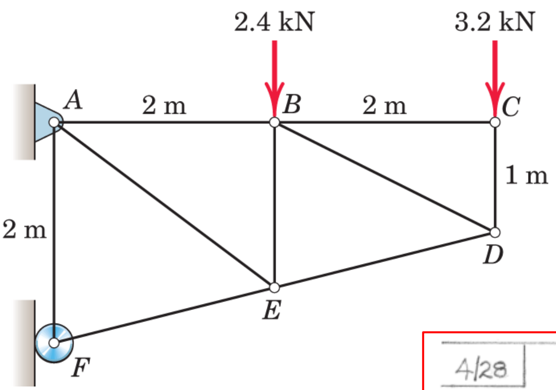


$$\Sigma F_y = 0 : CG \sin 45° - 20 - 20 = 0$$

$$\underline{CG = 56.6 \text{ kN T}}$$

4. Determine the force in member AE of the loaded truss. [20 Points]



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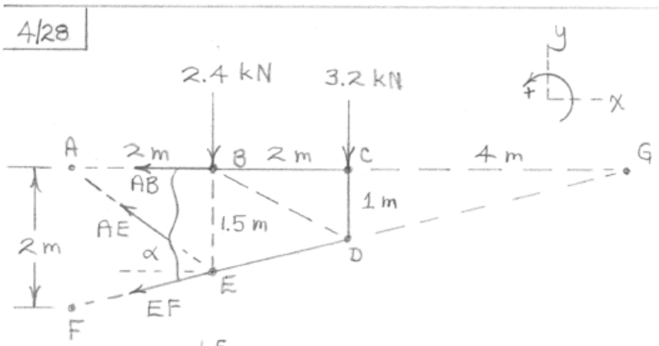


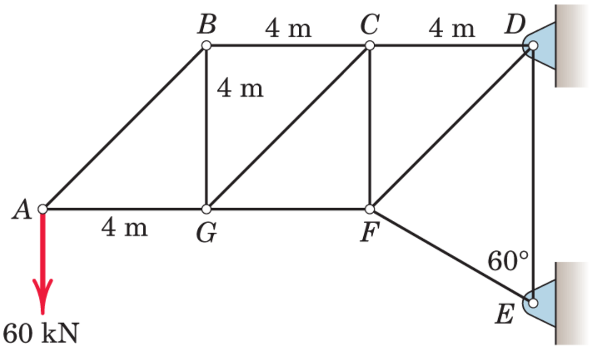
$$\alpha = \tan^{-1} \frac{1.5}{2} = 36.9°$$

$$\Sigma M_G = 0 : 3.2(4) + 2.4(6) - AE \cos\alpha \, (1.5)$$

$$- AE \sin\alpha \, (6) = 0$$

$$\underline{AE = 5.67 \text{ kN T}}$$

5. Determine the forces in members BC and CG. [20 Points]



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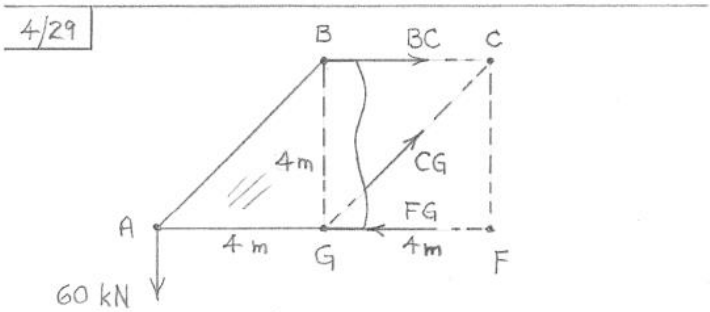


$+\uparrow \Sigma F = 0: \quad CG \frac{\sqrt{2}}{2} - 60 = 0, \qquad \underline{CG = 84.9 \text{ kN T}}$

$\curvearrowleft\!+ \Sigma M_G = 0: \quad 60(4) - BC(4) = 0, \qquad \underline{BC = 60 \text{ kN T}}$

APPENDIX A
AREA MOMENTS OF INERTIA

Jake Wyman/Photonica/Getty Images

APPENDIX OUTLINE

Article A/1 Introduction

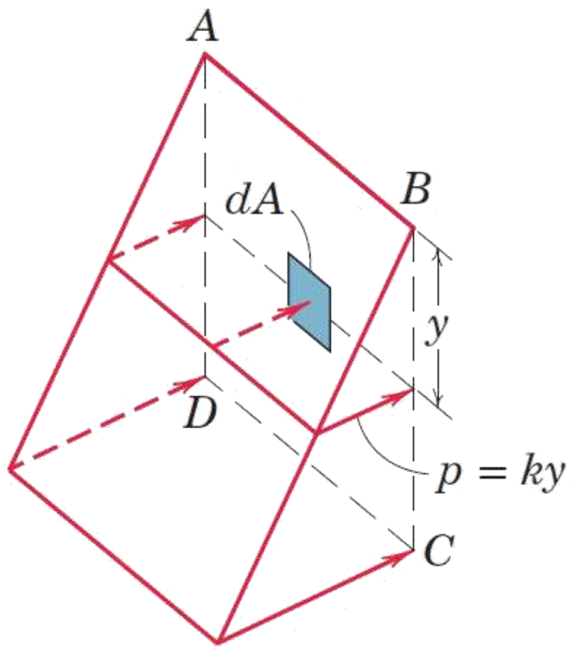
- Origin of the Area Moment of Inertia Calculation

  When forces are distributed continuously over an area on which they act, it is often necessary to calculate the moment of these forces about some axis either in or perpendicular to the plane of the area. When this occurs, an area moment of inertia calculation appears which has the form of an integral of a distance squared over an area.

- Form of the Calculation
  - $\int(\text{distance})^2 \, d(\text{area})$

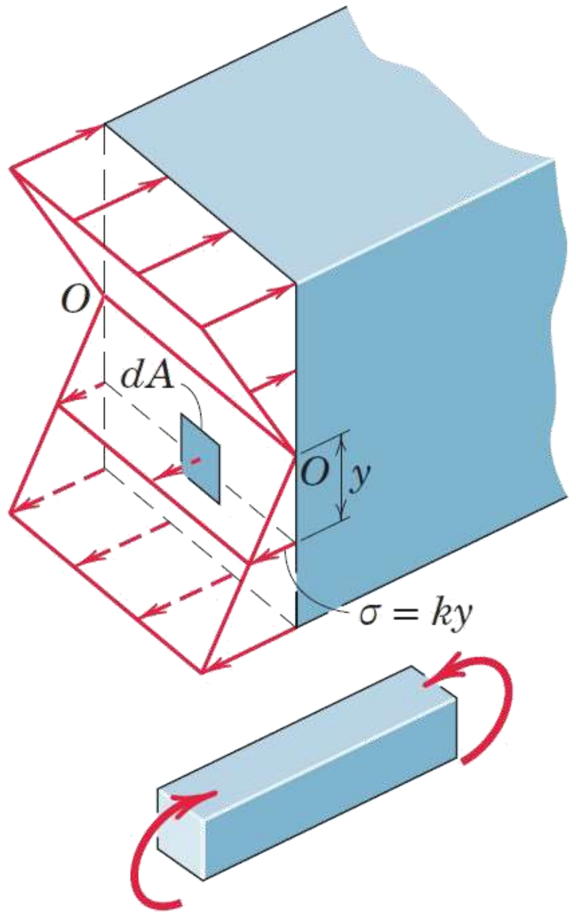
Article A/1 – Physical Origin of the Calculation (1 of 3)

- Example with Fluid Pressure
  - Consider the surface area *ABCD* which is subjected to a distributed pressure p whose intensity is proportional to the distance y from the axis *AB*. The pressure at depth in a fluid, $p = \rho g y$, where ρ is the density of the fluid and the pressure is considered relative to the atmospheric pressure.
  - The pressure at a depth y is equal to some constant k multiplied by the depth. Thus, $p = ky$.
  - The pressure p acting over a differential area dA will produce a differential force $dF = p\,dA = ky\,dA$.
  - The moment of this differential force about the axis *AB* can be written as $dM = y\,dF = yp\,dA = ky^2\,dA$.
  - The total moment $M = \int dM = \int ky^2\,dA$, which has the form of an integral of a distance squared over an area.



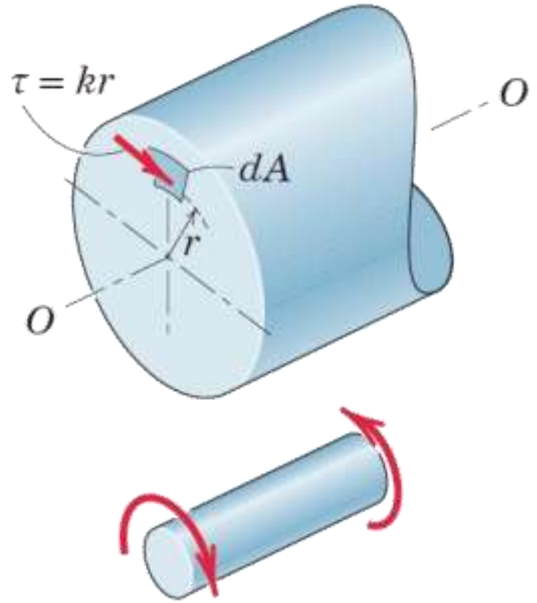
Article A/1 – Physical Origin of the Calculation (2 of 3)

- Example with Beams in Bending
  - Consider the normal stress distribution acting on a transverse section of a simple elastic beam bent by equal and opposite couples applied to its ends.
  - At any section of the beam, a linear distribution of force intensity or stress σ, given by $\sigma = ky$, is present.
  - The stress is positive (tensile) below the axis *O*-*O* and negative (compressive) above the axis. Axis *O*-*O* is termed the neutral axis.
  - The stress σ acting over a differential area dA will produce a differential force $dF = \sigma\, dA = ky\, dA$.
  - The moment of this differential force about the axis *O*-*O* can be written as $dM = y\, dF = y\sigma\, dA = ky^2\, dA$.
  - The total moment $M = \int dM = \int ky^2\, dA$, which has the form of an integral of a distance squared over an area.



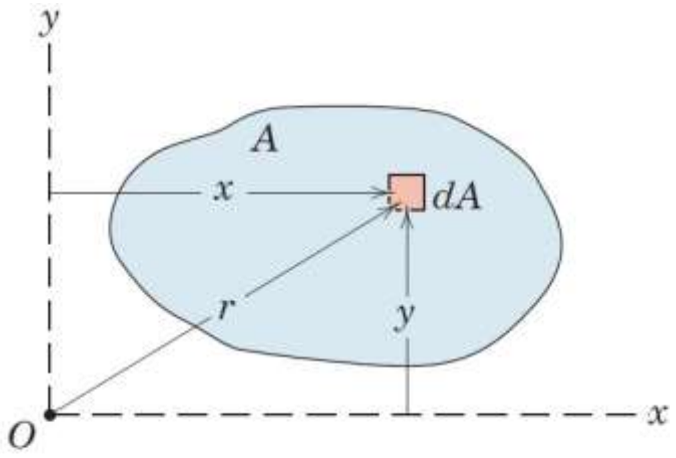
Article A/1 – Physical Origin of the Calculation (3 of 3)

- Example with Shafts in Torsion
  - Consider the shear stress distribution acting on a transverse section of a shaft subjected to equal and opposite torsional moments applied to its ends.
  - Within the elastic limit of the material, this moment is resisted at each cross section of the shaft by a distribution of tangential or shear stress τ, which is proportional to the radial distance r from the center of the shaft. Thus, $\tau = kr$.
  - The shear stress τ acting over a differential area dA will produce a differential force $dF = \tau\, dA = kr\, dA$.
  - The moment of this differential force about the shaft axis O-O can be written as $dM = r\, dF = r\tau\, dA = kr^2\, dA$.
  - The total moment $M = \int dM = \int kr^2\, dA$, which has the form of an integral of a distance squared over an area.



Article A/2 Definitions

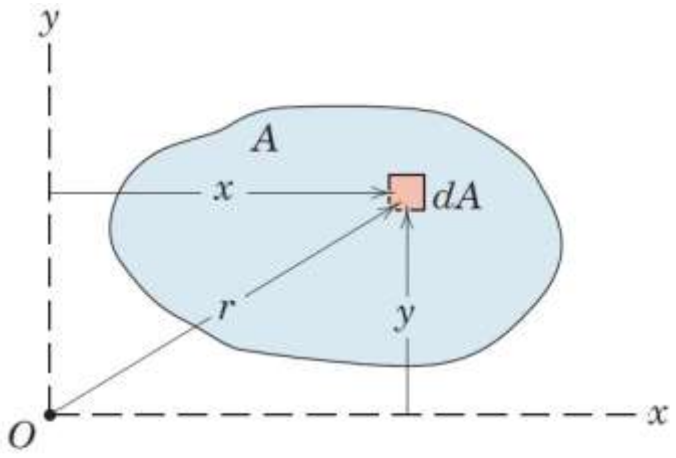
- Rectangular Moments of Inertia
  - $I_x = \int y^2\, dA$
  - $I_y = \int x^2\, dA$



- Polar Moments of Inertia
  - $I_z = \int r^2\, dA = \int x^2\, dA + \int y^2\, dA = I_x + I_y$

Article A/2 – Comments about Moments of Inertia

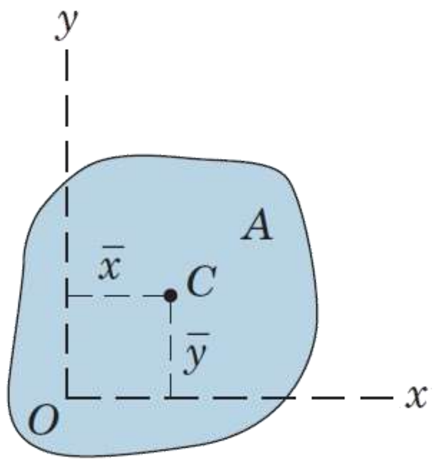
- Dimensions and Units
  - Length to the 4th Power, L^4
  - SI Units: mm^4 (most common)
  - U.S. Customary Units: $in.^4$ (most common)



- Things to Note
  - A moment of inertia is always positive since it involves the square of a distance from the inertia axis to the element of integration.
  - The choice of coordinates (rectangular or polar) to use for the calculation of moments of inertia is important.

Article A/2 – Radius of Gyration (1 of 3)

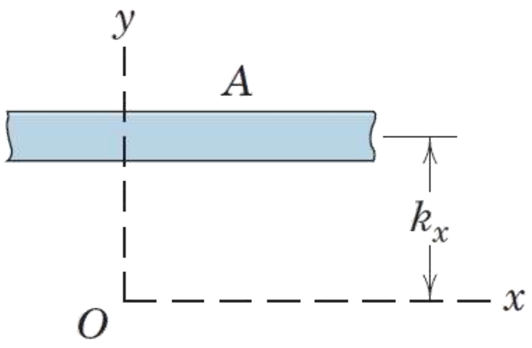
- Concept
  - The radius of gyration is a measure of the distribution of an area from a particular axis.



- Illustration with x-Axis
  - Consider the area A with moment of inertia I_x.
  - Concentrate the area into a thin strip which is parallel to the x-axis and located a distance k_x from the x-axis.
  - The moment of inertia of the strip about the x-axis is…

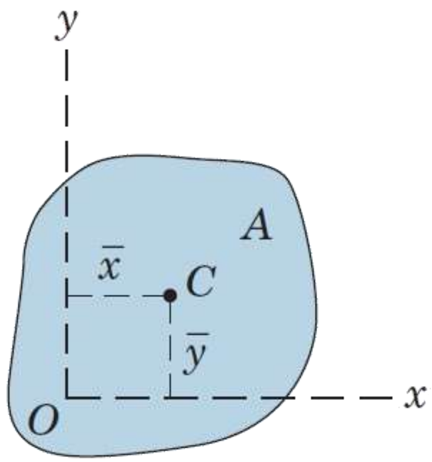
$$I_x = \int y^2\, dA = \int k_x^2\, dA = k_x^2 \int dA = k_x^2 A$$

  - The radius of gyration $k_x = \sqrt{I_x/A}$

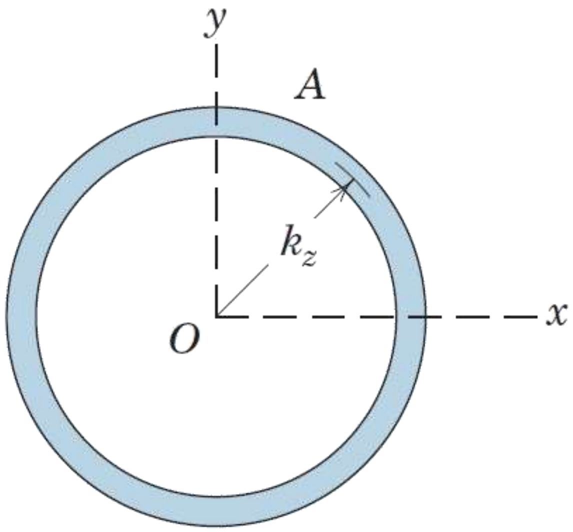


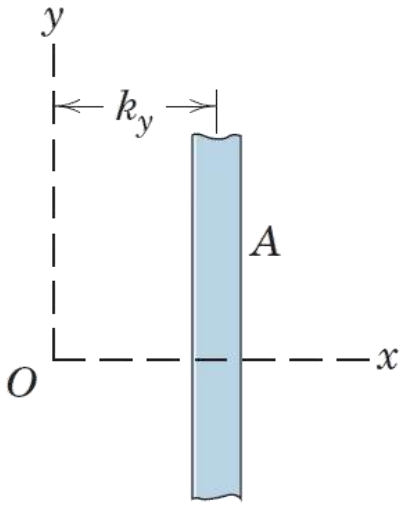
Article A/2 – Radius of Gyration (2 of 3)

- Illustration with *y*-Axis



- Illustration with the *z*-Axis

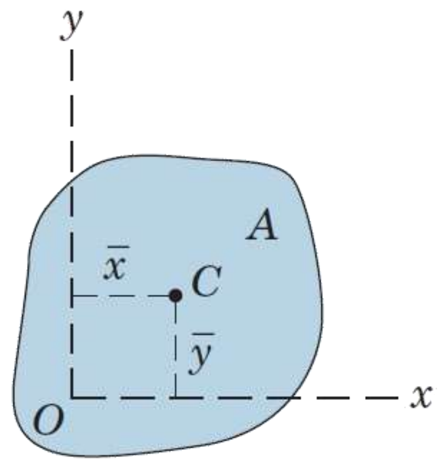




Article A/2 – Radius of Gyration (3 of 3)

- Summary of Equations

$$\boxed{\begin{aligned} I_x &= k_x^2 A \\ I_y &= k_y^2 A \\ I_z &= k_z^2 A \end{aligned}} \quad \text{or} \quad \boxed{\begin{aligned} k_x &= \sqrt{I_x/A} \\ k_y &= \sqrt{I_y/A} \\ k_z &= \sqrt{I_z/A} \end{aligned}}$$



- Alternative Equation for k_z

$$\boxed{k_z^2 = k_x^2 + k_y^2}$$

- Final Comment
  - Do not confuse the centroid coordinate C with the radii of gyration for the shape. They are not the same. For example, the moment of inertia about the x-axis is not equal to $A\bar{y}^2$.

Article A/2 – Transfer of Axes (1 of 2)

- Overview

- Illustration with the *x*-Axis

$$dI_x = (y_0 + d_x)^2\, dA$$

Expanding and integrating give us

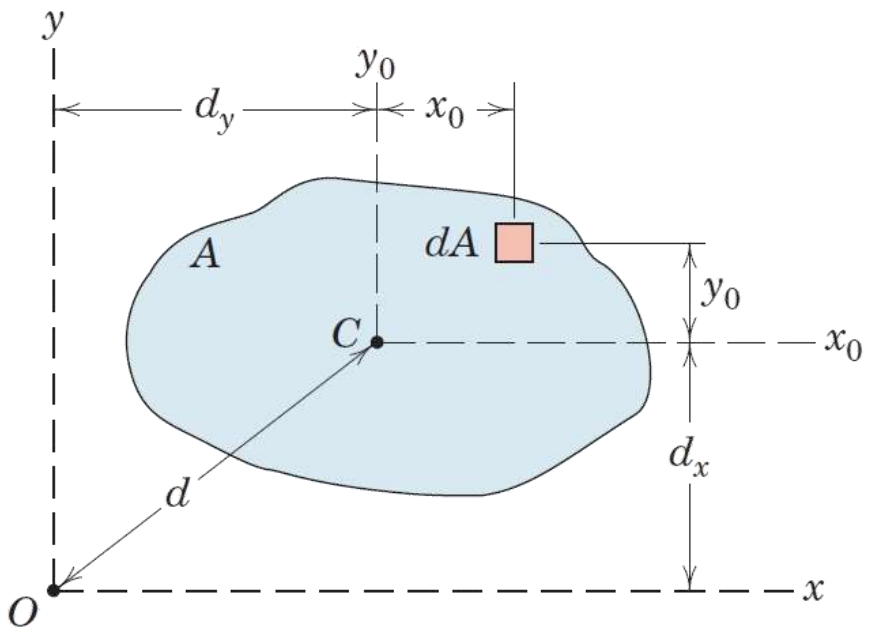
$$I_x = \int y_0^{\,2}\, dA + 2d_x \int y_0\, dA + d_x^{\,2} \int dA$$

- Result

$$\boxed{\begin{aligned} I_x &= \bar{I}_x + Ad_x^{\,2} \\ I_y &= \bar{I}_y + Ad_y^{\,2} \end{aligned}}$$

and

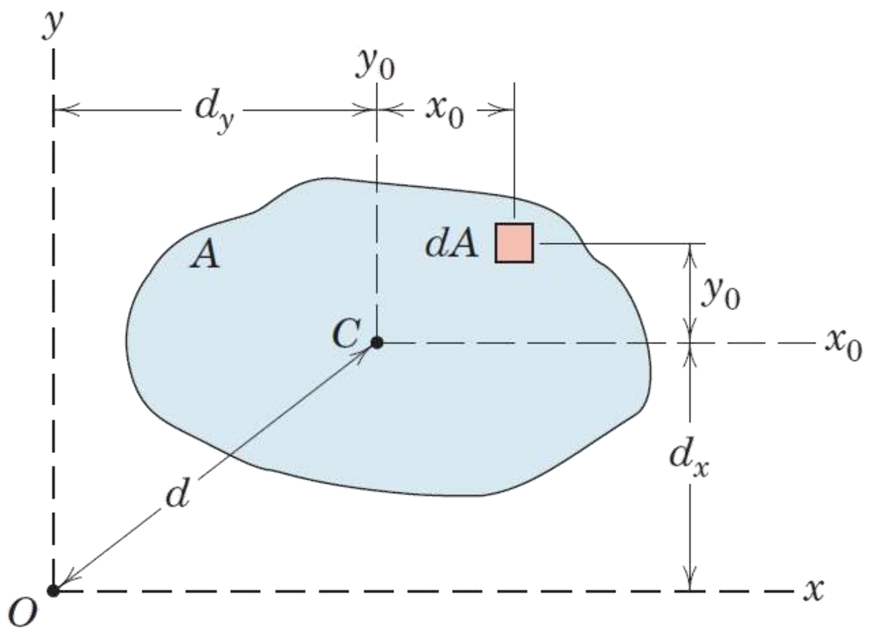
$$I_z = \bar{I}_z + Ad^2$$



Article A/2 – Transfer of Axes (2 of 2)

- Important Comments
  - The axes between which the transfer is made *must be parallel*.
  - One of the axes *must pass through the centroid of the area*.
- Effect on the Radius of Gyration

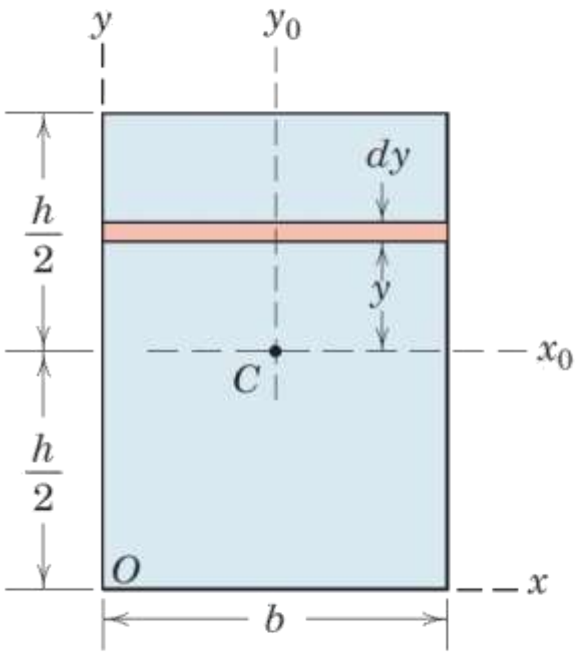
$$k^2 = \bar{k}^2 + d^2$$



Article A/2 – Sample Problem A/1 (1 of 2)

- Problem Statement

  Determine the moments of inertia of the rectangular area about the centroidal x_0- and y_0-axes, the centroidal polar axis z_0 through C, the x-axis, and the polar axis z through O.



Article A/2 – Sample Problem A/1 (2 of 2)

- Horizontal Strip of Area $dA = b\ dy$

$$[I_x = \int y^2\, dA] \qquad \bar{I}_x = \int_{-h/2}^{h/2} y^2 b\, dy = \tfrac{1}{12} bh^3 \qquad \textit{Ans.}$$

By interchange of symbols, the moment of inertia about the centroidal y_0-axis is

$$\bar{I}_y = \tfrac{1}{12} hb^3 \qquad \textit{Ans.}$$

The centroidal polar moment of inertia is

$$[\bar{I}_z = \bar{I}_x + \bar{I}_y] \qquad \bar{I}_z = \tfrac{1}{12}(bh^3 + hb^3) = \tfrac{1}{12}A(b^2 + h^2) \qquad \textit{Ans.}$$

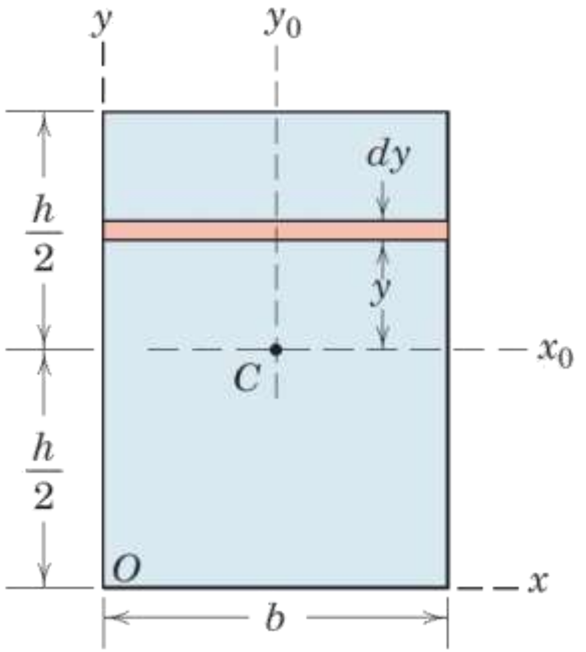
By the parallel-axis theorem, the moment of inertia about the x-axis is

$$[I_x = \bar{I}_x + Ad_x^{\,2}] \qquad I_x = \tfrac{1}{12} bh^3 + bh\left(\frac{h}{2}\right)^2 = \tfrac{1}{3} bh^3 = \tfrac{1}{3} Ah^2 \qquad \textit{Ans.}$$

We also obtain the polar moment of inertia about O by the parallel-axis theorem, which gives us

$$[I_z = \bar{I}_z + Ad^2] \qquad I_z = \tfrac{1}{12}A(b^2 + h^2) + A\left[\left(\frac{b}{2}\right)^2 + \left(\frac{h}{2}\right)^2\right]$$

$$I_z = \tfrac{1}{3}A(b^2 + h^2) \qquad \textit{Ans.}$$

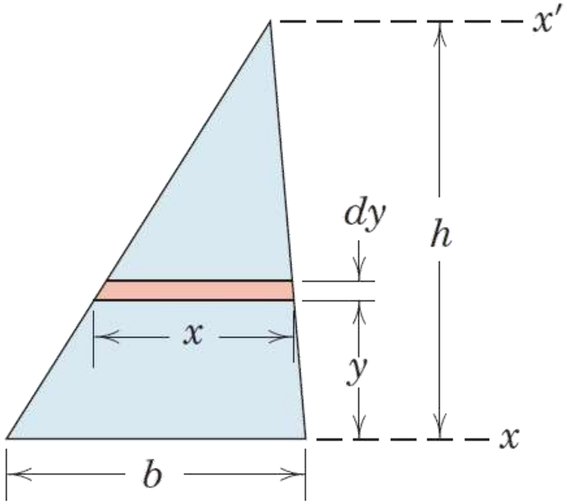


① If we had started with the second-order element $dA = dx\ dy$, integration with respect to x holding y constant amounts simply to multiplication by b and gives us the expression $y^2 b\ dy$, which we chose at the outset.

Article A/2 – Sample Problem A/2 (1 of 2)

- Problem Statement

  Determine the moments of inertia of the triangular area about its base and about parallel axes through its centroid and vertex.



Article A/2 – Sample Problem A/2 (2 of 2)

- Solution

A strip of area parallel to the base is selected as shown in the figure, and it has the area $dA = x\,dy = [(h - y)b/h]\,dy$. ① ② By definition

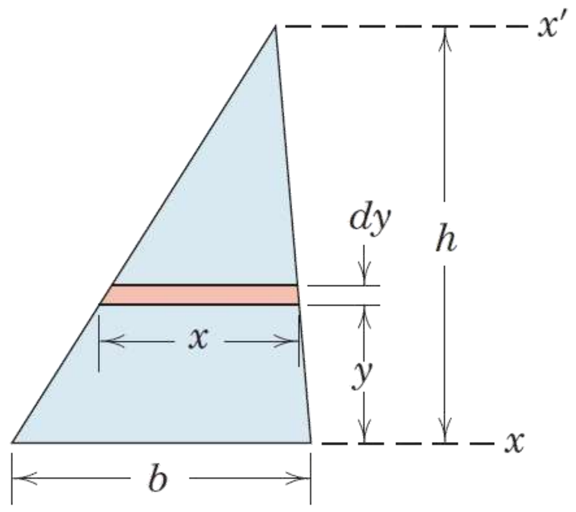
$$[I_x = \int y^2\,dA] \qquad I_x = \int_0^h y^2 \frac{h-y}{h} b\,dy = b\left[\frac{y^3}{3} - \frac{y^4}{4h}\right]_0^h = \frac{bh^3}{12} \qquad \textit{Ans.}$$

By the parallel-axis theorem, the moment of inertia $\bar{I}$ about an axis through the centroid, a distance $h/3$ above the x-axis, is

$$[\bar{I} = I - Ad^2] \qquad \bar{I} = \frac{bh^3}{12} - \left(\frac{bh}{2}\right)\left(\frac{h}{3}\right)^2 = \frac{bh^3}{36} \qquad \textit{Ans.}$$

A transfer from the centroidal axis to the x'-axis through the vertex gives

$$[I = \bar{I} + Ad^2] \qquad I_{x'} = \frac{bh^3}{36} + \left(\frac{bh}{2}\right)\left(\frac{2h}{3}\right)^2 = \frac{bh^3}{4} \qquad \textit{Ans.}$$



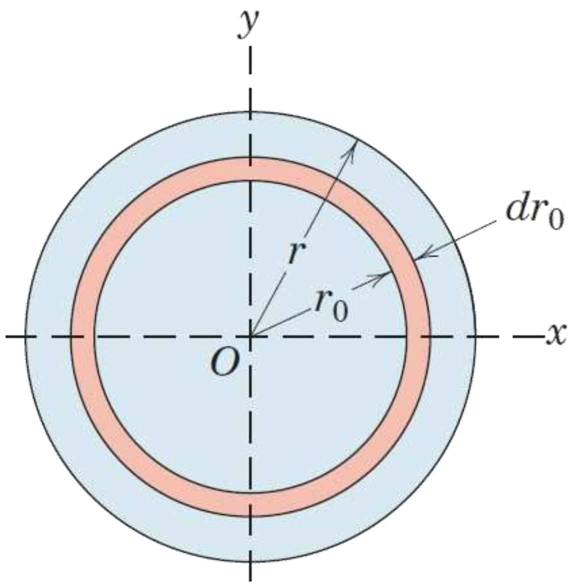
① Here again we choose the simplest possible element. If we had chosen $dA = dx\,dy$, we would have to integrate $y^2\,dx\,dy$ with respect to x first. This gives us $y^2x\,dy$, which is the expression we chose at the outset.

② Expressing x in terms of y should cause no difficulty if we observe the proportional relationship between the similar triangles.

Article A/2 – Sample Problem A/3 (1 of 3)

- Problem Statement

  Calculate the moments of inertia of the area of a circle about a diametral axis and about the polar axis through the center. Specify the radii of gyration.



Article A/2 – Sample Problem A/3 (2 of 3)

- Circular Ring of Area $dA = 2\pi r_0 dr_0$

$$[I_z = \int r^2\, dA] \qquad I_z = \int_0^r r_0^{\,2}(2\pi r_0\, dr_0) = \frac{\pi r^4}{2} = \frac{1}{2}Ar^2 \qquad \textit{Ans.}$$

The polar radius of gyration is

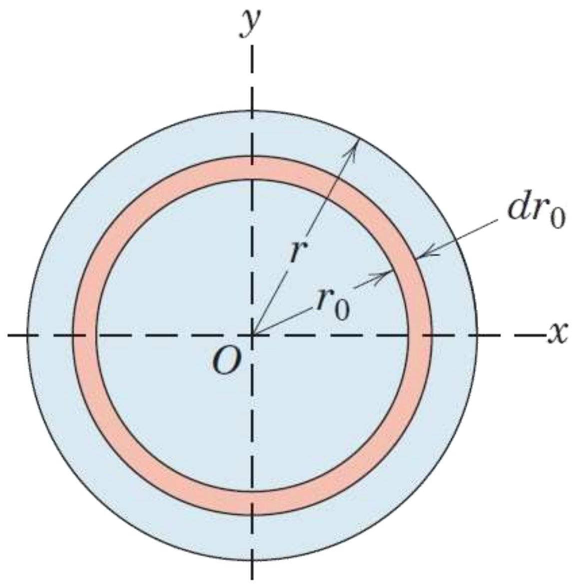
$$\left[k = \sqrt{\frac{I}{A}}\right] \qquad k_z = \frac{r}{\sqrt{2}} \qquad \textit{Ans.}$$

By symmetry $I_x = I_y$, so that from Eq. A/3

$$[I_z = I_x + I_y] \qquad I_x = \frac{1}{2}I_z = \frac{\pi r^4}{4} = \frac{1}{4}Ar^2 \qquad \textit{Ans.}$$

The radius of gyration about the diametral axis is

$$\left[k = \sqrt{\frac{I}{A}}\right] \qquad k_x = \frac{r}{2} \qquad \textit{Ans.}$$

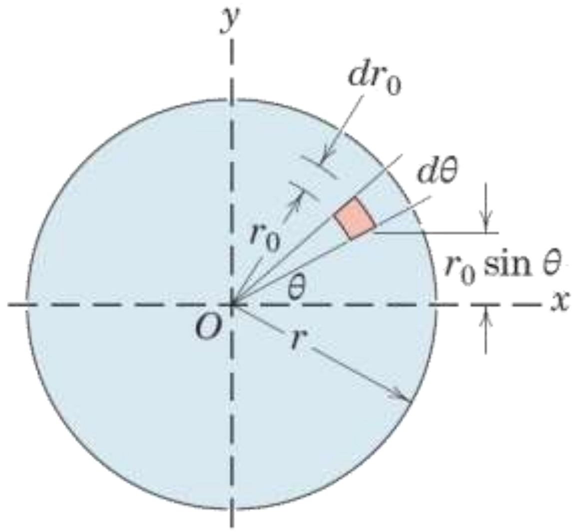


① Polar coordinates are certainly indicated here. Also, as before, we choose the simplest and lowest-order element possible, which is the differential ring. It should be evident immediately from the definition that the polar moment of inertia of the ring is its area $2\pi r_0\, dr_0$ times $r_0^{\,2}$.

Article A/2 – Sample Problem A/3 (3 of 3)

- Alternative Solution for Inertia about x-Axis

  Element of Area $dA = \pi r_0 dr_0 d\theta$



$$[I_x = \int y^2\, dA] \qquad I_x = \int_0^{2\pi} \int_0^{r} (r_0 \sin\theta)^2 r_0\, dr_0\, d\theta$$

$$= \int_0^{2\pi} \frac{r^4 \sin^2\theta}{4}\, d\theta$$

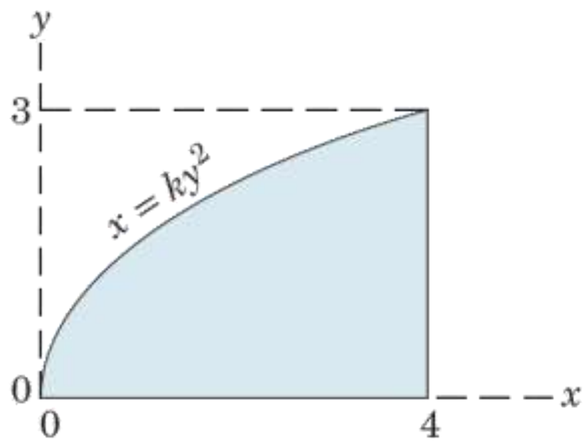
$$= \frac{r^4}{4}\frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} = \frac{\pi r^4}{4} \quad ② \qquad Ans.$$

② This integration is straightforward, but the use of Eq. A/3 along with the result for I_z is certainly simpler.

Article A/2 – Sample Problem A/4 (1 of 3)

- Problem Statement

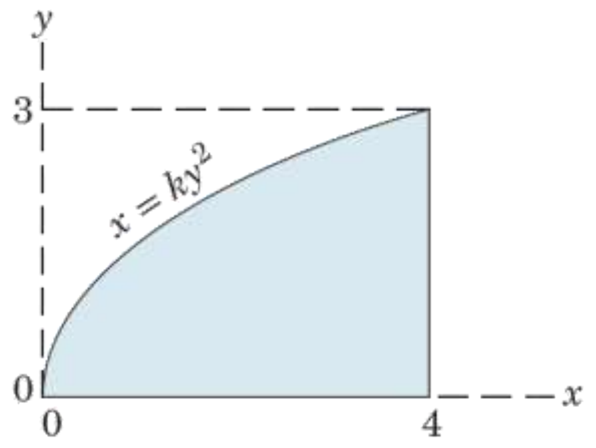
  Determine the moment of inertia of the area under the parabola about the *x*-axis. Solve by using (*a*) a horizontal strip of area and (*b*) a vertical strip of area.



Article A/2 – Sample Problem A/4 (2 of 3)

- Equation of the Parabola

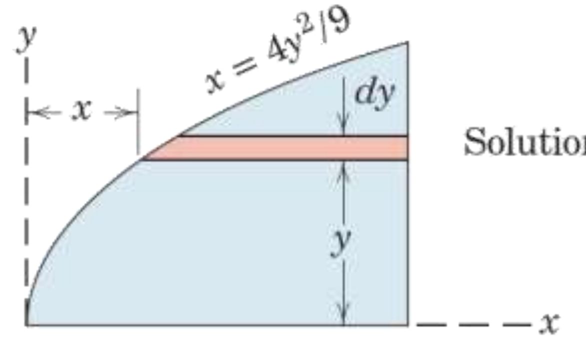
  The constant $k = 4/9$ is obtained first by substituting $x = 4$ and $y = 3$ into the equation for the parabola.



- (a) Solution using a Horizontal Strip

Since all parts of the horizontal strip are the same distance from the x-axis, the moment of inertia of the strip about the x-axis is $y^2\,dA$ where $dA = (4 - x)\,dy = 4(1 - y^2/9)\,dy$. Integrating with respect to y gives us

$$\left[I_x = \int y^2\,dA\right] \qquad I_x = \int_0^3 4y^2\left(1 - \frac{y^2}{9}\right)dy = \frac{72}{5} = 14.4\ (\text{units})^4 \qquad \textit{Ans.}$$



Solution (a)

Article A/2 – Sample Problem A/4 (3 of 3)

- (*b*) Solution using a Vertical Strip

Here all parts of the element are at different distances from the x-axis, so we must use the correct expressions for the moment of inertia of the elemental rectangle about its base, which, from Sample Problem A/1, is $bh^3/3$. For the width dx and the height y the expression becomes

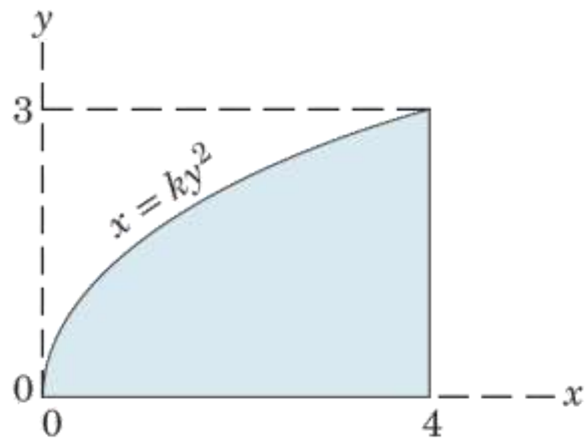
$$dI_x = \frac{1}{3}(dx)y^3$$

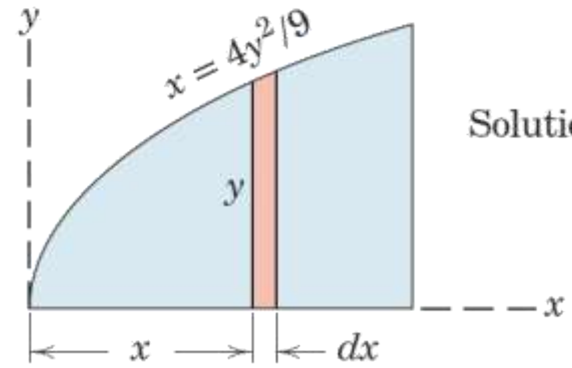
To integrate with respect to x, we must express y in terms of x, which gives $y = 3\sqrt{x}/2$, and the integral becomes

$$I_x = \frac{1}{3}\int_0^4 \left(\frac{3\sqrt{x}}{2}\right)^3 dx = \frac{72}{5} = 14.4\ (\text{units})^4 \quad ①$$

Ans.

① There is little preference between Solutions (*a*) and (*b*). Solution (*b*) requires knowing the moment of inertia for a rectangular area about its base.



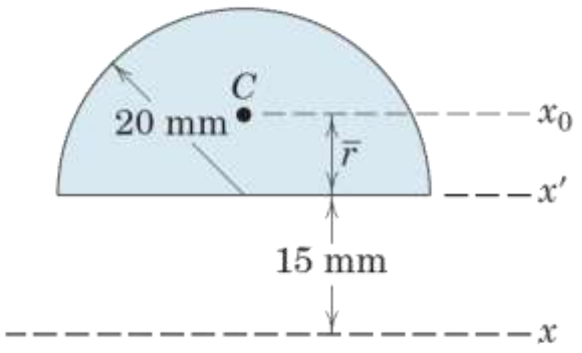


Solution (*b*)

Article A/2 – Sample Problem A/5 (1 of 2)

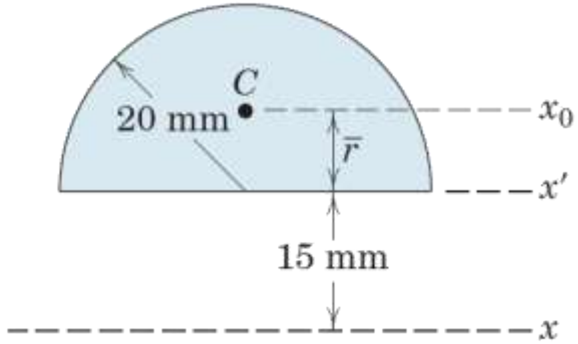
- Problem Statement

  Find the moment of inertia about the *x*-axis of the semicircular area.



Article A/2 – Sample Problem A/5 (2 of 2)

- Solution



The moment of inertia of the semicircular area about the x'-axis is one-half of that for a complete circle about the same axis. Thus, from the results of Sample Problem A/3,

$$I_{x'} = \frac{1}{2}\frac{\pi r^4}{4} = \frac{20^4\pi}{8} = 2\pi(10^4)\ \text{mm}^4$$

We obtain the moment of inertia $\bar{I}$ about the parallel centroidal axis x_0 next. Transfer is made through the distance $\bar{r} = 4r/(3\pi) = (4)(20)/(3\pi) = 80/(3\pi)$ mm by the parallel-axis theorem. Hence,

$$[\bar{I} = I - Ad^2] \qquad \bar{I} = 2(10^4)\pi - \left(\frac{20^2\pi}{2}\right)\left(\frac{80}{3\pi}\right)^2 = 1.755(10^4)\ \text{mm}^4$$

Finally, we transfer from the centroidal x_0-axis to the x-axis. ① Thus,

$$[I = \bar{I} + Ad^2] \qquad I_x = 1.755(10^4) + \left(\frac{20^2\pi}{2}\right)\left(15 + \frac{80}{3\pi}\right)^2$$

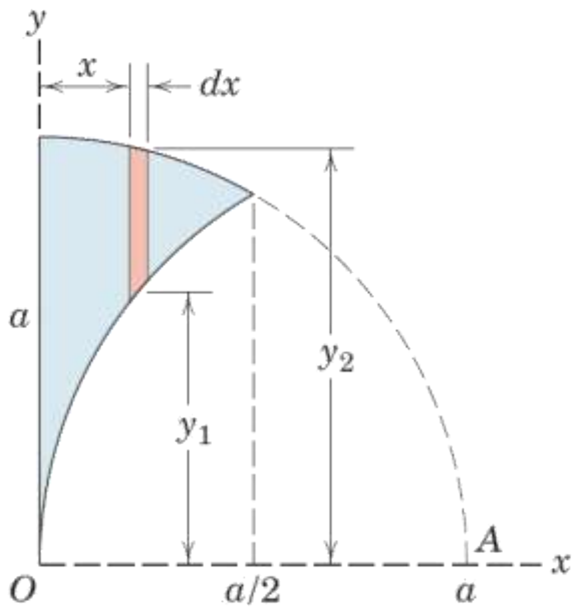
$$= 1.755(10^4) + 34.7(10^4) = 36.4(10^4)\ \text{mm}^4 \qquad \textit{Ans.}$$

① This problem illustrates the caution we should observe in using a double transfer of axes since neither the x'- nor the x-axis passes through the centroid C of the area. If the circle were complete with the centroid on the x' axis, only one transfer would be needed.

Article A/2 – Sample Problem A/6 (1 of 4)

- Problem Statement

  Calculate the moment of inertia about the *x*-axis of the area enclosed between the *y*-axis and the circular arcs of radius *a* whose centers are at *O* and *A*.



Article A/2 – Sample Problem A/6 (2 of 4)

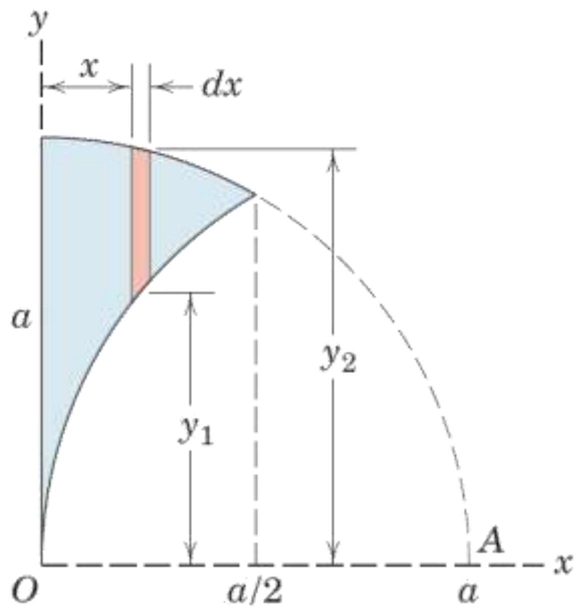
- Solution

The choice of a vertical differential strip of area permits one integration to cover the entire area. A horizontal strip would require two integrations with respect to y by virtue of the discontinuity. The moment of inertia of the strip about the x-axis is that of a strip of height y_2 minus that of a strip of height y_1. Thus, from the results of Sample Problem A/1 we write

$$dI_x = \frac{1}{3}(y_2\,dx)y_2^2 - \frac{1}{3}(y_1\,dx)y_1^2 = \frac{1}{3}(y_2^3 - y_1^3)\,dx$$

The values of y_2 and y_1 are obtained from the equations of the two curves, which are $x^2 + y_2^2 = a^2$ and $(x-a)^2 + y_1^2 = a^2$, and which give $y_2 = \sqrt{a^2 - x^2}$ and $y_1 = \sqrt{a^2 - (x-a)^2}$. ① Thus,

$$I_x = \frac{1}{3}\int_0^{a/2}\left\{(a^2 - x^2)\sqrt{a^2 - x^2} - [a^2 - (x-a)^2]\sqrt{a^2 - (x-a)^2}\right\}dx$$



① We choose the positive signs for the radicals here since both y_1 and y_2 lie above the x-axis.

Article A/2 – Sample Problem A/6 (3 of 4)

- Solution (cont.)

Simultaneous solution of the two equations which define the two circles gives the x-coordinate of the intersection of the two curves, which, by inspection, is $a/2$. Evaluation of the integrals gives

$$\int_0^{a/2} a^2\sqrt{a^2 - x^2}\, dx = \frac{a^4}{4}\left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right)$$

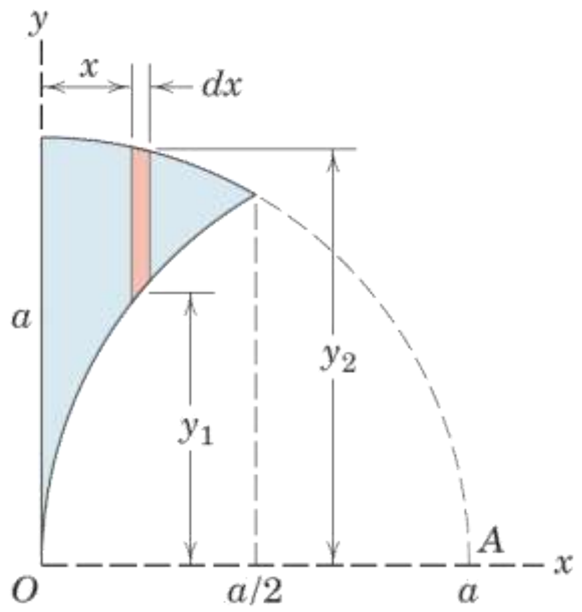
$$-\int_0^{a/2} x^2\sqrt{a^2 - x^2}\, dx = \frac{a^4}{16}\left(\frac{\sqrt{3}}{4} + \frac{\pi}{3}\right)$$

$$-\int_0^{a/2} a^2\sqrt{a^2 - (x-a)^2}\, dx = \frac{a^4}{4}\left(\frac{\sqrt{3}}{2} + \frac{2\pi}{3}\right)$$

$$\int_0^{a/2} (x-a)^2\sqrt{a^2 - (x-a)^2}\, dx = \frac{a^4}{8}\left(\frac{\sqrt{3}}{8} + \frac{\pi}{3}\right)$$

Collection of the integrals with the factor of $\frac{1}{3}$ gives

$$I_x = \frac{a^4}{96}(9\sqrt{3} - 2\pi) = 0.0969a^4 \qquad \textit{Ans.}$$



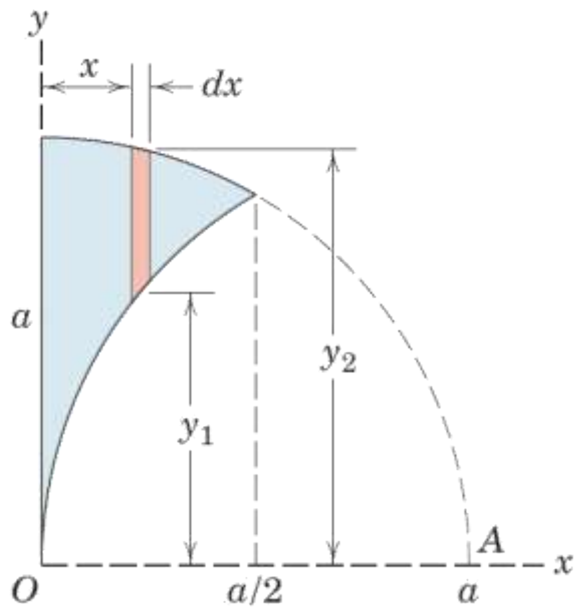
Article A/2 – Sample Problem A/6 (4 of 4)

- Alternative Element Form

If we had started from a second-order element $dA = dx\,dy$, we would write $y^2\,dx\,dy$ for the moment of inertia of the element about the x-axis. Integrating from y_1 to y_2 holding x constant produces for the vertical strip

$$dI_x = \left[\int_{y_1}^{y_2} y^2\,dy\right] dx = \frac{1}{3}({y_2}^3 - {y_1}^3)\,dx$$

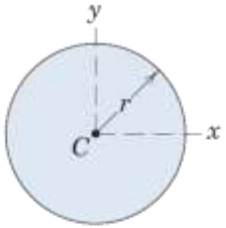
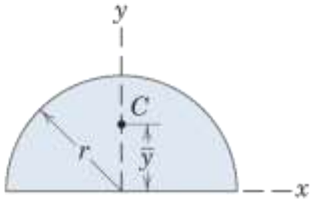
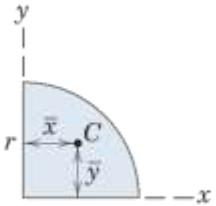
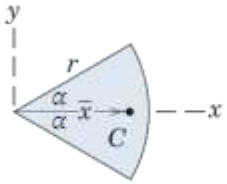
which is the expression we started with by having the moment-of-inertia result for a rectangle in mind.



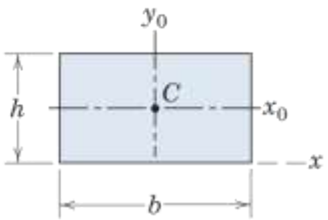
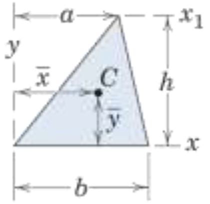
Article A/3 Composite Areas

- Introduction
  - It is frequently necessary to calculate the moment of inertia of an area composed of a number of distinct parts of simple and calculable geometric shape.
  - The moment of inertia of a composite area about a particular axis is simply the sum of the moments of inertia of its component parts about the same axis.
  - It is convenient to regard a composite area as being composed of positive and negative parts. We may then treat the moment of inertia of a negative area (hole or cutout) as a negative quantity.
  - Tabulated inertias and a systematic approach prove quite useful for these calculations.

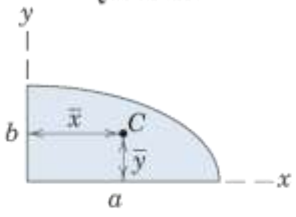
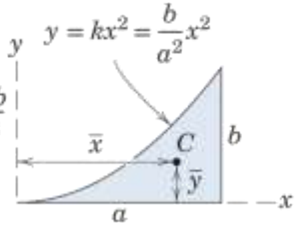
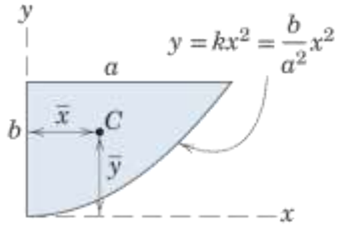
Article A/3 – Inertias of Common Shapes (1 of 3)

Figure	Centroid	Area Moments of Inertia
Circular Area 	—	$I_x = I_y = \frac{\pi r^4}{4}$ $I_z = \frac{\pi r^4}{2}$
Semicircular Area 	$\bar{y} = \frac{4r}{3\pi}$	$I_x = I_y = \frac{\pi r^4}{8}$ $\bar{I}_x = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right) r^4$ $I_z = \frac{\pi r^4}{4}$
Quarter-Circular Area 	$\bar{x} = \bar{y} = \frac{4r}{3\pi}$	$I_x = I_y = \frac{\pi r^4}{16}$ $\bar{I}_x = \bar{I}_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) r^4$ $I_z = \frac{\pi r^4}{8}$
Area of Circular Sector 	$\bar{x} = \frac{2}{3} \frac{r \sin \alpha}{\alpha}$	$I_x = \frac{r^4}{4}\left(\alpha - \frac{1}{2} \sin 2\alpha\right)$ $I_y = \frac{r^4}{4}\left(\alpha + \frac{1}{2} \sin 2\alpha\right)$ $I_z = \frac{1}{2} r^4 \alpha$

Article A/3 – Inertias of Common Shapes (2 of 3)

Figure	Centroid	Area Moments of Inertia
Rectangular Area 	—	$I_x = \frac{bh^3}{3}$ $\bar{I}_x = \frac{bh^3}{12}$ $\bar{I}_z = \frac{bh}{12}(b^2 + h^2)$
Triangular Area 	$\bar{x} = \frac{a + b}{3}$ $\bar{y} = \frac{h}{3}$	$I_x = \frac{bh^3}{12}$ $\bar{I}_x = \frac{bh^3}{36}$ $I_{x_1} = \frac{bh^3}{4}$

Article A/3 – Inertias of Common Shapes (3 of 3)

<table>
<tr><th>Figure</th><th>Centroid</th><th>Area Moments of Inertia</th></tr>
<tr><td>Area of Elliptical Quadrant
</td><td>$\bar{x} = \frac{4a}{3\pi}$
$\bar{y} = \frac{4b}{3\pi}$</td><td>$I_x = \frac{\pi a b^3}{16}, \bar{I}_x = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) ab^3$
$I_y = \frac{\pi a^3 b}{16}, \bar{I}_y = \left(\frac{\pi}{16} - \frac{4}{9\pi}\right) a^3 b$
$I_z = \frac{\pi ab}{16}(a^2 + b^2)$</td></tr>
<tr><td>Subparabolic Area
Area $A = \frac{ab}{3}$
</td><td>$\bar{x} = \frac{3a}{4}$
$\bar{y} = \frac{3b}{10}$</td><td>$I_x = \frac{ab^3}{21}$
$I_y = \frac{a^3 b}{5}$
$I_z = ab\left(\frac{a^2}{5} + \frac{b^2}{21}\right)$</td></tr>
<tr><td>Parabolic Area
Area $A = \frac{2ab}{3}$
</td><td>$\bar{x} = \frac{3a}{8}$
$\bar{y} = \frac{3b}{5}$</td><td>$I_x = \frac{2ab^3}{7}$
$I_y = \frac{2a^3 b}{15}$
$I_z = 2ab\left(\frac{a^2}{15} + \frac{b^2}{7}\right)$</td></tr>
</table>

Article A/3 – Typical Solution Process

- Tabulated Approach

Part	Area, A	d_x	d_y	Ad_x^2	Ad_y^2	$\bar{I}_x$	$\bar{I}_y$
Sums	ΣA			ΣAd_x^2	ΣAd_y^2	$\Sigma \bar{I}_x$	$\Sigma \bar{I}_y$

From the sums of the four columns, then, the moments of inertia for the composite area about the x- and y-axes become

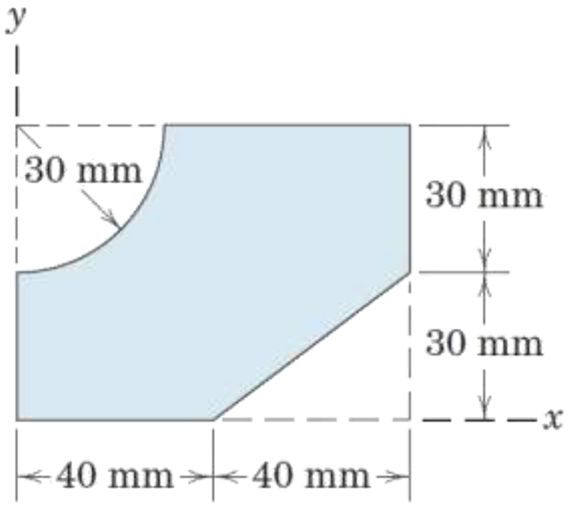
$$I_x = \Sigma \bar{I}_x + \Sigma Ad_x^2$$

$$I_y = \Sigma \bar{I}_y + \Sigma Ad_y^2$$

Article A/3 – Sample Problem A/7 (1 of 2)

- Problem Statement

  Determine the moments of inertia about the *x*- and *y*-axes for the shaded area. Make direct use of the expressions given in Table D/3 for the centroidal moments of inertia of the constituent parts.



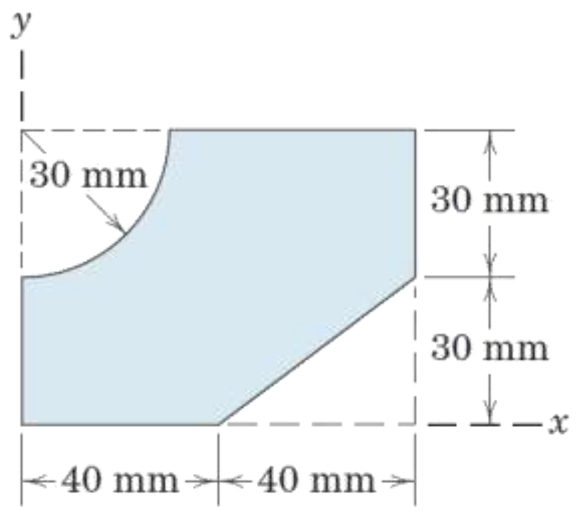
Article A/3 – Sample Problem A/7 (2 of 2)

- Composite Shapes

  Rectangular Area (1)

  Quarter-Circular Cutout (2)

  Triangular Cutout (3)

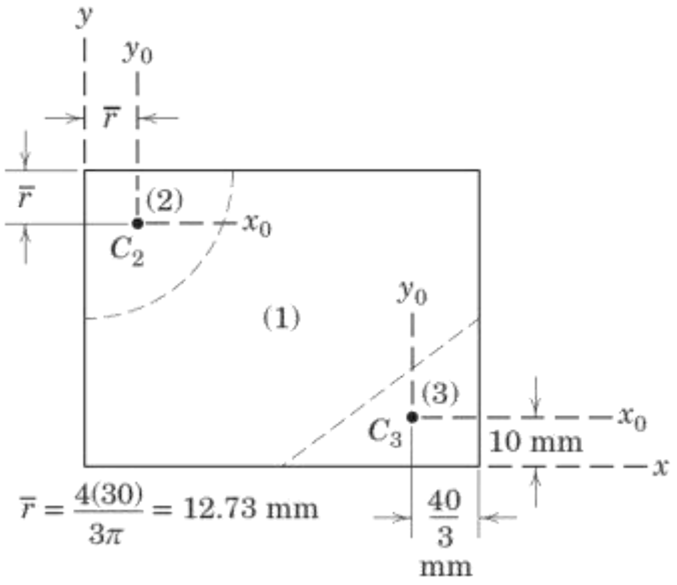


- Tabulated Values

PART	A mm^2	d_x mm	d_y mm	$Ad_x{}^2$ mm^4	$Ad_y{}^2$ mm^4	$\bar{I}_x$ mm^4	$\bar{I}_y$ mm^4
1	80(60)	30	40	$4.32(10^6)$	$7.68(10^6)$	$\frac{1}{12}(80)(60)^3$	$\frac{1}{12}(60)(80)^3$
2	$-\frac{1}{4}\pi(30)^2$	(60 − 12.73)	12.73	$-1.579(10^6)$	$-0.1146(10^6)$	$-\left(\frac{\pi}{16}-\frac{4}{9\pi}\right)30^4$	$-\left(\frac{\pi}{16}-\frac{4}{9\pi}\right)30^4$
3	$-\frac{1}{2}(40)(30)$	$\frac{30}{3}$	$\left(80-\frac{40}{3}\right)$	$-0.06(10^6)$	$-2.67(10^6)$	$-\frac{1}{36}40(30)^3$	$-\frac{1}{36}(30)(40)^3$
TOTALS	3490			$2.68(10^6)$	$4.90(10^6)$	$1.366(10^6)$	$2.46(10^6)$

$$[I_x = \Sigma \bar{I}_x + \Sigma A d_x{}^2] \quad I_x = 1.366(10^6) + 2.68(10^6) = 4.05(10^6) \text{ mm}^4 \; \textit{Ans.}$$

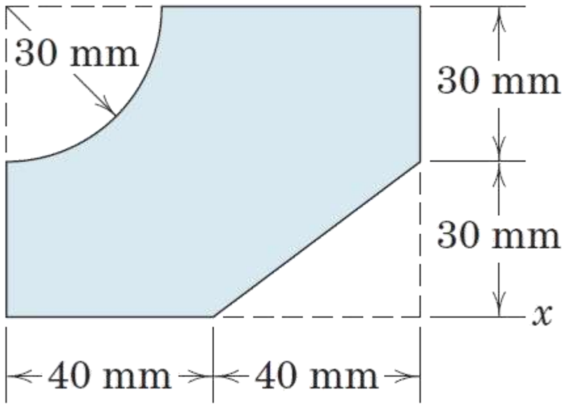
$$[I_y = \Sigma \bar{I}_y + \Sigma A d_y{}^2] \quad I_y = 2.46(10^6) + 4.90(10^6) = 7.36(10^6) \text{ mm}^4 \; \textit{Ans.}$$



Article A/3 – Sample Problem A/8 (1 of 4)

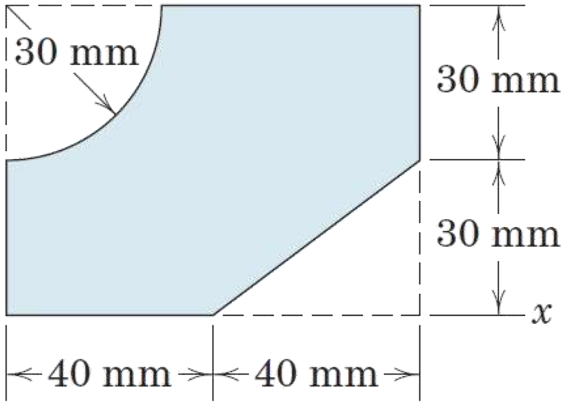
- Problem Statement

  Calculate the moment of inertia and radius of gyration about the *x*-axis for the shaded area shown. Wherever possible, make expedient use of tabulated moments of inertia.



Article A/3 – Sample Problem A/8 (2 of 4)

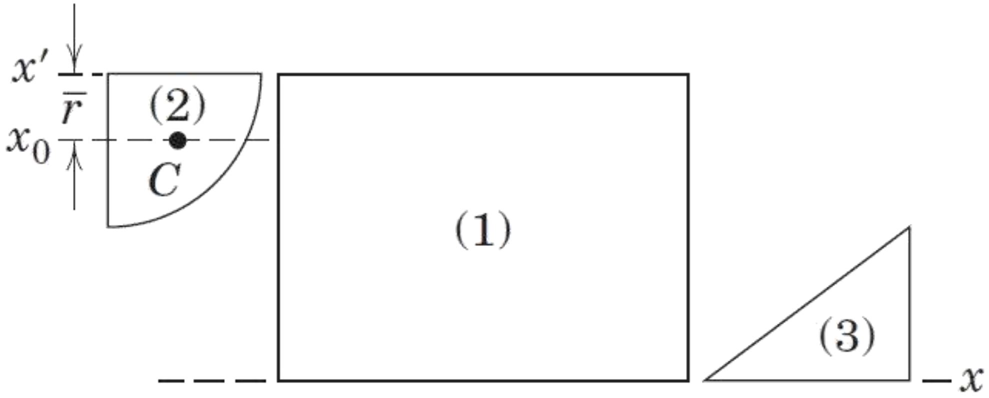
- Composite Shapes
  - Rectangular Area (1)
  - Quarter-Circular Cutout (2)
  - Triangular Cutout (3)



- Inertia for the Rectangular Area

The composite area is composed of the positive area of the rectangle (1) and the negative areas of the quarter circle (2) and triangle (3). For the rectangle the moment of inertia about the x-axis, from Sample Problem A/1 (or Table D/3), is

$$I_x = \frac{1}{3}Ah^2 = \frac{1}{3}(80)(60)(60)^2 = 5.76(10^6)\ \text{mm}^4$$



Article A/3 – Sample Problem A/8 (3 of 4)

- Inertia for the Quarter-Circular Cutout

From Sample Problem A/3 (or Table D/3), the moment of inertia of the negative quarter-circular area about its base axis x' is

$$I_{x'} = -\frac{1}{4}\left(\frac{\pi r^4}{4}\right) = -\frac{\pi}{16}(30)^4 = -0.1590(10^6)\ \text{mm}^4$$

We now transfer this result through the distance $\bar{r} = 4r/(3\pi) = 4(30)/(3\pi) = 12.73$ mm by the transfer-of-axis theorem to get the centroidal moment of inertia of part (2) (or use Table D/3 directly).

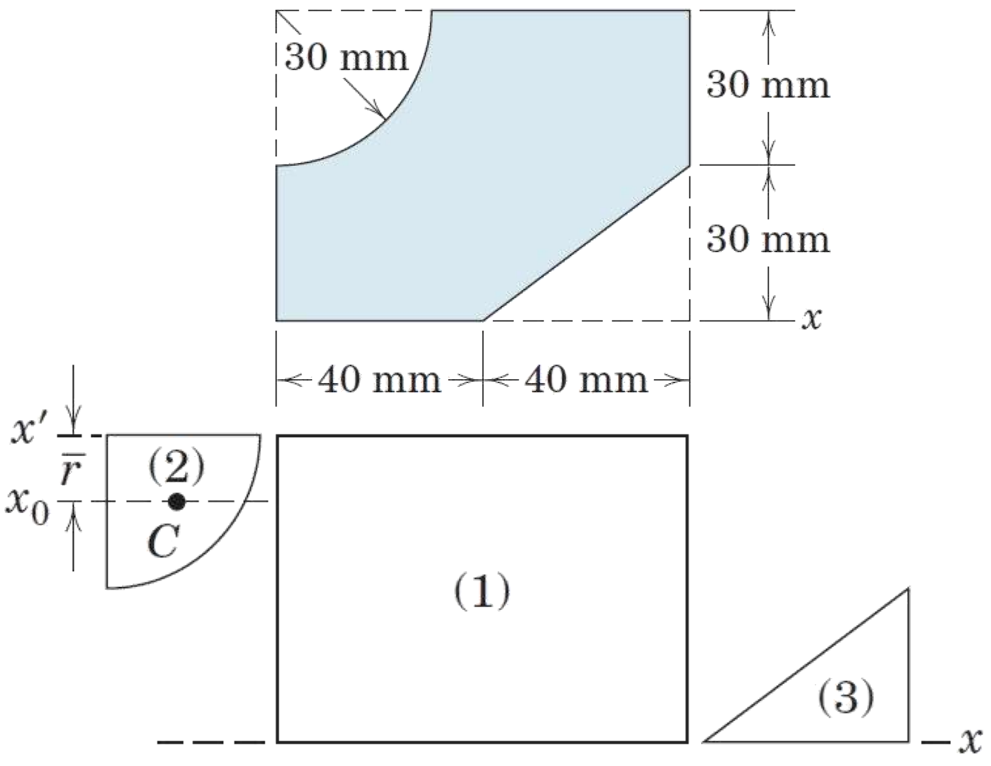
$$[\bar{I} = I - Ad^2] \qquad \bar{I}_x = -0.1590(10^6) - \left[-\frac{\pi(30)^2}{4}(12.73)^2\right] \quad ①$$

$$= -0.0445(10^6)\ \text{mm}^4$$

The moment of inertia of the quarter-circular part about the x-axis is now

$$[I = \bar{I} + Ad^2] \quad I_x = -0.0445(10^6) + \left[-\frac{\pi(30)^2}{4}\right](60 - 12.73)^2 \quad ②$$

$$= -1.624(10^6)\ \text{mm}^4$$



① Note that we must transfer the moment of inertia for the quarter-circular area to its centroidal axis x_0 before we can transfer it to the x-axis, as was done in Sample Problem A/5.

② We watch our signs carefully here. Since the area is negative, both $\bar{I}$ and A carry negative signs.

Article A/3 – Sample Problem A/8 (4 of 4)

- Inertia for the Triangular Cutout

Finally, the moment of inertia of the negative triangular area (3) about its base, from Sample Problem A/2 (or Table D/3), is

$$I_x = -\frac{1}{12}bh^3 = -\frac{1}{12}(40)(30)^3 = -0.90(10^6)\ \text{mm}^4$$

- Total Inertia and Radius of Gyration

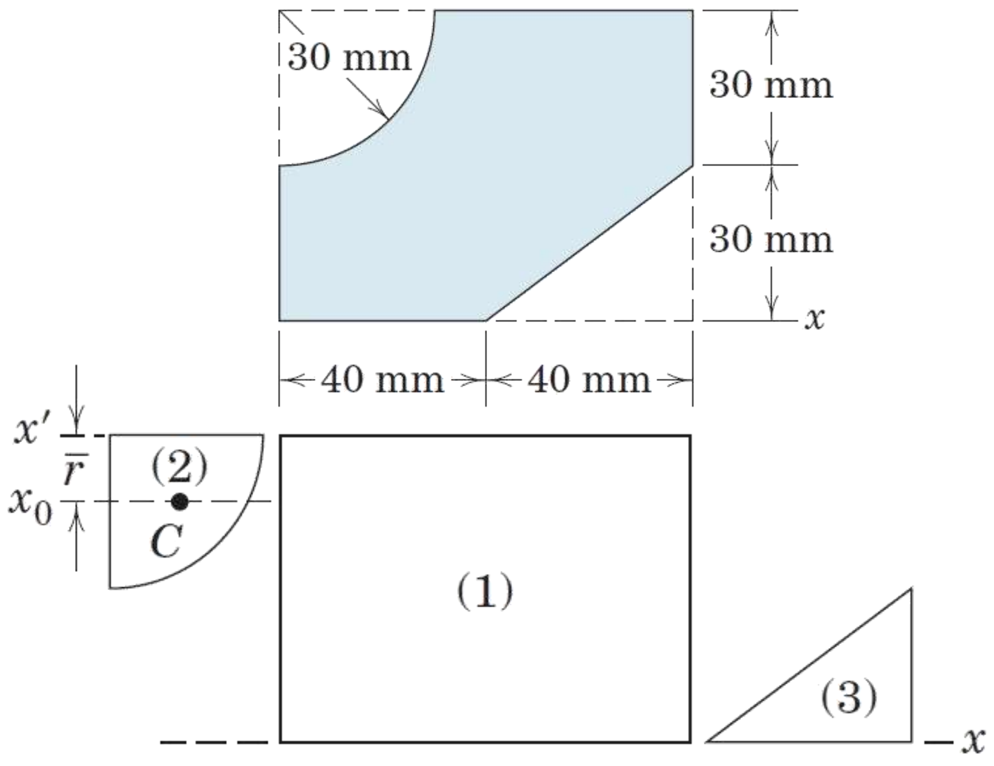
The total moment of inertia about the x-axis of the composite area is, consequently,

$$I_x = 5.76(10^6) - 1.624(10^6) - 0.09(10^6) = 4.05(10^6)\ \text{mm}^4 \quad ③ \quad \textit{Ans.}$$

This result agrees with that of Sample Problem A/7.

The net area of the figure is $A = 60(80) - \frac{1}{4}\pi(30)^2 - \frac{1}{2}(40)(30) = 3490\ \text{mm}^2$ so that the radius of gyration about the x-axis is

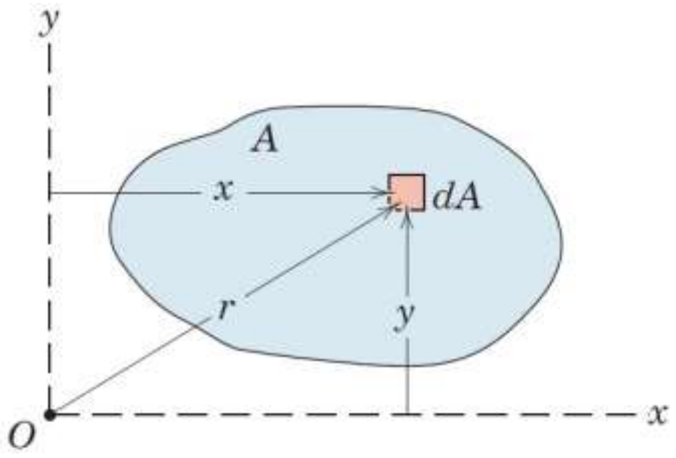
$$k_x = \sqrt{I_x/A} = \sqrt{4.05(10^6)/3490} = 34.0\ \text{mm} \qquad \textit{Ans.}$$



③ Always use common sense at key points such as this. The two minus signs are consistent with the fact that subareas (2) and (3) reduce the numerical value of the moment of inertia of the basic rectangular area.

Article A/4 Products of Inertia and Rotation of Axes

- Definition
  - $I_{xy} = \int xy \, dA$

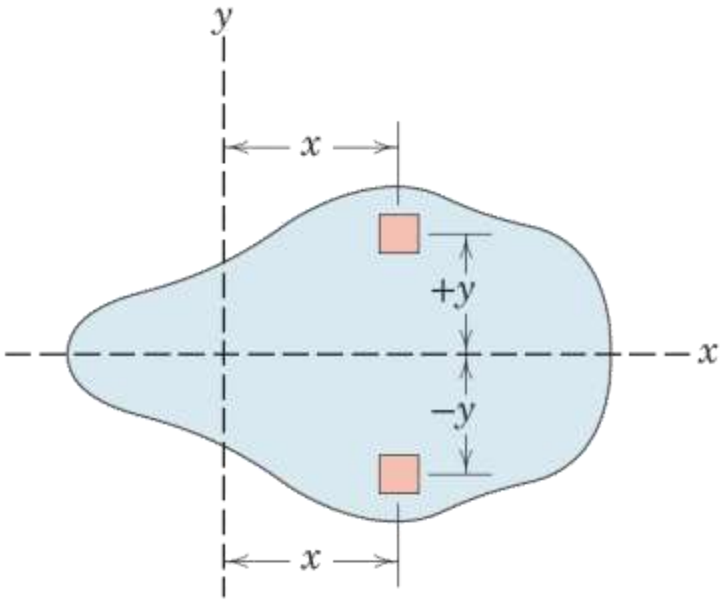


- Dimensions and Units
  - Length to the 4th Power, L^4
  - SI Units: mm^4 (most common)
  - U.S. Customary Units: $in.^4$ (most common)

Article A/4 – Comments About Products of Inertia

- A product of inertia can be positive, negative, or zero.

- The product of inertia is zero whenever either of the reference axes is an axis of symmetry. This is illustrated in the figure at right where the area is symmetric about the *x*-axis.



Article A/4 – Transfer of Axes

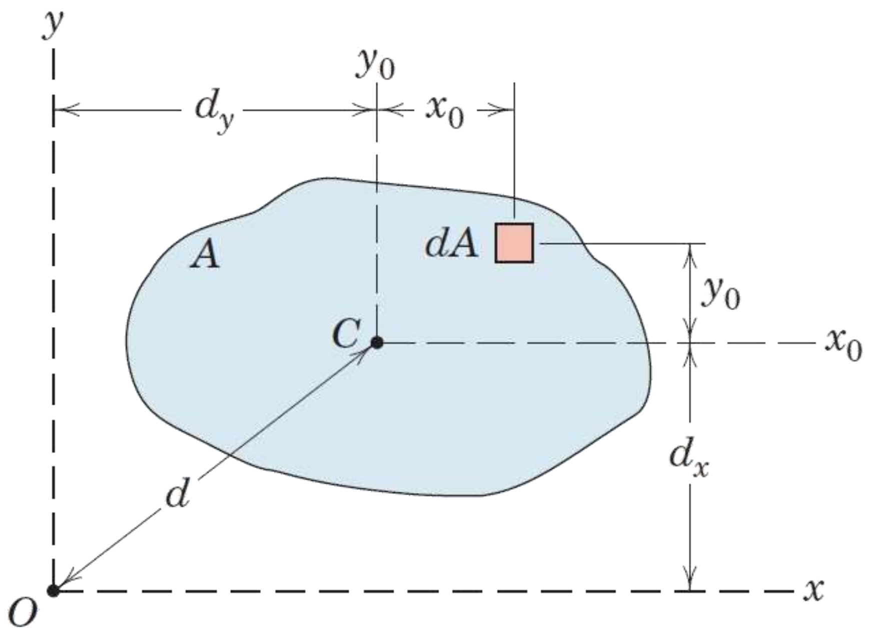
- Overview

- Mathematics

$$I_{xy} = \int (x_0 + d_y)(y_0 + d_x)\, dA$$
$$= \int x_0 y_0\, dA + d_x \int x_0\, dA + d_y \int y_0\, dA + d_x d_y \int dA$$

- Result

$$\boxed{I_{xy} = \bar{I}_{xy} + d_x d_y A}$$



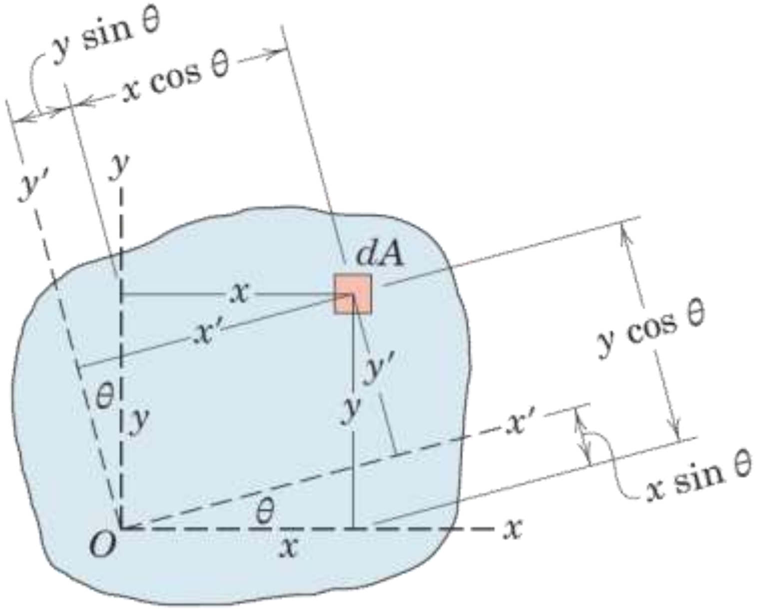
Article A/4 – Rotation of Axes (1 of 4)

- Introduction

- Moments of Inertia about Tilted Axes

$$I_{x'} = \int y'^2 \, dA = \int (y \cos \theta - x \sin \theta)^2 \, dA$$

$$I_{y'} = \int x'^2 \, dA = \int (y \sin \theta + x \cos \theta)^2 \, dA$$



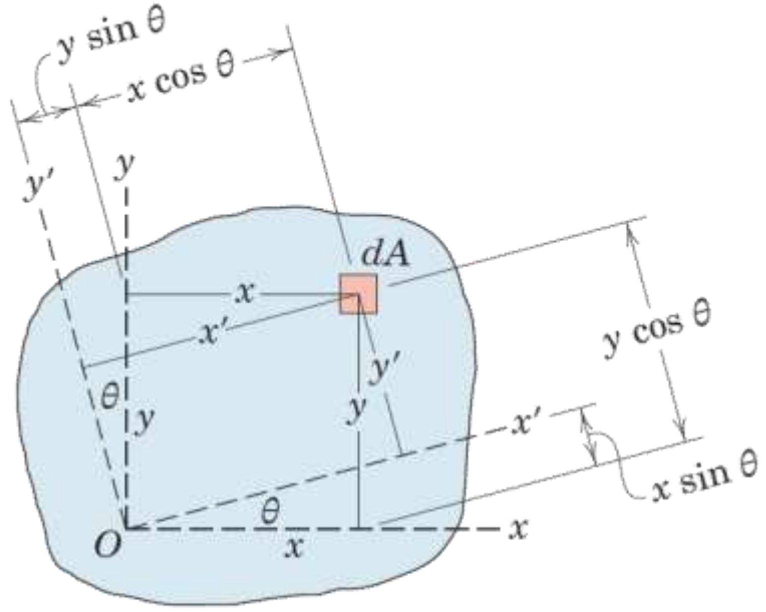
Article A/4 – Rotation of Axes (2 of 4)

- Trigonometric Identities of Importance

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

- Substitution and Simplification

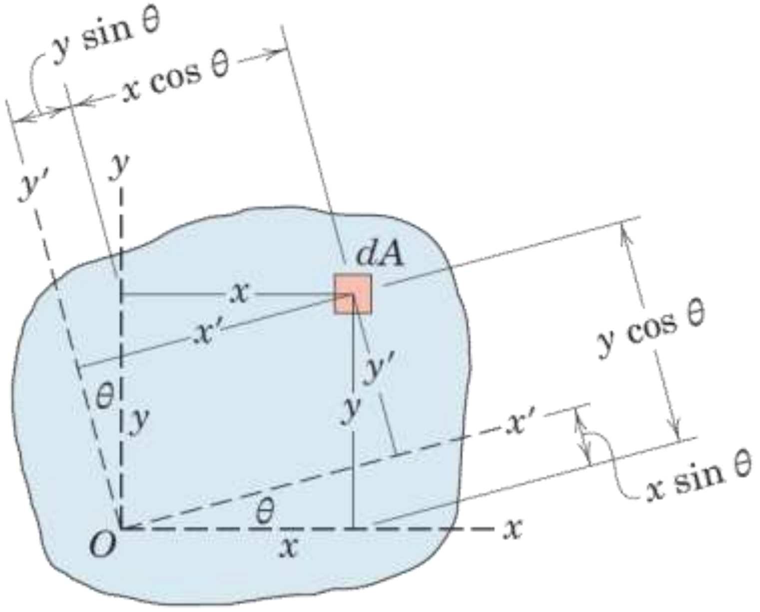
$$\boxed{\begin{aligned} I_{x'} &= \frac{I_x+I_y}{2} + \frac{I_x-I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta \\ I_{y'} &= \frac{I_x+I_y}{2} - \frac{I_x-I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta \end{aligned}}$$



Article A/4 – Rotation of Axes (3 of 4)

- Product of Inertia about Tilted Axes

$$I_{x'y'} = \int x'y' \, dA = \int (y \sin\theta + x \cos\theta)(y \cos\theta - x \sin\theta) \, dA$$



- Trigonometric Identities of Importance

$$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta \qquad \cos^2\theta - \sin^2\theta = \cos 2\theta$$

- Substitution and Simplification

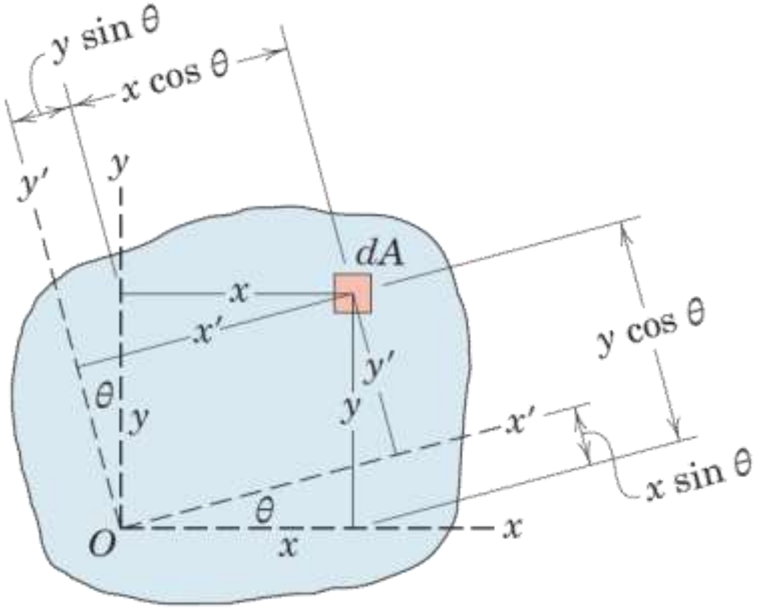
$$\boxed{I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta}$$

Article A/4 – Rotation of Axes (4 of 4)

- Angle for Principal Axes of Inertia

$$\frac{dI_{x'}}{d\theta} = (I_y - I_x)\sin 2\theta - 2I_{xy}\cos 2\theta = 0$$

$$\boxed{\tan 2\alpha = \frac{2I_{xy}}{I_y - I_x}}$$

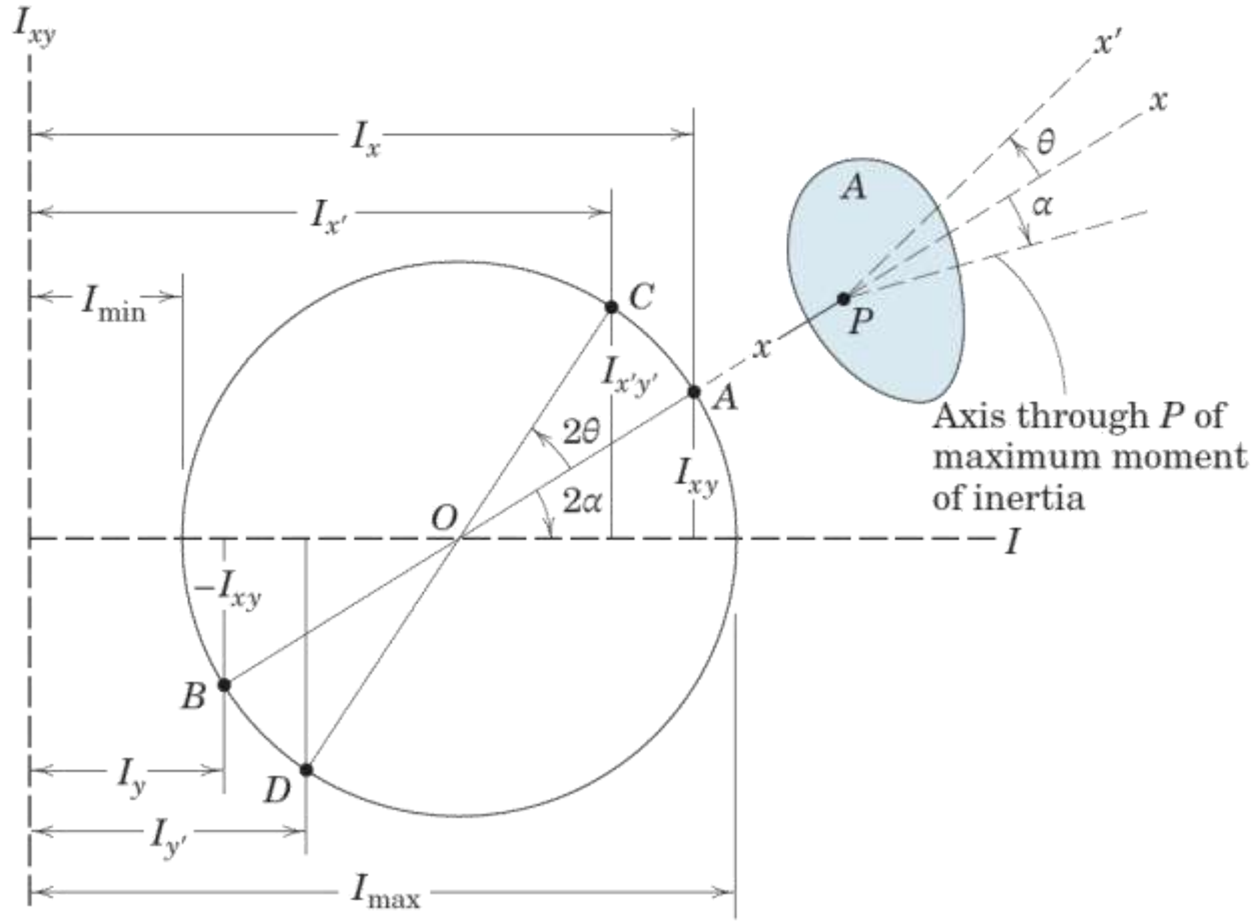


- Principal Moments of Inertia

$$\boxed{\begin{aligned} I_{\max} &= \frac{I_x + I_y}{2} + \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4I_{xy}{}^2} \\ I_{\min} &= \frac{I_x + I_y}{2} - \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4I_{xy}{}^2} \end{aligned}}$$

Article A/4 – Mohr's Circle of Inertia (1 of 2)

- Illustration



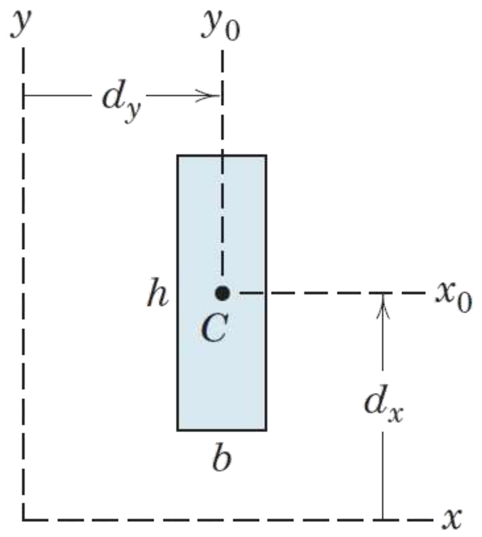
Article A/4 – Mohr’s Circle of Inertia (2 of 2)

- Procedure
  1. Sketch a horizontal axis for the measurement of moments of inertia and a vertical axis for the measurement of products of inertia.
  2. Plot point *A* which has coordinates (I_x, I_{xy}) and point *B* which has coordinates (I_y, -I_{xy}).
  3. Sketch a line between points *A* and *B* and then draw a circle with these two points as the extremities of a diameter.
- Calculated Quantities
  1. The angle from radius *OA* to the horizontal axis is 2α or twice the angle from the *x*-axis of the area in question to the axis of maximum moment of inertia. The angle on the diagram and the angle on the area are both measured in the same sense.
  2. The coordinates of any point *C* are ($I_{x'}$, $I_{x'y'}$), and those of corresponding point *D* are ($I_{y'}$, -$I_{x'y'}$), and can be found by rotating the diametral line sketched previously through the angle 2θ in the correct sense.
  3. The radius of the circle, maximum and minimum moments of inertia, and intermediate moments of inertia can be calculated using the geometry of the sketch.

Article A/4 – Sample Problem A/9 (1 of 2)

- Problem Statement

  Determine the product of inertia of the rectangular area with centroid at *C* with respect to the *x*-*y* axes parallel to its sides.



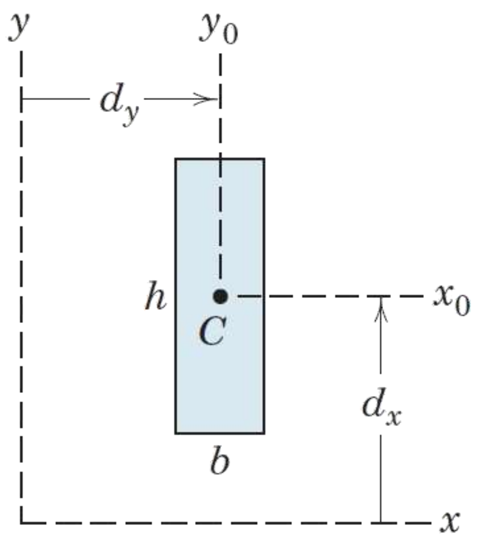
Article A/4 – Sample Problem A/9 (2 of 2)

- Solution

Since the product of inertia $\bar{I}_{xy}$ about the axes x_0-y_0 is zero by symmetry, the transfer-of-axis theorem gives us

$[I_{xy} = \bar{I}_{xy} + d_x d_y A]$ $\qquad I_{xy} = d_x d_y bh$ $\qquad$ *Ans.*

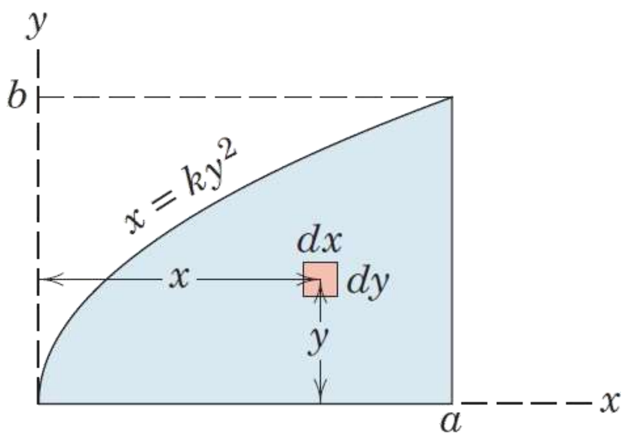
In this example both d_x and d_y are shown positive. We must be careful to be consistent with the positive directions of d_x and d_y as defined, so that their proper signs are observed.



Article A/4 – Sample Problem A/10 (1 of 3)

- Problem Statement

    Determine the product of inertia about the *x*-*y* axes for the area under the parabola.



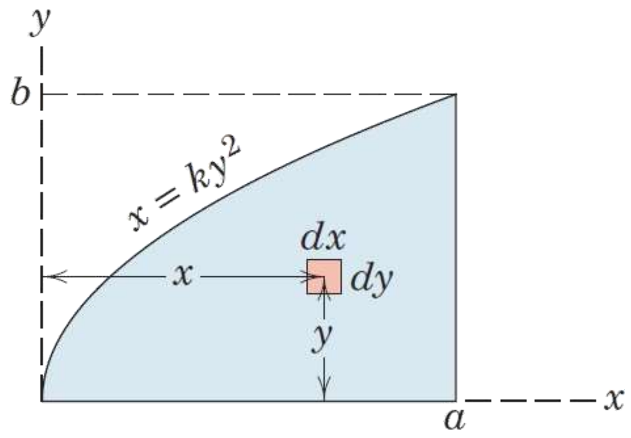
Article A/4 – Sample Problem A/10 (2 of 3)

- Equation of the Parabola

With the substitution of $x = a$ when $y = b$, the equation of the curve becomes $x = ay^2/b^2$.

- Second-Order Element $dA = dxdy$

$$I_{xy} = \int_0^b \int_{ay^2/b^2}^a xy\,dx\,dy = \int_0^b \frac{1}{2}\left(a^2 - \frac{a^2y^4}{b^4}\right) y\,dy = \frac{1}{6}a^2b^2 \qquad Ans.$$

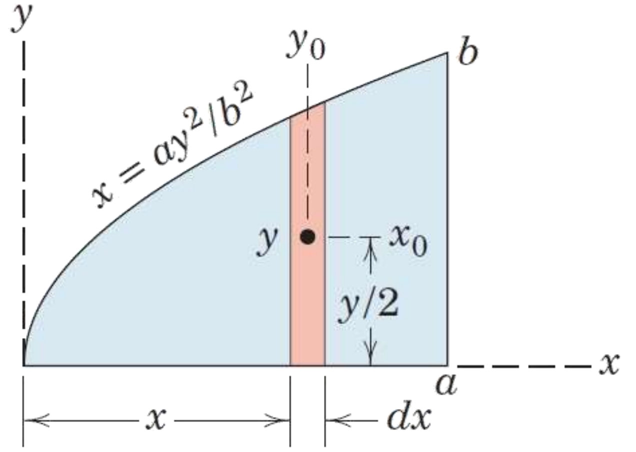


Article A/4 – Sample Problem A/10 (3 of 3)

- First Order Element, Vertical Strip

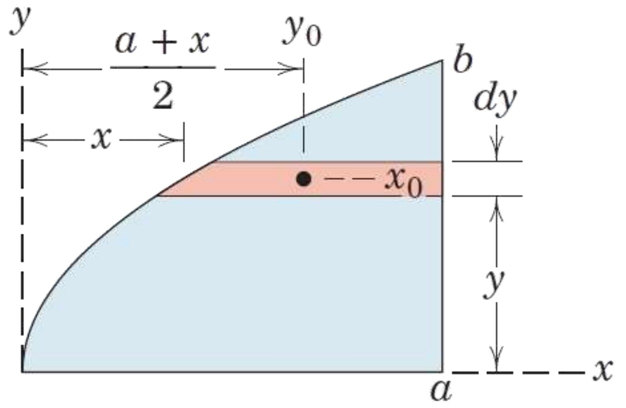
Alternatively, we can start with a first-order elemental strip and save one integration by using the results of Sample Problem A/9. Taking a vertical strip $dA = y\ dx$ gives $dI_{xy} = 0 + (\frac{1}{2}y)(x)(y\ dx)$, where the distances to the centroidal axes of the elemental rectangle are $d_x = y/2$ and $d_y = x$. ① Now we have

$$I_{xy} = \int_0^a \frac{y^2}{2} x\ dx = \int_0^a \frac{xb^2}{2a} x\ dx = \frac{b^2}{6a} x^3 \Big|_0^a = \frac{1}{6}a^2b^2 \qquad \textit{Ans.}$$



- First Order Element, Horizontal Strip

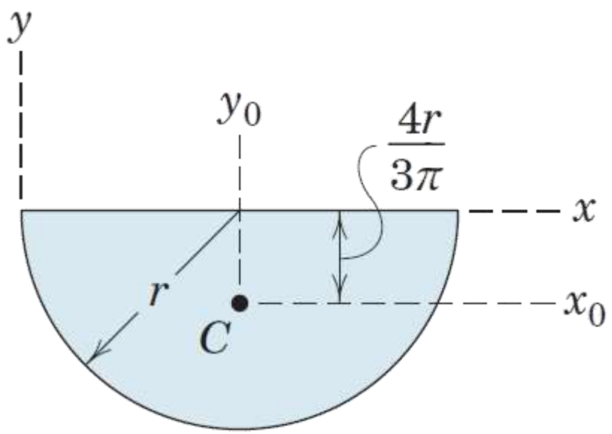
① If we had chosen a horizontal strip, our expression would have become $dI_{xy} = y\frac{1}{2}(a + x)[(a - x)\ dy]$, which when integrated, of course, gives us the same result as before.



Article A/4 – Sample Problem A/11 (1 of 2)

- Problem Statement

  Determine the product of inertia of the semicircular area with respect to the *x*-*y* axes.



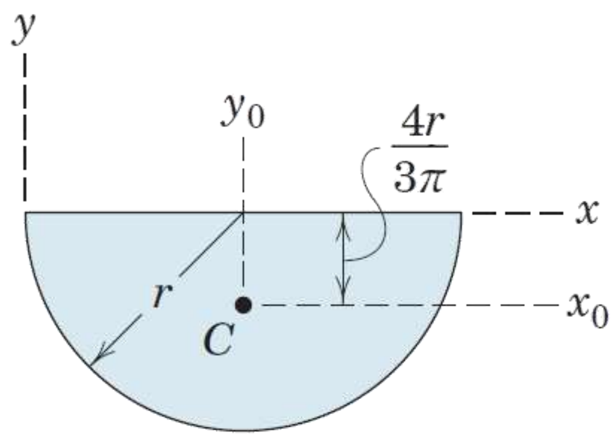
Article A/4 – Sample Problem A/11 (2 of 2)

- Solution

$$[I_{xy} = \bar{I}_{xy} + d_x d_y A] \qquad I_{xy} = 0 + \left(-\frac{4r}{3\pi}\right)(r)\left(\frac{\pi r^2}{2}\right) = -\frac{2r^4}{3} \qquad \textit{Ans.}$$

where the x- and y-coordinates of the centroid C are $d_y = +r$ and $d_x = -4r/(3\pi)$. Because y_0 is an axis of symmetry, $\bar{I}_{xy} = 0$.

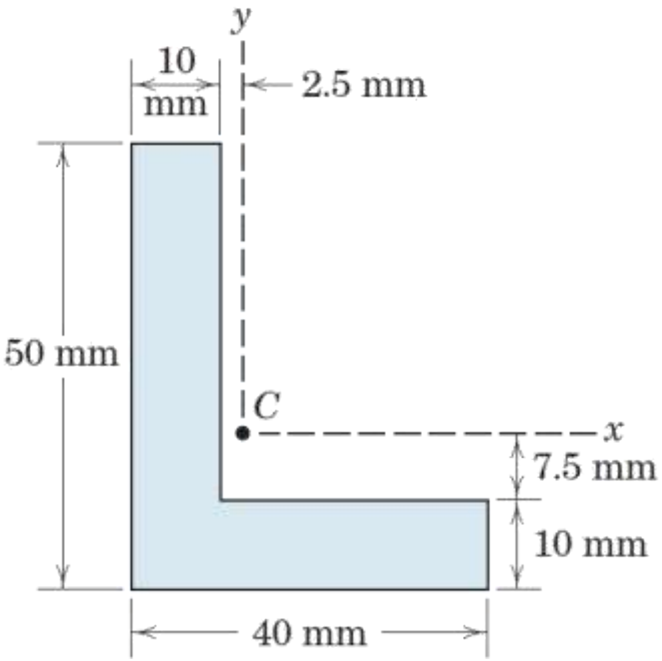
① Proper use of the transfer-of-axis theorem saves a great deal of labor in computing products of inertia.



Article A/4 – Sample Problem A/12 (1 of 5)

- Problem Statement

  Determine the orientation of the principal axes of inertia through the centroid of the angle section and determine the corresponding maximum and minimum moments of inertia.



Article A/4 – Sample Problem A/12 (2 of 5)

- Products of Inertia

Products of Inertia for the rectangles about their own centroidal axes are zero.

$[I_{xy} = \bar{I}_{xy} + d_x d_y A] \qquad I_{xy} = 0 + (-12.5)(+7.5)(400) = -3.75(10^4)\ \text{mm}^4$

where $\qquad d_x = -(7.5 + 5) = -12.5\ \text{mm}$

and $\qquad d_y = +(20 - 10 - 2.5) = 7.5\ \text{mm}$

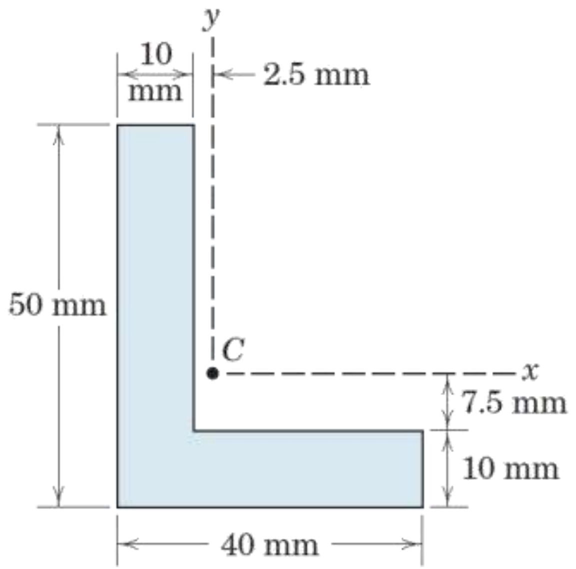
Likewise for part II,

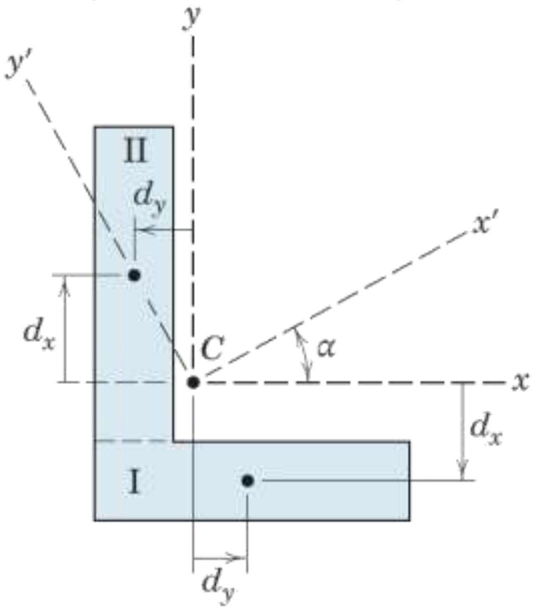
$[I_{xy} = \bar{I}_{xy} + d_x d_y A] \qquad I_{xy} = 0 + (12.5)(-7.5)(400) = -3.75(10^4)\ \text{mm}^4$

where $\quad d_x = +(20 - 7.5) = 12.5\ \text{mm}, \qquad d_y = -(5 + 2.5) = -7.5\ \text{mm}$

For the complete angle,

$$I_{xy} = -3.75(10^4) - 3.75(10^4) = -7.5(10^4)\ \text{mm}^4$$





Article A/4 – Sample Problem A/12 (3 of 5)

- Moments of Inertia

$[I = \bar{I} + Ad^2]$ $\quad I_x = \frac{1}{12}(40)(10)^3 + (400)(12.5)^2 = 6.58(10^4)\text{ mm}^4$

$$I_y = \frac{1}{12}(10)(40)^3 + (400)(7.5)^2 = 7.58(10^4)\text{ mm}^4$$

and the moments of inertia for part II about these same axes are

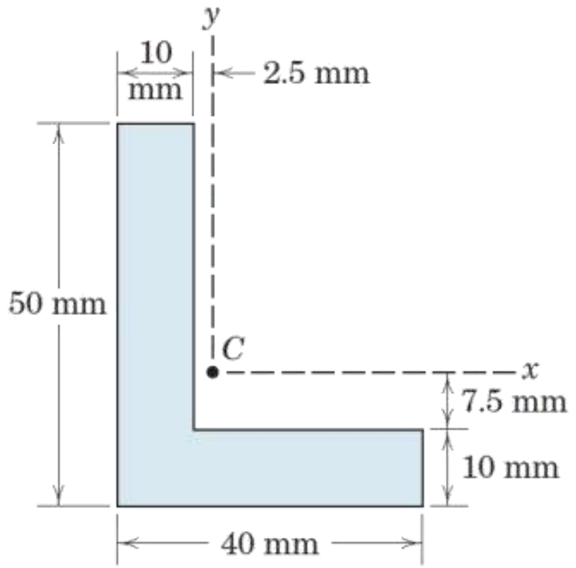
$[I = \bar{I} + Ad^2]$ $\quad I_x = \frac{1}{12}(10)(40)^3 + (400)(12.5)^2 = 11.58(10^4)\text{ mm}^4$

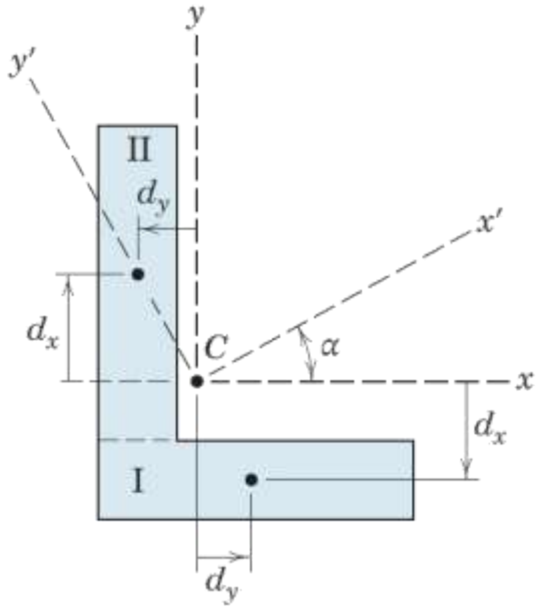
$$I_y = \frac{1}{12}(40)(10)^3 + (400)(7.5)^2 = 2.58(10^4)\text{ mm}^4$$

Thus, for the entire section we have

$$I_x = 6.58(10^4) + 11.58(10^4) = 18.17(10^4)\text{ mm}^4$$

$$I_y = 7.58(10^4) + 2.58(10^4) = 10.17(10^4)\text{ mm}^4$$





Article A/4 – Sample Problem A/12 (4 of 5)

- Principal Axes

$$\left[\tan 2\alpha = \frac{2I_{xy}}{I_y - I_x}\right] \qquad \tan 2\alpha = \frac{2(-7.50)}{10.17 - 18.17} = 1.875$$

$$2\alpha = 61.9^\circ \qquad \alpha = 31.0^\circ \qquad \textit{Ans.}$$

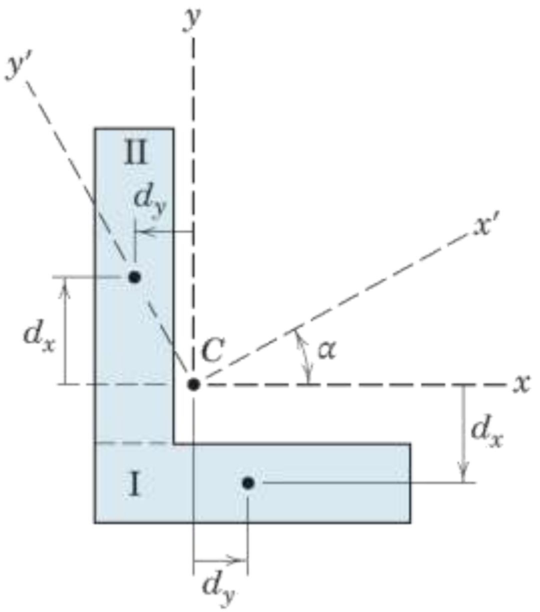
We now compute the principal moments of inertia from Eqs. A/9 using α for θ and get I_{max} from $I_{x'}$ and I_{min} from $I_{y'}$. Thus,

$$I_{max} = \left[\frac{18.17 + 10.17}{2} + \frac{18.17 - 10.17}{2}(0.471) + (7.50)(0.882)\right](10^4)$$

$$= 22.7(10^4)\ \text{mm}^4 \qquad \textit{Ans.}$$

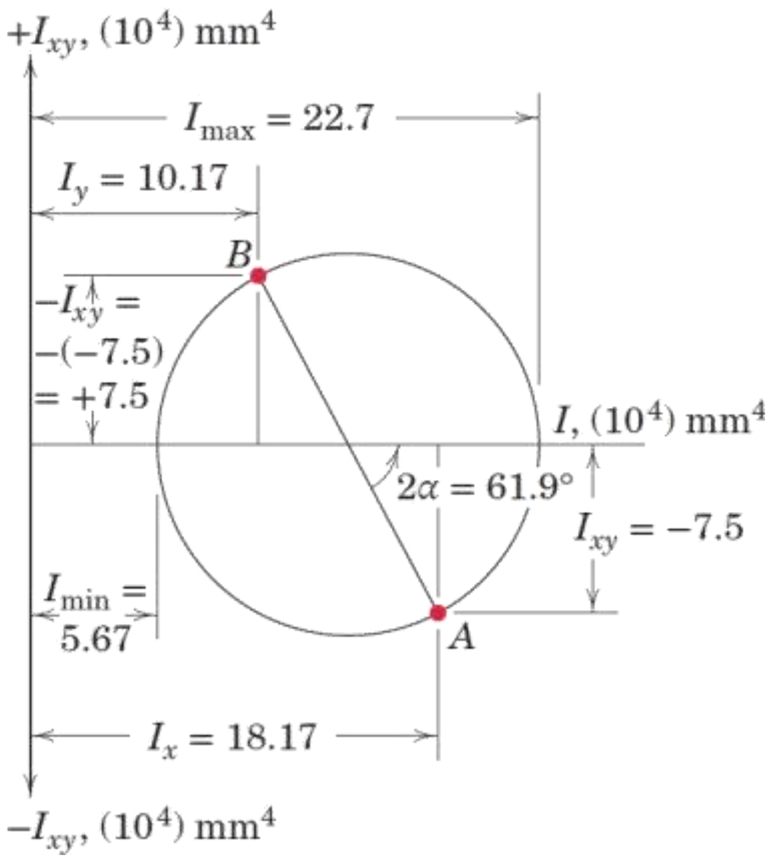
$$I_{min} = \left[\frac{18.17 + 10.17}{2} - \frac{18.17 - 10.17}{2}(0.471) - (7.50)(0.882)\right](10^4)$$

$$= 5.67(10^4)\ \text{mm}^4 \qquad \textit{Ans.}$$

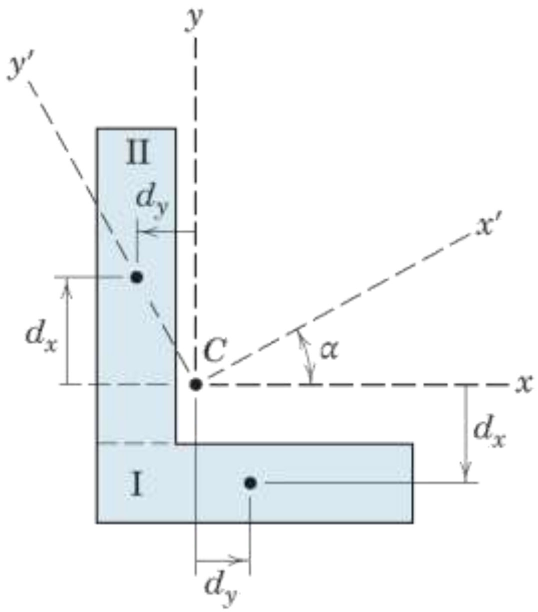


Article A/4 – Sample Problem A/12 (5 of 5)

- Mohr's Circle of Inertia Plot



Mohr's circle. Alternatively, we could use Eqs. A/11 to obtain the results for I_{max} and I_{min}, or we could construct the Mohr's circle from the calculated values of I_x, I_y, and I_{xy}. These values are spotted on the diagram to locate points A and B, which are the extremities of the diameter of the circle. The angle 2α and I_{max} and I_{min} are obtained from the figure, as shown.



CHAPTER 1
INTRODUCTION TO STATICS

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CHAPTER OUTLINE

Article 1/1 Mechanics

- Mechanics deals with the effects of forces on objects.
- No other subject plays a greater role in engineering analysis.
- Mechanics is the oldest of the physical sciences.
- This course deals with the development and application of the principles of mechanics.
- Statics is concerned with the equilibrium of bodies under the action of forces.

Article 1/2 Basic Concepts

- Space
- Time
- Mass
- Force
- Particle
- Rigid Body

Article 1/3 Scalars and Vectors

- Scalar Quantity
  - A quantity with only a magnitude.
  - Examples: time, volume, density, speed, energy, and mass

- Vector Quantity
  - Quantity with both a magnitude and a direction.
  - Obeys the parallelogram law of addition.
  - Examples: displacement, velocity, acceleration, force, moment, and momentum.

Article 1/3 – Types of Vectors in Mechanics

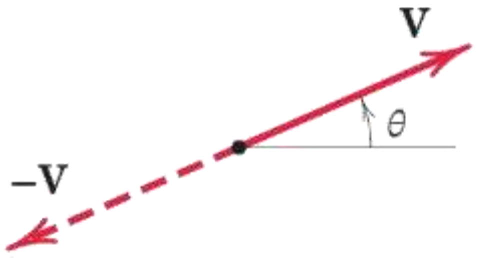
- Types of Vectors

  - Free Vector – one whose action is not confined to or associated with a unique line in space.

  - Sliding Vector – has a unique line of action in space but not a unique point of application.

  - Fixed Vector – has a unique line of action and point of application.

Article 1/3 – Conventions for Equations and Diagrams

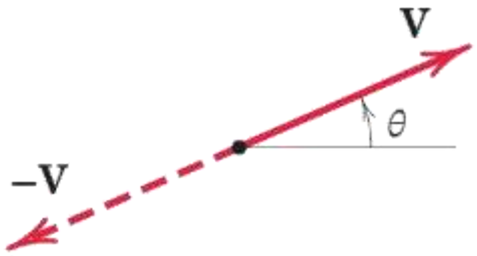


- Vector Representation
  - Line Segment with an Arrowhead to Indicate Direction
  - Written in Bold, Roman Type, e.g., **V**
  - Magnitude is Written in Lightface, Italic type, e.g., *V*

- Always Distinguish between Scalars and Vectors
  - Use an underline, over-arrow, under-squiggle, etc., to represent vectors.
  - Failure to do so causes many mistakes in mechanics.

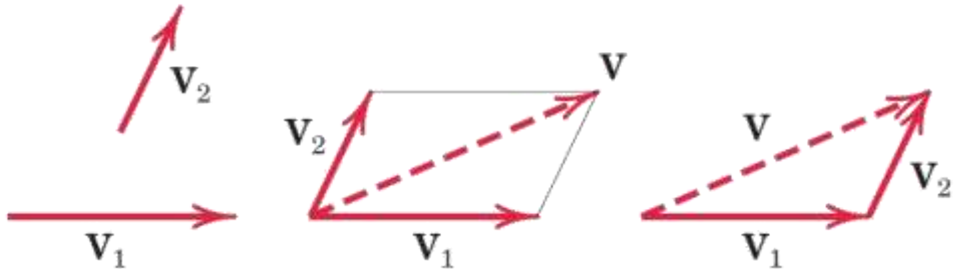
Article 1/3 – Working with Vectors (1 of 4)

- Drawing Vectors
  - Line Segment with an Arrowhead to Indicate Direction
  - Reference Angle θ

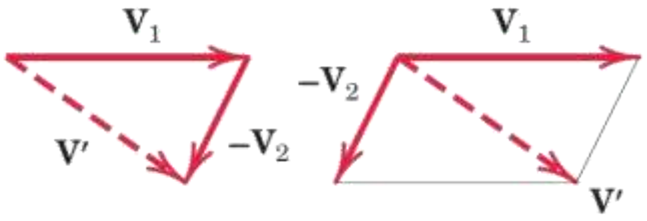


Article 1/3 – Working with Vectors (2 of 4)

- Parallelogram Law of Addition – Vector Sum $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$
  - Two Vectors, $\mathbf{V}_1$ and $\mathbf{V}_2$, treated as free vectors, may be replaced by their equivalent vector $\mathbf{V}$, which is the diagonal of the parallelogram formed by $\mathbf{V}_1$ and $\mathbf{V}_2$. This is called a *vector sum*.

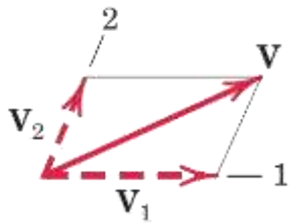


- Vector Difference $\mathbf{V}' = \mathbf{V}_1 – \mathbf{V}_2$ (Adding a Negative)

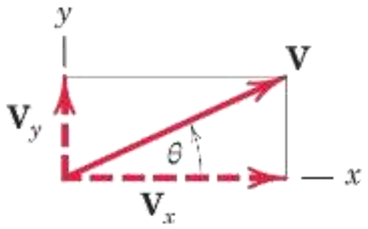


Article 1/3 – Working with Vectors (3 of 4)

- Vector Components

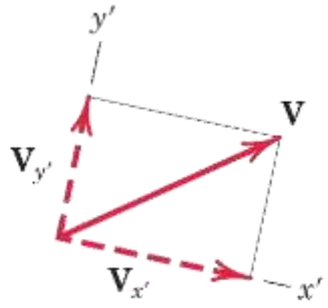


- Rectangular Components (*x*-*y*)



$$\theta = \tan^{-1}\frac{V_y}{V_x}$$

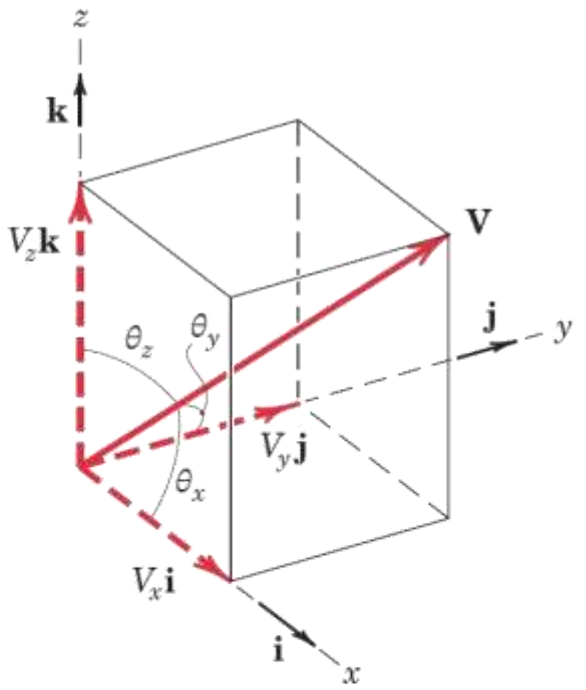
- Axis Orientation



Article 1/3 – Working with Vectors (4 of 4)

- Unit Vector Representation, $\mathbf{V} = V\mathbf{n}$
  - A unit vector **n** has a magnitude of one (unity) and points in the direction of a vector.

- Three-Dimensional Vectors and Direction Cosines
  - $\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}$
    - $V_x = V\cos\theta_x = Vl$ where $l = \cos\theta_x$
    - $V_y = V\cos\theta_y = Vm$ where $m = \cos\theta_y$
    - $V_z = V\cos\theta_z = Vn$ where $n = \cos\theta_z$
  - Pythagorean Theorem (Vector Magnitude)
    - $V^2 = V_x^2 + V_y^2 + V_z^2$
    - $l^2 + m^2 + n^2 = 1$



Article 1/4 Newton’s Laws

- Law I

  A particle remains at rest or continues to move with *uniform velocity* (in a straight line with a constant speed) if there is no unbalanced force acting on it.

- Law II

  The acceleration of a particle is proportional to the vector sum of forces acting on it and is in the direction of this vector sum.

- Law III

  The forces of action and reaction between interacting bodies are equal in magnitude, opposite in direction, and *collinear* (they lie on the same line).

Article 1/5 Units

- Fundamental Quantities of Mechanics and their Units

Quantity	Dimensional Symbol	SI Units			U.S. Customary Units		
			Unit	Symbol		Unit	Symbol
Mass	M	Base units	kilogram	kg		slug	—
Length	L	Base units	meter	m	Base units	foot	ft
Time	T	Base units	second	s	Base units	second	sec
Force	F		newton	N	Base units	pound	lb

Article 1/5 – SI Units

- Base Units
  - kilogram (kg)
  - meter (m)
  - second (s)

- Derived Unit: Newton (N)
  - Force Unit
  - $N = kg \cdot m/s^2$

Article 1/5 – U.S. Customary Units

- Base Units
  - pound (lb)
  - foot (ft)
  - second (sec)

- Derived Unit: slug (slug)
  - Mass Unit
  - slug = lb-sec^2/ft
  - Gravitational System

Article 1/5 – Primary Standards (1 of 2)

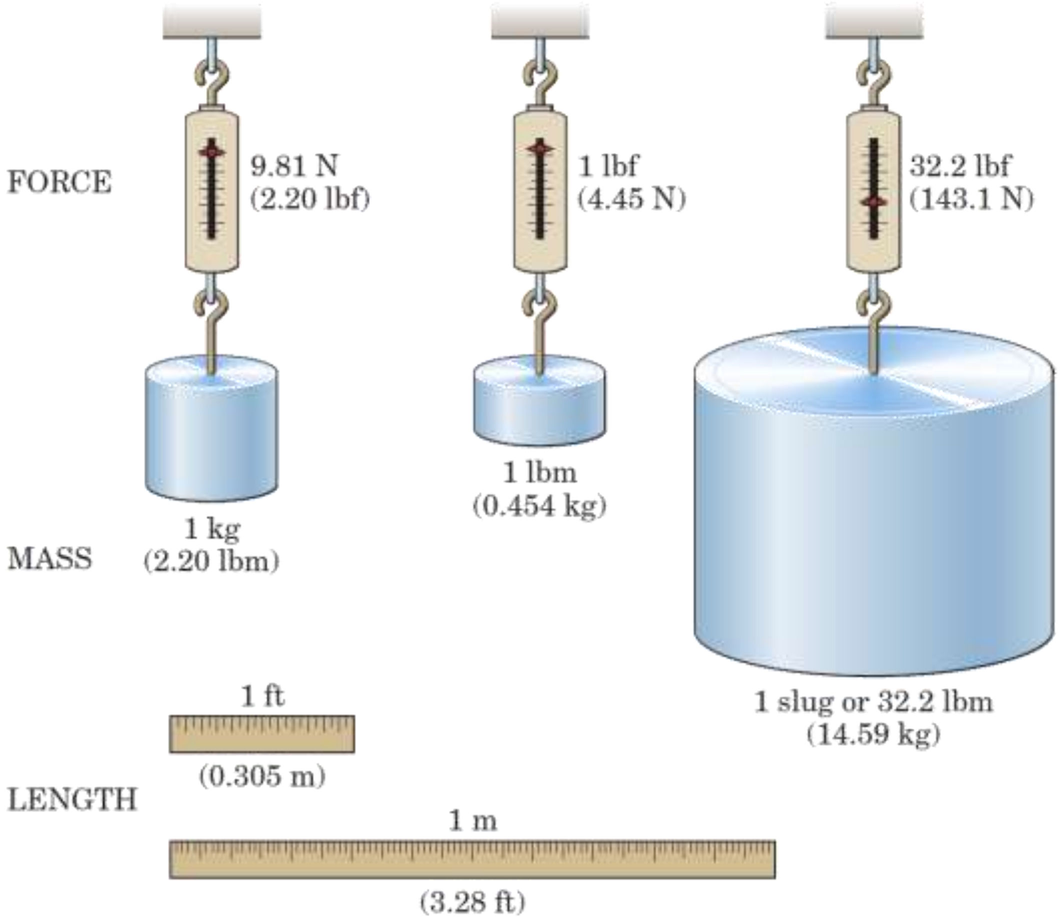
- Mass
  - Mass of a specific platinum iridium cylinder kept at the International Bureau of Weights and Measures near Paris, France.
- Length
  - The distance traveled by light in a vacuum in (1/299 792 458) second.
- Time
  - The duration of 9 192 631 770 periods of the radiation of a specific state of a cesium-133 atom.

Article 1/5 – Primary Standards (1 of 2)

- Acceleration of Gravity
  - SI Units: $g = 9.806\ 65\ \text{m/s}^2$
  - U.S. Units: $g = 32.1740\ \text{ft/sec}^2$
- Values for Most Problems in Mechanics
  - SI Units: $g = 9.81\ \text{m/s}^2$
  - U.S. Units: $g = 32.2\ \text{ft/sec}^2$

Article 1/5 – Unit Conversions

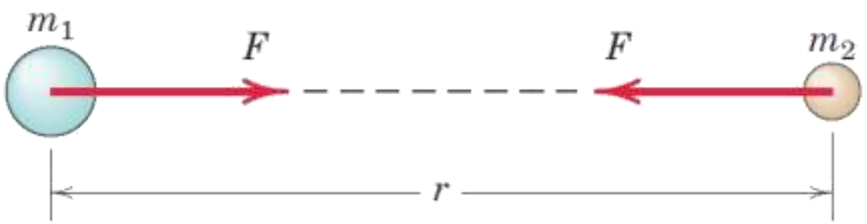
- Visual Comparison of Force, Mass, and Length



A comprehensive list of unit conversion is available in Table D/5 of Appendix D.

Article 1/6 Law of Gravitation

- Mathematical Expression



$$\boxed{F = G\frac{m_1 m_2}{r^2}}$$

F = the mutual force of attraction between two particles

G = a universal constant known as the constant of gravitation

m_1, m_2 = the masses of the two particles

r = the distance between the centers of the particles

- Constant of Gravitation, G
  - SI Units: $G = 6.673(10^{-11})\ m^3/(kg{\cdot}s^2)$
  - U.S. Units: $G = 3.439(10^{-8})\ ft^4/(lb\text{-}sec^4)$

Article 1/6 – Gravitational Attraction of the Earth

- Apparent Weight, $W = mg$

- SI Problems
  - Mass m is always in kilograms (kg) and is almost always provided in the book.
  - Acceleration of gravity g = 9.81 m/s^2 (unless stated otherwise).
  - Weight W is in newtons (N).
  - Kilogram (kg) is not a force!

- U.S. Problems
  - Mass m is always in slugs (slugs) and is almost never provided in the book.
  - Acceleration of gravity g = 32.2 ft/sec^2 (unless stated otherwise).
  - Weight W is in pounds (lb) and is usually what you are provided.
  - Pound (lb) is not a mass!

Article 1/7 Accuracy, Limits, and Approximations

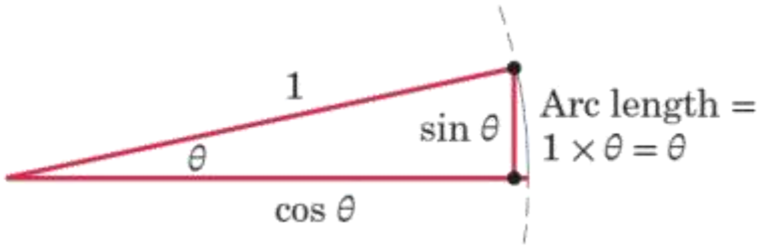
- Significant Figure Use in the Textbook
  - The textbook assumes that all given numbers are exact for simplicity.
  - If the first non-zero digit is a one (1), the textbook will record four (4) significant figures in the answer.
  - If the first non-zero digit is a two through nine (2-9), the textbook will record three (3) significant figures in the answer.
  - Retain all significant figures on intermediate calculations in a calculator, but only record answers or calculations according to the convention listed above.

Article 1/7 – Differentials

- Order of Differential Quantities

Article 1/7 – Small-Angle Approximations

- Small Angle Approximations (with radians)



  - $\sin\theta \cong \tan\theta \cong \theta$ and $\sin d\theta \cong \tan d\theta \cong d\theta$
  - $\cos\theta \cong 1$ and $\cos d\theta \cong 1$
  - Sample Calculation
    - 1° = 0.017 453 rad, sin 1° = 0.017452, tan 1° = 0.017455, cos 1° = 0.999848
    - The percent error for the sine function is only 0.51% at 10°.
  - If more accuracy is required, retain the first two terms in the series expansion of the function.
  - To convert from degrees to radians, multiply the angle in degrees by $\pi/180°$.

Article 1/8 Problem Solving in Statics (1 of 4)

- Dual Thought Process in Statics
  - Think about the **physical situation** and the corresponding **mathematical description**.

- Make Appropriate Assumptions

- Use Graphics
  1. Representing a problem geometrically helps us with its physical interpretation. This is especially true for three-dimensional problems.
  2. Graphical solutions can often be obtained more readily than with a direct mathematical solution.
  3. Charts and graphs are valuable aids for representing results.

Article 1/8 Problem Solving in Statics (2 of 4)

- Formulating Problems and Obtaining Solutions

1. Formulate the problem
   a) State the given data.
   b) State the desired result.
   c) State your assumptions and approximations.
2. Develop the solution
   a) Draw any diagrams you need to understand the relationships.
   b) State the governing principles to be applied to your solution.
   c) Make your calculations.
   d) Ensure that your calculations are consistent with the accuracy justified by the data.
   e) Be sure that you have used consistent units throughout your calculations.
   f) Ensure that your answers are reasonable in terms of magnitudes, directions, common sense, etc.
   g) Draw conclusions.

Article 1/8 Problem Solving in Statics (3 of 4)

- The Free-Body Diagram
  - Isolation of a Body from all other Interacting Bodies
  - Developed Fully in Chapter 3
  - Single Most Important Step in Equilibrium Problems

- Numerical Values versus Symbols
  - Symbolic Solutions Advantages
    - Helps to focus attention on the connection between the physical situation and its related mathematical description.
    - Can be used repeatedly for obtaining answers to the same type or problem but having different units or numerical values.
    - Enables dimensional checks at every step to ensure dimensional homogeneity.

Article 1/8 Problem Solving in Statics (4 of 4)

- Solution Methods

  1. Obtain mathematical solutions by hand, using either algebraic symbols or numerical values. We can solve most problems this way.

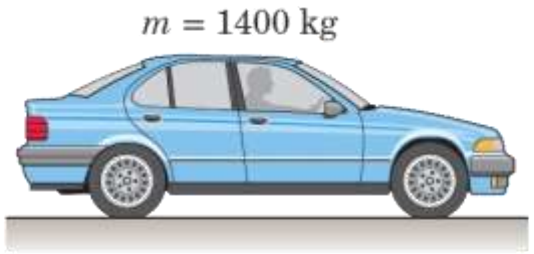
  2. Obtain graphical solutions for certain problems.

  3. Solve problems by computer. This is useful when a large number of equations must be solved, when a parameter variation must be studied, or when an intractable equation must be solved.

Article 1/9 – Sample Problem 1/1 (1 of 2)

- Problem Statement

  Determine the weight in newtons of a car whose mass is 1400 kg. Convert the mass of the car to slugs and then determine its weight in pounds.



Article 1/9 – Sample Problem 1/1 (2 of 2)

- Solution

$$W = mg = 1400(9.81) = 13\ 730 \text{ N} \quad ① \qquad Ans.$$

From the table of conversion factors in Table D/5 of Appendix D, we see that 1 slug is equal to 14.594 kg. Thus, the mass of the car in slugs is

$$m = 1400 \text{ kg}\left[\frac{1 \text{ slug}}{14.594 \text{ kg}}\right] = 95.9 \text{ slugs} \quad ② \qquad Ans.$$

Finally, its weight in pounds is

$$W = mg = (95.9)(32.2) = 3090 \text{ lb} \quad ③ \qquad Ans.$$

As another route to the last result, we can convert from kg to lbm. Again using Table D/5, we have

$$m = 1400 \text{ kg}\left[\frac{1 \text{ lbm}}{0.45359 \text{ kg}}\right] = 3090 \text{ lbm}$$

③ Note that we are using a previously calculated result (95.9 slugs). We must be sure that when a calculated number is needed in subsequent calculations, it is retained in the calculator to its full accuracy, (95.929834 . . .), until it is needed. This may require storing it in a register upon its initial calculation and recalling it later. We must not merely punch 95.9 into our calculator and proceed to multiply by 32.2—this practice will result in loss of numerical accuracy. Some individuals like to place a small indication of the storage register used in the right margin of the work paper, directly beside the number stored.

$m = 1400$ kg

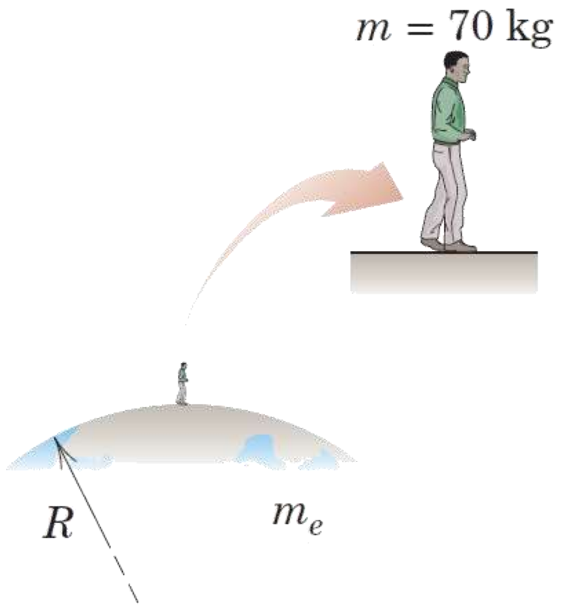
① Our calculator indicates a result of 13 734 N. Using the rules of significant-figure display used in this textbook, we round the written result to four significant figures, or 13 730 N. Had the number begun with any digit other than 1, we would have rounded to three significant figures.

② A good practice with unit conversion is to multiply by a factor such as $\left[\frac{1 \text{ slug}}{14.594 \text{ kg}}\right]$, which has a value of 1, because the numerator and the denominator are equivalent. Make sure that cancellation of the units leaves the units desired; here the units of kg cancel, leaving the desired units of slug.

Article 1/9 – Sample Problem 1/2 (1 of 2)

- Problem Statement

  Use Newton's law of universal gravitation to calculate the weight of a 70-kg person standing on the surface of the earth. Then repeat the calculation by using $W = mg$ and compare your two results. Use Table D/2 as needed.



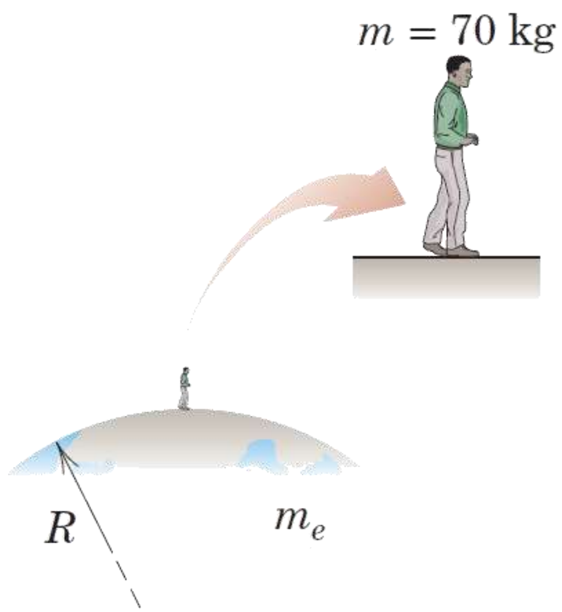
Article 1/9 – Sample Problem 1/2 (2 of 2)

- Solution

$$W = \frac{Gm_e m}{R^2} = \frac{(6.673 \cdot 10^{-11})(5.976 \cdot 10^{24})(70)}{[6371 \cdot 10^3]^2} = 688 \text{ N} \quad ① \qquad \textit{Ans.}$$

$$W = mg = 70(9.81) = 687 \text{ N} \qquad \textit{Ans.}$$

① The effective distance between the mass centers of the two bodies involved is the radius of the earth.

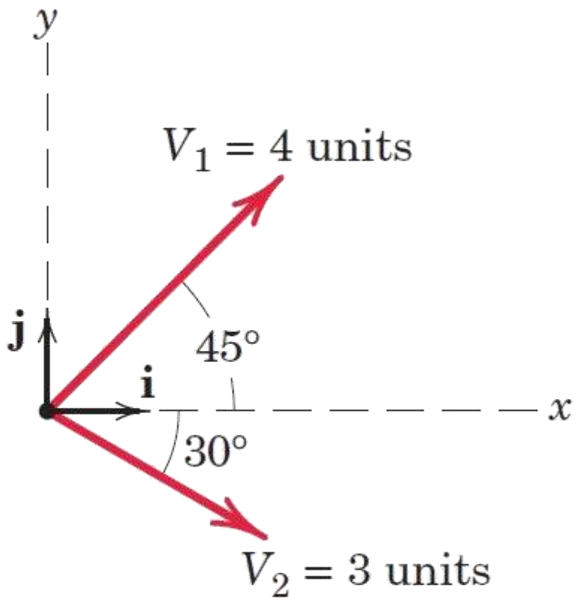


Article 1/9 – Sample Problem 1/3 (1 of 3)

- Problem Statement

For the vectors $\mathbf{V}_1$ and $\mathbf{V}_2$ shown in the figure,

a) determine the magnitude S of their vector sum $\mathbf{S} = \mathbf{V}_1 + \mathbf{V}_2$
b) determine the angle α between $\mathbf{S}$ and the positive x-axis
c) write $\mathbf{S}$ as a vector in terms of the unit vectors $\mathbf{i}$ and $\mathbf{j}$ and then write a unit vector $\mathbf{n}$ along the vector sum $\mathbf{S}$
d) determine the vector difference $\mathbf{D} = \mathbf{V}_1 - \mathbf{V}_2$



Article 1/9 – Sample Problem 1/3 (2 of 3)

- Part *a*) – Magnitude of **S**

$$S^2 = 3^2 + 4^2 - 2(3)(4)\cos 105°$$

$$S = 5.59 \text{ units} \qquad \textit{Ans.}$$

- Part *b*) – Angle Between **S** and *x*-axis

$$\frac{\sin 105°}{5.59} = \frac{\sin(\alpha + 30°)}{4}$$

$$\sin(\alpha + 30°) = 0.692$$

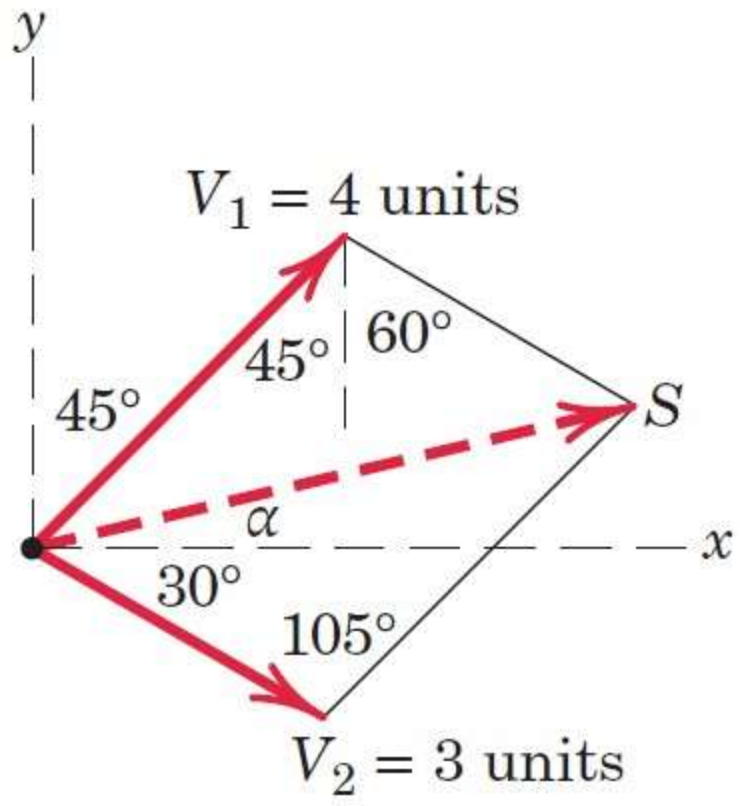
$$(\alpha + 30°) = 43.8° \qquad \alpha = 13.76° \qquad \textit{Ans.}$$

- Part *c*) – Vector Expression for **S** and **n**

$$\mathbf{S} = S[\mathbf{i}\cos\alpha + \mathbf{j}\sin\alpha]$$

$$= 5.59[\mathbf{i}\cos 13.76° + \mathbf{j}\sin 13.76°] = 5.43\mathbf{i} + 1.328\mathbf{j} \text{ units} \quad \textit{Ans.}$$

Then $$\mathbf{n} = \frac{\mathbf{S}}{S} = \frac{5.43\mathbf{i} + 1.328\mathbf{j}}{5.59} = 0.971\mathbf{i} + 0.238\mathbf{j} \quad ② \qquad \textit{Ans.}$$



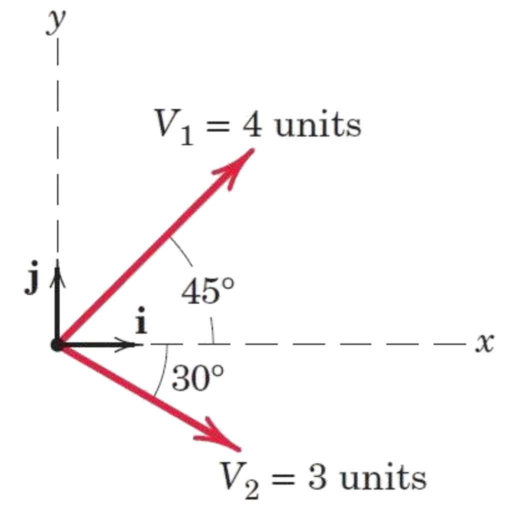
② A unit vector may always be formed by dividing a vector by its magnitude. Note that a unit vector is dimensionless.

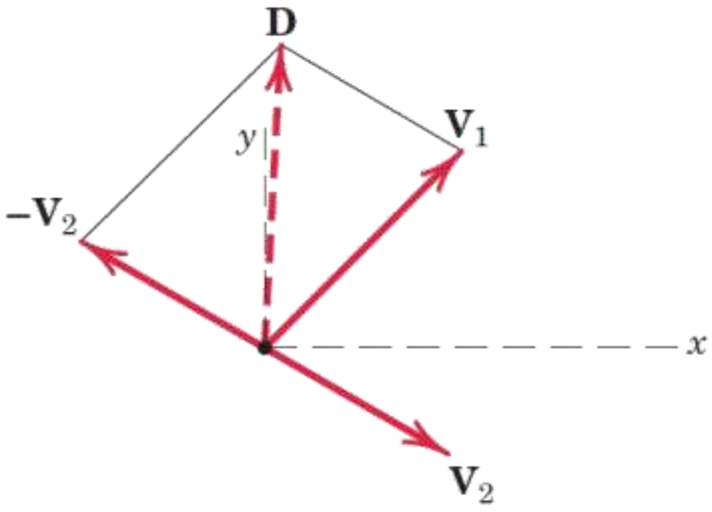
Article 1/9 – Sample Problem 1/3 (of 3)

- Part *d*) – Vector Difference

$$\mathbf{D} = \mathbf{V}_1 - \mathbf{V}_2 = 4(\mathbf{i}\cos 45^\circ + \mathbf{j}\sin 45^\circ) - 3(\mathbf{i}\cos 30^\circ - \mathbf{j}\sin 30^\circ)$$

$$= 0.230\mathbf{i} + 4.33\mathbf{j} \text{ units} \qquad \textit{Ans.}$$





CHAPTER 2
FORCE SYSTEMS

CHAPTER OUTLINE

2/1 Introduction

2/2 Force

SECTION A Two-Dimensional Force Systems

2/3 Rectangular Components

2/4 Moment

2/5 Couple

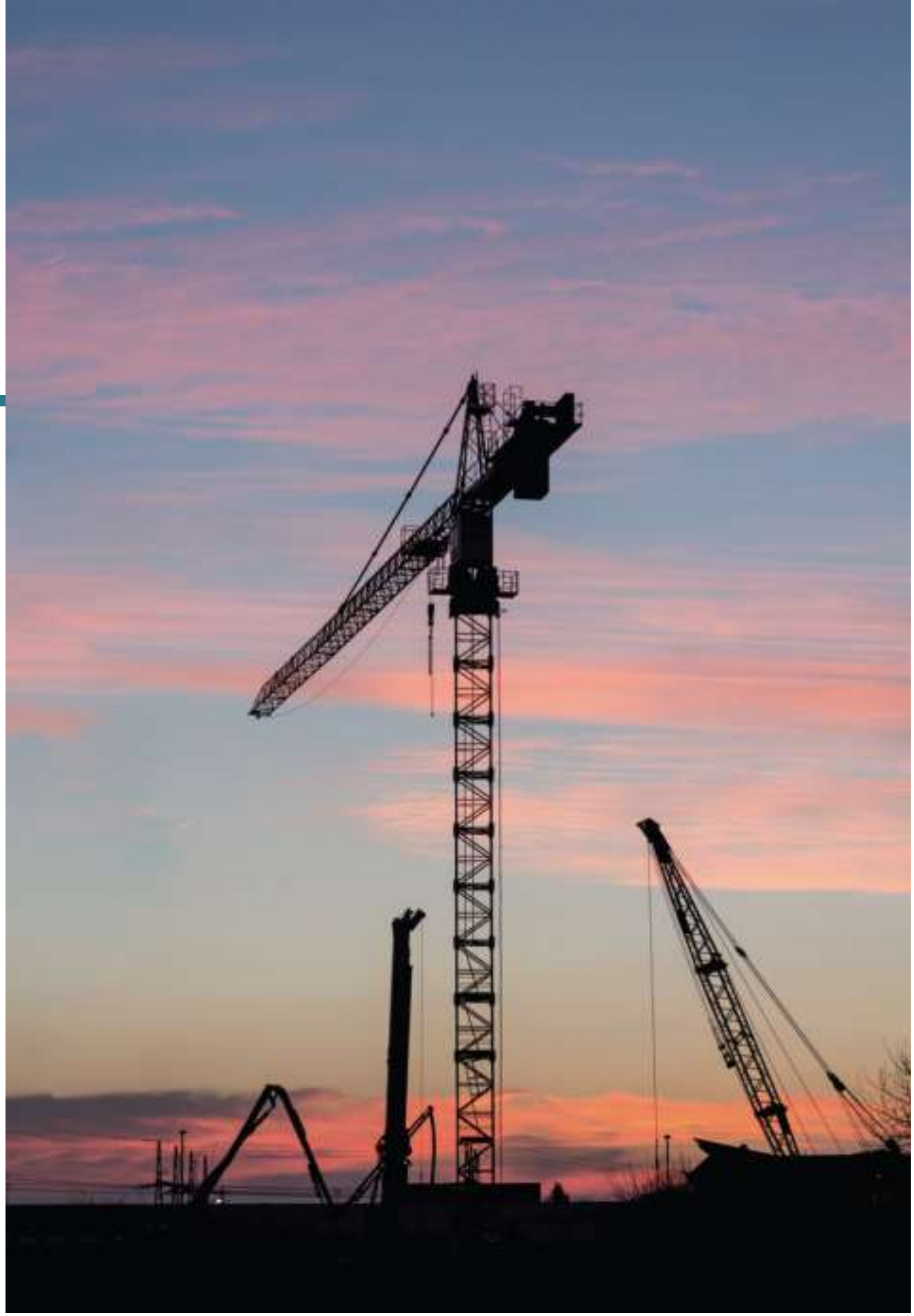
2/6 Resultants

SECTION B Three-Dimensional Force Systems

2/7 Rectangular Components

2/8 Moment and Couple

2/9 Resultants



Anze Bizjan/Shutterstock

Article 2/1 Introduction

- Chapter Purpose

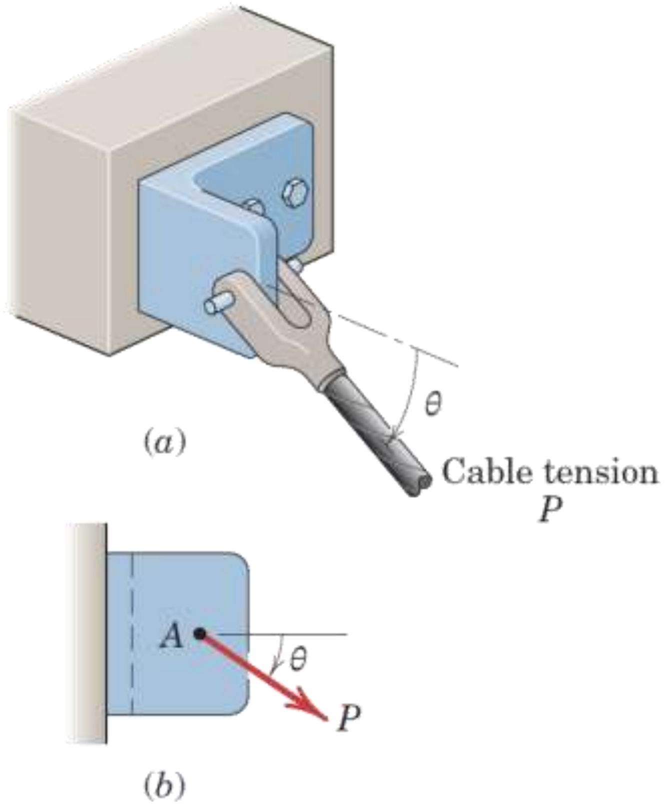
- Importance of Chapter 2 Concepts

Article 2/2 Force

- A Reminder and Illustration

- Vector Quantity

- External and Internal Effects

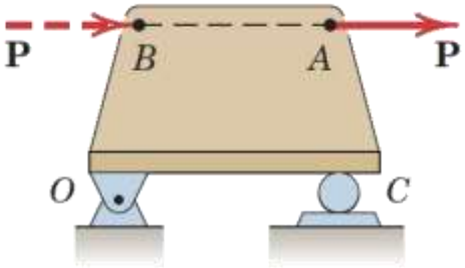


Article 2/2 – Principle of Transmissibility

- Statement of the Principle

  A force may be applied at any point on its given line of action without altering the resultant effects of the force external to the rigid body on which it acts.

- Illustration



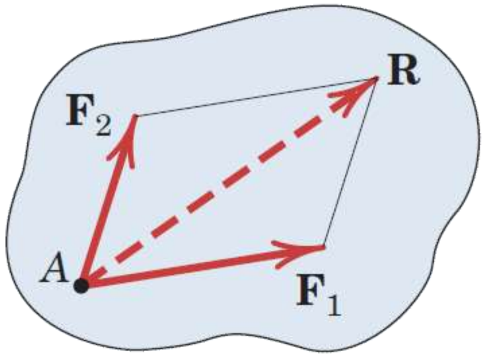
- Sliding Vectors

Article 2/2 – Force Classification

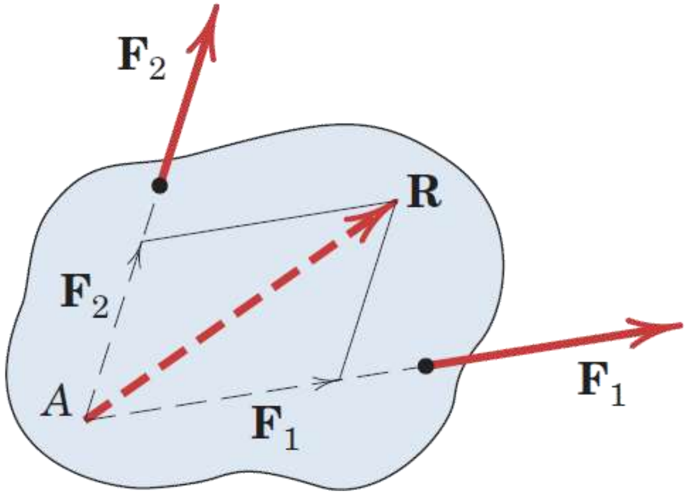
- Contact Force
- Body Force
- Concentrated Force
- Distributed Force
- Action and Reaction Pairs

Article 2/2 – Concurrent Forces (1 of 2)

- Definition and Vector Sum

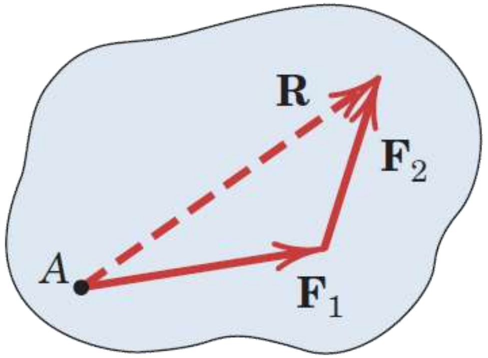


- Application of Transmissibility

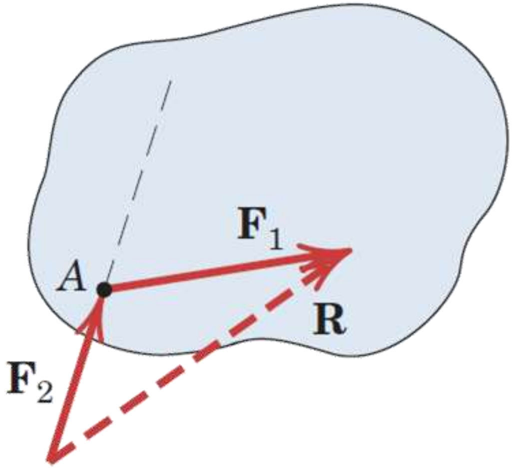


Article 2/2 – Concurrent Forces (2 of 2)

- Parallelogram Law of Vector Addition

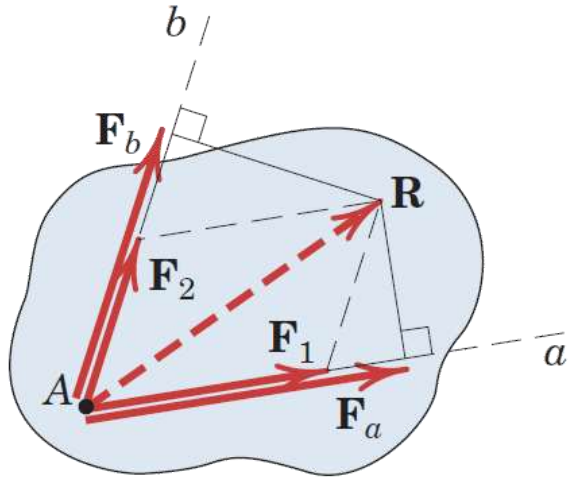


- Order of Addition



Article 2/2 – Vector Components

- Vector Components: $\mathbf{F}_1$ and $\mathbf{F}_2$
  - Obey the Parallelogram Law of Addition
  - Directed along the Axes of Choice

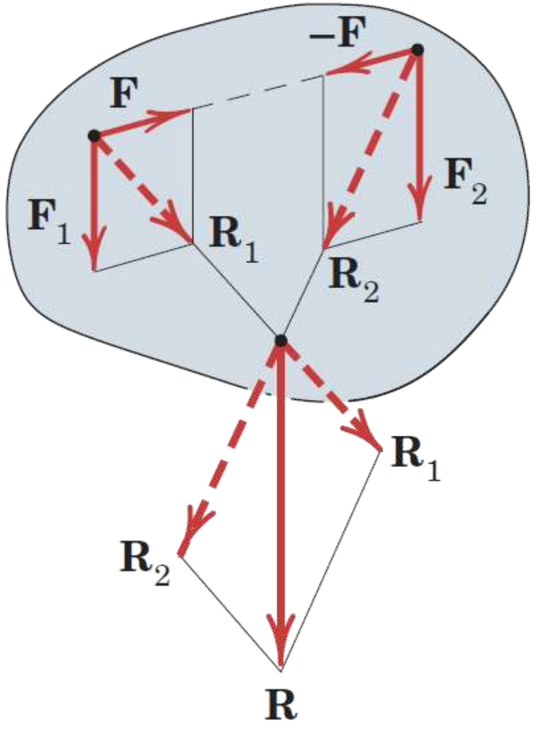


- Vector Projections: $\mathbf{F}_a$ and $\mathbf{F}_b$
  - Do not obey the Parallelogram Law of Addition
  - Directed along the Axes of Choice

Article 2/2 – Special Case of Vector Addition

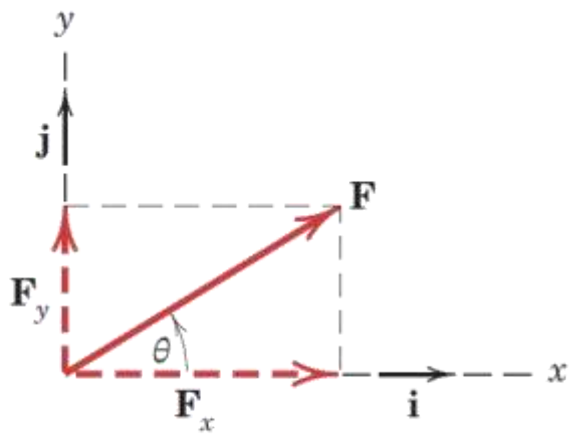
- Parallel Vectors: $\mathbf{F}_1$ and $\mathbf{F}_2$

- Procedure
  - Add Equal, Opposite, Collinear Forces **F** and **-F**
  - Find the Resultants $\mathbf{R}_1$ and $\mathbf{R}_2$
  - Locate the Point of Concurrency for the Resultants
  - Add the Resultants at the Point of Concurrency



2/3 Rectangular Components

- Illustration



- Vector Components: $\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y$

- Scalar Components: $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j}$

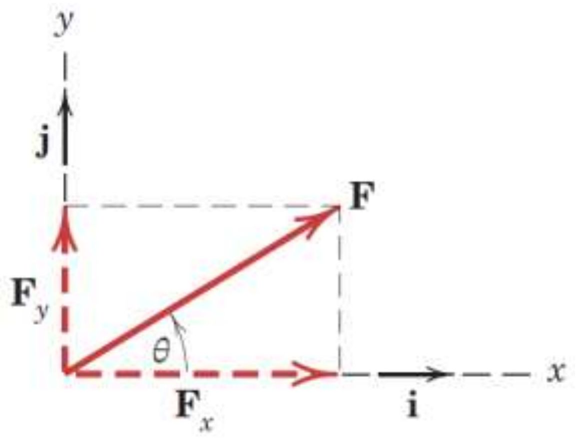
- Other Useful Relationships

$$F_x = F\cos\theta \qquad F = \sqrt{F_x^{\,2} + F_y^{\,2}}$$

$$F_y = F\sin\theta \qquad \theta = \tan^{-1}\frac{F_y}{F_x}$$

2/3 Conventions for Describing Vector Components

- Vector Magnitude, F
  - Lightface, Italic Font
  - Always Positive
- Scalar Component, F_x
  - Lightface, Italic Font
  - Positive or Negative
- Force Vector Depiction
  - Solid, Red Arrow
- Component Vector Depiction
  - Dashed, Red Arrow



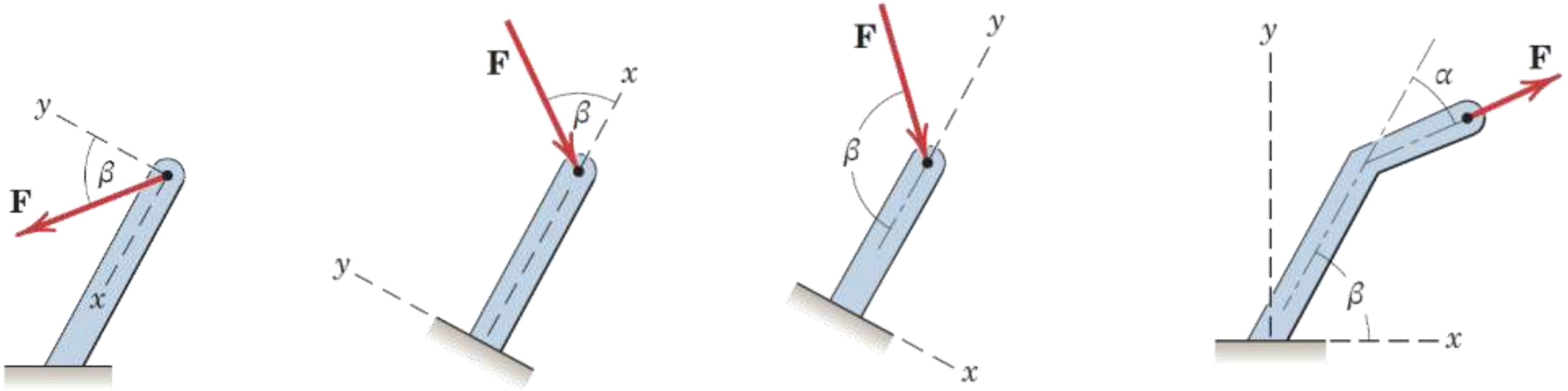
2/3 Determining the Components of a Force (1 of 2)

- It is often the case that…
  - Dimensions or axes are not given in horizontal and vertical directions.
  - Angles are not measured counterclockwise from the x-axis.
  - Coordinates do not originate from the line of action of a force.

- We still need to be able to find the components of a force vector!

2/3 Determining the Components of a Force (2 of 2)

- Some Illustrations



$F_x = F \sin \beta$
$F_y = F \cos \beta$

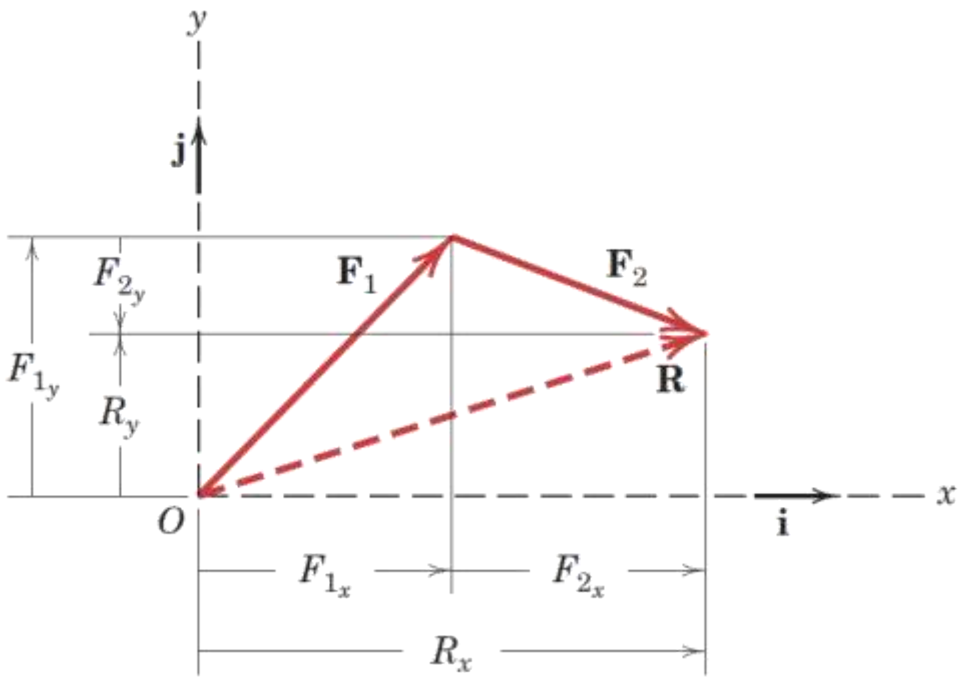
$F_x = -F \cos \beta$
$F_y = -F \sin \beta$

$F_x = F \sin(\pi - \beta)$
$F_y = -F \cos(\pi - \beta)$

$F_x = F \cos(\beta - \alpha)$
$F_y = F \sin(\beta - \alpha)$

2/3 Finding Resultants using Components

- Illustration



- Mathematics

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = (F_{1_x}\mathbf{i} + F_{1_y}\mathbf{j}) + (F_{2_x}\mathbf{i} + F_{2_y}\mathbf{j})$$

$$R_x\mathbf{i} + R_y\mathbf{j} = (F_{1_x} + F_{2_x})\mathbf{i} + (F_{1_y} + F_{2_y})\,\mathbf{j}$$

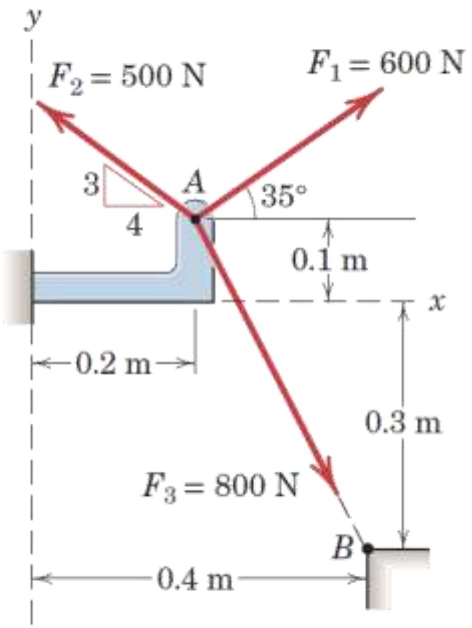
$$R_x = F_{1_x} + F_{2_x} = \Sigma F_x$$

$$R_y = F_{1_y} + F_{2_y} = \Sigma F_y$$

Article 2/3 – Sample Problem 2/1 (1 of 3)

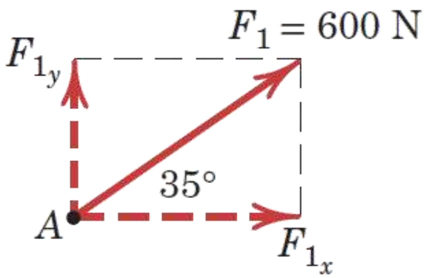
- Problem Statement

  The forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, all of which act on point A of the bracket, are specified in three different ways. Determine the x and y scalar components of each of the three forces.



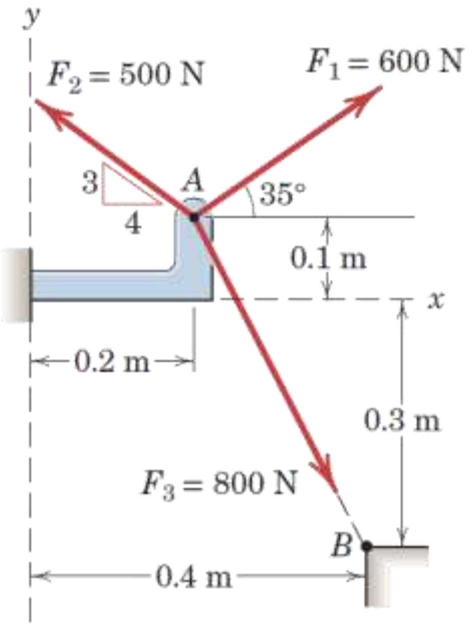
Article 2/3 – Sample Problem 2/1 (2 of 3)

- Scalar Components of $\mathbf{F}_1$

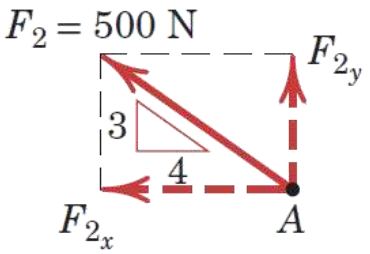


$F_{1_x} = 600 \cos 35° = 491$ N *Ans.*

$F_{1_y} = 600 \sin 35° = 344$ N *Ans.*



- Scalar Components of $\mathbf{F}_2$

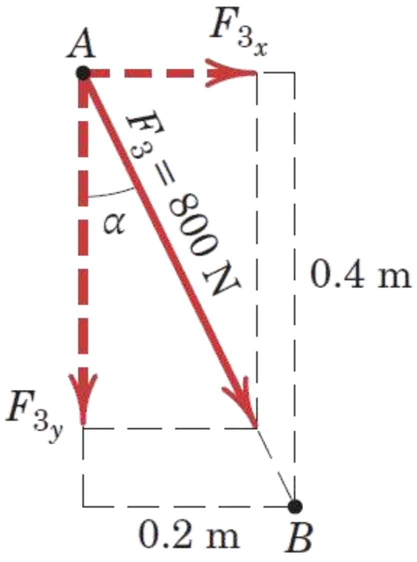


$F_{2_x} = -500(\frac{4}{5}) = -400$ N *Ans.*

$F_{2_y} = 500(\frac{3}{5}) = 300$ N *Ans.*

Article 2/3 – Sample Problem 2/1 (3 of 3)

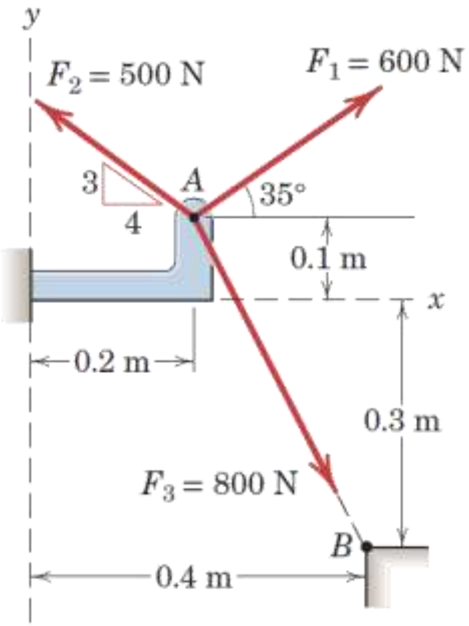
- Scalar Components of $\mathbf{F}_3$



$$\alpha = \tan^{-1}\left[\frac{0.2}{0.4}\right] = 26.6^\circ$$

$$F_{3_x} = F_3 \sin\alpha = 800 \sin 26.6^\circ = 358 \text{ N} \quad ① \qquad \textit{Ans.}$$

$$F_{3_y} = -F_3 \cos\alpha = -800 \cos 26.6^\circ = -716 \text{ N} \qquad \textit{Ans.}$$



- Alternative Calculation

$$\mathbf{F}_3 = F_3\mathbf{n}_{AB} = F_3\frac{\overrightarrow{AB}}{\overline{AB}} = 800\left[\frac{0.2\mathbf{i} - 0.4\mathbf{j}}{\sqrt{(0.2)^2 + (-0.4)^2}}\right]$$

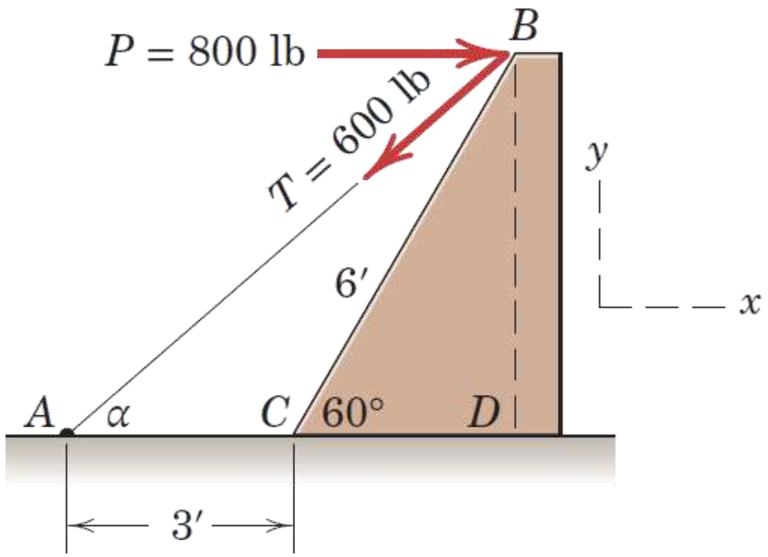
$$= 800\,[0.447\mathbf{i} - 0.894\mathbf{j}]$$

$$= 358\mathbf{i} - 716\mathbf{j} \text{ N}$$

Article 2/3 – Sample Problem 2/2 (1 of 4)

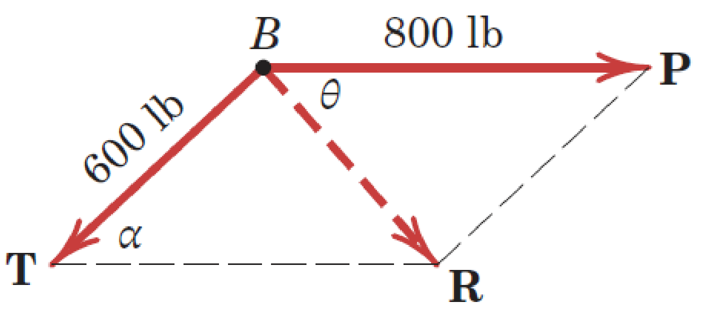
- Problem Statement

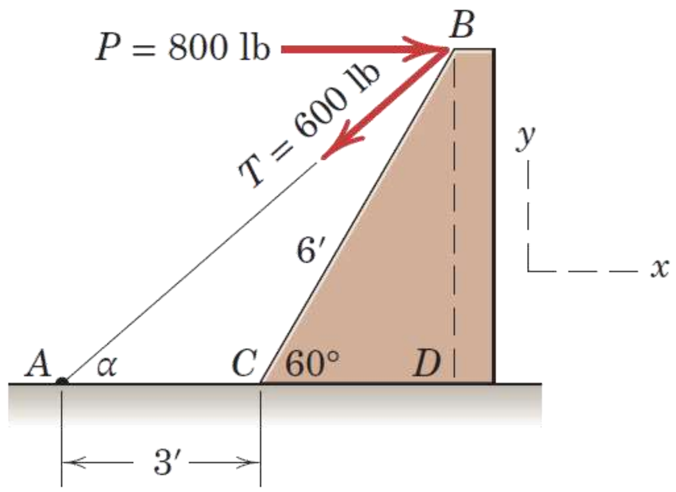
  Combine the two forces **P** and **T**, which act on the fixed structure at *B*, into a single equivalent force **R**.



Article 2/3 – Sample Problem 2/2 (2 of 4)

- Graphical Solution (Scaled Drawing)





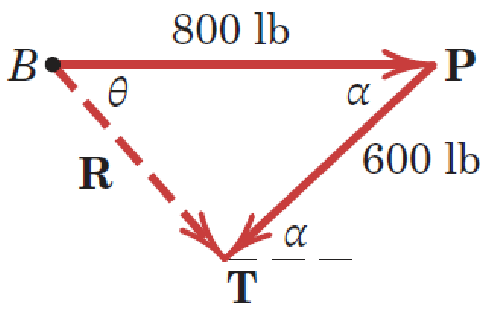
$$\tan\alpha = \frac{\overline{BD}}{\overline{AD}} = \frac{6\sin 60°}{3 + 6\cos 60°} = 0.866 \qquad \alpha = 40.9°$$

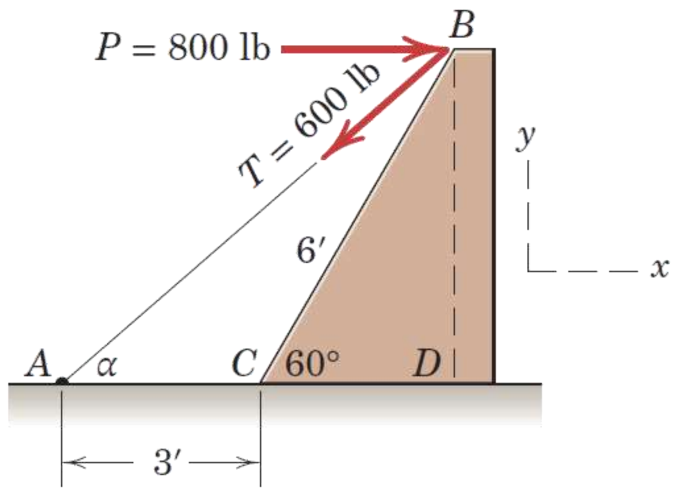
Measurement of the length R and direction θ of the resultant force $\mathbf{R}$ yields the approximate results

$$R = 525\text{ lb} \qquad \theta = 49° \qquad \textit{Ans.}$$

Article 2/3 – Sample Problem 2/2 (3 of 4)

- Geometric Solution





$$R^2 = (600)^2 + (800)^2 - 2(600)(800)\cos 40.9° = 274{,}300$$

$$R = 524 \text{ lb} \qquad\qquad \textit{Ans.}$$

From the law of sines, we may determine the angle θ which orients **R**. Thus,

$$\frac{600}{\sin\theta} = \frac{524}{\sin 40.9°} \qquad \sin\theta = 0.750 \qquad \theta = 48.6° \qquad \textit{Ans.}$$

Article 2/3 – Sample Problem 2/2 (4 of 4)

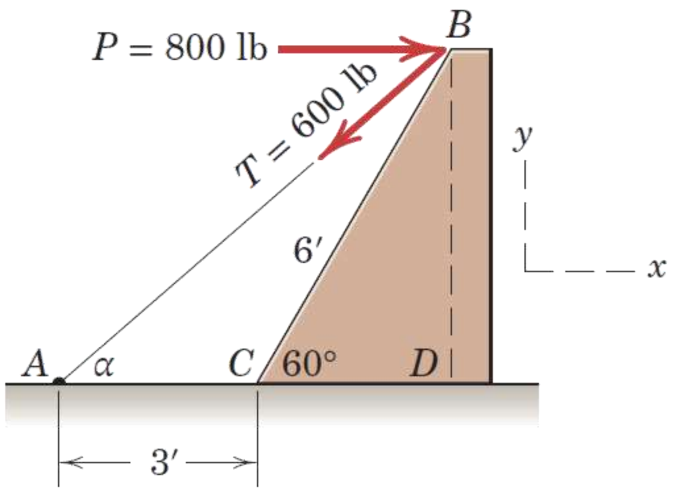
- Algebraic Solution

$$R_x = \Sigma F_x = 800 - 600 \cos 40.9° = 346 \text{ lb}$$

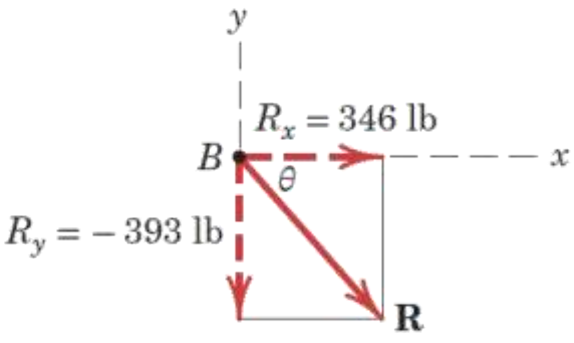
$$R_y = \Sigma F_y = -600 \sin 40.9° = -393 \text{ lb}$$

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(346)^2 + (-393)^2} = 524 \text{ lb} \qquad \textit{Ans.}$$

$$\theta = \tan^{-1}\frac{|R_y|}{|R_x|} = \tan^{-1}\frac{393}{346} = 48.6° \qquad \textit{Ans.}$$



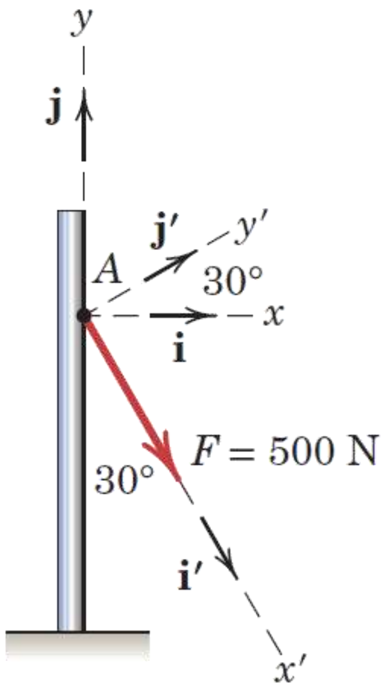
- Vector Representation



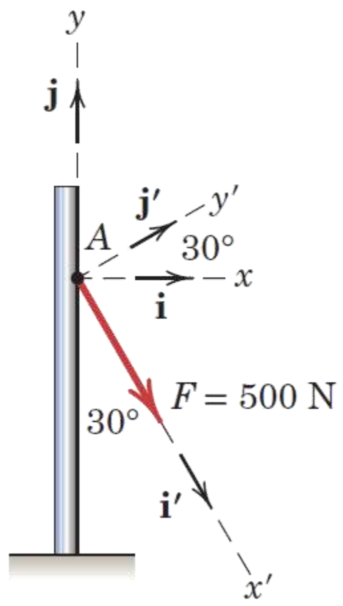
Article 2/3 – Sample Problem 2/3 (1 of 3)

- Problem Statement

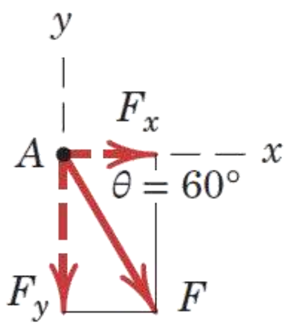
  The 500-N force **F** is applied to the vertical pole as shown. (1) Write **F** in terms of the unit vectors **i** and **j** and identify both its vector and scalar components. (2) Determine the scalar components of the force vector **F** along the x'- and y'-axes. (3) Determine the scalar components of **F** along the x- and y'-axes.



Article 2/3 – Sample Problem 2/3 (2 of 3)



- Part 1 Solution



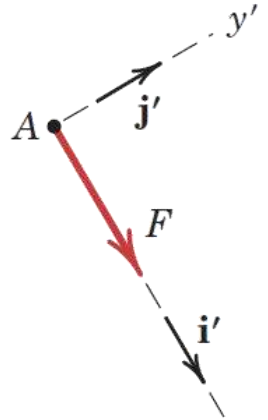
$$\mathbf{F} = (F\cos\theta)\mathbf{i} - (F\sin\theta)\mathbf{j}$$

$$= (500\cos 60°)\mathbf{i} - (500\sin 60°)\mathbf{j}$$

$$= (250\mathbf{i} - 433\mathbf{j})\ \text{N} \qquad \textit{Ans.}$$

The scalar components are $F_x = 250$ N and $F_y = -433$ N. The vector components are $\mathbf{F}_x = 250\mathbf{i}$ N and $\mathbf{F}_y = -433\mathbf{j}$ N.

- Part 2 Solution

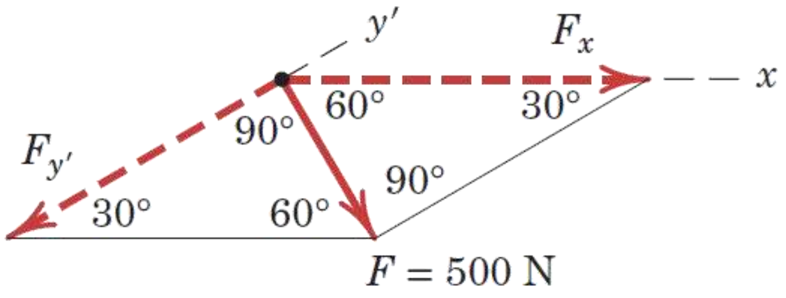


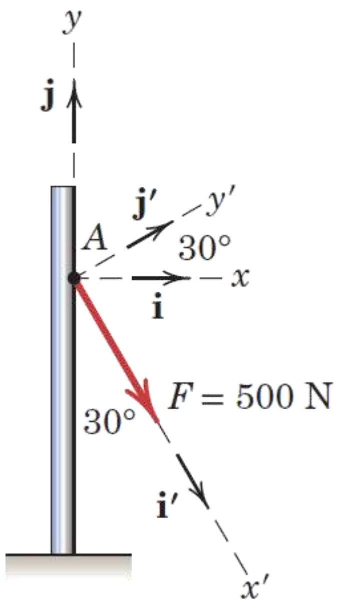
(b)

$$F_{x'} = 500\ \text{N} \qquad F_{y'} = 0 \qquad \textit{Ans.}$$

Article 2/3 – Sample Problem 2/3 (3 of 3)

- Part 3 Solution





$$\frac{|F_x|}{\sin 90°} = \frac{500}{\sin 30°} \qquad |F_x| = 1000 \text{ N} \quad ①$$

$$\frac{|F_{y'}|}{\sin 60°} = \frac{500}{\sin 30°} \qquad |F_{y'}| = 866 \text{ N}$$

The required scalar components are then

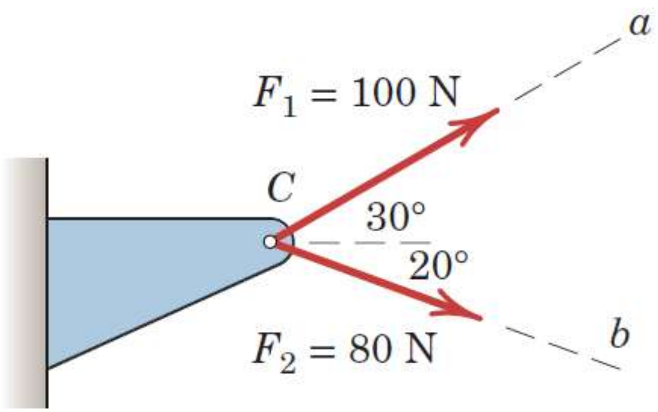
$$F_x = 1000 \text{ N} \qquad F_{y'} = -866 \text{ N} \qquad \textit{Ans.}$$

① Obtain F_x and $F_{y'}$ graphically and compare your results with the calculated values.

Article 2/3 – Sample Problem 2/4 (1 of 2)

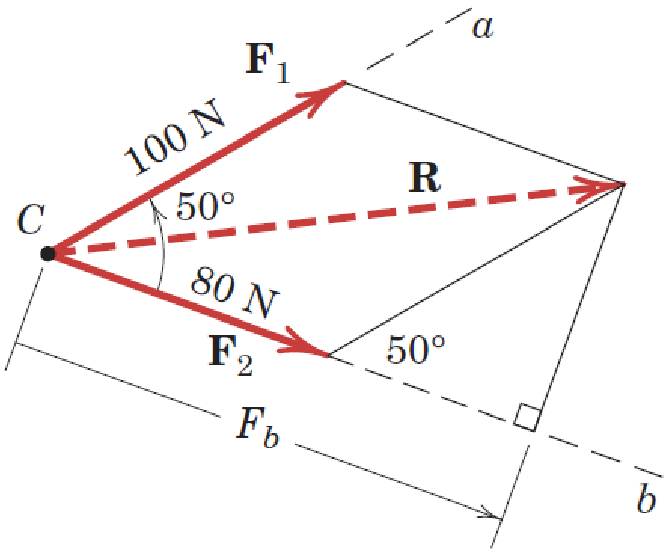
- Problem Statement

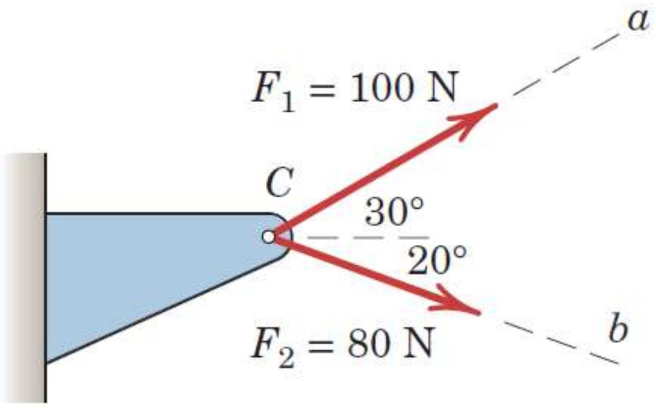
  Forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on the bracket as shown. Determine the projection F_b of their resultant $\mathbf{R}$ onto the b-axis.



Article 2/3 – Sample Problem 2/4 (1 of 2)

- Solution





$$R^2 = (80)^2 + (100)^2 - 2(80)(100)\cos 130° \qquad R = 163.4 \text{ N}$$

The figure also shows the orthogonal projection $\mathbf{F}_b$ of $\mathbf{R}$ onto the b-axis. Its length is

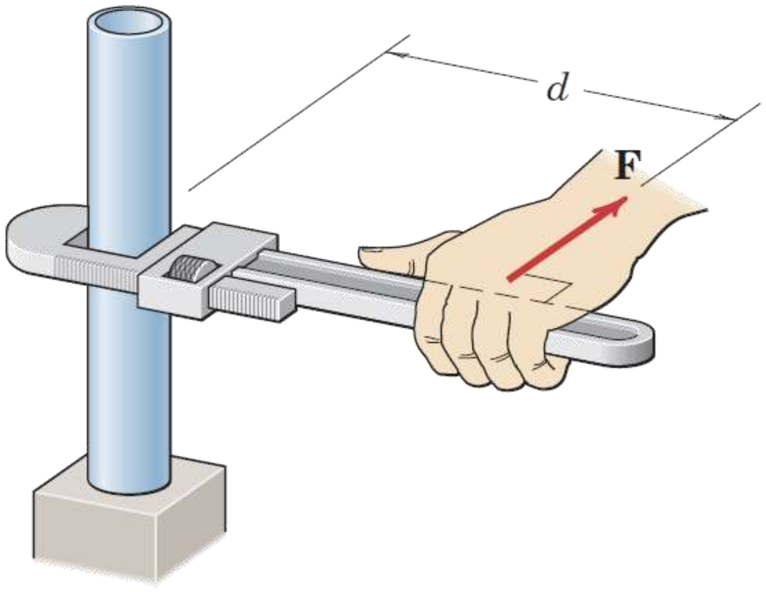
$$F_b = 80 + 100 \cos 50° = 144.3 \text{ N} \qquad \textit{Ans.}$$

Article 2/4 Moment

- A moment is the tendency of a force to rotate a body about an axis.

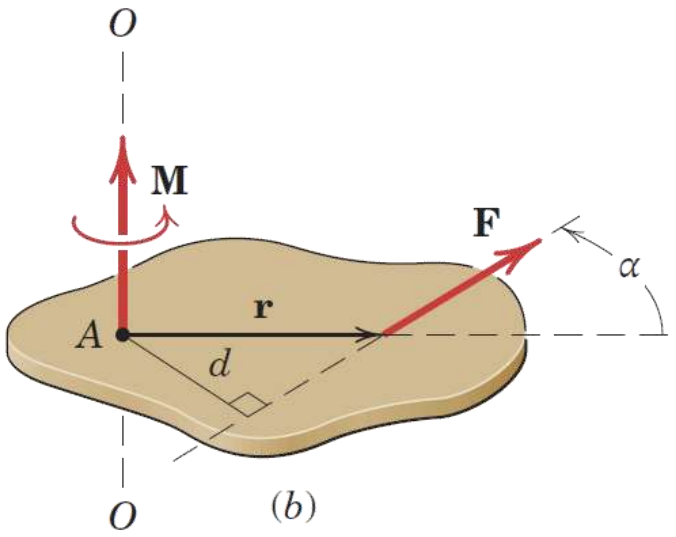
- Illustration: Pipe Wrench

- Things to Note
  - Direction and Orientation of the Force
  - Axis of Rotation
  - Direction of Rotation
  - Effective Length, *d*



Article 2/4 – Moment about a Point (1 of 3)

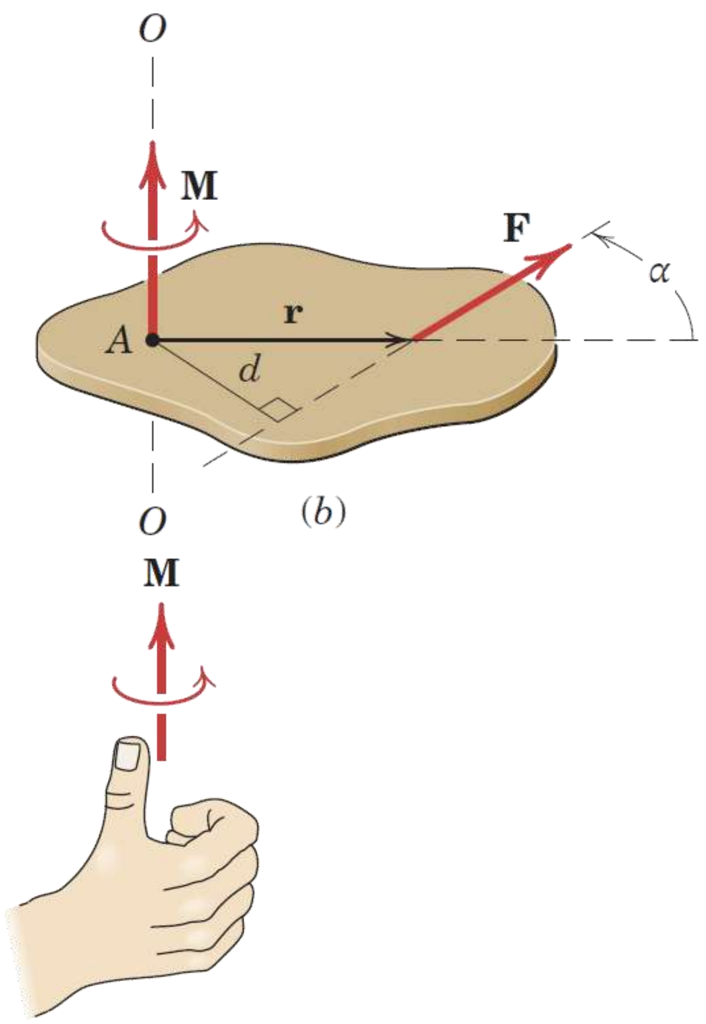
- Scalar Development
  - Moment Arm, *d*
  - Moment Vector, **M**
  - Axis of Rotation, *O*-*O*
  - Moment Magnitude, $M = Fd$
  - Direction of Rotation
  - Units



(b)

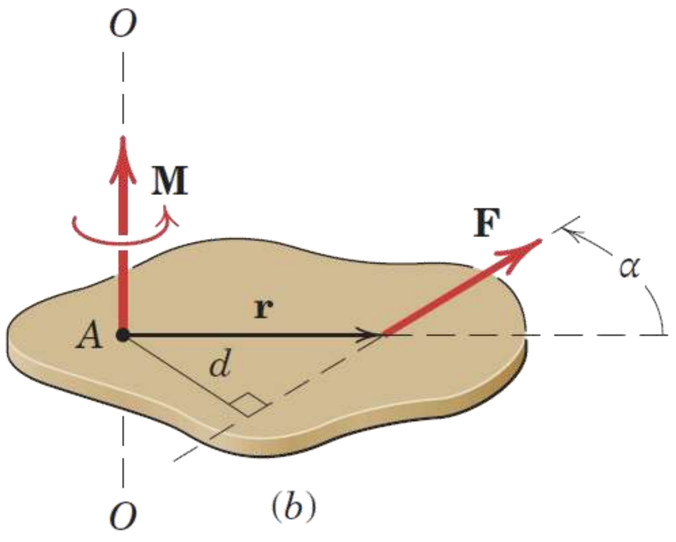
Article 2/4 – Moment about a Point (2 of 3)

- The Right-Hand Rule
  1. Position your right hand such that your fingers point in the same direction as the force.
  2. Orient your hand such that the point you are computing the moment about is on the same side as your palm. From the figure at right, your hand is positioned such that the moment arm *d* intersects the middle of your palm.
  3. Close your fingers to make a fist and extend your thumb straight up. From the figure at right, imagine closing your fist around line *O*-*O*, and your thumb would point in the direction of the moment vector. Curling your fingers about this line would represent the rotation of the moment about the axis.



Article 2/4 – Moment about a Point (3 of 3)

- Planar View
- Sign Conventions
  - Counterclockwise, CCW
  - Clockwise, CW
  - User-Defined
- Two-Dimensional Representation



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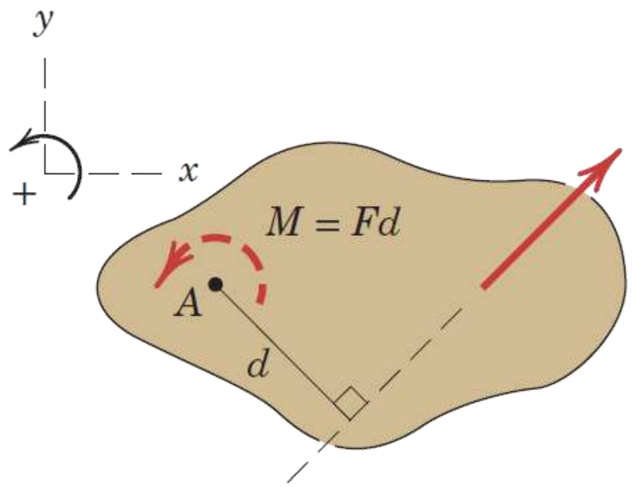
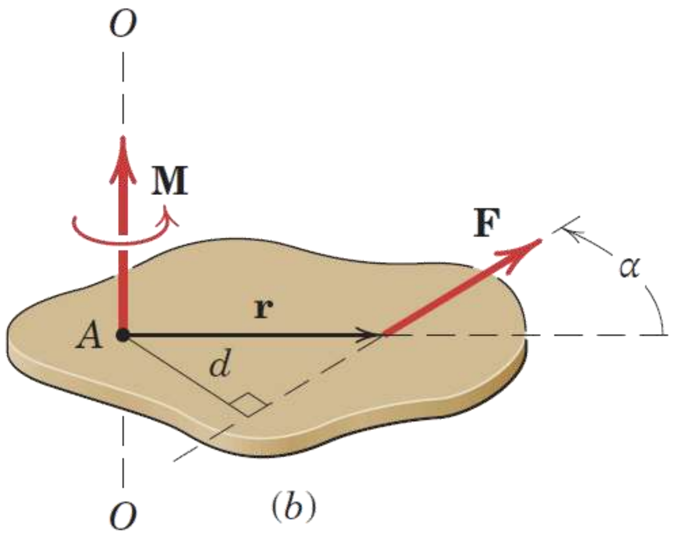


Article 2/4 – The Cross Product

- Vector Expression for Moments
  - Position Vector, **r**
  - Moment Vector, $\mathbf{M} = \mathbf{r} \times \mathbf{F}$
- Advantages of the Cross Product
- Disadvantages of the Cross Product



Article 2/4 – Varignon's Theorem (1 of 2)

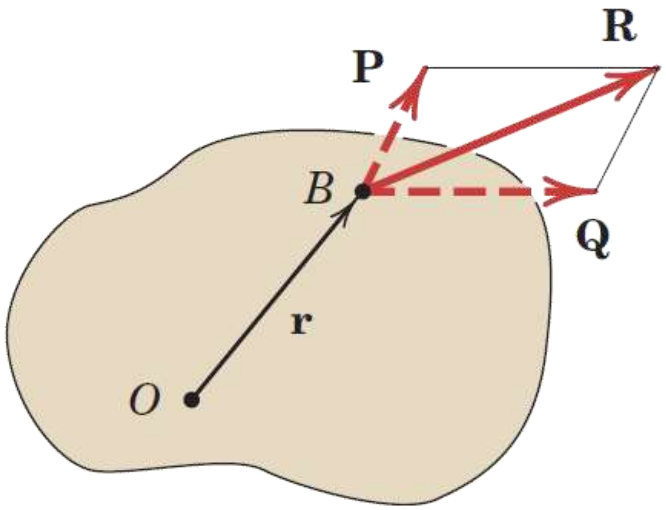
- The Theorem Stated

  The moment of a force about any point is equal to the sum of the moments of the components of the force about the same point.

- The Theorem Illustrated – Vectors

$$\mathbf{R} = \mathbf{P} + \mathbf{Q}$$

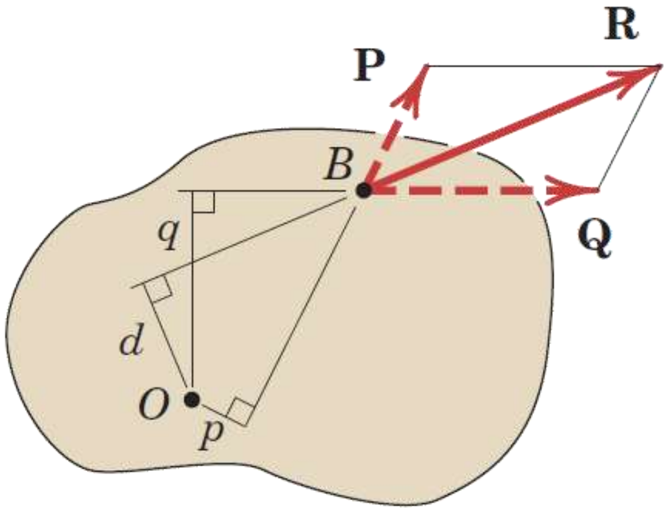
$$\mathbf{M}_O = \mathbf{r} \times \mathbf{R} = \mathbf{r} \times (\mathbf{P} + \mathbf{Q}) = \mathbf{r} \times \mathbf{P} + \mathbf{r} \times \mathbf{Q}$$



Article 2/4 – Varignon’s Theorem (1 of 2)

- The Theorem Illustrated – Scalars

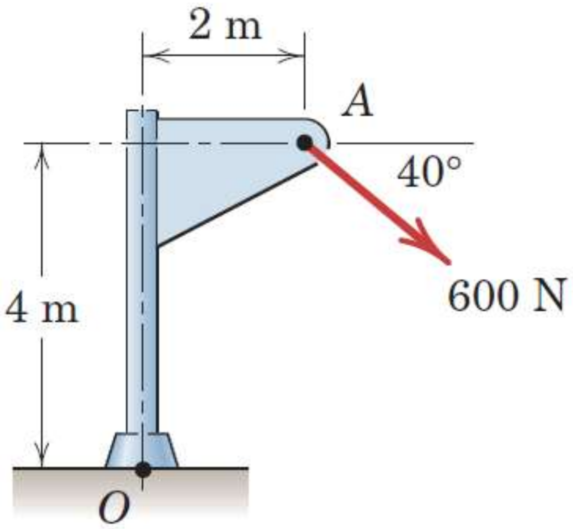
$M_O = Rd = -pP + qQ$ (Assumes CW +)



Article 2/4 – Sample Problem 2/5 (1 of 5)

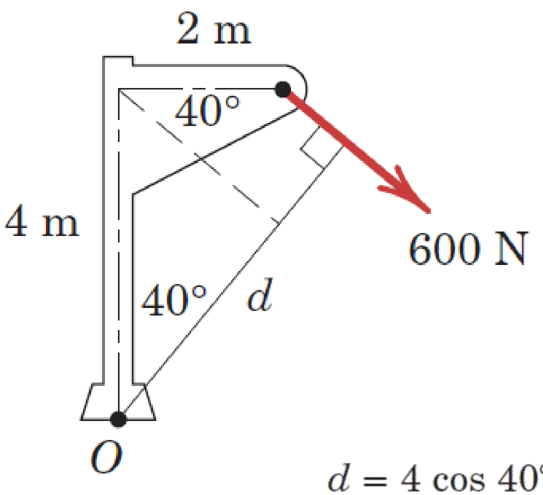
- Problem Statement

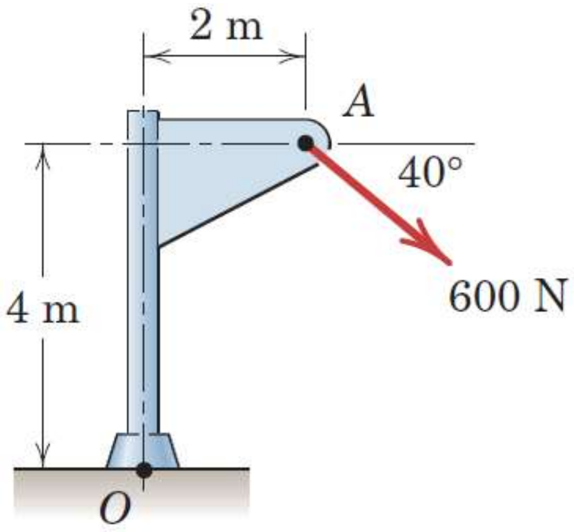
  Calculate the magnitude of the moment about the base point *O* of the 600-N force in five different ways.



Article 2/4 – Sample Problem 2/5 (2 of 5)

- Method 1: Use the Moment Arm (CW is +)





$$d = 4 \cos 40° + 2 \sin 40° = 4.35 \text{ m}$$

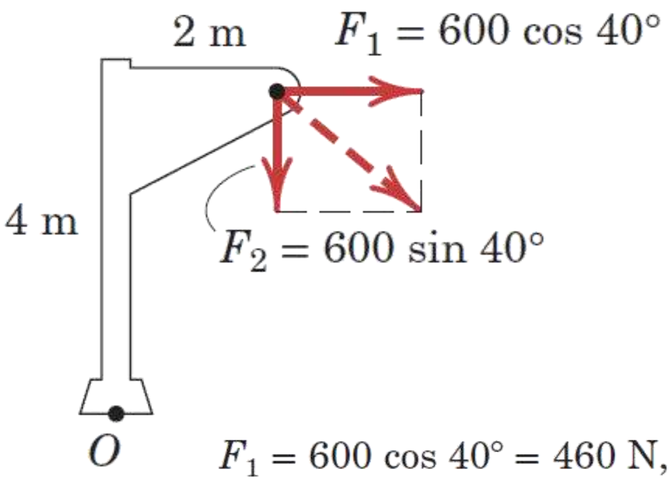
By $M = Fd$ the moment is clockwise and has the magnitude ①

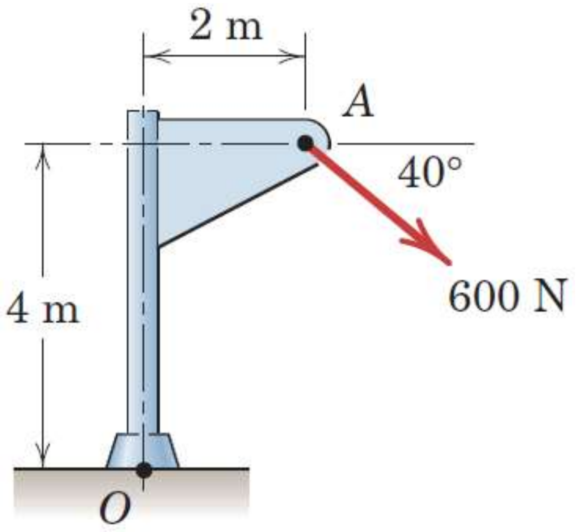
$$M_O = 600(4.35) = 2610 \text{ N}\cdot\text{m} \qquad \textit{Ans.}$$

① The required geometry here and in similar problems should not cause difficulty if the sketch is carefully drawn.

Article 2/4 – Sample Problem 2/5 (3 of 5)

- Method 2: Use Components at *A* (CW is +)





$F_1 = 600 \cos 40° = 460$ N, $F_2 = 600 \sin 40° = 386$ N

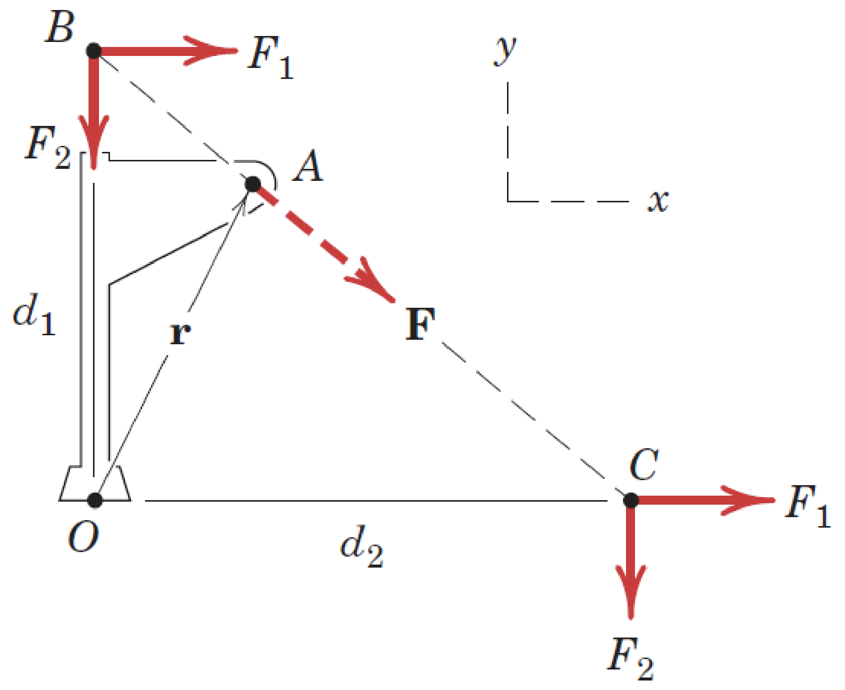
By Varignon's theorem, the moment becomes

$$M_O = 460(4) + 386(2) = 2610 \text{ N}\cdot\text{m} \quad ② \qquad \textit{Ans.}$$

② This procedure is frequently the shortest approach.

Article 2/4 – Sample Problem 2/5 (4 of 5)

- Methods 3 and 4: Alternative Moment Arms

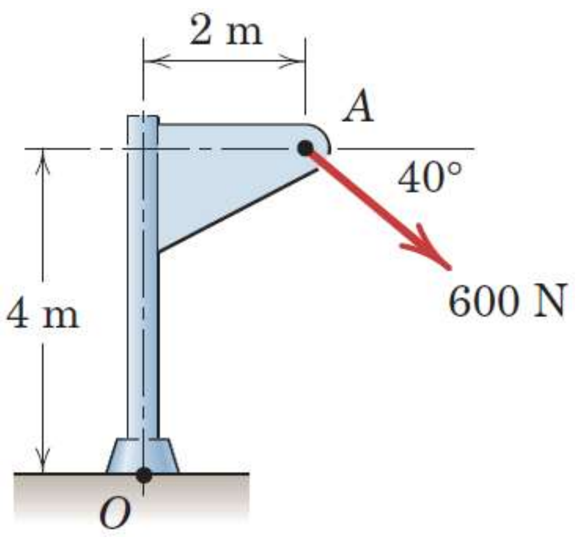


$$d_1 = 4 + 2 \tan 40° = 5.68 \text{ m}$$

$$M_O = 460(5.68) = 2610 \text{ N·m} \qquad \textit{Ans.}$$

$$d_2 = 2 + 4 \cot 40° = 6.77 \text{ m}$$

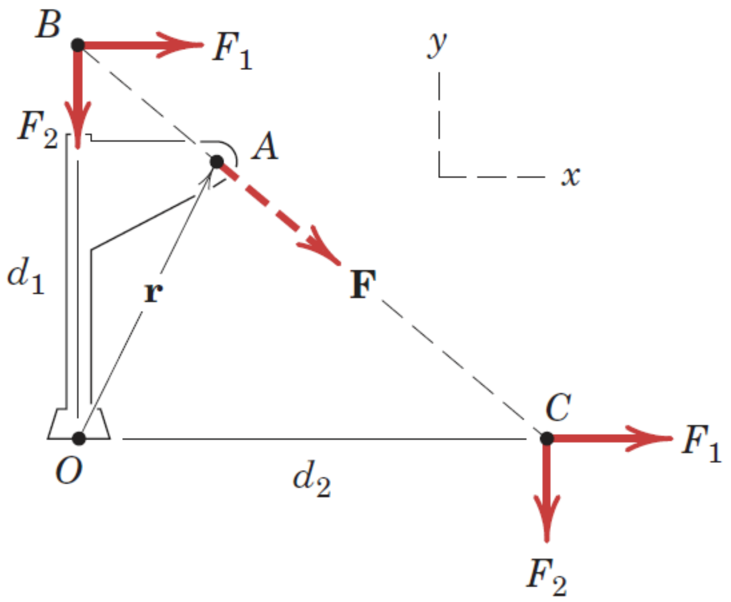
$$M_O = 386(6.77) = 2610 \text{ N·m} \qquad \textit{Ans.}$$

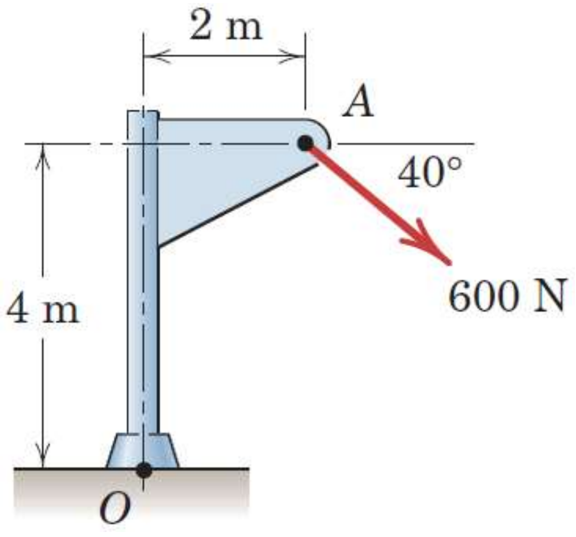


③ The fact that points B and C are not on the body proper should not cause concern, as the mathematical calculation of the moment of a force does not require that the force be on the body.

Article 2/4 – Sample Problem 2/5 (5 of 5)

- Method 5: Vector Approach





$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = (2\mathbf{i} + 4\mathbf{j}) \times 600(\mathbf{i} \cos 40° - \mathbf{j} \sin 40°) \quad ④$$

$$= -2610\mathbf{k}\ \mathrm{N \cdot m}$$

④ Alternative choices for the position vector $\mathbf{r}$ are $\mathbf{r} = d_1\mathbf{j} = 5.68\mathbf{j}$ m and $\mathbf{r} = d_2\mathbf{i} = 6.77\mathbf{i}$ m.

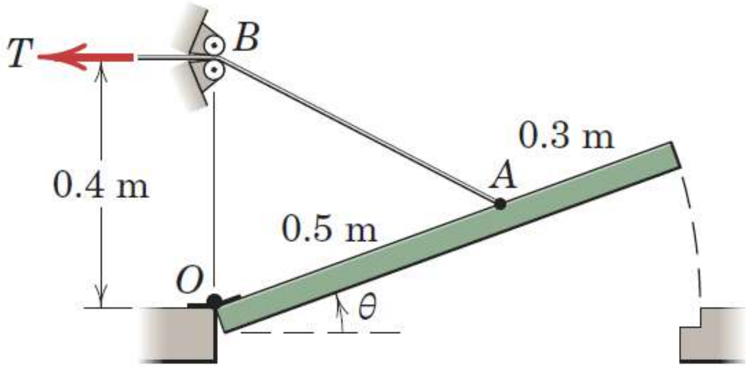
The minus sign indicates that the vector is in the negative z-direction. The magnitude of the vector expression is

$$M_O = 2610\ \mathrm{N \cdot m} \qquad \textit{Ans.}$$

Article 2/4 – Sample Problem 2/6 (1 of 4)

- Problem Statement

  The trap door *OA* is raised by the cable *AB*, which passes over the small frictionless guide pulleys at *B*. The tension everywhere in the cable is *T*, and this tension applied at *A* causes a moment M_O about the hinge at *O*. Plot the quantity M_O / T as a function of the door elevation angle θ over the range $0 \leq \theta \leq 90°$ and note minimum and maximum values. What is the physical significance of this ratio?



Article 2/4 – Sample Problem 2/6 (2 of 4)

- Tension Vector

$$\mathbf{n}_{AB} = \frac{\mathbf{r}_{AB}}{r_{AB}} = \frac{\mathbf{r}_{OB} - \mathbf{r}_{OA}}{r_{AB}} \quad ①$$

Using the x-y coordinates of our figure, we can write

$$\mathbf{r}_{OB} = 0.4\mathbf{j}\text{ m} \qquad \mathbf{r}_{OA} = 0.5(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})\text{ m} \quad ②$$

$$\mathbf{r}_{AB} = \mathbf{r}_{OB} - \mathbf{r}_{OA} = 0.4\mathbf{j} - (0.5)(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

$$= -0.5\cos\theta\mathbf{i} + (0.4 - 0.5\sin\theta)\mathbf{j}\text{ m}$$

So

$$r_{AB} = \sqrt{(0.5\cos\theta)^2 + (0.4 - 0.5\sin\theta)^2}$$

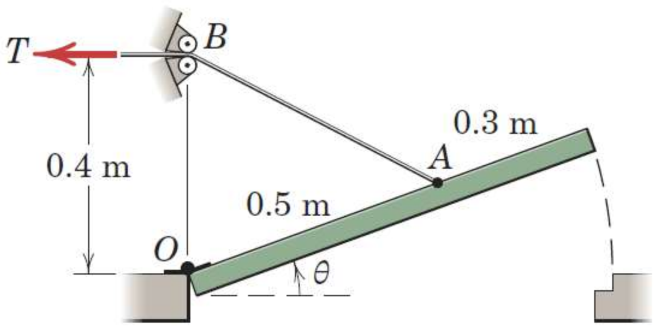
$$= \sqrt{0.41 - 0.4\sin\theta}\text{ m}$$

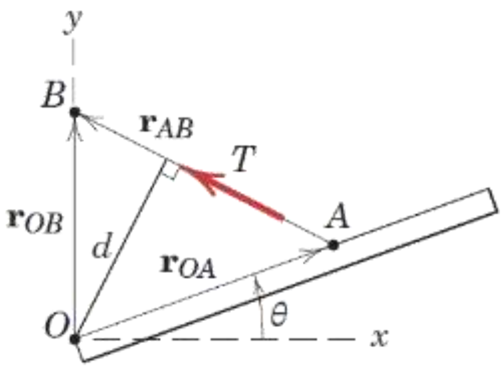
The desired unit vector is

$$\mathbf{n}_{AB} = \frac{\mathbf{r}_{AB}}{r_{AB}} = \frac{-0.5\cos\theta\mathbf{i} + (0.4 - 0.5\sin\theta)\mathbf{j}}{\sqrt{0.41 - 0.4\sin\theta}}$$

Our tension vector can now be written as

$$\mathbf{T} = T\mathbf{n}_{AB} = T\left[\frac{-0.5\cos\theta\mathbf{i} + (0.4 - 0.5\sin\theta)\mathbf{j}}{\sqrt{0.41 - 0.4\sin\theta}}\right]$$





① Recall that any unit vector can be written as a vector divided by its magnitude. In this case the vector in the numerator is a position vector.

② Recall that any vector may be written as a magnitude times an "aiming" unit vector.

Article 2/4 – Sample Problem 2/6 (3 of 4)

- Moment Vector

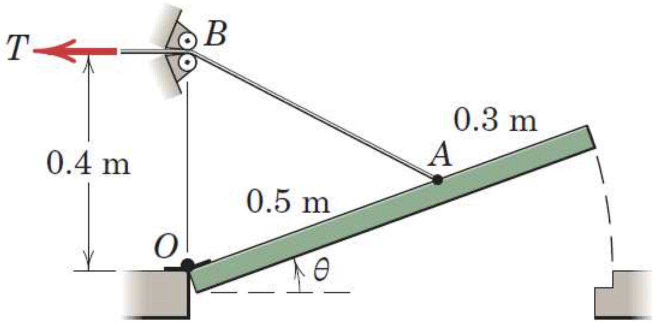
The moment of $\mathbf{T}$ about point O, as a vector, is $\mathbf{M}_O = \mathbf{r}_{OB} \times \mathbf{T}$, where $\mathbf{r}_{OB} = 0.4\mathbf{j}$ m, or ③

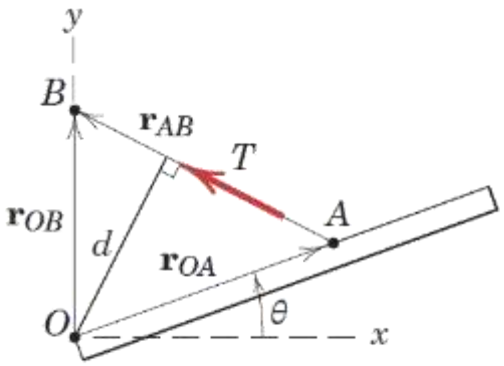
$$\mathbf{M}_O = 0.4\mathbf{j} \times T\left[\frac{-0.5\cos\theta\mathbf{i} + (0.4 - 0.5\sin\theta)\mathbf{j}}{\sqrt{0.41 - 0.4\sin\theta}}\right]$$

$$= \frac{0.2T\cos\theta}{\sqrt{0.41 - 0.4\sin\theta}}\mathbf{k}$$

The magnitude of $\mathbf{M}_O$ is

$$M_O = \frac{0.2T\cos\theta}{\sqrt{0.41 - 0.4\sin\theta}}$$





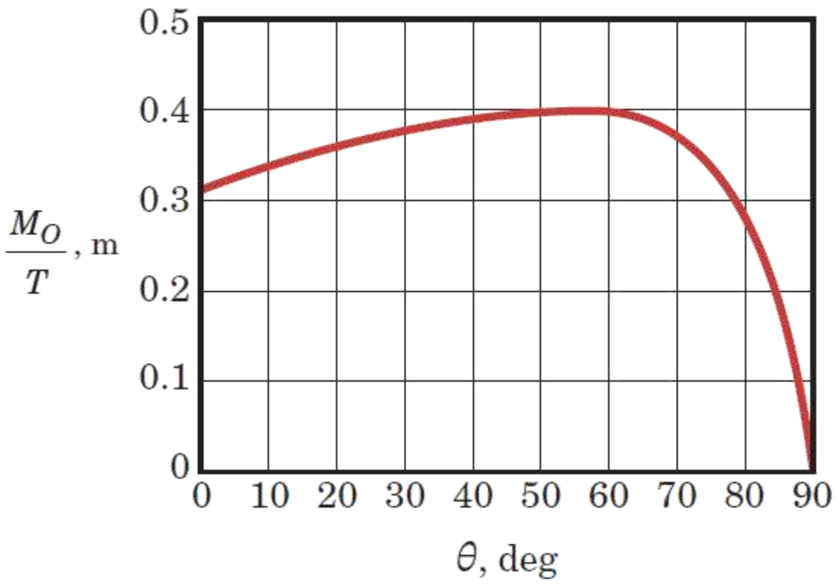
③ In the expression $\mathbf{M} = \mathbf{r} \times \mathbf{F}$, the position vector $\mathbf{r}$ runs *from* the moment center *to* any point on the line of action of $\mathbf{F}$. Here, $\mathbf{r}_{OB}$ is more convenient than $\mathbf{r}_{OA}$.

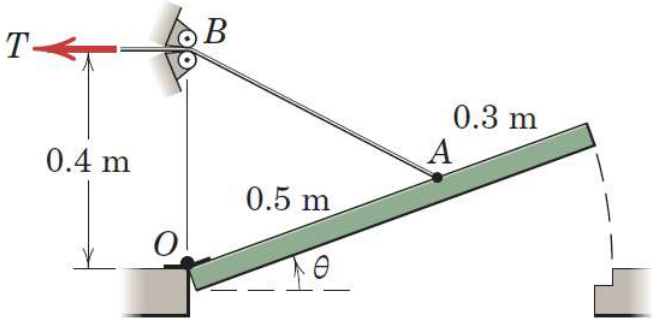
Article 2/4 – Sample Problem 2/6 (4 of 4)

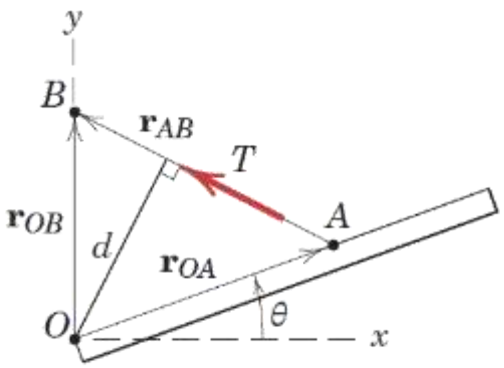
- Desired Expression and Plot

$$\frac{M_O}{T} = \frac{0.2 \cos \theta}{\sqrt{0.41 - 0.4 \sin \theta}} \qquad \textit{Ans.}$$

which is plotted in the accompanying graph. The expression M_O/T is the moment arm d (in meters) which runs from O to the line of action of $\mathbf{T}$. It has a maximum value of 0.4 m at $\theta = 53.1°$ (at which point $\mathbf{T}$ is horizontal) and a minimum value of 0 at $\theta = 90°$ (at which point $\mathbf{T}$ is vertical). The expression is valid even if T varies.







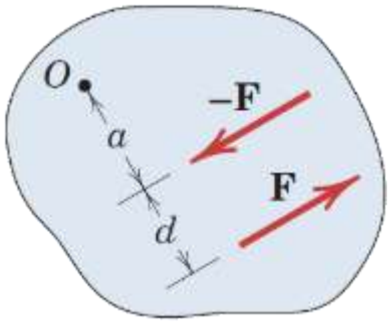
Article 2/5 Couple

- Definition

  The moment produced by two equal, opposite, and noncollinear forces is called a couple.

- Illustration and Derivation (Scalars)

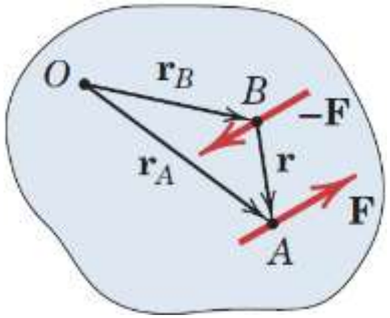
  $$M_O = F(a + d) - Fa = Fd$$



- Irrelevance of the Moment Center *O*

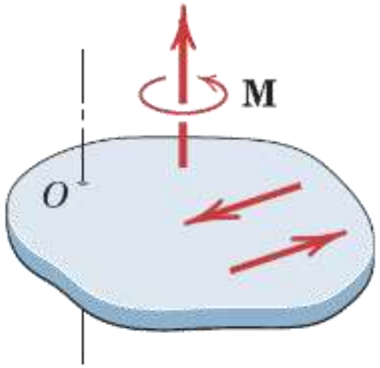
Article 2/5 – Vector Algebra Method

- Illustration and Derivation (Vectors)



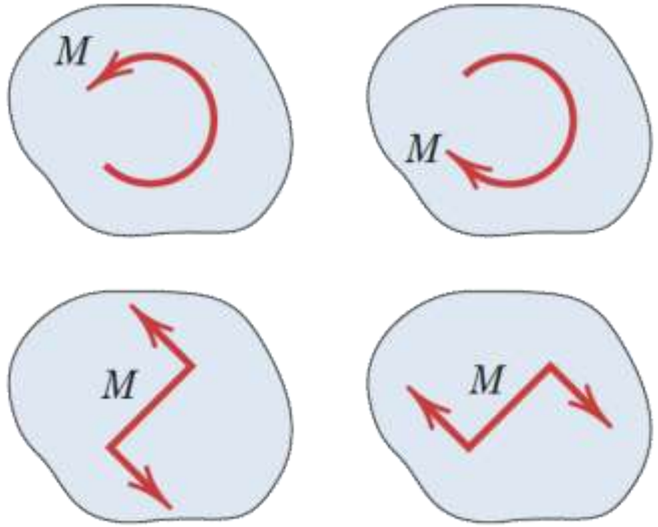
$$\mathbf{M}_O = \mathbf{r}_A \times \mathbf{F} + \mathbf{r}_B \times (-\mathbf{F}) = (\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F} = \mathbf{r} \times \mathbf{F}$$

- The Couple is a Free Vector



Article 2/5 – Alternative Representations of Couples

- Two-Dimensional Representations

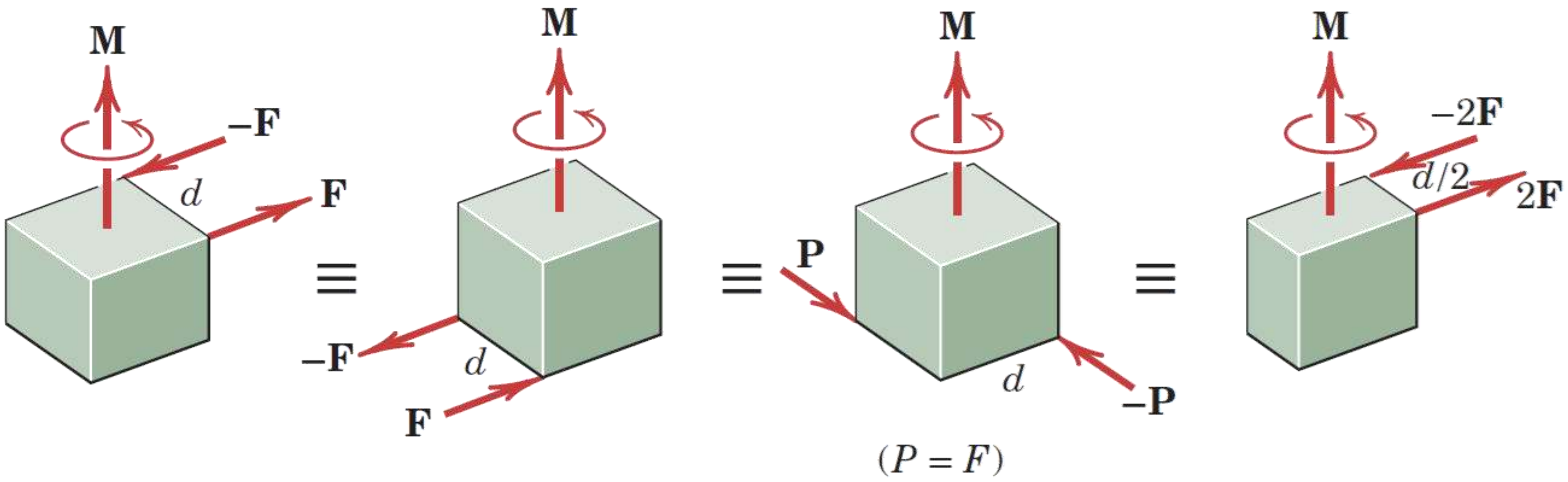


Counterclockwise couple

Clockwise couple

Article 2/5 – Equivalent Couples

- Illustration

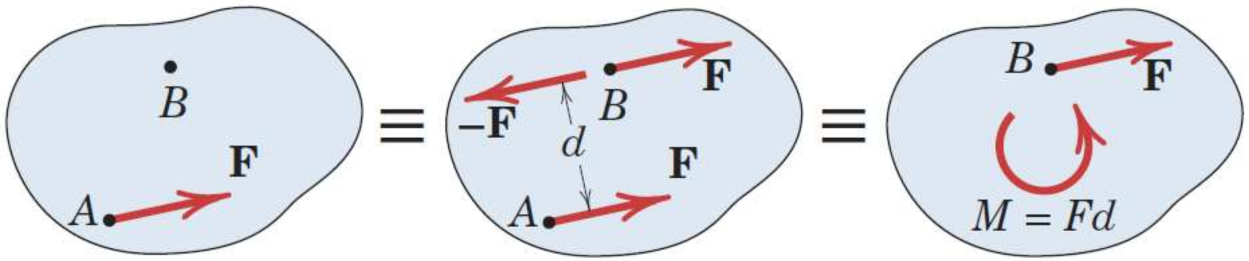


Article 2/5 – Force-Couple Systems (1 of 2)

- Principle of the Force-Couple System

  Any force which acts at a particular location on a body can be replaced by an equivalent force which acts at a different location and a couple.

- Illustration of the Process



Article 2/5 – Force-Couple Systems (2 of 2)

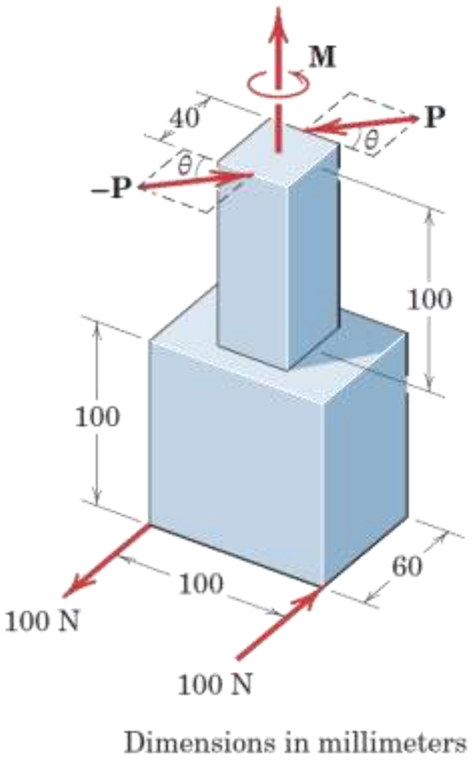
- Steps to Create a Force-Couple System
  1. Write the force as a vector.
  2. Compute the moment or couple which the force creates about the point.
  3. Redraw the force acting at the new location.
  4. Sketch the couple acting at the new location.

- Important Reminder
  - The force-couple system has the same effect on the body which the original force had. It is simply a different way to visualize the effect of the force acting at a new location.

Article 2/5 – Sample Problem 2/7 (1 of 2)

- Problem Statement

  The rigid structural member is subjected to a couple consisting of the two 100-N forces. Replace this couple by an equivalent couple consisting of the two forces **P** and −**P**, each of which has a magnitude of 400 N. Determine the proper angle θ.



Article 2/5 – Sample Problem 2/7 (2 of 2)

- Solution

$[M = Fd]$ $\qquad M = 100(0.1) = 10\ \text{N}\cdot\text{m}$

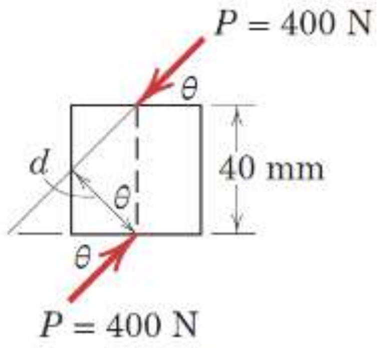
The forces **P** and **–P** produce a counterclockwise couple

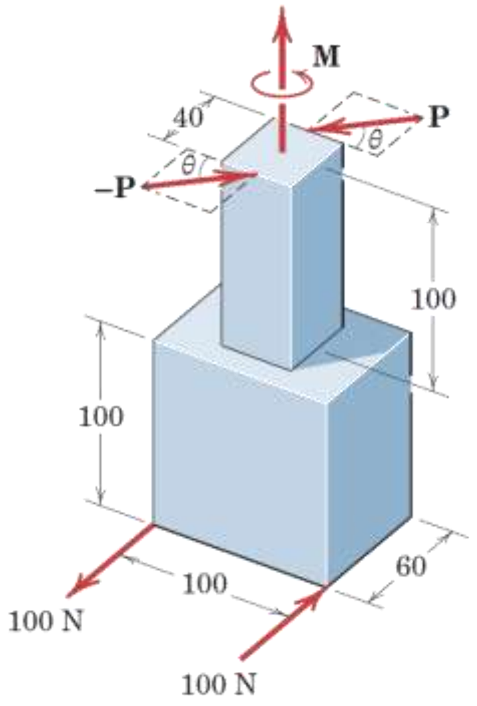
$$M = 400(0.040)\cos\theta$$

Equating the two expressions gives ①

$$10 = (400)(0.040)\cos\theta$$

$$\theta = \cos^{-1}\frac{10}{16} = 51.3^\circ \qquad \textit{Ans.}$$





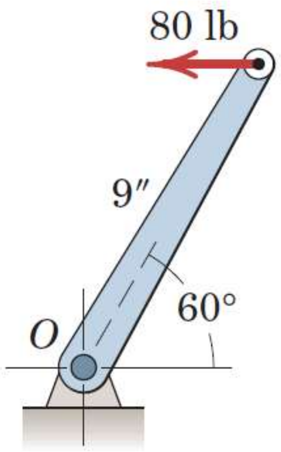
Dimensions in millimeters

① Since the two equal couples are parallel free vectors, the only dimensions which are relevant are those which give the perpendicular distances between the forces of the couples.

Article 2/5 – Sample Problem 2/8 (1 of 2)

- Problem Statement

  Replace the horizontal 80-lb force acting on the lever by an equivalent system consisting of a force at *O* and a couple.



Article 2/5 – Sample Problem 2/8 (2 of 2)

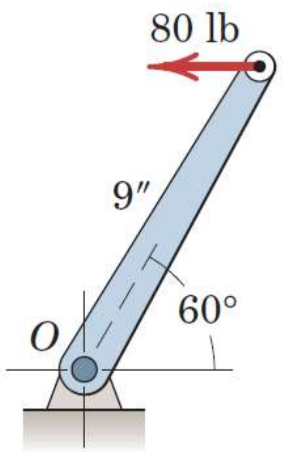
- Solution

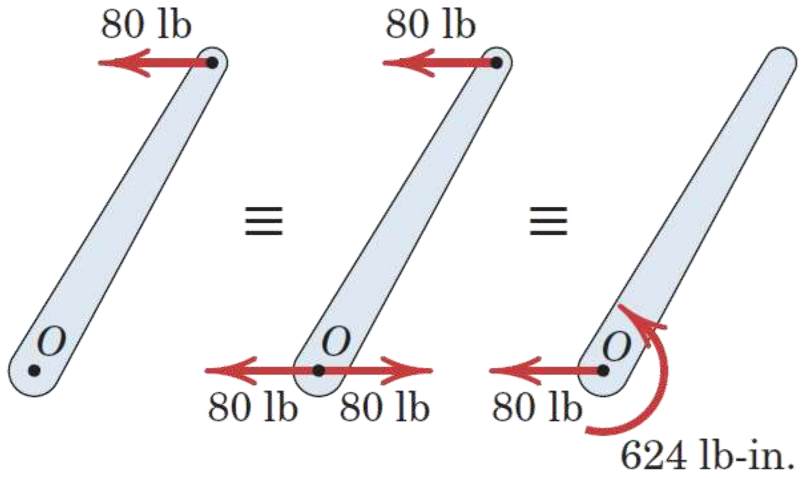
$[M = Fd]$ $\qquad M = 80(9 \sin 60°) = 624$ lb-in. $\qquad$ *Ans.*

Thus, the original force is equivalent to the 80-lb force at O and the 624-lb-in. couple as shown in the third of the three equivalent figures. ①

HELPFUL HINT

① The reverse of this problem is often encountered, namely, the replacement of a force and a couple by a single force. Proceeding in reverse is the same as replacing the couple by two forces, one of which is equal and opposite to the 80-lb force at O. The moment arm to the second force would be $M/F = 624/80 = 7.79$ in., which is 9 sin 60°, thus determining the line of action of the single resultant force of 80 lb.





Article 2/6 Resultants

- Definition

  The *resultant* of a system of forces is the simplest force combination which can replace the original forces without altering the external effect on the rigid body to which the forces are applied.

- Equilibrium Condition

- Nonequilibrium Condition

Article 2/6 – Planar Force System

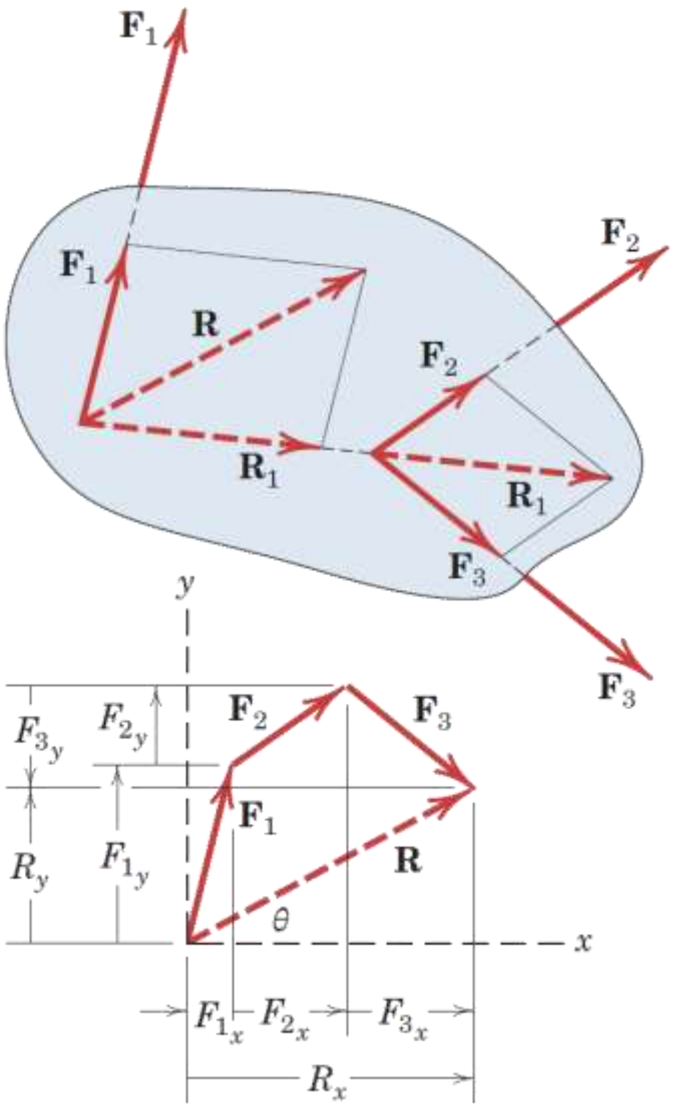
- Illustration

- Equations of Interest

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots = \Sigma\mathbf{F}$$

$$R_x = \Sigma F_x \qquad R_y = \Sigma F_y \qquad R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

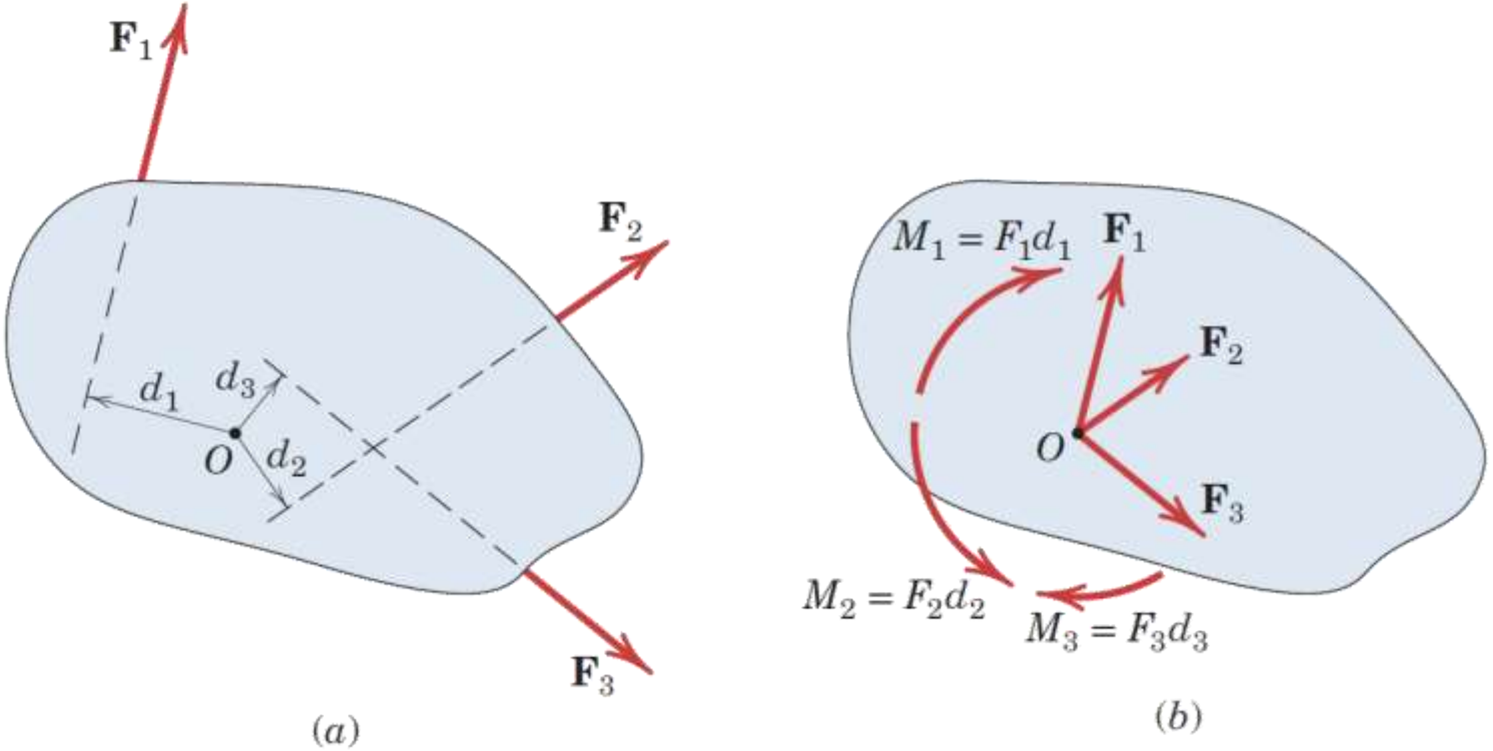
$$\theta = \tan^{-1}\frac{R_y}{R_x} = \tan^{-1}\frac{\Sigma F_y}{\Sigma F_x}$$



Article 2/6 – Algebraic Method (1 of 3)

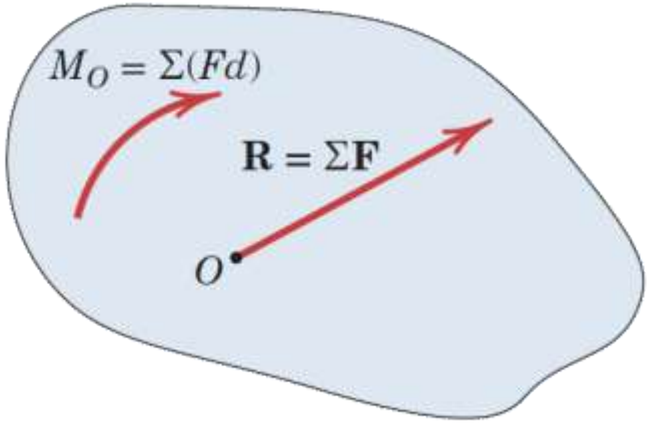
- Finding the Resultant and Line of Action

  1. Choose a convenient reference point and move all forces to that point. This process is depicted for a three-force system in Figs. (*a*) and (*b*) below where M_1, M_2, and M_3 are the couples resulting from the transfer of forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ from their respective original lines of action to lines of action through point O.

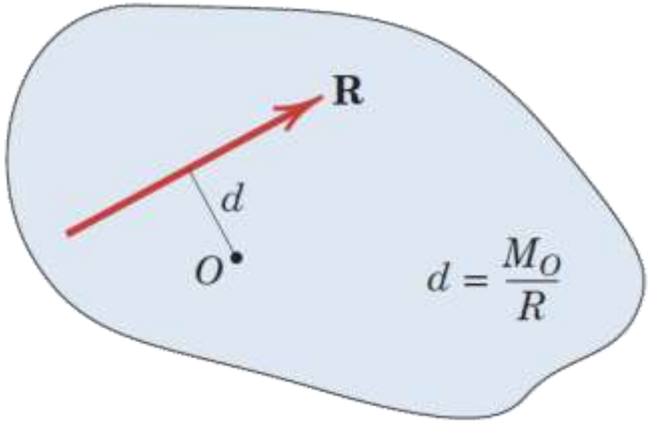


Article 2/6 – Algebraic Method (2 of 3)

- Finding the Resultant and Line of Action (cont.)
  2. Add all forces at O to form the resultant force **R**, and add all couples to form the resultant couple M_O. We now have the single force–couple system, as shown below in Fig. (*c*).
  3. Find the line of action of **R** by requiring **R** to have a moment of M_O about point O. Note that the force system in Fig. (*d*) is equivalent to the initial force system from Fig. (a) and that $\Sigma(Fd)$ in Fig. (*a*) is equal to Rd in Fig. (*d*).



(*c*)



(*d*)

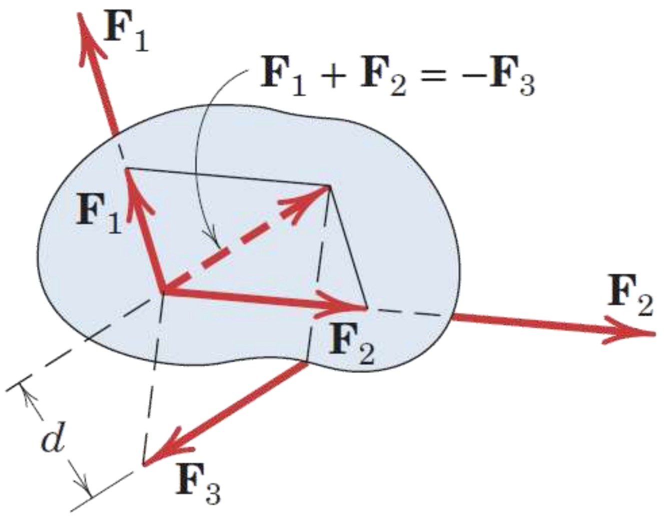
Article 2/6 – Algebraic Method (3 of 3)

- Equations of Interest
- Concurrent Force System
- Parallel Force System
- Zero-Resultant Force System

$$\mathbf{R} = \Sigma\mathbf{F}$$

$$M_O = \Sigma M = \Sigma(Fd)$$

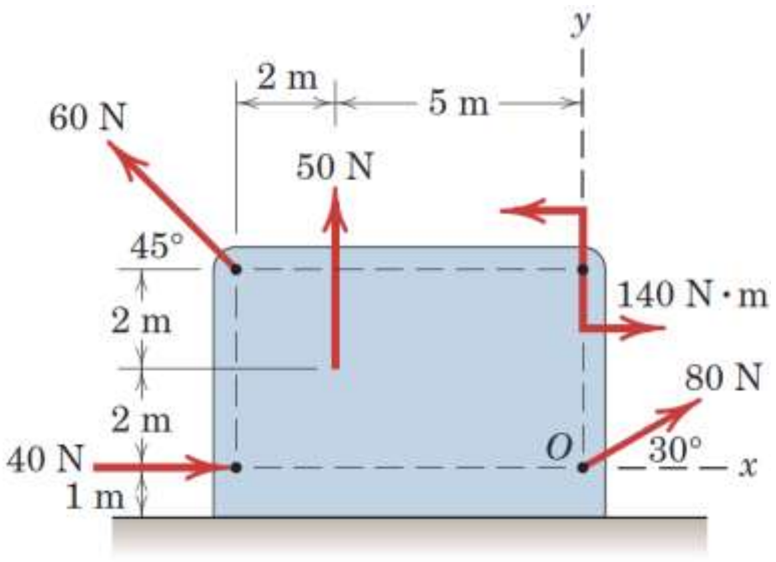
$$Rd = M_O$$



Article 2/6 – Sample Problem 2/9 (1 of 4)

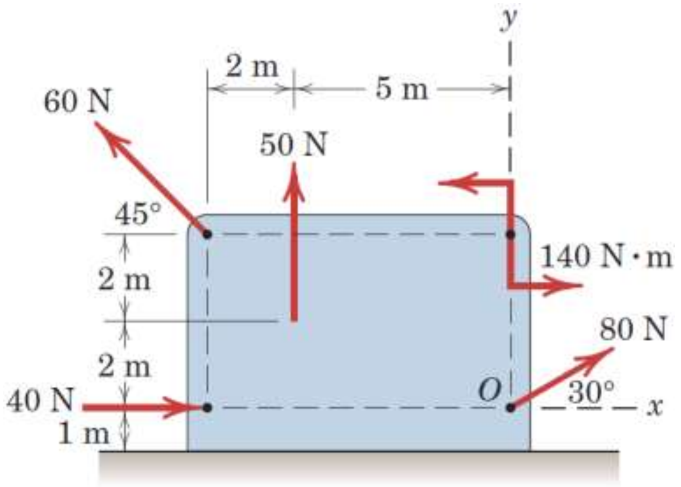
- Problem Statement

  Determine the resultant of the four forces and one couple which act on the plate shown.



Article 2/6 – Sample Problem 2/9 (2 of 4)

- Equivalent Force-Couple System



$[R_x = \Sigma F_x]$ $\quad R_x = 40 + 80 \cos 30° - 60 \cos 45° = 66.9 \text{ N}$

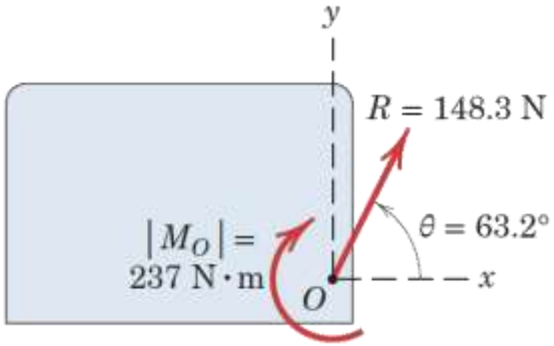
$[R_y = \Sigma F_y]$ $\quad R_y = 50 + 80 \sin 30° + 60 \cos 45° = 132.4 \text{ N}$

$\left[R = \sqrt{R_x{}^2 + R_y{}^2}\right]$ $\quad R = \sqrt{(66.9)^2 + (132.4)^2} = 148.3 \text{ N}$ *Ans.*

$\left[\theta = \tan^{-1} \dfrac{R_y}{R_x}\right]$ $\quad \theta = \tan^{-1} \dfrac{132.4}{66.9} = 63.2°$ *Ans.*

$[M_O = \Sigma(Fd)]$ $\quad M_O = 140 - 50(5) + 60 \cos 45°(4) - 60 \sin 45°(7)$ ①

$$= -237 \text{ N·m}$$



① We note that the choice of point O as a moment center eliminates any moments due to the two forces which pass through O. Had the clockwise sign convention been adopted, M_O would have been +237 N·m, with the plus sign indicating a sense which agrees with the sign convention. Either sign convention, of course, leads to the conclusion of a clockwise moment M_O.

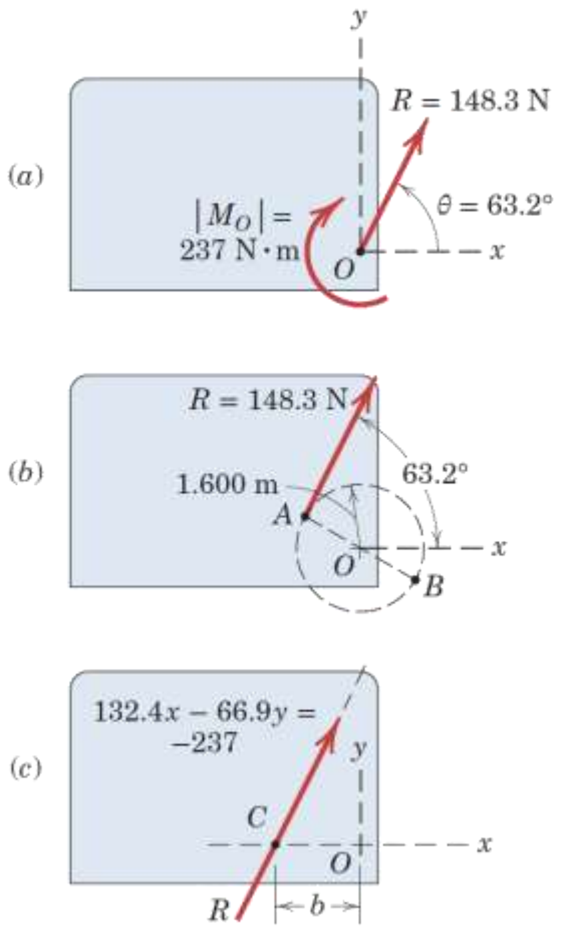
Article 2/6 – Sample Problem 2/9 (3 of 4)

- Line of Action for the Resultant

$[Rd = |M_O|] \qquad 148.3d = 237 \qquad d = 1.600 \text{ m} \qquad \textit{Ans.}$

- Alternative Solution (Point *C* on *x*-axis)

$$R_y b = |M_O| \qquad \text{and} \qquad b = \frac{237}{132.4} = 1.792 \text{ m}$$



Article 2/6 – Sample Problem 2/9 (4 of 4)

- Vector Approach for the Line of Action

$$\mathbf{r} \times \mathbf{R} = \mathbf{M}_O$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ is a position vector running from point O to any point on the line of action of $\mathbf{R}$. Substituting the vector expressions for $\mathbf{r}$, $\mathbf{R}$, and $\mathbf{M}_O$ and carrying out the cross product result in

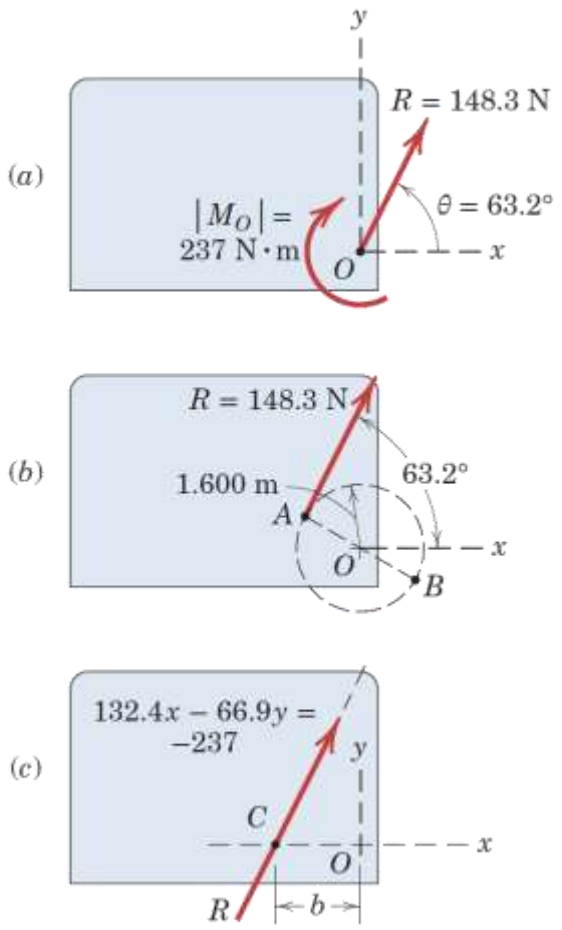
$$(x\mathbf{i} + y\mathbf{j}) \times (66.9\mathbf{i} + 132.4\mathbf{j}) = -237\mathbf{k}$$

$$(132.4x - 66.9y)\mathbf{k} = -237\mathbf{k}$$

Thus, the desired line of action, Fig. c, is given by

$$132.4x - 66.9y = -237$$

By setting $y = 0$, we obtain $x = -1.792$ m, which agrees with our earlier calculation of the distance b. ②

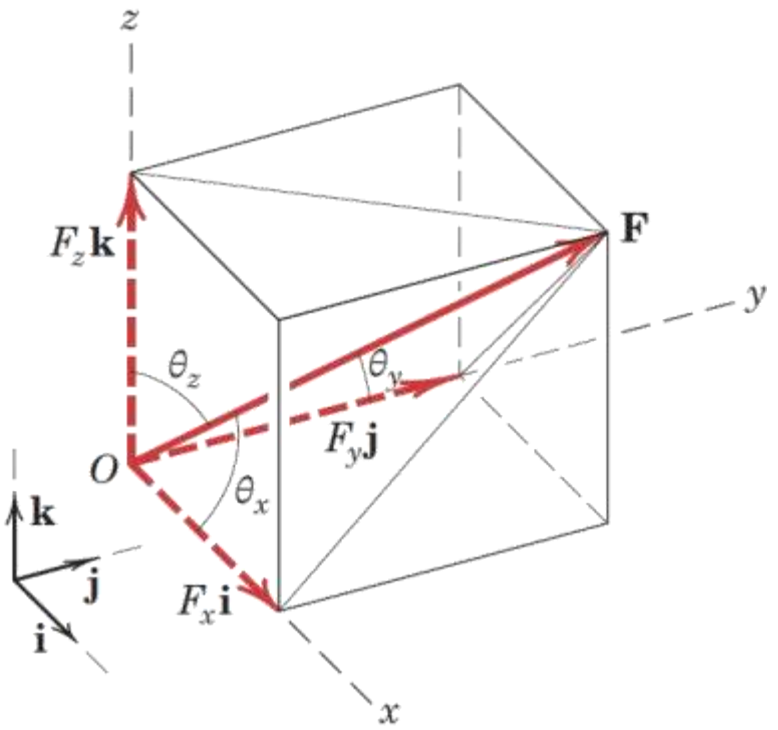


② Note that the vector approach yields sign information automatically, whereas the scalar approach is more physically oriented. You should master both methods.

Article 2/7 Rectangular Components (3D)

- Illustration and Equations of Interest

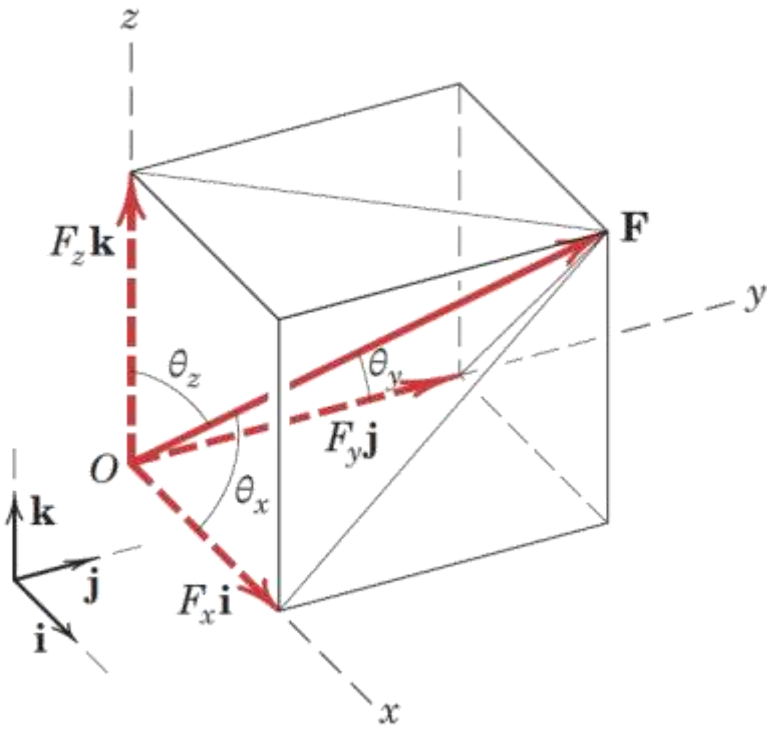
$F_x = F \cos \theta_x$	$F = \sqrt{F_x^{\,2} + F_y^{\,2} + F_z^{\,2}}$
$F_y = F \cos \theta_y$	$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$
$F_z = F \cos \theta_z$	$\mathbf{F} = F(\mathbf{i} \cos \theta_x + \mathbf{j} \cos \theta_y + \mathbf{k} \cos \theta_z)$



Article 2/7 – Rectangular Components (cont.)

- Magnitude and Direction Format
  - $\mathbf{F} = F\mathbf{n}_f$ where $\mathbf{n}_f$ is a unit vector in the direction of $\mathbf{F}$.
  - $\mathbf{n}_f = \cos\theta_x \mathbf{i} + \cos\theta_y \mathbf{j} + \cos\theta_z \mathbf{k}$

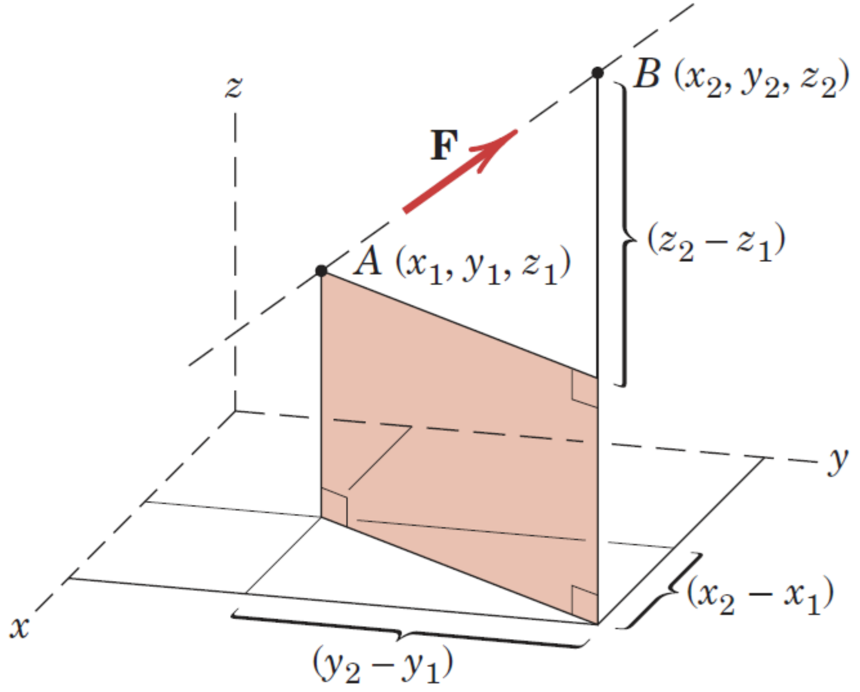
- Direction Cosine Format
  - $\mathbf{F} = F\mathbf{n}_f = F(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$
  - $l = \cos\theta_x$
  - $m = \cos\theta_y$
  - $n = \cos\theta_z$
  - $l^2 + m^2 + n^2 = 1$



Article 2/7 – Writing Vector Components (1 of 2)

- Specification by two points on the line of action of the force.

$$\mathbf{F} = F\mathbf{n}_F = F\frac{\overrightarrow{AB}}{AB} = F\frac{(x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$



Article 2/7 – Writing Vector Components (2 of 2)

- Specification by two angles which orient the line of action of the force.
  - Horizontal and Vertical Components

$$F_{xy} = F \cos \phi$$

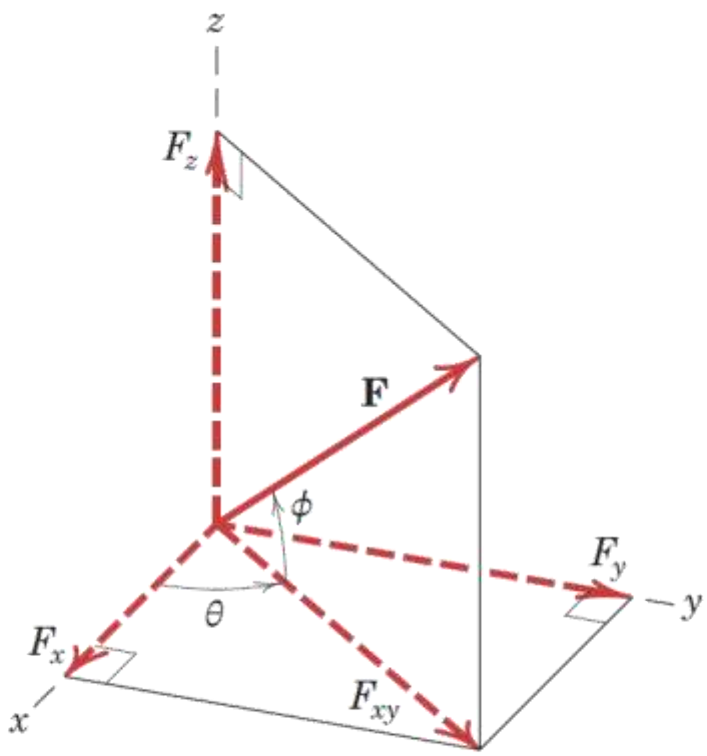
$$F_z = F \sin \phi$$

  - *x*- and *y*-Components

$$F_x = F_{xy} \cos \theta = F \cos \phi \cos \theta$$

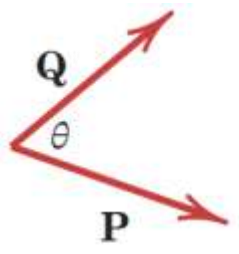
$$F_y = F_{xy} \sin \theta = F \cos \phi \sin \theta$$

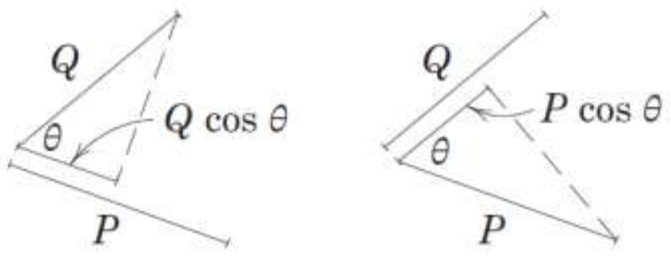
  - Other Combinations of Angles



Article 2/7 – The Dot Product

- Definitions and Illustration
  - $\mathbf{P}\cdot\mathbf{Q} = \mathbf{Q}\cdot\mathbf{P} = PQ\cos\,\theta$
  - $\theta = \cos^{-1}(\mathbf{P}\cdot\mathbf{Q}\,/PQ)$





- Mathematical Features of the Dot Product

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$$

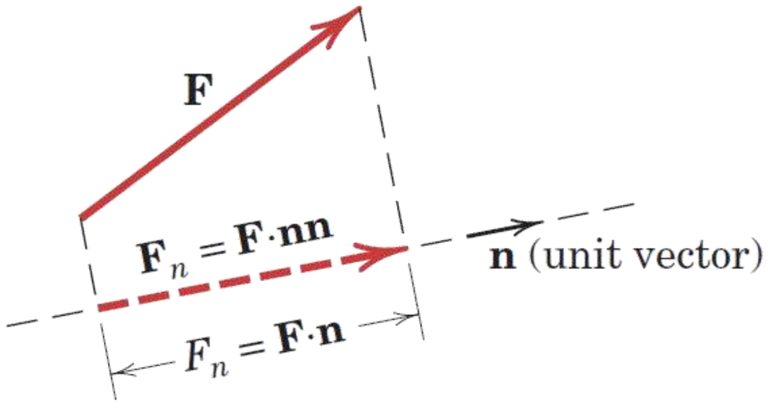
$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{i} = \mathbf{i}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{j} = 0$$

$$\begin{aligned}\mathbf{P}\cdot\mathbf{Q} &= (P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k})\cdot(Q_x\mathbf{i} + Q_y\mathbf{j} + Q_z\mathbf{k}) \\ &= P_xQ_x + P_yQ_y + P_zQ_z\end{aligned}$$

$$\mathbf{P}\cdot\mathbf{P} = P_x{}^2 + P_y{}^2 + P_z{}^2$$

Article 2/7 – Finding Projections of Forces onto Lines

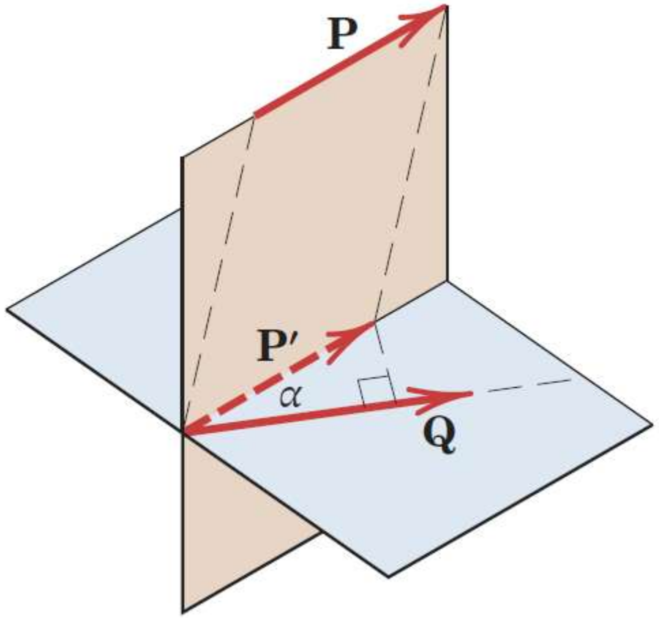
- Scalar Projection of a Force onto a line, F_n
  1. Write the force as a vector.
  2. Write a unit vector in the direction of the line.
  3. Dot the force vector with the unit vector.



- Vector Projection of a Force, $\mathbf{F}_n$
  1. Write the scalar projection of the force onto the line.
  2. Multiply the scalar projection by the unit vector for the line.

Article 2/7 – Finding the Angle between Two Vectors

- Dot Product Reminder
  - $\mathbf{P}\cdot\mathbf{Q} = \mathbf{Q}\cdot\mathbf{P} = PQ\cos\alpha$
  - $\alpha = \cos^{-1}(\mathbf{P}\cdot\mathbf{Q}\,/PQ)$



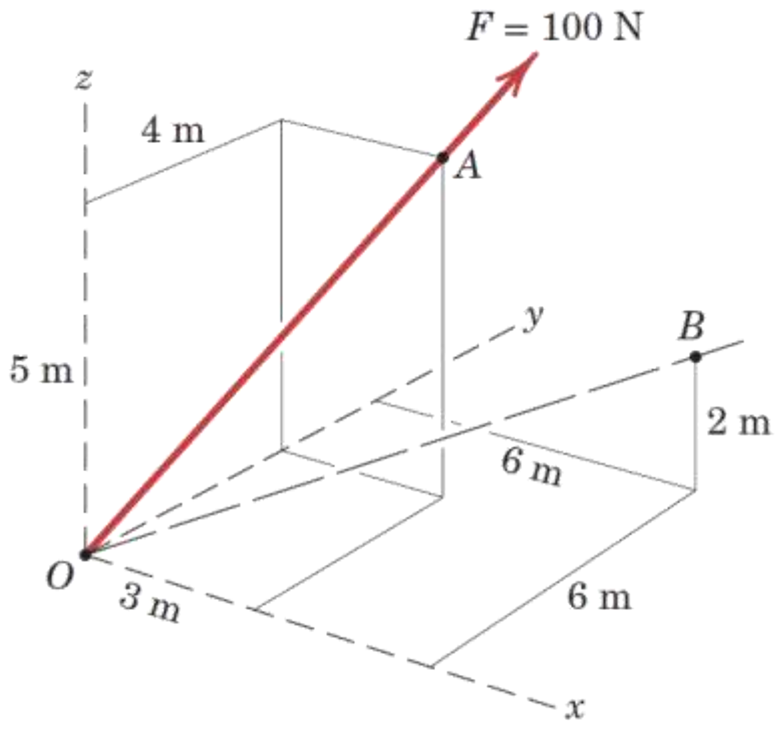
- Solution Steps
  1. Write each vector.
  2. Take a dot product between the vectors.
  3. Divide the dot product by the product of the magnitudes of the vectors.
  4. Take the inverse cosine of this ratio.

- This process is more easily carried out with unit vectors.

Article 2/7 – Sample Problem 2/10 (1 of 4)

- Problem Statement

  A force **F** with a magnitude of 100 N is applied at the origin *O* of the axes *x*-*y*-*z* as shown. The line of action of **F** passes through a point *A* whose coordinates are 3 m, 4 m, and 5 m. Determine (*a*) the *x*, *y*, and *z* scalar components of **F**, (*b*) the projection F_{xy} of **F** on the *x*-*y* plane, and (*c*) the projection F_{OB} of **F** along the line *OB*.



Article 2/7 – Sample Problem 2/10 (2 of 4)

- Scalar Components of *F*

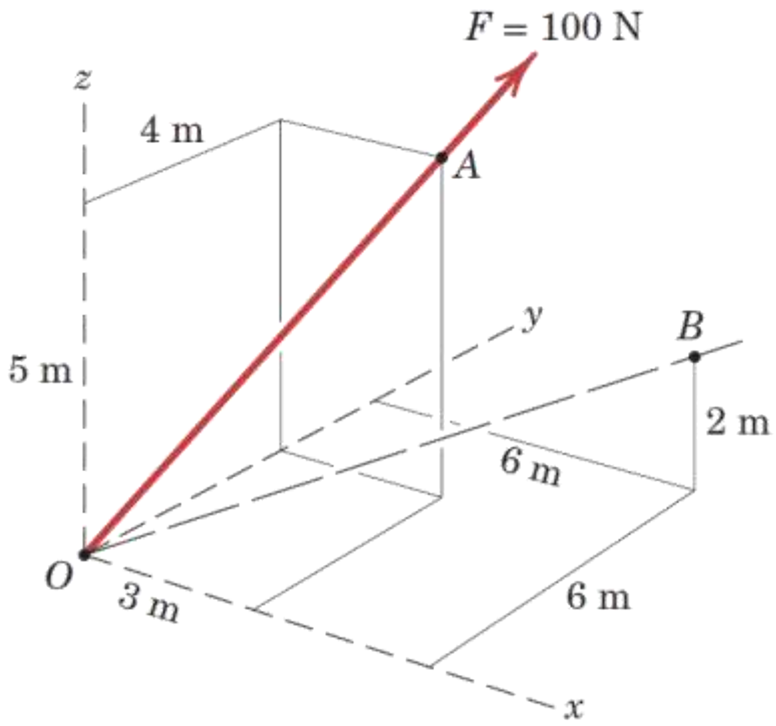
$$\mathbf{F} = F\mathbf{n}_{OA} = F\frac{\overrightarrow{OA}}{\overline{OA}} = 100\left[\frac{3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}}{\sqrt{3^2 + 4^2 + 5^2}}\right]$$

$$= 100[0.424\mathbf{i} + 0.566\mathbf{j} + 0.707\mathbf{k}]$$

$$= 42.4\mathbf{i} + 56.6\mathbf{j} + 70.7\mathbf{k}\ \text{N}$$

The desired scalar components are thus

$$F_x = 42.4\ \text{N} \qquad F_y = 56.6\ \text{N} \qquad F_z = 70.7\ \text{N} \quad ① \qquad \textit{Ans.}$$



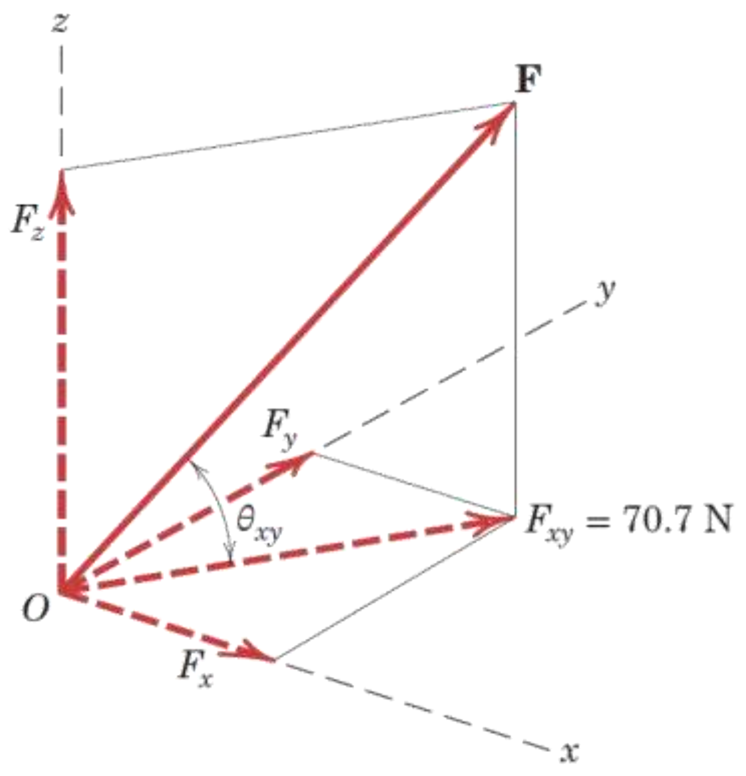
① In this example all scalar components are positive. Be prepared for the case where a direction cosine, and hence the scalar component, is negative.

Article 2/7 – Sample Problem 2/10 (3 of 4)

- Projection of F into the x-y Plane

$$\cos\theta_{xy} = \frac{\sqrt{3^2+4^2}}{\sqrt{3^2+4^2+5^2}} = 0.707$$

so that $F_{xy} = F\cos\theta_{xy} = 100(0.707) = 70.7 \text{ N}$ *Ans.*



Article 2/7 – Sample Problem 2/10 (4 of 4)

- Projection of F onto Line OB

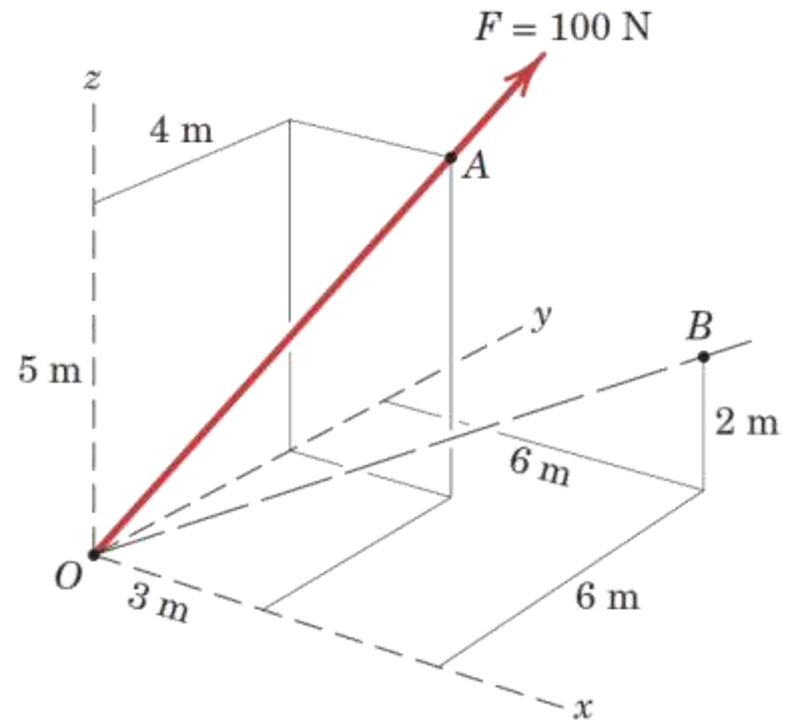
$$\mathbf{n}_{OB} = \frac{\overrightarrow{OB}}{OB} = \frac{6\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}}{\sqrt{6^2 + 6^2 + 2^2}} = 0.688\mathbf{i} + 0.688\mathbf{j} + 0.229\mathbf{k}$$

The scalar projection of **F** on OB is

$$F_{OB} = \mathbf{F}\cdot\mathbf{n}_{OB} = (42.4\mathbf{i} + 56.6\mathbf{j} + 70.7\mathbf{k})\cdot(0.688\mathbf{i} + 0.688\mathbf{j} + 0.229\mathbf{k}) \quad ②$$
$$= (42.4)(0.688) + (56.6)(0.688) + (70.7)(0.229)$$
$$= 84.4 \text{ N} \qquad \textit{Ans.}$$

If we wish to express the projection as a vector, we write

$$\mathbf{F}_{OB} = \mathbf{F}\cdot\mathbf{n}_{OB}\mathbf{n}_{OB}$$
$$= 84.4(0.688\mathbf{i} + 0.688\mathbf{j} + 0.229\mathbf{k})$$
$$= 58.1\mathbf{i} + 58.1\mathbf{j} + 19.35\mathbf{k} \text{ N}$$



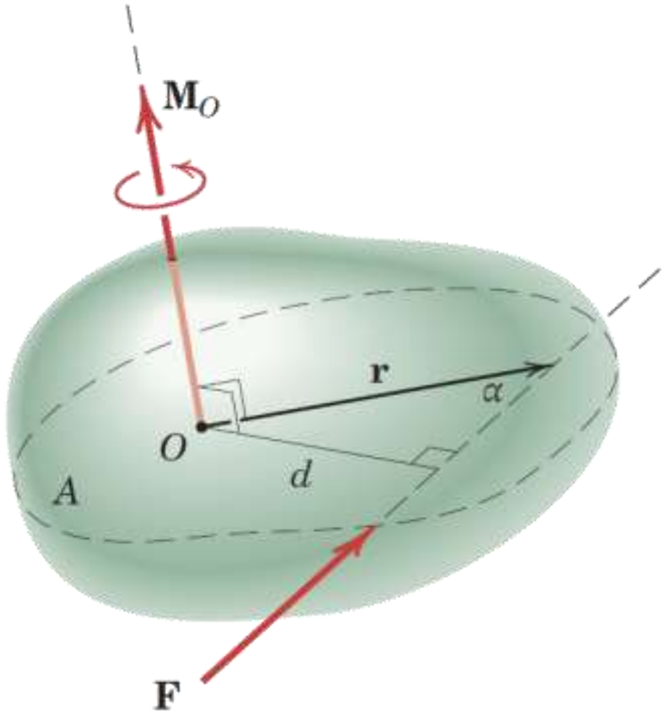
② The dot product automatically finds the projection or scalar component of **F** along line OB as shown.

Article 2/8 Moment and Couple (3D)

- Moments in Three Dimensions
  - Operate Identically to Moments in Two Dimensions
  - More Complicated to Visualize

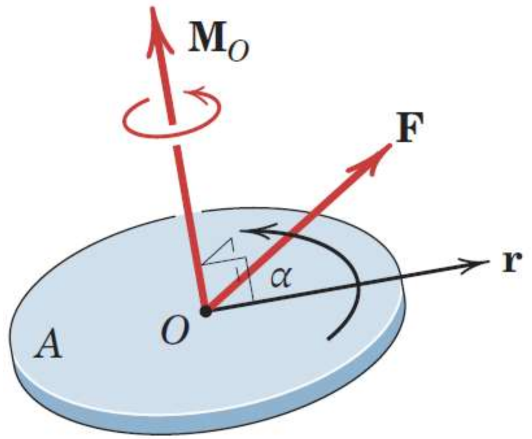
- Scalar Approach: $M_O = Fd$
  - More Difficult to Accomplish
  - Lacks Sign Information

- Vector Approach: $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$
  - Easy to Compute
  - Sign Information is Included Automatically



Article 2/8 – Right-Hand Rule Reminder

- Direction and Sense of the Moment
  - Established by Right-Hand Rule
  - Perpendicular to the Plane which Contains **r** and **F**
  - Cross Product Order is Essential

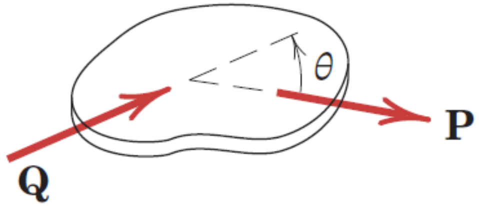


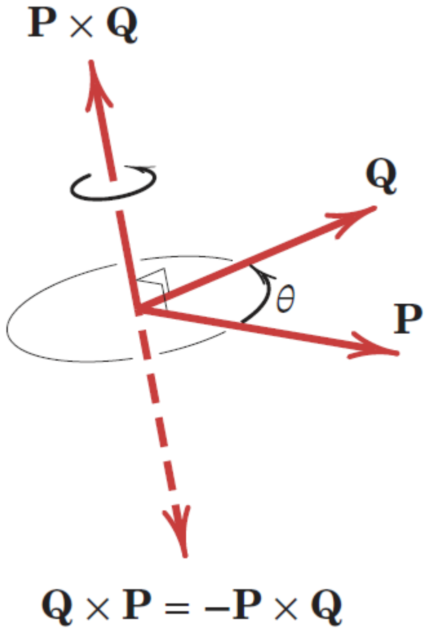
Article 2/8 – Cross Products (1 of 2)

- Definitions and Illustration

$$\mathbf{P} \times \mathbf{Q} = (P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k}) \times (Q_x\mathbf{i} + Q_y\mathbf{j} + Q_z\mathbf{k})$$
$$= (P_yQ_z - P_zQ_y)\mathbf{i} + (P_zQ_x - P_xQ_z)\mathbf{j} + (P_xQ_y - P_yQ_x)\mathbf{k}$$

$$|\mathbf{P} \times \mathbf{Q}| = PQ \sin\theta$$





- Mathematical Features of the Cross Product

Distributive law $\quad \mathbf{P} \times (\mathbf{Q} + \mathbf{R}) = \mathbf{P} \times \mathbf{Q} + \mathbf{P} \times \mathbf{R}$

From the definition of the cross product, using a *right-handed coordinate system*, we get

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$
$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$
$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$$

Article 2/8 – Cross Products (2 of 2)

- Calculation via Determinant

$$\mathbf{P} \times \mathbf{Q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}$$

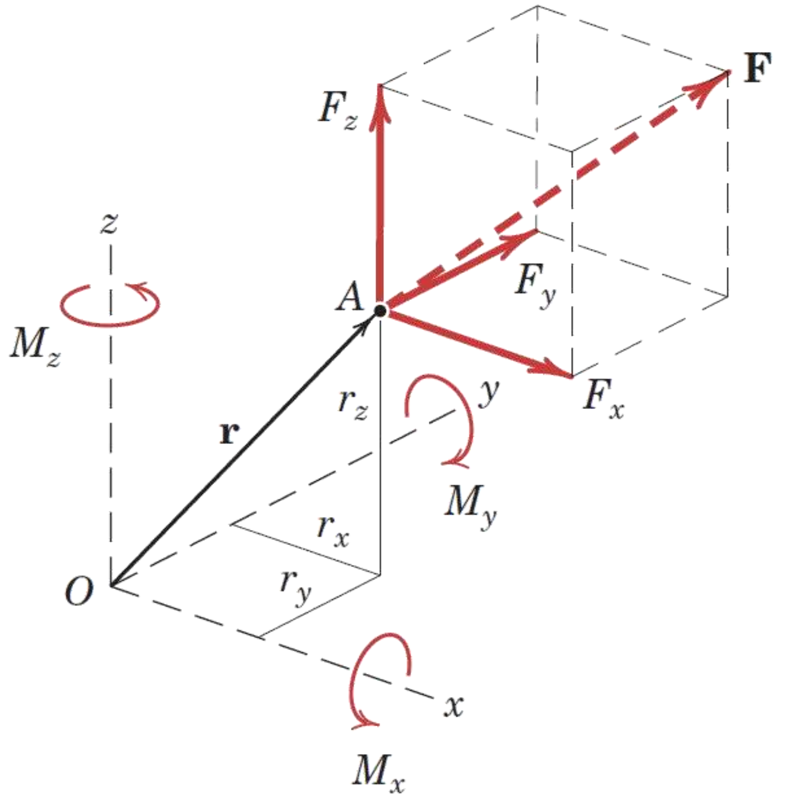
$$\begin{aligned} \mathbf{P} \times \mathbf{Q} &= (P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k}) \times (Q_x\mathbf{i} + Q_y\mathbf{j} + Q_z\mathbf{k}) \\ &= (P_yQ_z - P_zQ_y)\mathbf{i} + (P_zQ_x - P_xQ_z)\mathbf{j} + (P_xQ_y - P_yQ_x)\mathbf{k} \end{aligned}$$

Article 2/8 – Moment made by a General Force

- Vector Components of $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$

$$\boxed{\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}}$$

$$\mathbf{M}_O = (r_y F_z - r_z F_y)\mathbf{i} + (r_z F_x - r_x F_z)\mathbf{j} + (r_x F_y - r_y F_x)\mathbf{k}$$

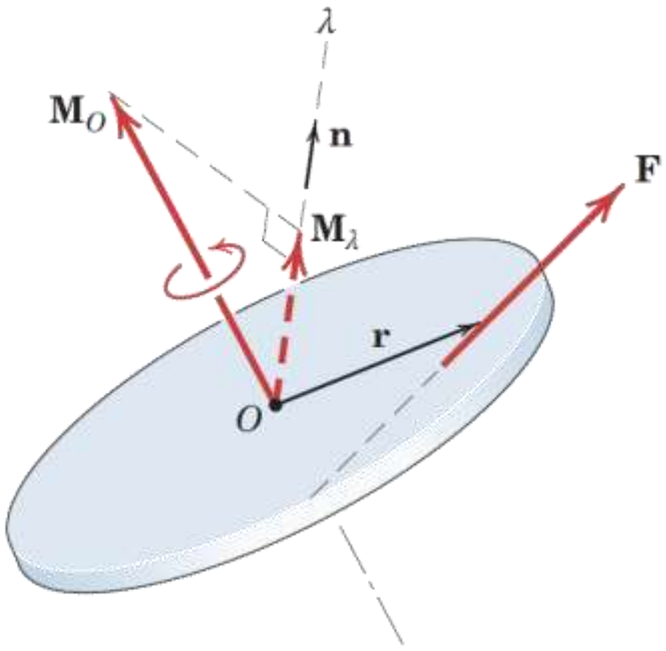


- Scalar Components

$$M_x = r_y F_z - r_z F_y \qquad M_y = r_z F_x - r_x F_z \qquad M_z = r_x F_y - r_y F_x$$

Article 2/8 – Moment about an Arbitrary Axis

- Illustration

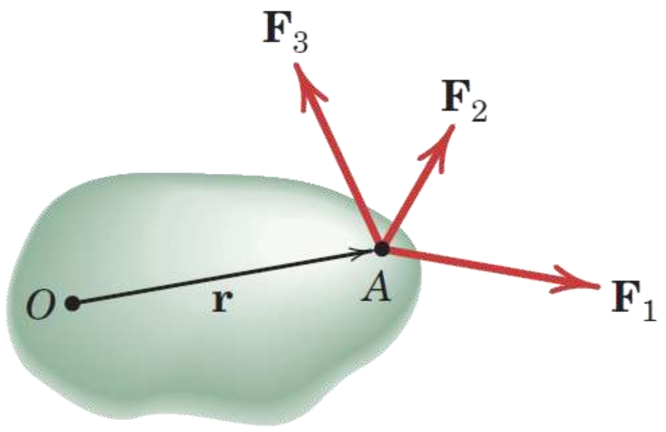


- Scalar Expression of the Moment about an Axis, M_λ
  1. Write the force as a vector.
  2. Write a position vector from any point on the axis to any point on the line of action of the force.
  3. Compute the moment of the force about the point.
  4. Write a unit vector in the direction of the axis.

- Vector Expression of the Moment about an Axis, $\mathbf{M}_\lambda$
  1. Write the scalar expression of the moment about the axis.
  2. Multiply the scalar expression of the moment about the axis by the unit vector in the direction of the axis.

Article 2/8 – Varignon's Theorem in Three Dimensions

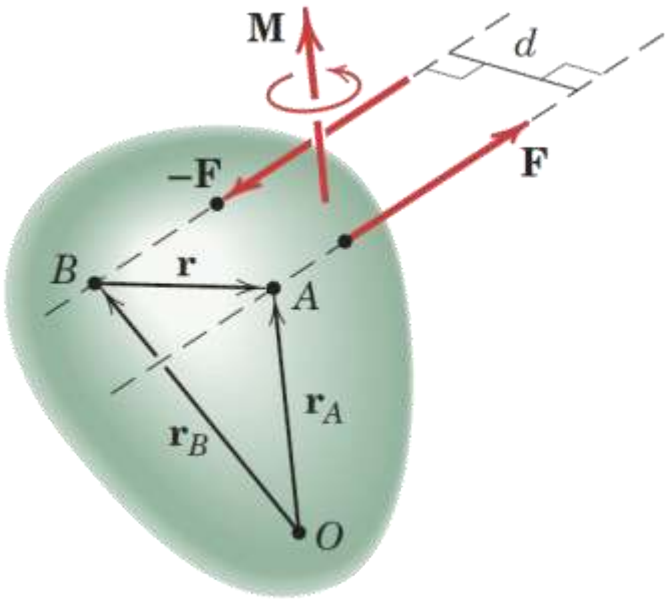
- Illustration



- Mathematics

$$\boxed{\mathbf{M}_O = \Sigma(\mathbf{r} \times \mathbf{F}) = \mathbf{r} \times \mathbf{R}}$$

Article 2/8 – Couples in Three Dimensions (1 of 3)



- Illustration

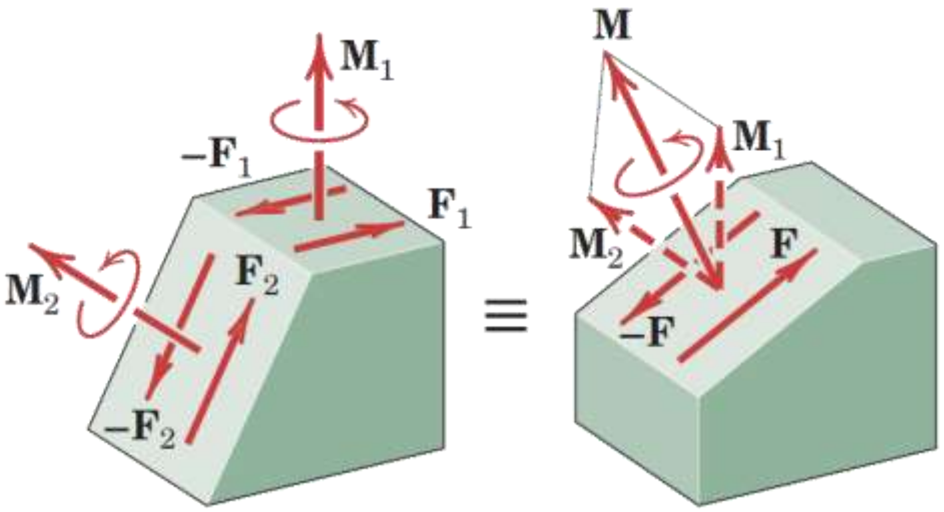
- Mathematics

$$\mathbf{M}_O = \mathbf{r}_A \times \mathbf{F} + \mathbf{r}_B \times (-\mathbf{F}) = (\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F} = \mathbf{r} \times \mathbf{F}$$

- Comments about Couples
  - Couples are free vectors.
  - You can simply compute the moment of one of the forces about any point on the line of action of the other force.
  - Couple vectors obey all of the corresponding mathematical rules which govern vector quantities.

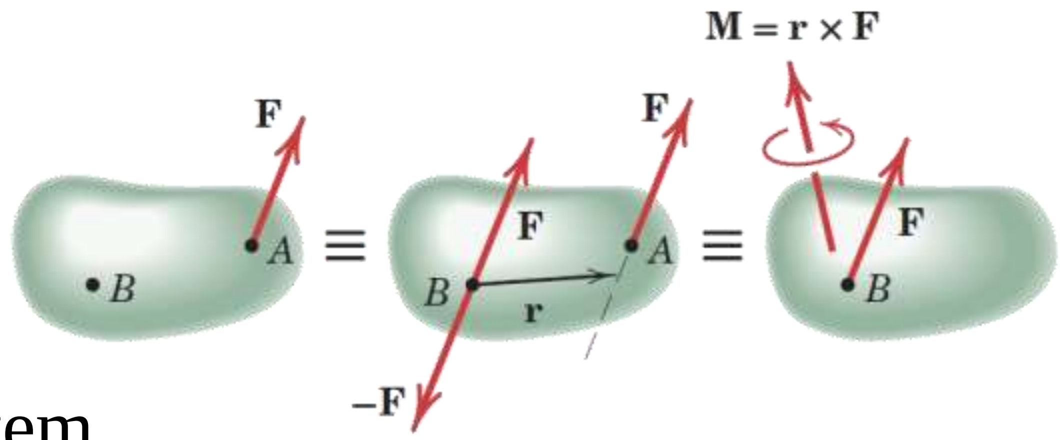
Article 2/8 – Couples in Three Dimensions (2 of 3)

- Adding Couples
  - Couples add with the Parallelogram Rule of Vector Addition



Article 2/8 – Force-Couple Systems

- Illustration of the Process

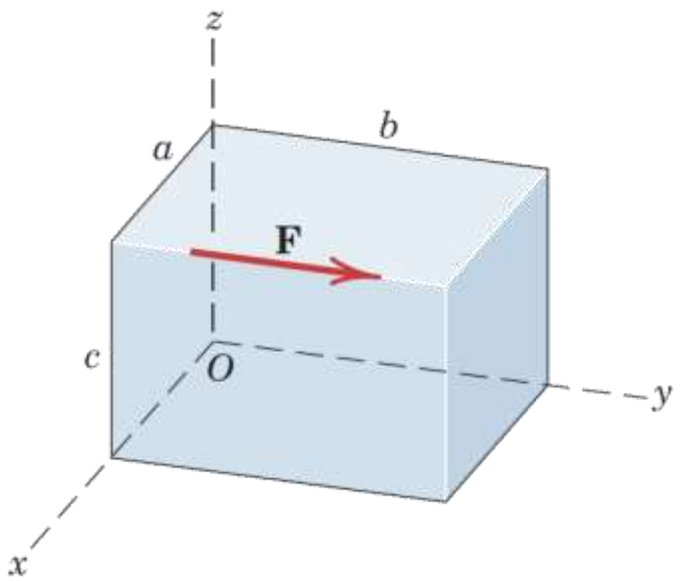


- Steps to Create a Force-Couple System
  1. Write the force as a vector.
  2. Compute the moment or couple which the force creates about the point.
  3. Redraw the force acting at the new location.
  4. Sketch the couple acting at the new location.
- Important Reminder
  - As with two-dimensional force-couple systems, the force-couple system has the same effect on the body which the original force had. It is simply a different way to visualize the effect of the force acting at a new location.

Article 2/8 – Sample Problem 2/11 (1 of 2)

- Problem Statement

  Determine the moment of force **F** about point O (a) by inspection and (b) by the formal cross-product definition $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$.



Article 2/8 – Sample Problem 2/11 (2 of 2)

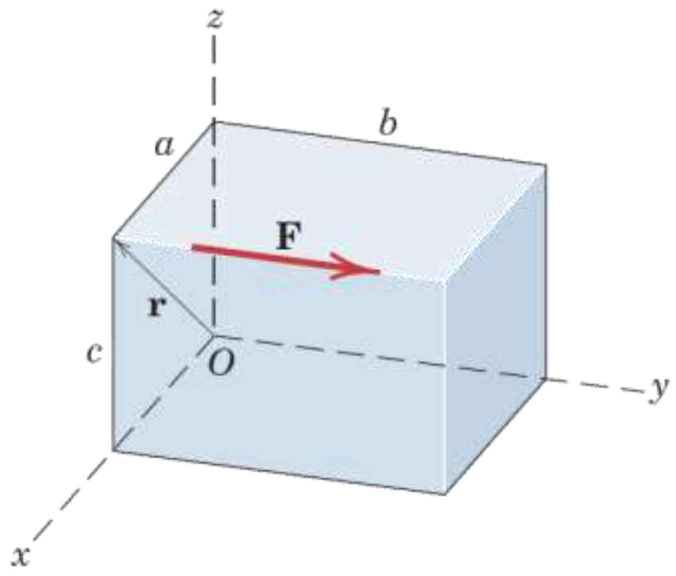
- Solution by Inspection

$$\mathbf{M}_O = -cF\mathbf{i} + aF\mathbf{k} = F(-c\mathbf{i} + a\mathbf{k}) \qquad \textit{Ans.}$$

- Solution by Cross Product

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = (a\mathbf{i} + c\mathbf{k}) \times F\mathbf{j} = aF\mathbf{k} - cF\mathbf{i} \qquad ①$$

$$= F(-c\mathbf{i} + a\mathbf{k}) \qquad \textit{Ans.}$$

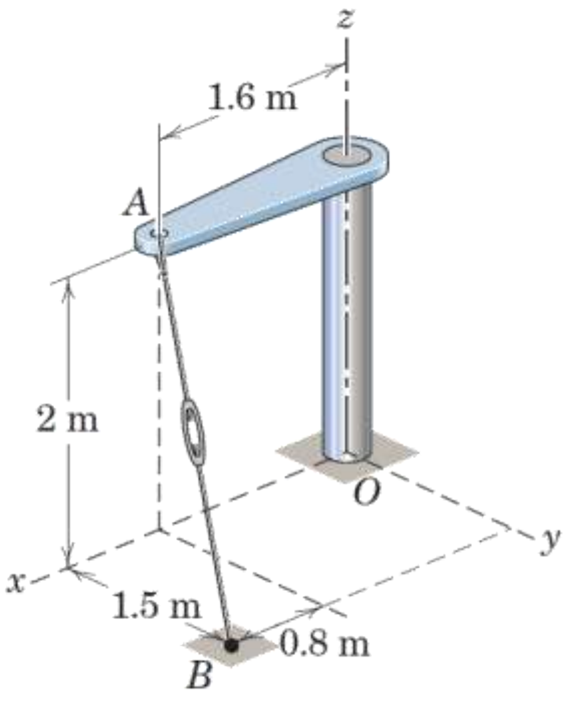


① Again we stress that **r** runs *from* the moment center *to* the line of action of **F**. Another permissible, but less convenient, position vector is $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

Article 2/8 – Sample Problem 2/12 (1 of 2)

- Problem Statement

  The turnbuckle is tightened until the tension in cable *AB* is 2.4 kN. Determine the moment about point *O* of the cable force acting on point *A* and the magnitude of this moment.



Article 2/8 – Sample Problem 2/12 (2 of 2)

- Tension Vector

$$\mathbf{T} = T\mathbf{n}_{AB} = 2.4\left[\frac{0.8\mathbf{i} + 1.5\mathbf{j} - 2\mathbf{k}}{\sqrt{0.8^2 + 1.5^2 + 2^2}}\right]$$

$$= 0.731\mathbf{i} + 1.371\mathbf{j} - 1.829\mathbf{k} \text{ kN}$$

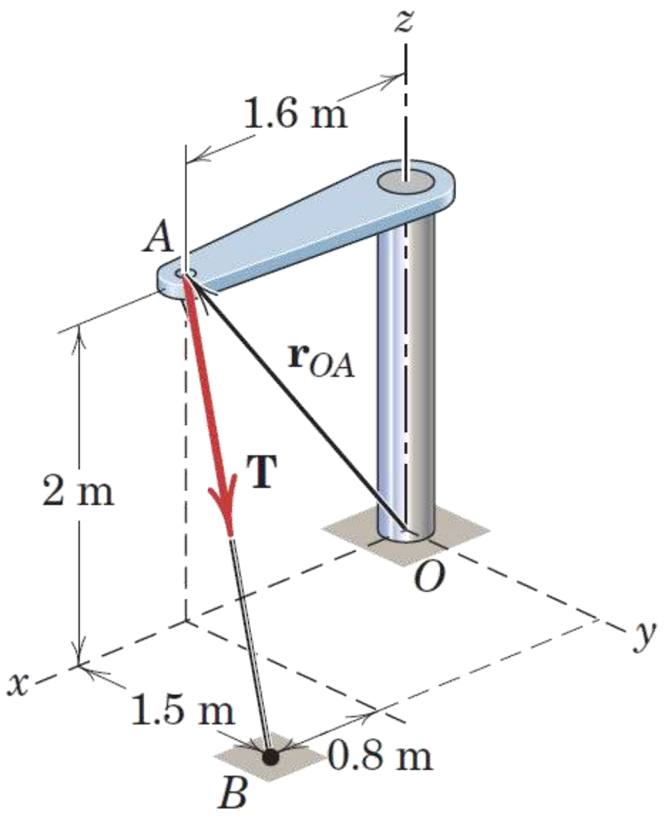
- Moment about Point O

$$\mathbf{M}_O = \mathbf{r}_{OA} \times \mathbf{T} = (1.6\mathbf{i} + 2\mathbf{k}) \times (0.731\mathbf{i} + 1.371\mathbf{j} - 1.829\mathbf{k})$$

$$= -2.74\mathbf{i} + 4.39\mathbf{j} + 2.19\mathbf{k} \text{ kN}\cdot\text{m} \quad ① \qquad \textit{Ans.}$$

- Magnitude of the Moment

$$M_O = \sqrt{2.74^2 + 4.39^2 + 2.19^2} = 5.62 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$



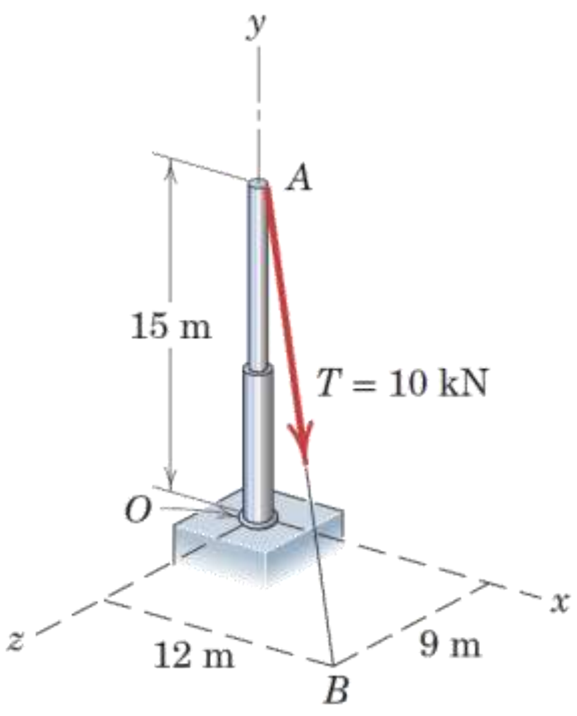
HELPFUL HINT

① The student should verify by inspection the signs of the moment components.

Article 2/8 – Sample Problem 2/13 (1 of 4)

- Problem Statement

  A tension **T** of magnitude 10 kN is applied to the cable attached to the top *A* of the rigid mast and secured to the ground at *B*. Determine the moment M_z of **T** about the *z*-axis passing through the base *O*.

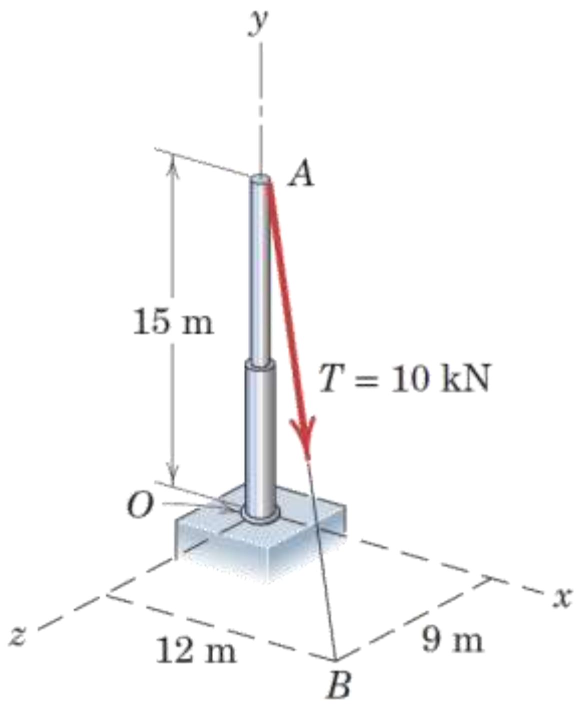


Article 2/8 – Sample Problem 2/13 (2 of 4)

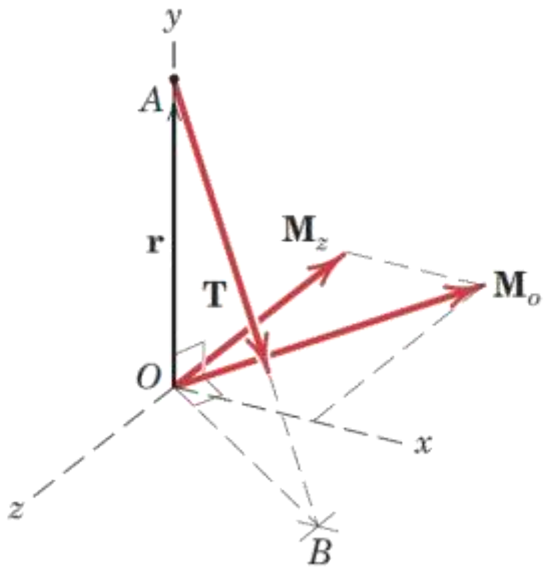
- Tension Vector

$$\mathbf{T} = T\mathbf{n}_{AB} = 10\left[\frac{12\mathbf{i} - 15\mathbf{j} + 9\mathbf{k}}{\sqrt{(12)^2 + (-15)^2 + (9)^2}}\right]$$

$$= 10(0.566\mathbf{i} - 0.707\mathbf{j} + 0.424\mathbf{k})\ \text{kN}$$



- Solution 1 – Cross Product



$[\mathbf{M}_O = \mathbf{r} \times \mathbf{F}]$ $\qquad \mathbf{M}_O = 15\mathbf{j} \times 10(0.566\mathbf{i} - 0.707\mathbf{j} + 0.424\mathbf{k})$

$$= 150(-0.566\mathbf{k} + 0.424\mathbf{i})\ \text{kN}\cdot\text{m}$$

The value M_z of the desired moment is the scalar component of $\mathbf{M}_O$ in the z-direction or $M_z = \mathbf{M}_O \cdot \mathbf{k}$. Therefore,

$$M_z = 150(-0.566\mathbf{k} + 0.424\mathbf{i}) \cdot \mathbf{k} = -84.9\ \text{kN}\cdot\text{m} \qquad \textit{Ans.}$$

Article 2/8 – Sample Problem 2/13 (3 of 4)

- Solution 2 – Two Scalar Components

$M_z = T_{xy}d$, where d is the perpendicular distance from T_{xy} to O. ③ The cosine of the angle between T and T_{xy} is $\sqrt{15^2 + 12^2}/\sqrt{15^2 + 12^2 + 9^2} = 0.906$, and therefore,

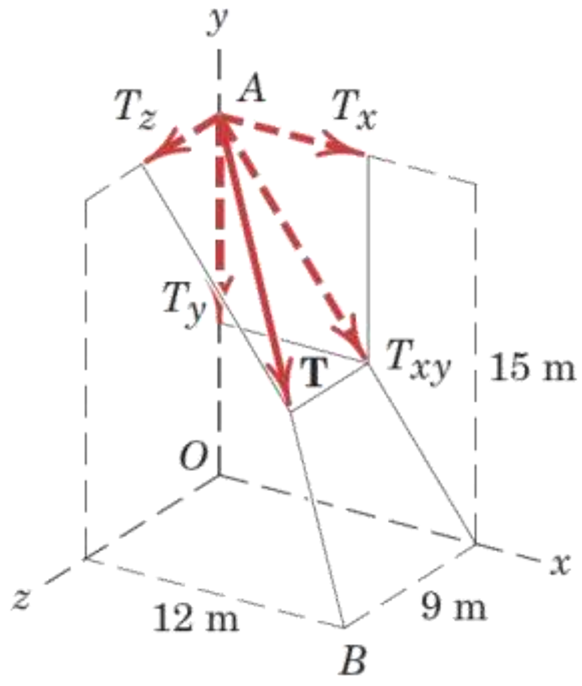
$$T_{xy} = 10(0.906) = 9.06 \text{ kN}$$

The moment arm d equals $\overline{OA}$ multiplied by the sine of the angle between T_{xy} and OA, or

$$d = 15\frac{12}{\sqrt{12^2 + 15^2}} = 9.37 \text{ m}$$

Hence, the moment of $\mathbf{T}$ about the z-axis has the magnitude

$$M_z = 9.06(9.37) = 84.9 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$



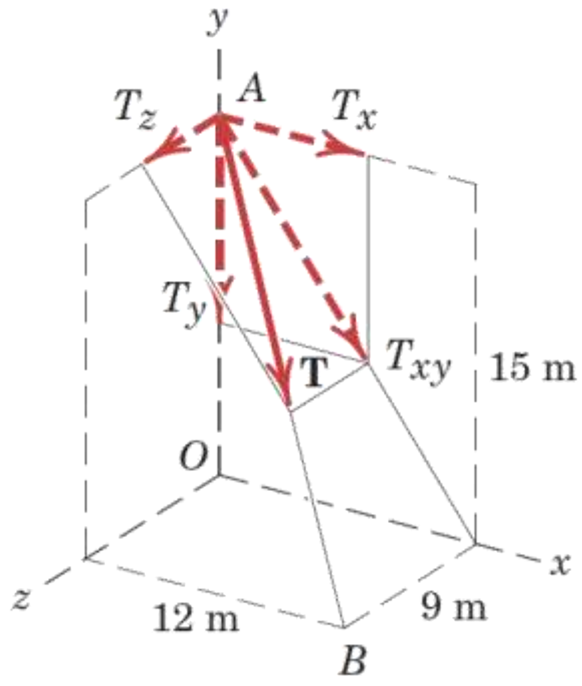
Article 2/8 – Sample Problem 2/13 (4 of 4)

- Solution 3 – Three Scalar Components

By inspection, only T_x makes a moment about point O. The y-component Intersects point O and the z-component is parallel to the z-axis.

From before, $T_x = 10(0.566) = 5.66$ kN.

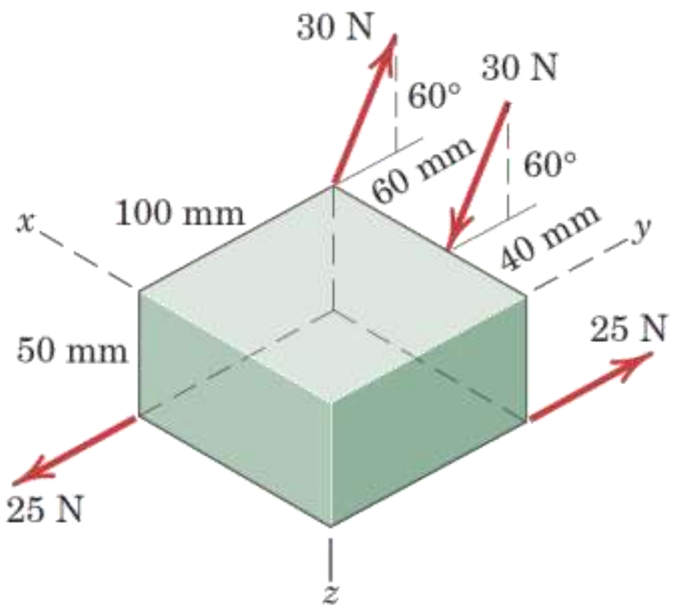
Therefore… $M_z = 5.66(15) = 84.9 \text{ kN} \cdot \text{m}$ *Ans.*



Article 2/8 – Sample Problem 2/14 (1 of 2)

- Problem Statement

  Determine the magnitude and direction of the couple **M** which will replace the two given couples and still produce the same external effect on the block. Specify the two forces **F** and -**F**, applied in the two faces of the block parallel to the *y*-*z* plane, which may replace the four given forces. The 30-N forces act parallel to the *y*-*z* plane.



Article 2/8 – Sample Problem 2/14 (2 of 2)

- Solution

The couple due to the 30-N forces has the magnitude $M_1 = 30(0.06) = 1.80\ \text{N}\cdot\text{m}$. The direction of $\mathbf{M}_1$ is normal to the plane defined by the two forces, and the sense, shown in the figure, is established by the right-hand convention. The couple due to the 25-N forces has the magnitude $M_2 = 25(0.10) = 2.50\ \text{N}\cdot\text{m}$ with the direction and sense shown in the same figure. The two couple vectors combine to give the components

$$M_y = 1.80 \sin 60° = 1.559\ \text{N}\cdot\text{m}$$

$$M_z = -2.50 + 1.80 \cos 60° = -1.600\ \text{N}\cdot\text{m}$$

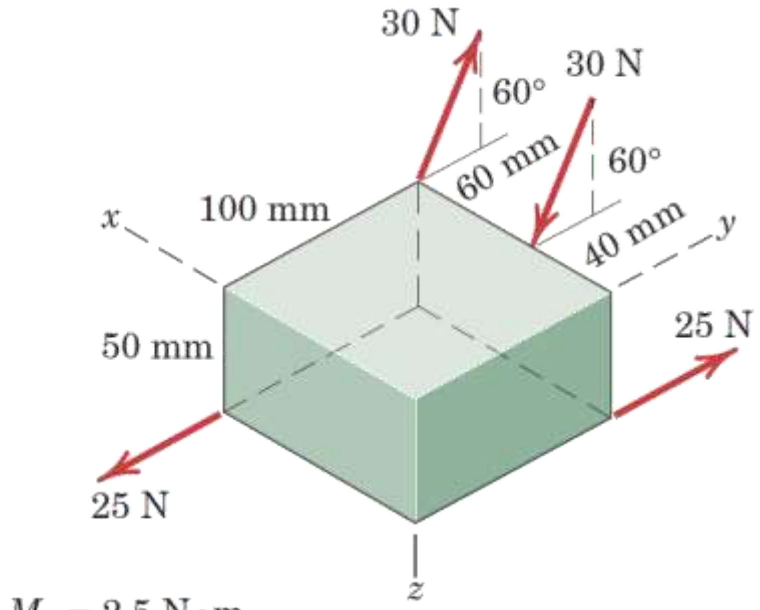
Thus, $$M = \sqrt{(1.559)^2 + (-1.600)^2} = 2.23\ \text{N}\cdot\text{m} \quad ① \qquad \textit{Ans.}$$

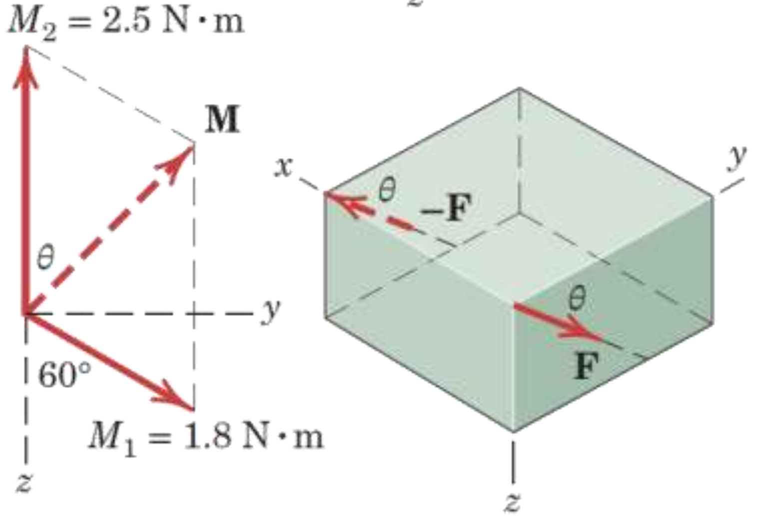
with $$\theta = \tan^{-1}\frac{1.559}{1.600} = \tan^{-1} 0.974 = 44.3° \qquad \textit{Ans.}$$

The forces **F** and –**F** lie in a plane normal to the couple **M**, and their moment arm as seen from the right-hand figure is 100 mm. Thus, each force has the magnitude

$[M = Fd]$ $$F = \frac{2.23}{0.10} = 22.3\ \text{N} \qquad \textit{Ans.}$$

and the direction $\theta = 44.3°$.

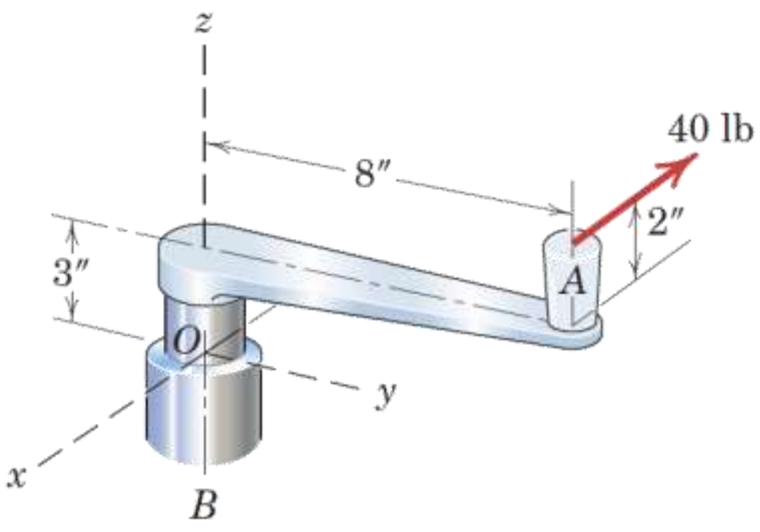




Article 2/8 – Sample Problem 2/15 (1 of 2)

- Problem Statement

  A force of 40 lb is applied at *A* to the handle of the control lever which is attached to the fixed shaft *OB*. In determining the effect of the force on the shaft at a cross section such as that at *O*, we may replace the force by an equivalent force at *O* and a couple. Describe this couple as a vector **M**.



Article 2/8 – Sample Problem 2/15 (2 of 2)

- Solution

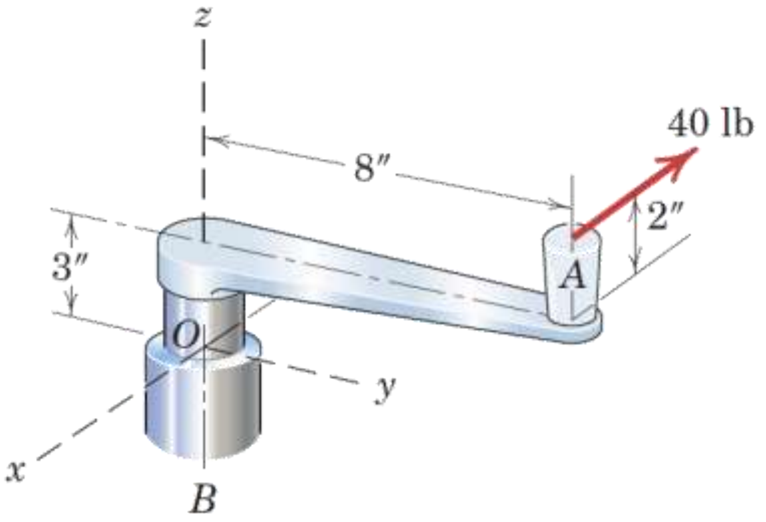
$$\mathbf{M} = (8\mathbf{j} + 5\mathbf{k}) \times (-40\mathbf{i}) = -200\mathbf{j} + 320\mathbf{k} \text{ lb-in.}$$

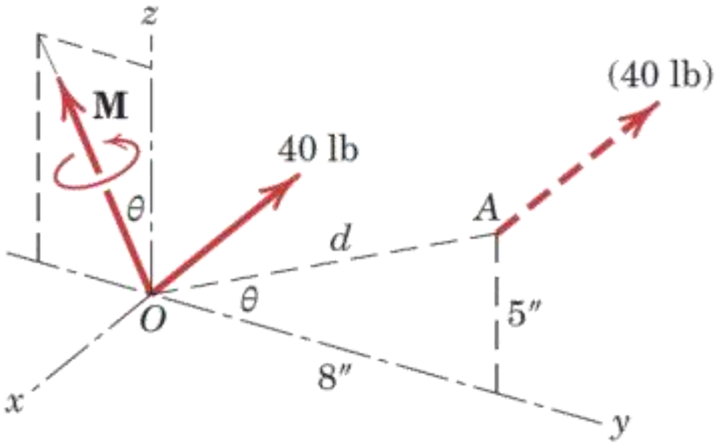
Alternatively we see that moving the 40-lb force through a distance $d = \sqrt{5^2 + 8^2} = 9.43$ in. to a parallel position through O requires the addition of a couple **M** whose magnitude is

$$M = Fd = 40(9.43) = 377 \text{ lb-in.} \qquad \textit{Ans.}$$

The couple vector is perpendicular to the plane in which the force is shifted, and its sense is that of the moment of the given force about O. The direction of **M** in the y-z plane is given by

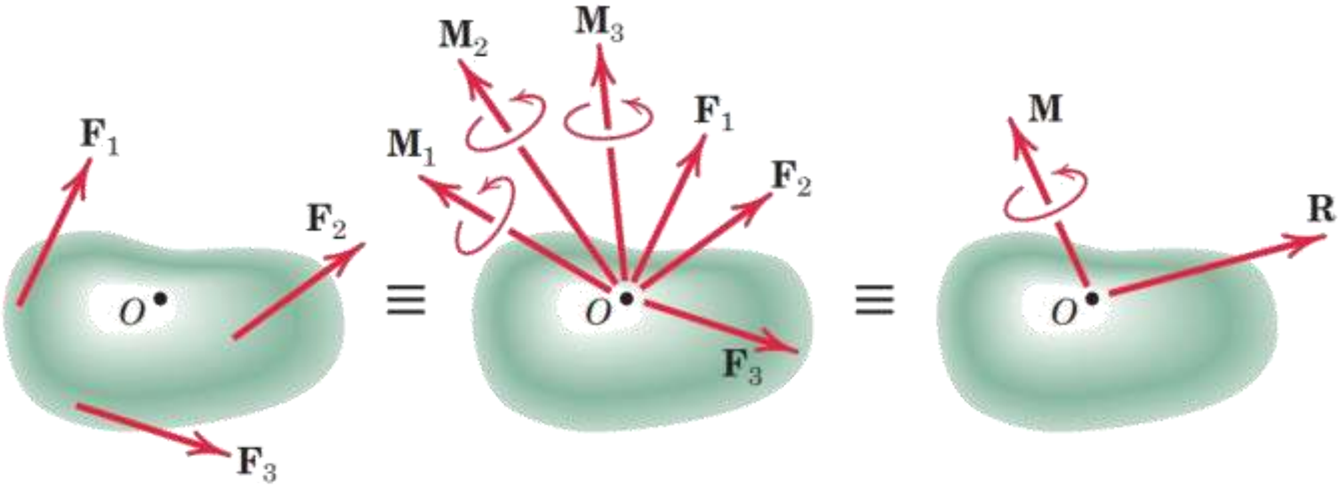
$$\theta = \tan^{-1}\frac{5}{8} = 32.0° \qquad \textit{Ans.}$$





Article 2/9 Resultants (3D)

- Illustration



- Equations of Interest

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots = \Sigma\mathbf{F}$$
$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 + \cdots = \Sigma(\mathbf{r} \times \mathbf{F})$$

$$R_x = \Sigma F_x \qquad R_y = \Sigma F_y \qquad R_z = \Sigma F_z$$
$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2 + (\Sigma F_z)^2}$$
$$\mathbf{M}_x = \Sigma(\mathbf{r} \times \mathbf{F})_x \qquad \mathbf{M}_y = \Sigma(\mathbf{r} \times \mathbf{F})_y \qquad \mathbf{M}_z = \Sigma(\mathbf{r} \times \mathbf{F})_z$$
$$M = \sqrt{M_x{}^2 + M_y{}^2 + M_z{}^2}$$

Article 2/9 – Types of Force Systems (1 of 2)

- Concurrent Forces
  - Because the forces are concurrent, there are no moments about the point of concurrency.
  - $\mathbf{R} = \Sigma \mathbf{F}$
- Coplanar Forces
  - Article 2/6 was devoted to this force system.

Article 2/9 – Types of Force Systems (2 of 2)

- Parallel Forces not in the Same Plane
  - Because the forces are parallel, the moment they produce about any point will be perpendicular to the line of action of the resultant.

- Calculation Steps
  1. Find the resultant, $\mathbf{R} = \Sigma\mathbf{F}$
  2. Find the couple at the point, $\mathbf{M}_O = \Sigma\mathbf{M}_O$ (from all forces or applied couples)
  3. Write a position vector $\mathbf{r}$ from the force-couple reference point to any point on the line of action of the resultant $\mathbf{R}$. Typically, the point will be specified in one of the three coordinate-axis planes.
     a. For a point in the x-y plane $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$
     b. For a point in the x-z plane, $\mathbf{r} = x\mathbf{i} + z\mathbf{k}$
     c. For a point in the y-z plane, $\mathbf{r} = y\mathbf{j} + z\mathbf{k}$
  4. Solve the equation $\mathbf{r} \times \mathbf{R} = \mathbf{M}_O$

Article 2/9 – Wrench Resultants (1 of 4)

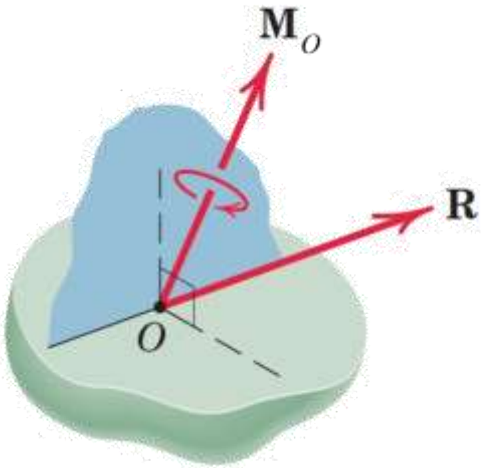
- Occurrence and Illustration
  - Wrenches occur for a system of forces which are not parallel or concurrent. In this case, the resultant couple vector, $\mathbf{M}_O$, will have a component that is parallel to the resultant force **R**. The resultant **R** is not able to produce this component of the moment regardless of its position relative to the force-couple system reference point.

  - The simplified force system will consist of two pieces.
    - A resultant **R** which equals the vector sum of all forces, and is positioned such that it can produce the part of the resultant couple vector which is perpendicular to its line of action.
    - A wrench moment **M** which is equal to the part of the resultant couple vector which is parallel to the line of action of the resultant **R**.

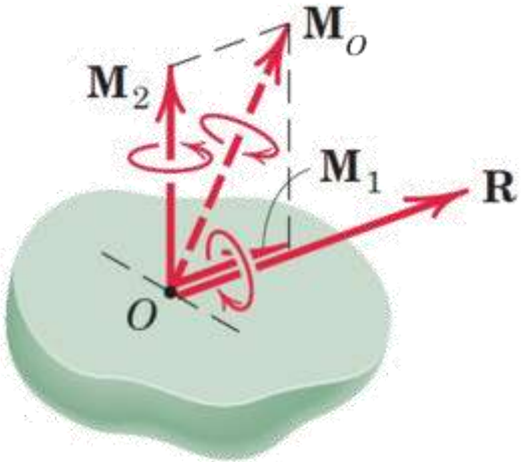
Positive wrench　　Negative wrench

Article 2/9 – Wrench Resultants (2 of 4)

- General Force-Couple System

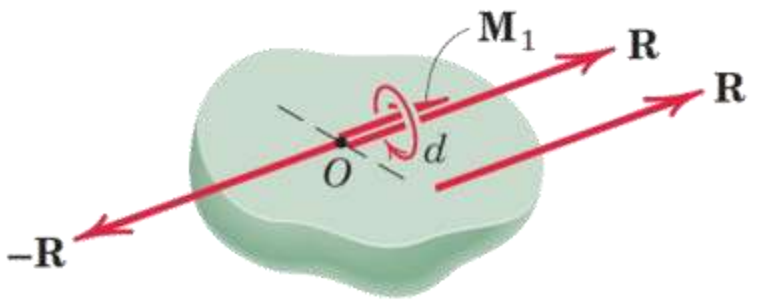


- Components of the Resultant Couple Vector
  - $\mathbf{M}_1$ is Parallel to $\mathbf{R}$
  - $\mathbf{M}_2$ is Perpendicular to $\mathbf{R}$

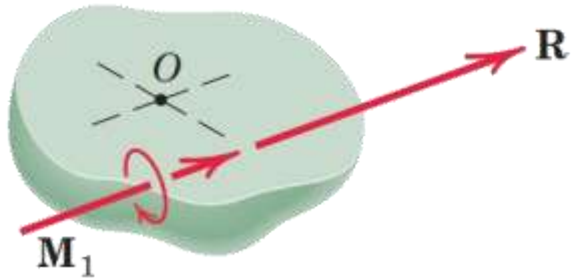


Article 2/9 – Wrench Resultants (3 of 4)

- Relocation of the Resultant by a Couple
  - **R** is moved a distance *d* from the reference point.
  - $Rd = M_2$



- Final Wrench Resultant
  - Force Resultant is Preserved
  - Moment Resultant is Preserved
    - $\mathbf{M}_1$ is Simply Added
    - **R** will Produce $\mathbf{M}_2$
    - $\mathbf{M}_O = \mathbf{M}_1 + \mathbf{M}_2$



Article 2/9 – Wrench Resultants (4 of 4)

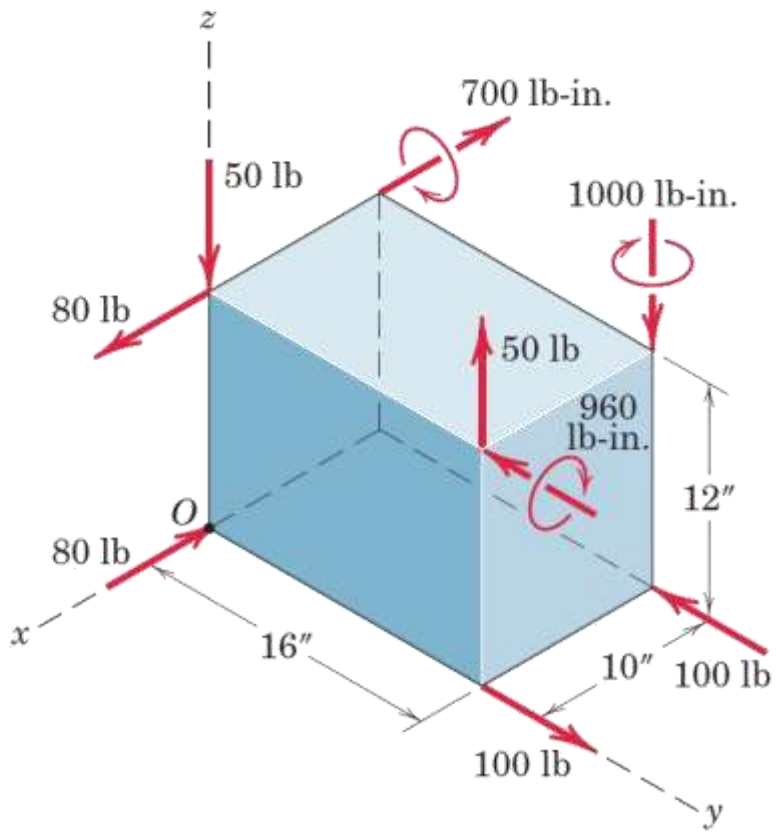
- Calculation Steps

1. Find the resultant, $\mathbf{R} = \Sigma\mathbf{F}$
2. Find the couple at the point, $\mathbf{M}_O = \Sigma\mathbf{M}_O$ (from all forces or applied couples)
3. Find the wrench moment, $\mathbf{M}_1$
   a. Write a unit vector $\mathbf{n}$ in the direction of the resultant $\mathbf{R}$, $\mathbf{n} = \mathbf{R}/\mathrm{R}$
   b. Take a Dot Product to find the scalar portion of $\mathbf{M}_O$ in the direction of $\mathbf{R}$, $M_1 = \mathbf{M}_O \cdot \mathbf{n}$
   c. The algebraic sign of M_1 will tell you if the wrench is in the positive or negative sense.
   d. Write the wrench-moment vector, $\mathbf{M}_1 = M_1\mathbf{n}$
4. Write a position vector $\mathbf{r}$ from the force-couple reference point to any point on the line of action of the resultant $\mathbf{R}$. Typically, the point will be specified in one of the three coordinate-axis planes.
   a. For a point in the x-y plane $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$
   b. For a point in the x-z plane, $\mathbf{r} = x\mathbf{i} + z\mathbf{k}$
   c. For a point in the y-z plane, $\mathbf{r} = y\mathbf{j} + z\mathbf{k}$
5. Solve the equation $\mathbf{r} \times \mathbf{R} + \mathbf{M}_1 = \mathbf{M}_O$

Article 2/9 – Sample Problem 2/16 (1 of 2)

- Problem Statement

Determine the resultant of the force and couple system which acts on the rectangular solid.



Article 2/9 – Sample Problem 2/16 (2 of 2)

- Solution

$$\mathbf{R} = \Sigma\mathbf{F} = (80 - 80)\mathbf{i} + (100 - 100)\mathbf{j} + (50 - 50)\mathbf{k} = \mathbf{0} \text{ lb} \quad ①$$

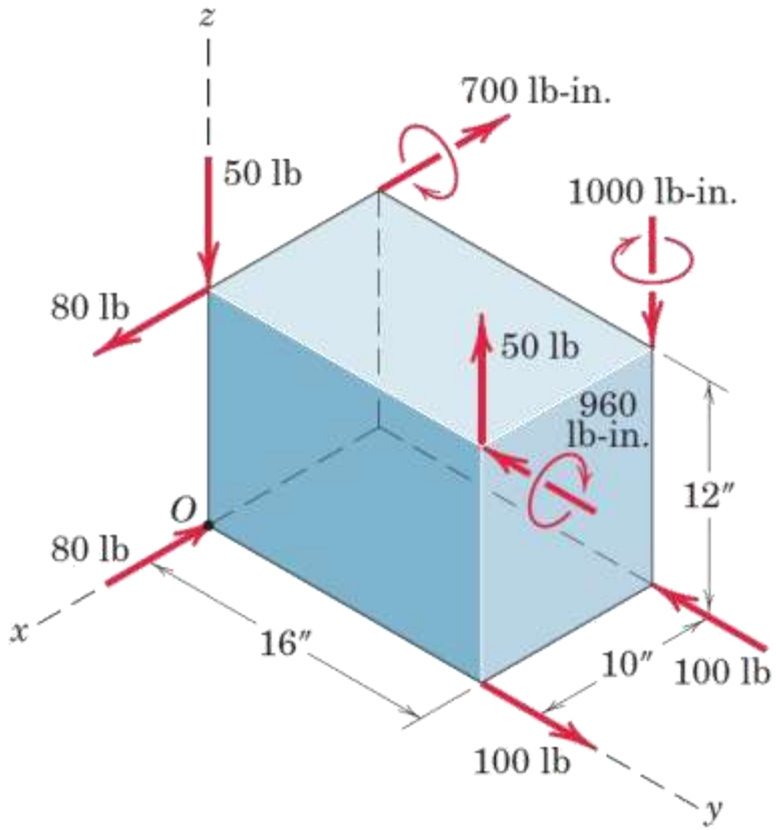
The sum of the moments about O is

$$\mathbf{M}_O = [50(16) - 700]\mathbf{i} + [80(12) - 960]\mathbf{j} + [100(10) - 1000]\mathbf{k} \text{ lb-in.}$$

$$= 100\mathbf{i} \text{ lb-in.} \quad ②$$

① Since the force summation is zero, we conclude that the resultant, if it exists, must be a couple.

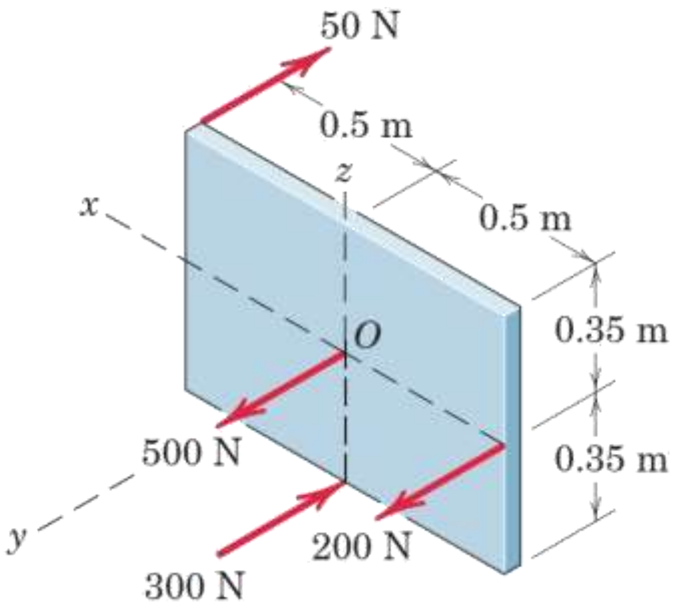
② The moments associated with the force pairs are easily obtained by using the $M = Fd$ rule and assigning the unit-vector direction by inspection. In many three-dimensional problems, this may be simpler than the $\mathbf{M} = \mathbf{r} \times \mathbf{F}$ approach.



Article 2/9 – Sample Problem 2/17 (1 of 2)

- Problem Statement

  Determine the resultant of the system of parallel forces which act on the plate. Solve with a vector approach.



Article 2/9 – Sample Problem 2/17 (2 of 2)

- Solution

$$\mathbf{R} = \Sigma\mathbf{F} = (200 + 500 - 300 - 50)\mathbf{j} = 350\mathbf{j} \text{ N}$$

$$\mathbf{M}_O = [50(0.35) - 300(0.35)]\mathbf{i} + [-50(0.50) - 200(0.50)]\mathbf{k}$$

$$= -87.5\mathbf{i} - 125\mathbf{k} \text{ N}\cdot\text{m}$$

The placement of **R** so that it alone represents the above force–couple system is determined by the principle of moments in vector form

$$\mathbf{r} \times \mathbf{R} = \mathbf{M}_O$$

$$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times 350\mathbf{j} = -87.5\mathbf{i} - 125\mathbf{k}$$

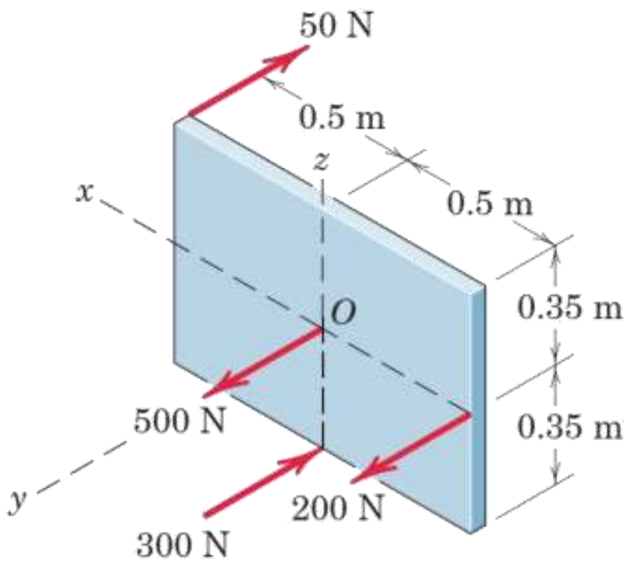
$$350x\mathbf{k} - 350z\mathbf{i} = -87.5\mathbf{i} - 125\mathbf{k}$$

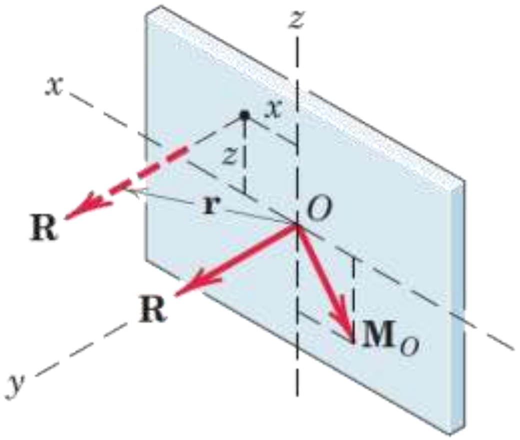
From the one vector equation we may obtain the two scalar equations

$$350x = -125 \qquad \text{and} \qquad -350z = -87.5$$

Hence, $x = -0.357$ m and $z = 0.250$ m are the coordinates through which the line of action of **R** must pass. The value of y may, of course, be any value, as permitted by the principle of transmissibility. Thus, as expected, the variable y drops out of the above vector analysis. ①

① You should also carry out a scalar solution to this problem.

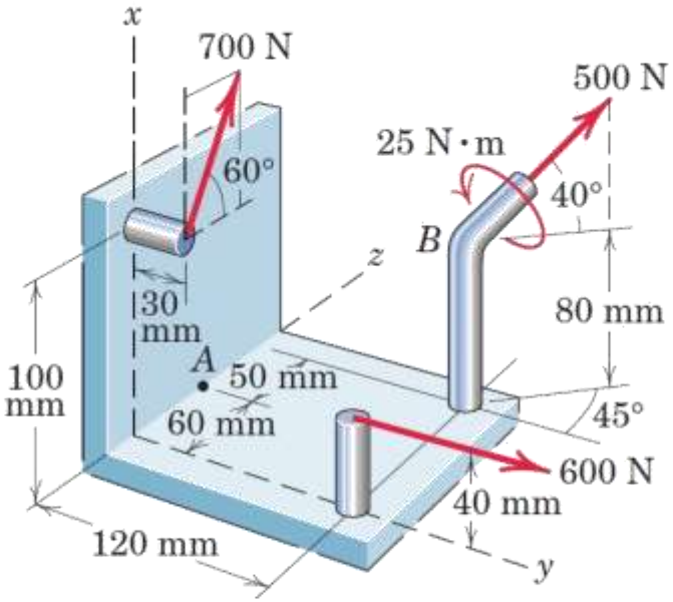




Article 2/9 – Sample Problem 2/18 (1 of 4)

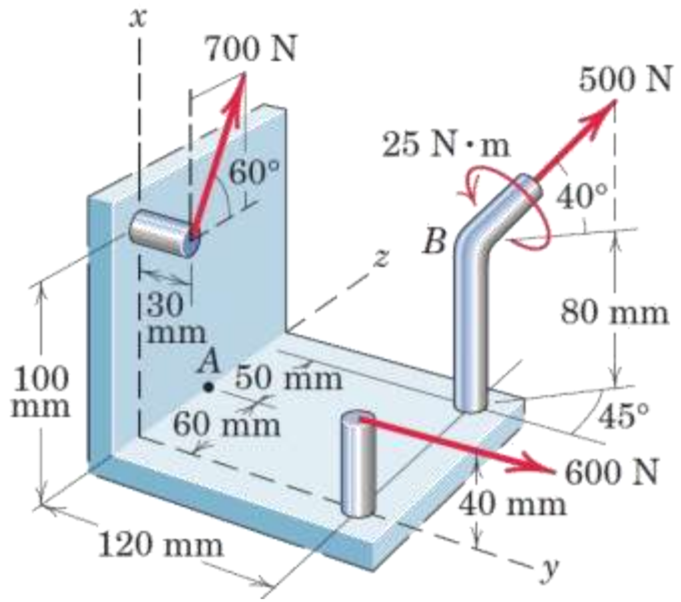
- Problem Statement

  Replace the two forces and the negative wrench by a single force **R** applied at *A* and the corresponding couple **M**.



Article 2/9 – Sample Problem 2/18 (2 of 4)

- Force Resultant



$[R_x = \Sigma F_x]$ $\quad R_x = 500 \sin 40° + 700 \sin 60° = 928$ N

$[R_y = \Sigma F_y]$ $\quad R_y = 600 + 500 \cos 40° \cos 45° = 871$ N

$[R_z = \Sigma F_z]$ $\quad R_z = 700 \cos 60° + 500 \cos 40° \sin 45° = 621$ N

Thus, $\quad \mathbf{R} = 928\mathbf{i} + 871\mathbf{j} + 621\mathbf{k}$ N

and $\quad R = \sqrt{(928)^2 + (871)^2 + (621)^2} = 1416$ N $\qquad$ *Ans.*

- Couple from the 500-N Force

$[\mathbf{M} = \mathbf{r} \times \mathbf{F}]$ $\quad \mathbf{M}_{500} = (0.08\mathbf{i} + 0.12\mathbf{j} + 0.05\mathbf{k}) \times 500(\mathbf{i} \sin 40° + \mathbf{j} \cos 40° \cos 45° + \mathbf{k} \cos 40° \sin 45°)$ ①

where $\mathbf{r}$ is the vector from A to B.

The term-by-term, or determinant, expansion gives

$$\mathbf{M}_{500} = 18.95\mathbf{i} - 5.59\mathbf{j} - 16.90\mathbf{k} \text{ N}\cdot\text{m}$$

① *Suggestion:* Check the cross-product results by evaluating the moments about A of the components of the 500-N force directly from the sketch.

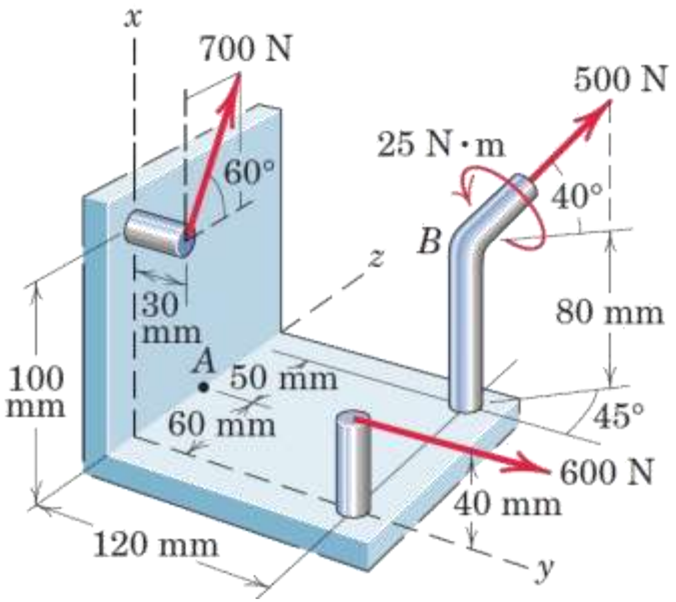
Article 2/9 – Sample Problem 2/18 (3 of 4)

- Couple from the 600-N Force

$$\mathbf{M}_{600} = (600)(0.060)\mathbf{i} + (600)(0.040)\mathbf{k}$$
$$= 36.0\mathbf{i} + 24.0\mathbf{k}\ \text{N}\cdot\text{m}$$

- Couple from the 700-N Force

$$\mathbf{M}_{700} = (700 \cos 60°)(0.030)\mathbf{i} - [(700 \sin 60°)(0.060) + (700 \cos 60°)(0.100)]\mathbf{j} - (700 \sin 60°)(0.030)\mathbf{k}$$
$$= 10.5\mathbf{i} - 71.4\mathbf{j} - 18.19\mathbf{k}\ \text{N}\cdot\text{m}$$



② For the 600-N and 700-N forces it is easier to obtain the components of their moments about the coordinate directions through A by inspection of the figure than it is to set up the cross-product relations.

Article 2/9 – Sample Problem 2/18 (4 of 4)

- Couple from the Wrench Moment

$$\mathbf{M}' = 25.0(-\mathbf{i}\sin 40° - \mathbf{j}\cos 40°\cos 45° - \mathbf{k}\cos 40°\sin 45°)$$
$$= -16.07\mathbf{i} - 13.54\mathbf{j} - 13.54\mathbf{k}\ \text{N}\cdot\text{m}$$

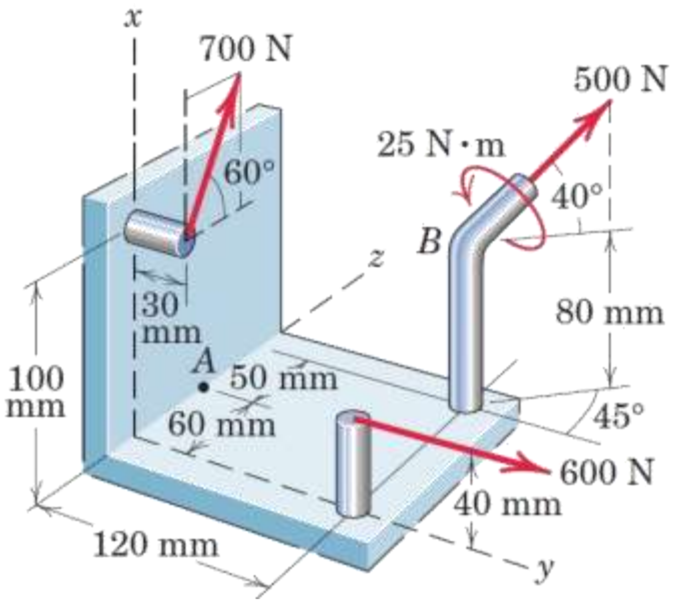
- Resultant Couple at *A*

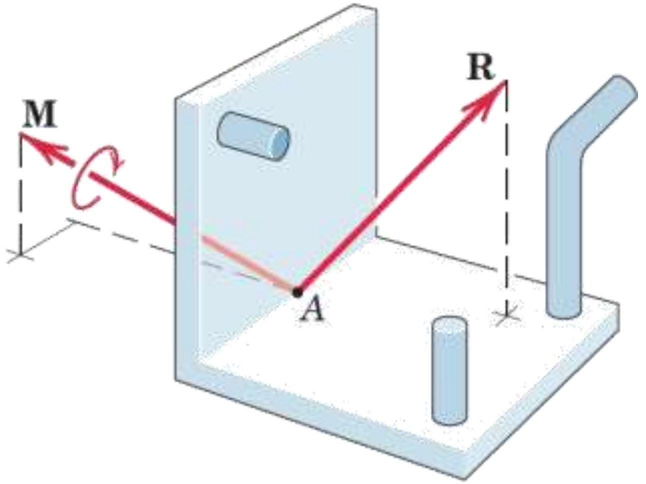
$$\mathbf{M} = 49.4\mathbf{i} - 90.5\mathbf{j} - 24.6\mathbf{k}\ \text{N}\cdot\text{m} \quad ④$$

and $\qquad M = \sqrt{(49.4)^2 + (90.5)^2 + (24.6)^2} = 106.0\ \text{N}\cdot\text{m}$ *Ans.*

③ The 25-N·m couple vector of the *wrench* points in the direction opposite to that of the 500-N force, and we must resolve it into its x-, y-, and z-components to be added to the other couple-vector components.

④ Although the resultant couple vector **M** in the sketch of the resultants is shown through A, we recognize that a couple vector is a free vector and therefore has no specified line of action.

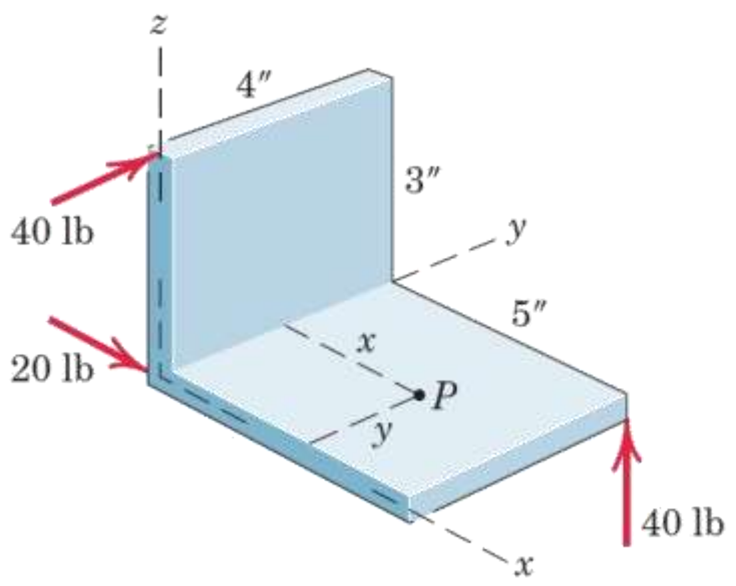




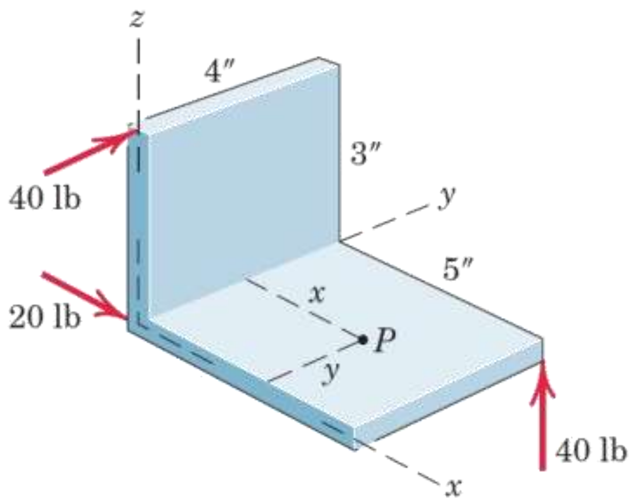
Article 2/9 – Sample Problem 2/19 (1 of 3)

- Problem Statement

  Determine the wrench resultant of the three forces acting on the bracket. Calculate the coordinates of the point P in the x-y plane through which the resultant force of the wrench acts. Also find the magnitude of the couple **M** of the wrench.



Article 2/9 – Sample Problem 2/19 (2 of 3)



- Resultant Force

$$\mathbf{R} = 20\mathbf{i} + 40\mathbf{j} + 40\mathbf{k} \text{ lb} \qquad R = \sqrt{(20)^2 + (40)^2 + (40)^2} = 60 \text{ lb}$$

and its direction cosines are

$$\cos\theta_x = 20/60 = 1/3 \qquad \cos\theta_y = 40/60 = 2/3 \qquad \cos\theta_z = 40/60 = 2/3$$

- Moment about *P*

The moment of the wrench couple must equal the sum of the moments of the given forces about point P through which $\mathbf{R}$ passes. The moments about P of the three forces are

$$(\mathbf{M})_{R_x} = 20y\mathbf{k} \text{ lb-in.}$$

$$(\mathbf{M})_{R_y} = -40(3)\mathbf{i} - 40x\mathbf{k} \text{ lb-in.}$$

$$(\mathbf{M})_{R_z} = 40(4 - y)\mathbf{i} - 40(5 - x)\mathbf{j} \text{ lb-in.}$$

and the total moment is

$$\mathbf{M} = (40 - 40y)\mathbf{i} + (-200 + 40x)\mathbf{j} + (-40x + 20y)\mathbf{k} \text{ lb-in.}$$

Article 2/9 – Sample Problem 2/19 (3 of 3)

- Final Solution

The direction cosines of **M** are

$$\cos\theta_x = (40 - 40y)/M$$

$$\cos\theta_y = (-200 + 40x)/M$$

$$\cos\theta_z = (-40x + 20y)/M$$

where M is the magnitude of **M**. Equating the direction cosines of **R** and **M** gives

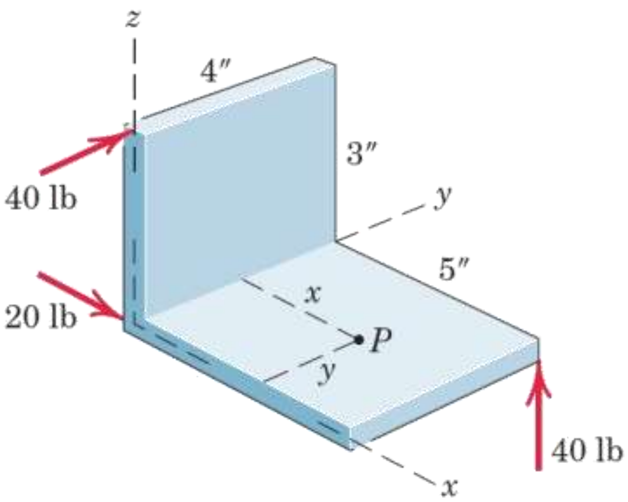
$$40 - 40y = \frac{M}{3}$$

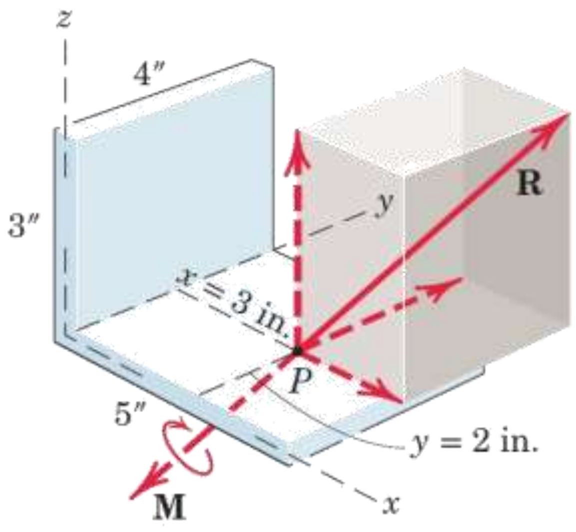
$$-200 + 40x = \frac{2M}{3}$$

$$-40x + 20y = \frac{2M}{3}$$

Solution of the three equations gives

$M = -120$ lb-in. $\quad x = 3$ in. $\quad y = 2$ in. $\quad$ *Ans.*





CHAPTER 3
EQUILIBRIUM

Bialobrzeski/Iaif/Redux Pictures

CHAPTER OUTLINE

3/1 Introduction

SECTION A Equilibrium in Two Dimensions

3/2 System Isolation and the Free-Body Diagram

3/3 Equilibrium Conditions

SECTION B Equilibrium in Three Dimensions

3/4 Equilibrium Conditions

Article 3/1 Introduction

- Equilibrium Conditions (Eq. 3/1)
  - Force Balance: $\Sigma\mathbf{F} = \mathbf{0}$
  - Moment Balance: $\Sigma\mathbf{M} = \mathbf{0}$

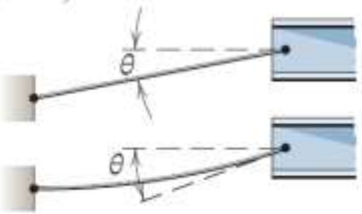
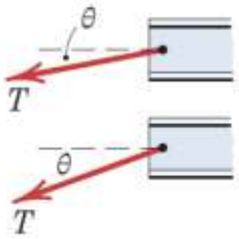
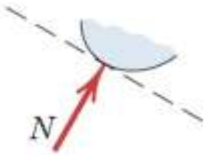
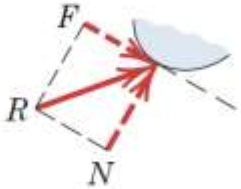
Article 3/2 System Isolation and the Free-Body Diagram

- System Isolation

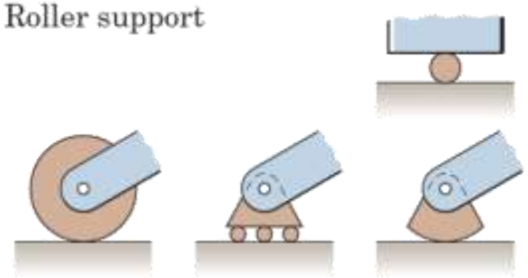
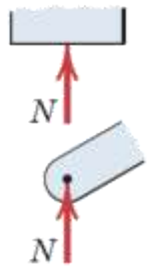
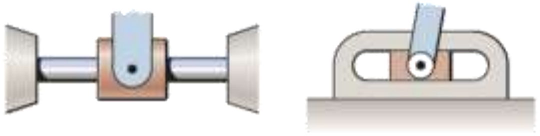
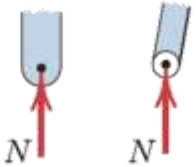
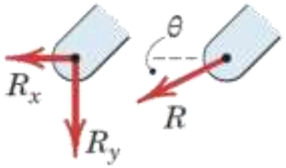
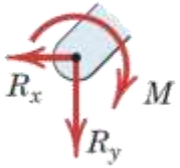
- Free-Body Diagram

  - The free-body diagram is the most important single step in the solution of problems in mechanics.

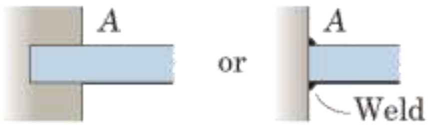
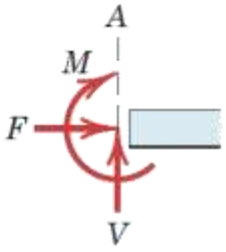
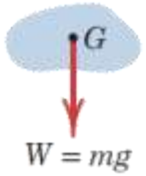
Article 3/2 – Modeling the Action of Forces (1 of 4)

MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS	
Type of Contact and Force Origin	**Action on Body to Be Isolated**
1. Flexible cable, belt, chain, or rope Weight of cable negligible Weight of cable not negligible 	 Force exerted by a flexible cable is always a tension away from the body in the direction of the cable.
2. Smooth surfaces 	 Contact force is compressive and is normal to the surface.
3. Rough surfaces 	 Rough surfaces are capable of supporting a tangential component F (frictional force) as well as a normal component N of the resultant contact force R.

Article 3/2 – Modeling the Action of Forces (2 of 4)

MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS	
Type of Contact and Force Origin	**Action on Body to Be Isolated**
4. Roller support 	 Roller, rocker, or ball support transmits a compressive force normal to the supporting surface.
5. Freely sliding guide 	 Collar or slider free to move along smooth guides; can support force normal to guide only.
6. Pin connection 	Pin free to turn  Pin not free to turn  A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the pin axis. We may either show two components R_x and R_y or a magnitude R and direction θ. A pin not free to turn also supports a couple M.

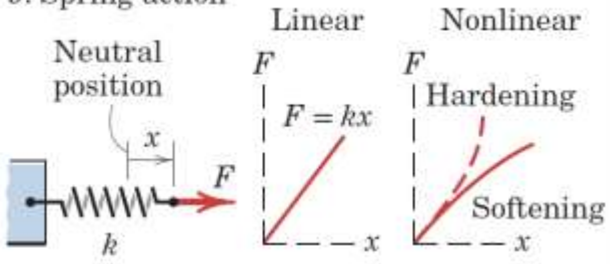
Article 3/2 – Modeling the Action of Forces (3 of 4)

MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS		
Type of Contact and Force Origin	Action on Body to Be Isolated	
7. Built-in or fixed support 		A built-in or fixed support is capable of supporting an axial force F, a transverse force V (shear force), and a couple M (bending moment) to prevent rotation.
8. Gravitational attraction 		The resultant of gravitational attraction on all elements of a body of mass m is the weight $W = mg$ and acts toward the center of the earth through the center of gravity G.

Article 3/2 – Modeling the Action of Forces (4 of 4)

<table>
<tr><th colspan="2">MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS</th></tr>
<tr><th>Type of Contact and Force Origin</th><th>Action on Body to Be Isolated</th></tr>
<tr><td>

9. Spring action



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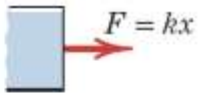

Spring force is tensile if the spring is stretched and compressive if compressed. For a linearly elastic spring the stiffness k is the force required to deform the spring a unit distance.

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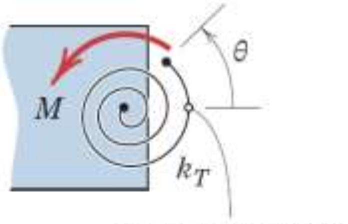
<tr><td>

10. Torsional spring action



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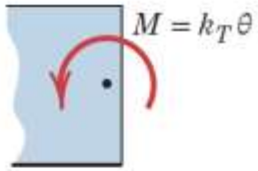

For a linear torsional spring, the applied moment M is proportional to the angular deflection θ from the neutral position. The stiffness k_T is the moment required to deform the spring one radian.

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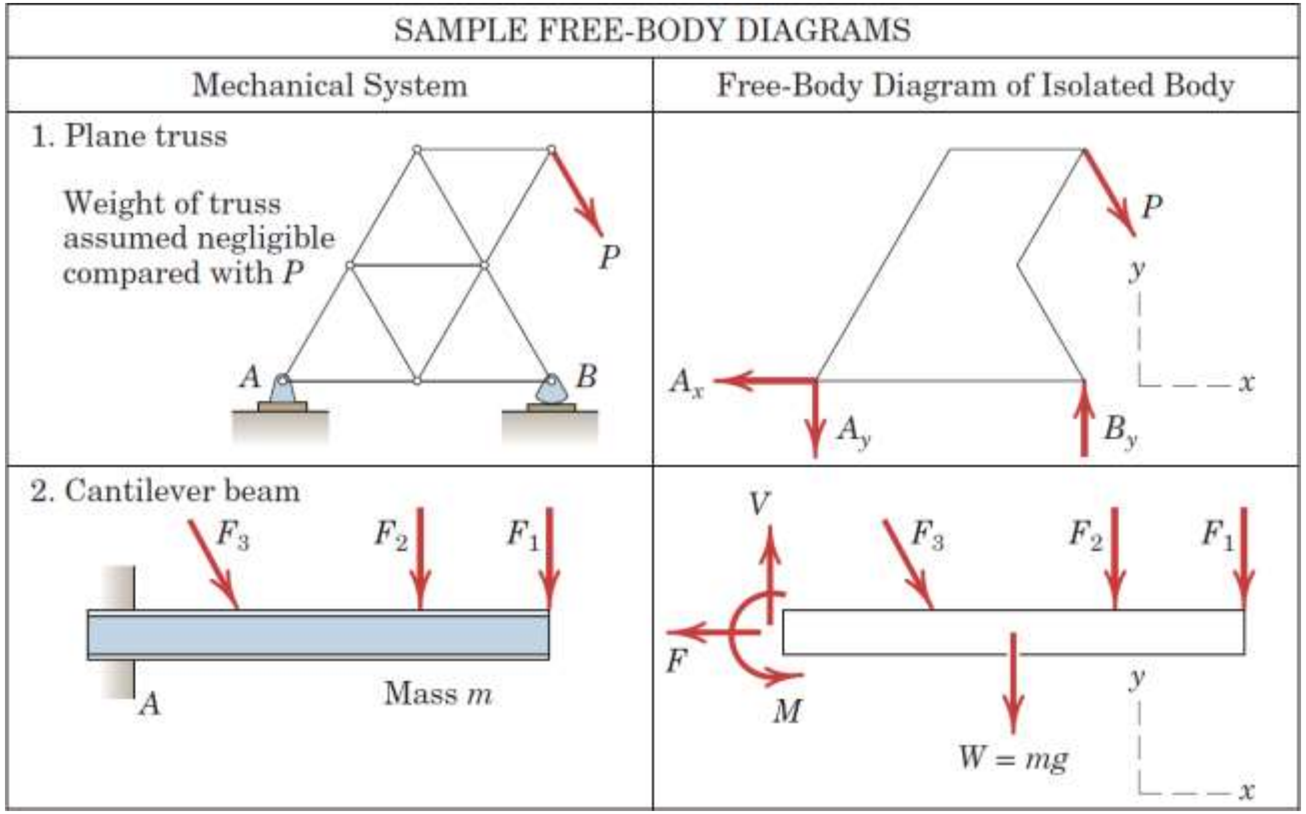
</table>

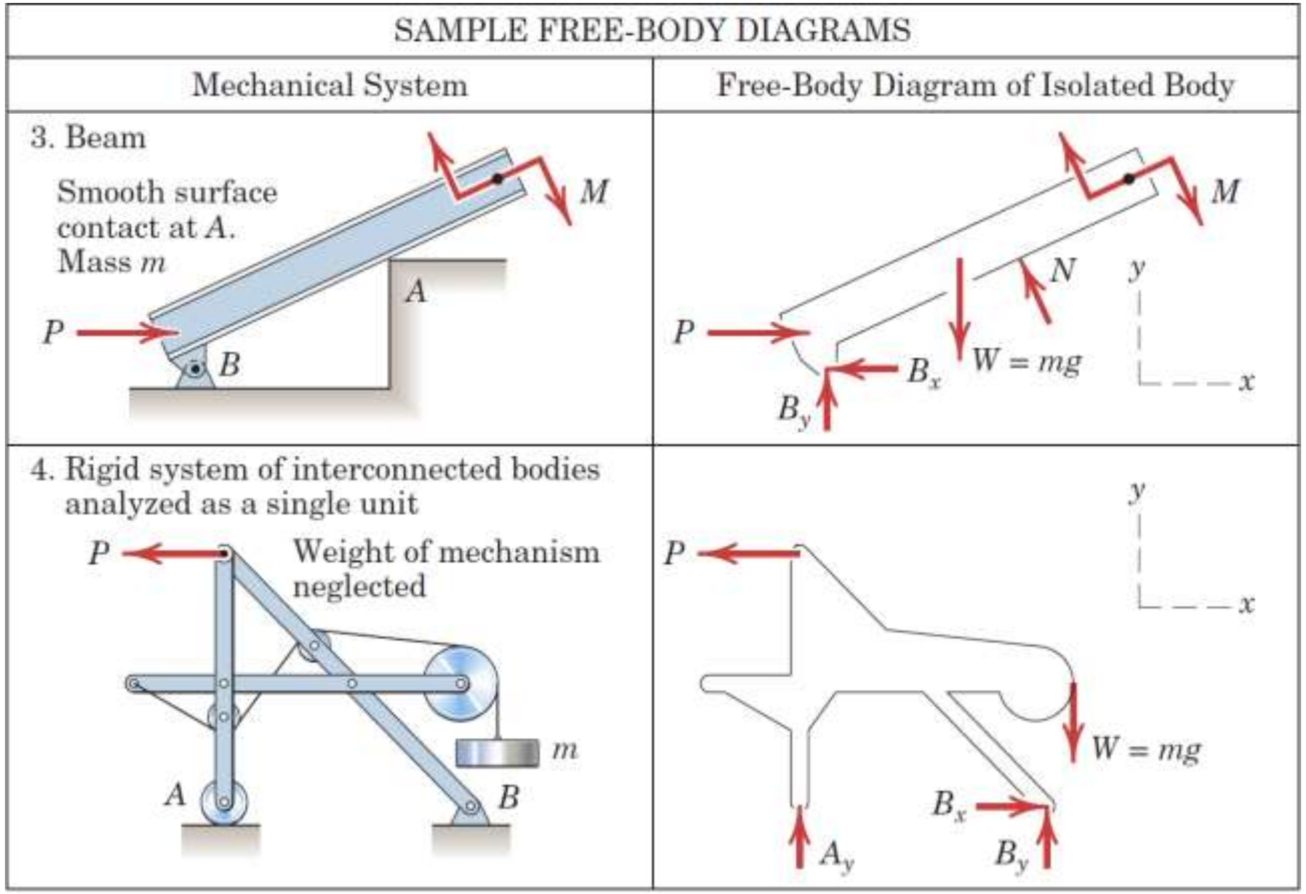
Article 3/2 – Construction of Free-Body Diagrams

- Step 1: Decide which system to isolate.
- Step 2: Isolate the system by drawing a diagram which represents its complete external boundary.
- Step 3: Identify all forces which act on the isolated system as applied by the removed contacting and attracting bodies, and represent them in their proper positions on the diagram of the isolated system.
- Step 4: Show the choice of coordinate axes directly on the diagram.

Article 3/2 – Examples of Free-Body Diagrams (1 of 2)



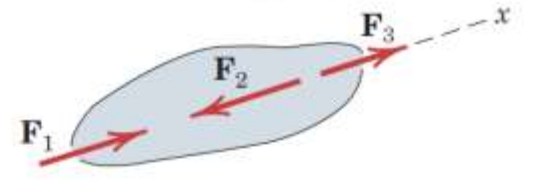
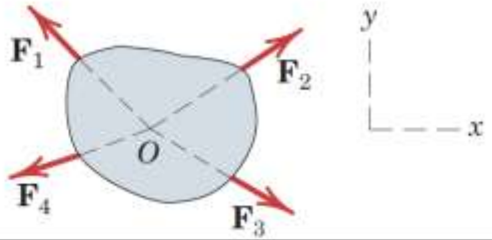
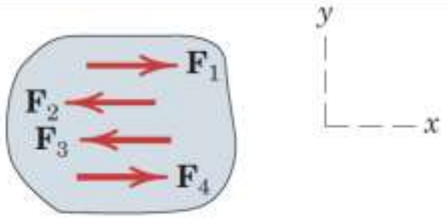
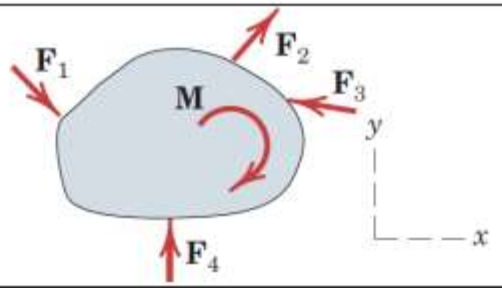
Article 3/2 – Examples of Free-Body Diagrams (2 of 2)



Article 3/3 Equilibrium Conditions

- Scalar Format (Eq. 3/2)
  - $\Sigma F_x = 0$
  - $\Sigma F_y = 0$
  - $\Sigma M_O = 0$

Article 3/3 – Categories of Equilibrium

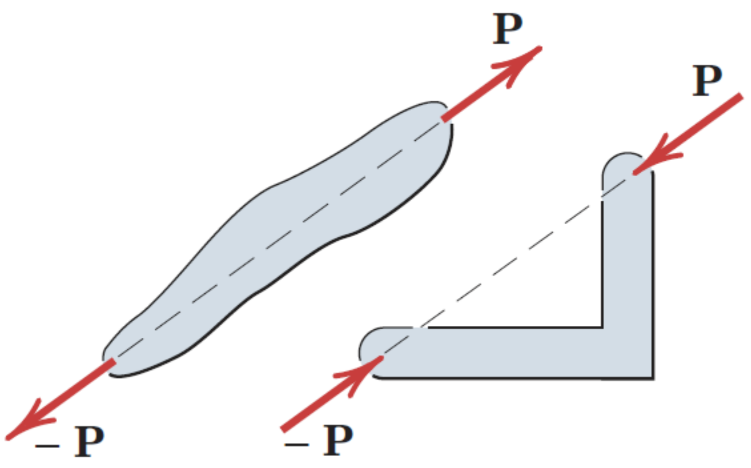
CATEGORIES OF EQUILIBRIUM IN TWO DIMENSIONS		
Force System	Free-Body Diagram	Independent Equations
1. Collinear		$\Sigma F_x = 0$
2. Concurrent at a point		$\Sigma F_x = 0$ $\Sigma F_y = 0$
3. Parallel		$\Sigma F_x = 0$ $\quad \Sigma M_z = 0$
4. General		$\Sigma F_x = 0$ $\quad \Sigma M_z = 0$ $\Sigma F_y = 0$

Article 3/3 – Two-Force Members

- Definition

  Occurs when a body is in equilibrium under the action of two forces only.

- Forces are…
  - Equal
  - Opposite
  - Collinear
  - Independent of Object Shape



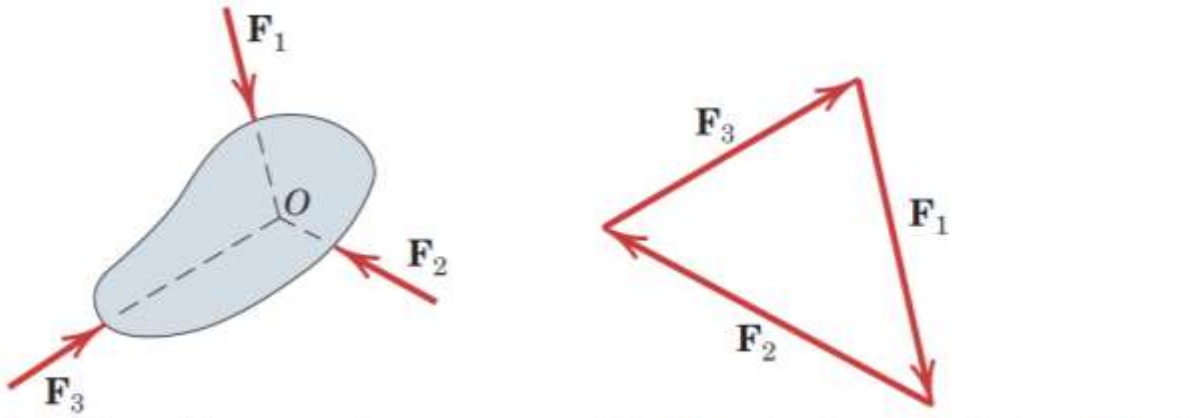
Article 3/3 – Three-Force Members

- Definition

  Occurs when a body is in equilibrium under the action of three forces only.

- Forces are Concurrent

- Case of Parallel Forces

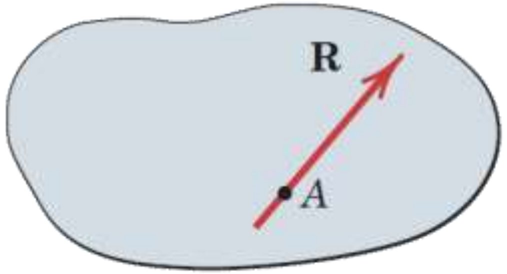


(*a*) Three-force member (*b*) Closed polygon satisfies $\Sigma\mathbf{F} = \mathbf{0}$

Article 3/3 – Alternative Equilibrium Equations (1 of 2)

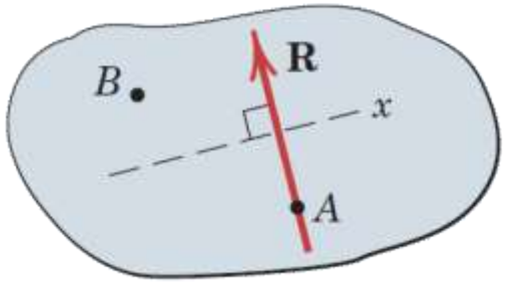
- Case 1: Use of Two Moment Equations
  - Case (*a*)
    - Resultant Intersects Point *A*
  - Case (*b*)
    - Resultant Intersects Point *A*
    - Resultant is Perpendicular to *x*-axis
  - Equilibrium Conditions
    - $\Sigma F_x = 0$
    - $\Sigma M_A = 0$
    - $\Sigma M_B = 0$

$\Sigma M_A = 0$ satisfied



(*a*)

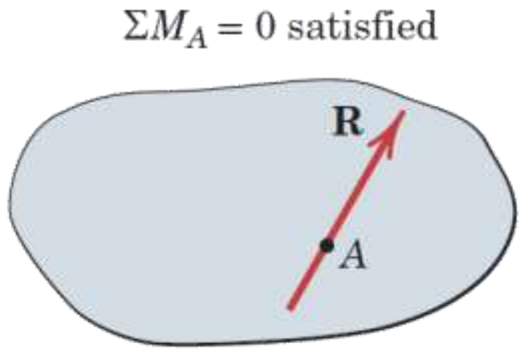
$\left.\begin{array}{l}\Sigma M_A = 0\\ \Sigma F_x = 0\end{array}\right\}$ satisfied



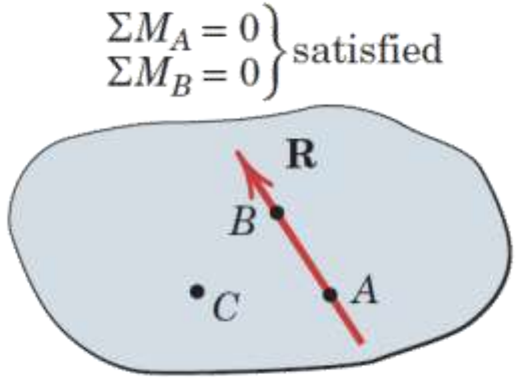
(*b*)

Article 3/3 – Alternative Equilibrium Equations (2 of 2)

- Case 2: Use of Three Moment Equations
  - Case (c)
    - Resultant Intersects Point A
  - Case (d)
    - Resultant Intersects Point A
    - Resultant Intersects Point B
  - Equilibrium Conditions
    - $\Sigma M_A = 0$
    - $\Sigma M_B = 0$
    - $\Sigma M_C = 0$



(c)



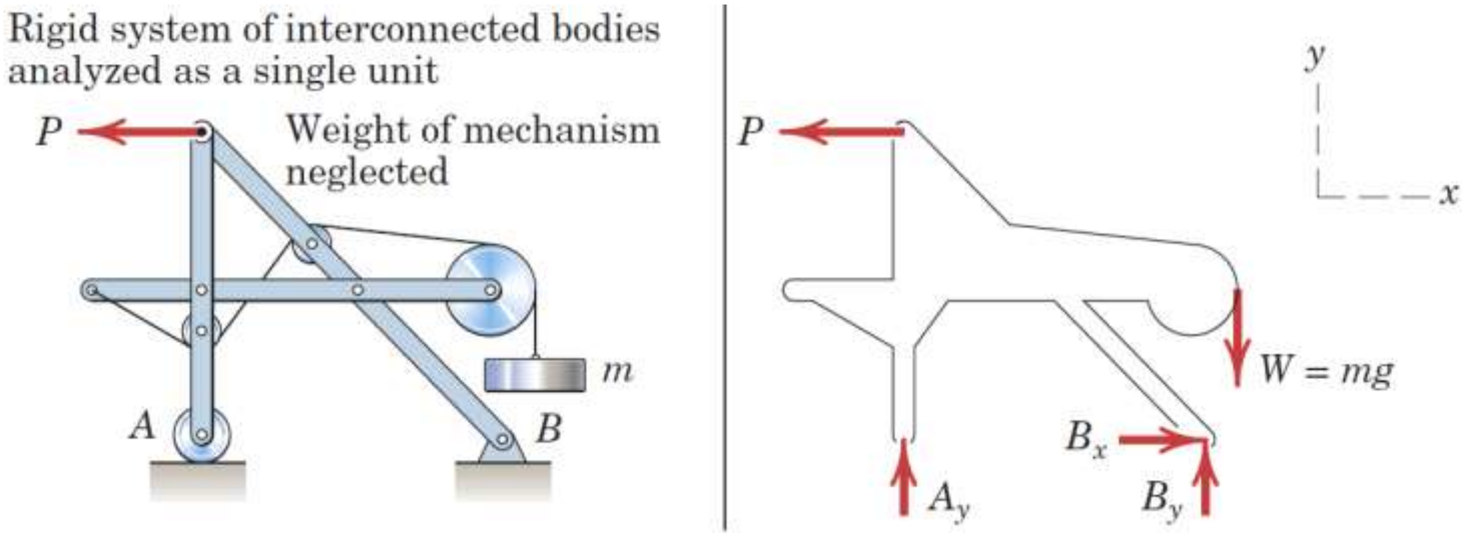
(d)

Article 3/3 – Constraints and Statical Determinacy

- Important Terms
  - Constraint
  - Statically Determinant
  - Statically Indeterminant
  - Redundancy
  - Degree of Statical Indeterminacy

Article 3/3 – Illustration of Determinacy

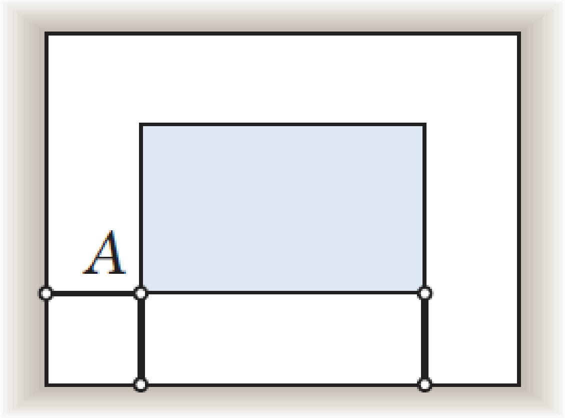
- Statically Determinant System



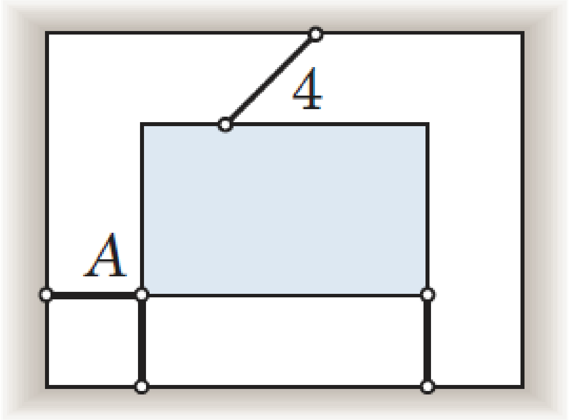
- Statically Indeterminant System – Replace Roller *A* with a Pin

Article 3/3 – Adequacy of Constraints (1 of 2)

- Complete Fixity (Adequate Constraints)

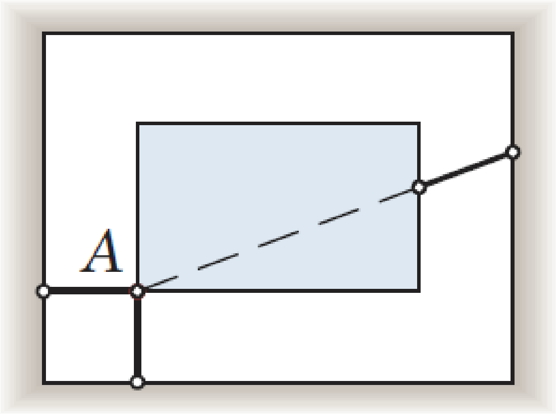


- Excessive Fixity (Redundant Constraints)

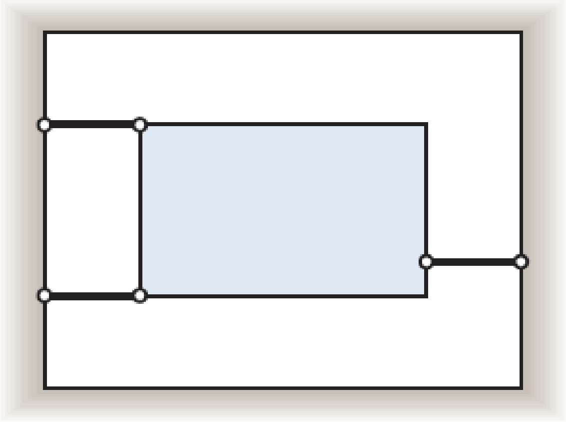


Article 3/3 – Adequacy of Constraints (2 of 2)

- Incomplete Fixity (Partial Constraints)
  - Concurrent Reactions



- Incomplete Fixity (Partial Constraints)
  - Parallel Reactions



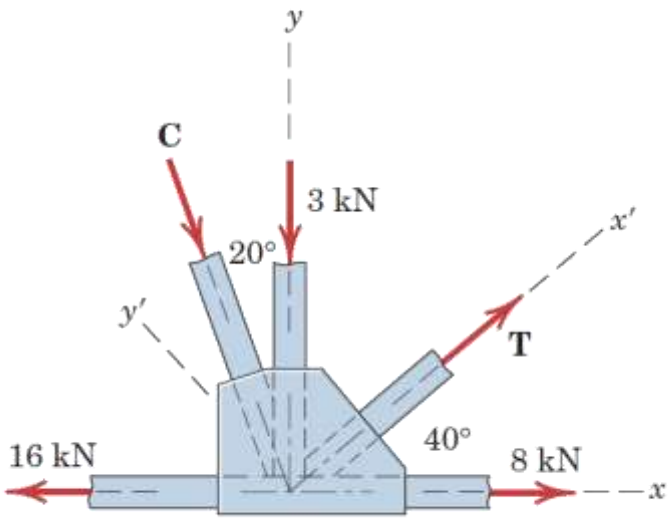
Article 3/3 – Approach to Solving Problems

- Identify clearly the quantities which are known and unknown.
- Make an unambiguous choice of the body (or system) to be isolated and draw its complete free-body diagram.
- Choose a convenient set of reference axes.
- Identify and state the applicable force and moment equations which govern the equilibrium conditions of the problem.
- Match the number of independent equations with the number of unknowns in the problem.
- Carry out the solution and check the results.

Article 3/3 – Sample Problem 3/1 (1 of 4)

- Problem Statement

  Determine the magnitudes of the forces **C** and **T**, which, along with the other three forces shown, act on the bridge-truss joint.



Article 3/3 – Sample Problem 3/1 (2 of 4)

- Solution I (Scalar Algebra) with *x*-*y* axes

$$[\Sigma F_x = 0] \qquad 8 + T\cos 40^\circ + C\sin 20^\circ - 16 = 0$$

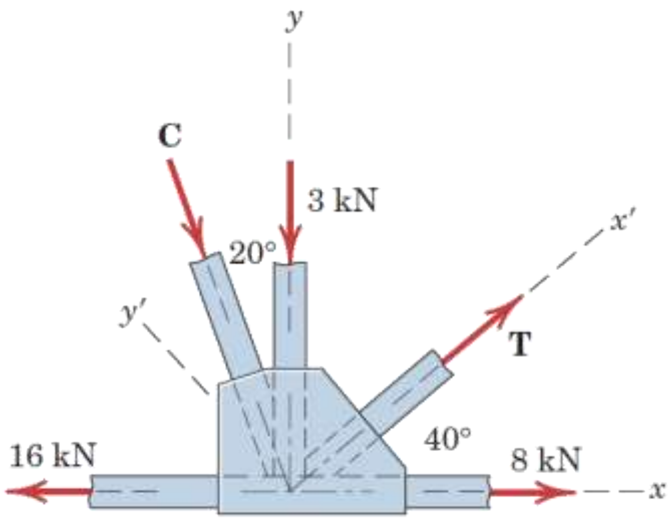
$$0.766T + 0.342C = 8 \qquad (a)$$

$$[\Sigma F_y = 0] \qquad T\sin 40^\circ - C\cos 20^\circ - 3 = 0$$

$$0.643T - 0.940C = 3 \qquad (b)$$

Simultaneous solution of Eqs. (a) and (b) produces

$$T = 9.09 \text{ kN} \qquad C = 3.03 \text{ kN} \qquad \textit{Ans.}$$



Article 3/3 – Sample Problem 3/1 (3 of 4)

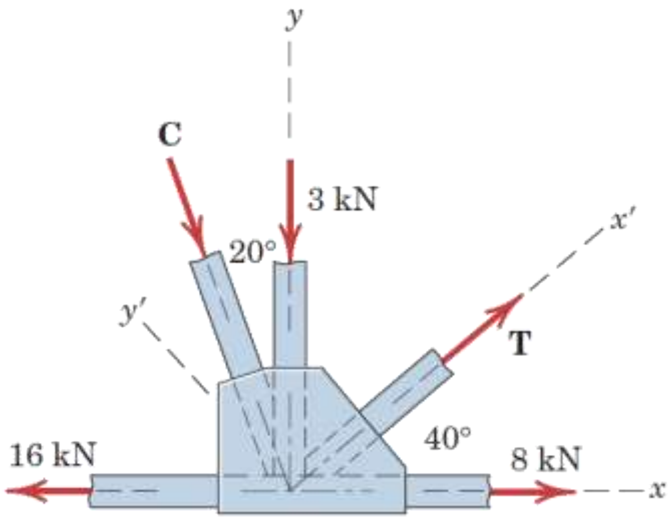
- Solution II (Scalar Algebra) with *x′*-*y′* axes

$[\Sigma F_{y'} = 0]$ $\quad -C \cos 20° - 3 \cos 40° - 8 \sin 40° + 16 \sin 40° = 0$

$C = 3.03 \text{ kN}$ *Ans.*

$[\Sigma F_{x'} = 0]$ $\quad T + 8 \cos 40° - 16 \cos 40° - 3 \sin 40° - 3.03 \sin 20° = 0$

$T = 9.09 \text{ kN}$ *Ans.*



Article 3/3 – Sample Problem 3/1 (4 of 4)

- Solution III (Vector Algebra) with *x*-*y* axes

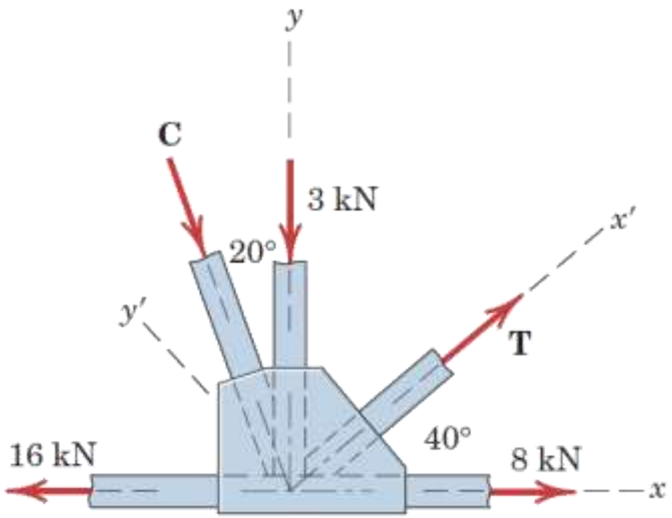
$$[\Sigma \mathbf{F} = \mathbf{0}] \qquad 8\mathbf{i} + (T\cos 40^\circ)\mathbf{i} + (T\sin 40^\circ)\mathbf{j} - 3\mathbf{j} + (C\sin 20^\circ)\mathbf{i} - (C\cos 20^\circ)\mathbf{j} - 16\mathbf{i} = \mathbf{0}$$

Equating the coefficients of the **i**- and **j**-terms to zero gives

$$8 + T\cos 40^\circ + C\sin 20^\circ - 16 = 0$$

$$T\sin 40^\circ - 3 - C\cos 20^\circ = 0$$

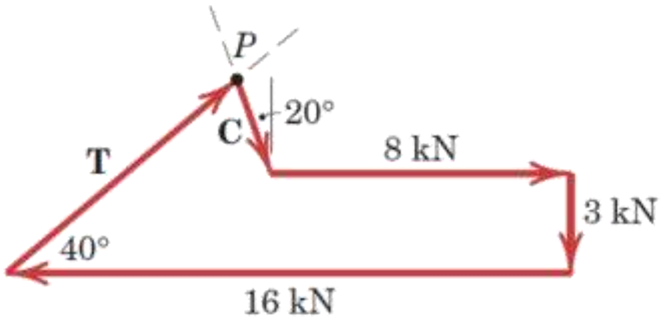
which are the same, of course, as Eqs. (*a*) and (*b*), which we solved above.

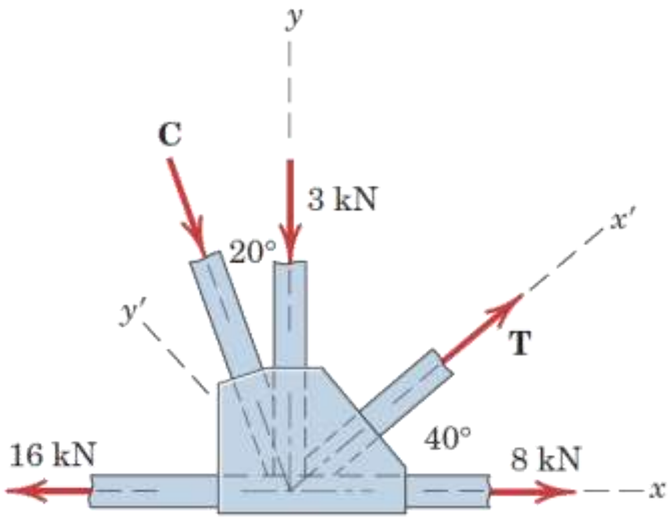


Article 3/3 – Sample Problem 3/1 (4 of 4)

- Solution IV (Geometric)

A graphical solution is easily obtained. The known vectors are laid off head-to-tail to some convenient scale, and the directions of **T** and **C** are then drawn to close the polygon. ③ The resulting intersection at point P completes the solution, thus enabling us to measure the magnitudes of **T** and **C** directly from the drawing to whatever degree of accuracy we incorporate in the construction.



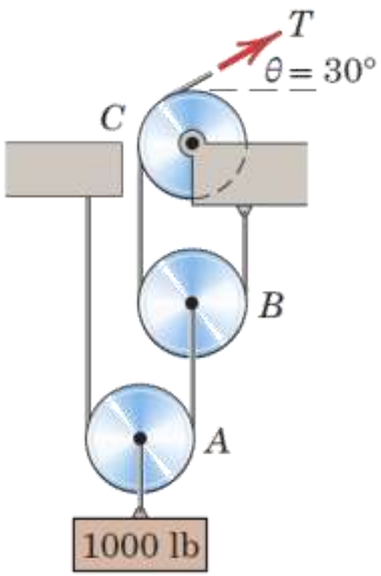


③ The known vectors may be added in any order desired, but they must be added before the unknown vectors.

Article 3/3 – Sample Problem 3/2 (1 of 3)

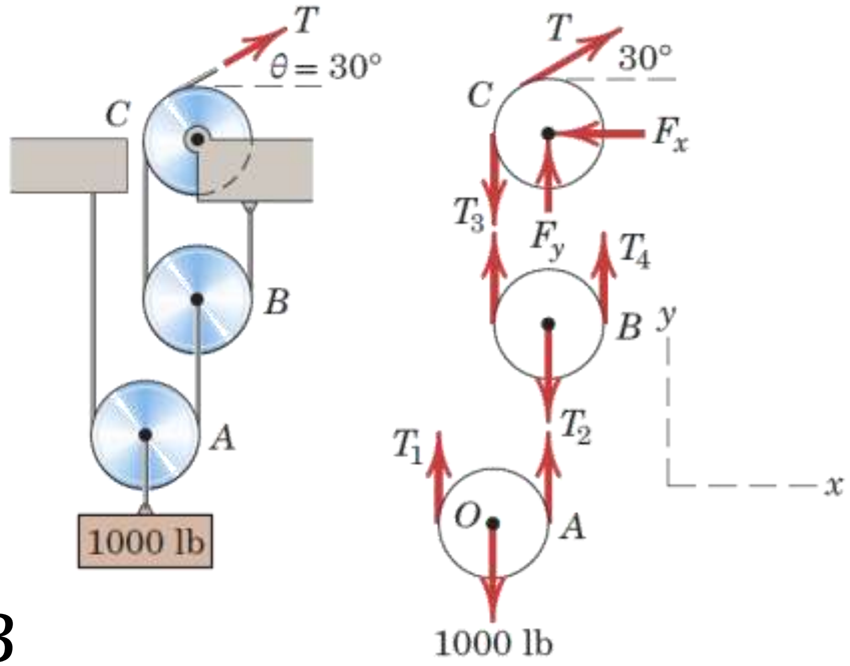
- Problem Statement

Calculate the tension T in the cable which supports the 1000-lb load with the pulley arrangement shown. Each pulley is free to rotate about its b earing, and the weights of all parts are small compared with the load. Find the magnitude of the total force on the bearing of pulley C.



Article 3/3 – Sample Problem 3/2 (2 of 3)

- Free-Body Diagrams



- Equilibrium Conditions for Pulleys *A* and *B*

$[\Sigma M_O = 0] \qquad T_1 r - T_2 r = 0 \qquad T_1 = T_2$ ①

$[\Sigma F_y = 0] \qquad T_1 + T_2 - 1000 = 0 \qquad 2T_1 = 1000 \qquad T_1 = T_2 = 500 \text{ lb}$

From the example of pulley *A* we may write the equilibrium of forces on pulley *B* by inspection as

$$T_3 = T_4 = T_2/2 = 250 \text{ lb}$$

① Clearly the radius r does not influence the results. Once we have analyzed a simple pulley, the results should be perfectly clear by inspection.

Article 3/3 – Sample Problem 3/2 (3 of 3)

- Equilibrium Conditions for Pulley *C*

For pulley C the angle $\theta = 30°$ in no way affects the moment of T about the center of the pulley, so that moment equilibrium requires

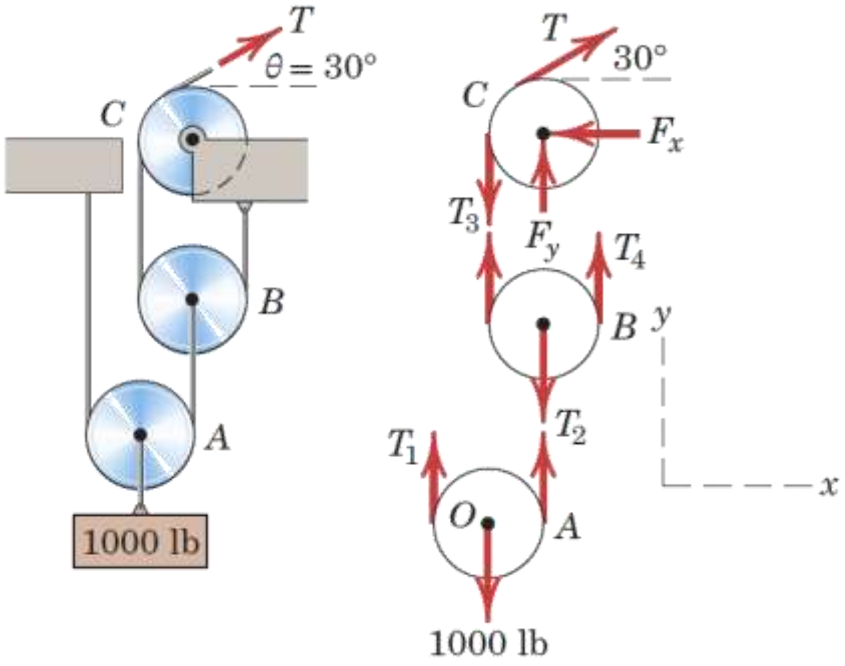
$$T = T_3 \qquad \text{or} \qquad T = 250 \text{ lb} \qquad \textit{Ans.}$$

Equilibrium of the pulley in the x- and y-directions requires

$$[\Sigma F_x = 0] \qquad 250 \cos 30° - F_x = 0 \qquad F_x = 217 \text{ lb}$$

$$[\Sigma F_y = 0] \qquad F_y + 250 \sin 30° - 250 = 0 \qquad F_y = 125 \text{ lb}$$

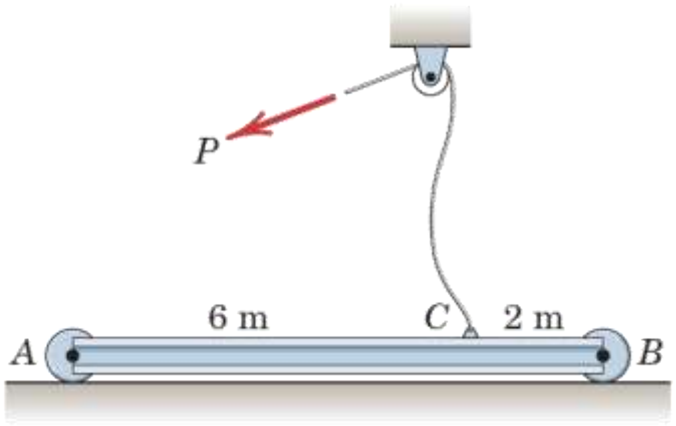
$$\left[F = \sqrt{F_x^{\,2} + F_y^{\,2}}\right] \qquad F = \sqrt{(217)^2 + (125)^2} = 250 \text{ lb} \qquad \textit{Ans.}$$



Article 3/3 – Sample Problem 3/3 (1 of 2)

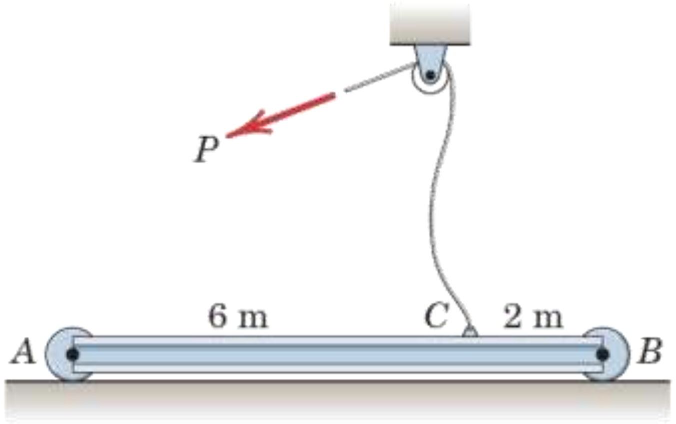
- Problem Statement

  The uniform 100-kg I-beam is supported initially by its end rollers on the horizontal surface at *A* and *B*. By means of the cable at *C*, it is desired to elevate end *B* to a position 3 m above end *A*. Determine the required tension *P*, the reaction at *A*, and the angle θ made by the beam with the horizontal in the elevated position.



Article 3/3 – Sample Problem 3/3 (2 of 2)

- Free-Body Diagram



- Equilibrium Conditions

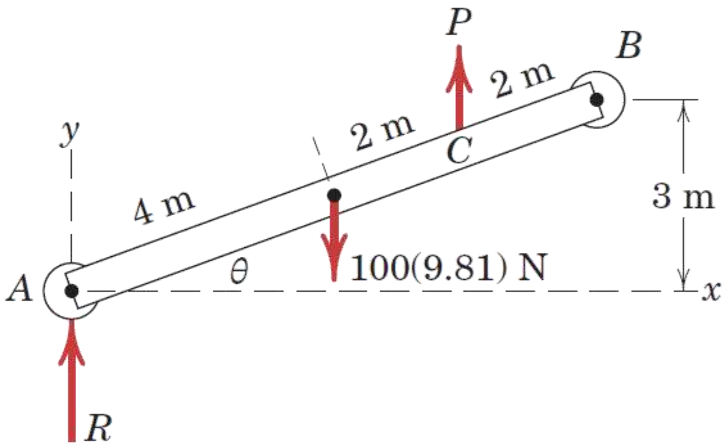
$[\Sigma M_A = 0]$ $\quad P(6 \cos \theta) - 981(4 \cos \theta) = 0 \qquad P = 654\text{ N}$ ① *Ans.*

Equilibrium of vertical forces requires

$[\Sigma F_y = 0]$ $\quad 654 + R - 981 = 0 \qquad R = 327\text{ N}$ *Ans.*

The angle θ depends only on the specified geometry and is

$$\sin \theta = 3/8 \qquad \theta = 22.0°$$ *Ans.*

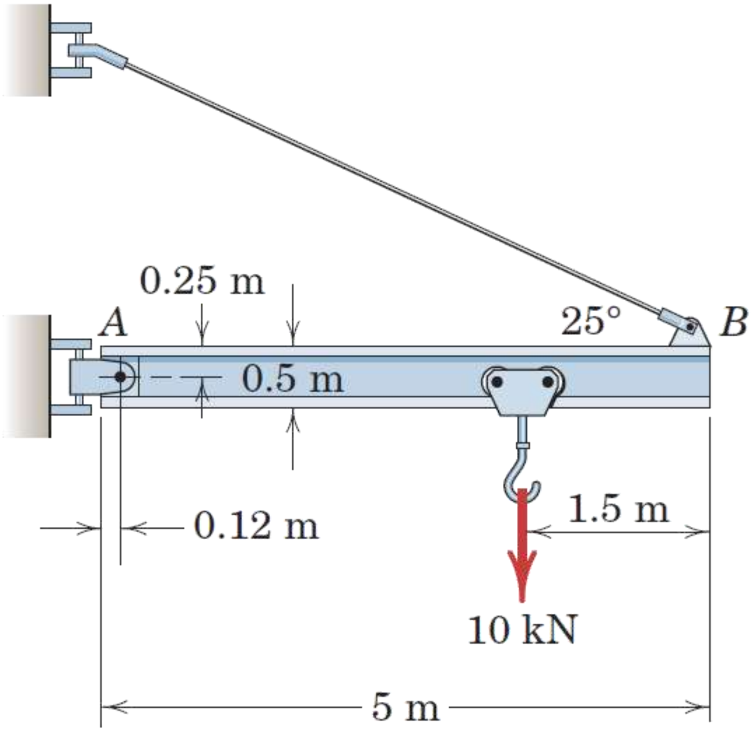


① Clearly the equilibrium of this parallel force system is independent of θ.

Article 3/3 – Sample Problem 3/4 (1 of 3)

- Problem Statement

Determine the magnitude T of the tension in the supporting cable and the magnitude of the force on the pin at A for the jib crane shown. The beam AB is a standard 0.5-m I-beam with a mass of 95 kg per meter of length.



Article 3/3 – Sample Problem 3/4 (2 of 3)

- Free-Body Diagram

- Equilibrium Conditions

$[\Sigma M_A = 0]$ $\quad (T\cos 25°)0.25 + (T\sin 25°)(5 - 0.12)$

$\qquad - 10(5 - 1.5 - 0.12) - 4.66(2.5 - 0.12) = 0$ ②

from which $\quad T = 19.61$ kN $\quad$ *Ans.*

Equating the sums of forces in the x- and y-directions to zero gives

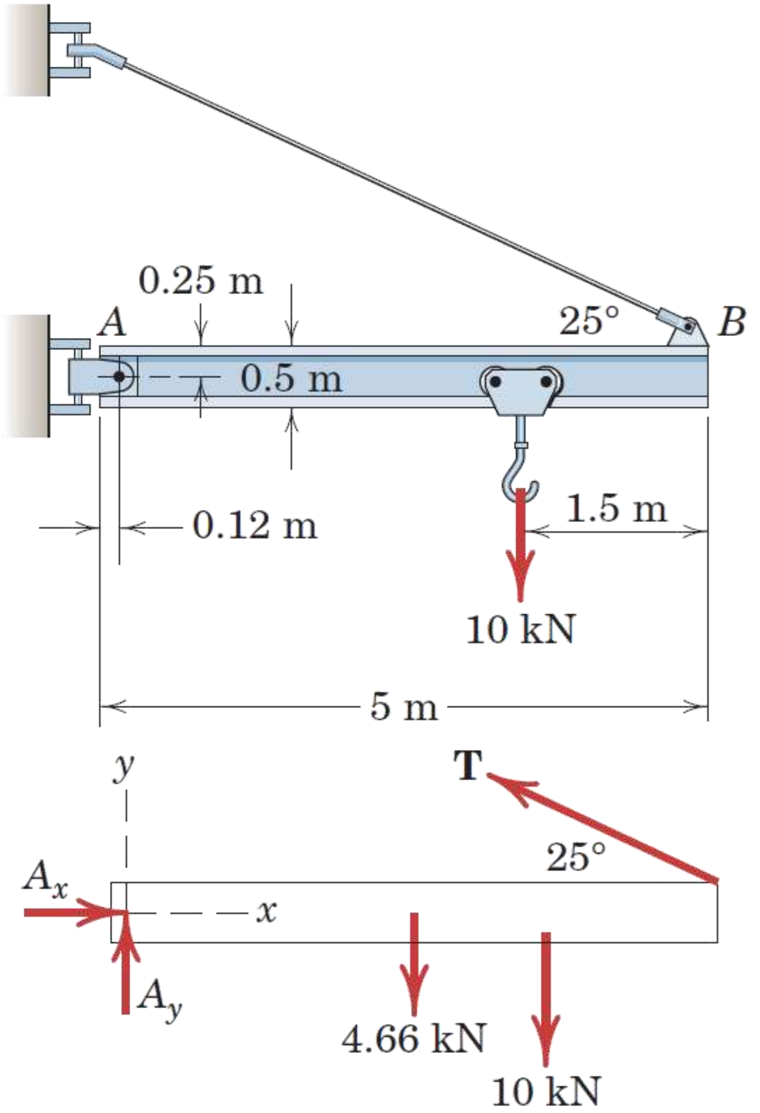
$[\Sigma F_x = 0]$ $\quad A_x - 19.61\cos 25° = 0 \quad A_x = 17.77$ kN

$[\Sigma F_y = 0]$ $\quad A_y + 19.61\sin 25° - 4.66 - 10 = 0 \quad A_y = 6.37$ kN

$[A = \sqrt{A_x^2 + A_y^2}]$ $\quad A = \sqrt{(17.77)^2 + (6.37)^2} = 18.88$ kN ③ $\quad$ *Ans.*

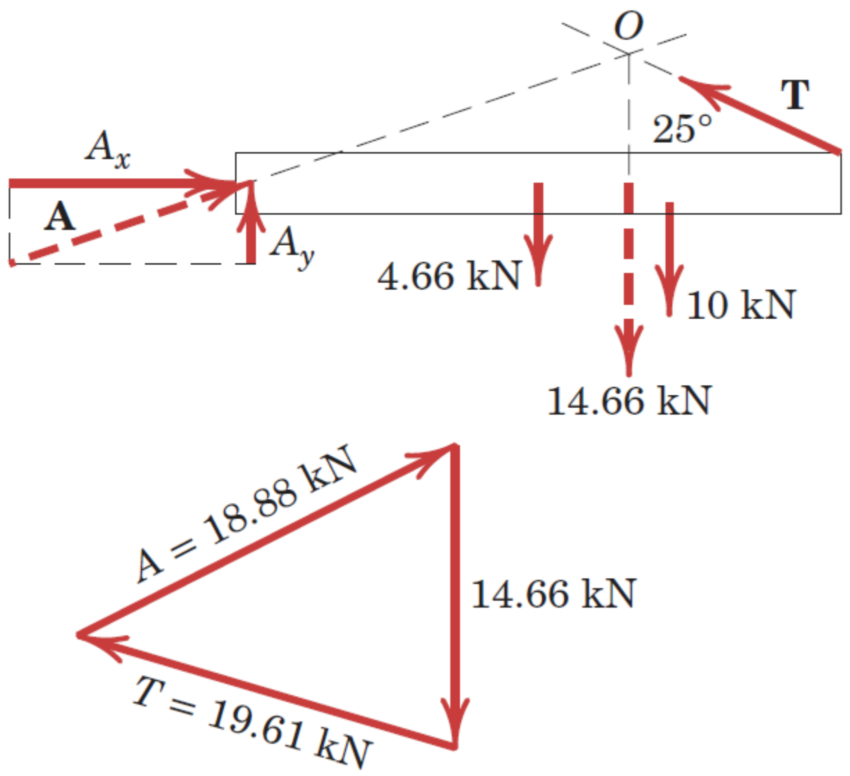
② The calculation of moments in two-dimensional problems is generally handled more simply by scalar algebra than by the vector cross product $\mathbf{r} \times \mathbf{F}$. In three dimensions, as we will see later, the reverse is often the case.

③ The direction of the force at A could be easily calculated if desired. However, in designing the pin A or in checking its strength, it is only the magnitude of the force that matters.



Article 3/3 – Sample Problem 3/4 (3 of 3)

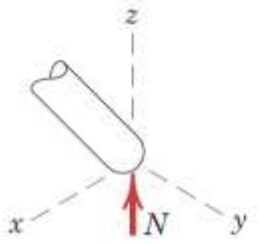
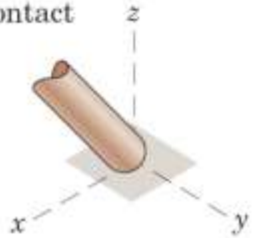
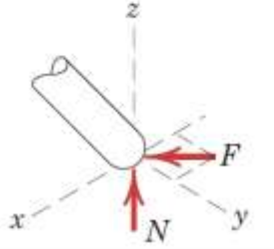
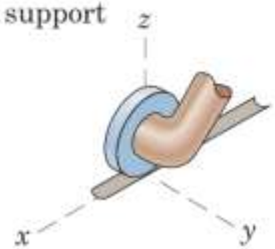
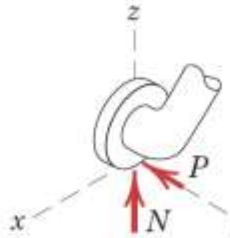
- Graphical Solution



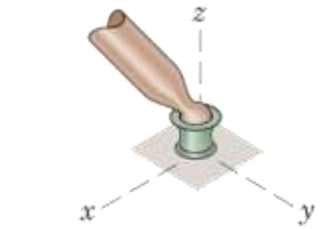
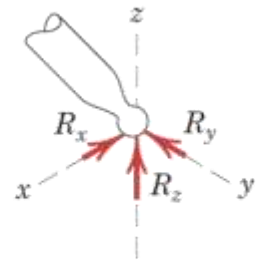
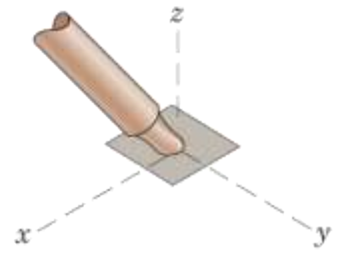
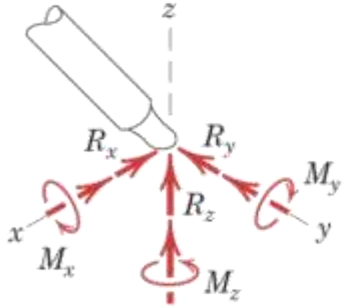
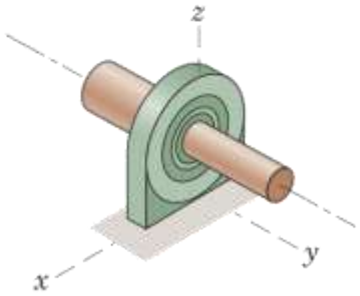
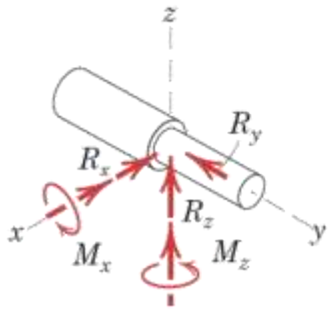
Article 3/4 Equilibrium Conditions (3D)

- Equilibrium Conditions (Eq. 3/1 extended)
  - Force Balance: $\Sigma\mathbf{F} = \mathbf{0}$
    - $\Sigma F_x = 0$
    - $\Sigma F_y = 0$
    - $\Sigma F_z = 0$
  - Moment Balance: $\Sigma\mathbf{M} = \mathbf{0}$
    - $\Sigma M_x = 0$
    - $\Sigma M_y = 0$
    - $\Sigma M_z = 0$
  - The moment sum is taken about any convenient reference point.

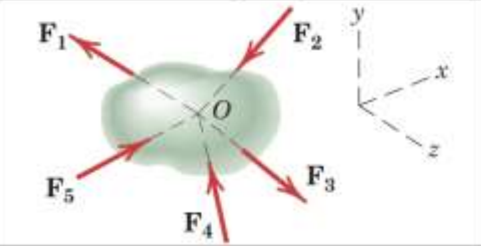
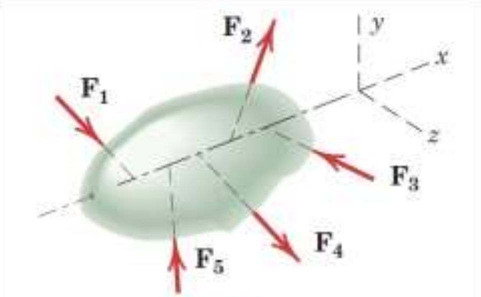
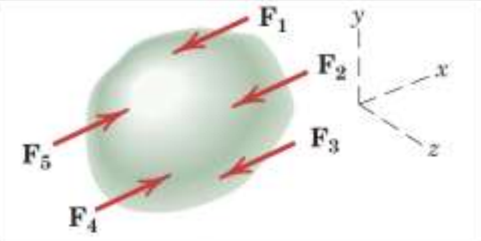
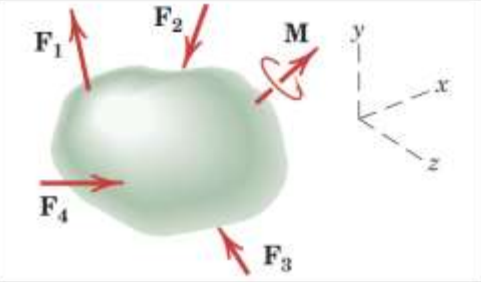
Article 3/4 – Modeling the Action of Forces (1 of 2)

MODELING THE ACTION OF FORCES IN THREE-DIMENSIONAL ANALYSIS		
Type of Contact and Force Origin	Action on Body to Be Isolated	
1. Member in contact with smooth surface, or ball-supported member 		Force must be normal to the surface and directed toward the member.
2. Member in contact with rough surface 		The possibility exists for a force F tangent to the surface (friction force) to act on the member, as well as a normal force N.
3. Roller or wheel support with lateral constraint 		A lateral force P exerted by the guide on the wheel can exist, in addition to the normal force N.

Article 3/4 – Modeling the Action of Forces (1 of 2)

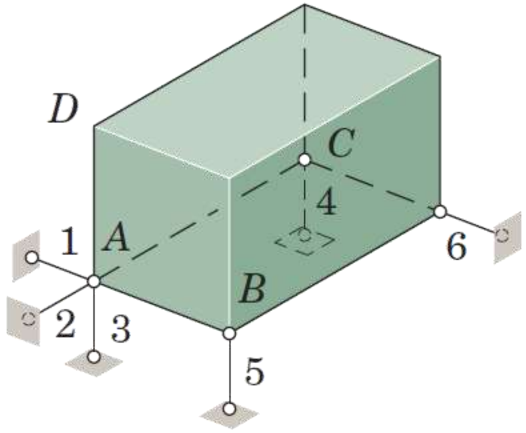
MODELING THE ACTION OF FORCES IN THREE-DIMENSIONAL ANALYSIS		
Type of Contact and Force Origin	**Action on Body to Be Isolated**	
4. Ball-and-socket joint 		A ball-and-socket joint free to pivot about the center of the ball can support a force **R** with all three components.
5. Fixed connection (embedded or welded) 		In addition to three components of force, a fixed connection can support a couple **M** represented by its three components.
6. Thrust-bearing support 		Thrust bearing is capable of supporting axial force R_y as well as radial forces R_x and R_z. Couples M_x and M_z must, in some cases, be assumed zero in order to provide statical determinacy.

Article 3/4 – Categories of Equilibrium

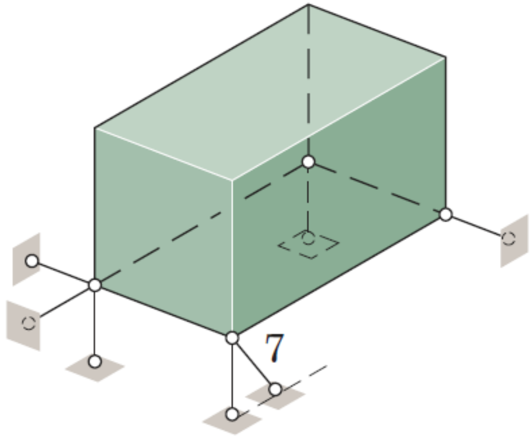
CATEGORIES OF EQUILIBRIUM IN THREE DIMENSIONS		
Force System	Free-Body Diagram	Independent Equations
1. Concurrent at a point		$\Sigma F_x = 0$ $\Sigma F_y = 0$ $\Sigma F_z = 0$
2. Concurrent with a line		$\Sigma F_x = 0$ $\Sigma M_y = 0$ $\Sigma F_y = 0$ $\Sigma M_z = 0$ $\Sigma F_z = 0$
3. Parallel		$\Sigma F_x = 0$ $\Sigma M_y = 0$ $\Sigma M_z = 0$
4. General		$\Sigma F_x = 0$ $\Sigma M_x = 0$ $\Sigma F_y = 0$ $\Sigma M_y = 0$ $\Sigma F_z = 0$ $\Sigma M_z = 0$

Article 3/4 – Constraints and Statical Determinacy

- Complete Fixity (Adequate Constraints)

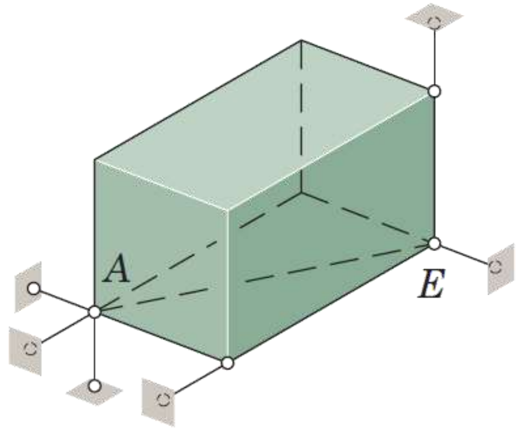


- Excessive Fixity (Redundant Constraints)

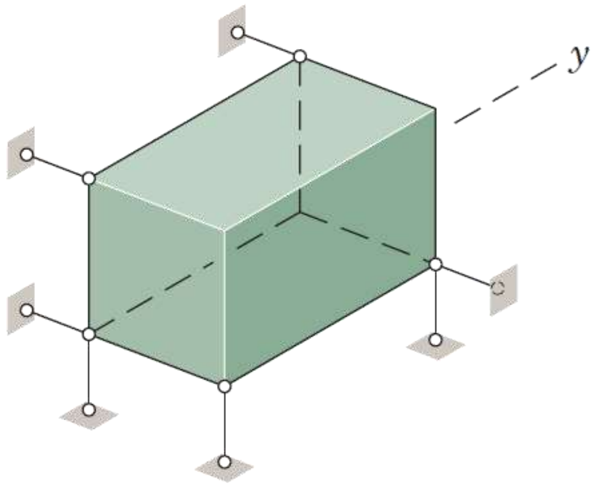


Article 3/4 – Constraints and Statical Determinacy

- Incomplete Fixity (Partial Constraints)
  - No Moment Resistance about Line *AE*



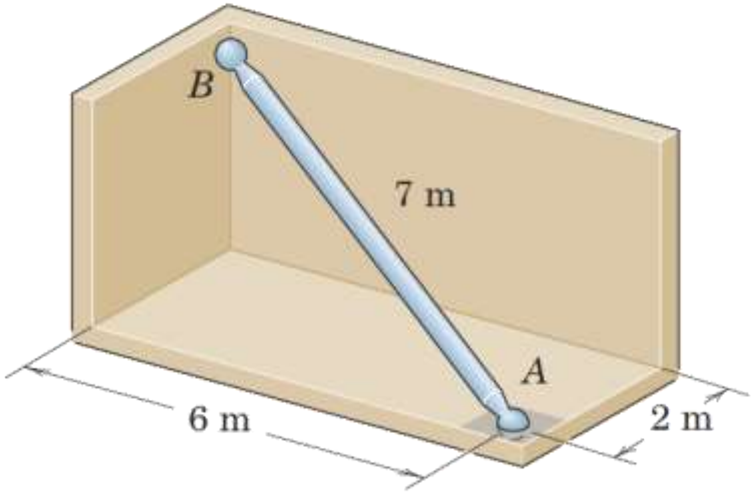
- Incomplete Fixity (Partial Constraints)
  - No Force Resistance along y-Axis



Article 3/4 – Sample Problem 3/5 (1 of 4)

- Problem Statement

  The uniform 7-m steel shaft has a mass of 200 kg and is supported by a ball-and-socket joint at *A* in the horizontal floor. The ball end *B* rests against the smooth vertical walls as shown. Compute the forces exerted by the walls and the floor on the ends of the shaft.



Article 3/4 – Sample Problem 3/5 (2 of 4)

- Free-Body Diagram
- Equilibrium Conditions – Moment Balance

Vector Solution We will use A as a moment center to eliminate reference to the forces at A. The position vectors needed to compute the moments about A are

$$\mathbf{r}_{AG} = -1\mathbf{i} - 3\mathbf{j} + 1.5\mathbf{k} \text{ m} \qquad \text{and} \qquad \mathbf{r}_{AB} = -2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k} \text{ m}$$

where the mass center G is located halfway between A and B.

The vector moment equation gives

$$[\Sigma \mathbf{M}_A = \mathbf{0}] \qquad \mathbf{r}_{AB} \times (\mathbf{B}_x + \mathbf{B}_y) + \mathbf{r}_{AG} \times \mathbf{W} = \mathbf{0}$$

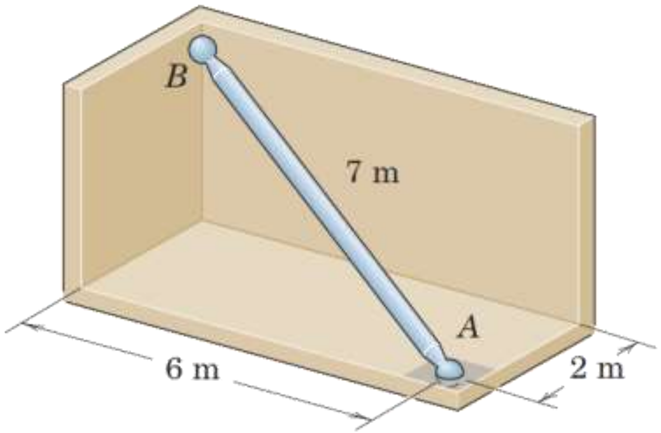
$$(-2\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}) \times (B_x\mathbf{i} + B_y\mathbf{j}) + (-\mathbf{i} - 3\mathbf{j} + 1.5\mathbf{k}) \times (-1962\mathbf{k}) = \mathbf{0}$$

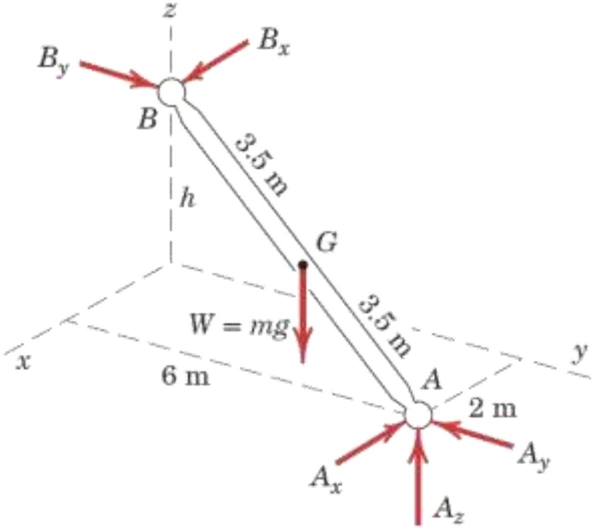
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & -6 & 3 \\ B_x & B_y & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -3 & 1.5 \\ 0 & 0 & -1962 \end{vmatrix} = \mathbf{0}$$

$$(-3B_y + 5890)\mathbf{i} + (3B_x - 1962)\mathbf{j} + (-2B_y + 6B_x)\mathbf{k} = \mathbf{0}$$

Equating the coefficients of **i**, **j**, and **k** to zero and solving give

$$B_x = 654 \text{ N} \qquad \text{and} \qquad B_y = 1962 \text{ N} \quad ② \qquad \qquad \textit{Ans.}$$





Article 3/4 – Sample Problem 3/5 (3 of 4)

- Equilibrium Conditions – Force Balance

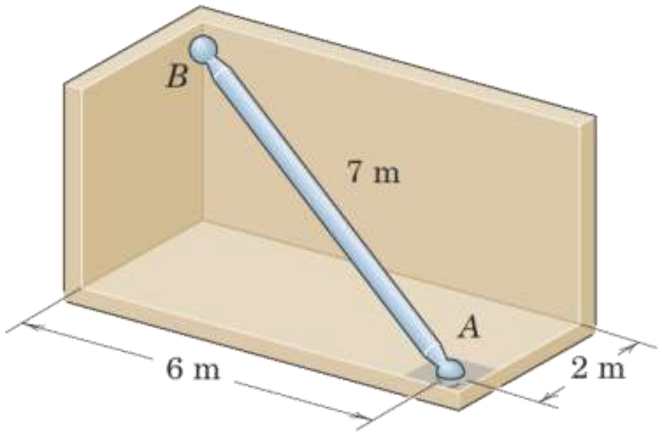
The forces at A are easily determined by

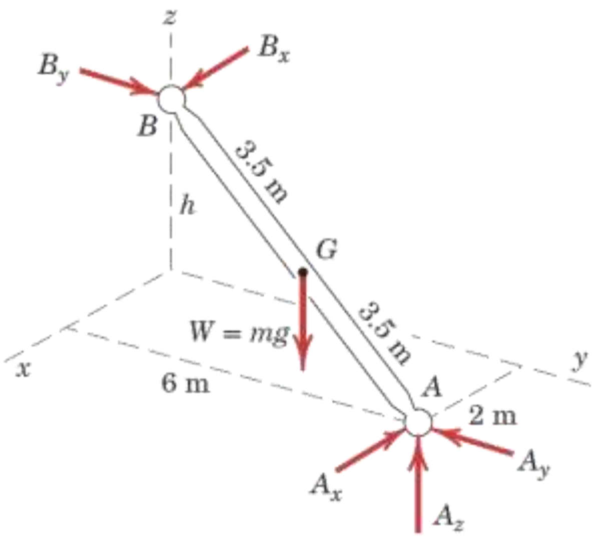
$[\Sigma \mathbf{F} = \mathbf{0}]$ $$(654 - A_x)\mathbf{i} + (1962 - A_y)\mathbf{j} + (-1962 + A_z)\mathbf{k} = \mathbf{0}$$

and $A_x = 654\text{ N}$ $\quad A_y = 1962\text{ N}$ $\quad A_z = 1962\text{ N}$

Finally, $$A = \sqrt{A_x^{\,2} + A_y^{\,2} + A_z^{\,2}}$$

$$= \sqrt{(654)^2 + (1962)^2 + (1962)^2} = 2850\text{ N} \qquad \textit{Ans.}$$





Article 3/4 – Sample Problem 3/5 (4 of 4)

- Equilibrium Conditions

Scalar Solution Evaluating the scalar moment equations about axes through A parallel, respectively, to the x- and y-axes, gives

$[\Sigma M_{A_x} = 0]$ $\quad 1962(3) - 3B_y = 0 \quad B_y = 1962 \text{ N}$

$[\Sigma M_{A_y} = 0]$ $\quad -1962(1) + 3B_x = 0 \quad B_x = 654 \text{ N}$ ③

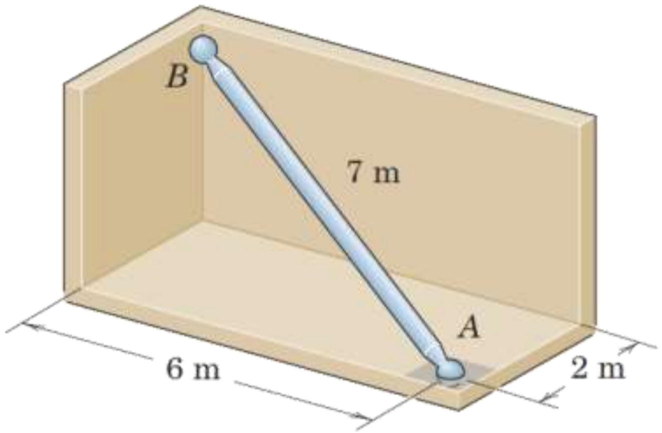
The force equations give, simply,

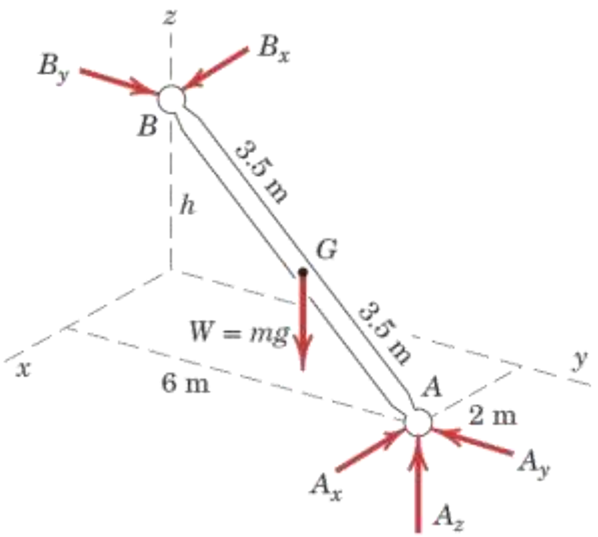
$[\Sigma F_x = 0]$ $\quad -A_x + 654 = 0 \quad A_x = 654 \text{ N}$

$[\Sigma F_y = 0]$ $\quad -A_y + 1962 = 0 \quad A_y = 1962 \text{ N}$

$[\Sigma F_z = 0]$ $\quad A_z - 1962 = 0 \quad A_z = 1962 \text{ N}$

③ We observe that a moment sum about an axis through A parallel to the z-axis merely gives us $6B_x - 2B_y = 0$, which serves only as a check as noted previously. Alternatively we could have first obtained A_z from $\Sigma F_z = 0$ and then taken our moment equations about axes through B to obtain A_x and A_y.

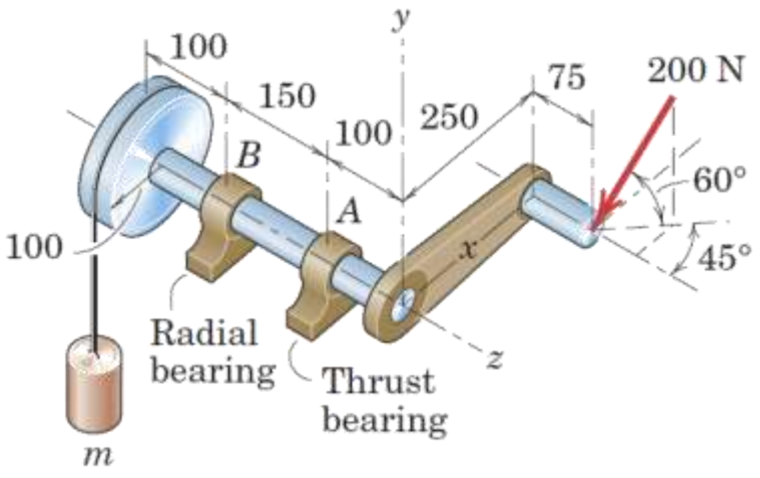




Article 3/4 – Sample Problem 3/6 (1 of 2)

- Problem Statement

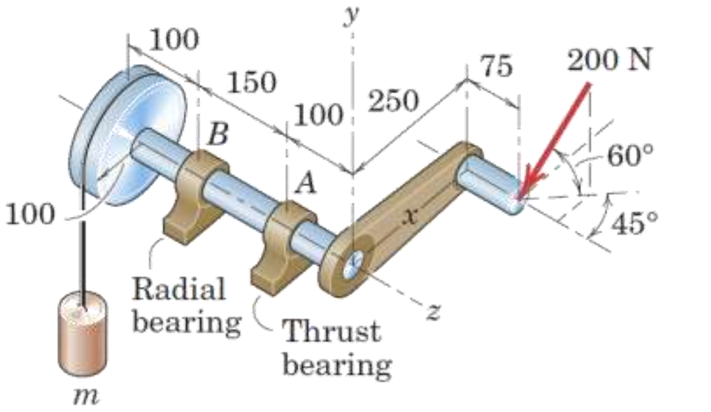
  A 200-N force is applied to the handle of the hoist in the direction shown. The bearing *A* supports the thrust (force in the direction of the shaft axis), while bearing *B* supports only radial load (load normal to the shaft axis). Determine the mass *m* which can be supported and the total radial force exerted on the shaft by each bearing. Assume neither bearing to be capable of supporting a moment about a line normal to the shaft axis.



Dimensions in millimeters

Article 3/4 – Sample Problem 3/6 (2 of 2)

- Free-Body Diagram (Orthographic Projections)
- Equilibrium Conditions



Dimensions in millimeters

From the x-y projection: ②

$[\Sigma M_O = 0]$ $\quad 100(9.81m) - 250(173.2) = 0 \quad m = 44.1\text{ kg}$ *Ans.*

From the x-z projection:

$[\Sigma M_A = 0]$ $\quad 150B_x + 175(70.7) - 250(70.7) = 0 \quad B_x = 35.4\text{ N}$

$[\Sigma F_x = 0]$ $\quad A_x + 35.4 - 70.7 = 0 \quad A_x = 35.4\text{ N}$

The y-z view gives ③

$[\Sigma M_A = 0]$ $\quad 150B_y + 175(173.2) - 250(44.1)(9.81) = 0 \quad B_y = 520\text{ N}$

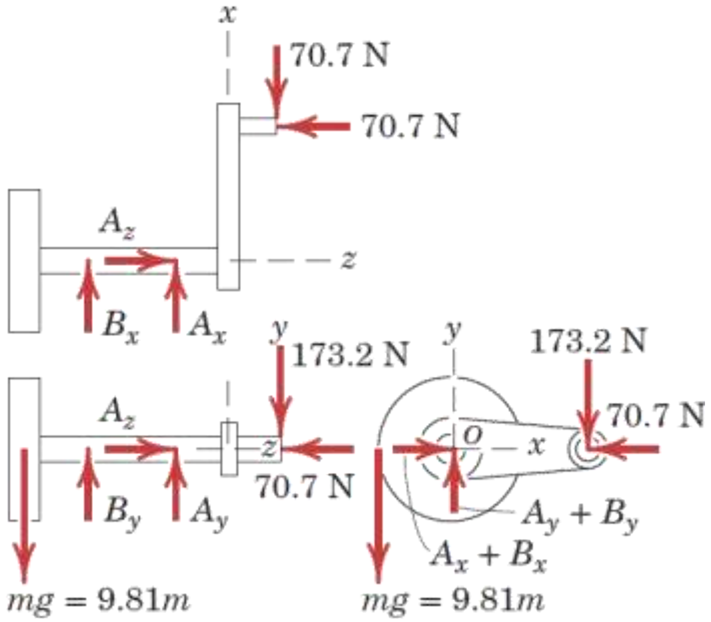
$[\Sigma F_y = 0]$ $\quad A_y + 520 - 173.2 - (44.1)(9.81) = 0 \quad A_y = 86.8\text{ N}$

$[\Sigma F_z = 0]$ $\quad A_z = 70.7\text{ N}$

The total radial forces on the bearings become

$[A_r = \sqrt{A_x^2 + A_y^2}]$ $\quad A_r = \sqrt{(35.4)^2 + (86.8)^2} = 93.5\text{ N}$ *Ans.*

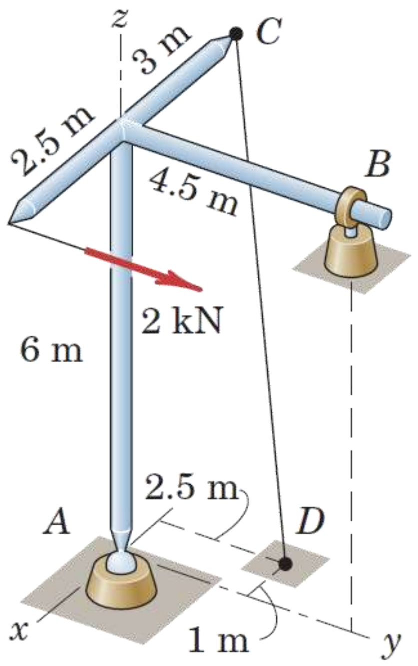
$[B = \sqrt{B_x^2 + B_y^2}]$ $\quad B = \sqrt{(35.4)^2 + (520)^2} = 521\text{ N}$ ④ *Ans.*



Article 3/4 – Sample Problem 3/7 (1 of 3)

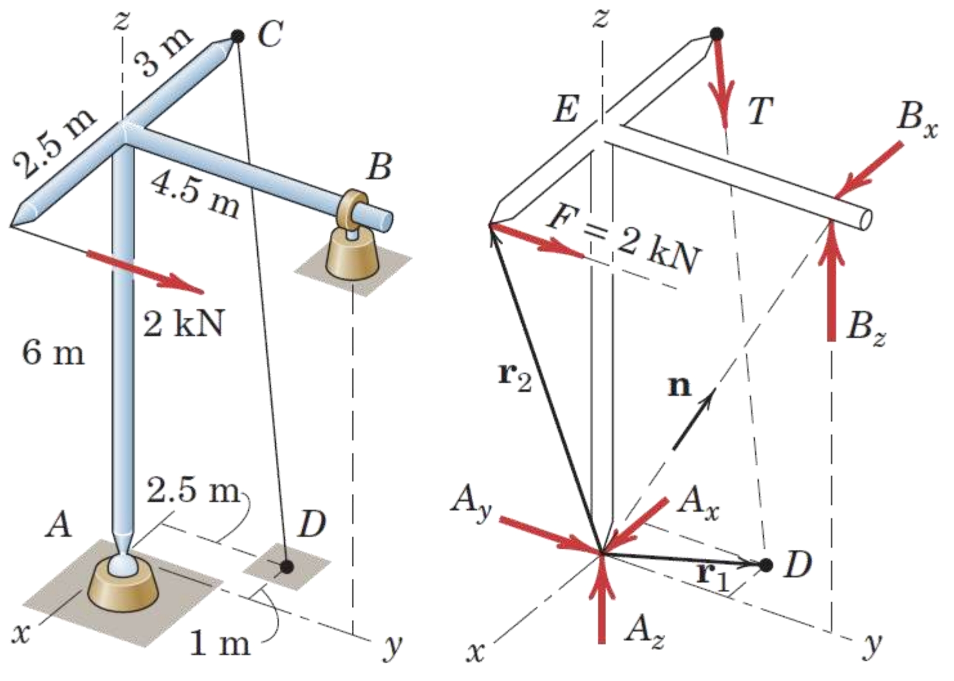
- Problem Statement

The welded tubular frame is secured to the horizontal *x*-*y* plane by a ball-and-socket joint at *A* and receives support from the loose-fitting ring at *B*. Under the action of the 2-kN load, rotation about a line from *A* to *B* is prevented by the cable *CD*, and the frame is stable in the position shown. Neglect the weight of the frame compared with the applied load and determine the tension *T* in the cable, the reaction at the ring, and the reaction components at *A*.



Article 3/4 – Sample Problem 3/7 (2 of 3)

- Free-Body Diagram
- Moment Sum about Line *AB*



$\mathbf{n} = \dfrac{1}{\sqrt{6^2 + 4.5^2}}(4.5\mathbf{j} + 6\mathbf{k}) = \frac{1}{5}(3\mathbf{j} + 4\mathbf{k})$. The moment of $\mathbf{T}$ about AB is the component in the direction of AB of the vector moment about the point A and equals $\mathbf{r}_1 \times \mathbf{T}\cdot\mathbf{n}$. Similarly the moment of the applied load F about AB is $\mathbf{r}_2 \times \mathbf{F}\cdot\mathbf{n}$. With $\overline{CD} = \sqrt{46.2}$ m, the vector expressions for $\mathbf{T}$, $\mathbf{F}$, $\mathbf{r}_1$, and $\mathbf{r}_2$ are

$$\mathbf{T} = \frac{T}{\sqrt{46.2}}(2\mathbf{i} + 2.5\mathbf{j} - 6\mathbf{k}) \qquad \mathbf{F} = 2\mathbf{j}\ \text{kN}$$

$$\mathbf{r}_1 = -\mathbf{i} + 2.5\mathbf{j}\ \text{m} \qquad \mathbf{r}_2 = 2.5\mathbf{i} + 6\mathbf{k}\ \text{m} \quad ②$$

The moment equation now becomes

$$[\Sigma M_{AB} = 0] \quad (-\mathbf{i} + 2.5\mathbf{j}) \times \frac{T}{\sqrt{46.2}}(2\mathbf{i} + 2.5\mathbf{j} - 6\mathbf{k})\cdot\frac{1}{5}(3\mathbf{j} + 4\mathbf{k})$$
$$+ (2.5\mathbf{i} + 6\mathbf{k}) \times (2\mathbf{j})\cdot\frac{1}{5}(3\mathbf{j} + 4\mathbf{k}) = 0$$

Completion of the vector operations gives

$$-\frac{48T}{\sqrt{46.2}} + 20 = 0 \qquad T = 2.83\ \text{kN} \qquad \textit{Ans.}$$

and the components of T become

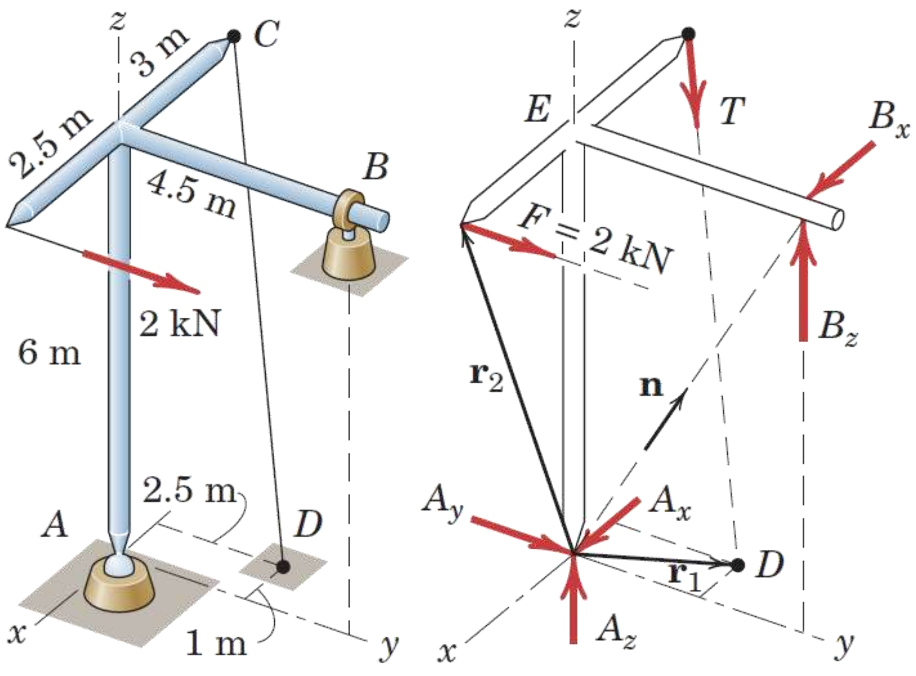
$$T_x = 0.833\ \text{kN} \qquad T_y = 1.042\ \text{kN} \qquad T_z = -2.50\ \text{kN}$$

Article 3/4 – Sample Problem 3/7 (3 of 3)

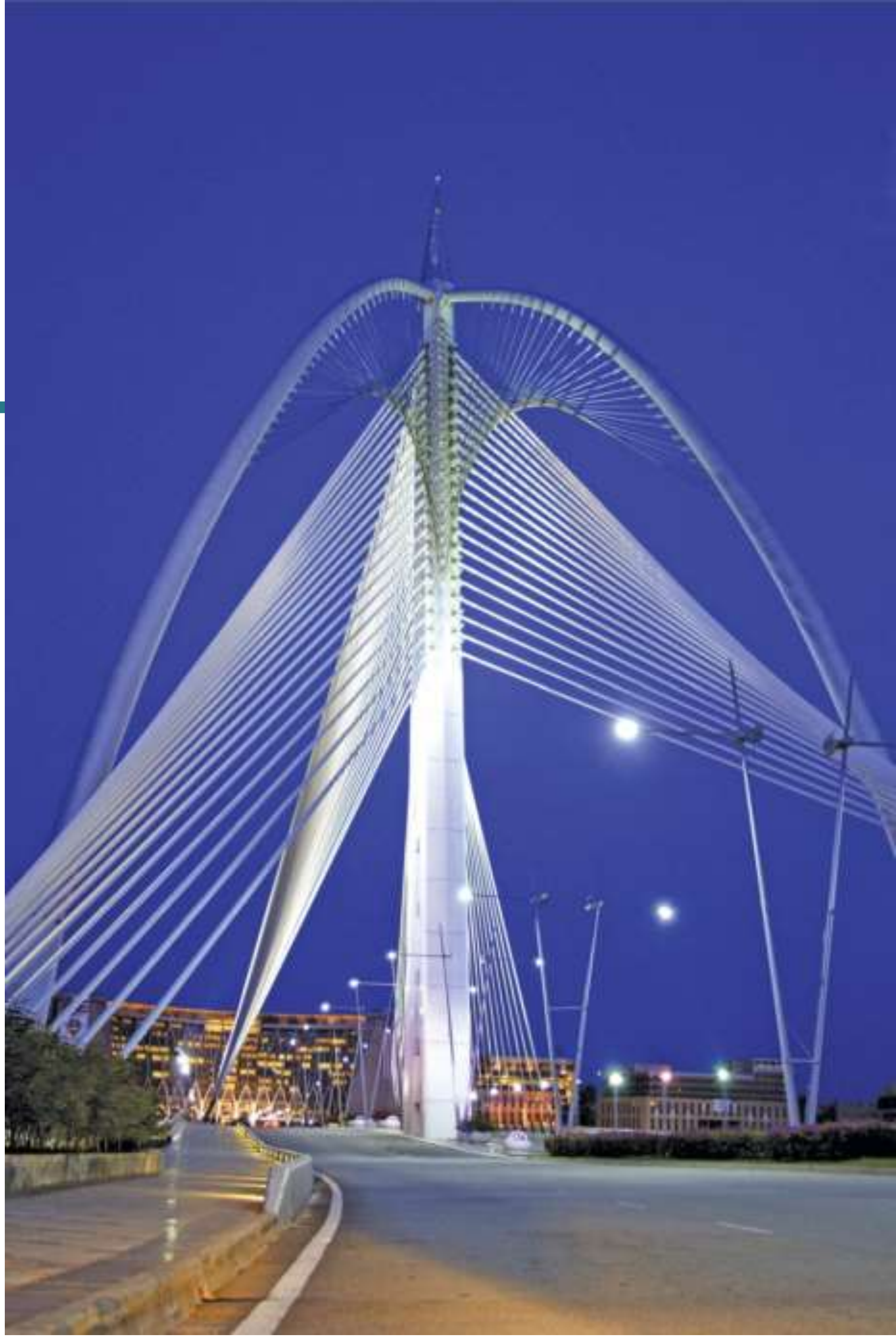
- Remaining Equilibrium Conditions

$[\Sigma M_z = 0]$	$2(2.5) - 4.5B_x - 1.042(3) = 0$	$B_x = 0.417$ kN	*Ans.*
$[\Sigma M_x = 0]$	$4.5B_z - 2(6) - 1.042(6) = 0$	$B_z = 4.06$ kN	*Ans.*
$[\Sigma F_x = 0]$	$A_x + 0.417 + 0.833 = 0$	$A_x = -1.250$ kN	*Ans.*
$[\Sigma F_y = 0]$	$A_y + 2 + 1.042 = 0$	$A_y = -3.04$ kN ③	*Ans.*
$[\Sigma F_z = 0]$	$A_z + 4.06 - 2.50 = 0$	$A_z = -1.556$ kN	*Ans.*

③ The negative signs associated with the *A*-components indicate that they are in the opposite direction to those shown on the free-body diagram.



CHAPTER 4
STRUCTURES



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CHAPTER OUTLINE

Article 4/1 Introduction

- Reminders from Chapter 3
  - Equilibrium of a Single Rigid Body
  - Free-Body Diagrams of an Entire Structure
  - Application of Force and Moment Equations of Equilibrium

- Purpose of Chapter 4
  - Equilibrium of Several Rigid Bodies
  - Free-Body Diagrams of Portions of a Structure
  - Application of Force and Moment Equations of Equilibrium
  - Application of Newton’s 3rd Law

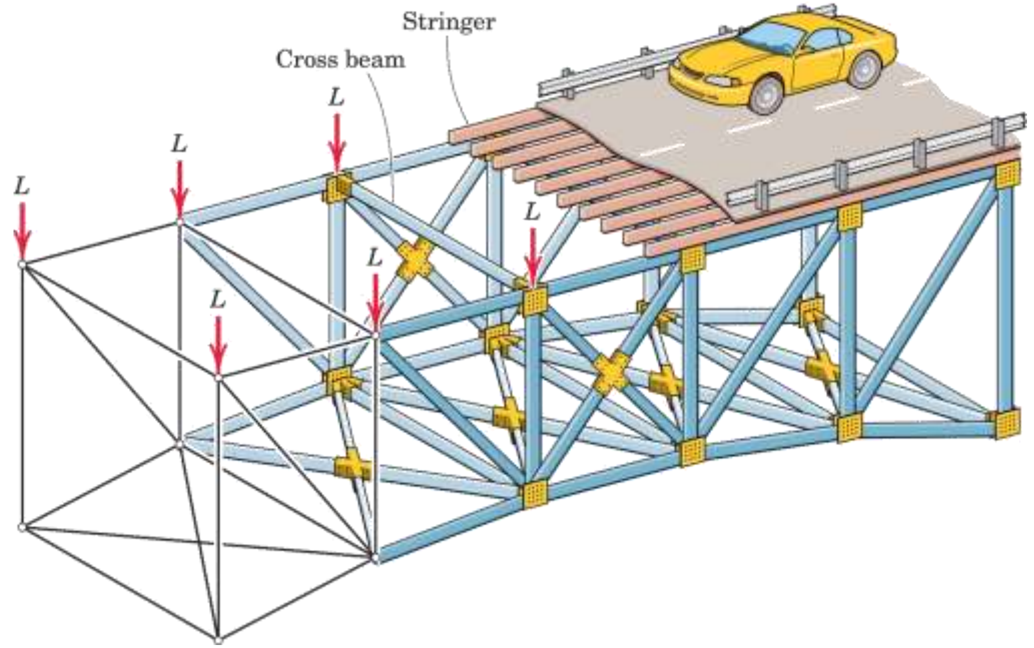
Article 4/2 Plane Trusses

- ## Introduction

Previously, we have been been concerned with the **external** forces acting on a body. In this section, however, we study the forces **internal** to a structure. Determining the load on each internal member of a truss or frame allows one to then **design** that particular member.

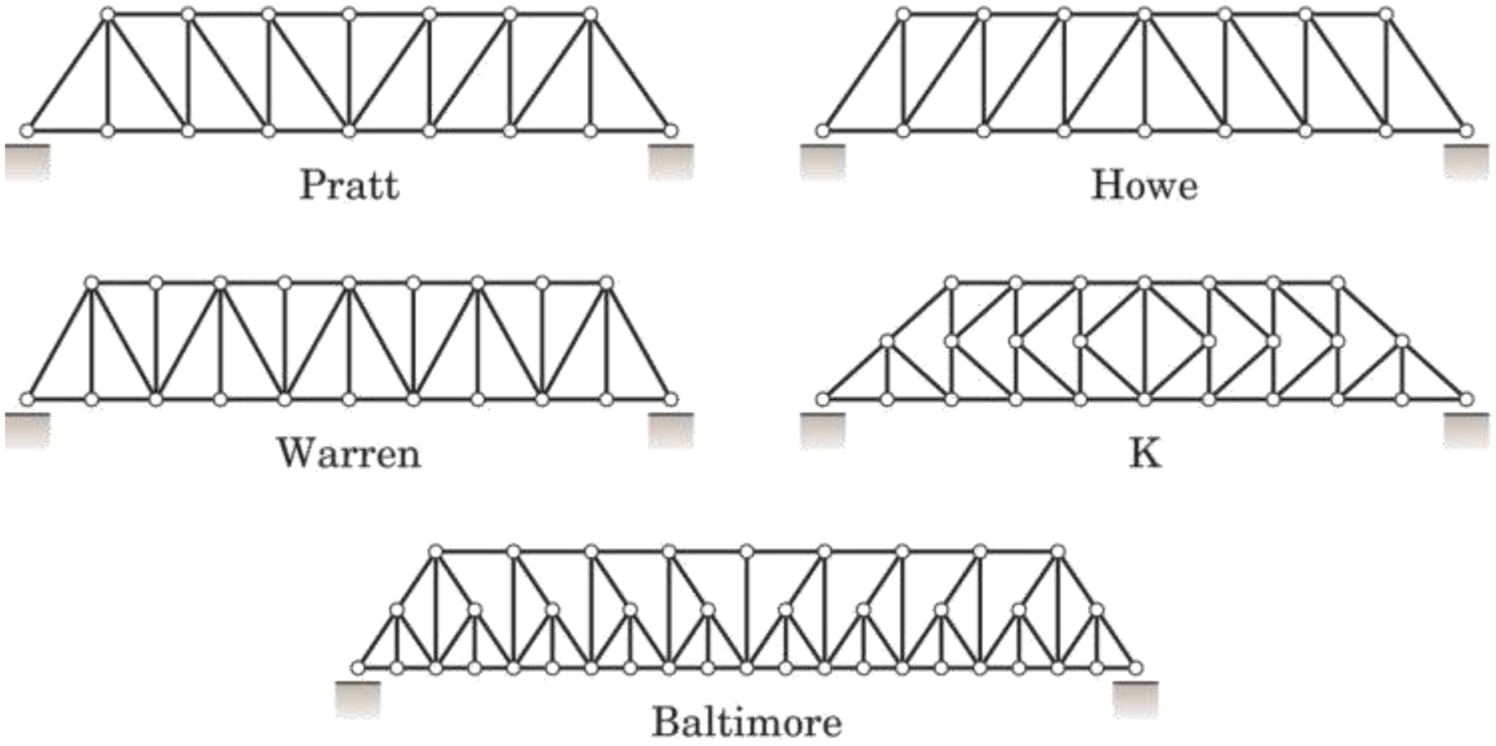
- ## Example Bridge Truss

Each vertical side of the structure is a plane truss, which is a framework composed of members joined at their ends. A simplified **model** is indicated at the left end of the illustration. The forces *L* represent the portions of the weights of the roadway, vehicles, stringers, and cross beams which are transferred to the truss joints. Note that joints are modeled as simple pins, even though they might actually be gusset plates or welded connections.



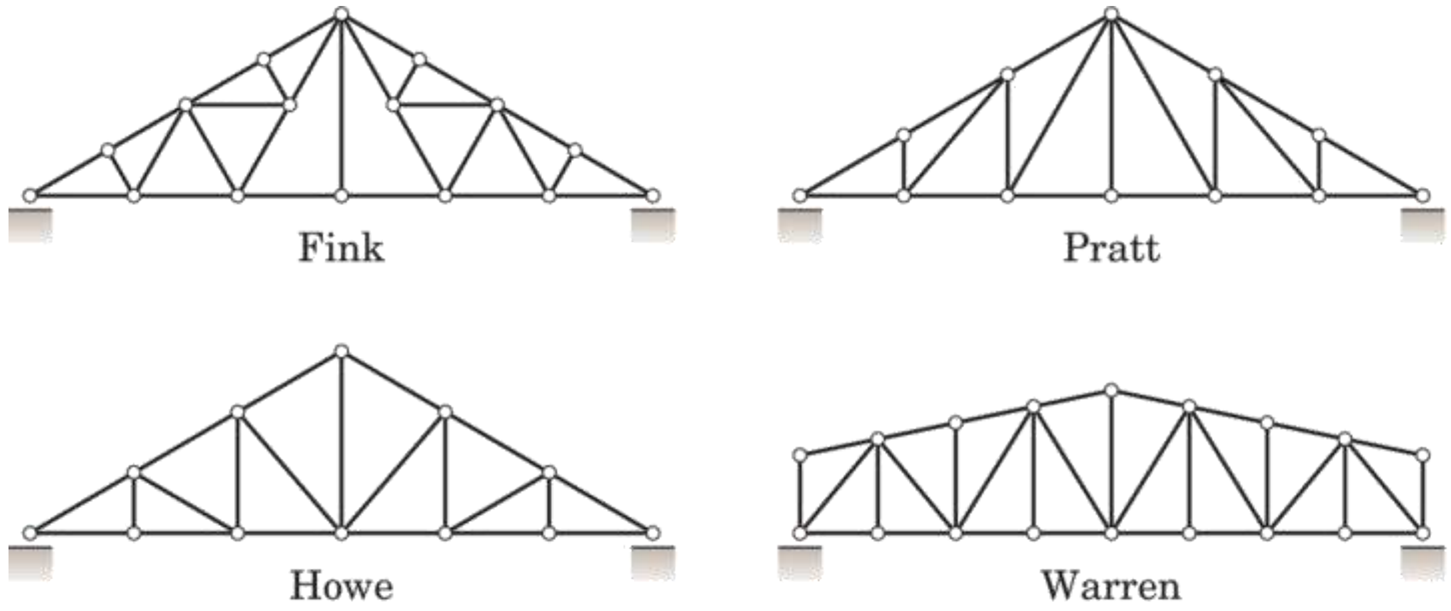
Article 4/2 – Common Truss Shapes (1 of 2)

- Commonly Used Bridge Trusses



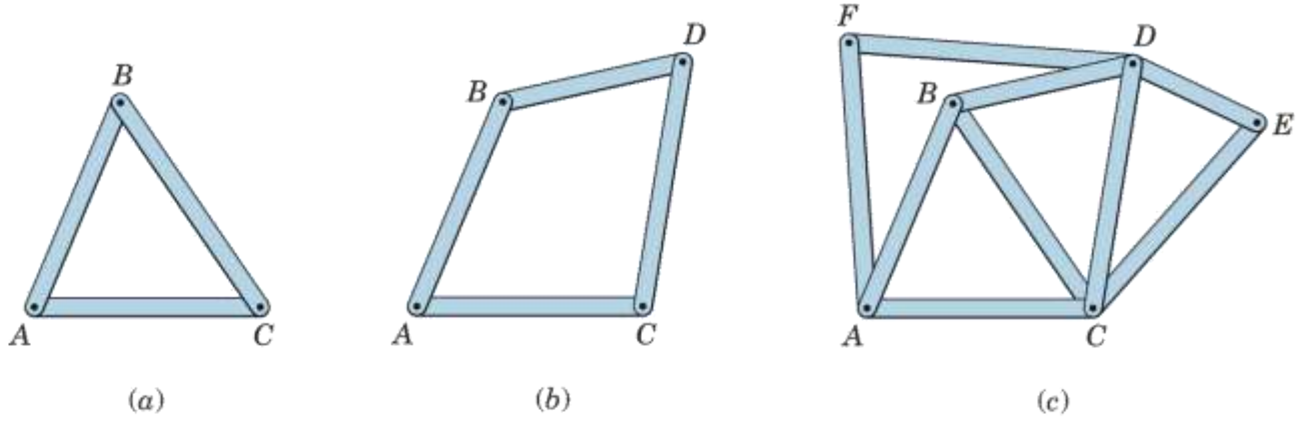
Article 4/2 – Common Truss Shapes (2 of 2)

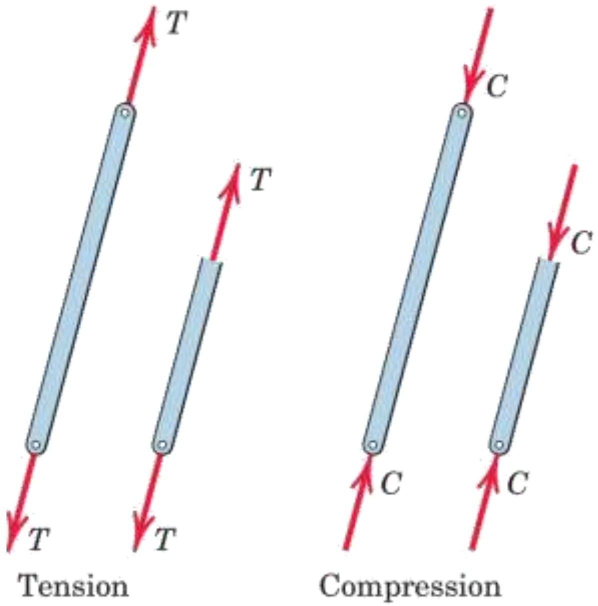
- Commonly Used Roof Trusses



Article 4/2 – Features of Simple Trusses (1 of 2)

- Simple Trusses
  - Rigid
  - Basic Element
  - Two-Force Members
  - Weight of Truss Members
  - Location of Applied Loads

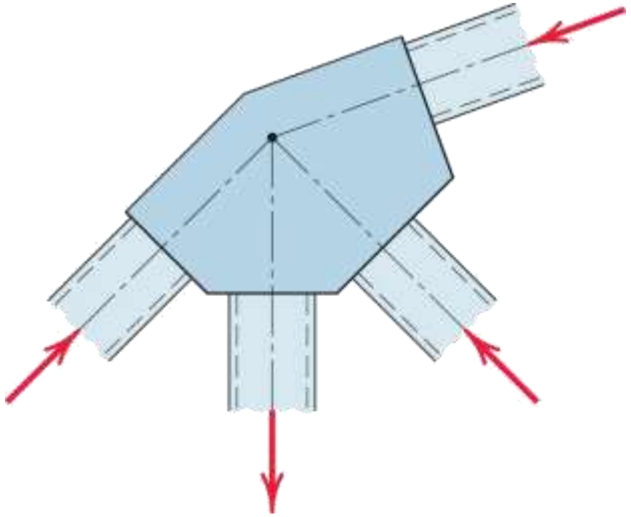




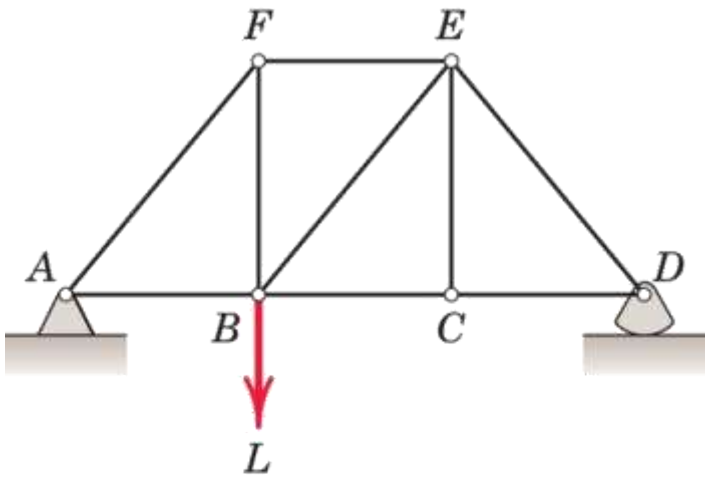
Two-Force Members

Article 4/2 – Features of Simple Trusses (2 of 2)

- Simple Trusses (cont.)
  - Connections between Members



  - Support Representation



The trusses in this text are statically determinant, which means a complete application of Newton's First Law will provide the solution for all unknown reactions and internal forces. Trusses that cannot be solved in this fashion are statically indeterminate, and information regarding the deformation of the members must be included to obtain the solution.

Article 4/3 Method of Joints

- Features of the Method
  - Looks at Equilibrium of Each Joint Connection
  - Limited to Force Summation Equations of Equilibrium at each Joint
  - Limited to Finding Two Unknown Reactions at a Time
  - Proceeds Systematically Across the Truss
  - Very Methodical and Easy to Utilize
  - Very Useful for Finding all of the Truss-Member Forces

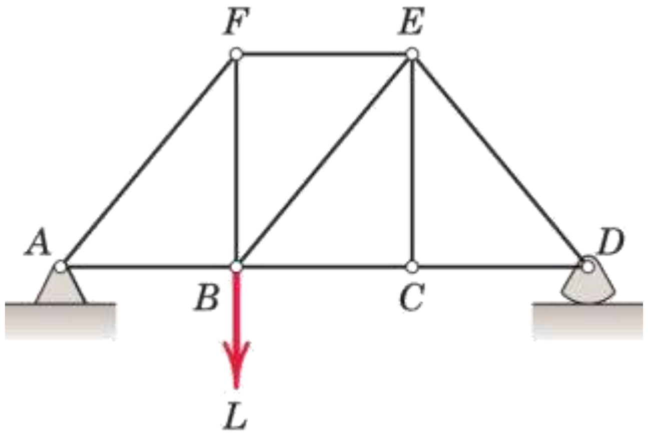
Article 4/3 – Overview of Method of Joints

- General Procedure
  1. In general, find the external reactions first (this may or may not be necessary).
  2. Start by analyzing a joint where at least one known applied load acts, and at most, only two unknown truss forces are present.
  3. Apply a sum of forces in an appropriate x- and y-direction at this joint.
  4. Draw members in tension or compression, and stay consistent with your choice until you move to the next joint.
  5. Solve the equilibrium equations for the unknown truss forces.
  6. Move to the next adjacent joint and repeat the process.

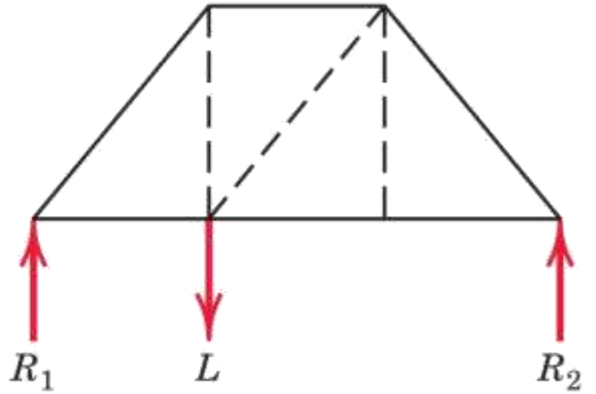
Article 4/3 – Illustration of Method of Joints (1 of 3)

- Determine the force in each member of the truss.

    1. Determine the external reactions



(a)

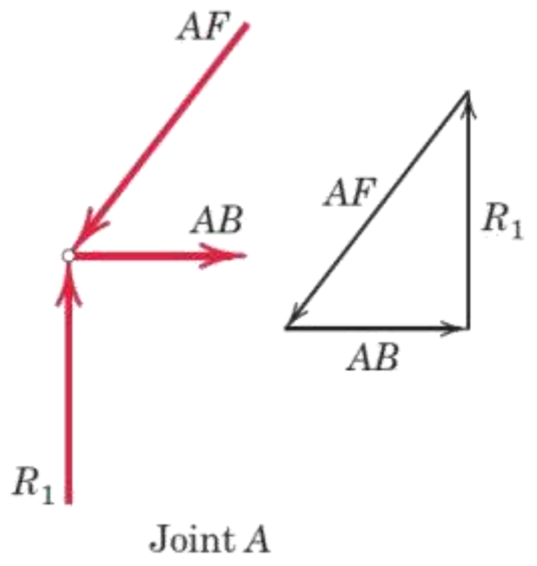


(b)

Article 4/3 – Illustration of Method of Joints (2 of 3)

- Determine the force in each member of the truss.

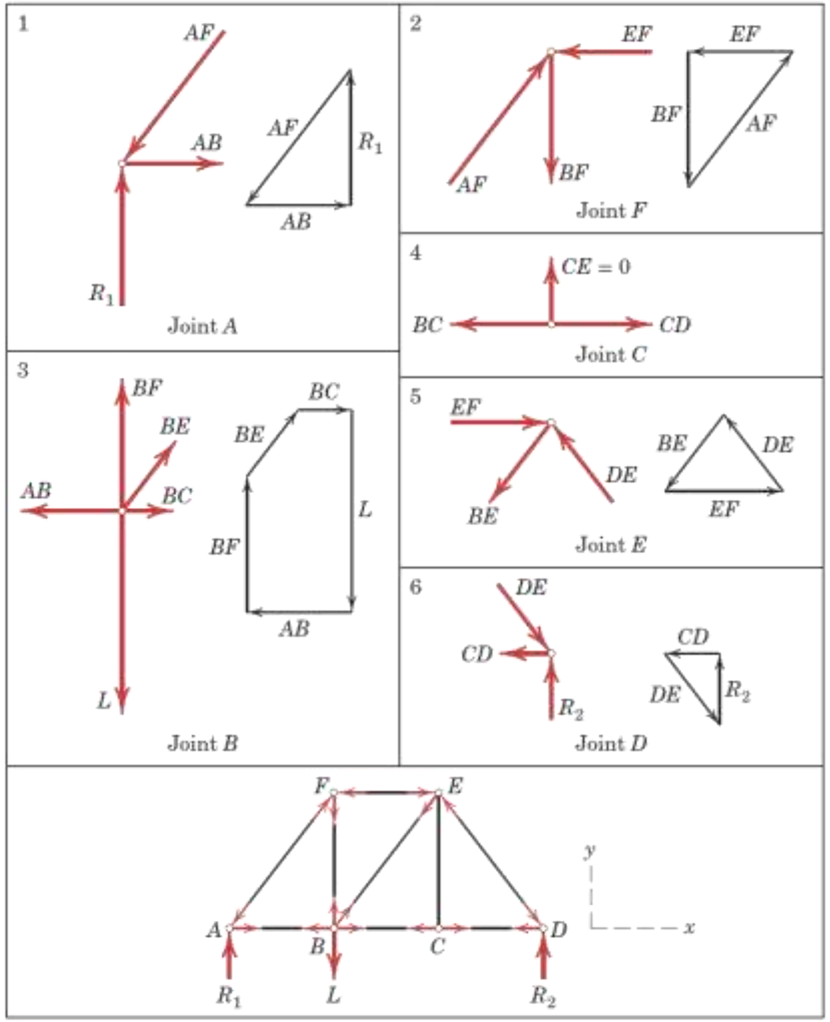
  1. Determine the external reactions.
  2. Start at a joint with only two unknowns, joint *A*.
  3. Draw the free-body diagram and apply equilibrium equations.



Article 4/3 – Illustration of Method of Joints (3 of 3)

- Determine the force in each member of the truss.

1. Determine the external reactions.
2. Start at a joint with only two unknowns, Joint *A*.
3. Draw the free-body diagram and apply equilibrium equations.
4. Proceed to the next joint and repeat the process.
5. Be careful with the signs of the members and indicate if they are in tension or compression.



Article 4/3 – Internal and External Redundancy

- External Redundancy

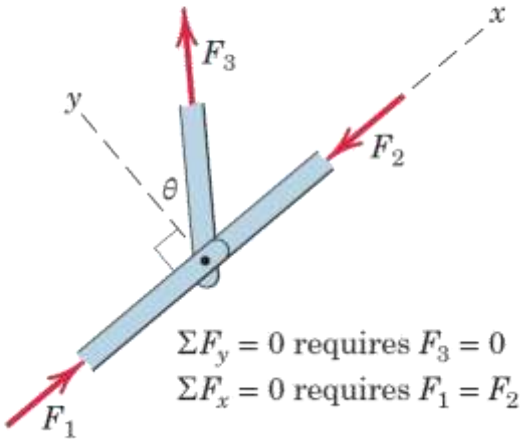
- Internal Redundancy

- Checking for Internal Statical Determinacy
  - Number of Internal Members, m
  - Number of Joint Connections, j
  - Truss is statically determinate internally if… $m + 3 = 2j$

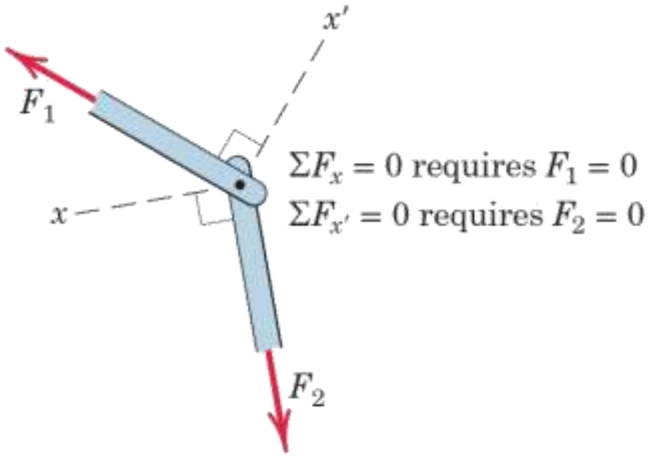
- Comment on Stability

Article 4/3 – Special Conditions in Trusses (1 of 3)

- Condition 1: Three Members at a Joint, Two are Collinear

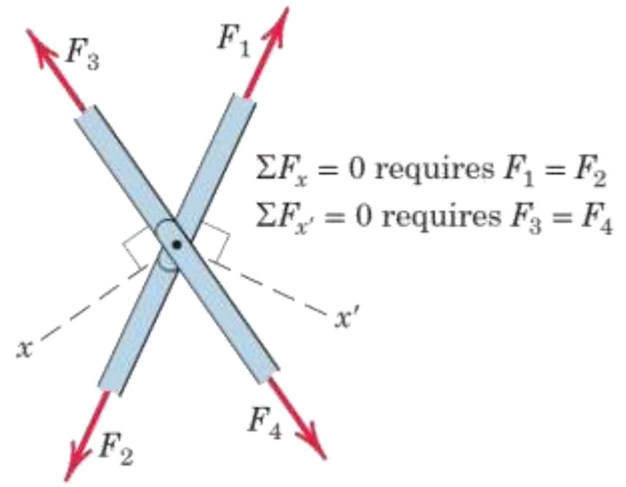


- Condition 2: Two Members at a Joint, Not Collinear

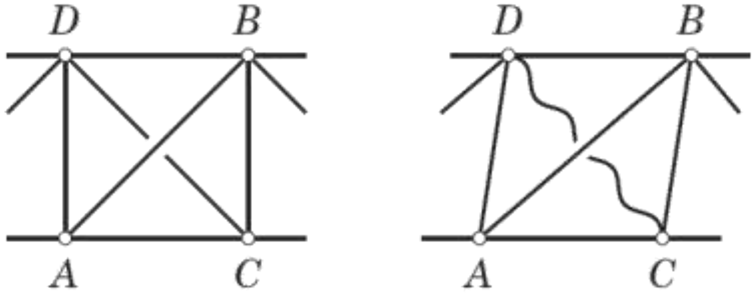


Article 4/3 – Special Conditions in Trusses (2 of 3)

- Condition 3: Two Pairs of Collinear Members at a Joint

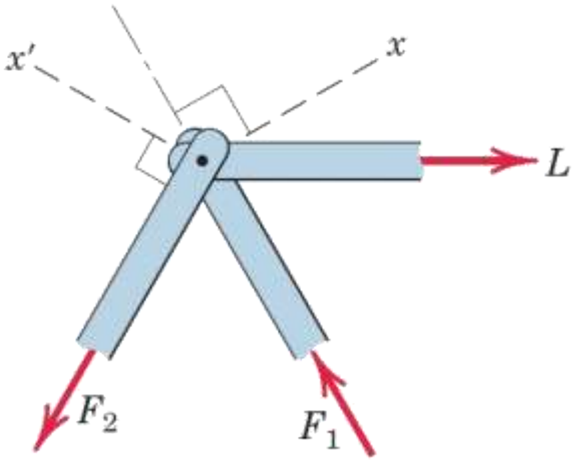


- Condition 4: Cross Bracing



Article 4/3 – Special Conditions in Trusses (3 of 3)

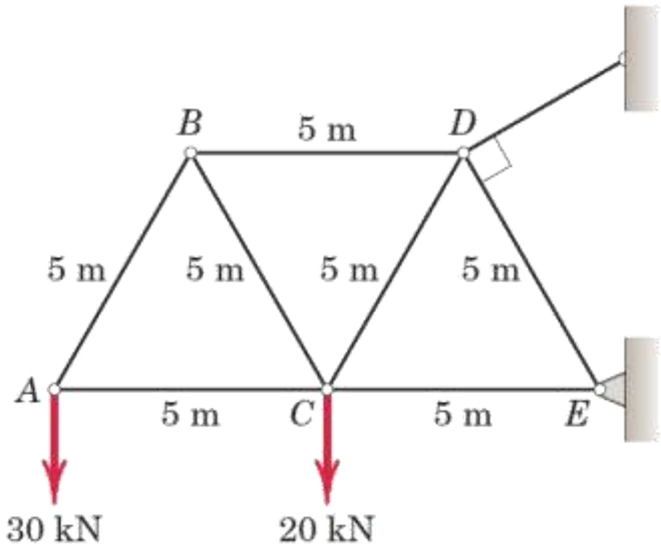
- Condition 5: Avoiding Simultaneous Equations
  - Summing Forces along x-Axis Eliminates F_1
  - Summing Forces along x'-Axis Eliminates F_2



Article 4/3 – Sample Problem 4/1 (1 of 4)

- Problem Statement

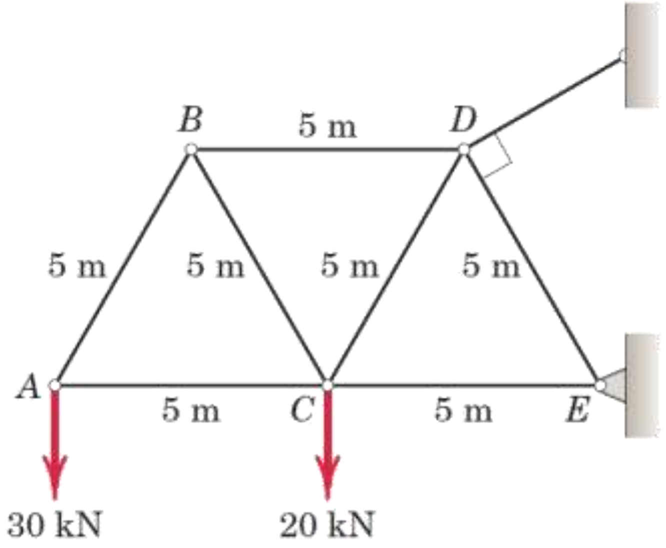
  Compute the force in each member of the loaded cantilever truss by the method of joints.

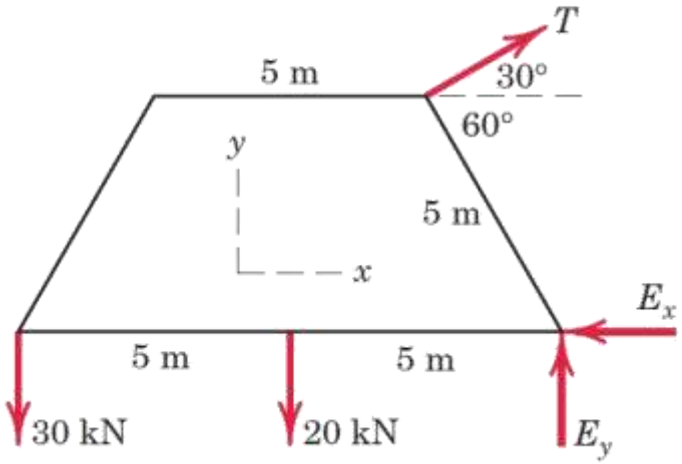


Article 4/3 – Sample Problem 4/1 (2 of 4)

- Find External Reactions

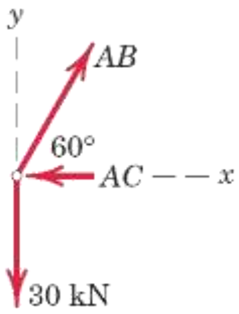
$[\Sigma M_E = 0]$	$5T - 20(5) - 30(10) = 0$	$T = 80 \text{ kN}$
$[\Sigma F_x = 0]$	$80 \cos 30° - E_x = 0$	$E_x = 69.3 \text{ kN}$
$[\Sigma F_y = 0]$	$80 \sin 30° + E_y - 20 - 30 = 0$	$E_y = 10 \text{ kN}$





Article 4/3 – Sample Problem 4/1 (3 of 4)

- Equilibrium of Joint *A*



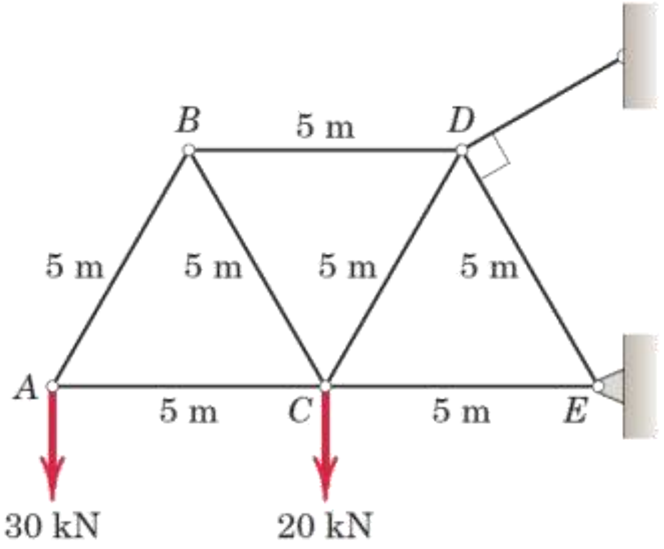
Joint *A*

Equilibrium requires

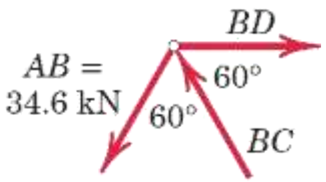
$[\Sigma F_y = 0]$ $\quad 0.866AB - 30 = 0 \quad AB = 34.6 \text{ kN } T$ *Ans.*

$[\Sigma F_x = 0]$ $\quad AC - 0.5(34.6) = 0 \quad AC = 17.32 \text{ kN } C$ *Ans.*

where *T* stands for tension and *C* stands for compression. ①



- Equilibrium of Joint *B*



Joint *B*

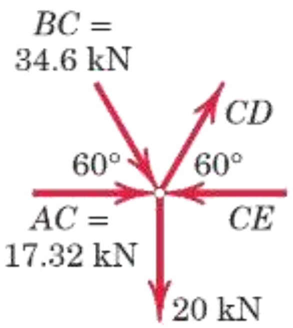
$[\Sigma F_y = 0]$ $\quad 0.866BC - 0.866(34.6) = 0 \quad BC = 34.6 \text{ kN } C$ *Ans.*

$[\Sigma F_x = 0]$ $\quad BD - 2(0.5)(34.6) = 0 \quad BD = 34.6 \text{ kN } T$ *Ans.*

① It should be stressed that the tension/compression designation refers to the *member*, not the joint. Note that we draw the force arrow on the same side of the joint as the member which exerts the force. In this way tension (arrow away from the joint) is distinguished from compression (arrow toward the joint).

Article 4/3 – Sample Problem 4/1 (4 of 4)

- Equilibrium of Joint C



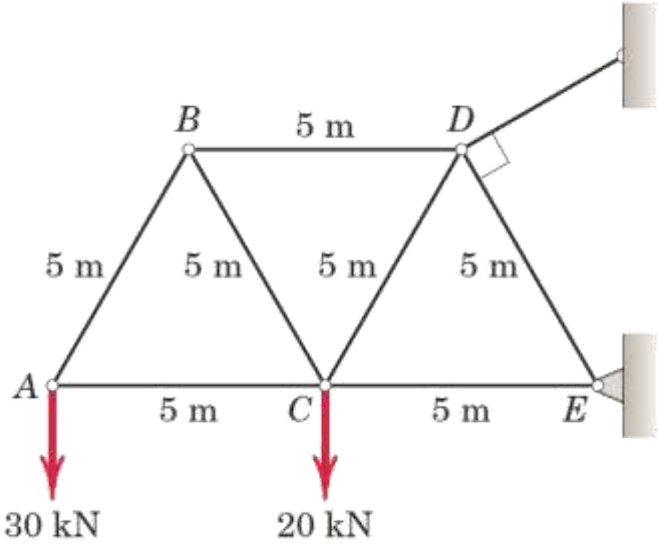
Joint C

$[\Sigma F_y = 0]$ $\quad 0.866CD - 0.866(34.6) - 20 = 0$

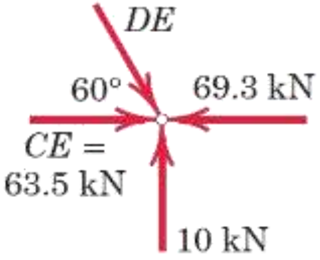
$CD = 57.7 \text{ kN } T$ *Ans.*

$[\Sigma F_x = 0]$ $\quad CE - 17.32 - 0.5(34.6) - 0.5(57.7) = 0$

$CE = 63.5 \text{ kN } C$ *Ans.*



- Equilibrium of Joint E



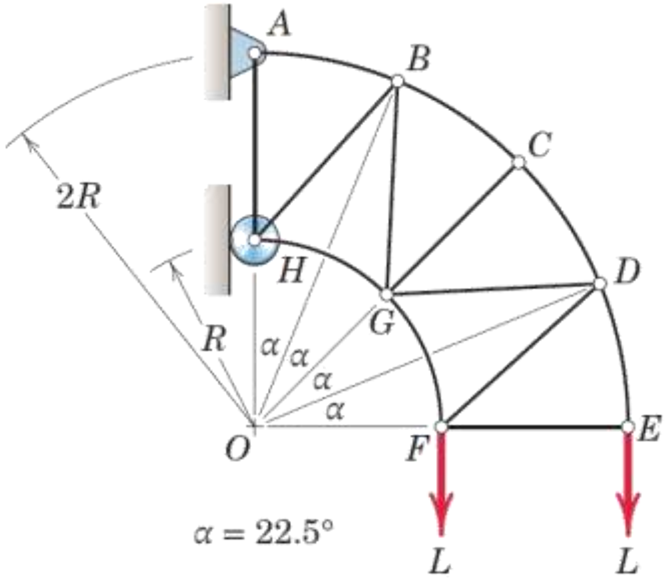
Joint E

$[\Sigma F_y = 0]$ $\quad 0.866DE = 10 \qquad DE = 11.55 \text{ kN } C$ *Ans.*

Article 4/3 – Sample Problem 4/2 (1 of 4)

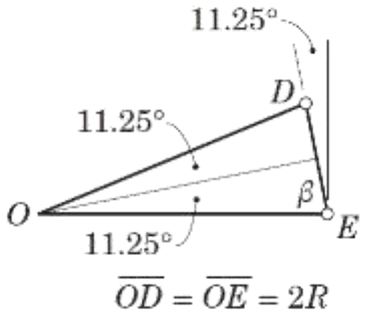
- Problem Statement

The simple truss shown supports the two loads, each of magnitude *L*. Determine the forces in members *DE*, *DF*, *DG*, and *CD*.

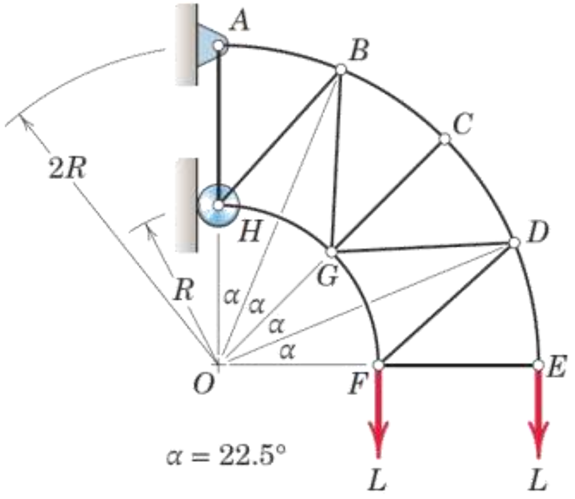


Article 4/3 – Sample Problem 4/2 (2 of 4)

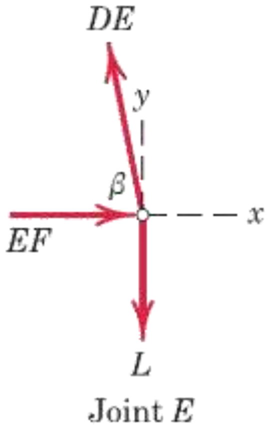
- Geometry of Joint *E*



We can begin with joint E because there are only two unknown member forces acting there. With reference to the free-body diagram and accompanying geometry for joint E, we note that $\beta = 180° - 11.25° - 90° = 78.8°$.



- Equilibrium of Joint *E*



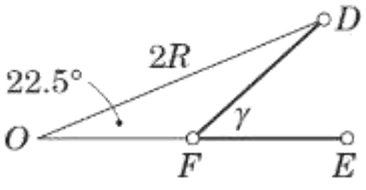
$[\Sigma F_y = 0]$ $\quad DE \sin 78.8° - L = 0 \quad DE = 1.020L\ T$ ① *Ans.*

$[\Sigma F_x = 0]$ $\quad EF - DE \cos 78.8° = 0 \quad EF = 0.1989L\ C$

① Rather than calculate and use the angle $\beta = 78.8°$ in the force equations, we could have used the 11.25° angle directly.

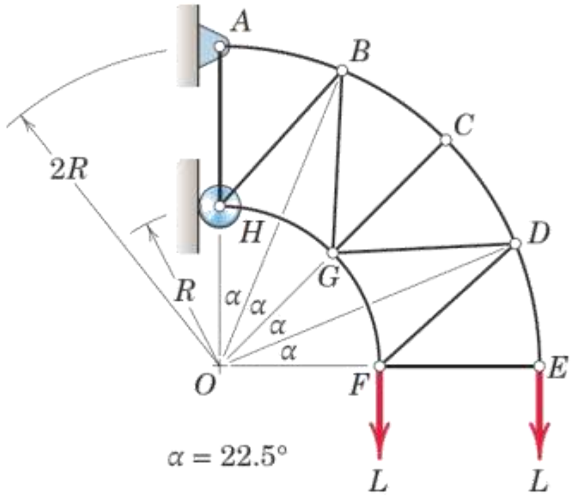
Article 4/3 – Sample Problem 4/2 (3 of 4)

- Geometry of Joint *F*

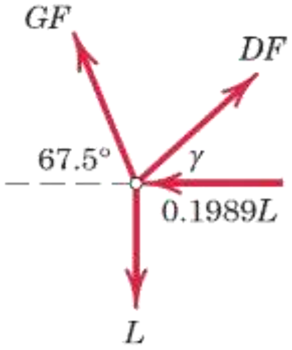


$\overline{OF} = \overline{FE} = R$

$$\gamma = \tan^{-1}\left[\frac{2R \sin 22.5°}{2R \cos 22.5° - R}\right] = 42.1°$$



- Equilibrium of Joint *F*



Joint *F*

$[\Sigma F_x = 0]$ $\quad -GF \cos 67.5° + DF \cos 42.1° - 0.1989L = 0$

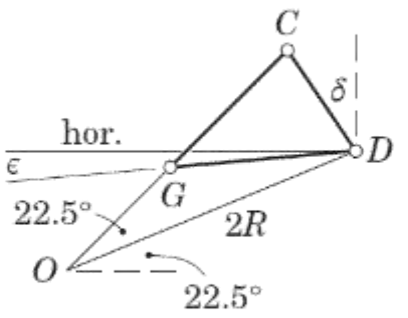
$[\Sigma F_y = 0]$ $\quad GF \sin 67.5° + DF \sin 42.1° - L = 0$

Simultaneous solution of these two equations yields

$GF = 0.646L\ T \qquad DF = 0.601L\ T$ *Ans.*

Article 4/3 – Sample Problem 4/2 (4 of 4)

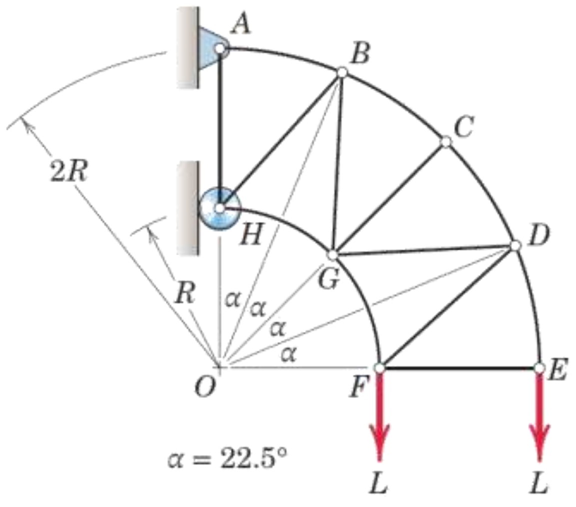
- Geometry of Joint *D*



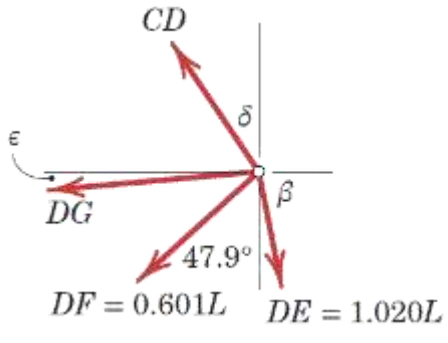
$\overline{OG} = \overline{CG} = R$

$$\delta = \tan^{-1}\left[\frac{2R\cos 22.5° - 2R\cos 45°}{2R\sin 45° - 2R\sin 22.5°}\right] = 33.8°$$

$$\epsilon = \tan^{-1}\left[\frac{2R\sin 22.5° - R\sin 45°}{2R\cos 22.5° - R\cos 45°}\right] = 2.92°$$



- Equilibrium of Joint *D*



$[\Sigma F_x = 0]$

$-DG\cos 2.92° - CD\sin 33.8° - 0.601L\sin 47.9° + 1.020L\cos 78.8° = 0$

$[\Sigma F_y = 0]$

$-DG\sin 2.92° + CD\cos 33.8° - 0.601L\cos 47.9° - 1.020L\sin 78.8° = 0$

The simultaneous solution is

$CD = 1.617L\ T \qquad DG = -1.147L$ or $DG = 1.147L\ C \qquad$ *Ans.*

Article 4/4 Method of Sections

- Differences with the Method of Joints
  - Can utilize the moment equation of equilibrium.
  - Can analyze an entire section of a truss.
  - Can determine the force in *almost any* desired member directly.
- A Word of Caution
  - Don’t abandon the method of joints.
  - Usually limited to cutting *at most* three members at one time.

Article 4/4 – Overview of Method of Sections

- General Procedure

1. In general, find the external reactions first (this may or may not be necessary).

2. If possible, pass a section (or cut) through the desired member and up to two (2) other unknown members, isolating a portion of the truss.

3. Apply two-dimensional rigid-body equilibrium equations to the isolated truss portion; e.g., $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma M_O = 0$.

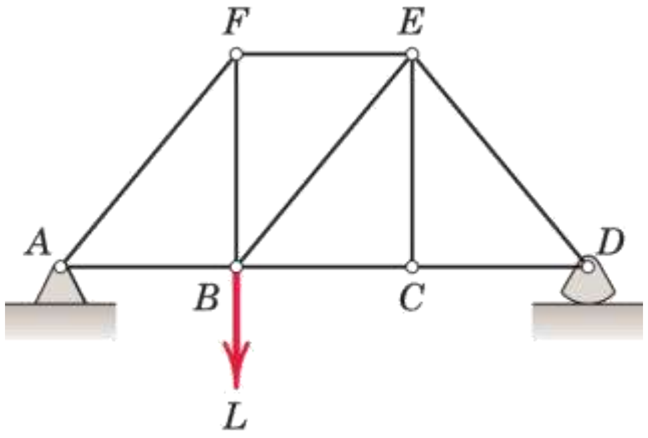
4. Solve for the unknowns.

5. Note that the methods of sections and joints may be used in combination.

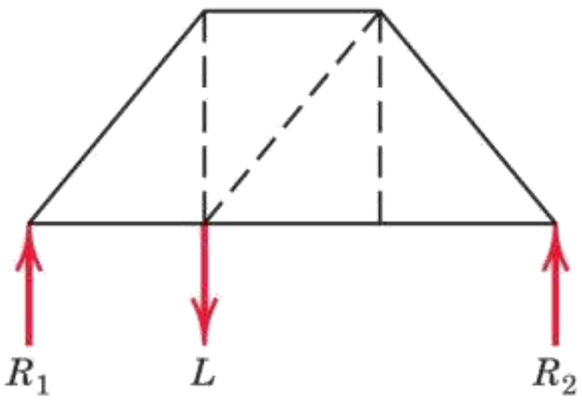
Article 4/4 – Illustration of Method of Sections (1 of 2)

- Determine the force in member *BE*.

  1. Determine the external reactions.



(a)

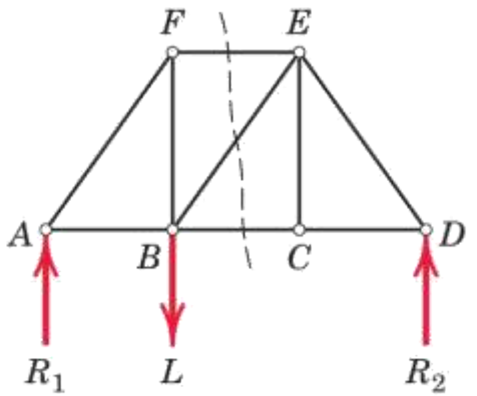


(b)

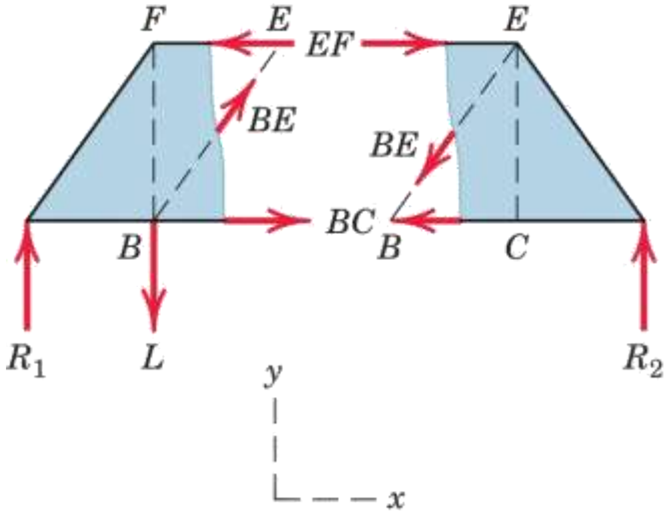
Article 4/4 – Illustration of Method of Sections (2 of 2)

- Determine the force in member *BE*.

  1. Determine the external reactions.
  2. Pass a section through member *BE*.
  3. Choose either side of the truss.
  4. Apply equilibrium equations.



(a)



(b)

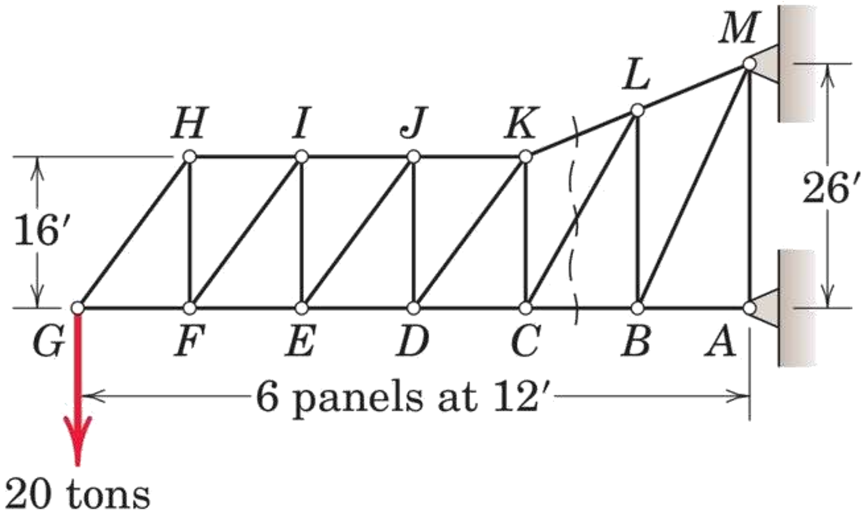
Article 4/4 – Additional Considerations

- Additional Considerations
  - Consider an entire portion of the truss in equilibrium.
  - Forces internal to the section are not considered.
  - Use either portion of the truss after cutting.
  - Combine with method of joints if necessary.
  - Select advantageous moment centers.
  - Stay mindful of assumed directions for unknown forces, tension or compression.

Article 4/4 – Sample Problem 4/3 (1 of 3)

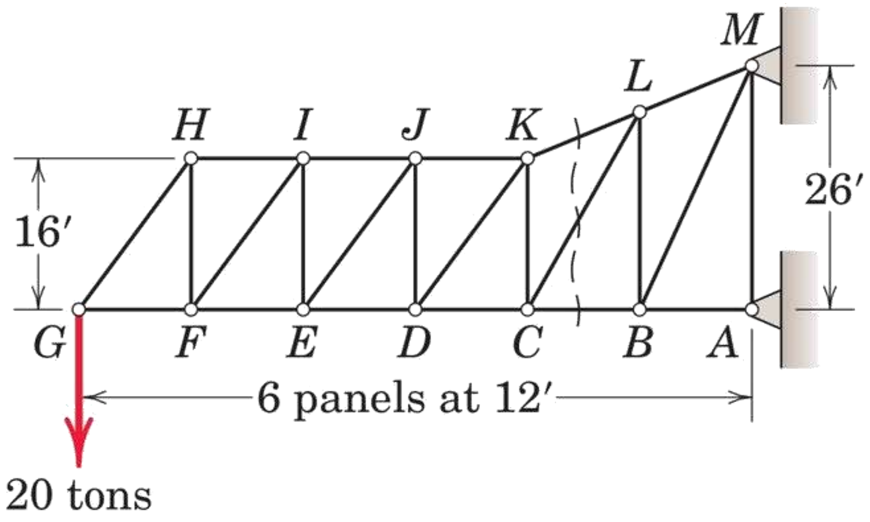
- Problem Statement

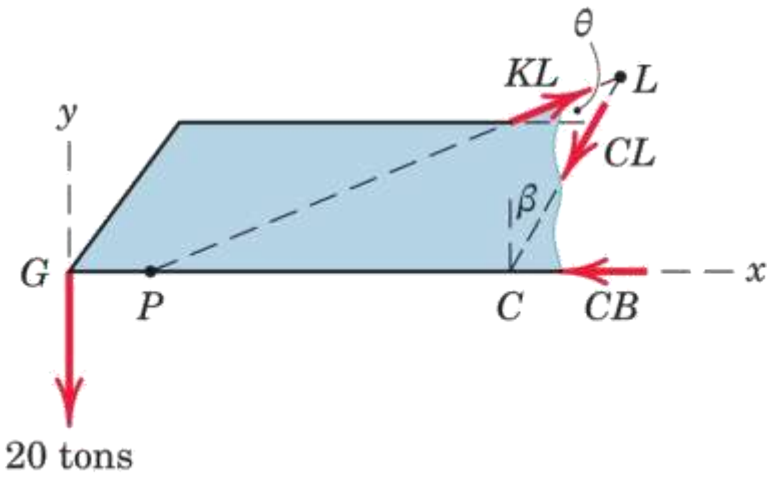
  Calculate the forces induced in members *KL*, *CL*, and *CB* by the 20-ton load on the cantilever truss.



Article 4/4 – Sample Problem 4/3 (2 of 3)

- FBD of Section





Article 4/4 – Sample Problem 4/3 (3 of 3)

- Equilibrium Conditions

Summing moments about L requires finding the moment arm $\overline{BL} = 16 + (26 - 16)/2 = 21$ ft. Thus,

$[\Sigma M_L = 0]$ $\quad 20(5)(12) - CB(21) = 0 \qquad CB = 57.1$ tons C *Ans.*

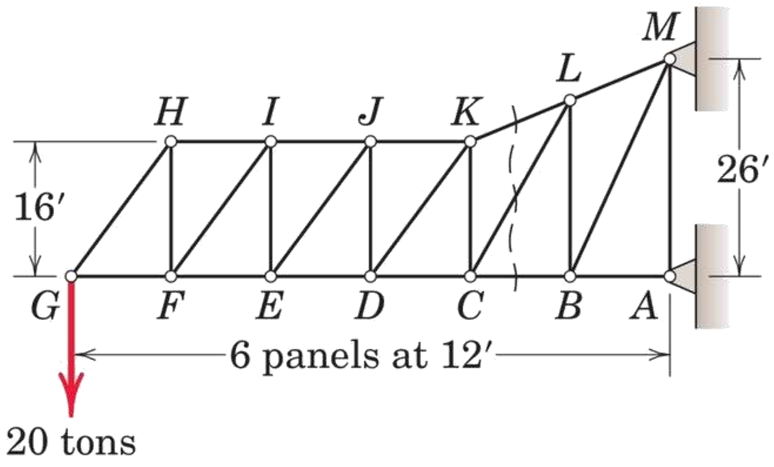
Next we take moments about C, which requires a calculation of $\cos\theta$. From the given dimensions we see $\theta = \tan^{-1}(5/12)$ so that $\cos\theta = 12/13$. Therefore,

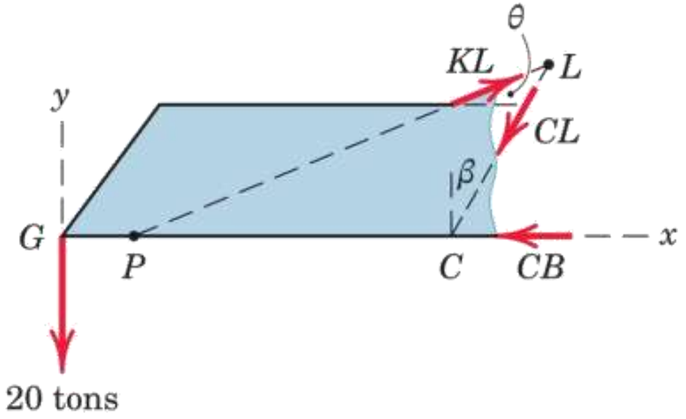
$[\Sigma M_C = 0]$ $\quad 20(4)(12) - \frac{12}{13}KL(16) = 0 \qquad KL = 65$ tons T *Ans.*

Finally, we may find CL by a moment sum about P, whose distance from C is given by $\overline{PC}/16 = 24/(26 - 16)$ or $\overline{PC} = 38.4$ ft. We also need β, which is given by $\beta = \tan^{-1}(\overline{CB}/\overline{BL}) = \tan^{-1}(12/21) = 29.7°$ and $\cos\beta = 0.868$. We now have

$[\Sigma M_P = 0]$ $\quad 20(48 - 38.4) - CL(0.868)(38.4) = 0$

$CL = 5.76$ tons C *Ans.*

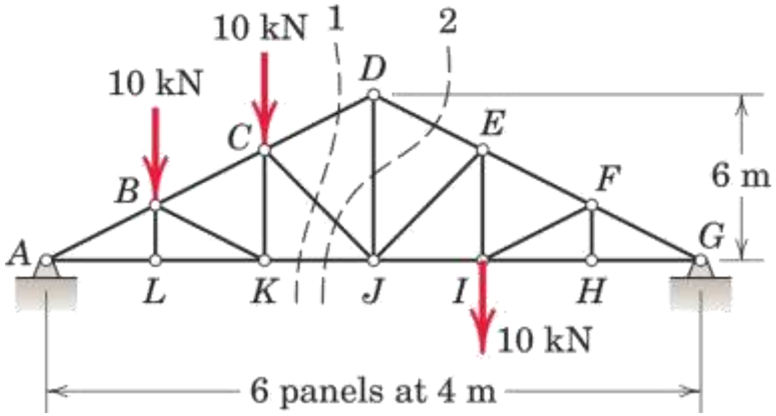




Article 4/4 – Sample Problem 4/4 (1 of 4)

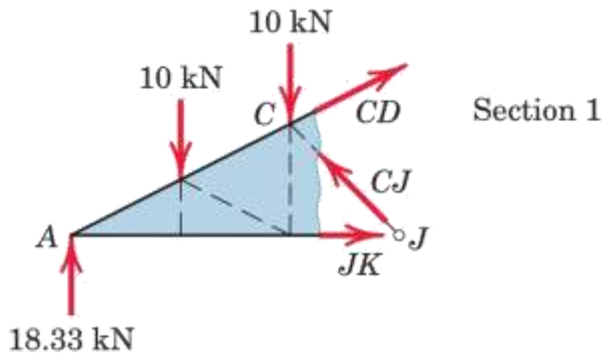
- Problem Statement

  Calculate the force in member *DJ* of the Howe roof truss illustrated. Neglect any horizontal components of force at the supports.

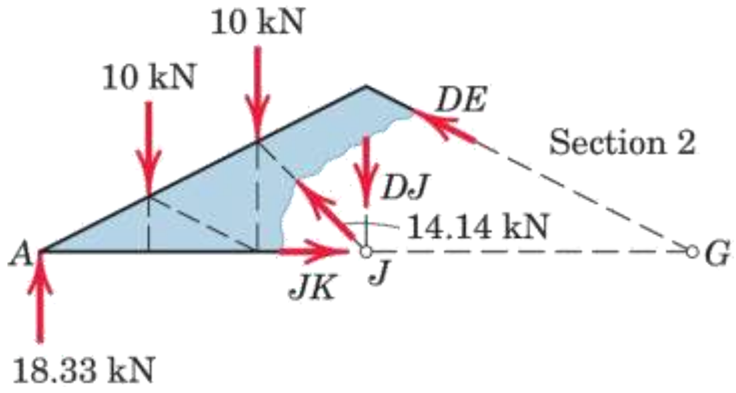


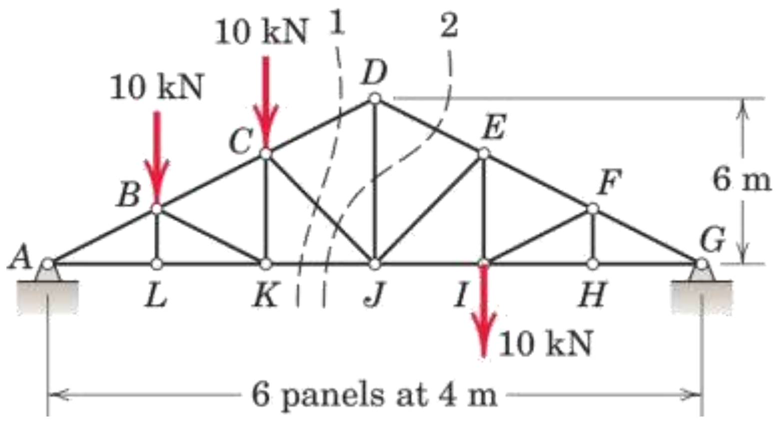
Article 4/4 – Sample Problem 4/4 (2 of 4)

- FBD of Section 1



- FBD of Section 2





Article 4/4 – Sample Problem 4/4 (3 of 4)

- Equilibrium Conditions

By the analysis of section 1, CJ is obtained from

$[\Sigma M_A = 0]$ $\quad 0.707CJ(12) - 10(4) - 10(8) = 0 \qquad CJ = 14.14 \text{ kN } C$

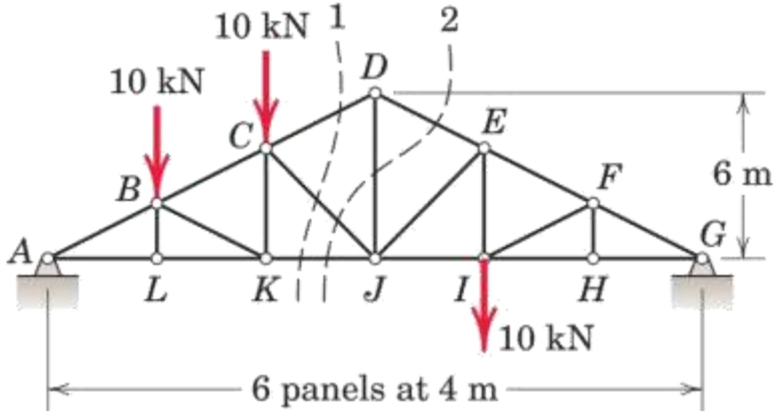
In this equation the moment of CJ is calculated by considering its horizontal and vertical components acting at point J. Equilibrium of moments about J requires

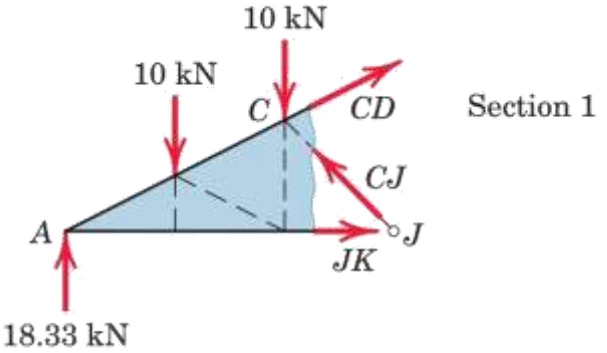
$[\Sigma M_J = 0]$ $\quad 0.894CD(6) + 18.33(12) - 10(4) - 10(8) = 0$

$$CD = -18.63 \text{ kN}$$

The moment of CD about J is calculated here by considering its two components as acting through D. The minus sign indicates that CD was assigned in the wrong direction.

Hence, $\qquad CD = 18.63 \text{ kN } C$





Article 4/4 – Sample Problem 4/4 (4 of 4)

- Equilibrium Conditions

From the free-body diagram of section 2, which now includes the known value of CJ, a balance of moments about G is seen to eliminate DE and JK. Thus,

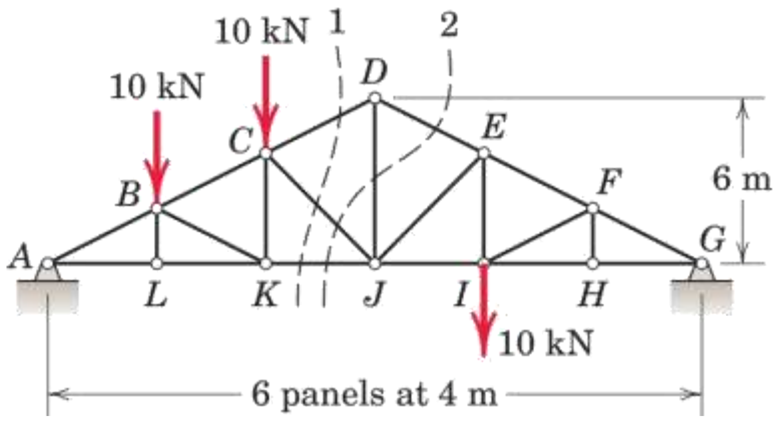
$[\Sigma M_G = 0]$

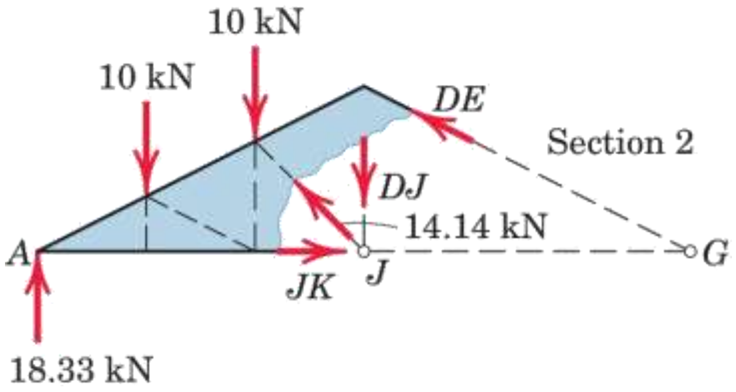
$$12DJ + 10(16) + 10(20) - 18.33(24) - 14.14(0.707)(12) = 0$$

$$DJ = 16.67 \text{ kN } T \qquad \textit{Ans.}$$

Again the moment of CJ is determined from its components considered to be acting at J. The answer for DJ is positive, so that the assumed tensile direction is correct.

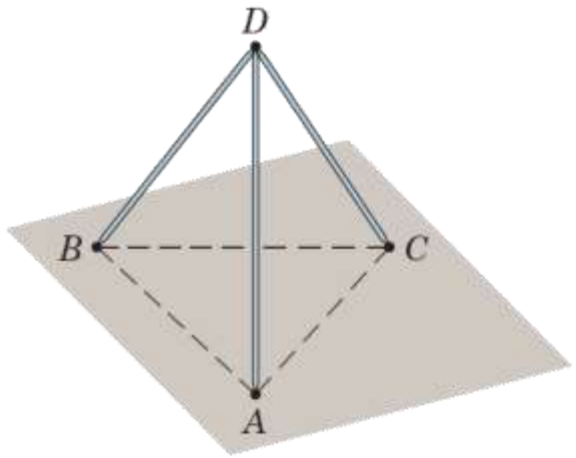
An alternative approach to the entire problem is to utilize section 1 to determine CD and then use the method of joints applied at D to determine DJ.

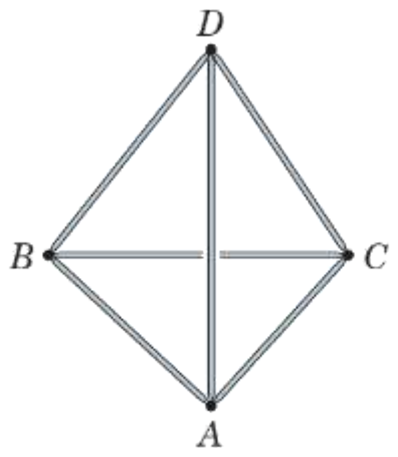




Article 4/5 Space Trusses (1 of 2)

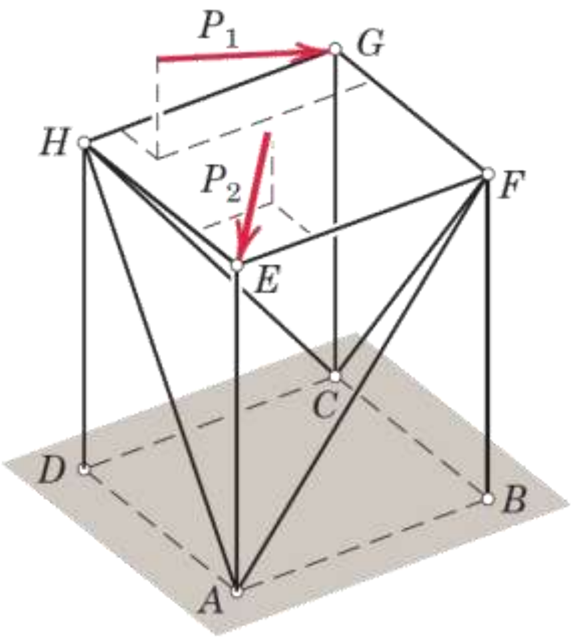
- Three-Dimensional Analogue of Plane Truss
- Basic Shape is a Tetrahedron
- Analysis Utilizes Method of Joints and Method of Sections extended to Three Dimensions
- Statically Determinant if… $m + 6 = 3j$





Article 4/5 Space Trusses (2 of 2)

- Method of Joints for Space Trusses
  - Same Basic Procedure as for Plane Trusses
  - $\Sigma \mathbf{F} = \mathbf{0}$ at each Joint ($\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma F_z = 0$)
  - Limited to Solving Three (3) Unknown Forces at a Time

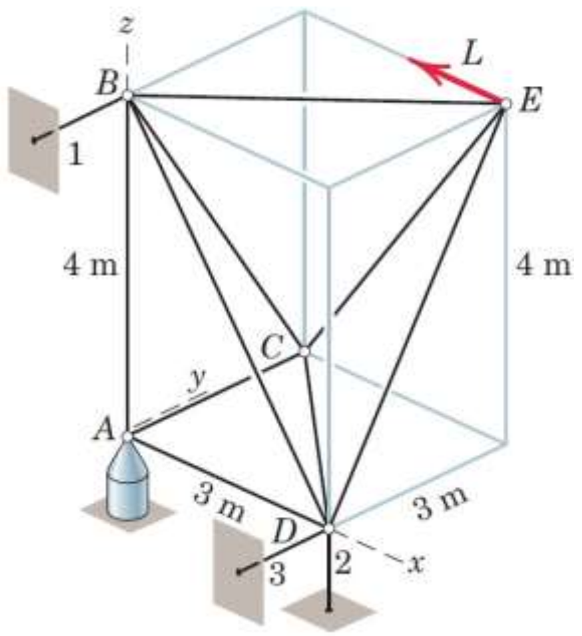


- Method of Joints for Space Trusses
  - Same Basic Procedure as for Plane Trusses
  - $\Sigma \mathbf{F} = \mathbf{0}$ for any Section ($\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma F_z = 0$)
  - $\Sigma \mathbf{M} = \mathbf{0}$ for any Section ($\Sigma M_x = 0$, $\Sigma M_y = 0$, and $\Sigma M_z = 0$)
  - Limited to Solving Six (6) Unknown Forces at a Time

Article 4/5 – Sample Problem 4/5 (1 of 3)

- Problem Statement

  The space truss consists of the rigid tetrahedron *ABCD* anchored by a ball-and-socket connection at *A* and prevented from any rotation about the *x*-, *y*-, or *z*-axes by the respective links 1, 2, and 3. The load *L* is applied to joint *E*, which is rigidly fixed to the tetrahedron by the three additional links. Solve for the forces in the members at joint *E* and indicate the procedure for the determination of the forces in the remaining members of the truss.

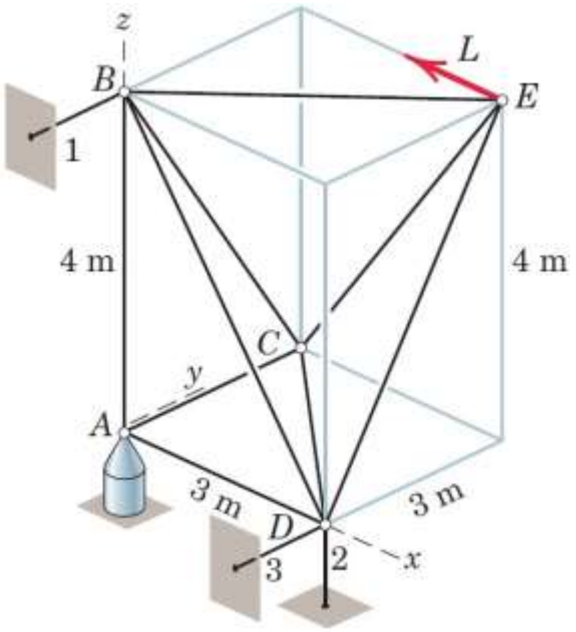


Article 4/5 – Sample Problem 4/5 (2 of 3)

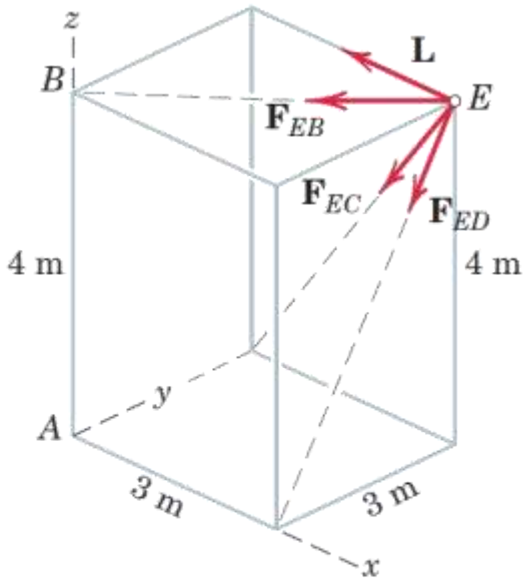
- Check Determinacy

  $m + 6 = 3j$

  $9 + 6 = 3(5)$ True



- Free-Body Diagram of Joint *E*



$$\mathbf{F}_{EB} = \frac{F_{EB}}{\sqrt{2}}(-\mathbf{i} - \mathbf{j}), \qquad \mathbf{F}_{EC} = \frac{F_{EC}}{5}(-3\mathbf{i} - 4\mathbf{k}), \qquad \mathbf{F}_{ED} = \frac{F_{ED}}{5}(-3\mathbf{j} - 4\mathbf{k})$$

Article 4/5 – Sample Problem 4/5 (3 of 3)

- Equilibrium of Joint *E*

$$[\Sigma \mathbf{F} = \mathbf{0}] \qquad \mathbf{L} + \mathbf{F}_{EB} + \mathbf{F}_{EC} + \mathbf{F}_{ED} = \mathbf{0} \qquad \text{or}$$

$$-L\mathbf{i} + \frac{F_{EB}}{\sqrt{2}}(-\mathbf{i} - \mathbf{j}) + \frac{F_{EC}}{5}(-3\mathbf{i} - 4\mathbf{k}) + \frac{F_{ED}}{5}(-3\mathbf{j} - 4\mathbf{k}) = \mathbf{0}$$

Rearranging terms gives

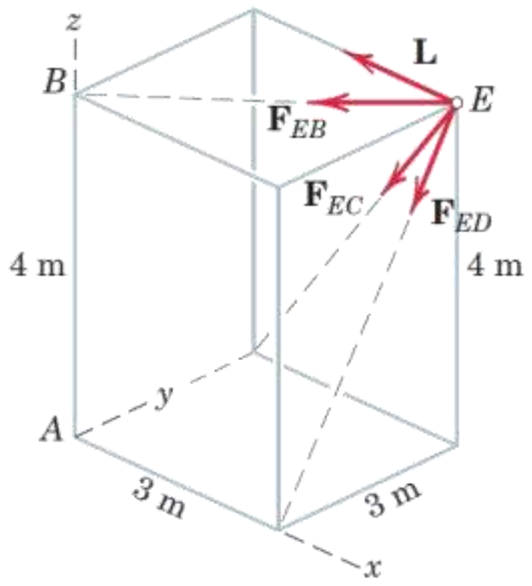
$$\left(-L - \frac{F_{EB}}{\sqrt{2}} - \frac{3F_{EC}}{5}\right)\mathbf{i} + \left(-\frac{F_{EB}}{\sqrt{2}} - \frac{3F_{ED}}{5}\right)\mathbf{j} + \left(-\frac{4F_{EC}}{5} - \frac{4F_{ED}}{5}\right)\mathbf{k} = \mathbf{0}$$

Equating the coefficients of the **i**-, **j**-, and **k**-unit vectors to zero gives the three equations

$$\frac{F_{EB}}{\sqrt{2}} + \frac{3F_{EC}}{5} = -L \qquad \frac{F_{EB}}{\sqrt{2}} + \frac{3F_{ED}}{5} = 0 \qquad F_{EC} + F_{ED} = 0$$

Solving the equations gives us

$$F_{EB} = -L/\sqrt{2} \qquad F_{EC} = -5L/6 \qquad F_{ED} = 5L/6 \qquad \textit{Ans.}$$



Article 4/6 Frames and Machines

- Definition

  A structure is called a *frame* or *machine* if at least one of its individual members is a *multiforce member*. A multiforce member is defined as one with three or more forces acting on it, or one with two or more forces and one or more couples acting on it.

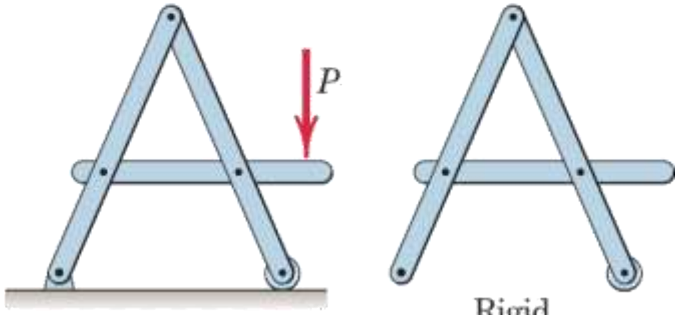
- Example – Jaws of Life

Billy Gadbury/Shutterstock

Two devices used by rescuers to free accident victims from wreckage. The "jaws of life" machine shown at the left is the subject of problems in this article and the chapter-review article.

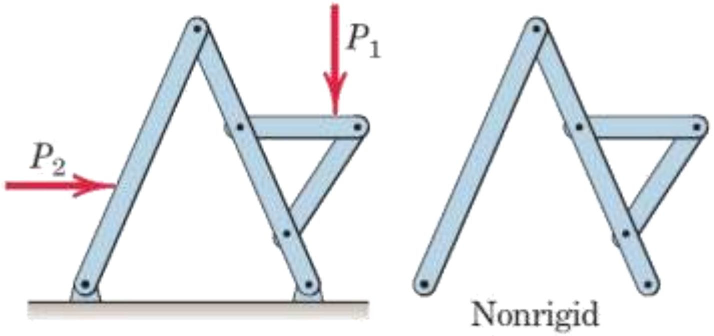
Article 4/6 – Types of Structures

- Interconnected Rigid Bodies with Multiforce Members
  - Rigid Noncollapsible
    - Find external reactions for the entire structure first.
    - Dismember to find internal reactions.
    - Look for two-force members.



Rigid
noncollapsible

  - Nonrigid Collapsible
    - Dismember the structure first to find internal reactions.
    - Compute external reactions later.
    - Look for two-force members.



Nonrigid
collapsible

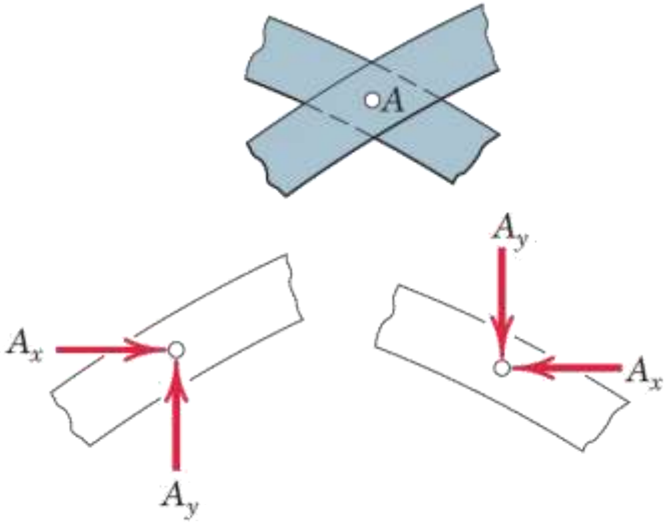
Article 4/6 – Force Representation and FBDs

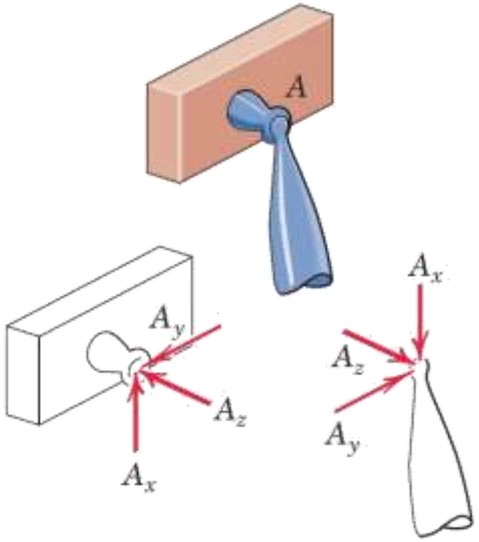
- Importance of Action-Reaction Pairs

- Assumed Force Directions

- Vector Notation

- Simultaneous Equations

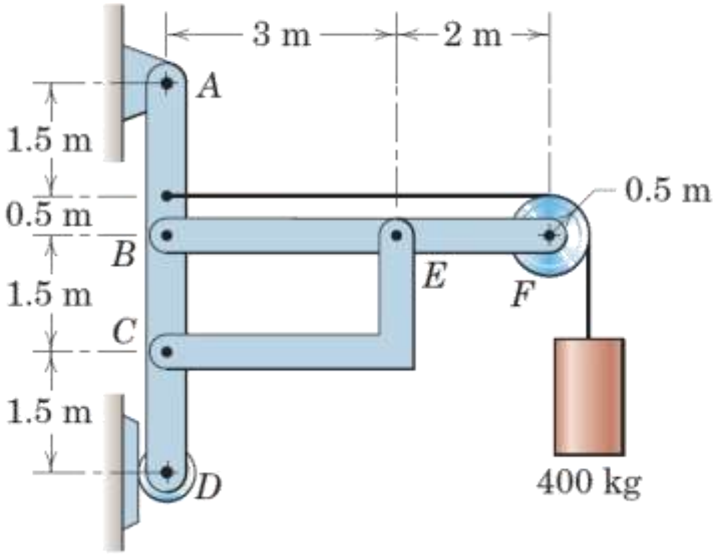




Article 4/6 – Sample Problem 4/6 (1 of 3)

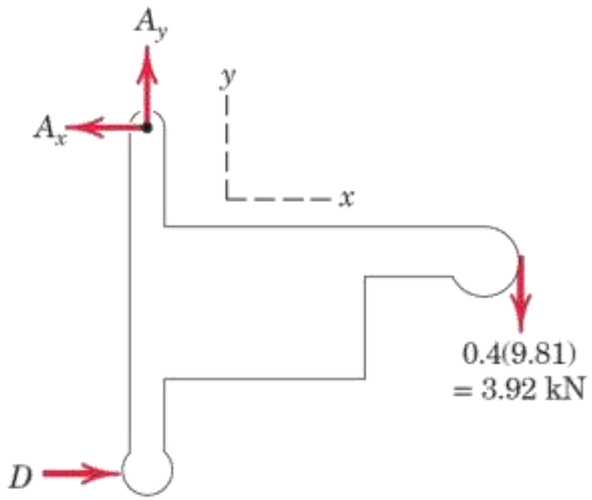
- Problem Statement

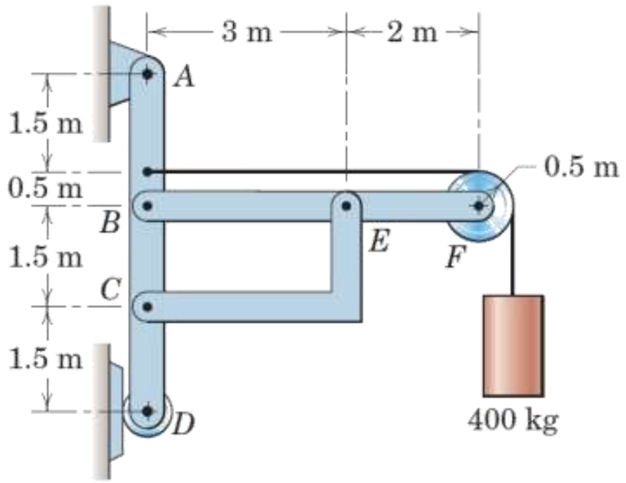
  The frame supports the 400-kg load in the manner shown. Neglect the weights of the members compared with the forces induced by the load and compute the horizontal and vertical components of all forces acting on each of the members.



Article 4/6 – Sample Problem 4/6 (2 of 3)

- Free-Body Diagram of Entire Structure (Rigid)



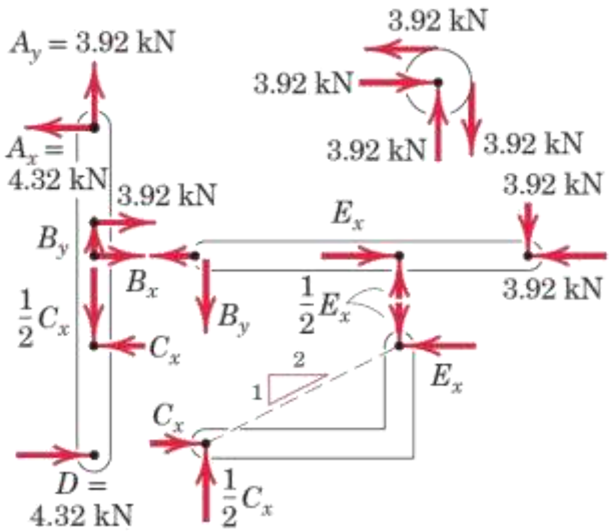


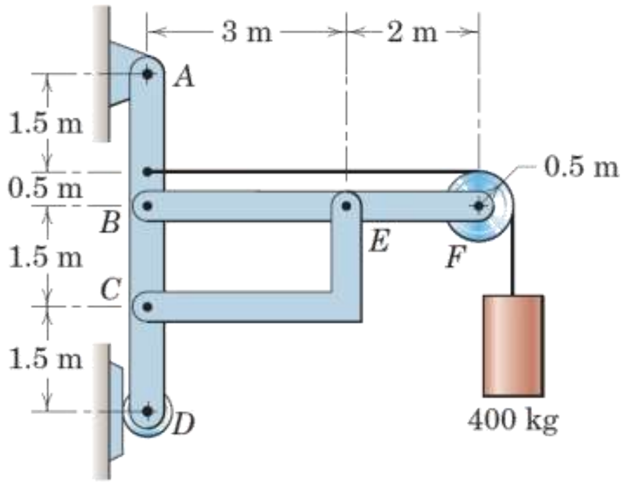
- Equilibrium Conditions

$[\Sigma M_A = 0]$	$5.5(0.4)(9.81) - 5D = 0$	$D = 4.32\text{ kN}$
$[\Sigma F_x = 0]$	$A_x - 4.32 = 0$	$A_x = 4.32\text{ kN}$
$[\Sigma F_y = 0]$	$A_y - 3.92 = 0$	$A_y = 3.92\text{ kN}$

Article 4/6 – Sample Problem 4/6 (3 of 3)

- Free-Body Diagrams of Individual Pieces





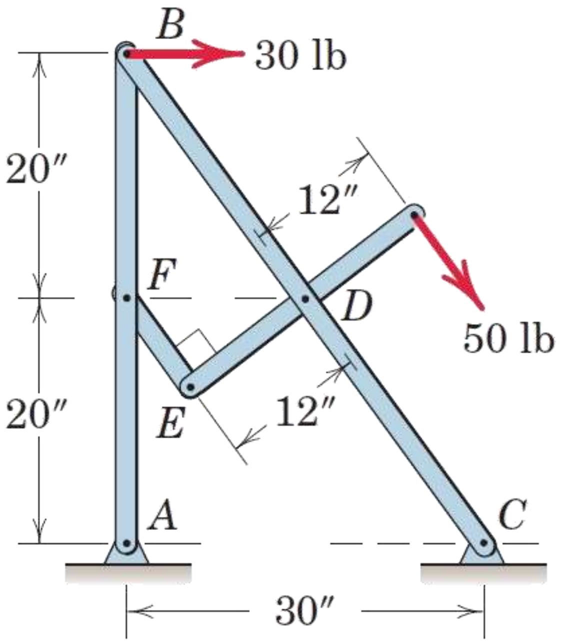
- Equilibrium of Member *BF*

$[\Sigma M_B = 0]$	$3.92(5) - \frac{1}{2}E_x(3) = 0$	$E_x = 13.08$ kN	*Ans.*
$[\Sigma F_y = 0]$	$B_y + 3.92 - 13.08/2 = 0$	$B_y = 2.62$ kN	*Ans.*
$[\Sigma F_x = 0]$	$B_x + 3.92 - 13.08 = 0$	$B_x = 9.15$ kN	*Ans.*

Article 4/6 – Sample Problem 4/7 (1 of 4)

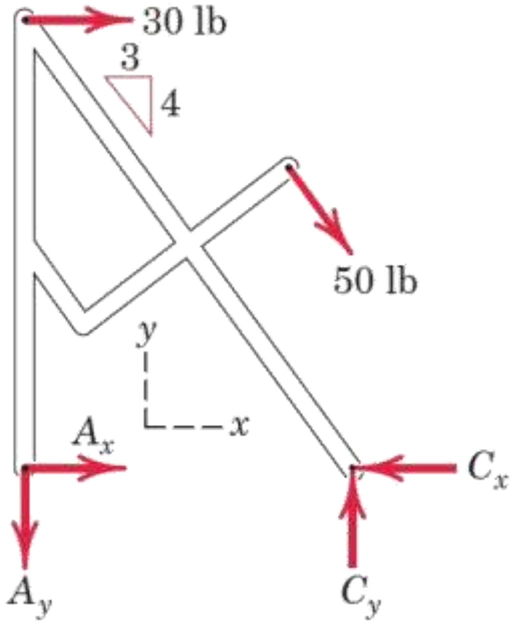
- Problem Statement

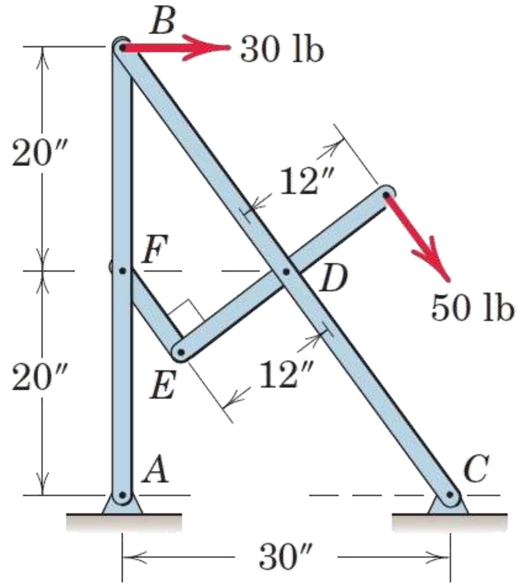
  Neglect the weight of the frame and compute the forces acting on all of its members.



Article 4/6 – Sample Problem 4/7 (2 of 4)

- Free-Body Diagram of Entire Structure (Nonrigid)





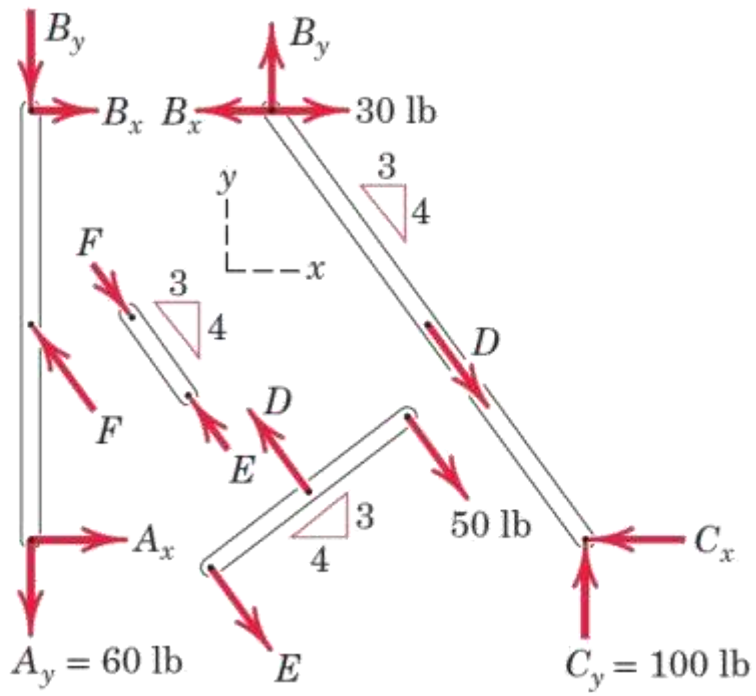
- Vertical Reactions at *A* and *C*

$[\Sigma M_C = 0]$ $\quad 50(12) + 30(40) - 30A_y = 0 \quad A_y = 60 \text{ lb}$ *Ans.*

$[\Sigma F_y = 0]$ $\quad C_y - 50(4/5) - 60 = 0 \quad C_y = 100 \text{ lb}$ *Ans.*

Article 4/6 – Sample Problem 4/7 (3 of 4)

- Free-Body Diagrams of Individual Pieces



- Equilibrium of Member *ED*

$[\Sigma M_D = 0]$	$50(12) - 12E = 0$	$E = 50$ lb	*Ans.*
$[\Sigma F = 0]$	$D - 50 - 50 = 0$	$D = 100$ lb	*Ans.*

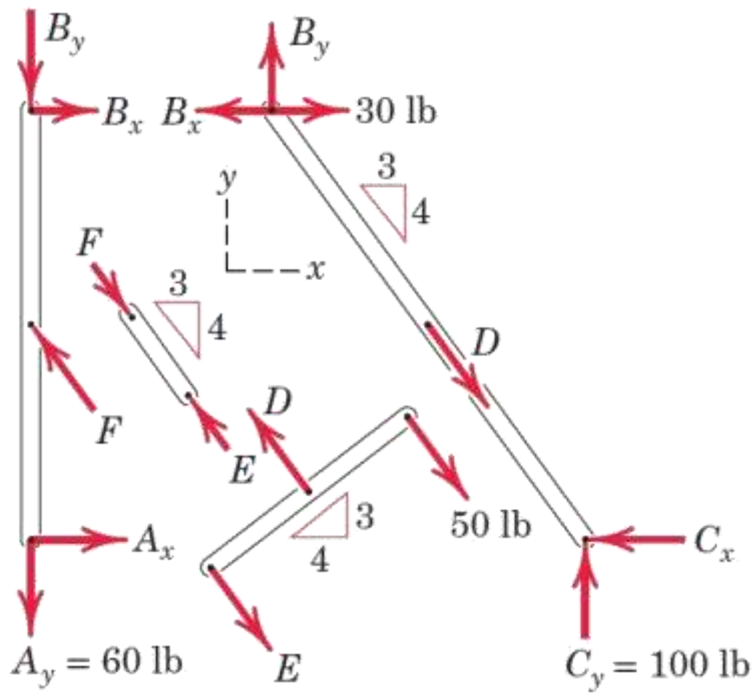
F is equal and opposite to *E* since *EF* is a two-force member.

- Equilibrium of Member *AB*

$[\Sigma M_A = 0]$	$50(3/5)(20) - B_x(40) = 0$	$B_x = 15$ lb	*Ans.*
$[\Sigma F_x = 0]$	$A_x + 15 - 50(3/5) = 0$	$A_x = 15$ lb	*Ans.*
$[\Sigma F_y = 0]$	$50(4/5) - 60 - B_y = 0$	$B_y = -20$ lb	*Ans.*

Article 4/6 – Sample Problem 4/7 (4 of 4)

- Free-Body Diagrams of Individual Pieces



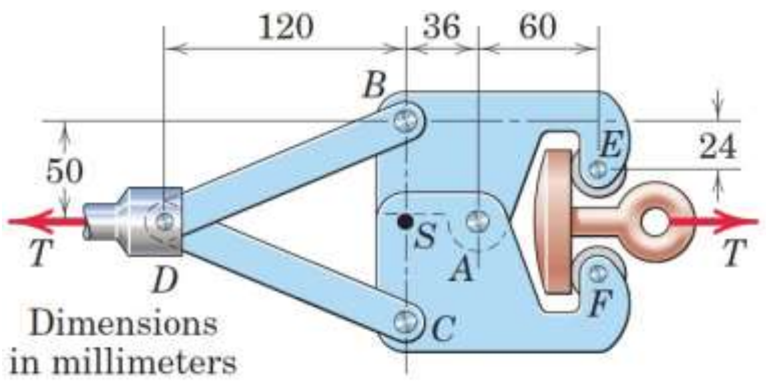
- Equilibrium of Member *BC*

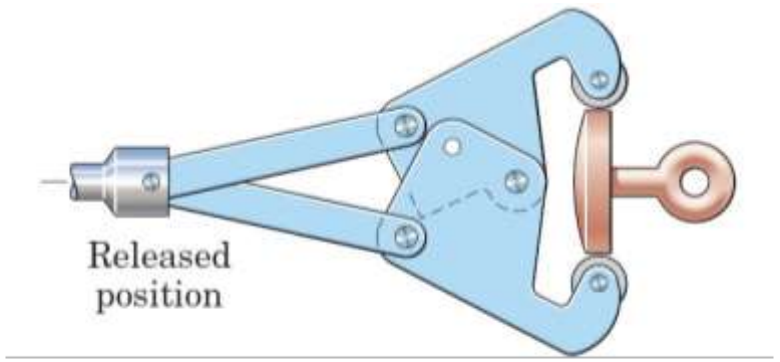
$[\Sigma F_x = 0] \qquad 30 + 100(3/5) - 15 - C_x = 0 \qquad C_x = 75 \text{ lb}$ ④ *Ans.*

Article 4/6 – Sample Problem 4/8 (1 of 3)

- Problem Statement

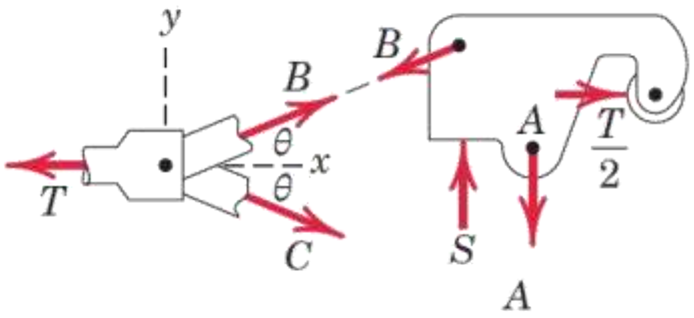
  The machine shown is designed as an overload protection device which releases the load when it exceeds a predetermined value *T*. A soft metal shear pin *S* is inserted in a hole in the lower half and is acted on by the upper half. When the total force on the pin exceeds its strength, it will break. The two halves then rotate about *A* under the action of the tensions in *BD* and *CD*, as shown in the second sketch, and rollers *E* and *F* release the eye bolt. Determine the maximum allowable tension *T* if the pin *S* will shear when the total force on it is 800 N. Also compute the corresponding force on the hinge pin *A*.

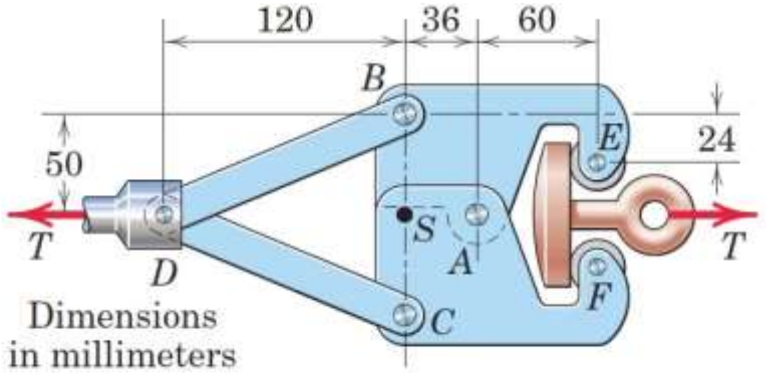




Article 4/6 – Sample Problem 4/8 (2 of 3)

- Individual Free-Body Diagrams





- Equilibrium of the Connection *D* (Note the Symmetry)

$[\Sigma F_x = 0]$ $\quad B\cos\theta + C\cos\theta - T = 0 \qquad 2B\cos\theta = T$

$$B = T/(2\cos\theta)$$

① It is always useful to recognize symmetry. Here it tells us that the forces acting on the two parts behave as mirror images of each other with respect to the x-axis. Thus, we cannot have an action on one member in the plus x-direction and its reaction on the other member in the negative x-direction. Consequently, the forces at S and A have no x-components.

Article 4/6 – Sample Problem 4/8 (3 of 3)

- Equilibrium of the Upper Part

From the free-body diagram of the upper part we express the equilibrium of moments about point A. Substituting $S = 800$ N and the expression for B gives

$[\Sigma M_A = 0]$

$$\frac{T}{2\cos\theta}(\cos\theta)(50) + \frac{T}{2\cos\theta}(\sin\theta)(36) - 36(800) - \frac{T}{2}(26) = 0 \quad ②$$

Substituting $\sin\theta/\cos\theta = \tan\theta = 5/12$ and solving for T give

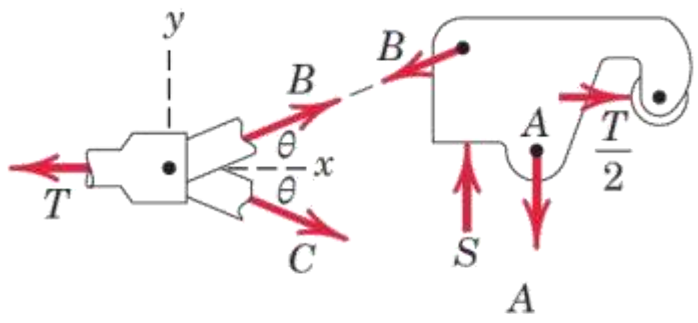
$$T\left(25 + \frac{5(36)}{2(12)} - 13\right) = 28\ 800$$

$$T = 1477 \text{ N} \qquad \text{or} \qquad T = 1.477 \text{ kN} \qquad \textit{Ans.}$$

Finally, equilibrium in the y-direction gives us

$[\Sigma F_y = 0]$ $\qquad S - B\sin\theta - A = 0$

$$800 - \frac{1477}{2(12/13)}\frac{5}{13} - A = 0 \qquad A = 492 \text{ N} \qquad \textit{Ans.}$$

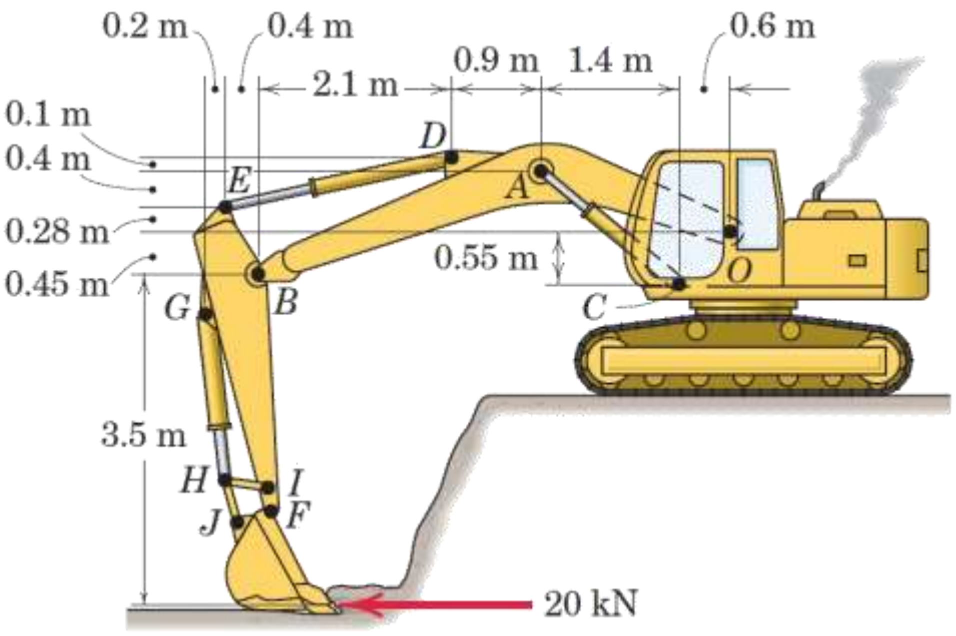


② Be careful not to forget the moment of the y-component of B. Note that our units here are newton-millimeters.

Article 4/6 – Sample Problem 4/9 (1 of 3)

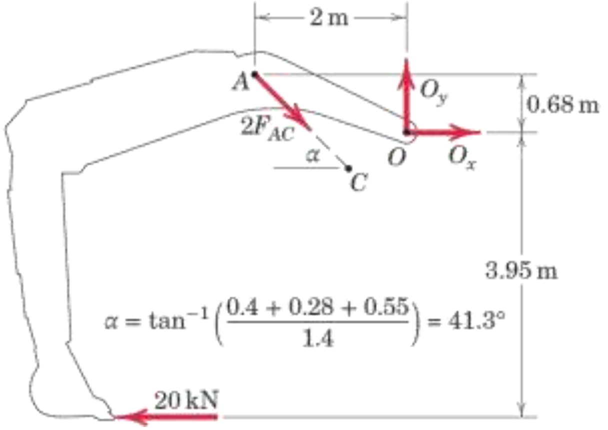
- Problem Statement

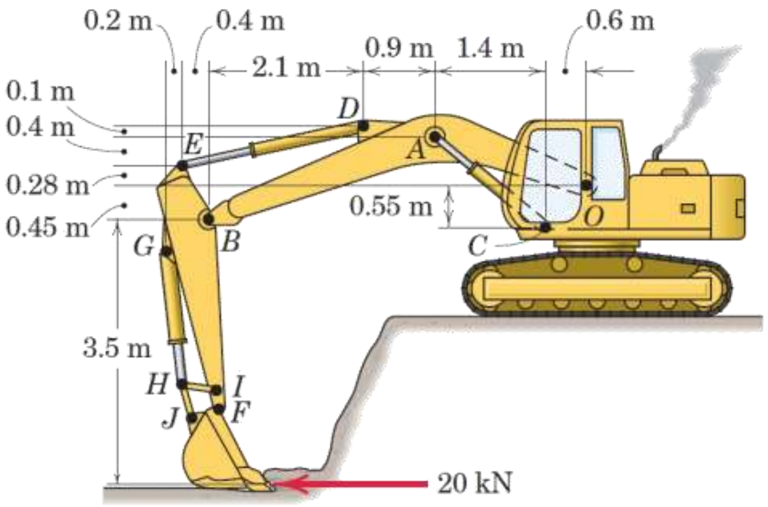
  In the particular position shown, the excavator applies a 20-kN force parallel to the ground. There are two hydraulic cylinders *AC* to control the arm *OAB* and a single cylinder *DE* to control arm *EBIF*. (*a*) Determine the force in the hydraulic cylinders *AC* and the pressure p_{AC} against their pistons, which have an effective diameter of 95 mm. (*b*) Also determine the force in hydraulic cylinder *DE* and the pressure p_{DE} against its 105-mm-diameter piston. Neglect the weights of the members compared with the effects of the 20-kN force.



Article 4/6 – Sample Problem 4/9 (2 of 3)

- Free-Body Diagram of Entire Arm Assembly





- Force and Pressure in Cylinder *AC*

$[\Sigma M_O = 0]$

$$-20\,000(3.95) - 2F_{AC}\cos 41.3°(0.68) + 2F_{AC}\sin 41.3°(2) = 0$$

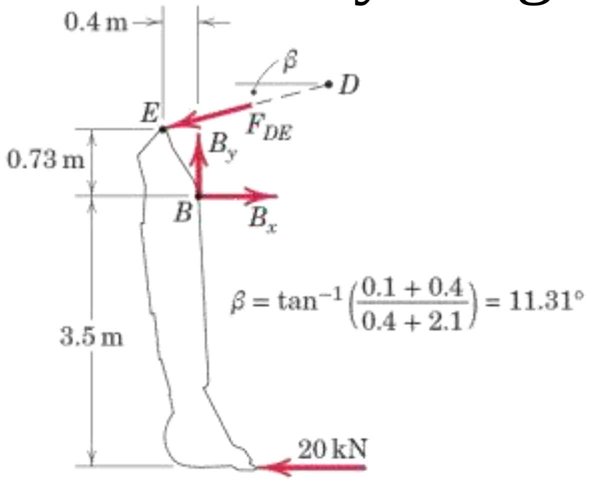
$$F_{AC} = 48\,800 \text{ N or } 48.8 \text{ kN} \qquad \textit{Ans.}$$

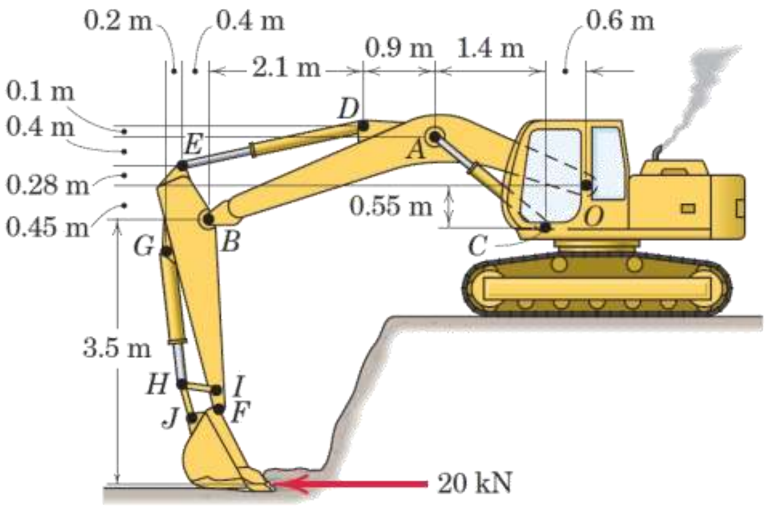
$$p_{AC} = \frac{F_{AC}}{A_{AC}} = \frac{48\,800}{\left(\pi\frac{0.095^2}{4}\right)} = 6.89(10^6)\text{ Pa or } 6.89\text{ MPa} \quad ① \qquad \textit{Ans.}$$

① Recall that force = (pressure)(area).

Article 4/6 – Sample Problem 4/9 (3 of 3)

- Free-Body Diagram of Lower Arm





- Force and Pressure in Cylinder *DE*

$[\Sigma M_B = 0]$

$$-20\,000(3.5) + F_{DE} \cos 11.31°(0.73) + F_{DE} \sin 11.31°(0.4) = 0$$

$$F_{DE} = 88\,100 \text{ N or } 88.1 \text{ kN} \qquad \textit{Ans.}$$

$$p_{DE} = \frac{F_{DE}}{A_{DE}} = \frac{88\,100}{\left(\pi \dfrac{0.105^2}{4}\right)} = 10.18(10^6) \text{ Pa or } 10.18 \text{ MPa} \qquad \textit{Ans.}$$

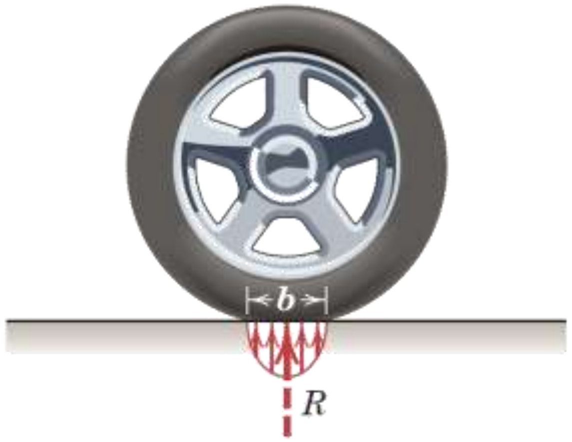
CHAPTER 5
DISTRIBUTED FORCES

Graham Oliver/Alamy Stock Photo

Article 5/1 Introduction

- Concentrated Forces

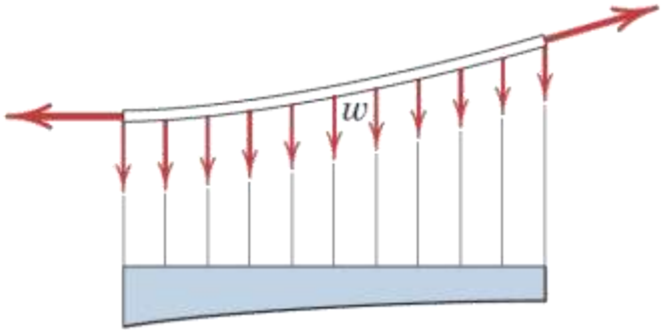
- Distributed Forces



  - Forces are spread out over a line, area, or volume.
  - Treated as concentrated forces when analyzing their external effects on a body.
  - Occur when a force is applied over a region whose dimensions are not negligible compared with other pertinent dimensions.

Article 5/1 – Categories of Distributed Forces (1 of 3)

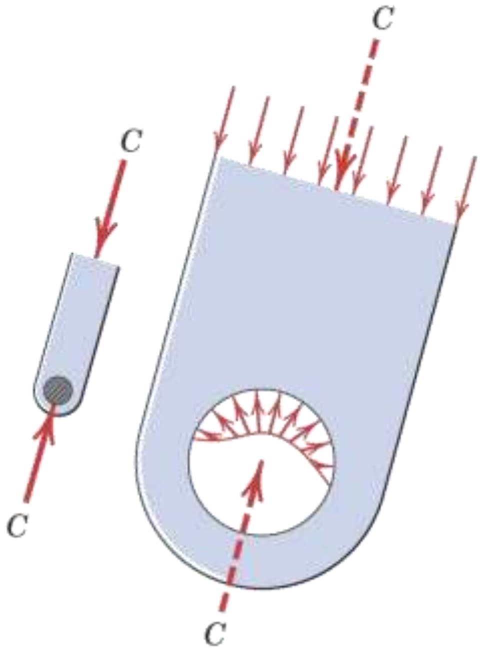
- Line Distribution
  - Illustration

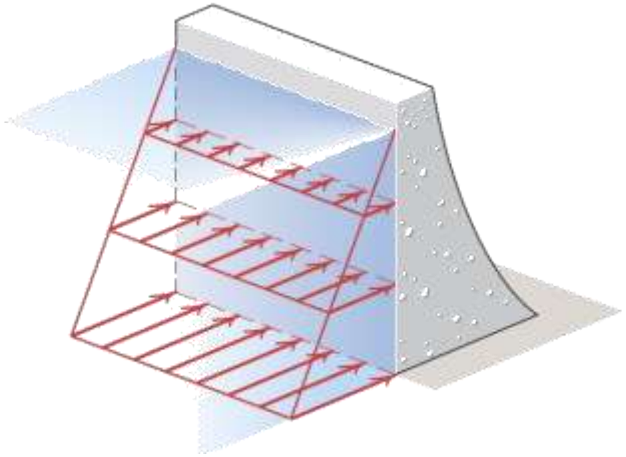


  - Intensity *w* is N/m or lb/ft

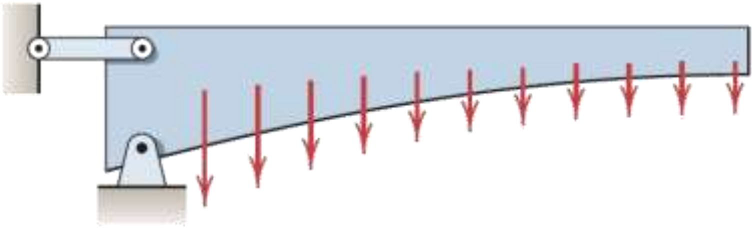
Article 5/1 – Categories of Distributed Forces (2 of 3)

- Area Distribution
  - Illustrations
  - Intensity is termed pressure for fluids and stress for solids.
    - SI unit is the Pascal (Pa).
    - 1 Pa = 1 N/m^2
    - 6895 Pa = 1 lb/in.^{2}
    - Pressure is commonly reported in kilopascals (kPa = 10^3 Pa).
    - Stress is commonly reported in megapascals (MPa = 10^6 Pa).





Article 5/1 – Categories of Distributed Forces (3 of 3)



- Volume Distribution (Body Force)
  - Illustration
  - Most common body force is gravitational attraction.
  - Intensity is termed specific weight, $\gamma = \rho g$.
    - ρ = density (mass per unit volume)
    - g = acceleration due to gravity
    - SI units are N/m^3
    - U.S. customary units are lb/ft^3 or $lb/in.^3$

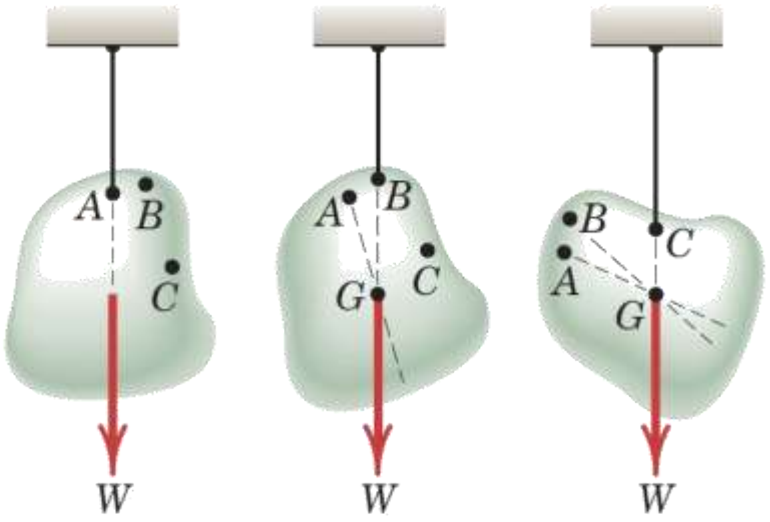
Article 5/2 Center of Mass

- Introduction

- Illustration with Hanging Mass

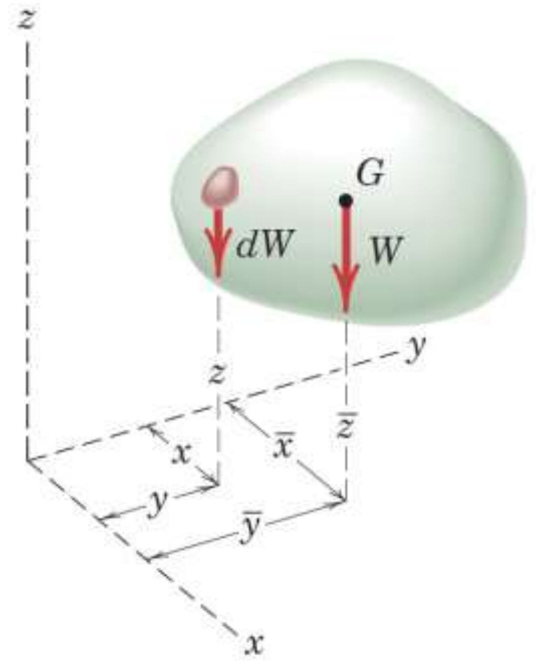
- Center of Gravity, *G*

- Assumptions



Article 5/2 – Determining the Center of Gravity (1 of 3)

- Principle of Moments
  - Body of Total Weight W
  - Center of Gravity Coordinates are $(\bar{x}, \bar{y}, \bar{z})$
  - Differential Element of the Body of Weight dW
  - Coordinates of the Element are (x, y, z)
  - The sum of the moments equals the moment of the sum.



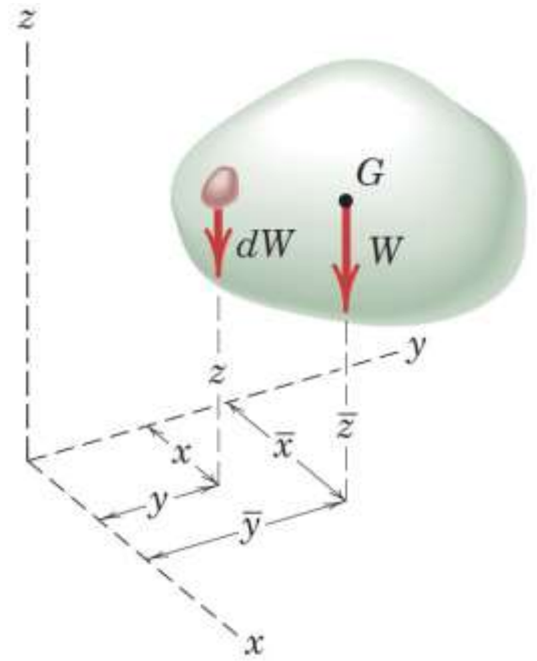
- Equations of Interest

$$\bar{x} = \frac{\int x\, dW}{W} \qquad \bar{y} = \frac{\int y\, dW}{W} \qquad \bar{z} = \frac{\int z\, dW}{W}$$

Article 5/2 – Determining the Center of Gravity (2 of 3)

- Substitute $W = mg$ and $dW = g\ dm$…

$$\boxed{\bar{x} = \frac{\int x\,dm}{m} \qquad \bar{y} = \frac{\int y\,dm}{m} \qquad \bar{z} = \frac{\int z\,dm}{m}}$$



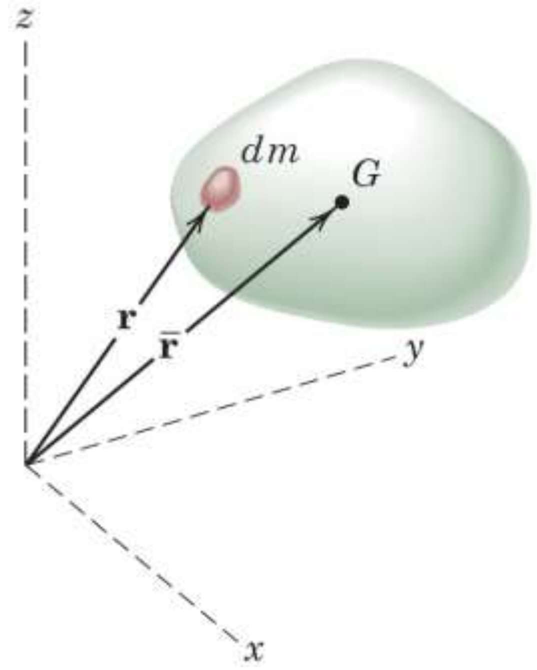
- Substitute $\rho\ = mV$ and $dm = \rho\ dV$, with ρ variable…

$$\bar{x} = \frac{\int x\rho\,dV}{\int \rho\,dV} \qquad \bar{y} = \frac{\int y\rho\,dV}{\int \rho\,dV} \qquad \bar{z} = \frac{\int z\rho\,dV}{\int \rho\,dV}$$

Article 5/2 – Determining the Center of Gravity (3 of 3)

- Vector Format
  - $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
  - $\bar{\boldsymbol{r}} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j} + \bar{z}\mathbf{k}$

$$\boxed{\bar{\mathbf{r}} = \frac{\int \mathbf{r}\, dm}{m}}$$

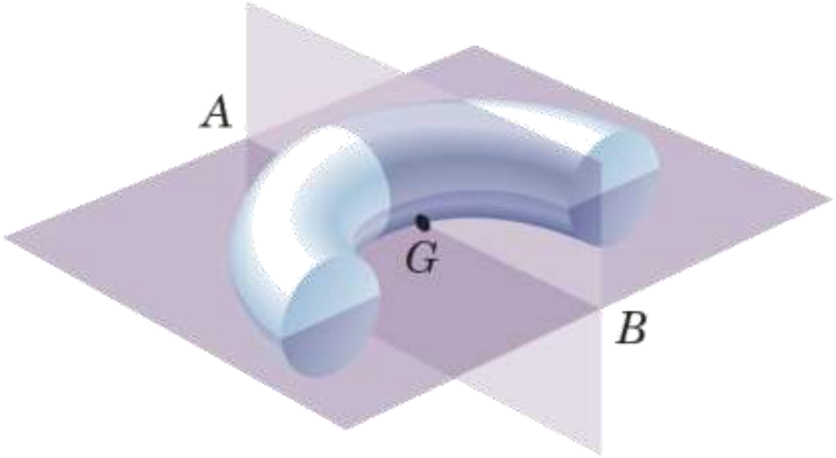


Article 5/2 – Center of Mass versus Center of Gravity

- Center of Mass is…
  - a unique point in the body which is only a function of the distribution of mass.
  - coincident with the center of gravity if the gravitational field is uniform and parallel.
  - still present when the body is removed from a gravitational field.
  - the preferred choice of reference.
  - important in calculating the dynamic response of a body to unbalanced forces.

Article 5/2 – Comments about the Center of Mass

- Tips and Hints
  - Choice of Axes
  - Lines and Planes of Symmetry



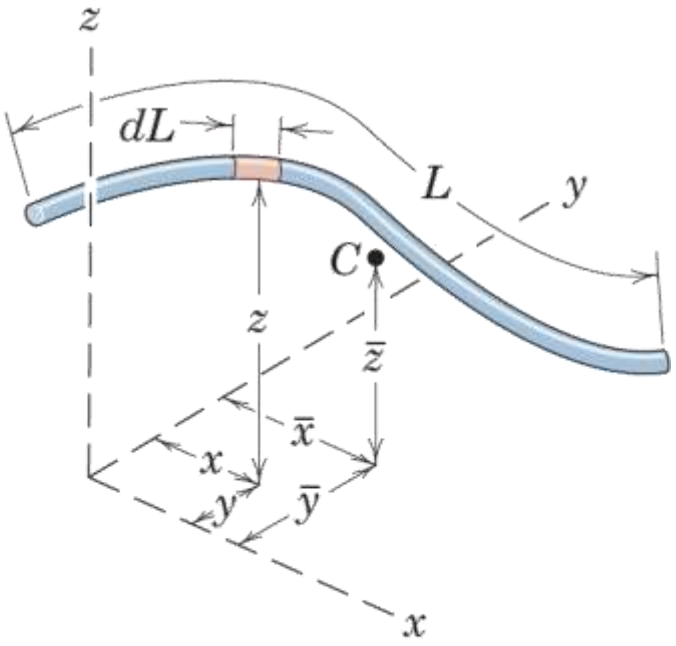
Article 5/3 Centroids of Lines, Areas, and Volumes

- Centroid

  Term used when a center of mass calculation concerns a geometric shape only.

- Centroids of Lines
  - Differential segment $dm = \rho A\, dL$
  - A = cross-sectional area of line
  - Equations of interest if ρ and A are constant…

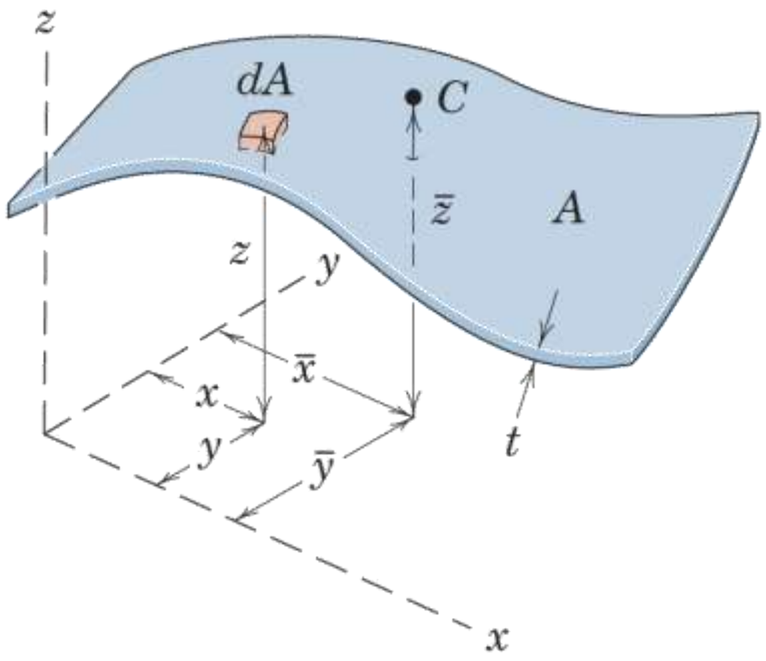
$$\bar{x} = \frac{\int x\, dL}{L} \qquad \bar{y} = \frac{\int y\, dL}{L} \qquad \bar{z} = \frac{\int z\, dL}{L}$$



Article 5/3 – Centroids (cont.)

- Centroids of Areas
  - Differential section $dm = \rho t \, dA$
  - t = thickness of area
  - Equations of interest if ρ and t are constant…

$$\bar{x} = \frac{\int x \, dA}{A} \qquad \bar{y} = \frac{\int y \, dA}{A} \qquad \bar{z} = \frac{\int z \, dA}{A}$$

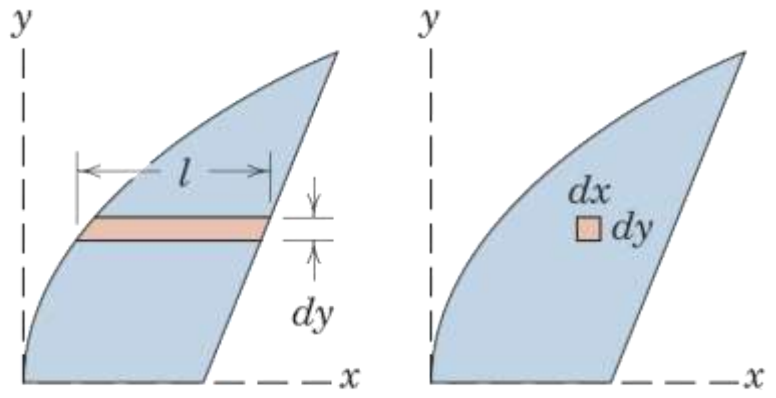


- Centroids of Volumes
  - Differential portion $dm = \rho \, dV$
  - Equations of interest if ρ is constant…

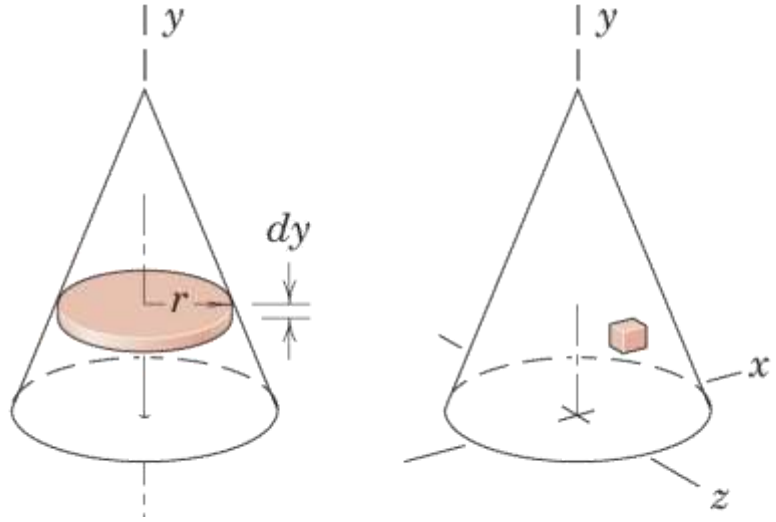
$$\bar{x} = \frac{\int x \, dV}{V} \qquad \bar{y} = \frac{\int y \, dV}{V} \qquad \bar{z} = \frac{\int z \, dV}{V}$$

Article 5/3 – Choice of Element for Integration (1 of 6)

- Order of Element
  - Whenever possible, select a first-order differential element to reduce the number of integrations.
  - Lower: Horizontal Strip, $dA = l\,dy$
  - Higher: Square, $dA = dx\,dy$

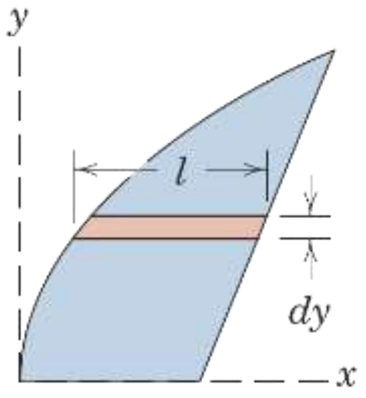


  - Lower: Circular Disk, $dV = \pi r^2\,dy$
  - Higher: Cube, $dV = dx\,dy\,dz$



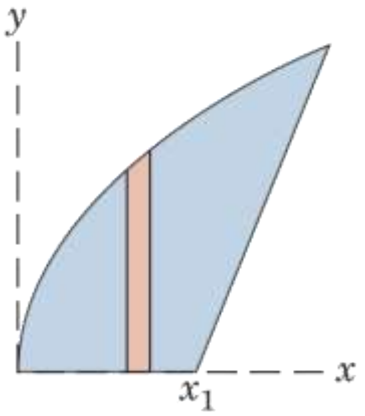
Article 5/3 – Choice of Element for Integration (2 of 6)

- Continuity
  - Whenever possible, choose an element which can be integrated in one continuous operation to cover the figure.



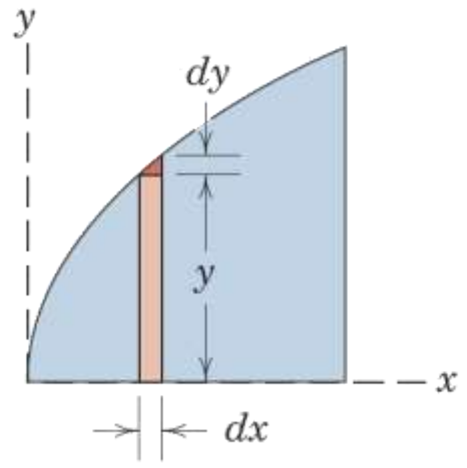
  - Horizontal strip requires one integration in *dy* because the boundaries for the right and left ends of the strip are continuous in the vertical direction

  - Vertical strip requires two integrations in *dx* because the boundaries for the top and bottom of the strip are not continuous in the horizontal direction.



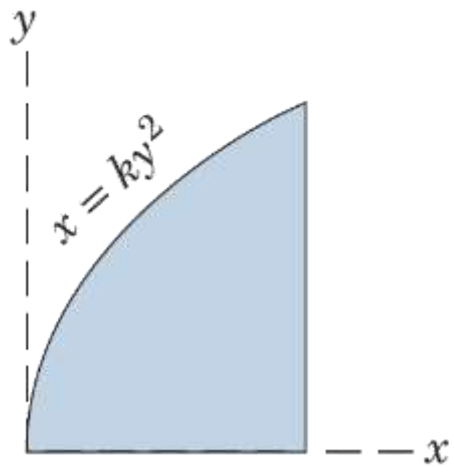
Article 5/3 – Choice of Element for Integration (3 of 6)

- Discarding Higher-Order Terms
  - Higher order terms may always be dropped when compared to lower-order terms.
  - First Order: $dA = y\, dx$
  - Second Order: $dA' = \frac{1}{2}\, dy\, dx$

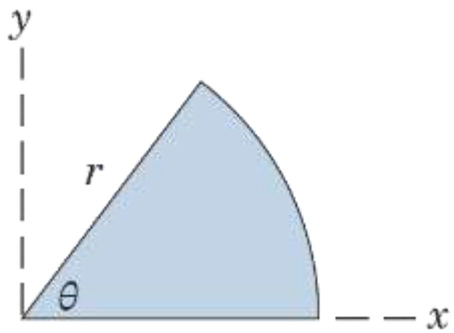


Article 5/3 – Choice of Element for Integration (4 of 6)

- Choice of Coordinates
  - Choose the coordinate system which best matches the boundaries of the figure.
  - Rectangular Coordinates: $x = ky^2$



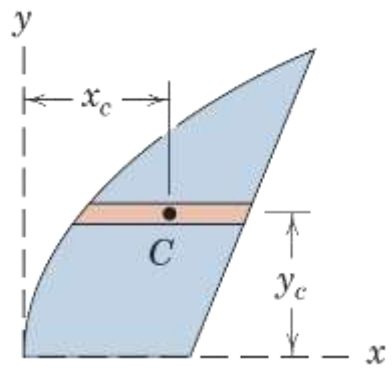
  - Polar Coordinates



Article 5/3 – Choice of Element for Integration (5 of 6)

- Centroidal Coordinate of Element
  - When a first- or second-order differential element is chosen, it is essential to use the coordinate of the centroid of the element for the moment arm in expressing the moment of the differential element.
  - Example with a Horizontal Strip
  - New Equations

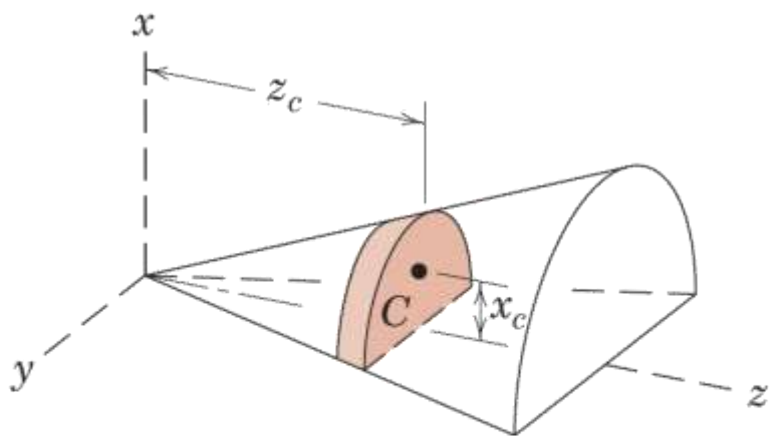
$$\bar{x} = \frac{\int x_c \, dA}{A} \qquad \bar{y} = \frac{\int y_c \, dA}{A} \qquad \bar{z} = \frac{\int z_c \, dA}{A}$$



Article 5/3 – Choice of Element for Integration (6 of 6)

- Centroidal Coordinate of Element (cont.)
  - Example with a Semicircular Slice
  - New Equations

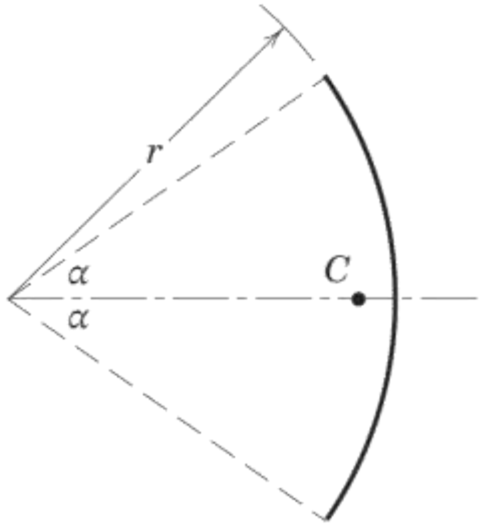
$$\bar{x} = \frac{\int x_c \, dV}{V} \qquad \bar{y} = \frac{\int y_c \, dV}{V} \qquad \bar{z} = \frac{\int z_c \, dV}{V}$$



Article 5/3 – Sample Problem 5/1 (1 of 2)

- Problem Statement

  Locate the centroid of a circular arc as shown in the figure.



Article 5/3 – Sample Problem 5/1 (2 of 2)

- Solution

Choosing the axis of symmetry as the x-axis makes $\bar{y} = 0$. A differential element of arc has the length $dL = r\,d\theta$ expressed in polar coordinates, and the x-coordinate of the element is $r\cos\theta$. ①

Applying the first of Eqs. 5/4 and substituting $L = 2\alpha r$ give

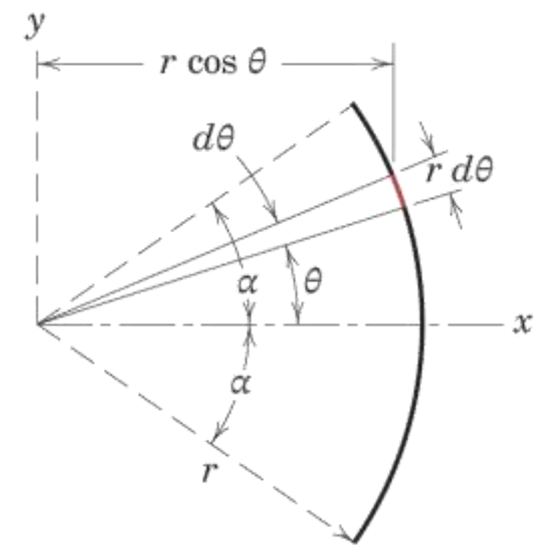
$$\left[L\bar{x} = \int x\,dL\right] \qquad (2\alpha r)\bar{x} = \int_{-\alpha}^{\alpha} (r\cos\theta)\,r\,d\theta$$

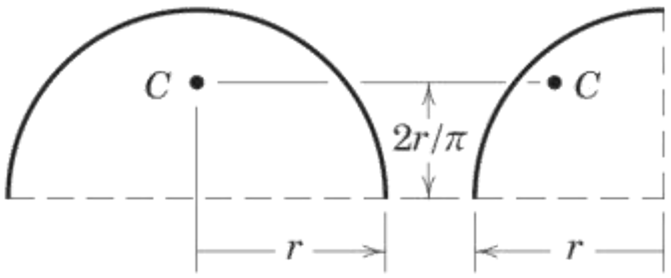
$$2\alpha r\bar{x} = 2r^2 \sin\alpha$$

$$\bar{x} = \frac{r\sin\alpha}{\alpha} \qquad \textit{Ans.}$$

For a semicircular arc $2\alpha = \pi$, which gives $\bar{x} = 2r/\pi$. By symmetry we see immediately that this result also applies to the quarter-circular arc when the measurement is made as shown.

① It should be perfectly evident that polar coordinates are preferable to rectangular coordinates to express the length of a circular arc.

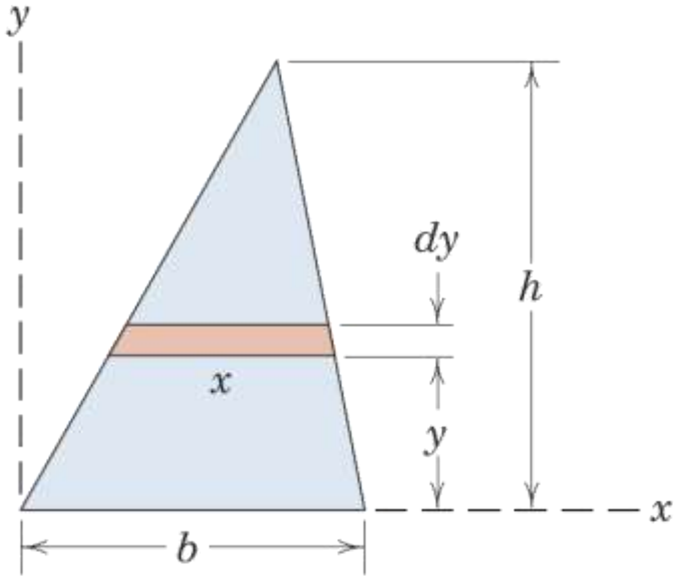




Article 5/3 – Sample Problem 5/2 (1 of 2)

- Problem Statement

  Determine the distance h from the base of a triangle of altitude h to the centroid of its area.



Article 5/3 – Sample Problem 5/2 (2 of 2)

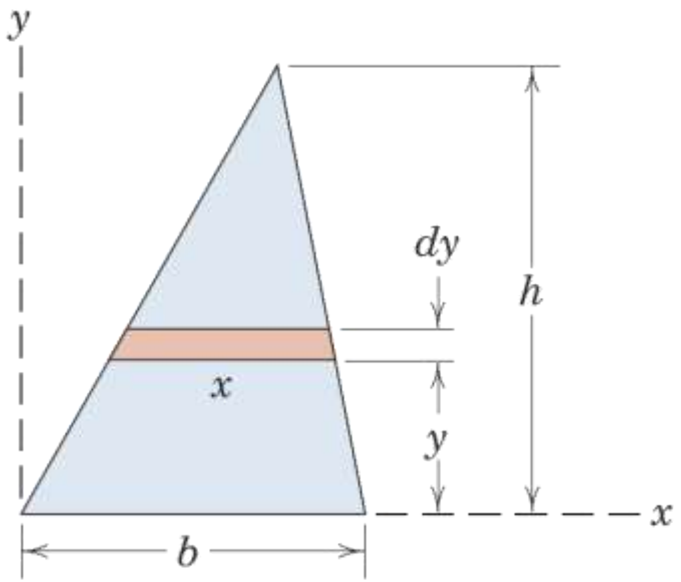
- Solution

The x-axis is taken to coincide with the base. A differential strip of area $dA = x\,dy$ is chosen. ① By similar triangles $x/(h - y) = b/h$. Applying the second of Eqs. 5/5a gives

$$[A\bar{y} = \int y_c\,dA] \qquad \frac{bh}{2}\bar{y} = \int_0^h y\,\frac{b(h-y)}{h}\,dy = \frac{bh^2}{6}$$

and $$\bar{y} = \frac{h}{3} \qquad \textit{Ans.}$$

This same result holds with respect to either of the other two sides of the triangle considered a new base with corresponding new altitude. Thus, the centroid lies at the intersection of the medians, since the distance of this point from any side is one-third the altitude of the triangle with that side considered the base.

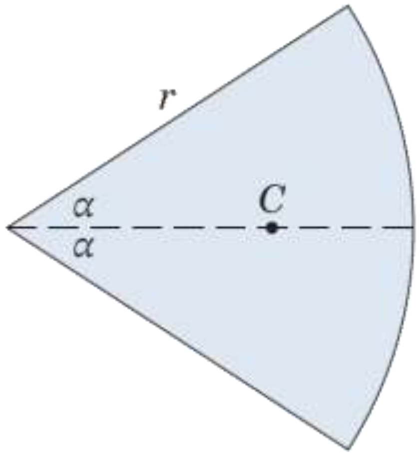


① We save one integration here by using the first-order element of area. Recognize that dA must be expressed in terms of the integration variable y; hence, $x = f(y)$ is required.

Article 5/3 – Sample Problem 5/3 (1 of 3)

- Problem Statement

  Locate the centroid of the area of a circular sector with respect to its vertex.



Article 5/3 – Sample Problem 5/3 (2 of 3)

- Solution I

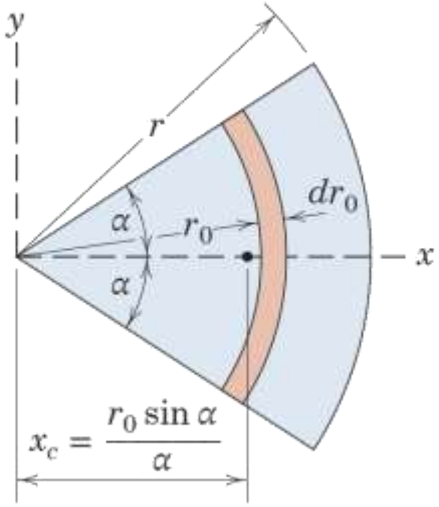
The x-axis is chosen as the axis of symmetry, and $\bar{y}$ is therefore automatically zero. We may cover the area by moving an element in the form of a partial circular ring, as shown in the figure, from the center to the outer periphery. The radius of the ring is r_0 and its thickness is dr_0, so that its area is $dA = 2r_0\alpha\, dr_0$. ①

The x-coordinate to the centroid of the element from Sample Problem 5/1 is $x_c = r_0 \sin\alpha/\alpha$, where r_0 replaces r in the formula. ② Thus, the first of Eqs. 5/5a gives

$$[A\bar{x} = \int x_c\, dA] \qquad \frac{2\alpha}{2\pi}(\pi r^2)\bar{x} = \int_0^r \left(\frac{r_0 \sin\alpha}{\alpha}\right)(2r_0\alpha\, dr_0)$$

$$r^2\alpha\bar{x} = \frac{2}{3}r^3 \sin\alpha$$

$$\bar{x} = \frac{2}{3}\frac{r\sin\alpha}{\alpha} \qquad \textit{Ans.}$$



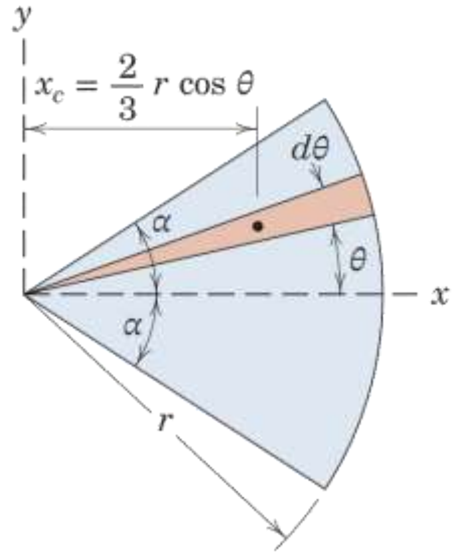
① Note carefully that we must distinguish between the variable r_0 and the constant r.

② Be careful not to use r_0 as the centroidal coordinate for the element.

Article 5/3 – Sample Problem 5/3 (3 of 3)

- Solution II

The area may also be covered by swinging a triangle of differential area about the vertex and through the total angle of the sector. This triangle, shown in the illustration, has an area $dA = (r/2)(r\,d\theta)$, where higher-order terms are neglected. From Sample Problem 5/2 the centroid of the triangular element of area is two-thirds of its altitude from its vertex, so that the x-coordinate to the centroid of the element is $x_c = \frac{2}{3}r\cos\theta$. Applying the first of Eqs. 5/5a gives



$$[A\bar{x} = \int x_c\,dA] \qquad (r^2\alpha)\bar{x} = \int_{-\alpha}^{\alpha} (\tfrac{2}{3}r\cos\theta)(\tfrac{1}{2}r^2\,d\theta)$$

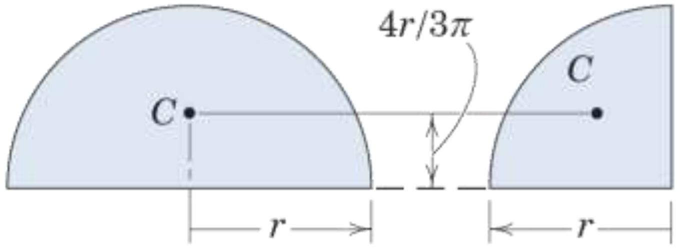
$$r^2\alpha\bar{x} = \tfrac{2}{3}r^3\sin\alpha$$

and as before

$$\bar{x} = \frac{2}{3}\frac{r\sin\alpha}{\alpha} \qquad \textit{Ans.}$$

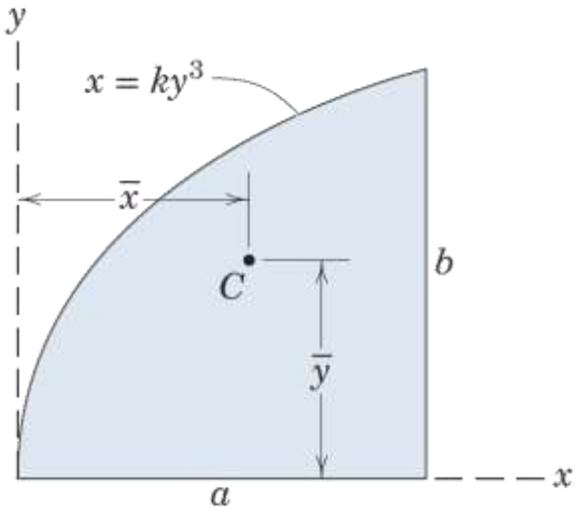
For a semicircular area $2\alpha = \pi$, which gives $\bar{x} = 4r/3\pi$. By symmetry we see immediately that this result also applies to the quarter-circular area where the measurement is made as shown.

It should be noted that, if we had chosen a second-order element $r_0\,dr_0\,d\theta$, one integration with respect to θ would yield the ring with which *Solution I* began. On the other hand, integration with respect to r_0 initially would give the triangular element with which *Solution II* began.



Article 5/3 – Sample Problem 5/4 (1 of 3)

- Problem Statement
  - Locate the centroid of the area under the curve $x = ky^3$ from $x = 0$ to $x = a$.



Article 5/3 – Sample Problem 5/4 (2 of 3)

- Solution I: Vertical Strip

A vertical element of area $dA = y\,dx$ is chosen as shown in the figure. The x-coordinate of the centroid is found from the first of Eqs. 5/5a. Thus,

$$[A\bar{x} = \int x_c\,dA] \qquad \bar{x}\int_0^a y\,dx = \int_0^a xy\,dx \quad ①$$

Substituting $y = (x/k)^{1/3}$ and $k = a/b^3$ and integrating give

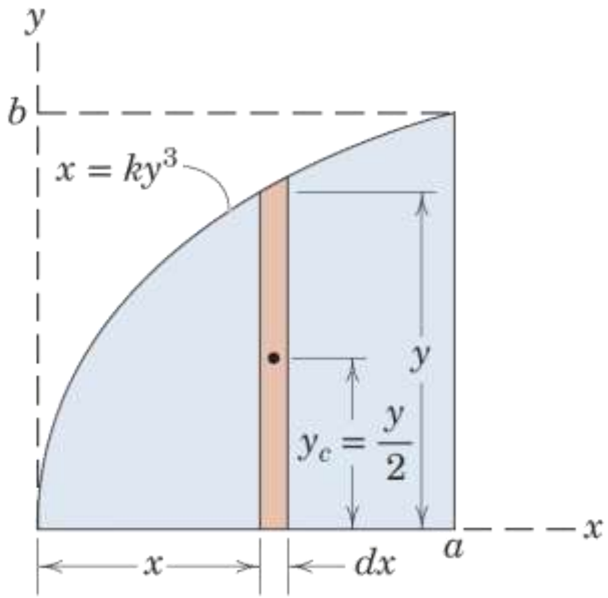
$$\frac{3ab}{4}\bar{x} = \frac{3a^2b}{7} \qquad \bar{x} = \tfrac{4}{7}a \qquad \textit{Ans.}$$

In the solution for $\bar{y}$ from the second of Eqs. 5/5a, the coordinate to the centroid of the rectangular element is $y_c = y/2$, where y is the height of the strip governed by the equation of the curve $x = ky^3$. Thus, the moment principle becomes

$$[A\bar{y} = \int y_c\,dA] \qquad \frac{3ab}{4}\bar{y} = \int_0^a \left(\frac{y}{2}\right) y\,dx$$

Substituting $y = b(x/a)^{1/3}$ and integrating give

$$\frac{3ab}{4}\bar{y} = \frac{3ab^2}{10} \qquad \bar{y} = \tfrac{2}{5}b \qquad \textit{Ans.}$$



① Note that $x_c = x$ for the vertical element.

Article 5/3 – Sample Problem 5/4 (3 of 3)

- Solution II: Horizontal Strip

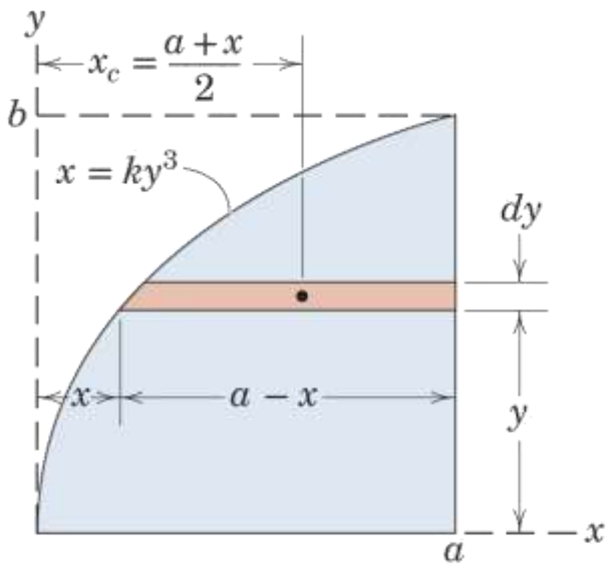
The horizontal element of area shown in the lower figure may be employed in place of the vertical element. The x-coordinate to the centroid of the rectangular element is seen to be $x_c = x + \frac{1}{2}(a - x) = (a + x)/2$, which is simply the average of the coordinates a and x of the ends of the strip. Hence,

$$\left[A\bar{x} = \int x_c \, dA\right] \qquad \bar{x}\int_0^b (a - x)\, dy = \int_0^b \left(\frac{a + x}{2}\right)(a - x)\, dy$$

The value of $\bar{y}$ is found from

$$\left[A\bar{y} = \int y_c \, dA\right] \qquad \bar{y}\int_0^b (a - x)\, dy = \int_0^b y(a - x)\, dy$$

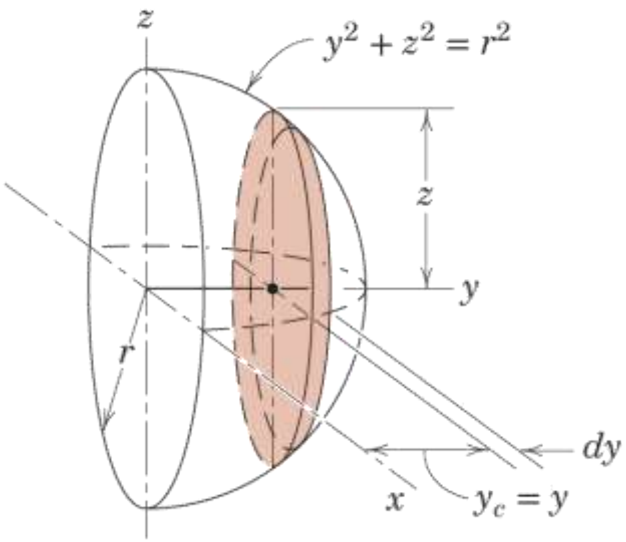
where $y_c = y$ for the horizontal strip. The evaluation of these integrals will check the previous results for $\bar{x}$ and $\bar{y}$.



Article 5/3 – Sample Problem 5/5 (1 of 4)

- Problem Statement

  Locate the centroid of the volume of a hemisphere of radius *r* with respect to its base.



Article 5/3 – Sample Problem 5/5 (2 of 4)

- Solution I: Circular Slice

With the axes chosen as shown in the figure, $\bar{x} = \bar{z} = 0$ by symmetry. The most convenient element is a circular slice of thickness dy parallel to the x-z plane. Since the hemisphere intersects the y-z plane in the circle $y^2 + z^2 = r^2$, the radius of the circular slice is $z = +\sqrt{r^2 - y^2}$. The volume of the elemental slice becomes

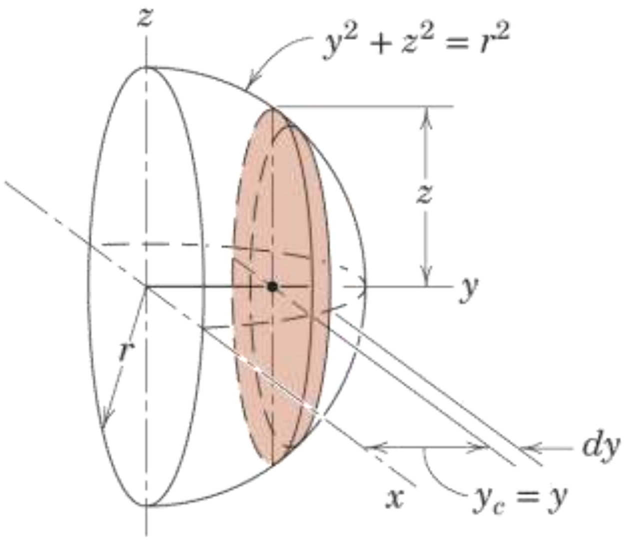
$$dV = \pi(r^2 - y^2)\, dy \quad ①$$

The second of Eqs. 5/6a requires

$$\left[V\bar{y} = \int y_c\, dV\right] \qquad \bar{y}\int_0^r \pi(r^2 - y^2)\, dy = \int_0^r y\pi(r^2 - y^2)\, dy$$

where $y_c = y$. Integrating gives

$$\frac{2}{3}\pi r^3 \bar{y} = \frac{1}{4}\pi r^4 \qquad \bar{y} = \frac{3}{8}r \qquad\qquad \textit{Ans.}$$



① Can you identify the higher-order element of volume which is omitted from the expression for dV?

Article 5/3 – Sample Problem 5/5 (3 of 4)

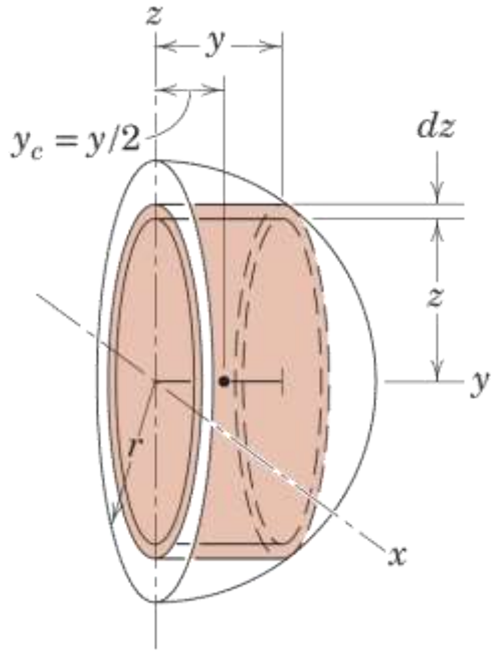
- Solution II: Cylindrical Shell

Alternatively we may use for our differential element a cylindrical shell of length y, radius z, and thickness dz, as shown in the lower figure. By expanding the radius of the shell from zero to r, we cover the entire volume. By symmetry the centroid of the elemental shell lies at its center, so that $y_c = y/2$. The volume of the element is $dV = (2\pi z\, dz)(y)$. Expressing y in terms of z from the equation of the circle gives $y = +\sqrt{r^2 - z^2}$. Using the value of $\frac{2}{3}\pi r^3$ computed in *Solution I* for the volume of the hemisphere and substituting in the second of Eqs. 5/6*a* give us

$$\left[V\bar{y} = \int y_c\, dV\right] \qquad (\tfrac{2}{3}\pi r^3)\bar{y} = \int_0^r \frac{\sqrt{r^2 - z^2}}{2}(2\pi z\sqrt{r^2 - z^2})\, dz$$

$$= \int_0^r \pi(r^2 z - z^3)\, dz = \frac{\pi r^4}{4}$$

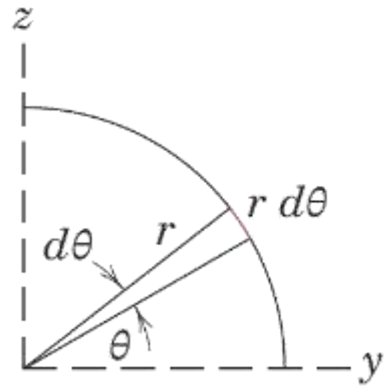
$$\bar{y} = \tfrac{3}{8}r \qquad \textit{Ans.}$$



Article 5/3 – Sample Problem 5/5 (4 of 4)

- Solution III: Use θ as the Variable

As an alternative, we could use the angle θ as our variable with limits of 0 and $\pi/2$. The radius of either element would become $r \sin \theta$, whereas the thickness of the slice in *Solution I* would be $dy = (r\, d\theta) \sin \theta$ and that of the shell in *Solution II* would be $dz = (r\, d\theta) \cos \theta$. The length of the shell would be $y = r \cos \theta$.

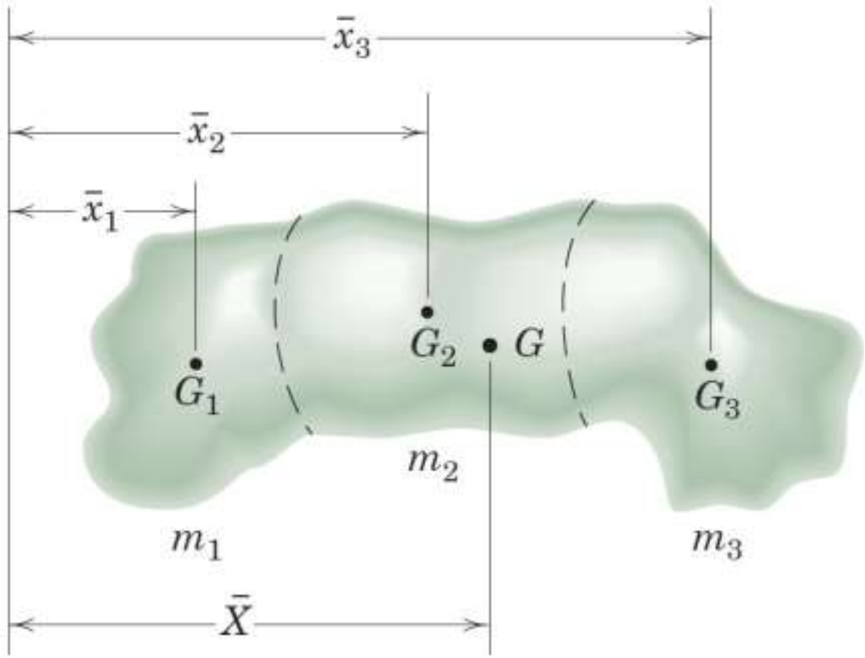


Article 5/4 Composite Bodies and Figures; Approximations

- Overview of Composites

- General Expressions

$$\bar{X} = \frac{\Sigma m\bar{x}}{\Sigma m} \qquad \bar{Y} = \frac{\Sigma m\bar{y}}{\Sigma m} \qquad \bar{Z} = \frac{\Sigma m\bar{z}}{\Sigma m}$$



- Analogous equations exist for lines, areas, and volumes by replacing the *m*'s with *L*'s, *A*'s, and *V*'s, respectively.

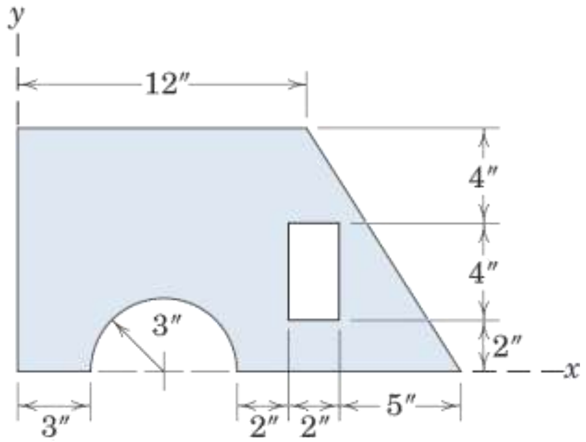
Article 5/4 – Comments about Composite Bodies

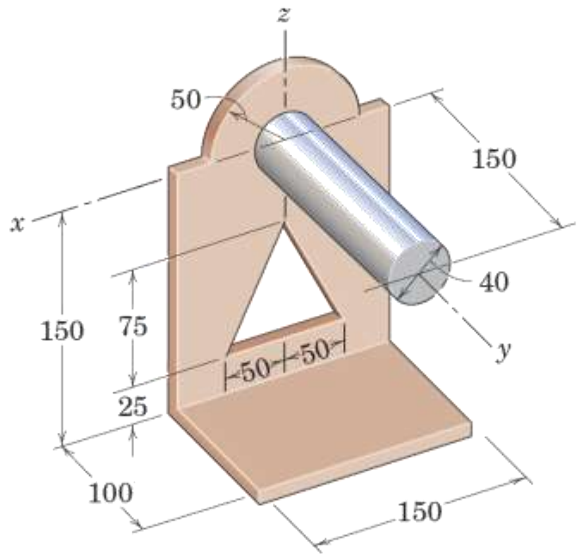
- Identify composite pieces based on Table D/3 and D/4 in Appendix D.

- Treat holes and cutouts as negative quantities. In the upper shape, the rectangular cutout has an area of -8 in.^{2}.

- Measure the centroidal coordinate for every piece from the origin of your reference axes. In the upper shape, the horizontal centroidal coordinate of the rectangular cutout is 12 in., not 1 in.

- Most objects consist of a single type of composite piece geometry (lines, areas, or volumes), but sometimes they are combined. For example, the upper shape consists of areas, while the lower shape consists of areas and volumes. To find the centroid of the lower shape, you must use the mass of the individual pieces.

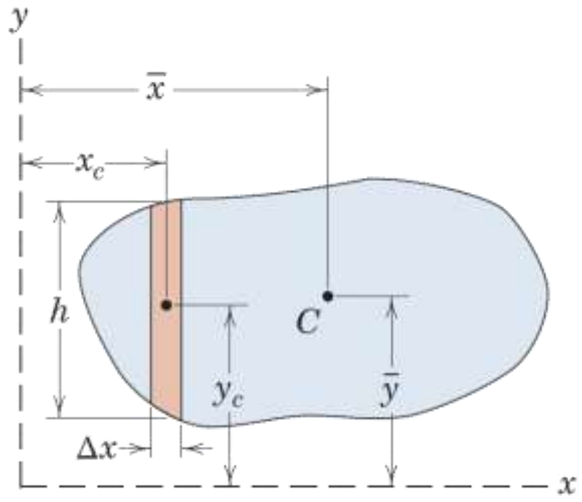




Dimensions in millimeters

Article 5/4 – An Approximation Method

When the boundaries of an area (or volume) are not expressible in terms of simple geometrical shapes or mathematical expressions…

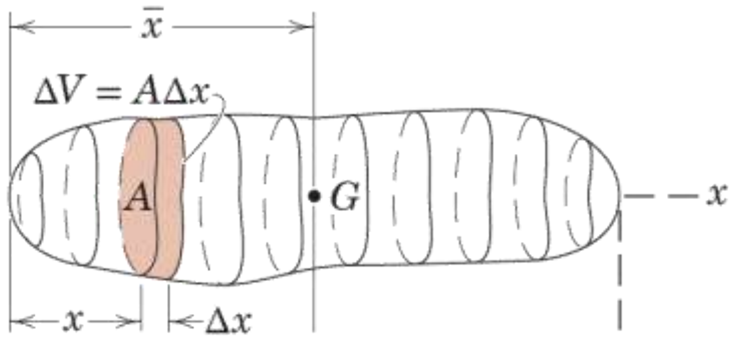


- Divide the body into individual pieces.
- Calculate the area $A = h\,\Delta x$ of each piece.
- Determine the centroidal coordinates, x_c and y_c for each piece.
- Evaluate the expression…

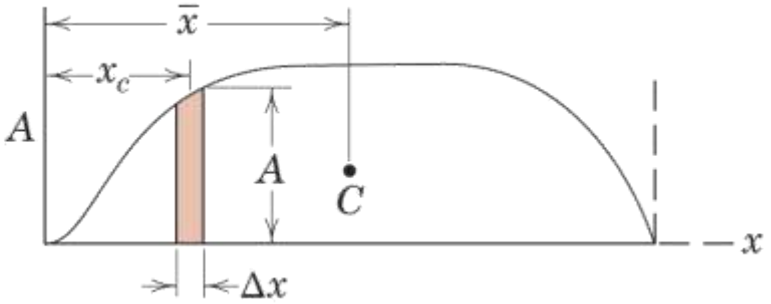
$$\bar{x} = \frac{\Sigma A x_c}{\Sigma A} \qquad \bar{y} = \frac{\Sigma A y_c}{\Sigma A}$$

Article 5/4 – Irregular Volumes

The centroidal location of an irregular volume may be simplified to one of locating the centroid of an area.



- Plot the magnitudes of the cross-sectional areas *A*.
- Calculate the volume $V = A\,\Delta x$ of individual slices.
- Determine the centroidal coordinates x_c of each slice.
- Evaluate the expression…



$$\bar{x} = \frac{\Sigma(A\ \Delta x)x_c}{\Sigma A\ \Delta x} \qquad \text{which equals} \qquad \bar{x} = \frac{\Sigma V x_c}{\Sigma V}$$

Article 5/4 – Centroids of Common Shapes (1 of 6)

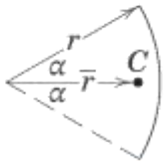
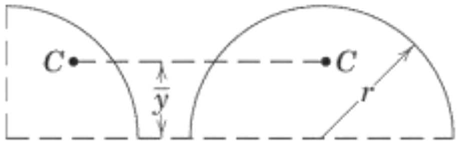
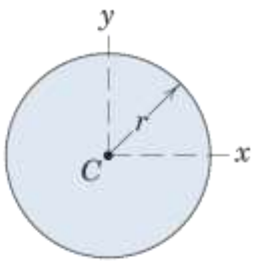
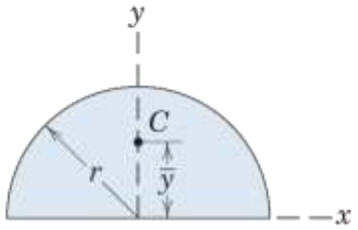
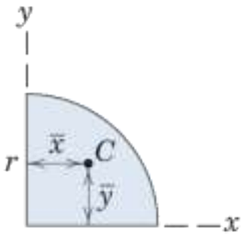
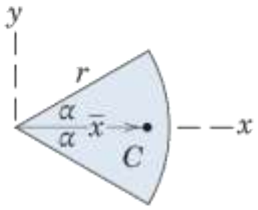
Figure	Centroid
Arc Segment 	$\bar{r} = \frac{r \sin \alpha}{\alpha}$
Quarter and Semicircular Arcs 	$\bar{y} = \frac{2r}{\pi}$
Circular Area 	—
Semicircular Area 	$\bar{y} = \frac{4r}{3\pi}$

Figure	Centroid
Quarter-Circular Area 	$\bar{x} = \bar{y} = \frac{4r}{3\pi}$
Area of Circular Sector 	$\bar{x} = \frac{2}{3}\frac{r \sin \alpha}{\alpha}$

Article 5/4 – Centroids of Common Shapes (2 of 6)

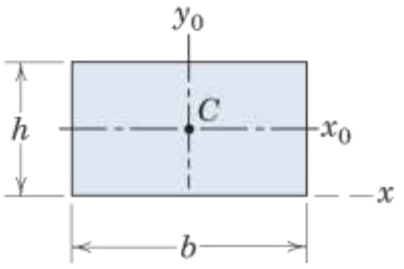
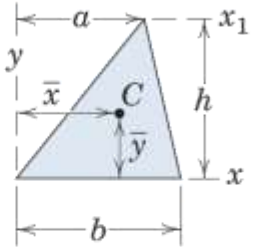
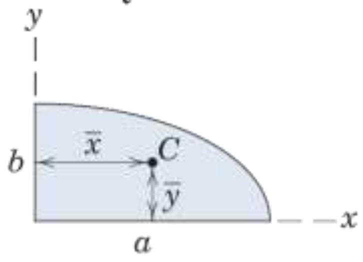
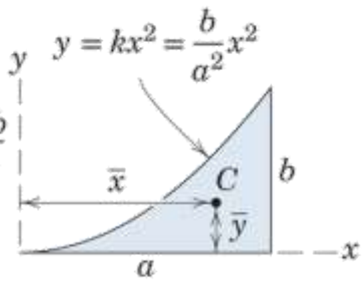
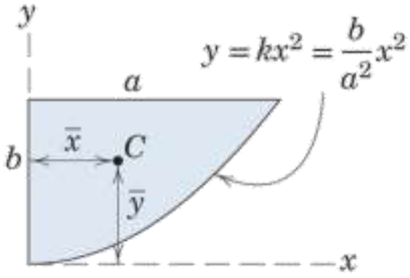
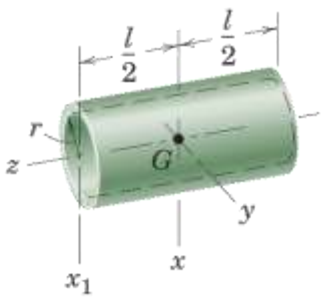
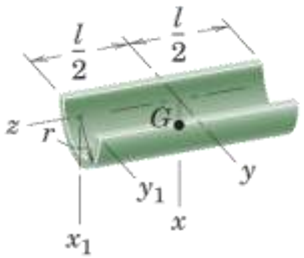
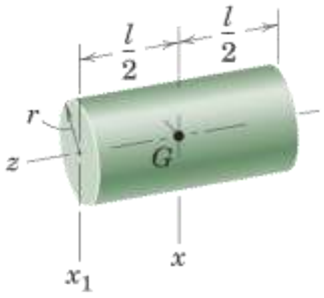
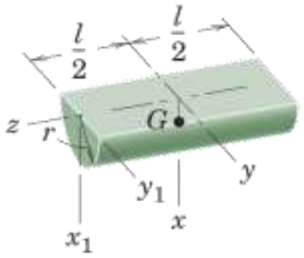
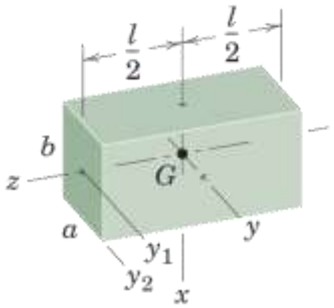
Figure	Centroid
Rectangular Area 	—
Triangular Area 	$\bar{x} = \dfrac{a+b}{3}$ $\bar{y} = \dfrac{h}{3}$
Area of Elliptical Quadrant 	$\bar{x} = \dfrac{4a}{3\pi}$ $\bar{y} = \dfrac{4b}{3\pi}$

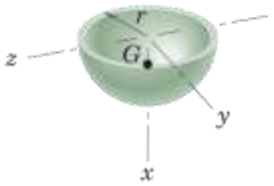
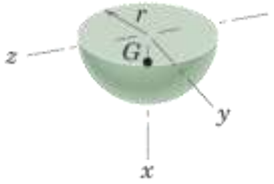
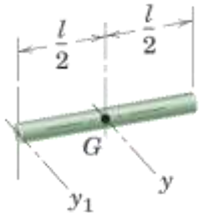
Figure	Centroid
Subparabolic Area Area $A = \dfrac{ab}{3}$ 	$\bar{x} = \dfrac{3a}{4}$ $\bar{y} = \dfrac{3b}{10}$
Parabolic Area Area $A = \dfrac{2ab}{3}$ 	$\bar{x} = \dfrac{3a}{8}$ $\bar{y} = \dfrac{3b}{5}$

Article 5/4 – Centroids of Common Shapes (3 of 6)

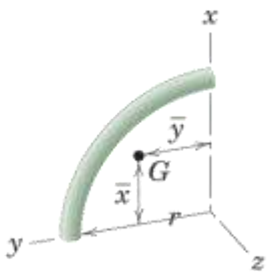
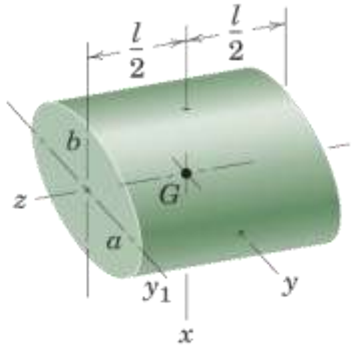
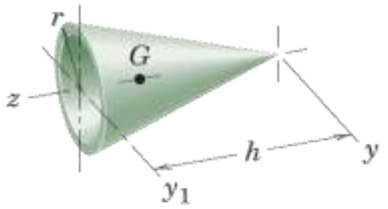
Body		Mass Center
	Circular Cylindrical Shell	—
	Half Cylindrical Shell	$\bar{x} = \frac{2r}{\pi}$
	Circular Cylinder	—

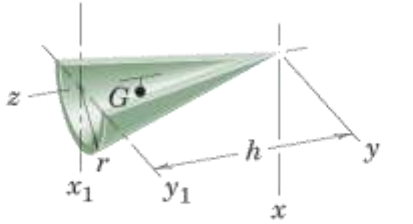
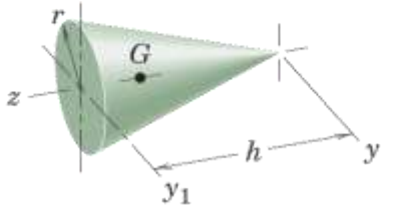
Body		Mass Center
	Semicylinder	$\bar{x} = \frac{4r}{3\pi}$
	Rectangular Parallelepiped	—

Article 5/4 – Centroids of Common Shapes (4 of 6)

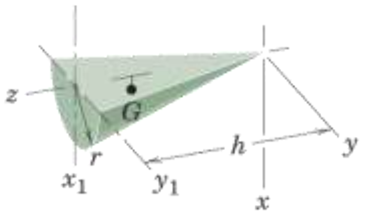
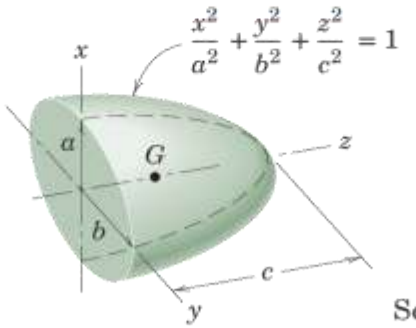
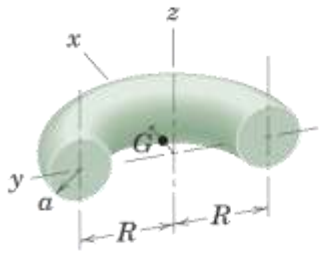
Body		Mass Center
	Spherical Shell	—
	Hemispherical Shell	$\bar{x} = \frac{r}{2}$
	Sphere	—
	Hemisphere	$\bar{x} = \frac{3r}{8}$
	Uniform Slender Rod	—

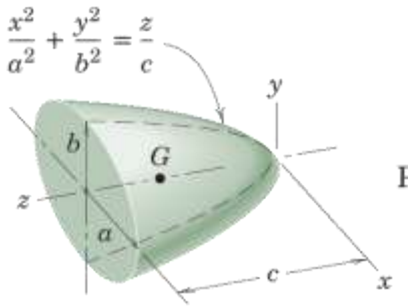
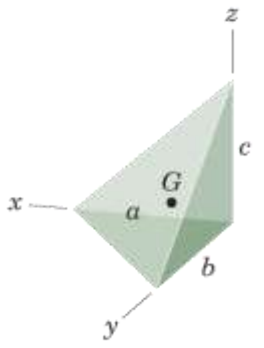
Article 5/4 – Centroids of Common Shapes (5 of 6)

Body		Mass Center
	Quarter-Circular Rod	$\bar{x} = \bar{y}$ $= \frac{2r}{\pi}$
	Elliptical Cylinder	—
	Conical Shell	$\bar{z} = \frac{2h}{3}$

Body		Mass Center
	Half Conical Shell	$\bar{x} = \frac{4r}{3\pi}$ $\bar{z} = \frac{2h}{3}$
	Right Circular Cone	$\bar{z} = \frac{3h}{4}$

Article 5/4 – Centroids of Common Shapes (6 of 6)

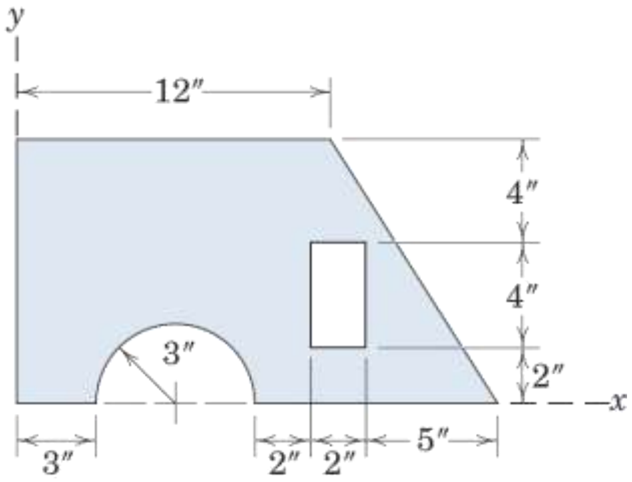
Body		Mass Center
	Half Cone	$\bar{x} = \frac{r}{\pi}$ $\bar{z} = \frac{3h}{4}$
	Semiellipsoid	$\bar{z} = \frac{3c}{8}$
	Half Torus	$\bar{x} = \frac{a^2 + 4R^2}{2\pi R}$

Body		Mass Center
	Elliptic Paraboloid	$\bar{z} = \frac{2c}{3}$
	Rectangular Tetrahedron	$\bar{x} = \frac{a}{4}$ $\bar{y} = \frac{b}{4}$ $\bar{z} = \frac{c}{4}$

Article 5/4 – Sample Problem 5/6 (1 of 3)

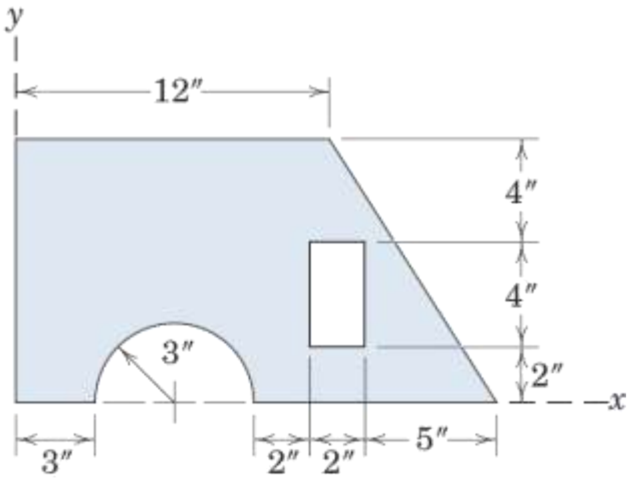
- Problem Statement

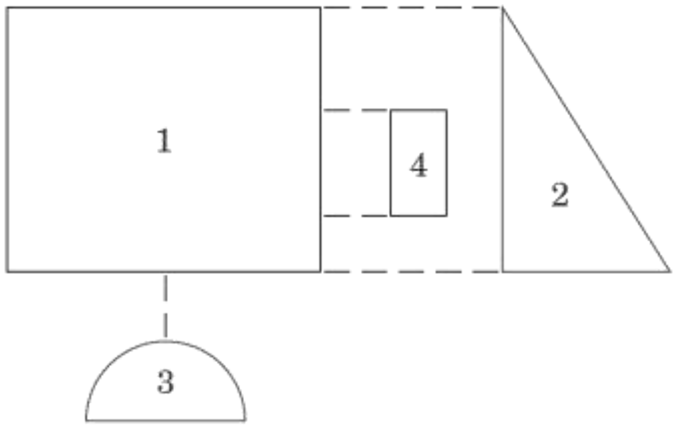
  Locate the centroid of the shaded area.



Article 5/4 – Sample Problem 5/6 (2 of 3)

- Composite Shapes
  - Rectangular Area (1)
  - Triangular Area (2)
  - Semicircular Cutout (3)
  - Square Cutout (4)





Article 5/4 – Sample Problem 5/6 (2 of 3)

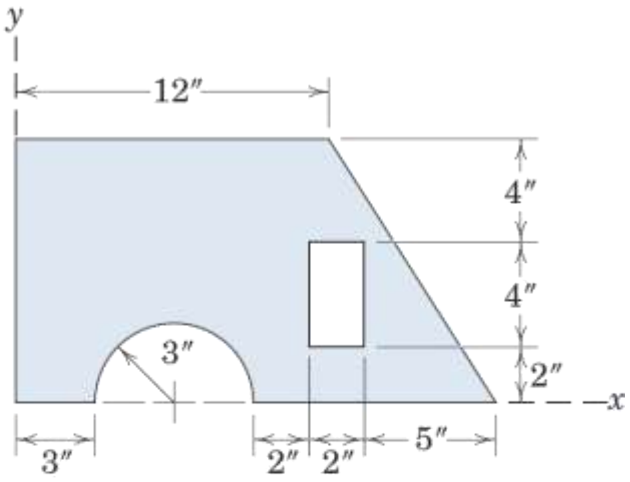
- Tabulated Values

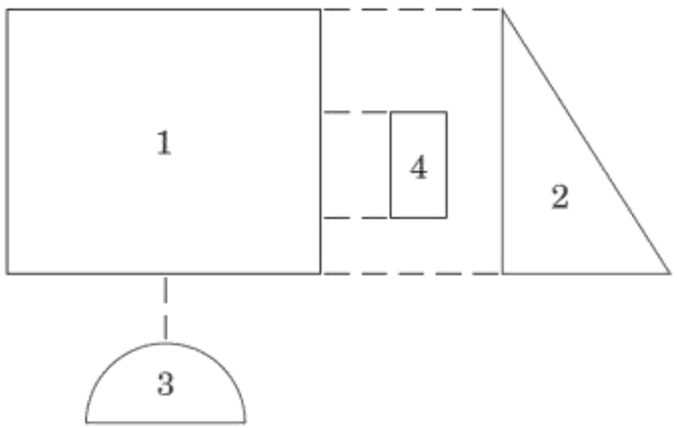
PART	A in.^{2}	$\bar{x}$ in.	$\bar{y}$ in.	$\bar{x}A$ in.^{3}	$\bar{y}A$ in.^{3}
1	120	6	5	720	600
2	30	14	10/3	420	100
3	–14.14	6	1.273	–84.8	–18
4	–8	12	4	–96	–32
TOTALS	127.9			959	650

The area counterparts to Eqs. 5/7 are now applied and yield

$$\left[\bar{X} = \frac{\Sigma A\bar{x}}{\Sigma A}\right] \qquad \bar{X} = \frac{959}{127.9} = 7.50 \text{ in.} \qquad \textit{Ans.}$$

$$\left[\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A}\right] \qquad \bar{Y} = \frac{650}{127.9} = 5.08 \text{ in.} \qquad \textit{Ans.}$$

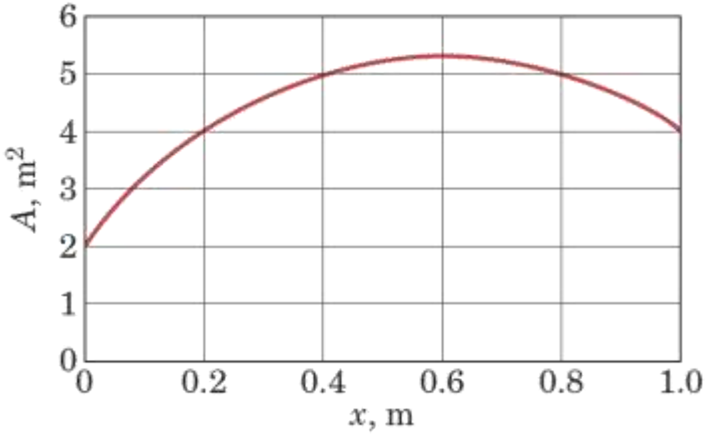




Article 5/4 – Sample Problem 5/7 (1 of 2)

- Problem Statement

  Approximate the *x*-coordinate of the volume centroid of a body whose length is 1 m and whose cross-sectional area varies with *x* as shown in the figure.



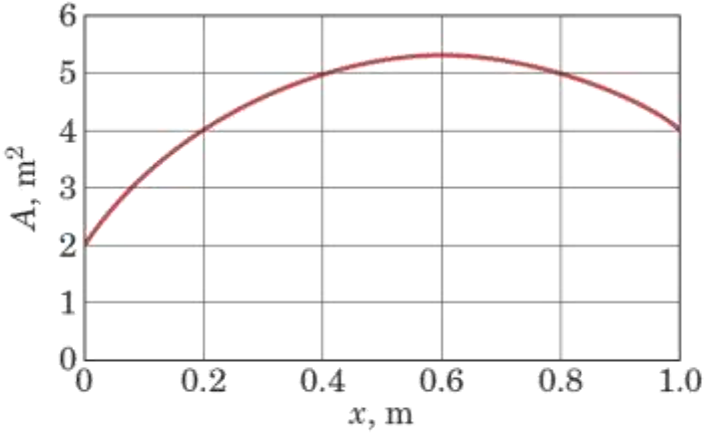
Article 5/4 – Sample Problem 5/7 (2 of 2)

- Solution

The body is divided into five sections. For each section, the average area, volume, and centroid location are determined and entered in the following table:

INTERVAL	A_{av} m^2	Volume V m^3	$\bar{x}$ m	$V\bar{x}$ m^4
0–0.2	3	0.6	0.1	0.060
0.2–0.4	4.5	0.90	0.3	0.270
0.4–0.6	5.2	1.04	0.5	0.520
0.6–0.8	5.2	1.04	0.7	0.728
0.8–1.0	4.5	0.90	0.9	0.810
TOTALS		4.48		2.388

$$\left[\bar{X} = \frac{\Sigma V\bar{x}}{\Sigma V}\right] \qquad \bar{X} = \frac{2.388}{4.48} = 0.533 \text{ m} \quad ① \qquad \textit{Ans.}$$

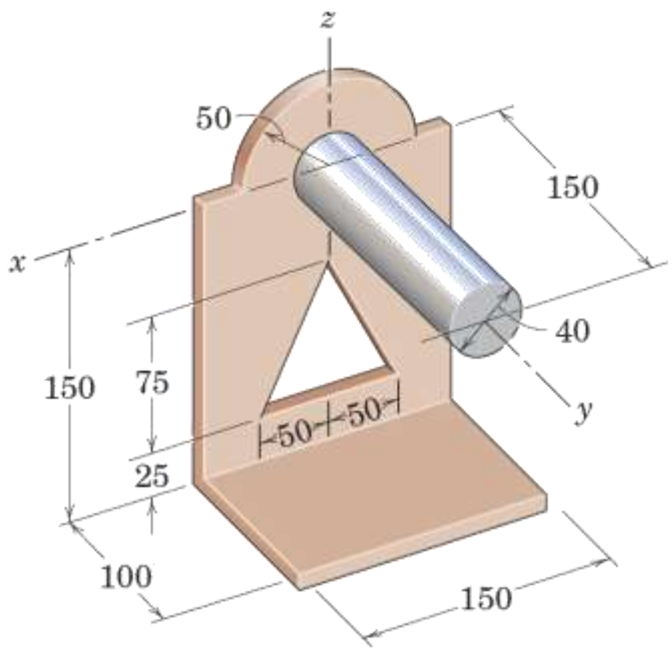


① Note that the shape of the body as a function of y and z does not affect $\bar{X}$.

Article 5/4 – Sample Problem 5/8 (1 of 4)

- Problem Statement

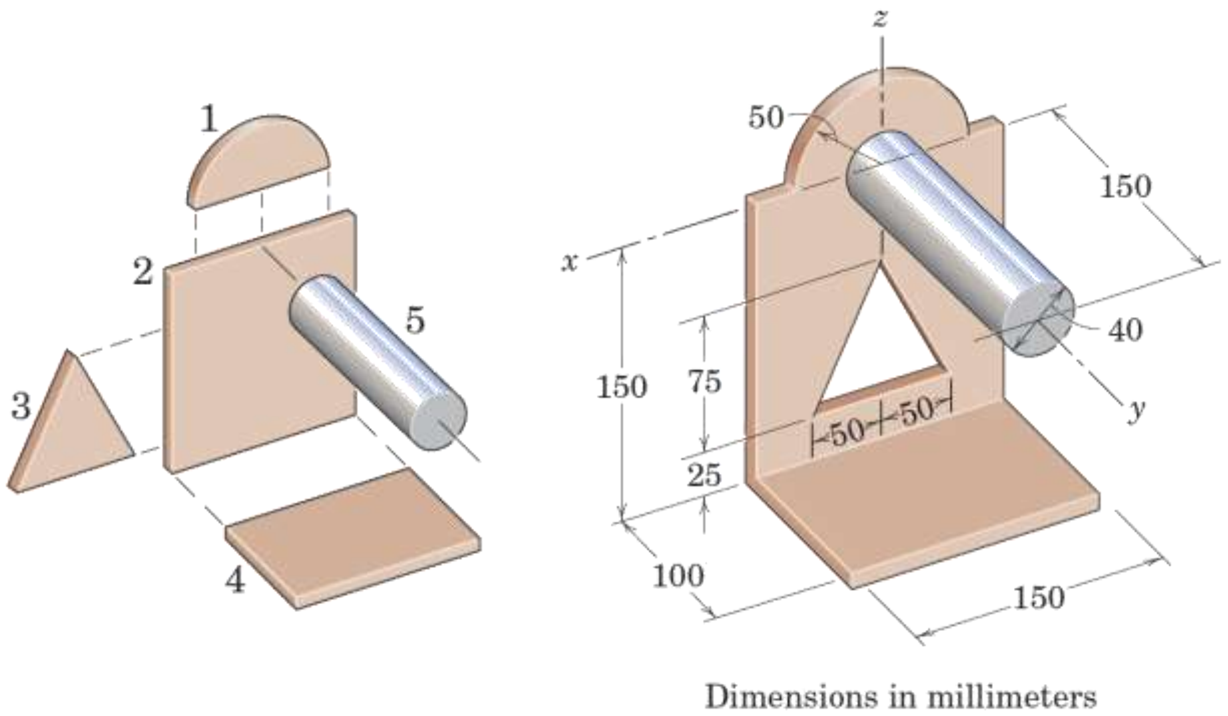
  Locate the center of mass of the bracket-and-shaft combination. The vertical face is made from sheet metal which has a mass of 25 kg/m^2. The material of the horizontal base has a mass of 40 kg/m^2, and the steel shaft has a density of 7.83 Mg/m^3.



Dimensions in millimeters

Article 5/4 – Sample Problem 5/8 (2 of 4)

- Composite Shapes
  - Semicircular Area (1)
  - Square Plate (2)
  - Triangular Cutout (3)
  - Rectangular Flat (4)
  - Cylindrical Shaft (5)



Article 5/4 – Sample Problem 5/8 (3 of 4)

- Solution Comments

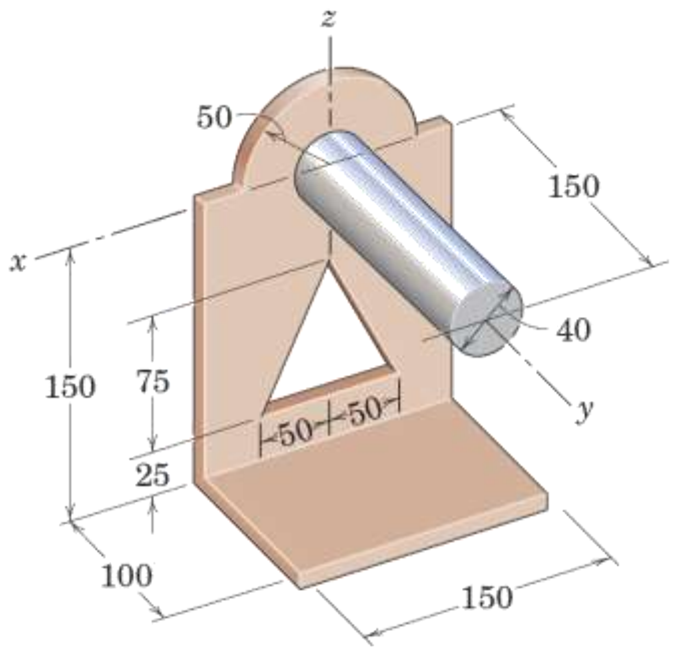
The composite body may be considered to be composed of the five elements shown in the lower portion of the illustration. The triangular part will be taken as a negative mass. For the reference axes indicated, it is clear by symmetry that the x-coordinate of the center of mass is zero.

The mass m of each part is easily calculated and should need no further explanation. For Part 1 we have from Sample Problem 5/3

$$\bar{z} = \frac{4r}{3\pi} = \frac{4(50)}{3\pi} = 21.2 \text{ mm}$$

For Part 3 we see from Sample Problem 5/2 that the centroid of the triangular mass is one-third of its altitude above its base. Measurement from the coordinate axes becomes

$$\bar{z} = -[150 - 25 - \tfrac{1}{3}(75)] = -100 \text{ mm}$$



Dimensions in millimeters

Article 5/4 – Sample Problem 5/8 (4 of 4)

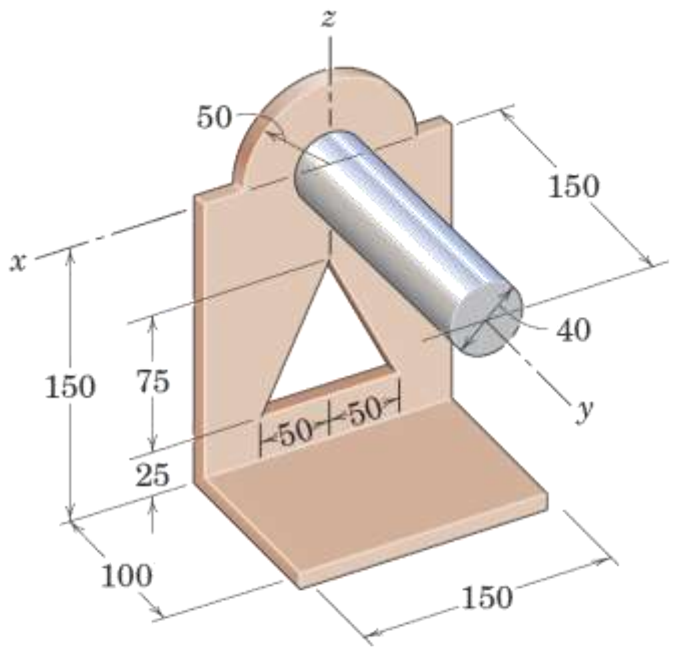
- Tabulated Values

PART	m kg	$\bar{y}$ mm	$\bar{z}$ mm	$m\bar{y}$ kg·mm	$m\bar{z}$ kg·mm
1	0.098	0	21.2	0	2.08
2	0.562	0	–75.0	0	–42.19
3	–0.094	0	–100.0	0	9.38
4	0.600	50.0	–150.0	30.0	–90.00
5	1.476	75.0	0	110.7	0
TOTALS	2.642			140.7	–120.73

Equations 5/7 are now applied and the results are

$$\left[\bar{Y} = \frac{\Sigma m\bar{y}}{\Sigma m}\right] \qquad \bar{Y} = \frac{140.7}{2.642} = 53.3 \text{ mm} \qquad \textit{Ans.}$$

$$\left[\bar{Z} = \frac{\Sigma m\bar{z}}{\Sigma m}\right] \qquad \bar{Z} = \frac{-120.73}{2.642} = -45.7 \text{ mm} \qquad \textit{Ans.}$$

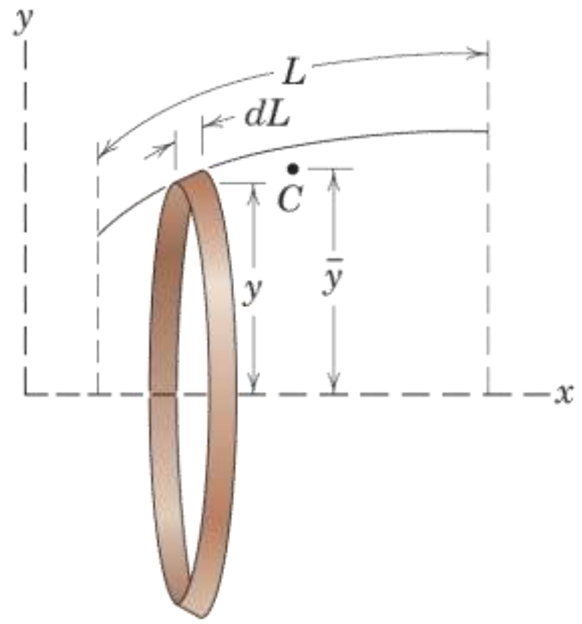


Dimensions in millimeters

Article 5/5 Theorems of Pappus

- Area of a Revolved Line Segment
  - L = length of a revolved line segment
  - $\bar{y}$ = y-coordinate of the centroid C for the line of length L

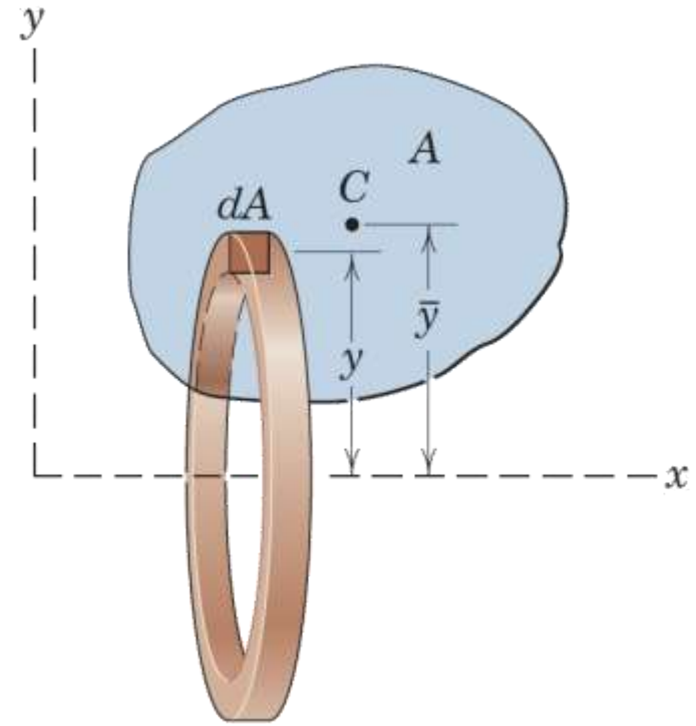
  - Equation of Interest: $A = 2\pi\bar{y}L$



- For a partial revolution through the angle θ…
  - $A = \theta\bar{y}L$ where θ is expressed in radians

Article 5/5 – Theorems of Pappus (cont.)

- Volume of a Solid of Revolution
  - A = area of a revolved section
  - $\bar{y}$ = y-coordinate of the centroid C for the area A
  - Equation of Interest: $V = 2\pi\bar{y}A$

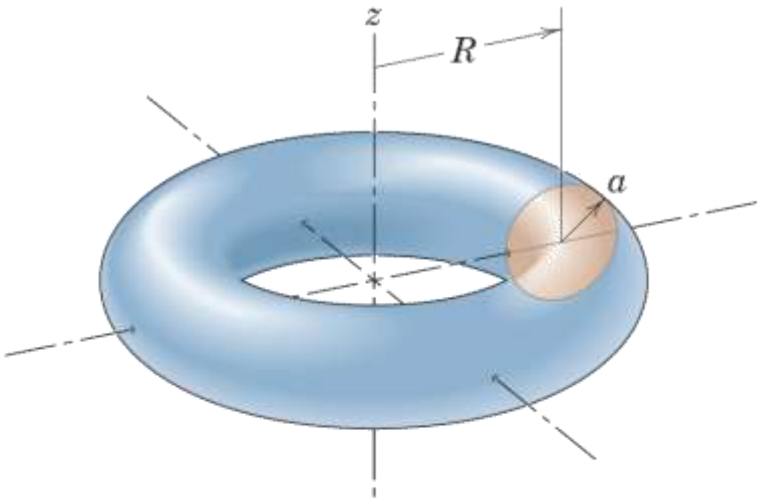


- For a partial revolution through the angle θ…
  - $V = \theta\bar{y}A$ where θ is expressed in radians

Article 5/5 – Sample Problem 5/9 (1 of 2)

- Problem Statement

Determine the volume V and surface area A of the complete torus of circular cross section.



Article 5/5 – Sample Problem 5/9 (2 of 2)

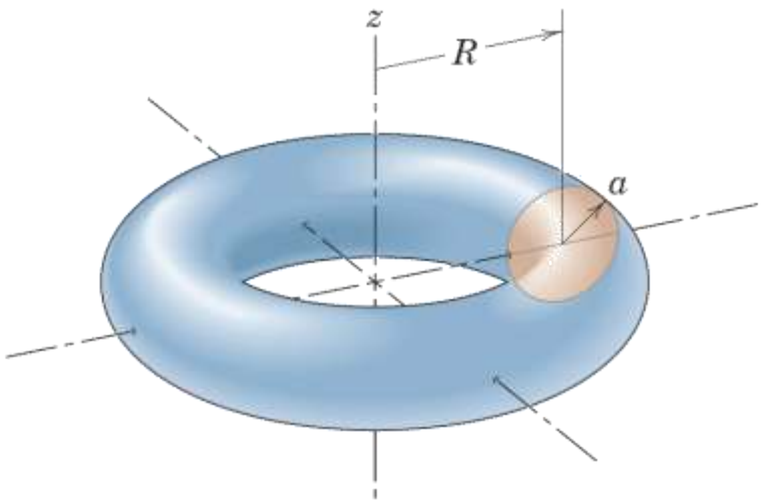
- Solution

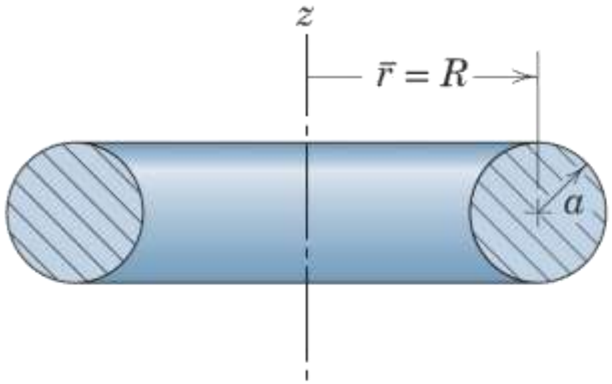
$$V = \theta\bar{r}A = 2\pi(R)(\pi a^2) = 2\pi^2 R a^2 \quad ① \qquad \textit{Ans.}$$

Similarly, using Eq. 5/8a gives

$$A = \theta\bar{r}L = 2\pi(R)(2\pi a) = 4\pi^2 R a \qquad \textit{Ans.}$$

① We note that the angle θ of revolution is 2π for the complete ring. This common but special-case result is given by Eq. 5/9.

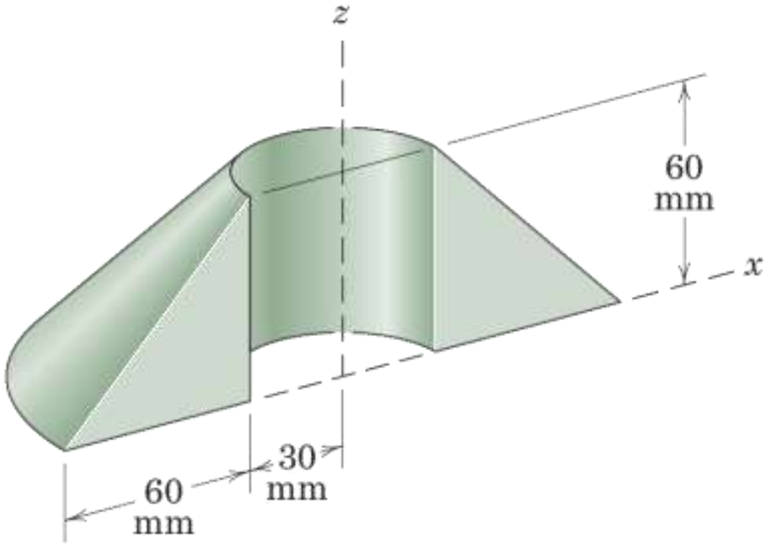




Article 5/5 – Sample Problem 5/10 (1 of 2)

- Problem Statement

Calculate the volume V of the solid generated by revolving the 60-mm right-triangular area through 180° about the z-axis. If this body were constructed of steel, what would be its mass m?



Article 5/5 – Sample Problem 5/10 (2 of 2)

- Solution

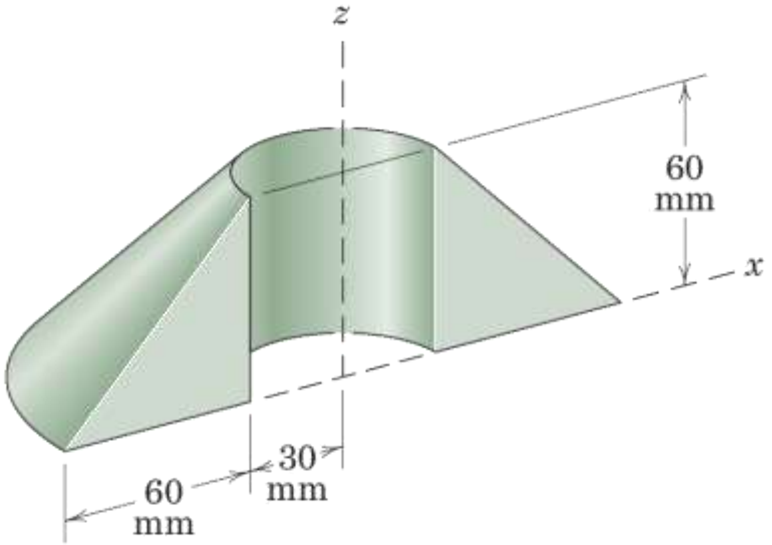
$$V = \theta \bar{r} A = \pi[30 + \tfrac{1}{3}(60)][\tfrac{1}{2}(60)(60)] = 2.83(10^5)\ \text{mm}^3 \quad ① \qquad \textit{Ans.}$$

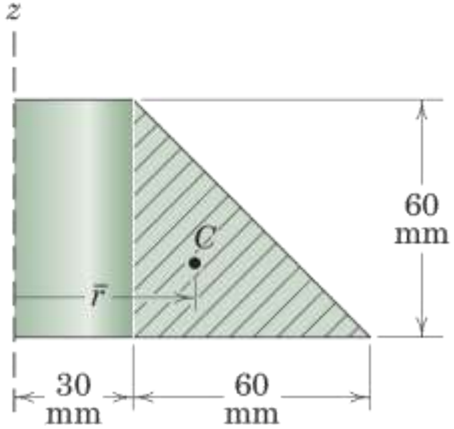
The mass of the body is then

$$m = \rho V = \left[7830\ \frac{\text{kg}}{\text{m}^3}\right][2.83(10^5)\text{mm}^3]\left[\frac{1\ \text{m}}{1000\ \text{mm}}\right]^3$$

$$= 2.21\ \text{kg} \qquad \textit{Ans.}$$

① Note that θ must be in radians.





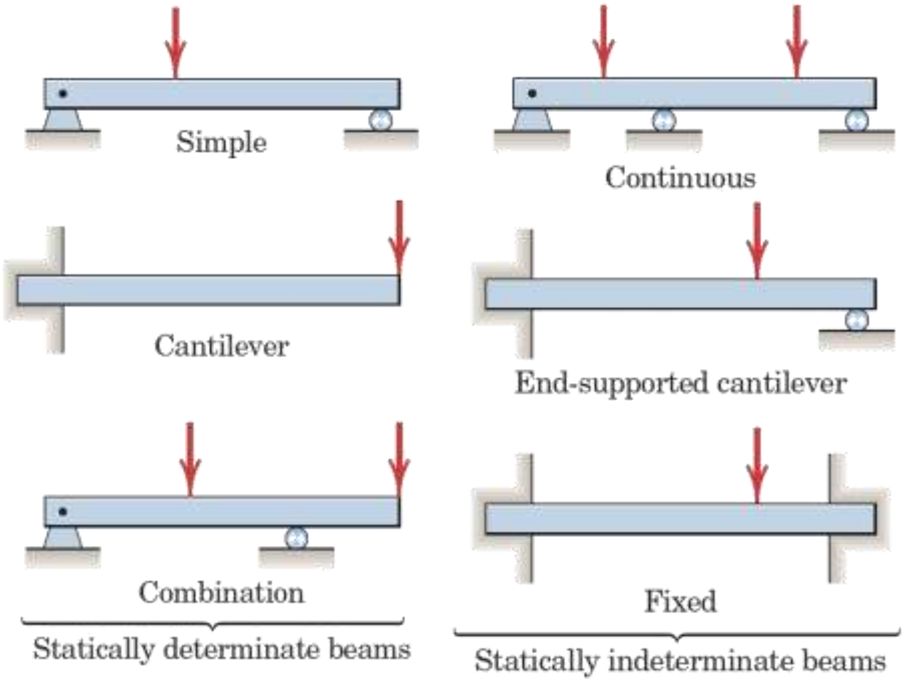
Article 5/6 Beams – External Effects

- Introduction

- Types of Beams
  - Determinate

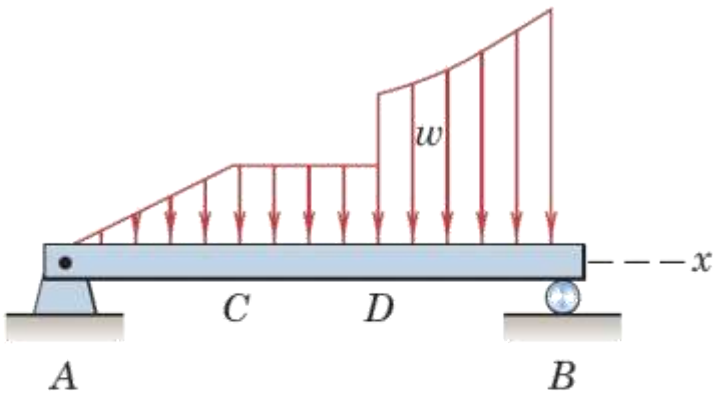
  - Indeterminate

- Types of Supports



Article 5/6 – Distributed Loads

- Illustration



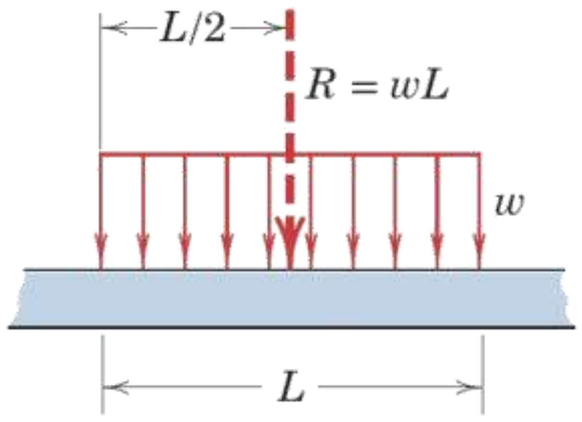
- Intensity of a Distributed Load, *w*
  - Units are Force/Length (N/m or lb/ft)
  - Various Types, e.g., Constant, Linear, Functional

- Question: How do we analyze the external reactions on a beam subjected to various distributed loads?

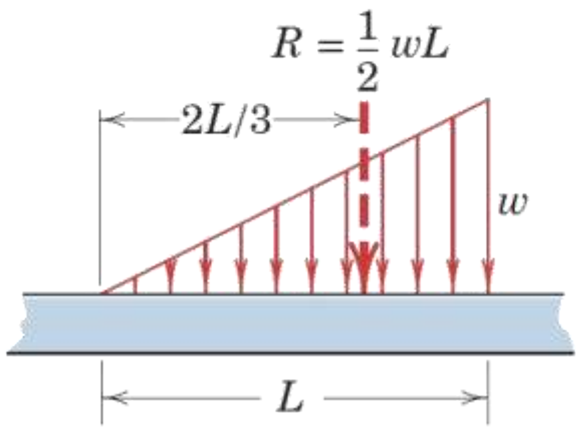
- Answer: We must replace the distributed load by its equivalent concentrated force which acts at the appropriate location.

Article 5/6 – Types of Distributed Loads (1 of 2)

- Uniform Loads

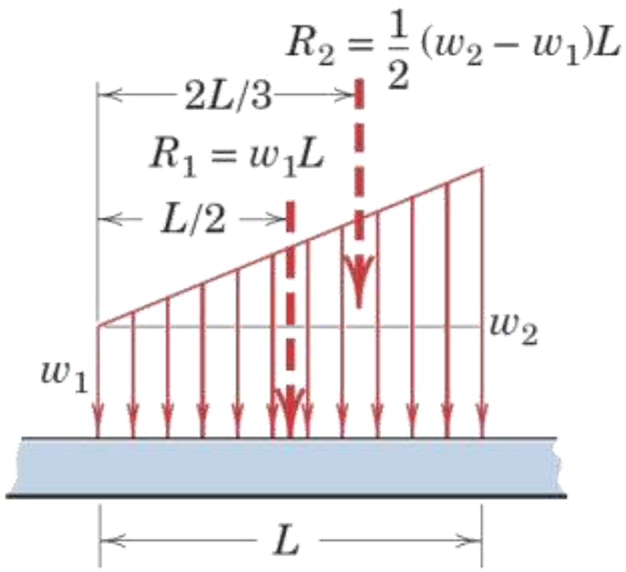


- Linear Loads



Article 5/6 – Types of Distributed Loads (2 of 2)

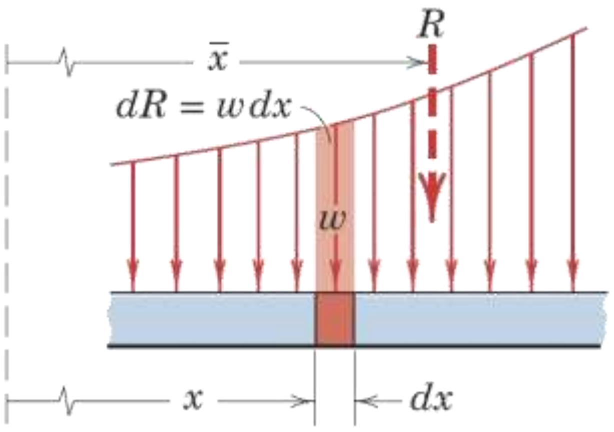
- Trapezoidal Loads



- Functional Loads

$$R = \int w\,dx$$

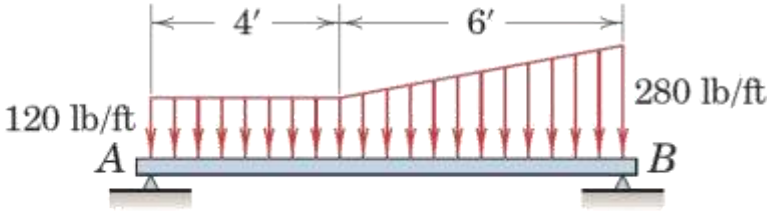
$$\bar{x} = \frac{\int xw\,dx}{R}$$



Article 5/6 – Sample Problem 5/11 (1 of 2)

- Problem Statement

  Determine the equivalent concentrated load(s) and external reactions for the simply supported beam which is subjected to the distributed load shown.



Article 5/6 – Sample Problem 5/11 (2 of 2)

- Solution

The area associated with the load distribution is divided into the rectangular and triangular areas shown. The concentrated-load values are determined by computing the areas, and these loads are located at the centroids of the respective areas. ①

Once the concentrated loads are determined, they are placed on the free-body diagram of the beam along with the external reactions at A and B. Using principles of equilibrium, we have

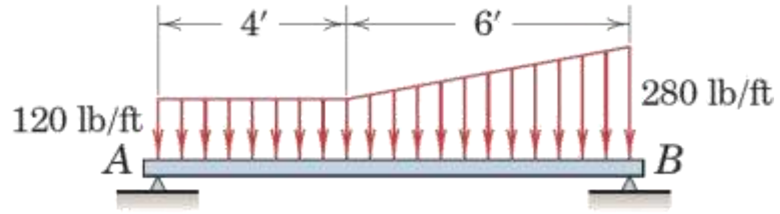
$[\Sigma M_A = 0] \qquad 1200(5) + 480(8) - R_B(10) = 0$

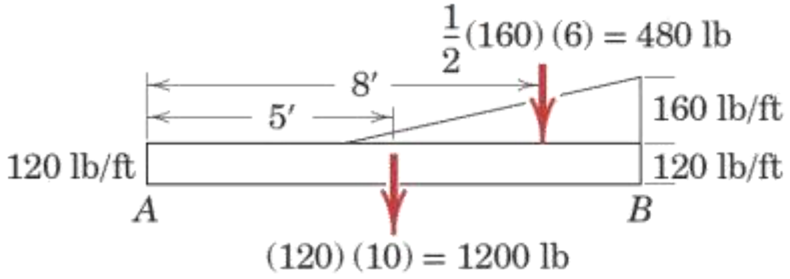
$$R_B = 984 \text{ lb} \qquad \textit{Ans.}$$

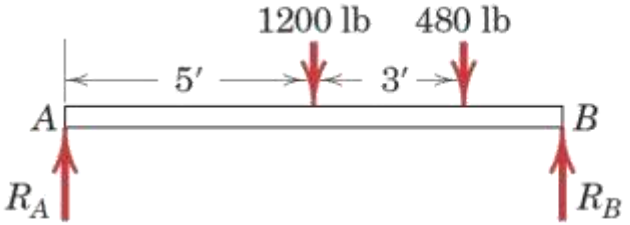
$[\Sigma M_B = 0] \qquad R_A(10) - 1200(5) - 480(2) = 0$

$$R_A = 696 \text{ lb} \qquad \textit{Ans.}$$

① Note that it is usually unnecessary to reduce a given distributed load to a *single* concentrated load.



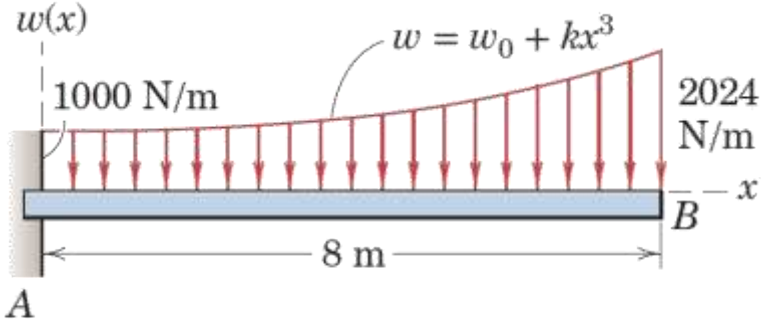




Article 5/6 – Sample Problem 5/12 (1 of 2)

- Problem Statement

  Determine the reaction at the support A of the loaded cantilever beam.



Article 5/6 – Sample Problem 5/12 (2 of 2)

- Solution

The constants in the load distribution are found to be $w_0 =$ 1000 N/m and $k = 2$ N/m^4. ① The load R is then

$$R = \int w\,dx = \int_0^8 (1000 + 2x^3)\,dx = \left(1000x + \frac{x^4}{2}\right)\Big|_0^8 = 10\,050\text{ N}$$

The x-coordinate of the centroid of the area is found by ②

$$\bar{x} = \frac{\int xw\,dx}{R} = \frac{1}{10\,050}\int_0^8 x(1000 + 2x^3)\,dx$$

$$= \frac{1}{10\,050}(500x^2 + \tfrac{2}{5}x^5)\Big|_0^8 = 4.49\text{ m}$$

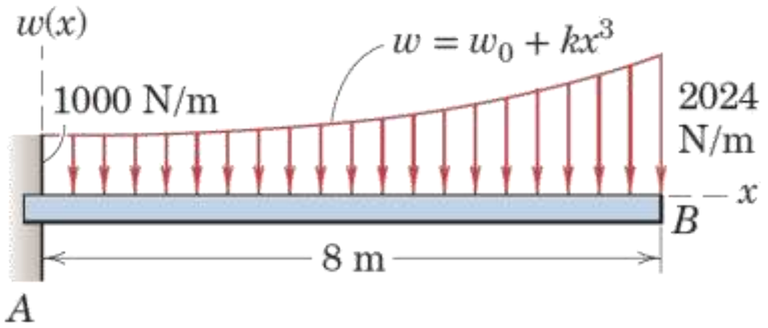
From the free-body diagram of the beam, we have

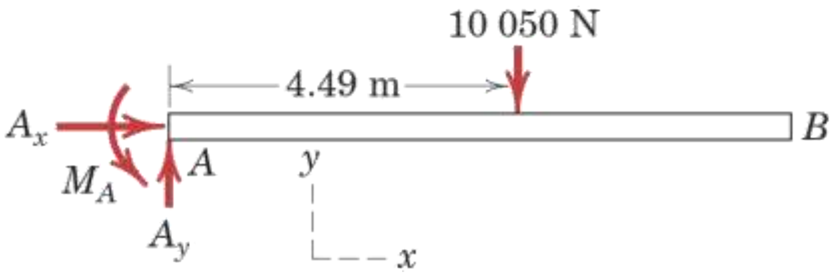
$[\Sigma M_A = 0]$ $\qquad M_A - (10\,050)(4.49) = 0$

$M_A = 45\,100$ N·m $\qquad$ *Ans.*

$[\Sigma F_y = 0]$ $\qquad A_y = 10\,050$ N $\qquad$ *Ans.*

Note that $A_x = 0$ by inspection.



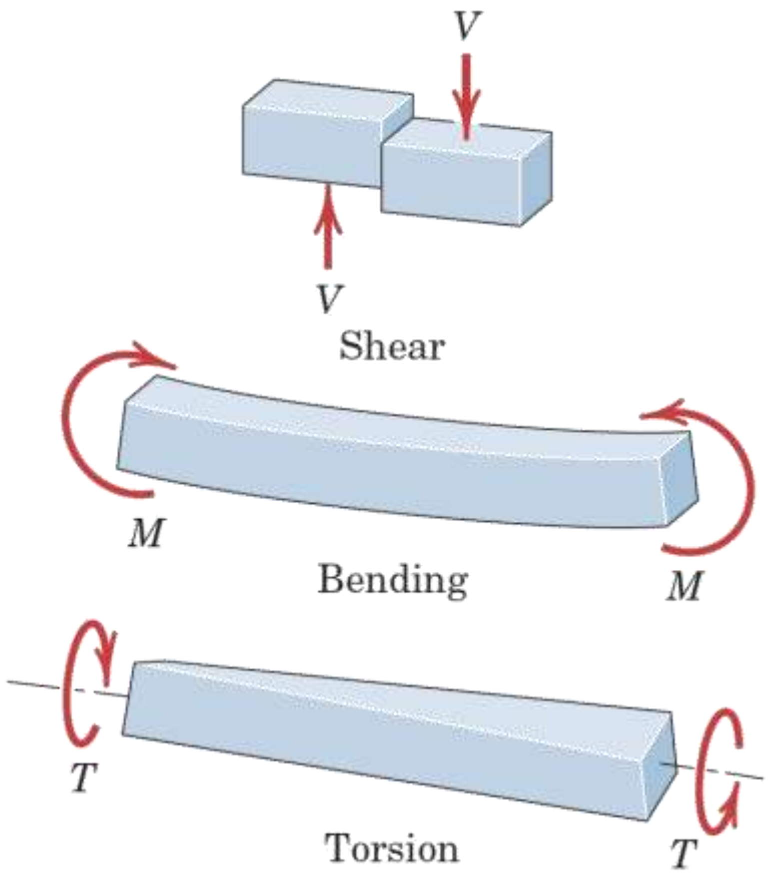


① Use caution with the units of the constants w_0 and k.

② The student should recognize that the calculation of R and its location $\bar{x}$ is simply an application of centroids as treated in Art. 5/3.

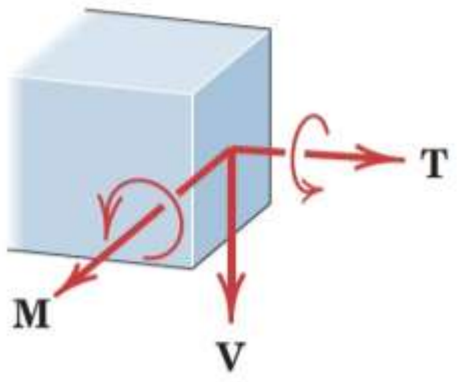
Article 5/7 Beams – Internal Effects

- Internal Loads in Beams
  - Shear Force
  - Bending Moment
  - Torsional Moment



Article 5/7 – Internal Loads (cont.)

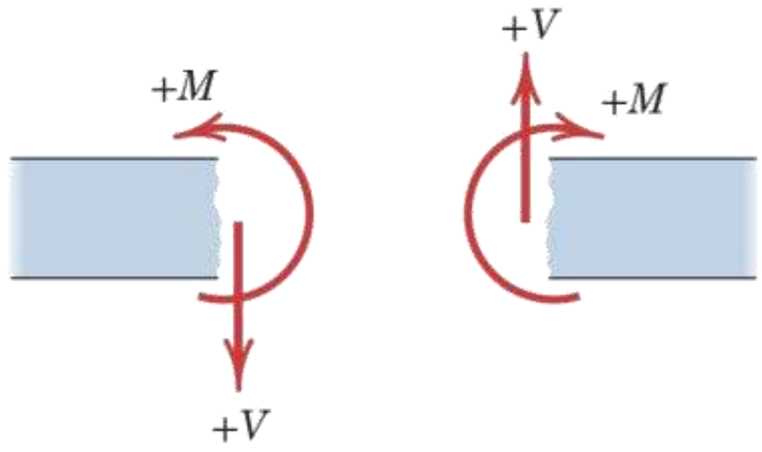
- Combined Loading Effect



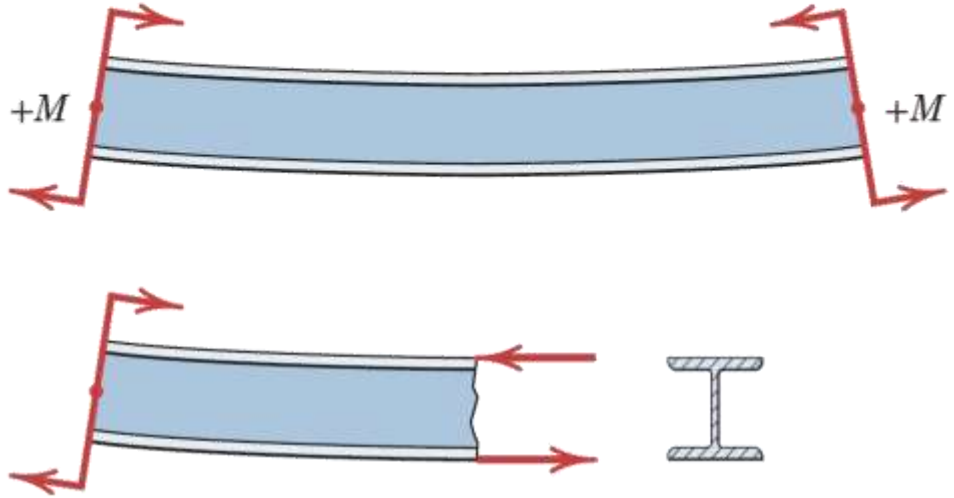
Combined loading

Article 5/7 – Shear and Bending Explored

- Sign Convention for *V* and *M*

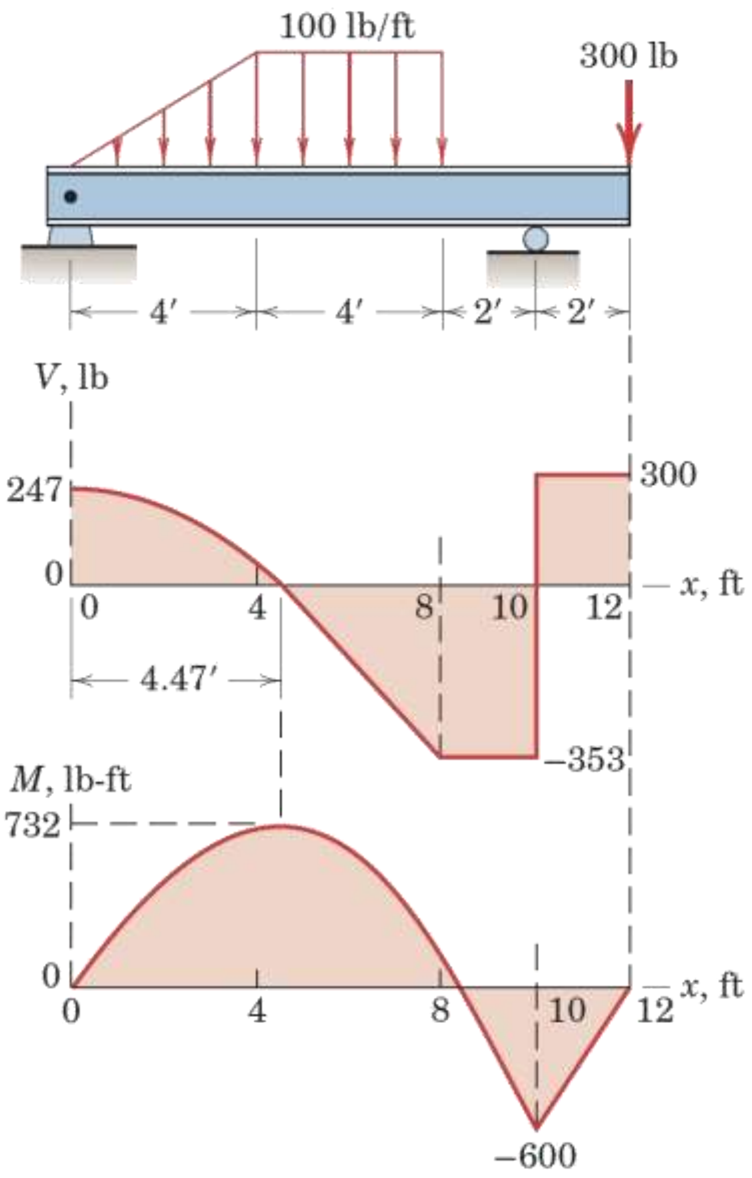


- Visualization of the Bending Moment



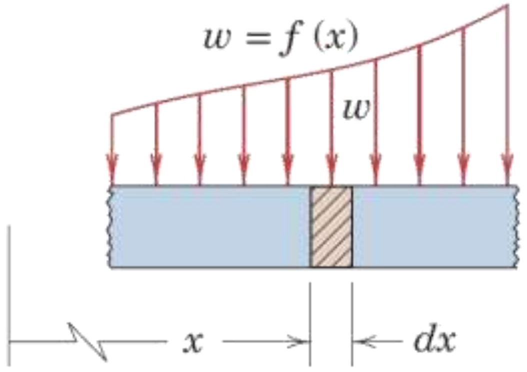
Article 5/7 – Shear-Force and Bending-Moment Diagrams

- Overview
- Solution Steps
  1. Determine the external reactions which act on the beam.
  2. Isolate a portion of the beam, either to the right or the left of an arbitrary transverse section, and construct a free-body diagram of the section. Sections cannot be taken at a discontinuity in the loading diagram, e.g., at a reaction or concentrated force, or at a discontinuity in a distributed load. Loads in the section should be shown in a positive sign convention.
  3. Apply the equations of equilibrium to the isolated portion to determine expressions for the shear force V and the bending moment M which act in that portion of the beam.
  4. Plot the resulting expressions over the section of the beam for which they are valid.
  5. Continue to the next section until the entire beam has been analyzed.

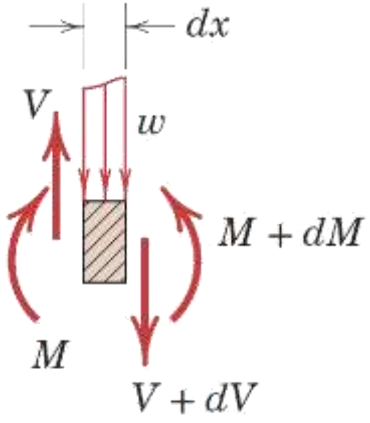


Article 5/7 – General Loading, Shear, and Moment Relationships

- Situation of Interest



- Free-Body Diagram of a Segment

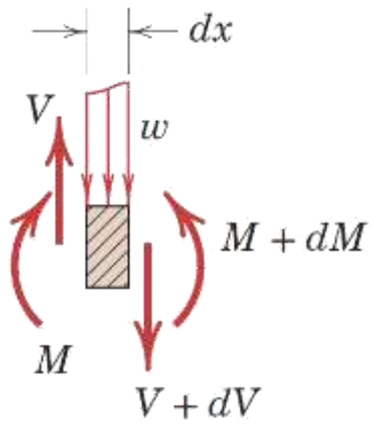


- Solution Process
  - Vertical Equilibrium yields *V*
  - Moment Equilibrium yields *M*

Article 5/7 – Relationships between Shear and Loading (1 of 3)

- Vertical Equilibrium (positive up)

$$V - w\,dx - (V + dV) = 0 \quad \rightarrow \quad dV = -w\,dx$$



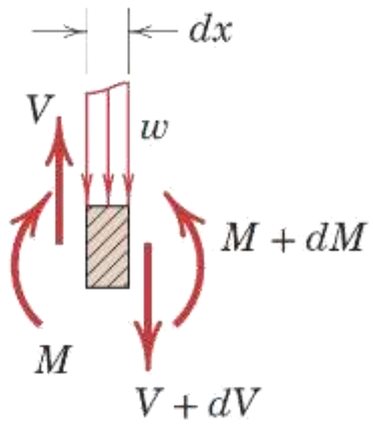
- Result 1: Indefinite integral of both sides.

$$\int dV = -\int w\,dx \quad \rightarrow \quad V = C_1 - \int w\,dx$$

  - The expression for the internal shear force in a region of the beam as a function of x is found by integrating the equation for the distributed loading within that region of the beam and adding an appropriate constant. The integral expression must be multiplied by a negative to obtain the correct relationship.

  - The constant C_1 is most often determined as the value of the shear force at the beginning of the section of the beam where the distributed loading is defined.

Article 5/7 – Relationships between Shear and Loading (2 of 3)



- Vertical Equilibrium

$$dV = -w\,dx$$

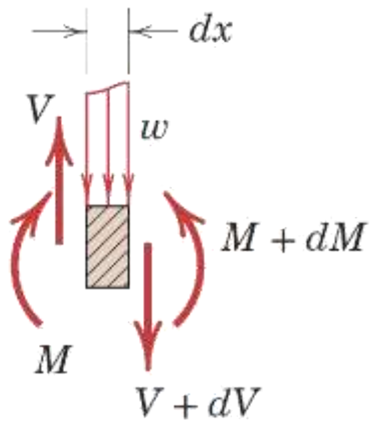
- Result 2: Definite integral of both sides.

$$\int_{V_1}^{V_2} dV = -\int_{x_1}^{x_2} w\,dx \quad \rightarrow \quad V_2 - V_1 = \Delta V_{1-2} = -\int_{x_1}^{x_2} w\,dx$$

  - The change in the value of the internal shear force between two points on a beam is the negative of the area under the distributed loading curve between those same two points.

  - Note that V_1 is the value of the shear force at the coordinate x_1, and V_2 is the value of the shear force at the coordinate x_2.

Article 5/7 – Relationships between Shear and Loading (3 of 3)



- Vertical Equilibrium

$$dV = -w\,dx$$

- Result 3: Divide both sides by *dx*.

$$w = -\frac{dV}{dx}$$

  - The slope of the internal shear-force diagram at any location on a beam is equal to the negative intensity of the distributed loading at that point.

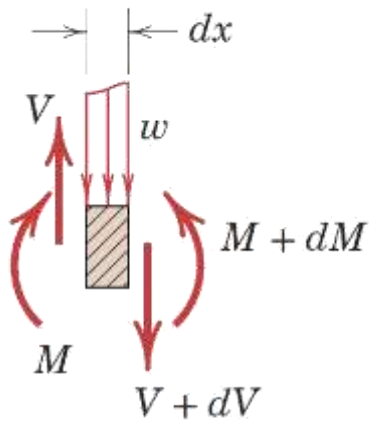
  - If there is no distributed loading at a location in the beam, then the slope is zero at that location meaning the shear force is temporarily constant.

Article 5/7 – Relationships between Moment and Loading (1 of 3)

- Moment Equilibrium (about Left Side, CCW +)

$$-M+\left(M+dM\right)-(V+dV)dx-w\,dx\frac{dx}{2}=0$$

$$dM=V\ dx$$



- Result 4: Indefinite integral of both sides.

$$\int dM=\int V\ dx\ \ \rightarrow\ \ M=C_2+\int V\ dx$$

  - The expression for the internal bending moment in a region of the beam is found by integrating the equation for the internal shear force within that region of the beam and adding an appropriate constant.
  - The constant C_2 is most often determined as the value of the bending moment at the beginning of the section of the beam where the distributed loading is defined.
  - Note that the expression for V was determined from the first integral of the loading function, so the bending moment is the second integral of the loading function.

Article 5/7 – Relationships between Moment and Loading (2 of 3)

- Moment Equilibrium (about Left Side, CCW +)

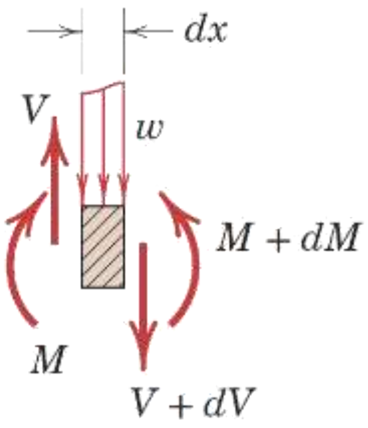
$$dM = V\ dx$$

- Result 5: Definite integral of both sides.

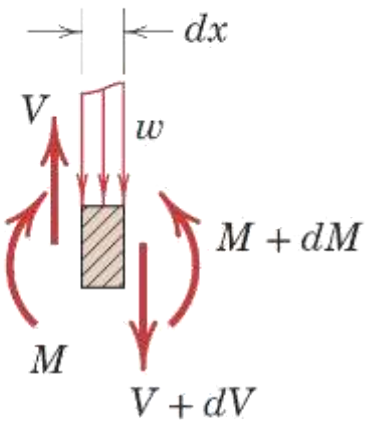
$$\int_{M_1}^{M_2} dM = \int_{x_1}^{x_2} V\ dx \quad \rightarrow \quad M_2 - M_1 = \Delta M_{1-2} = \int_{x_1}^{x_2} V\ dx$$

  - The change in the value of the internal bending moment between two points on a beam is the area under the shear force diagram between those same two points.

  - Note that M_1 is the value of the bending moment at the coordinate x_1, and M_2 is the value of the bending moment at the coordinate x_2.



Article 5/7 – Relationships between Moment and Loading (3 of 3)



- Moment Equilibrium (about Left Side, CCW +)

$$dM = V\ dx$$

- Result 6: Divide both sides by *dx*.

$$V = \frac{dM}{dx}$$

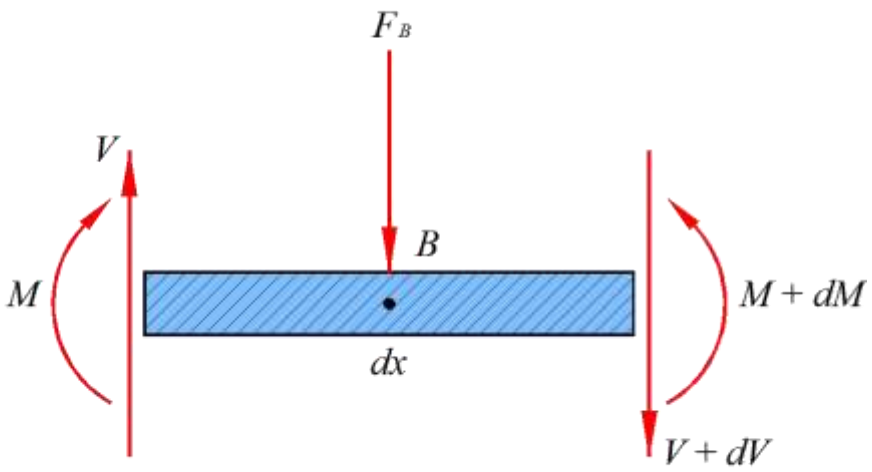
  - The slope of the internal bending moment diagram at any location on a beam is equal to the value of the internal shear force at that point.

  - If there is no shear force at a location in the beam, then the slope is zero at that location meaning the bending moment is temporarily constant.

  - When V passes through zero and is continuous function of x with $dV/dx \neq 0$, the bending moment M will be a maximum or a minimum. Critical values of M also occur when V crosses the zero axis discontinuously which occurs at a concentrated load.

Article 5/7 – Effect of a Concentrated Force (1 of 2)

- Situation of Interest



- Vertical Equilibrium (positive up)

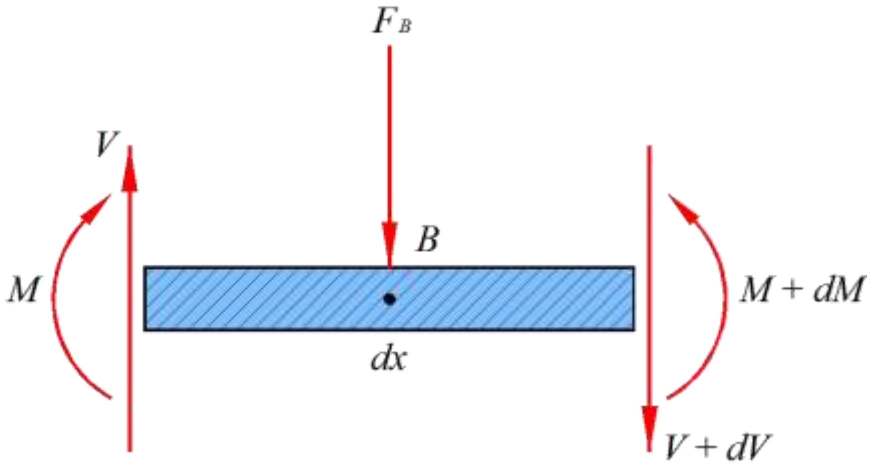
$$V-(V+dV)-F_B=0 \quad \rightarrow \quad dV=-F_B$$

- Result 7
  - An externally applied, concentrated force creates a change in the internal shear force that is consistent with both the magnitude and direction of the concentrated force.
  - A downward force creates an instantaneous *drop* in the internal shear force and an upward force creates an instantaneous *rise* in the internal shear force. The drop or rise is equal to the magnitude of the concentrated force.

Article 5/7 – Effect of a Concentrated Force (2 of 2)

- Moment Equilibrium (at *B*, CCW+, $dx \to 0$)

$$-M - V\frac{dx}{2} - (V + dV)\frac{dx}{2} + (M + dM) = 0 \quad \to \quad dM = 0$$

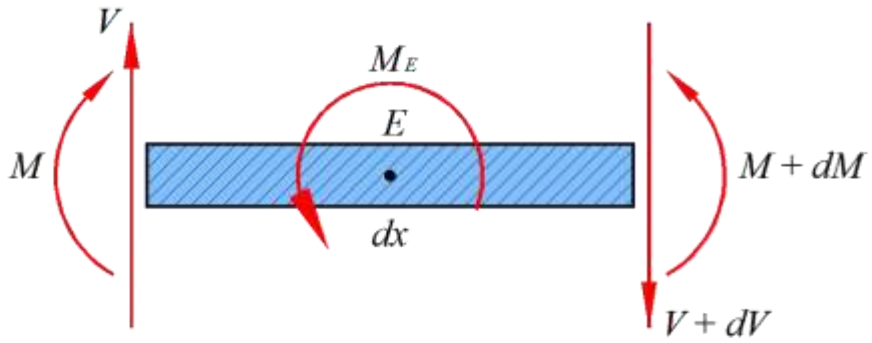


- Result 8
  - An externally applied, concentrated force has no effect on the value of the internal bending moment at the point of application.

  - An externally applied, concentrated force will create a sudden change in the slope of the bending-moment diagram at the point of application.

Article 5/7 – Effect of a Concentrated Moment (1 of 2)

- Situation of Interest



- Vertical Equilibrium (positive up)

$$V - (V + dV) = 0 \quad \rightarrow \quad dV = 0$$

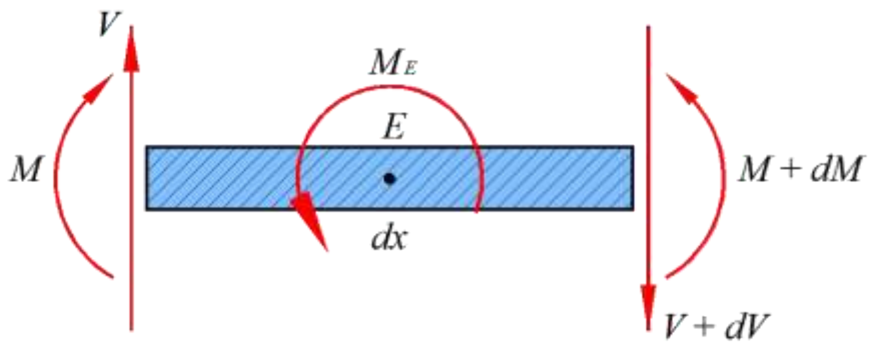
- Result 9
  - An externally applied, concentrated moment has no effect on the value of the internal shear force at the point of application for the moment.

Article 5/7 – Effect of a Concentrated Moment (2 of 2)

- Moment Equilibrium (at *B*, CCW+, *dx*→0)

$$-M - V\frac{dx}{2} - (V + dV)\frac{dx}{2} + (M + dM) + M_E = 0$$

$$dM = -M_E$$

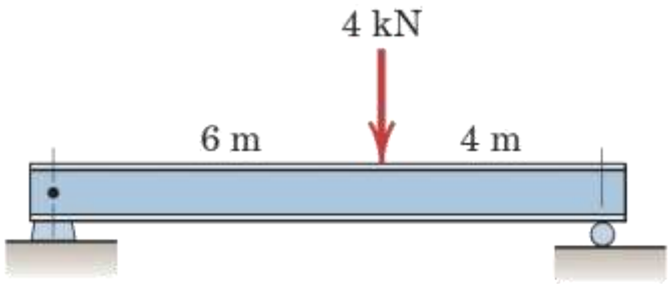


- Result 10
  - An externally applied, concentrated moment creates a change in the internal bending moment that is consistent with the magnitude of the concentrated moment, but is inconsistent with the direction of the concentrated moment.
  - An externally applied *counterclockwise* moment creates an instantaneous *drop* in the internal bending moment and an externally applied *clockwise* moment creates an instantaneous *rise* in the internal bending moment. The *drop* or *rise* is equal to the magnitude of the concentrated moment.

Article 5/7 – Sample Problem 5/13 (1 of 4)

- Problem Statement

  Determine the shear and moment distributions produced in the simple beam by the 4-kN concentrated load.

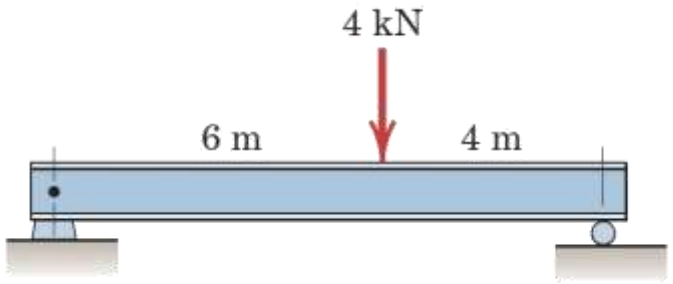


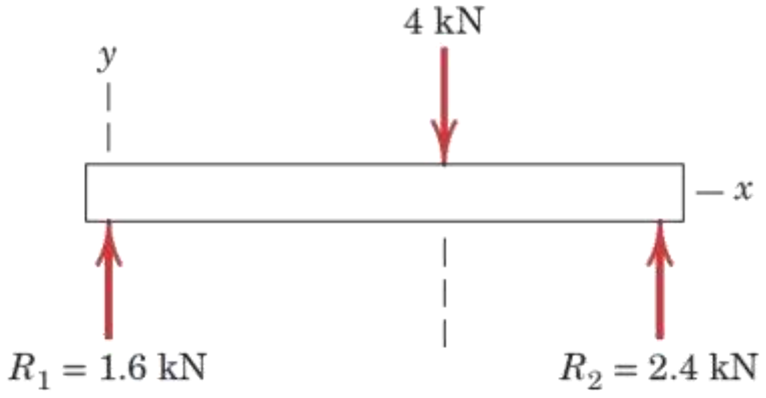
Article 5/7 – Sample Problem 5/13 (2 of 4)

- Support Reactions

From the free-body diagram of the entire beam we find the support reactions, which are

$$R_1 = 1.6 \text{ kN} \qquad R_2 = 2.4 \text{ kN}$$





Article 5/7 – Sample Problem 5/13 (3 of 4)

- Beean Segment Equilibrium

A section of the beam of length x is next isolated with its free-body diagram on which we show the shear V and the bending moment M in their positive directions. Equilibrium gives

$[\Sigma F_y = 0]$ $\quad 1.6 - V = 0 \qquad V = 1.6\text{ kN}$

$[\Sigma M_{R_1} = 0]$ $\quad M - 1.6x = 0 \qquad M = 1.6x$

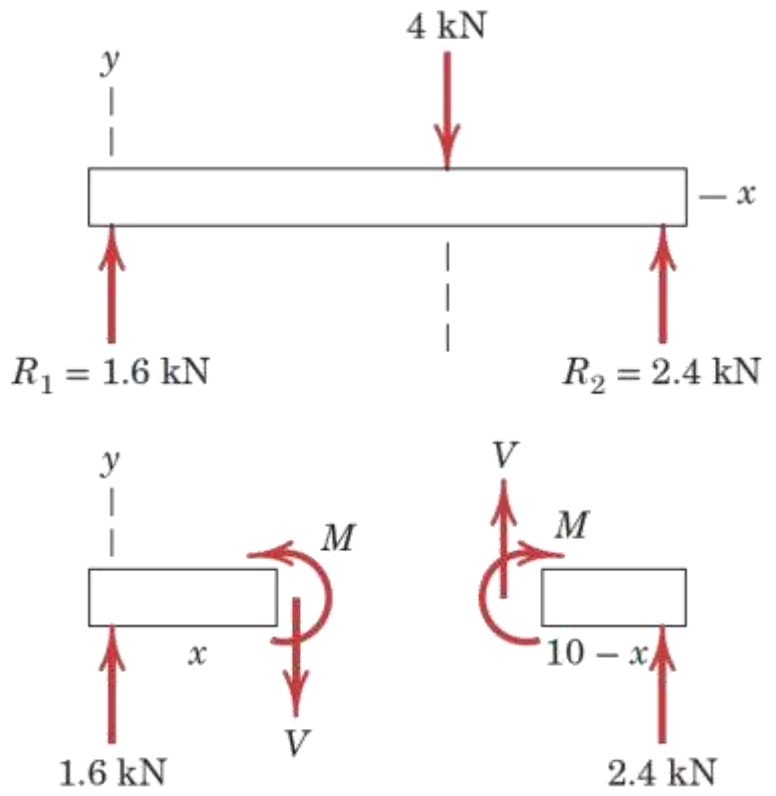
These values of V and M apply to all sections of the beam to the left of the 4-kN load. ①

A section of the beam to the right of the 4-kN load is next isolated with its free-body diagram on which V and M are shown in their positive directions. Equilibrium requires

$[\Sigma F_y = 0]$ $\quad V + 2.4 = 0 \qquad V = -2.4\text{ kN}$

$[\Sigma M_{R_2} = 0]$ $\quad -(2.4)(10 - x) + M = 0 \qquad M = 2.4(10 - x)$

These results apply only to sections of the beam to the right of the 4-kN load.

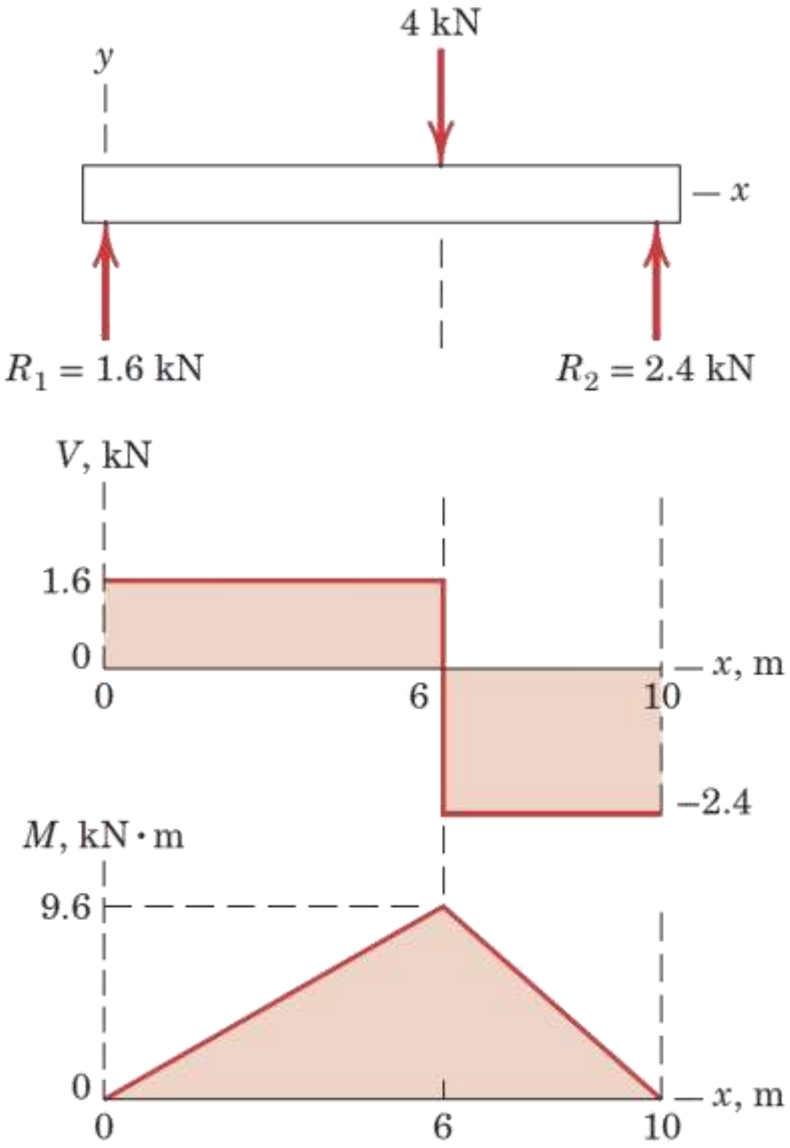


① We must be careful not to take our section at a concentrated load (such as $x = 6$ m) since the shear and moment relations involve discontinuities at such positions.

Article 5/7 – Sample Problem 5/13 (4 of 4)

- *V* and *M* Diagrams

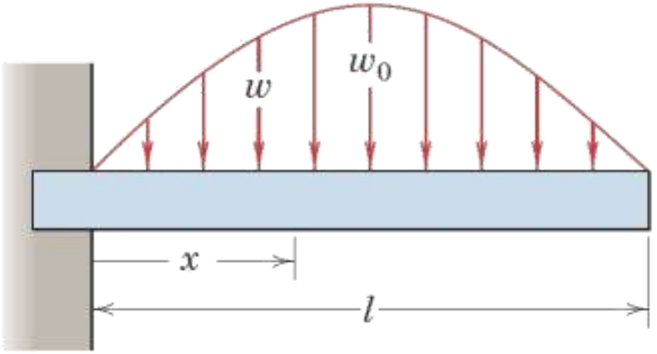
The values of V and M are plotted as shown. The maximum bending moment occurs where the shear changes direction. As we move in the positive x-direction starting with $x = 0$, we see that the moment M is merely the accumulated area under the shear diagram.



Article 5/7 – Sample Problem 5/14 (1 of 5)

- Problem Statement

  The cantilever beam is subjected to the load intensity (force per unit length) which varies as $w = w_0 \sin(\pi x/l)$. Determine the shear force V and bending moment M as functions of the ratio x/l.



Article 5/7 – Sample Problem 5/14 (2 of 5)

- Support Reactions

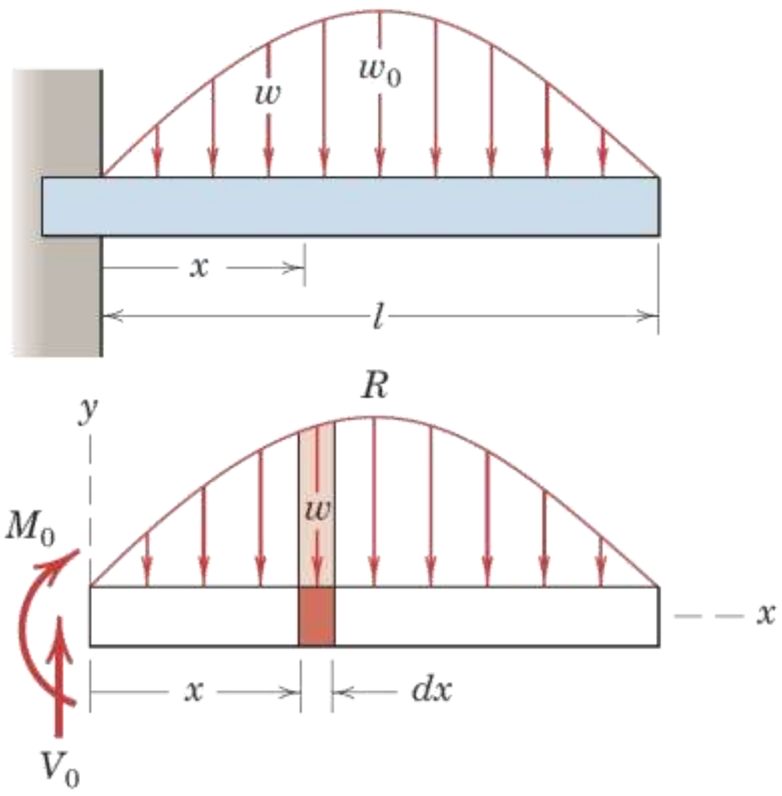
The free-body diagram of the entire beam is drawn first so that the shear force V_0 and bending moment M_0 which act at the supported end at $x = 0$ can be computed. By convention V_0 and M_0 are shown in their positive mathematical senses. A summation of vertical forces for equilibrium gives

$$[\Sigma F_y = 0] \qquad V_0 - \int_0^l w\,dx = 0 \qquad V_0 = \int_0^l w_0 \sin\frac{\pi x}{l}\,dx = \frac{2w_0 l}{\pi}$$

A summation of moments about the left end at $x = 0$ for equilibrium gives ①

$$[\Sigma M = 0] \qquad -M_0 - \int_0^l x(w\,dx) = 0 \qquad M_0 = -\int_0^l w_0 x \sin\frac{\pi x}{l}\,dx$$

$$M_0 = \frac{-w_0 l^2}{\pi^2}\left[\sin\frac{\pi x}{l} - \frac{\pi x}{l}\cos\frac{\pi x}{l}\right]_0^l = -\frac{w_0 l^2}{\pi}$$



① In this case of symmetry, it is clear that the resultant $R = V_0 = 2w_0 l/\pi$ of the load distribution acts at midspan, so that the moment requirement is simply $M_0 = -Rl/2 = -w_0 l^2/\pi$. The minus sign tells us that physically the bending moment at $x = 0$ is opposite to that represented on the free-body diagram.

Article 5/7 – Sample Problem 5/14 (3 of 5)

- Shear-Force Expression

From a free-body diagram of an arbitrary section of length x, integration of Eq. 5/10 permits us to find the shear force internal to the beam. Thus,

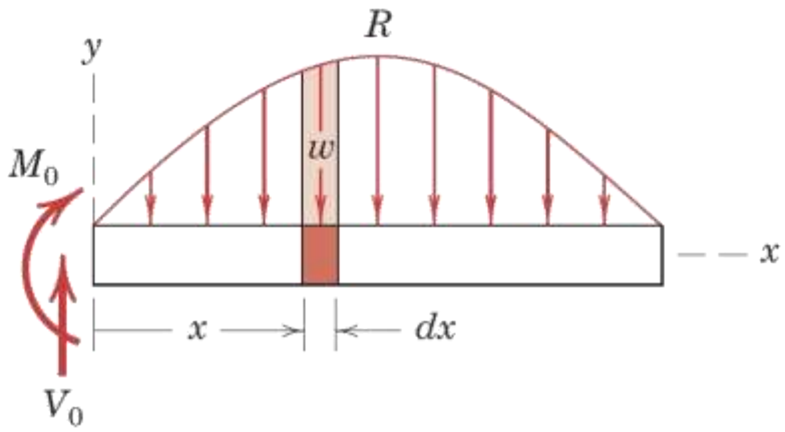
$$[dV = -w\,dx] \qquad \int_{V_0}^{V} dV = -\int_0^x w_0 \sin\frac{\pi x}{l}\,dx \quad ②$$

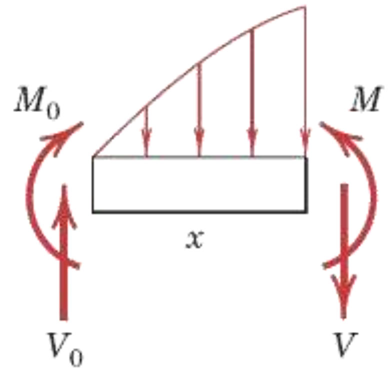
$$V - V_0 = \left[\frac{w_0 l}{\pi}\cos\frac{\pi x}{l}\right]_0^x \qquad V - \frac{2w_0 l}{\pi} = \frac{w_0 l}{\pi}\left(\cos\frac{\pi x}{l} - 1\right)$$

or in dimensionless form

$$\frac{V}{w_0 l} = \frac{1}{\pi}\left(1 + \cos\frac{\pi x}{l}\right) \qquad \textit{Ans.}$$

② The free-body diagram serves to remind us that the integration limits for V as well as for x must be accounted for. We see that the expression for V is positive, so that the shear force is as represented on the free-body diagram.





Article 5/7 – Sample Problem 5/14 (4 of 5)

- Bending-Moment Expression

The bending moment is obtained by integration of Eq. 5/11, which gives

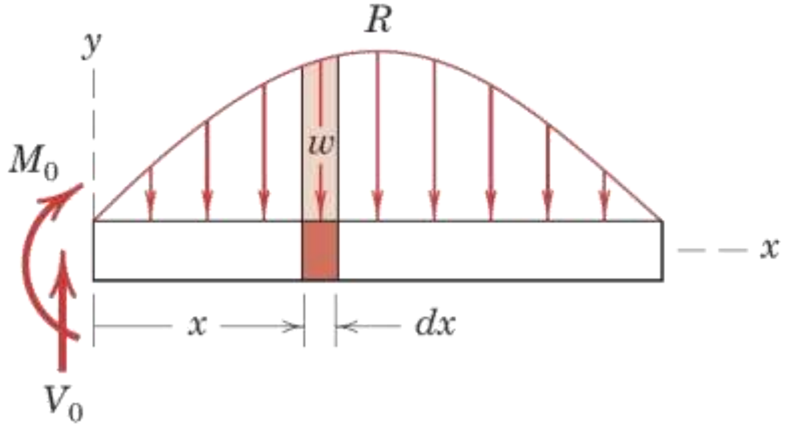
$$[dM = V\,dx] \qquad \int_{M_0}^{M} dM = \int_0^x \frac{w_0 l}{\pi}\left(1 + \cos\frac{\pi x}{l}\right) dx$$

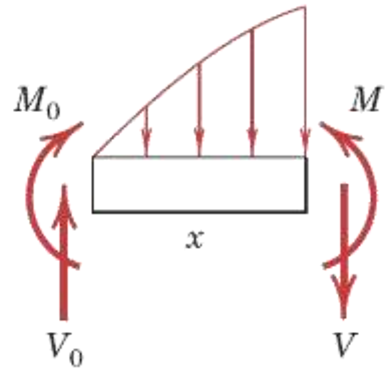
$$M - M_0 = \frac{w_0 l}{\pi}\left[x + \frac{l}{\pi}\sin\frac{\pi x}{l}\right]_0^x$$

$$M = -\frac{w_0 l^2}{\pi} + \frac{w_0 l}{\pi}\left[x + \frac{l}{\pi}\sin\frac{\pi x}{l} - 0\right]$$

or in dimensionless form

$$\frac{M}{w_0 l^2} = \frac{1}{\pi}\left(\frac{x}{l} - 1 + \frac{1}{\pi}\sin\frac{\pi x}{l}\right) \qquad \textit{Ans.}$$

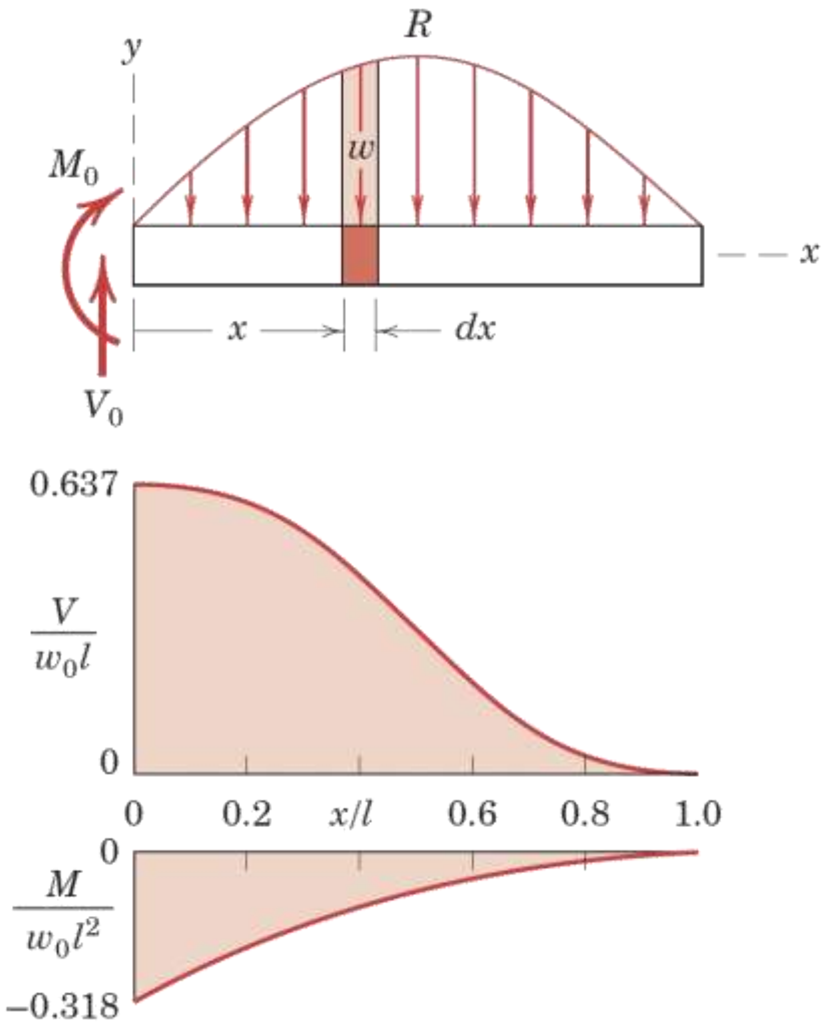




Article 5/7 – Sample Problem 5/14 (5 of 5)

- *V* and *M* Diagrams

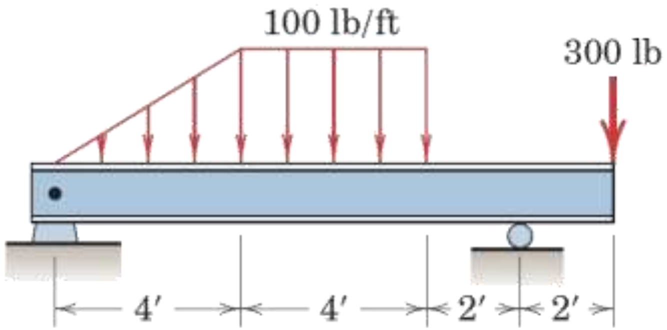
The variations of $V/w_0 l$ and $M/w_0 l^2$ with x/l are shown in the bottom figures. The negative values of $M/w_0 l^2$ indicate that physically the bending moment is in the direction opposite to that shown.



Article 5/7 – Sample Problem 5/15 (1 of 6)

- Problem Statement

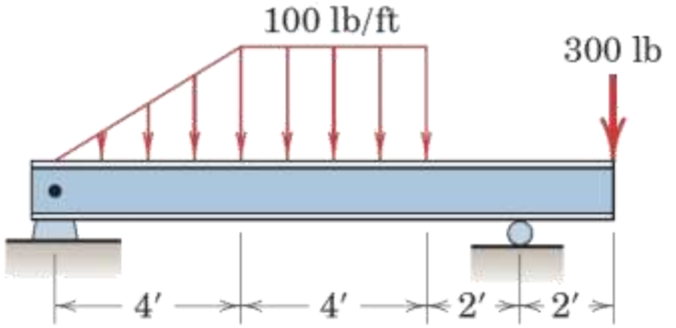
Draw the shear-force and bending-moment diagrams for the loaded beam and determine the maximum moment M and its location x from the left end.

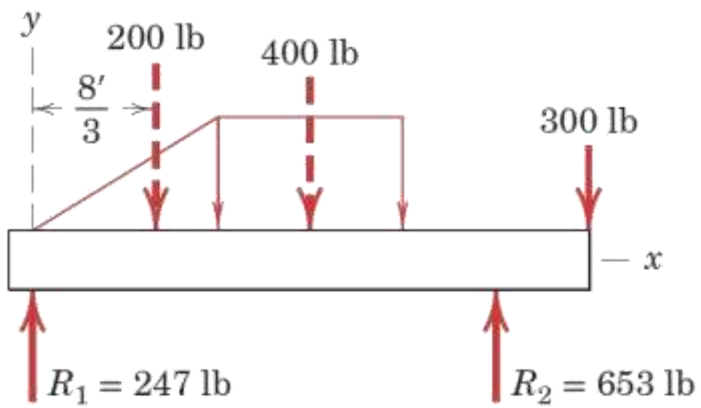


Article 5/7 – Sample Problem 5/15 (2 of 6)

- Support Reactions

  The support reactions are most easily obtained by considering the resultants of the distributed loads as shown on the free-body diagram of the beam as a whole.



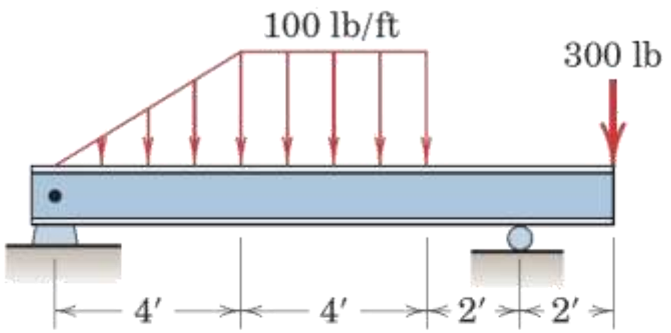


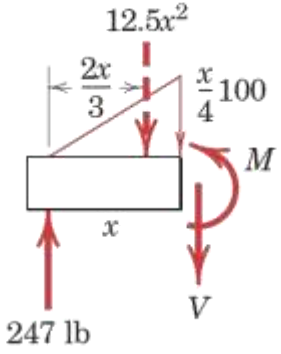
Article 5/7 – Sample Problem 5/15 (3 of 6)

- First Beam Segment: $0 < x < 4$ ft

$[\Sigma F_y = 0]$ $\qquad V = 247 - 12.5x^2$

$[\Sigma M = 0]$ $\qquad M + (12.5x^2)\dfrac{x}{3} - 247x = 0 \qquad M = 247x - 4.17x^3$





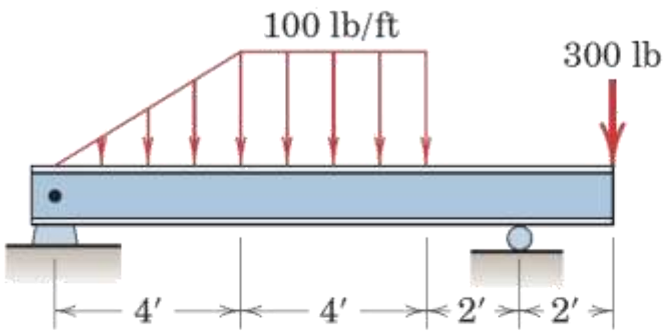
Article 5/7 – Sample Problem 5/15 (4 of 6)

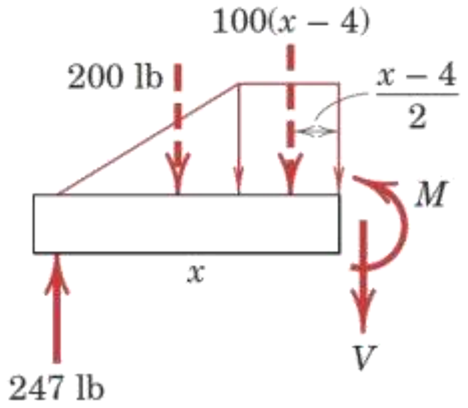
- Second Beam Segment: $4 < x < 8$ ft

$[\Sigma F_y = 0] \qquad V + 100(x - 4) + 200 - 247 = 0 \qquad V = 447 - 100x$

$$[\Sigma M = 0] \qquad M + 100(x - 4)\frac{x - 4}{2} + 200[x - \tfrac{2}{3}(4)] - 247x = 0$$

$$M = -267 + 447x - 50x^2$$



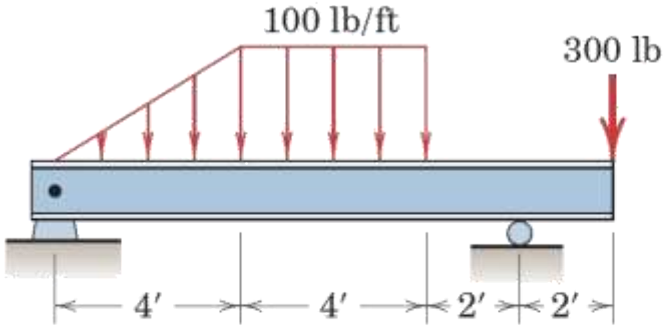


Article 5/7 – Sample Problem 5/15 (5 of 6)

- Third Beam Segment: $8 < x < 10$ ft

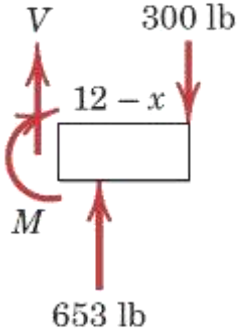
$[\Sigma F_y = 0]$ $V + 653 - 300 = 0$ $V = -353$ lb

$[\Sigma M = 0]$ $-M + 653(12 - x - 2) - 300(12 - x) = 0$ $M = 2930 - 353x$



- Fourth Beam Segment: $10 < x < 12$ ft

The last interval may be analyzed by inspection. The shear is constant at +300 lb, and the moment follows a straight-line relation beginning with zero at the right end of the beam.



Article 5/7 – Sample Problem 5/15 (6 of 6)

- *V* and *M* Diagrams

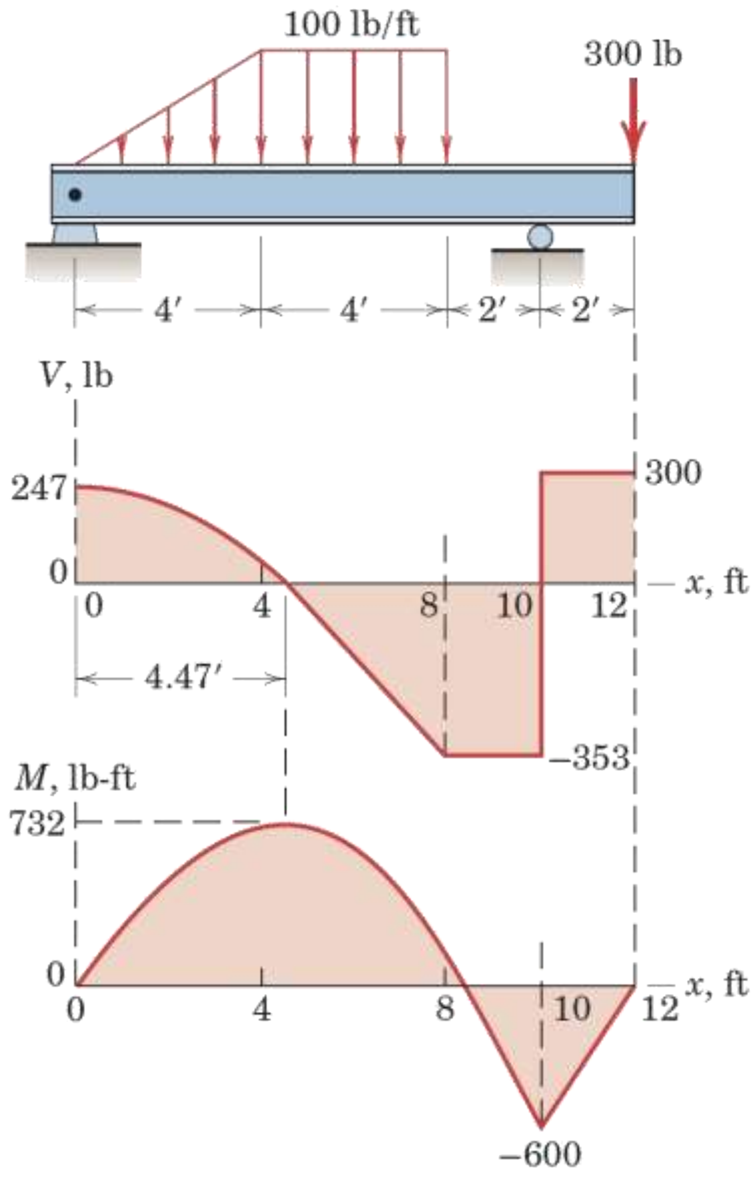
The maximum moment occurs at $x = 4.47$ ft, where the shear curve crosses the zero axis, and the magnitude of M is obtained for this value of x by substitution into the expression for M for the second interval. The maximum moment is

$$M = 732 \text{ lb-ft} \qquad \textit{Ans.}$$

As before, note that the change in moment M up to any section equals the area under the shear diagram up to that section. For instance, for $x < 4$ ft,

$$\left[\Delta M = \int V\,dx\right] \qquad M - 0 = \int_0^x (247 - 12.5x^2)\,dx$$

and, as above, $\qquad M = 247x - 4.17x^3$

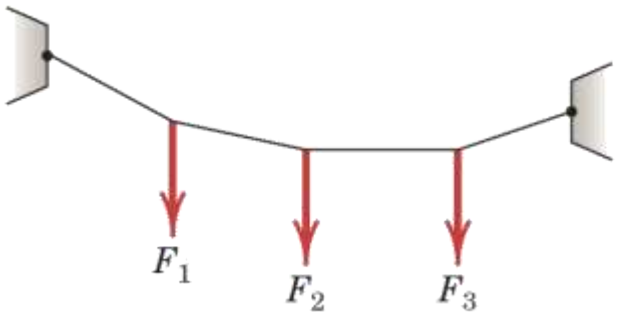


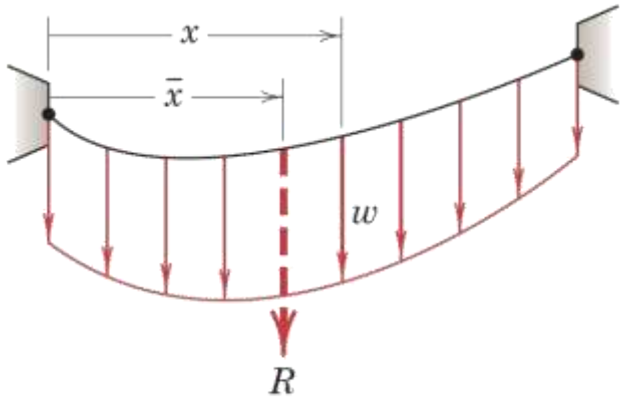
Article 5/8 Flexible Cables

- Introduction

- Examples

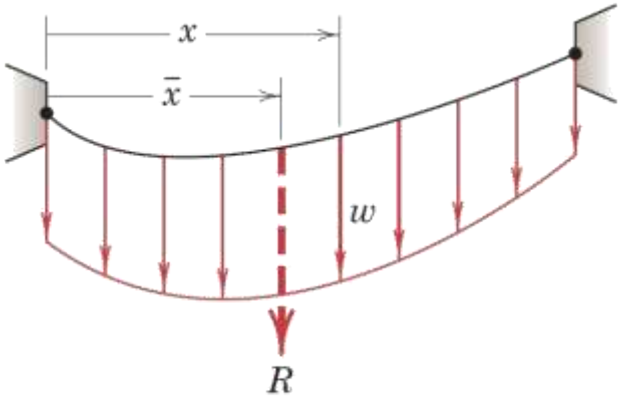
- Parameters of Importance
  - Tension
  - Span
  - Sag
  - Length
  - Load Intensity





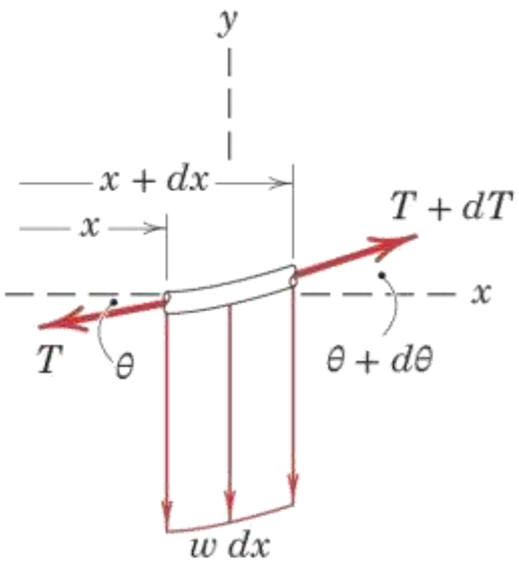
Article 5/8 – General Relationships (1 of 3)

- Variable and Continuous Load, w



- Resultant, $R = \int w\, dx$

- Location, $R\bar{x} = \int xw\, dx$

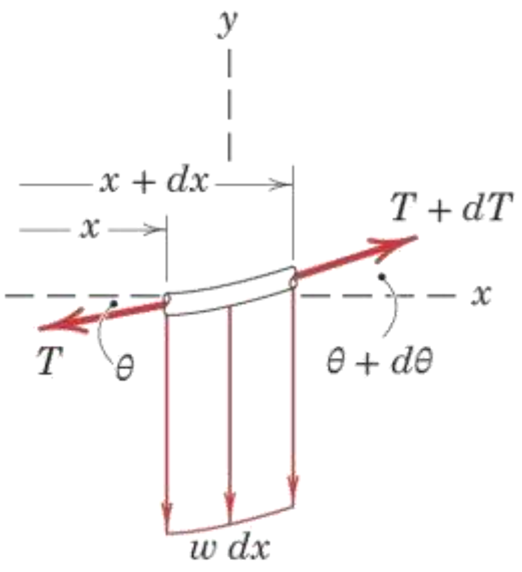


Article 5/8 – General Relationships (2 of 3)

- Equilibrium of a Cable Segment

$$\Sigma F_x = 0: \quad (T + dT)\cos(\theta + d\theta) - T\cos\theta = 0$$

$$\Sigma F_y = 0: \quad (T + dT)\sin(\theta + d\theta) - T\sin\theta - w\,dx = 0$$



- Trigonometric and Small-Angle Substitutions

$$\sin(\theta + d\theta) = \sin\theta\cos d\theta + \cos\theta\sin d\theta \qquad \sin d\theta \cong d\theta$$

$$\cos(\theta + d\theta) = \cos\theta\cos d\theta - \sin\theta\sin d\theta \qquad \cos d\theta \cong 1$$

- Result

$$T\cos\theta\; d\theta + dT\sin\theta = w\;dx$$

$$-T\sin\theta\; d\theta + dT\cos\theta = 0$$

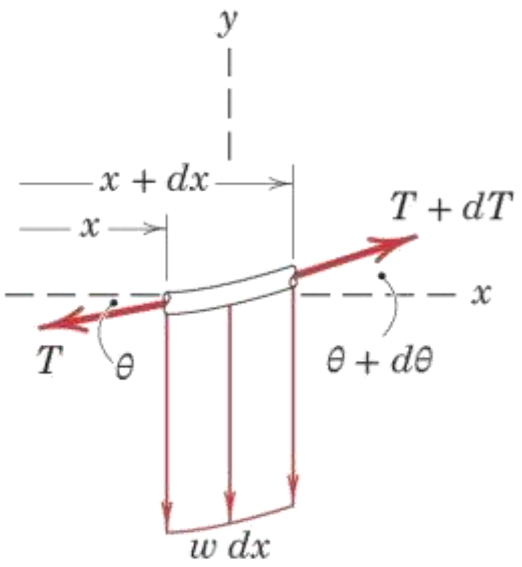
Article 5/8 – General Relationships (3 of 3)

- Rewrite Equations

$$d\left(T\sin\theta\right) = w\,dx$$

$$d\left(T\cos\theta\right) = 0$$

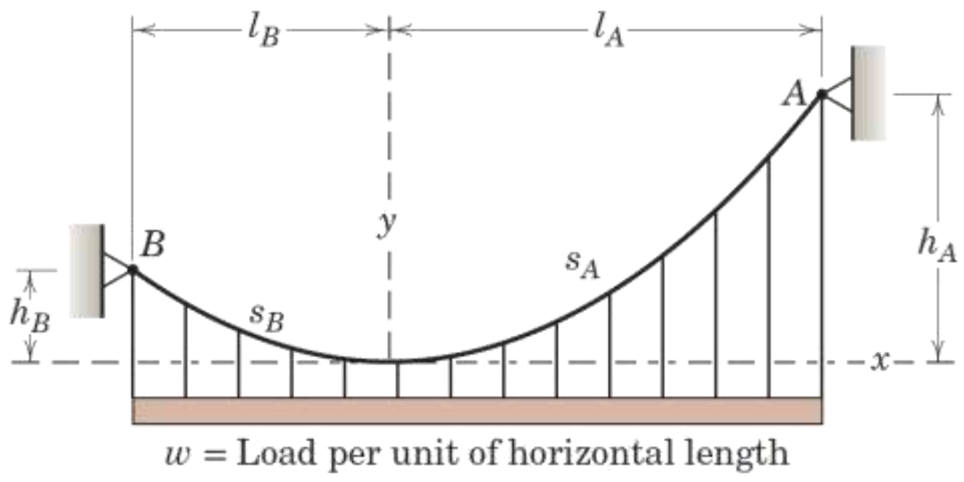
- Result
  - Horizontal component of the cable tension remains unchanged, so we introduce the variable $T_0 = T\cos\theta$ to represent it.
  - Substitute $T = T_0/\cos\theta$ into the first equation to obtain the differential equation for the flexible cable.
  - Note: $\tan\theta = dy/dx$

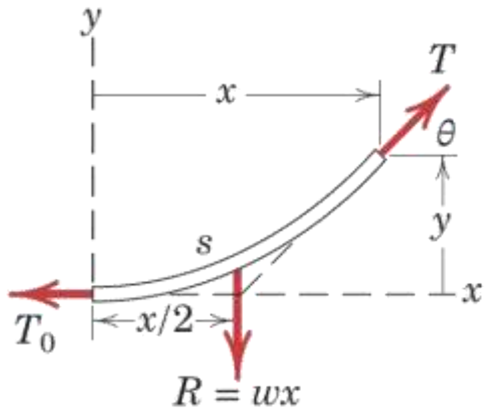


$$\boxed{\frac{d^2y}{dx^2} = \frac{w}{T_0}}$$

Article 5/8 – Parabolic Cables (1 of 3)

- Characteristics
  - Cable Weight is Negligible
  - Vertical Loading, *w*, is Constant
  - Span, *l*
  - Sag, *h*
  - Length, *s*
  - Shape is a Parabolic Arch
  - Coordinate Origin at Lowest Point
- Free-Body Diagram of a Cable Segment





Article 5/8 – Parabolic Cables (2 of 3)

- Integrate General Relationship Twice…

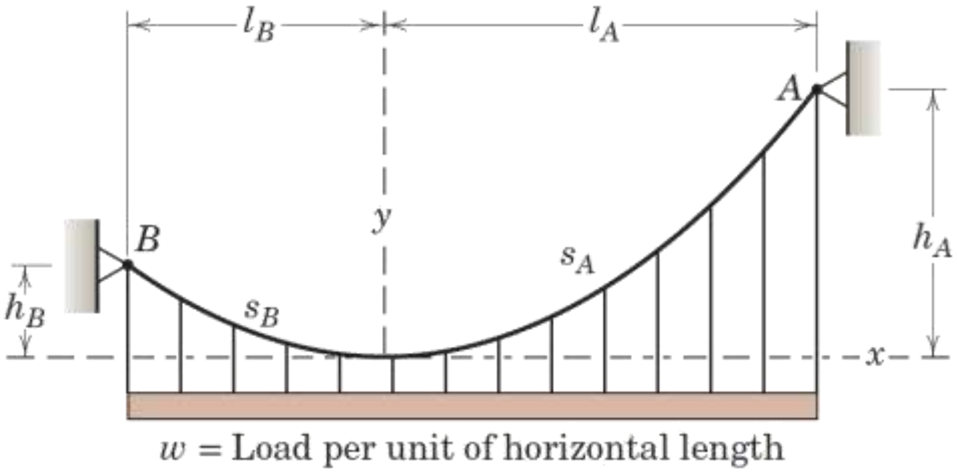
$$\boxed{y = \frac{wx^2}{2T_0}}$$

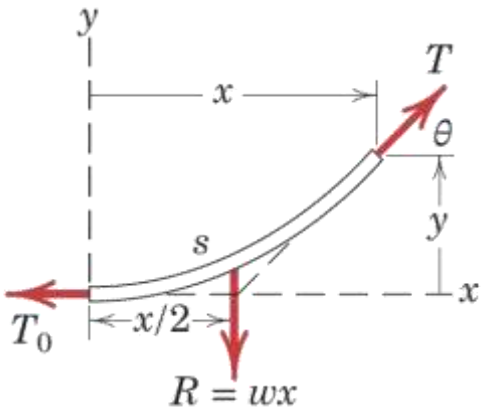
- Equations of Interest

$$T_0 = \frac{wl_A{}^2}{2h_A} = \frac{wl_B{}^2}{2h_B} \qquad \text{and} \qquad T = w\sqrt{x^2 + \left(\frac{l_A{}^2}{2h_A}\right)^2}$$

$$T_A = wl_A\sqrt{1 + \left(\frac{l_A{}^2}{2h_A}\right)^2} \qquad \text{and} \qquad T_B = wl_B\sqrt{1 + \left(\frac{l_B{}^2}{2h_B}\right)^2}$$

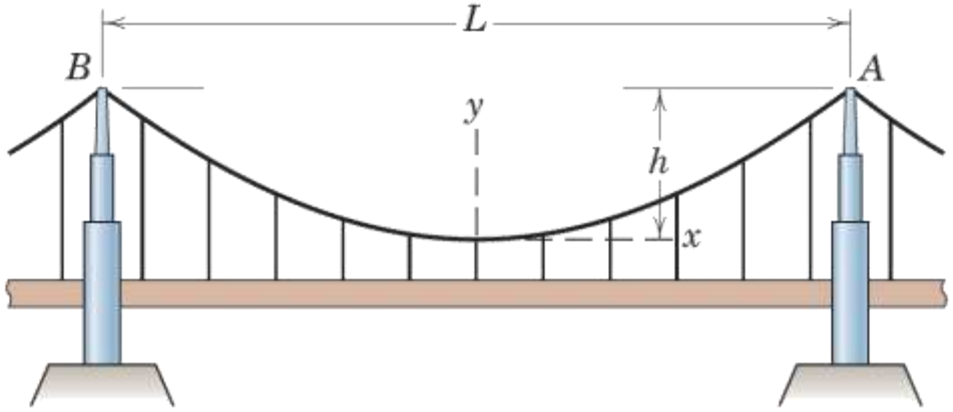
$$s_A = \int_0^{l_A}\sqrt{1 + \left(\frac{wx}{T_0}\right)^2}\,dx \qquad \text{and} \qquad s_B = \int_0^{l_B}\sqrt{1 + \left(\frac{wx}{T_0}\right)^2}\,dx$$





Article 5/8 – Parabolic Cables (3 of 3)

- Suspension Bridge
  - Total Span, $L = 2l_A$
  - Sag, $h = h_A$
  - Total Length, $S = 2s_A$



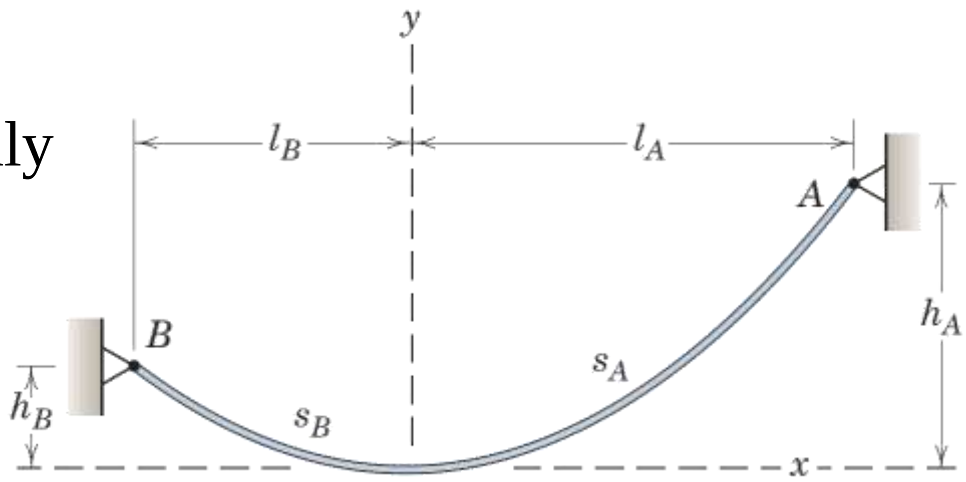
- Equations of Interest

$$T_{\max} = \frac{wL}{2}\sqrt{1 + (L/4h)^2}$$

$$S = L\left[1 + \frac{8}{3}\left(\frac{h}{L}\right)^2 - \frac{32}{5}\left(\frac{h}{L}\right)^4 + \cdots\right]$$

Article 5/8 – Catenary Cables (1 of 3)

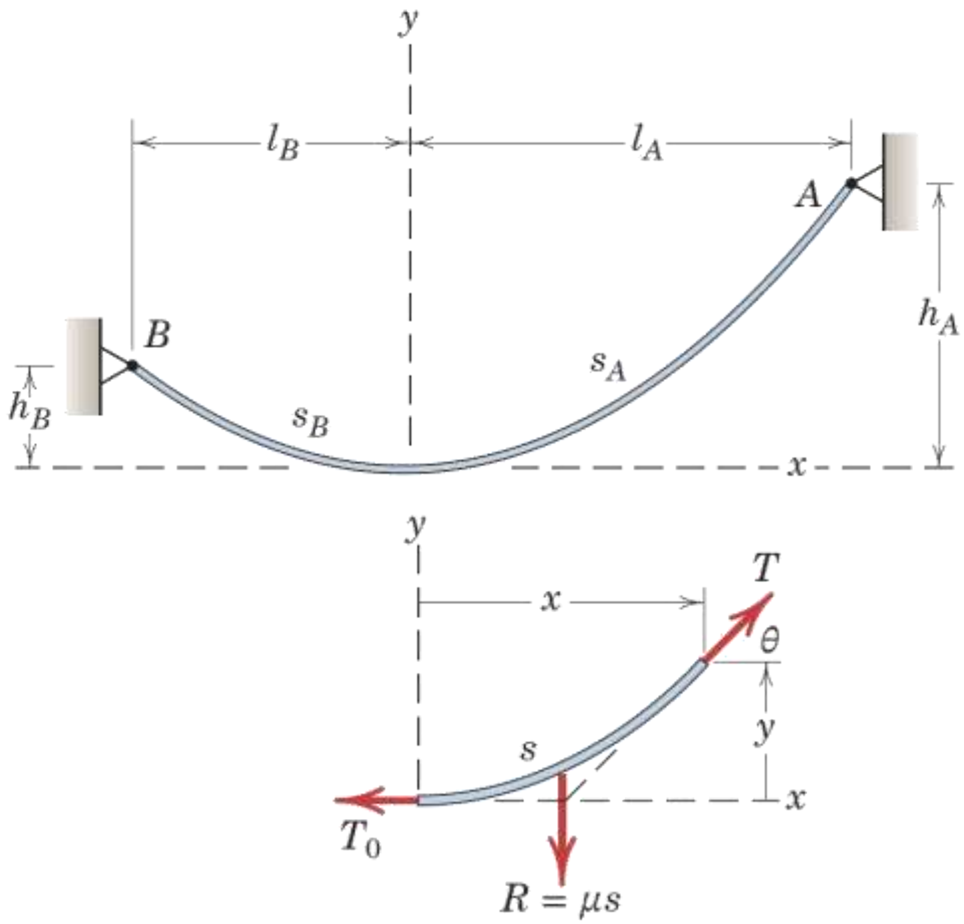
- Characteristics
  - Cable Hangs Under Action of its Weight Only
  - Cable Weight per Unit Length, μ
  - Span, l
  - Sag, h
  - Length, s
  - Shape is a Catenary
  - Coordinate Origin at Lowest Point



Article 5/8 – Catenary Cables (2 of 3)

- Free-Body Diagram of a Cable Segment
  - Resultant, $R = \mu s$
  - From Before, $w\,dx = \mu\,ds$
  - Substitute into the General Expression…

$$\boxed{\frac{d^2y}{dx^2} = \frac{w}{T_0}} \longrightarrow \boxed{\frac{d^2y}{dx^2} = \frac{\mu}{T_0}\frac{ds}{dx}}$$



Article 5/8 – Catenary Cables (3 of 3)

- Equations of Interest

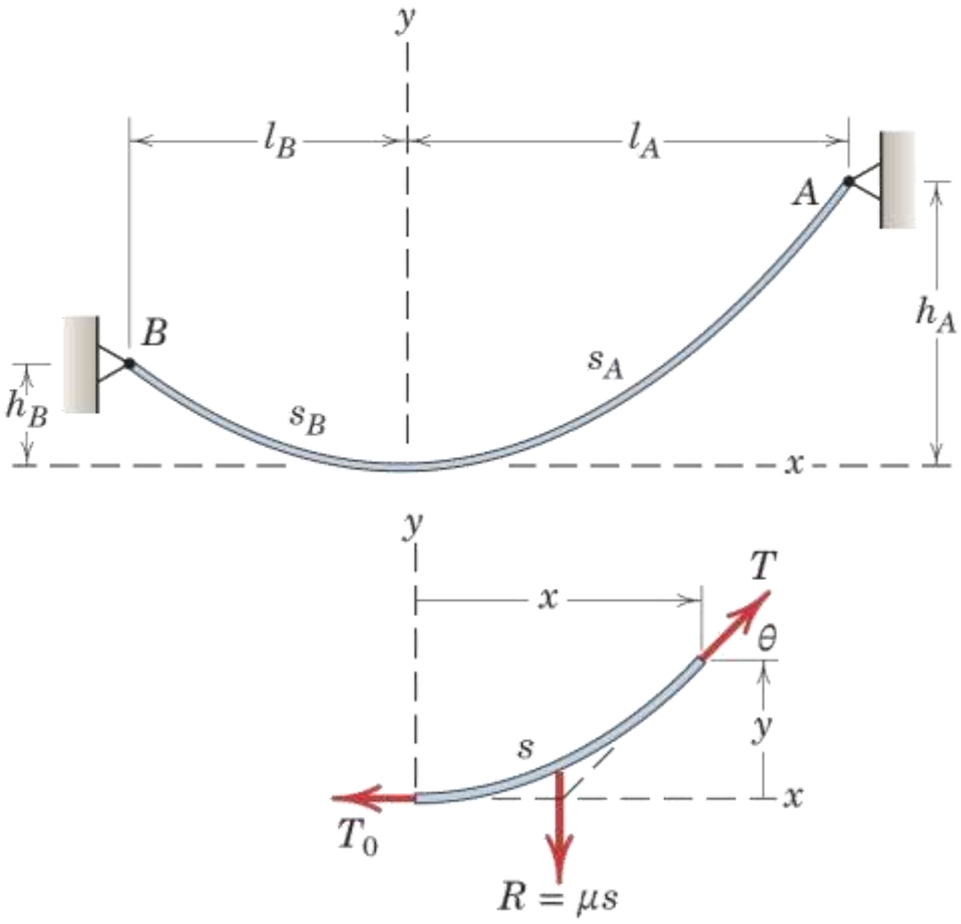
$$y = \frac{T_0}{\mu}\left(\cosh\frac{\mu x}{T_0} - 1\right)$$

$$s = \frac{T_0}{\mu}\sinh\frac{\mu x}{T_0}$$

$$T = T_0 \cosh\frac{\mu x}{T_0}$$

$$T = T_0 + \mu y$$

$$T^2 = \mu^2 s^2 + T_0^{\,2}$$

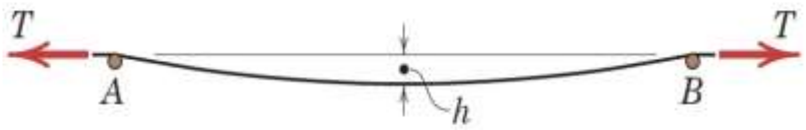


- Note: Catenary cable problems generally require numerical solutions!

Article 5/8 – Sample Problem 5/16 (1 of 2)

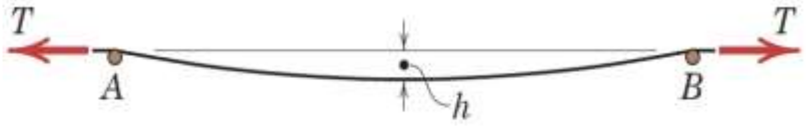
- Problem Statement

  A 100-ft length of surveyor's tape weighs 0.6 lb. When the tape is stretched between two points on the same level by a tension of 10 lb at each end, calculate the sag h in the middle.



Article 5/8 – Sample Problem 5/16 (2 of 2)

- Solution



The weight per unit length is $\mu = 0.6/100 = 0.006$ lb/ft. The total length is $2s = 100$ or $s = 50$ ft.

$$[T^2 = \mu^2 s^2 + T_0^2] \qquad 10^2 = (0.006)^2(50)^2 + T_0^2$$

$$T_0 = 9.995 \text{ lb} \quad ①$$

$$[T = T_0 + \mu y] \qquad 10 = 9.995 + 0.006h$$

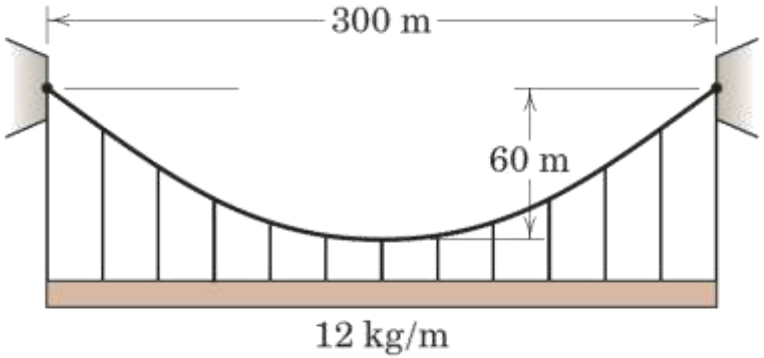
$$h = 0.750 \text{ ft} \quad \text{or} \quad 9.00 \text{ in.} \qquad \textit{Ans.}$$

① An extra significant figure is displayed here for clarity.

Article 5/8 – Sample Problem 5/17 (1 of 3)

- Problem Statement

  The light cable supports a mass of 12 kg per meter of horizontal length and is suspended between the two points on the same level 300 m apart. If the sag is 60 m, find the tension at midlength, the maximum tension, and the total length of the cable.



Article 5/8 – Sample Problem 5/17 (2 of 3)

- Midlength Tension

With a uniform horizontal distribution of load, the solution of part (*b*) of Art. 5/8 applies, and we have a parabolic shape for the cable. For $h = 60$ m, $L = 300$ m, and $w = 12(9.81)(10^{-3})$ kN/m, the relation following Eq. 5/14 with $l_A = L/2$ gives for the midlength tension

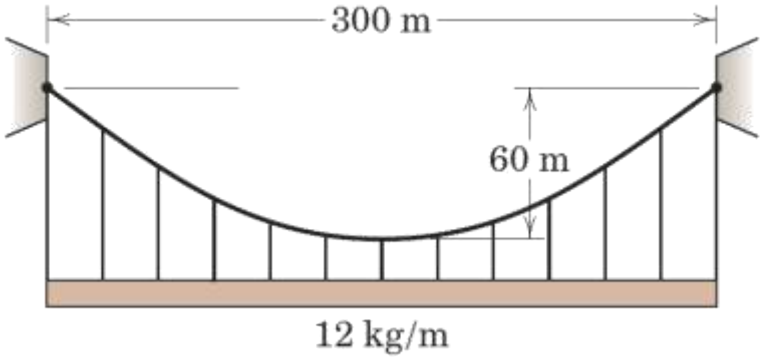
$$\left[T_0 = \frac{wL^2}{8h}\right] \qquad T_0 = \frac{0.1177(300)^2}{8(60)} = 22.1 \text{ kN} \qquad \textit{Ans.}$$

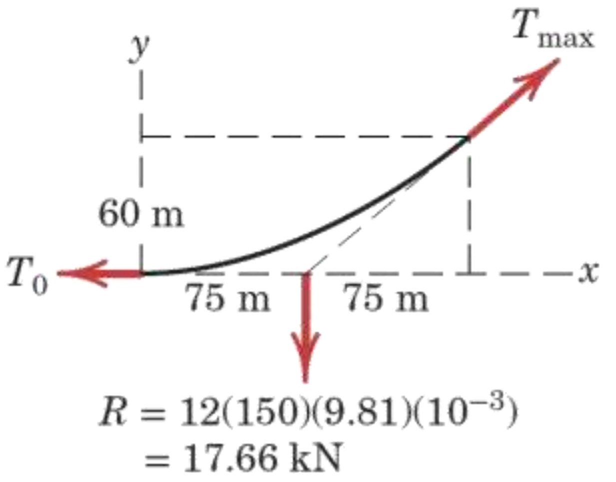
- Maximum Tension

The maximum tension occurs at the supports and is given by Eq. 5/15*b*. Thus,

$$\left[T_{max} = \frac{wL}{2}\sqrt{1 + \left(\frac{L}{4h}\right)^2}\right]$$

$$T_{max} = \frac{12(9.81)(10^{-3})(300)}{2}\sqrt{1 + \left(\frac{300}{4(60)}\right)^2} = 28.3 \text{ kN} \quad ① \quad \textit{Ans.}$$





① *Suggestion:* Check the value of T_{max} directly from the free-body diagram of the right-hand half of the cable, from which a force polygon may be drawn.

Article 5/8 – Sample Problem 5/17 (3 of 3)

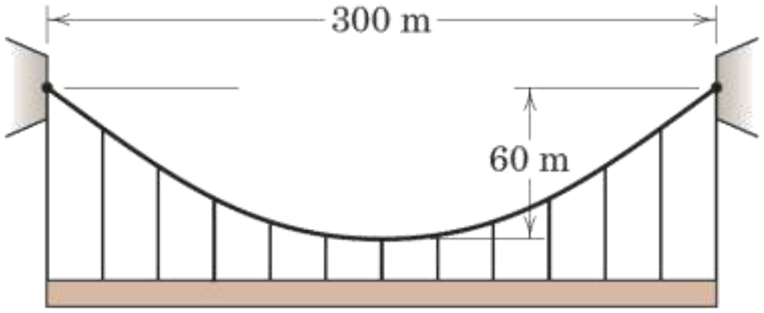
- Cable Length

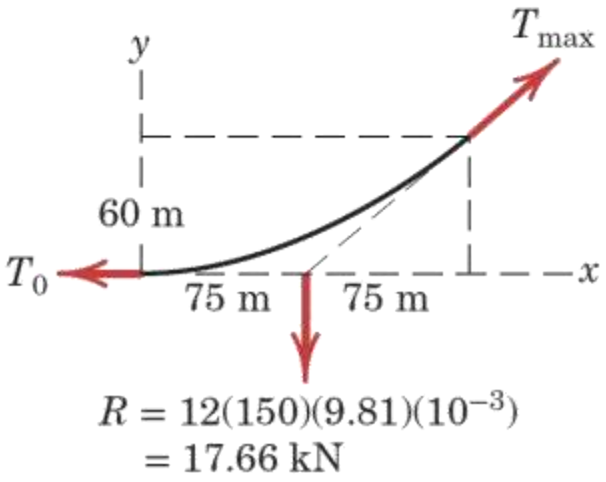
The sag-to-span ratio is 60/300 = 1/5 < 1/4. Therefore, the series expression developed in Eq. 5/16*a* is convergent, and we may write for the total length

$$S = 300\left[1 + \frac{8}{3}\left(\frac{1}{5}\right)^2 - \frac{32}{5}\left(\frac{1}{5}\right)^4 + \cdots\right]$$

$$= 300[1 + 0.1067 - 0.01024 + \cdots]$$

$$= 329 \text{ m} \qquad \textit{Ans.}$$

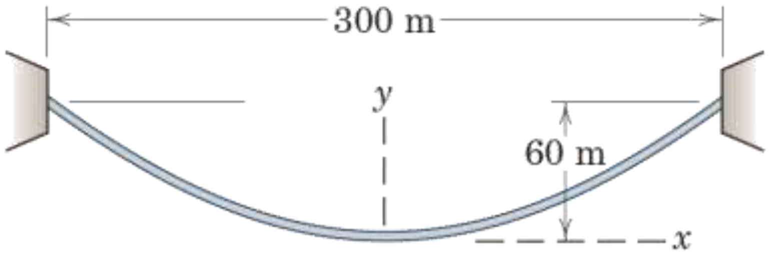




Article 5/8 – Sample Problem 5/18 (1 of 3)

- Problem Statement

  Replace the cable of Sample Problem 5/17, which is loaded uniformly along the horizontal, by a cable which has a mass of 12 kg per meter of its own length and supports its own weight only. The cable is suspended between two points on the same level 300 m apart and has a sag of 60 m. Find the tension at midlength, the maximum tension, and the total length of the cable.



Article 5/8 – Sample Problem 5/18 (2 of 3)

- Midlength Tension

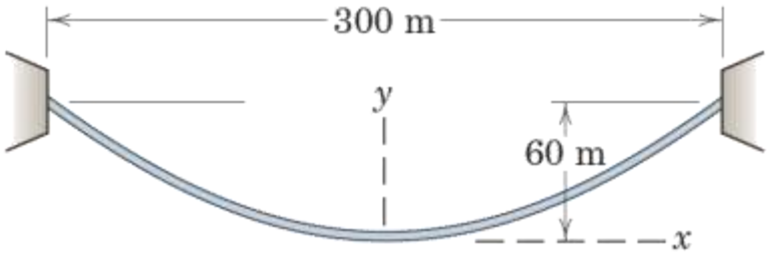
With a load distributed uniformly along the length of the cable, the solution of part (*c*) of Art. 5/8 applies, and we have a catenary shape of the cable. Equations 5/20 and 5/21 for the cable length and tension both involve the minimum tension T_0 at midlength, which must be found from Eq. 5/19. Thus, for $x = 150$ m, $y = 60$ m, and $\mu = 12(9.81)(10^{-3}) = 0.1177$ kN/m, we have

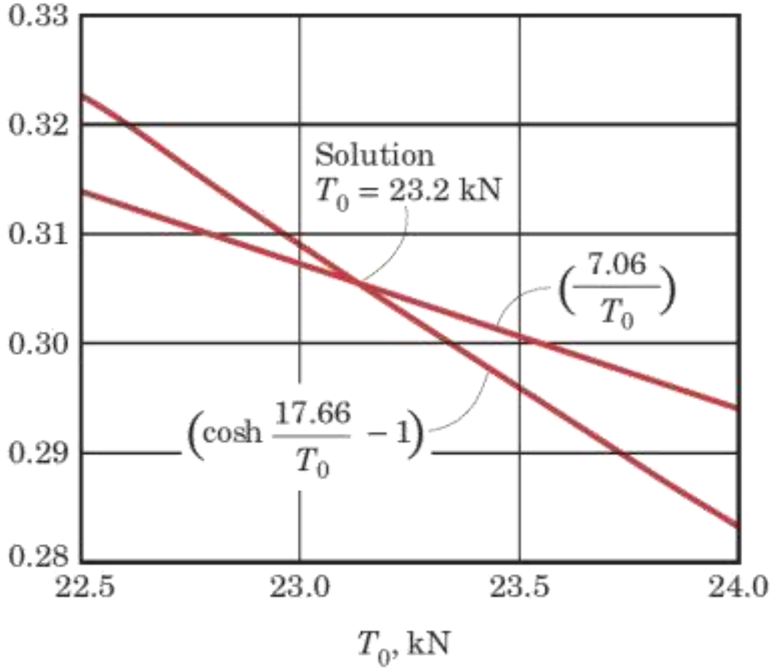
$$60 = \frac{T_0}{0.1177}\left[\cosh\frac{(0.1177)(150)}{T_0} - 1\right]$$

or
$$\frac{7.06}{T_0} = \cosh\frac{17.66}{T_0} - 1$$

This equation can be solved graphically. We compute the expression on each side of the equals sign and plot it as a function of T_0. The intersection of the two curves establishes the equality and determines the correct value of T_0. This plot is shown in the figure accompanying this problem and yields the solution

$$T_0 = 23.2 \text{ kN}$$



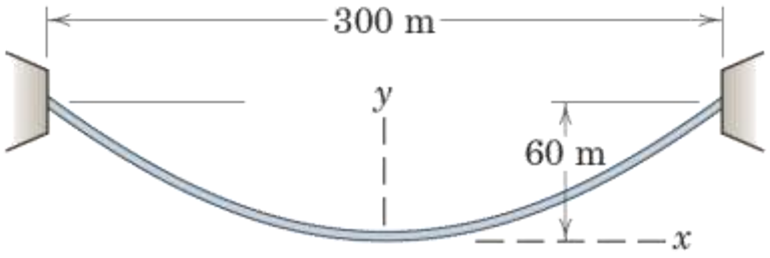


Article 5/8 – Sample Problem 5/18 (3 of 3)

- Maximum Tension

The maximum tension occurs for maximum y and from Eq. 5/22 is

$$T_{max} = 23.2 + (0.1177)(60) = 30.2 \text{ kN} \qquad \textit{Ans.}$$



- Cable Length

From Eq. 5/20 the total length of the cable becomes

$$2s = 2\frac{23.2}{0.1177}\sinh\frac{(0.1177)(150)}{23.2} = 330 \text{ m} \quad ① \qquad \textit{Ans.}$$

① Note that the solution of Sample Problem 5/17 for the parabolic cable gives a very close approximation to the values for the catenary even though we have a fairly large sag. The approximation is even better for smaller sag-to-span ratios.

Article 5/9 Fluid Statics

- Introduction

- Important Concepts
  - A fluid is any continuous substance which, when at rest, is unable to support a shear force. Fluids are either gaseous or liquid.
  - A shear force is any force which is tangent to the surface on which it acts.
  - Fluids at rest can only exert normal forces on a bounding surface.
  - The statics of fluids is called *hydrostatics* when the fluid is a liquid and *aerostatics* when the fluid is a gas.

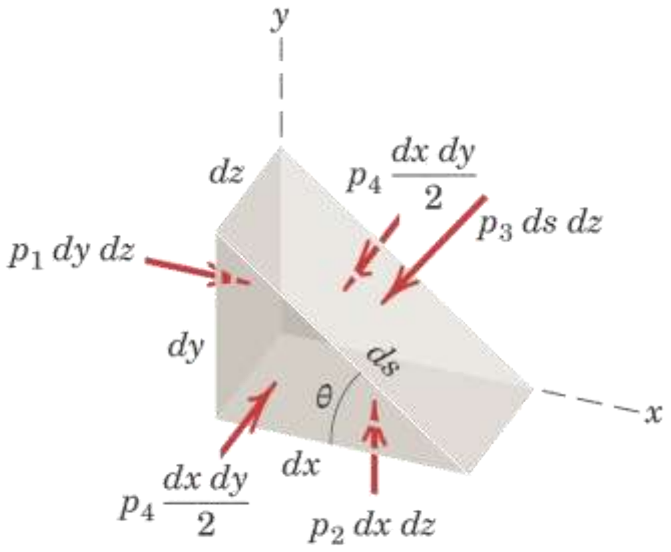
Article 5/9 – Fluid Pressure at a Point (1 of 3)

- Pascal's Law
  - The pressure at any given point in a fluid is the same in all directions.

- Equilibrium of a Fluid Prism

$$\Sigma F_x = 0: \quad p_1 dy\ dz - p_3 ds\ dz \sin\theta = 0$$

$$\Sigma F_y = 0: \quad p_2 dx\ dz - p_3 ds\ dz \cos\theta = 0$$



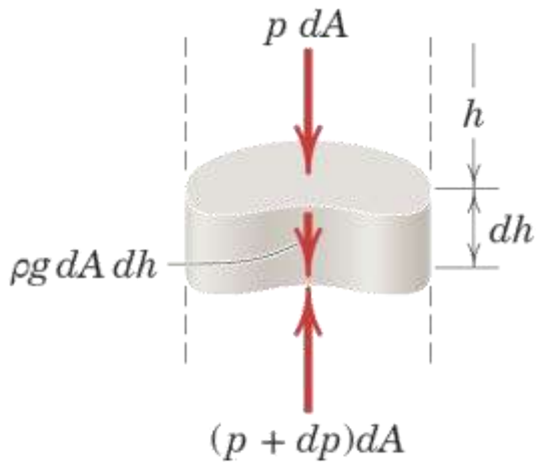
- Since $ds \sin\theta = dy$ and $ds \cos\theta = dx$…

$$p_1 = p_2 = p_3 = p$$

- Rotating the element 90° would show the same thing for p_4.

Article 5/9 – Fluid Pressure at a Point (2 of 3)

- Effect of Depth on Pressure in a Fluid at Rest



- Vertical Equilibrium of a Fluid Slug, +*h* is Down

$$\Sigma F_h = 0: \quad \rho g \; dh \; dA + p \; dA - (p + dp) dA = 0$$

$$dp = \rho g \; dh$$

- Generalized Result: $p = p_0 + \rho g h$
  - p = the pressure at some depth h in a fluid
  - p_0 = the pressure on the surface of the fluid
  - ρ = the density of the fluid, constant if the fluid is incompressible
  - g = the acceleration of gravity, constant if the fluid depth is small
  - h = the depth below the surface of the fluid

Article 5/9 – Atmospheric, Absolute, and Gage Pressure

- Atmospheric Pressure
  - Represented by the weight of a column of air which extends from the surface of the earth to the top of the atmosphere.
  - Standard value is 14.7 lb/in.^{2} or 101.3 kPa.
  - In many problems, p_0 is atmospheric pressure.

- Gage Pressure
  - Is a measurement of pressure which is made relative to atmospheric pressure.
  - Gage pressure in fluid statics is computed as ρgh.

- Absolute Pressure
  - Is the sum of atmospheric and gage pressure.
  - In fluid statics, absolute pressure is computed as $p_0 + \rho gh$.

Article 5/9 – Units of Pressure

- Pressure is a force divided by an area.

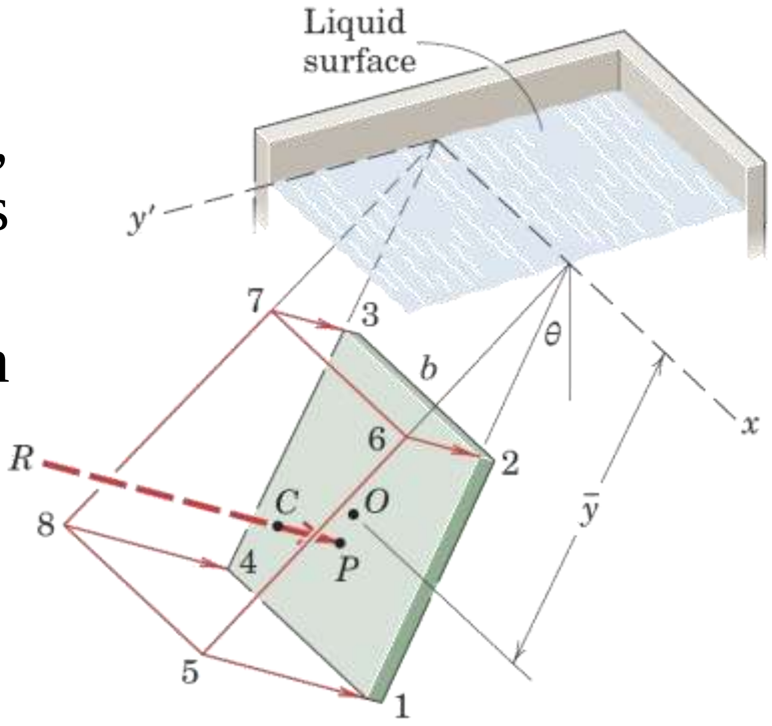
- SI Units
  - The most common unit is the Pascal (Pa), which is a newton per square meter.
  - $Pa = N/m^2$
  - kilopascal (kPa) = 10^3 Pa

- U.S. Units
  - The most common unit is a pound per square inch (psi or lb/in.^{2}) or a pound per square foot (psf or lb/ft^2).

- Conversion: 1 lb/in.^{2} = 6895 Pa

Article 5/9 – Hydrostatic Pressure on Submerged Rectangular Surfaces (1 of 4)

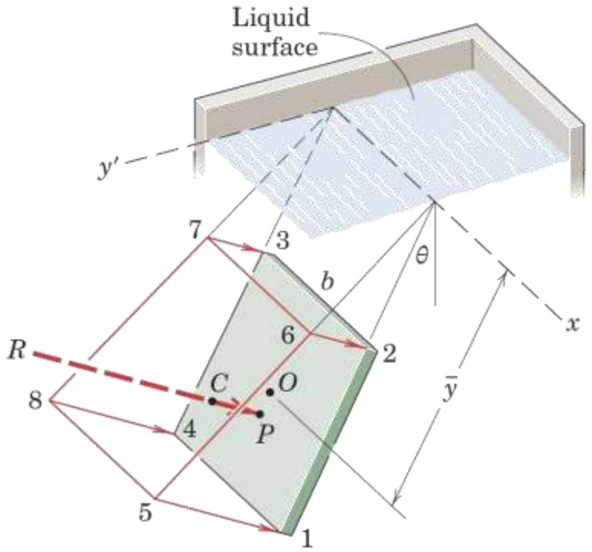
- Pressure
  - Varies linearly with depth according to ρgh, where the pressure at the liquid surface is atmospheric.
  - The pressure forms a trapezoidal prism which acts over the surface of the submerged plate.

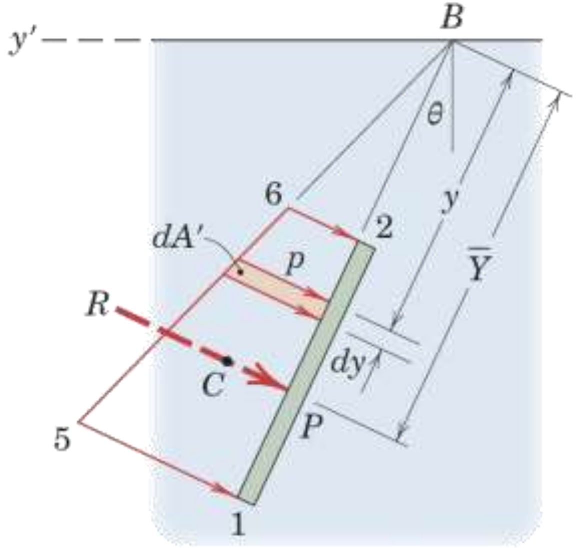


- Resultant R
  - Equal to the volume of the pressure prism.
  - Acts at the center of pressure P, which is the centroid of the pressure prism, not the centroid of the plate.

Article 5/9 – Hydrostatic Pressure on Submerged Rectangular Surfaces (2 of 4)

- Calculation of R – Method 1
  - $R = bA'$
    - b = plate width normal to the plane of the figure.
    - A' = geometrical area defined by the trapezoidal distribution of pressure, 1-2-6-5.
    - Acts at the center of pressure P.
- Calculation of R – Method 2
  - $R = p_{av}A = \rho g\bar{h}A$
    - p_{av} = average pressure acting on the plate, ½ $(p_1 + p_2)$, or the pressure which exists at the average depth, measured to the centroid O of the plate.
    - A = area of the plate.
    - $\bar{h} = \bar{y}\cos\theta$, is the average depth of the plate, which is measured to the plate centroid O.
    - Acts at the center of pressure P.

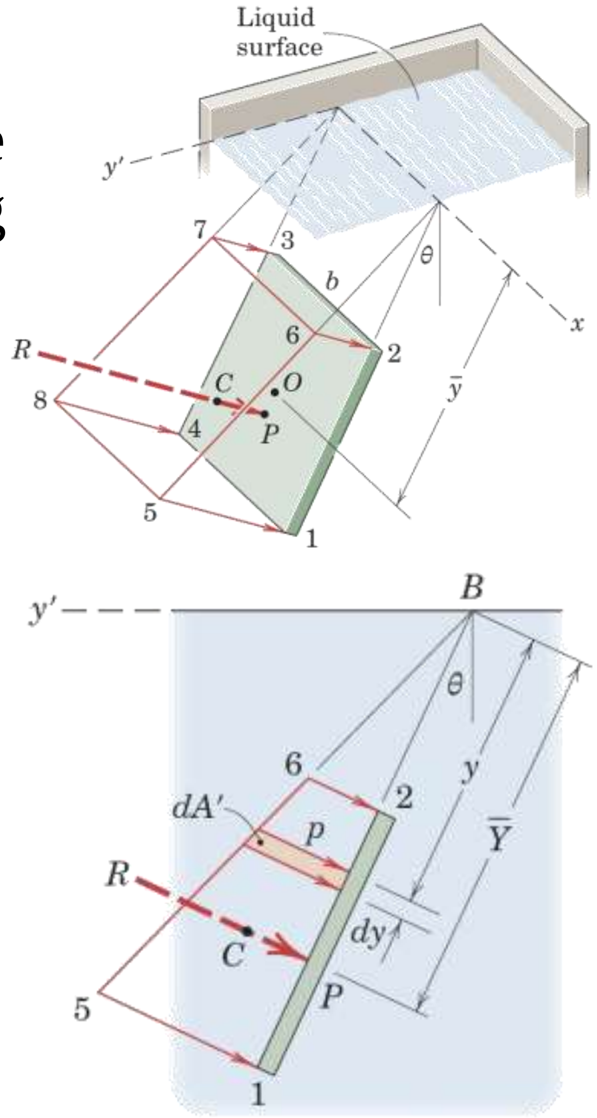




Article 5/9 – Hydrostatic Pressure on Submerged Rectangular Surfaces (3 of 4)

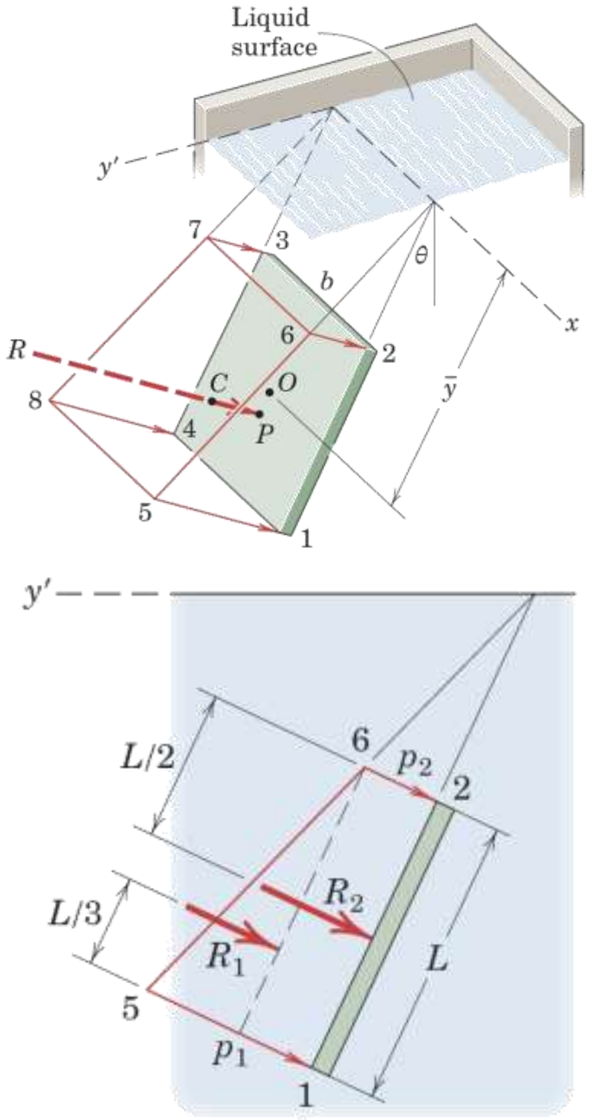
- Center of Pressure Location
  - The center of pressure P is located at the centroid of the trapezoidal pressure prism, which is easily found using composite techniques or evaluating the integral…

$$\overline{Y} = \frac{\int y \, dA'}{\int dA'}$$



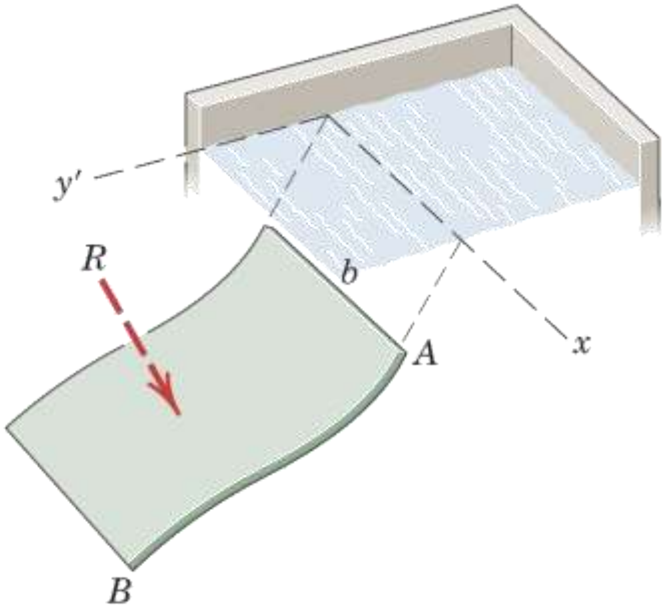
Article 5/9 – Hydrostatic Pressure on Submerged Rectangular Surfaces (4 of 4)

- Calculation of R – Method 3
  - $R_1 = ½ (p_1 - p_2)A$
  - Acts at the centroid of the triangular portion of the pressure prism.
  - $A = bL$ is the area of the plate
  - $R_2 = p_2A$
  - Acts at the centroid of the rectangular portion of the pressure prism.



Article 5/9 – Hydrostatic Pressure on Submerged Cylindrical Surfaces (1 of 2)

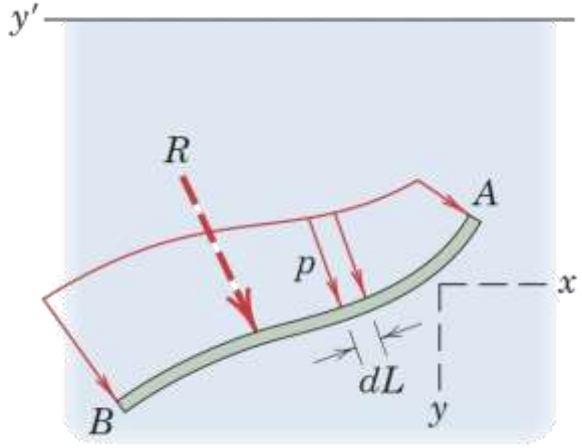
- Illustration



- Two-Dimensional Representation

- Calculation Procedure – Method 1

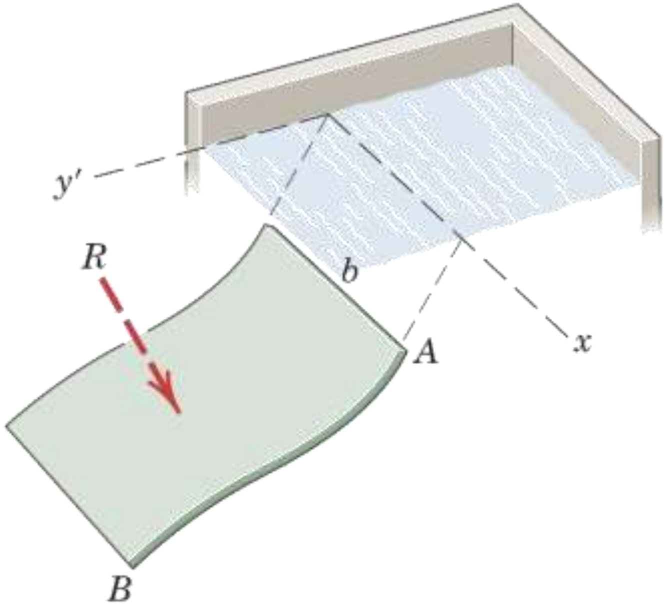
$$R_x = b\int (p\,dL)_x = b\int p\,dy \qquad \text{and} \qquad R_y = b\int (p\,dL)_y = b\int p\,dx$$

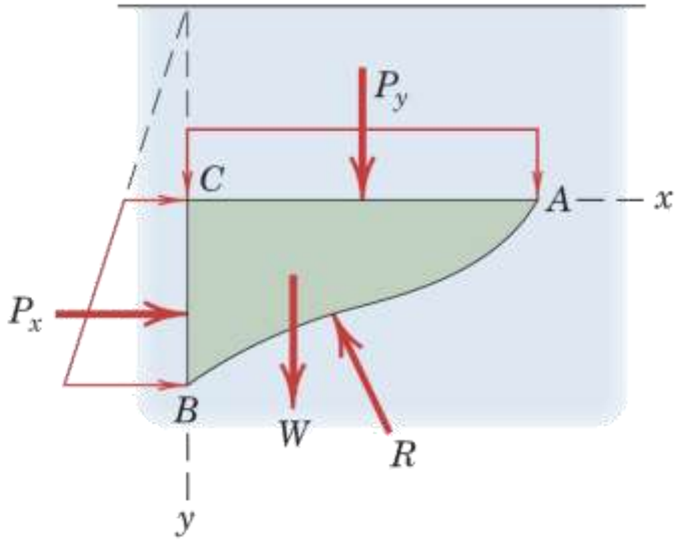


- Moment Equation will give Location

Article 5/9 – Hydrostatic Pressure on Submerged Cylindrical Surfaces (2 of 2)

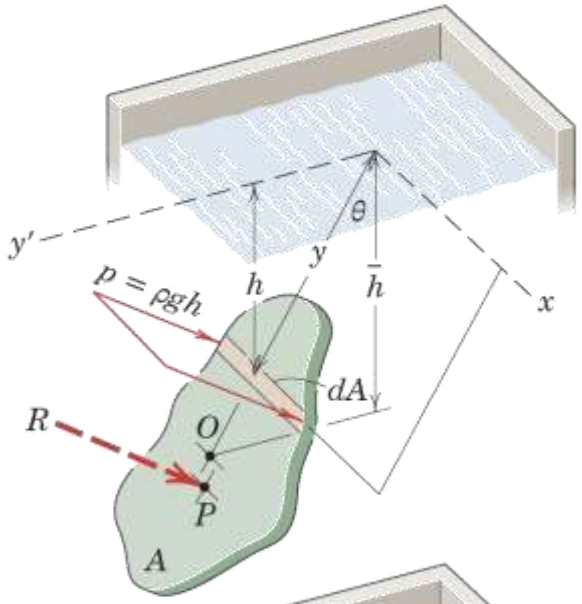
- Calculation Procedure – Method 2
  - P_x = pressure along *CB* and can be found using the techniques established for flat rectangular plates.
  - P_y = pressure along *AC* and can be found using the techniques established for flat rectangular plates.
  - *W* = weight of the fluid block and is the area *ABC* times the plate width *b*. The weight passes through the centroid of area *ABC*.

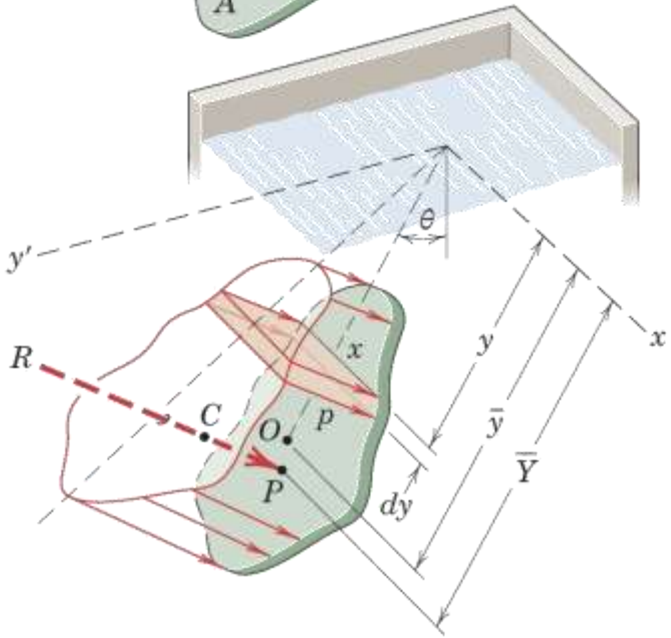




Article 5/9 – Hydrostatic Pressure on Submerged Irregular Flat Surfaces (1 of 2)

- Illustration





- Resultant
  - $R = \int \rho g h x \, dy$
  - $R = \rho g \bar{h} A$
  - A = area of the plate.
  - $\bar{h} = \bar{y} \cos \theta$, is the average depth of the plate, which is measured to the plate centroid O.

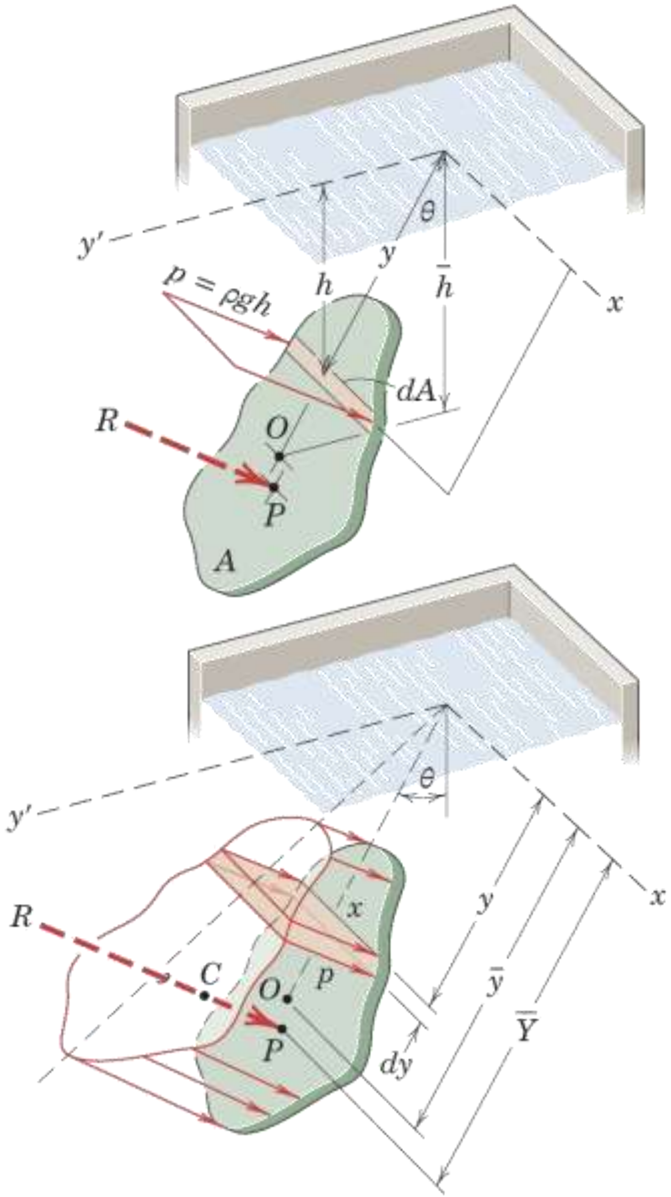
- Location
  - Acts at the center of pressure P which is the centroid of the irregular pressure volume.

Article 5/9 – Hydrostatic Pressure on Submerged Irregular Flat Surfaces (2 of 2)

- Center of Pressure
  - The center of pressure *P* is located at the centroid of the trapezoidal pressure prism, which is easily found using composite techniques or evaluating the integral…

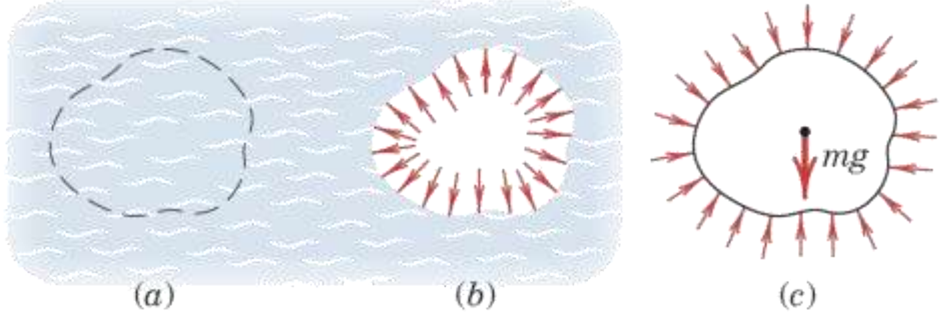
$$\overline{Y} = \frac{\int y(px\,dy)}{\int px\,dy}$$

  - Note that the pressure $p = \rho g h = \rho g y \cos\theta$.
  - Also note that the center of pressure *P* and the centroid of the plate *O* are not the same.



Article 5/9 – Buoyancy (1 of 4)

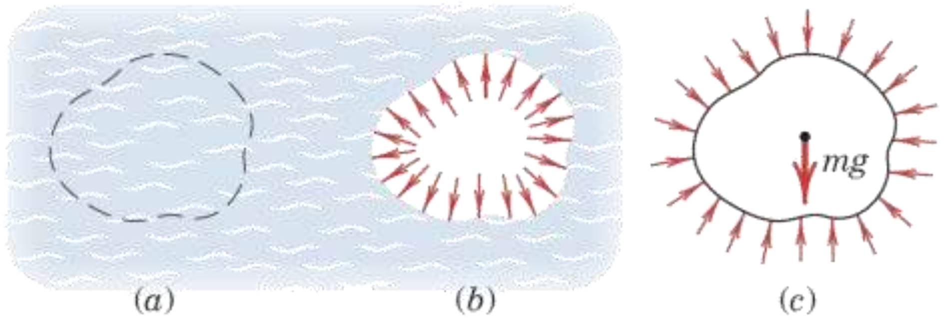
- Illustration



- Description
  - Fluid Portion (*a*)
  - Replaced Fluid Portion Force Distribution (*b*)
  - Free-Body Diagram of Fluid Portion (*c*)
  - Equilibrium Requirements

Article 5/9 – Buoyancy (2 of 4)

- Buoyant Force, $F = \rho g V$
  - ρ = density of the fluid
  - g = acceleration of gravity
  - V = volume of the displaced fluid

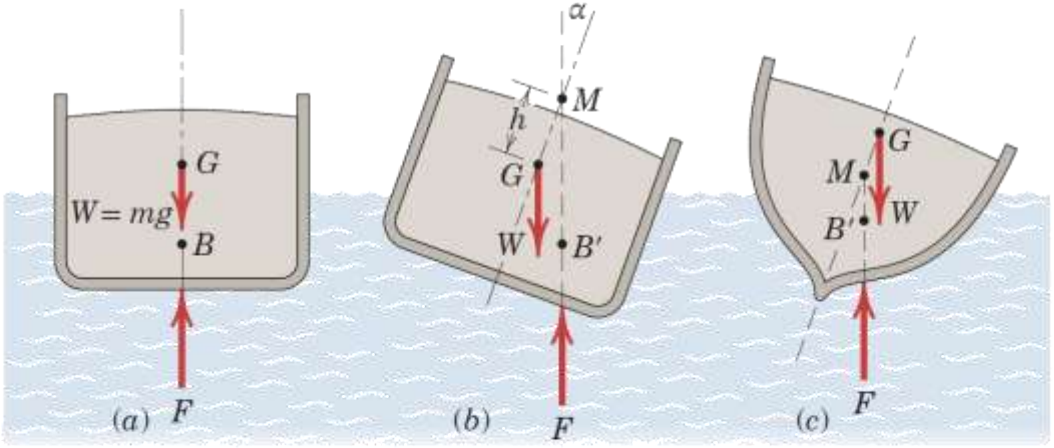


- The buoyant force is equal, opposite, and collinear with the weight of the fluid portion.

- For objects at rest in a fluid, the buoyant force will be equal in magnitude to the weight of the fluid displaced by the object.

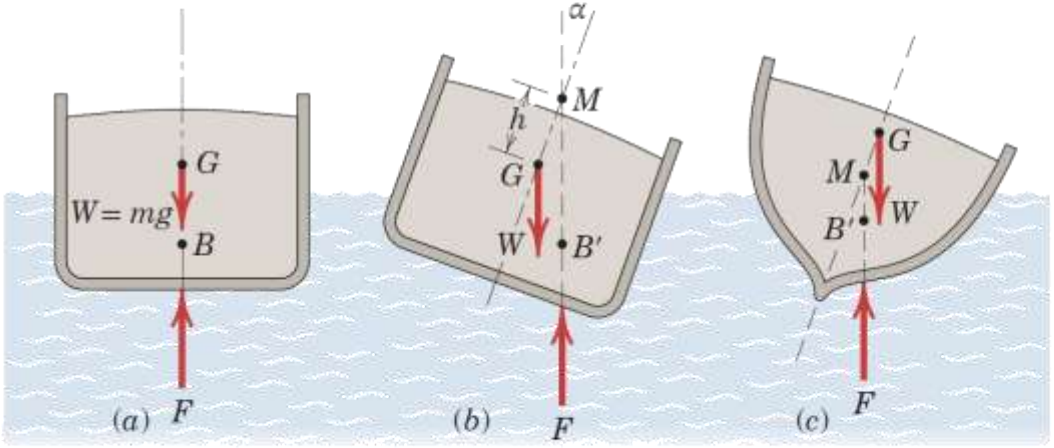
Article 5/9 – Buoyancy (3 of 4)

- Stability of a Floating Object
  - B = center of buoyancy, which is the centroid of displaced fluid volume.
  - The buoyant force F will pass through the center of buoyancy, and is equal and opposite to the weight of the ship, as shown in Figure (*a*).
  - If the ship lists through an angle α, the shape of the displaced volume will change, and the center of buoyancy will shift to B'.



Article 5/9 – Buoyancy (4 of 4)

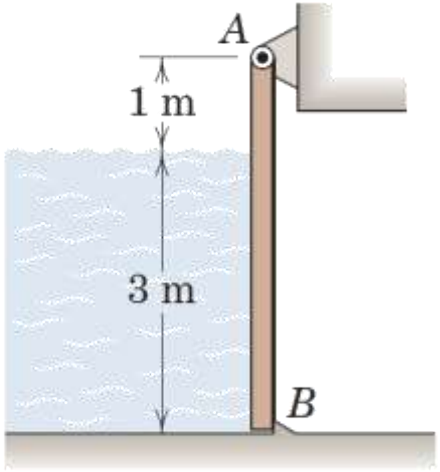
- Stability of a Floating Object (cont.)
  - Metacenter *M* is the intersection of a vertical line through the center of buoyancy and the ship centerline.
  - Metacentric height *h* is the distance of the metacenter from the mass center of the ship. For most hull shapes *h* is roughly constant for lists up to 20°.
  - If *M* is above *G*, the buoyant force creates a restoring moment against listing.
  - If *M* is below *G*, the buoyant force will cause the list to increase.



Article 5/9 – Sample Problem 5/19 (1 of 2)

- Problem Statement

  A rectangular plate, shown in vertical section *AB*, is 4 m high and 6 m wide (normal to the plane of the paper) and blocks the end of a fresh-water channel 3 m deep. The plate is hinged about a horizontal axis along its upper edge through *A* and is restrained from opening by the fixed ridge *B* which bears horizontally against the lower edge of the plate. Find the force *B* exerted on the plate by the ridge.



Article 5/9 – Sample Problem 5/19 (2 of 2)

- Solution

The free-body diagram of the plate is shown in section and includes the vertical and horizontal components of the force at A, the unspecified weight $W = mg$ of the plate, the unknown horizontal force B, and the resultant R of the triangular distribution of pressure against the vertical face.

The density of fresh water is $\rho = 1.000\ \mathrm{Mg/m^3}$ so that the average pressure is

$$[p_{av} = \rho g\bar{h}] \qquad p_{av} = 1.000(9.81)(\tfrac{3}{2}) = 14.72\ \mathrm{kPa} \quad ①$$

The resultant R of the pressure forces against the plate becomes

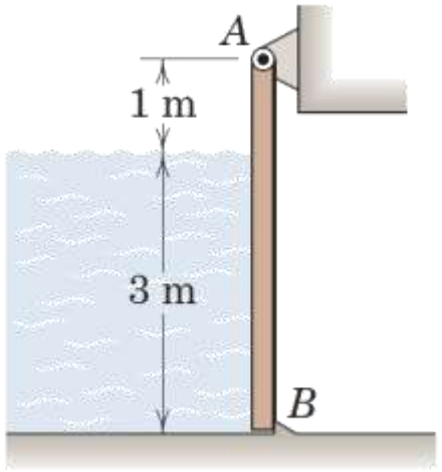
$$[R = p_{av}A] \qquad R = (14.72)(3)(6) = 265\ \mathrm{kN}$$

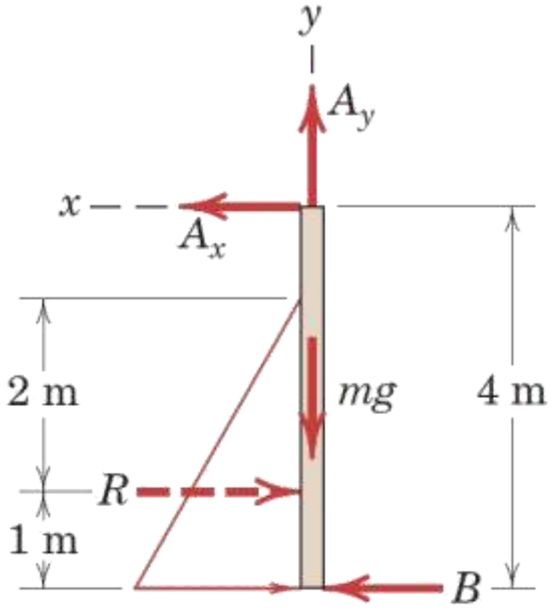
This force acts through the centroid of the triangular distribution of pressure, which is 1 m above the bottom of the plate. A zero moment summation about A establishes the unknown force B. Thus,

$$[\Sigma M_A = 0] \qquad 3(265) - 4B = 0 \qquad B = 198.7\ \mathrm{kN} \qquad \textit{Ans.}$$

① Note that the units of pressure ρgh are

$$\left(10^3\frac{\mathrm{kg}}{\mathrm{m^3}}\right)\left(\frac{\mathrm{m}}{\mathrm{s^2}}\right)(\mathrm{m}) = \left(10^3\frac{\mathrm{kg\cdot m}}{\mathrm{s^2}}\right)\left(\frac{1}{\mathrm{m^2}}\right)$$
$$= \mathrm{kN/m^2} = \mathrm{kPa}.$$

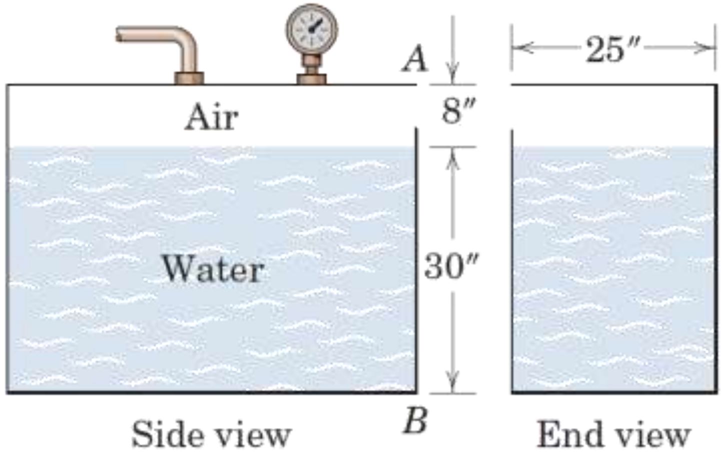




Article 5/9 – Sample Problem 5/20 (1 of 2)

- Problem Statement

  The air space in the closed fresh-water tank is maintained at a pressure of 0.80 lb/in.^{2} (above atmospheric). Determine the resultant force R exerted by the air and water on the end of the tank.



Article 5/9 – Sample Problem 5/20 (2 of 2)

- Solution

The pressure distribution on the end surface is shown, where $p_0 = 0.80$ lb/in.^{2} The specific weight of fresh water is $\mu = \rho g = 62.4/1728 = 0.0361$ lb/in.^{3} so that the increment of pressure Δp due to the water is

$$\Delta p = \mu\,\Delta h = 0.0361(30) = 1.083 \text{ lb/in.}^2$$

The resultant forces R_1 and R_2 due to the rectangular and triangular distributions of pressure, respectively, are ①

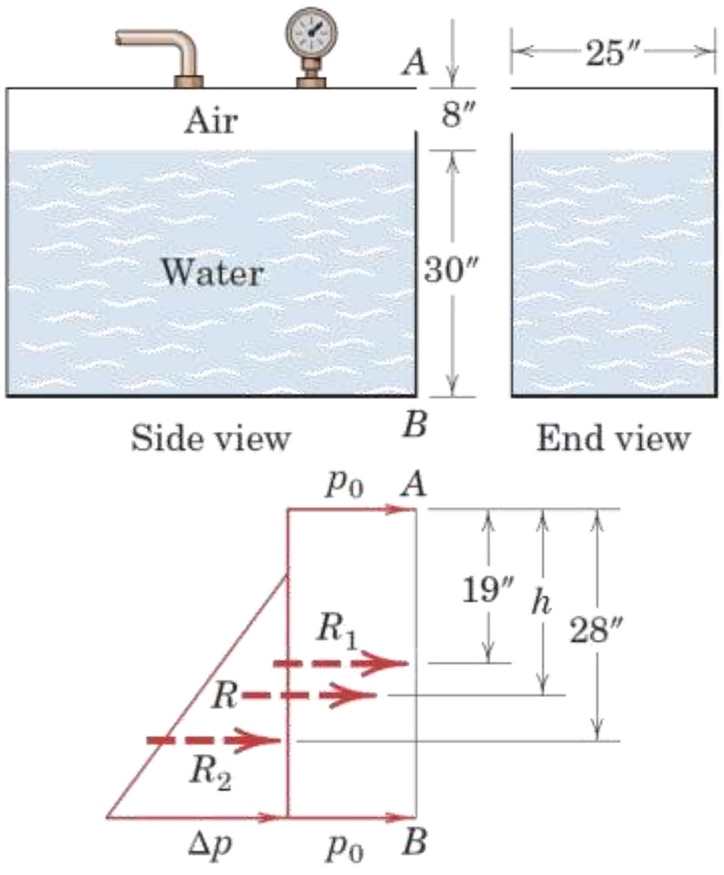
$$R_1 = p_0 A_1 = 0.80(38)(25) = 760 \text{ lb}$$

$$R_2 = \Delta p_{av} A_2 = \frac{1.083}{2}(30)(25) = 406 \text{ lb}$$

The resultant is then $R = R_1 + R_2 = 760 + 406 = 1166$ lb. *Ans.*

We locate R by applying the moment principle about A noting that R_1 acts through the center of the 38-in. depth and that R_2 acts through the centroid of the triangular pressure distribution 20 in. below the surface of the water and $20 + 8 = 28$ in. below A. Thus,

$[Rh = \Sigma M_A]$ $\quad 1166h = 760(19) + 406(28) \quad h = 22.1$ in. *Ans.*

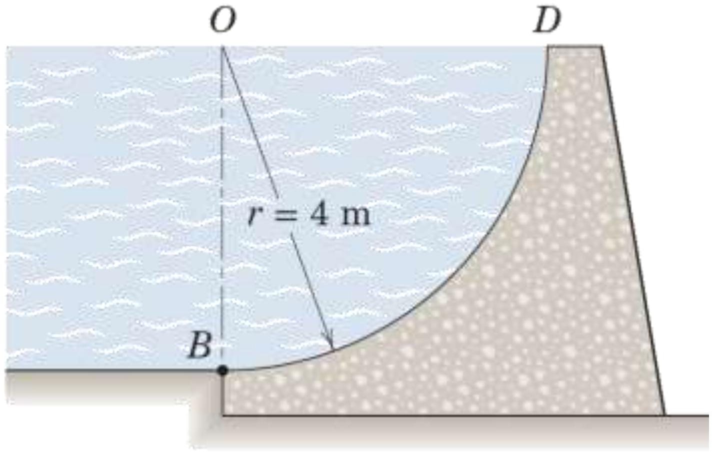


① Dividing the pressure distribution into these two parts is decidedly the simplest way in which to make the calculation.

Article 5/9 – Sample Problem 5/21 (1 of 3)

- Problem Statement

  Determine completely the resultant force *R* exerted on the cylindrical dam surface by the water. The density of fresh water is 1.000 Mg/m^3, and the dam has a length *b*, normal to the paper, of 30 m.



Article 5/9 – Sample Problem 5/21 (2 of 3)

- Solution I

The circular block of water BDO is isolated and its free-body diagram is drawn. The force P_x is

$$P_x = \rho g \bar{h} A = \frac{\rho g r}{2} br = \frac{(1.000)(9.81)(4)}{2}(30)(4) = 2350 \text{ kN} \quad ①$$

The weight W of the water passes through the mass center G of the quarter-circular section and is

$$mg = \rho g V = (1.000)(9.81)\frac{\pi(4)^2}{4}(30) = 3700 \text{ kN}$$

Equilibrium of the section of water requires

$[\Sigma F_x = 0]$ $\qquad R_x = P_x = 2350$ kN

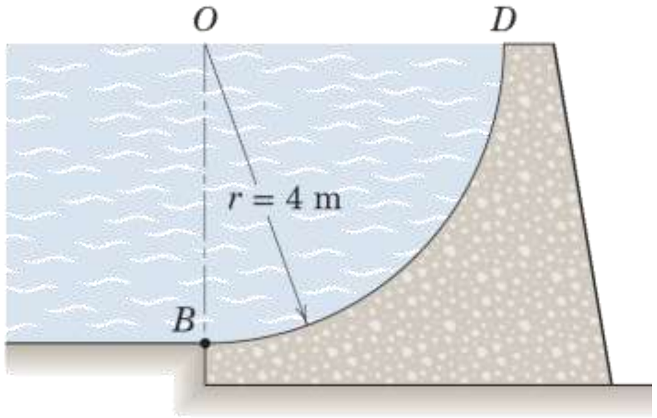
$[\Sigma F_y = 0]$ $\qquad R_y = mg = 3700$ kN

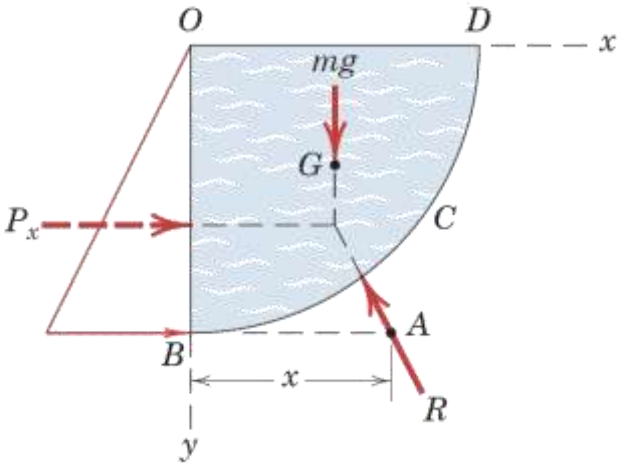
The resultant force R exerted by the fluid on the dam is equal and opposite to that shown acting on the fluid and is

$[R = \sqrt{R_x^{\,2} + R_y^{\,2}}]$ $\qquad R = \sqrt{(2350)^2 + (3700)^2} = 4380$ kN $\qquad$ *Ans.*

The x-coordinate of the point A through which R passes may be found from the principle of moments. Using B as a moment center gives

$$P_x \frac{r}{3} + mg\frac{4r}{3\pi} - R_y x = 0, \quad x = \frac{2350\left(\frac{4}{3}\right) + 3700\left(\frac{16}{3\pi}\right)}{3700} = 2.55 \text{ m} \quad \textit{Ans.}$$





① See note ① in Sample Problem 5/19 if there is any question about the units for $\rho g \bar{h}$.

Article 5/9 – Sample Problem 5/21 (3 of 3)

- Alternative Solution

The force acting on the dam surface may be obtained by a direct integration of the components ②

$$dR_x = p\,dA\cos\theta \qquad \text{and} \qquad dR_y = p\,dA\sin\theta$$

where $p = \rho gh = \rho gr\sin\theta$ and $dA = b(r\,d\theta)$. Thus,

$$R_x = \int_0^{\pi/2} \rho gr^2 b\sin\theta\cos\theta\,d\theta = -\rho gr^2 b\left[\frac{\cos 2\theta}{4}\right]_0^{\pi/2} = \tfrac{1}{2}\rho gr^2 b$$

$$R_y = \int_0^{\pi/2} \rho gr^2 b\sin^2\theta\,d\theta = \rho gr^2 b\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = \tfrac{1}{4}\pi\rho gr^2 b$$

Thus, $R = \sqrt{R_x{}^2 + R_y{}^2} = \frac{1}{2}\rho gr^2 b\sqrt{1+\pi^2/4}$. Substituting the numerical values gives

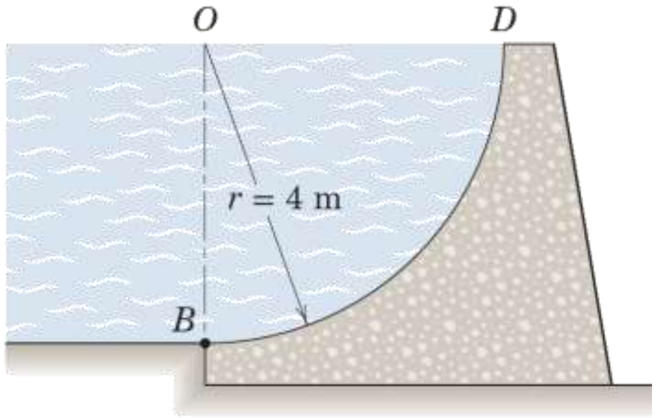
$$R = \tfrac{1}{2}(1.000)(9.81)(4^2)(30)\sqrt{1+\pi^2/4} = 4380 \text{ kN} \qquad \textit{Ans.}$$

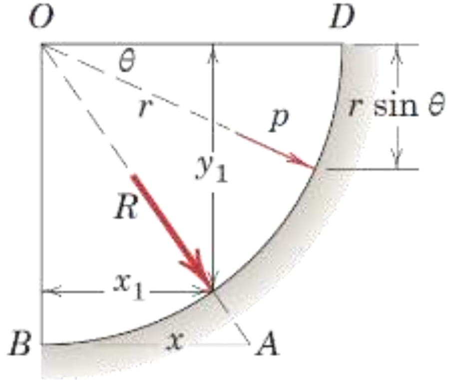
Since dR always passes through point O, we see that R also passes through O and, therefore, the moments of R_x and R_y about O must cancel. So we write $R_x y_1 = R_y x_1$, which gives us

$$x_1/y_1 = R_x/R_y = (\tfrac{1}{2}\rho gr^2 b)/(\tfrac{1}{4}\pi\rho gr^2 b) = 2/\pi$$

By similar triangles we see that

$$x/r = x_1/y_1 = 2/\pi \qquad \text{and} \qquad x = 2r/\pi = 2(4)/\pi = 2.55 \text{ m} \qquad \textit{Ans.}$$



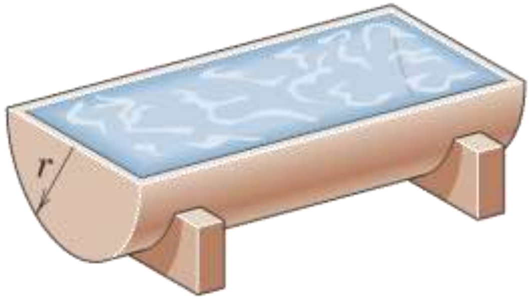


② This approach by integration is feasible here mainly because of the simple geometry of the circular arc.

Article 5/9 – Sample Problem 5/22 (1 of 3)

- Problem Statement

  Determine the resultant force R exerted on the semicircular end of the water tank shown in the figure if the tank is filled to capacity. Express the result in terms of the radius r and the water density ρ.



Article 5/9 – Sample Problem 5/22 (2 of 3)

- Solution I

We will obtain R first by a direct integration. With a horizontal strip of area $dA = 2x\,dy$ acted on by the pressure $p = \rho g y$, the increment of the resultant force is $dR = p\,dA$ so that

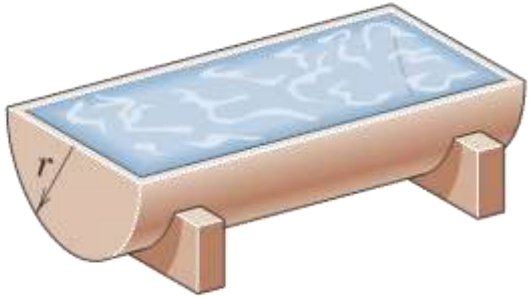
$$R = \int p\,dA = \int \rho g y(2x\,dy) = 2\rho g \int_0^r y\sqrt{r^2 - y^2}\,dy$$

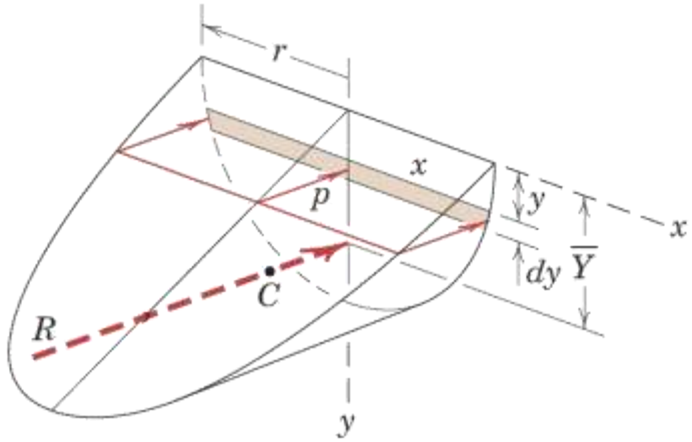
Integrating gives $\qquad R = \frac{2}{3}\rho g r^3 \qquad$ *Ans.*

The location of R is determined by using the principle of moments. Taking moments about the x-axis gives

$$\left[R\bar{Y} = \int y\,dR\right] \qquad \frac{2}{3}\rho g r^3 \bar{Y} = 2\rho g \int_0^r y^2\sqrt{r^2 - y^2}\,dy$$

Integrating gives $\qquad \frac{2}{3}\rho g r^3 \bar{Y} = \frac{\rho g r^4}{4}\frac{\pi}{2} \qquad$ and $\qquad \bar{Y} = \frac{3\pi r}{16} \qquad$ *Ans.*





Article 5/9 – Sample Problem 5/22 (3 of 3)

- Solution II

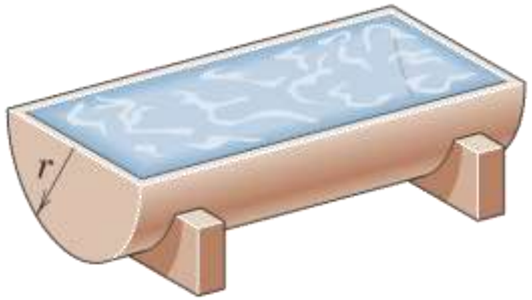
We may use Eq. 5/25 directly to find R, where the average pressure is $\rho g\bar{h}$ and $\bar{h}$ is the coordinate to the centroid of the area over which the pressure acts. For a semicircular area, $\bar{h} = 4r/(3\pi)$.

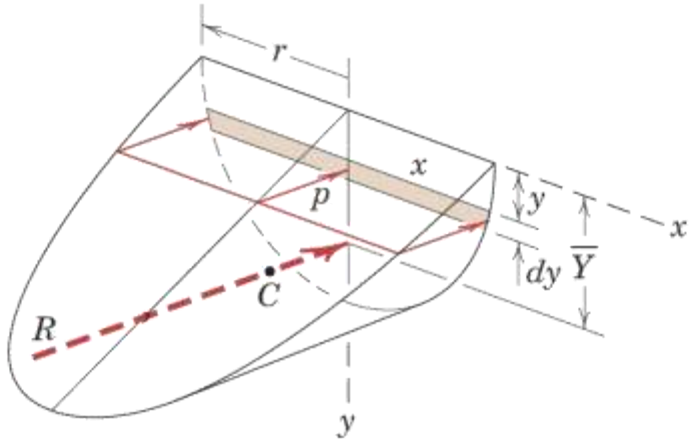
$$[R = \rho g\bar{h}A] \qquad R = \rho g \frac{4r}{3\pi}\frac{\pi r^2}{2} = \frac{2}{3}\rho g r^3 \qquad \textit{Ans.}$$

which is the volume of the pressure-area figure.

The resultant R acts through the centroid C of the volume defined by the pressure-area figure. ① Calculation of the centroidal distance $\bar{Y}$ involves the same integral obtained in *Solution I*.

① Be very careful not to make the mistake of assuming that R passes through the centroid of the area over which the pressure acts.

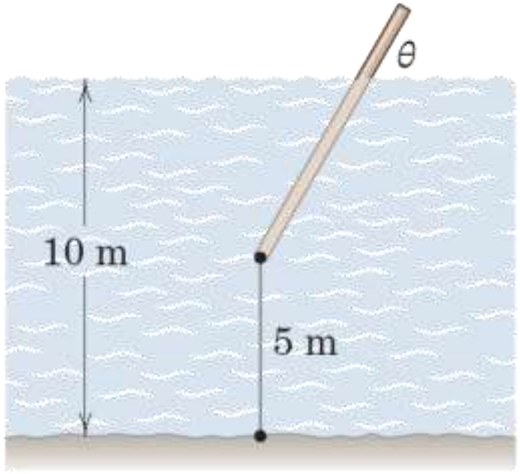




Article 5/9 – Sample Problem 5/23 (1 of 2)

- Problem Statement

  A buoy in the form of a uniform 8-m pole 0.2 m in diameter has a mass of 200 kg and is secured at its lower end to the bottom of a fresh-water lake with 5 m of cable. If the depth of the water is 10 m, calculate the angle θ made by the pole with the horizontal.



Article 5/9 – Sample Problem 5/23 (2 of 2)

- Solution

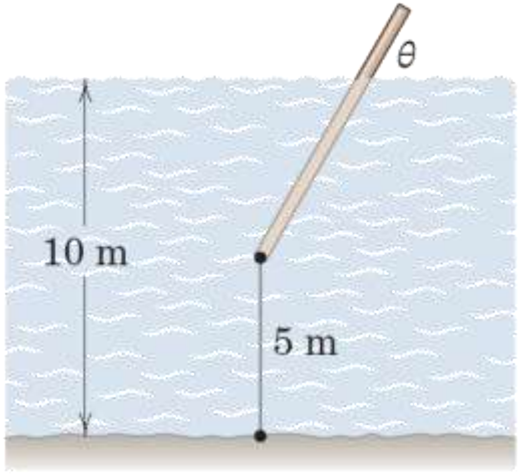
The free-body diagram of the buoy shows its weight acting through G, the vertical tension T in the anchor cable, and the buoyancy force B which passes through centroid C of the submerged portion of the buoy. Let x be the distance from G to the waterline. The density of fresh water is $\rho = 10^3$ kg/m^3, so that the buoyancy force is

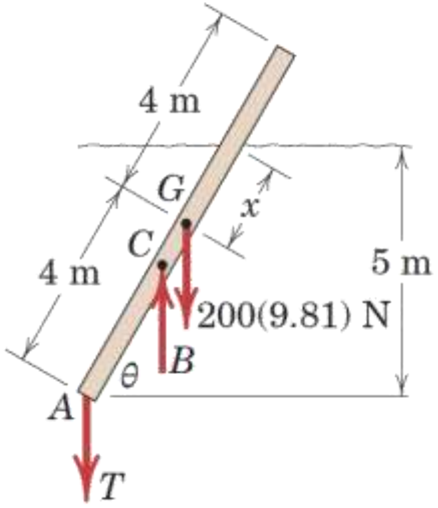
$[B = \rho g V]$ $\quad B = 10^3(9.81)\pi(0.1)^2(4 + x)$ N

Moment equilibrium, $\Sigma M_A = 0$, about A gives

$$200(9.81)(4 \cos \theta) - [10^3(9.81)\pi(0.1)^2(4 + x)]\frac{4 + x}{2} \cos \theta = 0$$

Thus, $\quad x = 3.14$ m $\quad$ and $\quad \theta = \sin^{-1}\left(\frac{5}{4 + 3.14}\right) = 44.5°$ $\quad$ *Ans.*





CHAPTER 6
FRICTION


Courtesy of Alyse Gagne

CHAPTER OUTLINE

Article 6/1 Introduction

- Introduction and Overview

Article 6/2 Types of Friction

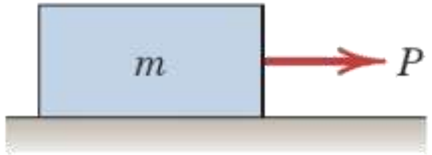
- Dry Friction

- Fluid Friction

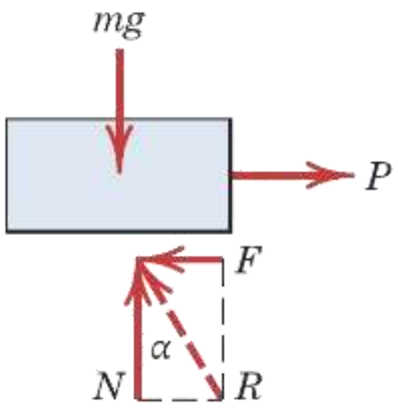
- Internal Friction

Article 6/3 Dry Friction

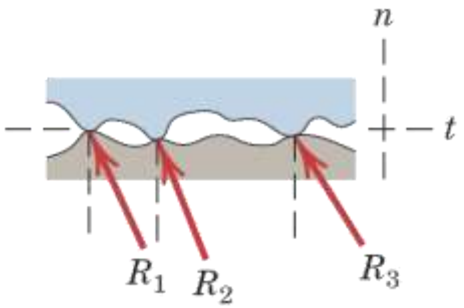
- Mechanism of Dry Friction
  - Block on a Rough Surface



  - Free-Body Diagram

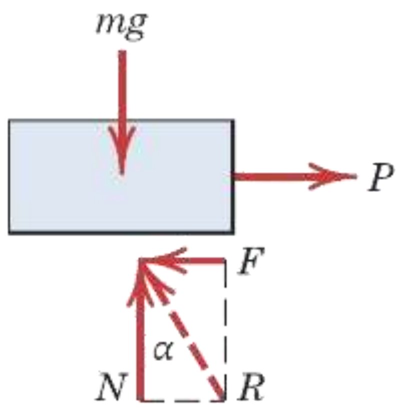


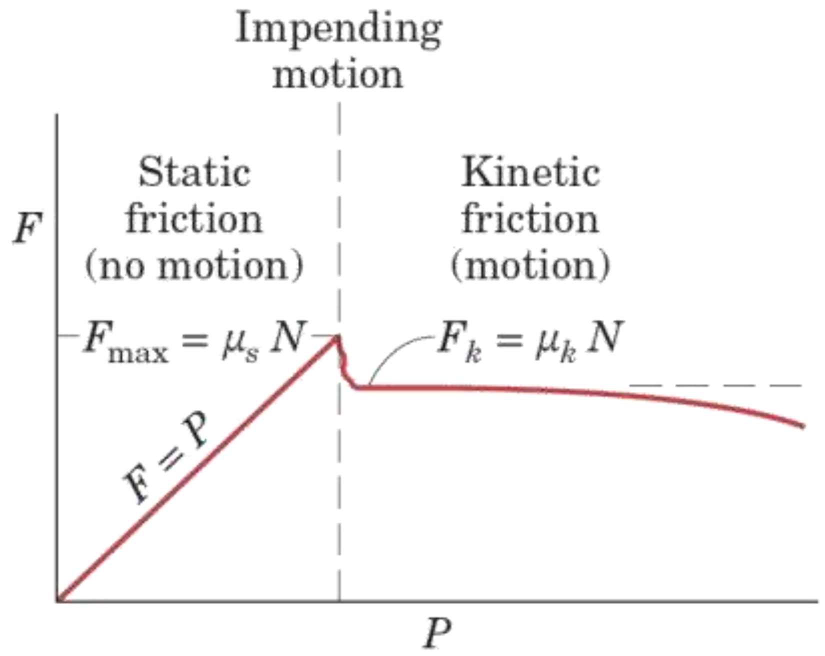
  - Close-Up of Mating Surfaces



Article 6/3 – Plot of Friction Force

- Regions of Significance
  - Static Friction Range, $F < F_{max}$
  - Impending Motion, $F = F_{max} = \mu_s N$
  - Kinetic Friction, $F = F_k = \mu_k N$





Article 6/3 – Friction Coefficients

- Coefficient of Static Friction, μ_s

- Coefficient of Kinetic Friction, μ_k

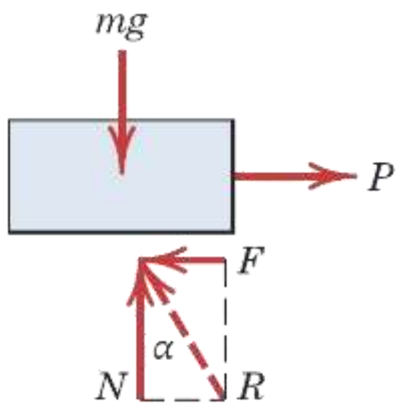
- Table of Values

  Exact values for a specific problem can vary significantly from the values presented.

Contacting Surface	Typical Values of Coefficient of Friction	
	Static, μ_s	Kinetic, μ_k
Steel on steel (dry)	0.6	0.4
Steel on steel (greasy)	0.1	0.05
Teflon on steel	0.04	0.04
Steel on babbitt (dry)	0.4	0.3
Steel on babbitt (greasy)	0.1	0.07
Brass on steel (dry)	0.5	0.4
Brake lining on cast iron	0.4	0.3
Rubber tires on smooth pavement (dry)	0.9	0.8
Wire rope on iron pulley (dry)	0.2	0.15
Hemp rope on metal	0.3	0.2
Metal on ice		0.02

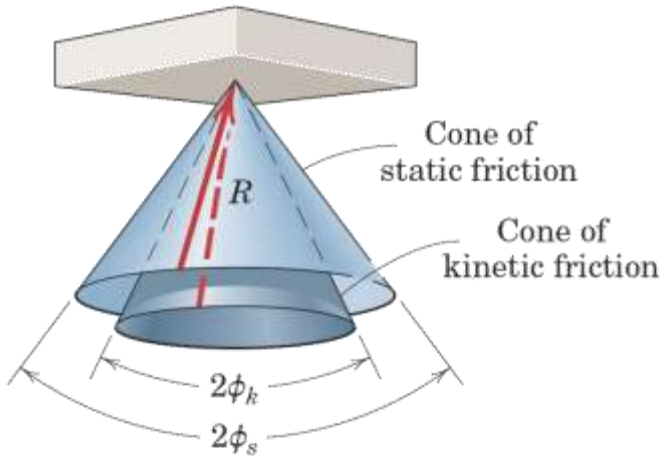
Article 6/3 – Friction Angles

- Static Friction Angle, ϕ_s
  - $\tan \phi_s = \mu_s$



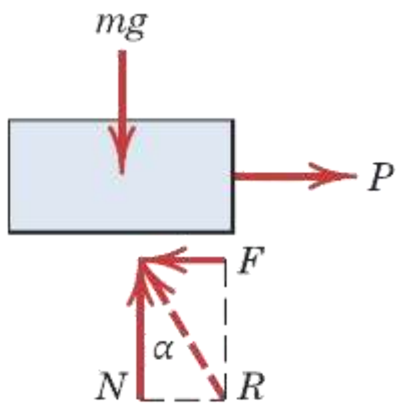
- Kinetic Friction Angle, ϕ_k
  - $\tan \phi_k = \mu_k$

- Friction Cones



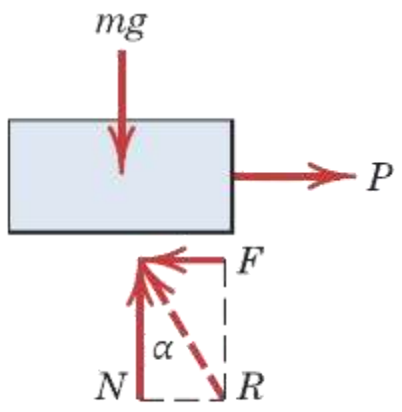
Article 6/3 – Factors Affecting Friction

- Contact Area
- Molecular Attraction
- Temperature
- Adhesion
- Relative Hardness of Mating Areas
- Surface Films



Article 6/3 – Types of Friction Problems (1 of 2)

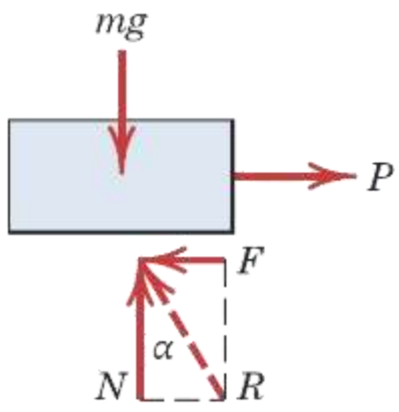
- Type I: Impending Motion
  - Body is on the verge of slipping.
  - Equilibrium holds for the body.
  - $F = F_{\text{max}} = \mu_s N$



- Type II: Relative Motion Exists
  - Body is slipping.
  - Equilibrium does not hold in the direction of slip.
  - $F = F_k = \mu_k N$

Article 6/3 – Types of Friction Problems (2 of 2)

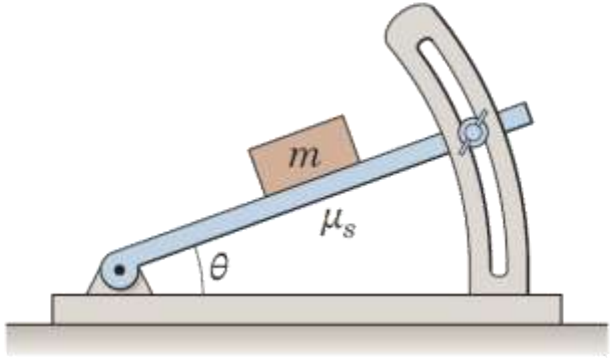
- Type III: Unknown – Body may or may not be slipping
  - Analysis Steps
    1. Assume equilibrium of the body.
    2. Solve for the necessary equilibrium friction force F.
    3. Check the assumption of equilibrium.
       a) If $F < (F_{max} = \mu_s N)$:
          Friction force necessary for equilibrium can be supported, and the body is in static equilibrium as assumed. The *actual* friction force F is *less than* the limiting value F_{max}.
       b) If $F = (F_{max} = \mu_s N)$:
          Motion is impending, but the assumption of equilibrium still holds.
       c) If $F > (F_{max} = \mu_s N)$:
          The surfaces cannot support more force than the maximum $\mu_s N$, so the assumption of equilibrium is invalid. Motion occurs and $F = (F_k = \mu_k N)$.



Article 6/3 – Sample Problem 6/1 (1 of 2)

- Problem Statement

  Determine the maximum angle θ which the adjustable incline may have with the horizontal before the block of mass m begins to slip. The coefficient of static friction between the block and the inclined surface is μ_s.



Article 6/3 – Sample Problem 6/1 (2 of 2)

- Equilibrium Conditions

Equilibrium in the x- and y-directions requires ①

$[\Sigma F_x = 0]$ $\quad mg \sin\theta - F = 0 \quad F = mg \sin\theta$

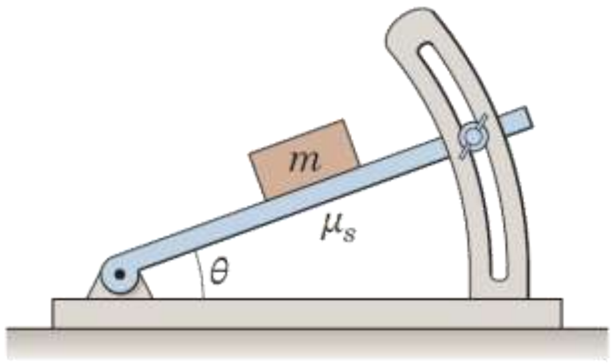
$[\Sigma F_y = 0]$ $\quad -mg \cos\theta + N = 0 \quad N = mg \cos\theta$

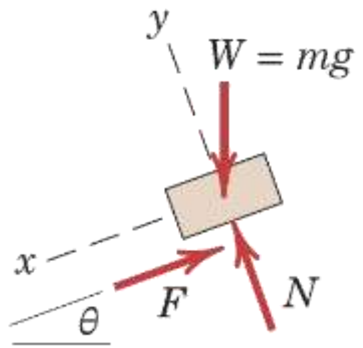
Dividing the first equation by the second gives $F/N = \tan\theta$. Since the maximum angle occurs when $F = F_{max} = \mu_s N$, for impending motion we have

$$\mu_s = \tan\theta_{max} \quad \text{or} \quad \theta_{max} = \tan^{-1}\mu_s \quad ② \qquad \textit{Ans.}$$

① We choose reference axes along and normal to the direction of F to avoid resolving both F and N into components.

② This problem describes a very simple way to determine a static coefficient of friction. The maximum value of θ is known as the *angle of repose*.

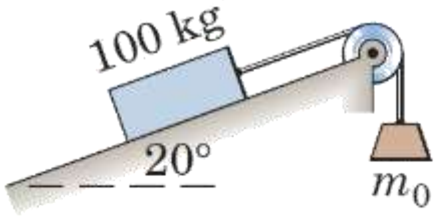




Article 6/3 – Sample Problem 6/2 (1 of 3)

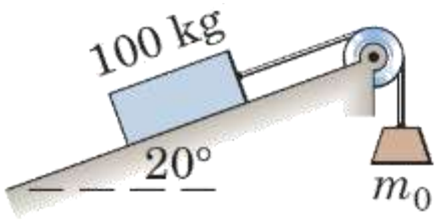
- Problem Statement

  Determine the range of values which the mass m_0 may have so that the 100-kg block shown in the figure will neither start moving up the plane nor slip down the plane. The coefficient of static friction for the contact surfaces is 0.30.



Article 6/3 – Sample Problem 6/2 (2 of 3)

- Case I: Motion Impends Up the Plane

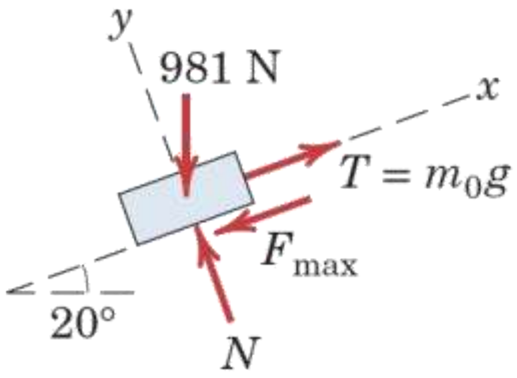


The maximum value of m_0 will be given by the requirement for motion impending up the plane. The friction force on the block therefore acts down the plane, as shown in the free-body diagram of the block for Case I in the figure. With the weight $mg = 100(9.81) = 981$ N, the equations of equilibrium give

$[\Sigma F_y = 0]$ $\qquad N - 981 \cos 20° = 0 \qquad N = 922$ N

$[F_{max} = \mu_s N]$ $\qquad F_{max} = 0.30(922) = 277$ N

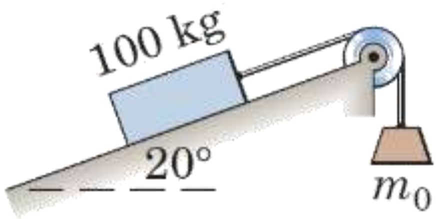
$[\Sigma F_x = 0]$ $\qquad m_0(9.81) - 277 - 981 \sin 20° = 0 \qquad m_0 = 62.4$ kg $\qquad$ *Ans.*



Case I

Article 6/3 – Sample Problem 6/2 (3 of 3)

- Case II: Motion Impends Down the Plane

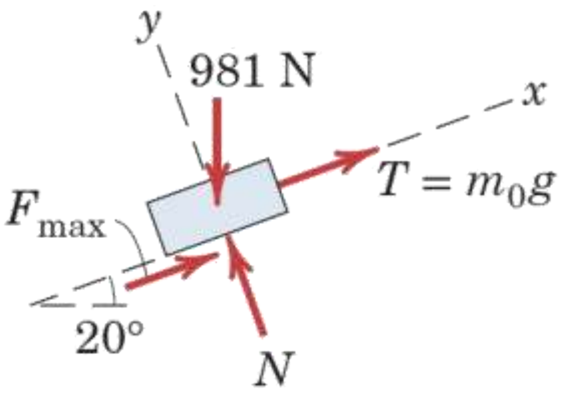


The minimum value of m_0 is determined when motion is impending down the plane. ① The friction force on the block will act up the plane to oppose the tendency to move, as shown in the free-body diagram for Case II. Equilibrium in the x-direction requires

$$[\Sigma F_x = 0] \qquad m_0(9.81) + 277 - 981 \sin 20° = 0 \qquad m_0 = 6.01 \text{ kg} \qquad \textit{Ans.}$$

Thus, m_0 may have any value from 6.01 to 62.4 kg, and the block will remain at rest.

In both cases equilibrium requires that the resultant of F_{max} and N be concurrent with the 981-N weight and the tension T.



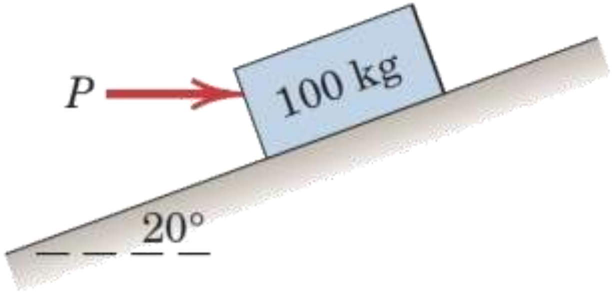
Case II

① We see from the results of Sample Problem 6/1 that the block would slide down the incline without the restraint of attachment to m_0 since $\tan 20° > 0.30$. Thus, a value of m_0 will be required to maintain equilibrium.

Article 6/3 – Sample Problem 6/3 (1 of 4)

- Problem Statement

  Determine the magnitude and direction of the friction force acting on the 100-kg block shown if, first, P = 500 N and, second, P = 100 N. The coefficient of static friction is 0.20, and the coefficient of kinetic friction is 0.17. The forces are applied with the block initially at rest.



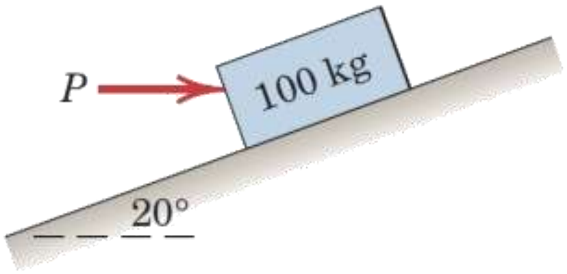
Article 6/3 – Sample Problem 6/3 (2 of 4)

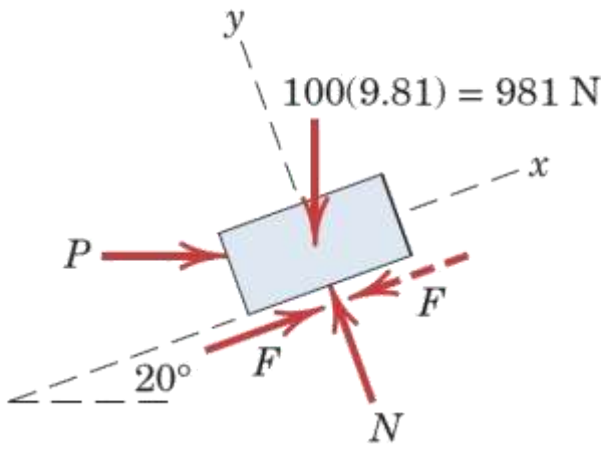
- Equilibrium Conditions

There is no way of telling from the statement of the problem whether the block will remain in equilibrium or whether it will begin to slip following the application of P. It is therefore necessary that we make an assumption, so we will take the friction force to be up the plane, as shown by the solid arrow. From the free-body diagram a balance of forces in both x- and y-directions gives

$[\Sigma F_x = 0]$ $\quad P \cos 20° + F - 981 \sin 20° = 0$

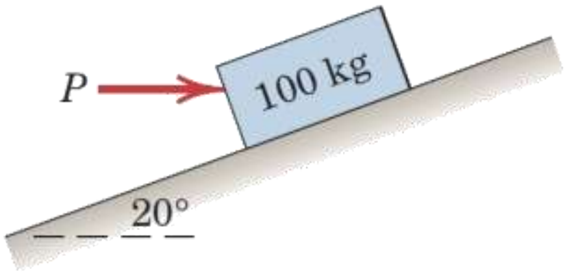
$[\Sigma F_y = 0]$ $\quad N - P \sin 20° - 981 \cos 20° = 0$





Article 6/3 – Sample Problem 6/3 (3 of 4)

- Case I: $P = 500$ N

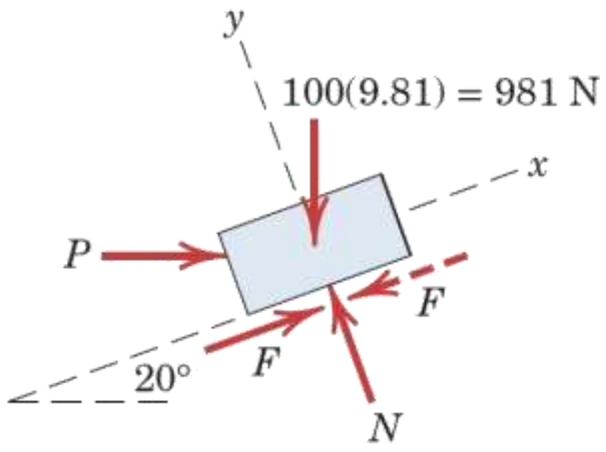


Substitution into the first of the two equations gives

$$F = -134.3 \text{ N}$$

The negative sign tells us that *if* the block is in equilibrium, the friction force acting on it is in the direction opposite to that assumed and therefore is down the plane, as represented by the dashed arrow. We cannot reach a conclusion on the magnitude of F, however, until we verify that the surfaces are capable of supporting 134.3 N of friction force. This may be done by substituting $P = 500$ N into the second equation, which gives

$$N = 1093 \text{ N}$$



The maximum static friction force which the surfaces can support is then

$$[F_{\max} = \mu_s N] \qquad F_{\max} = 0.20(1093) = 219 \text{ N}$$

Since this force is greater than that required for equilibrium, we conclude that the assumption of equilibrium was correct. The answer is, then,

$$F = 134.3 \text{ N down the plane} \qquad \textit{Ans.}$$

Article 6/3 – Sample Problem 6/3 (4 of 4)

- Case II: $P = 100$ N

Substitution into the two equilibrium equations gives

$$F = 242 \text{ N} \qquad N = 956 \text{ N}$$

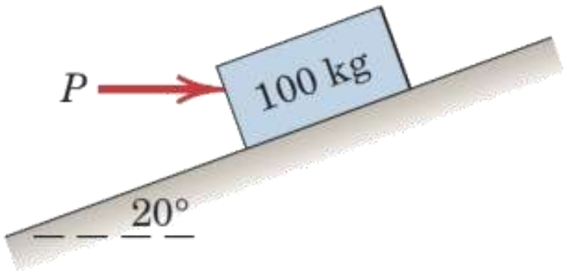
But the maximum possible static friction force is

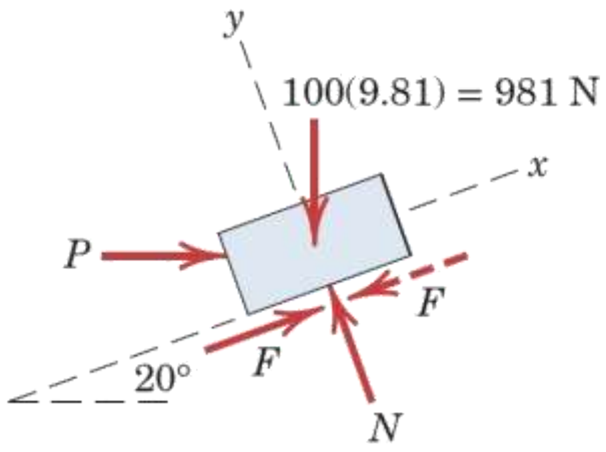
$[F_{max} = \mu_s N]$ $\qquad F_{max} = 0.20(956) = 191.2$ N

It follows that 242 N of friction cannot be supported. Therefore, equilibrium cannot exist, and we obtain the correct value of the friction force by using the kinetic coefficient of friction accompanying the motion down the plane. Hence, the answer is

$[F_k = \mu_k N]$ $\qquad F = 0.17(956) = 162.5$ N up the plane ① $\qquad$ *Ans.*

① We should note that even though ΣF_x is no longer equal to zero, equilibrium does exist in the y-direction, so that $\Sigma F_y = 0$. Therefore, the normal force N is 956 N whether or not the block is in equilibrium.

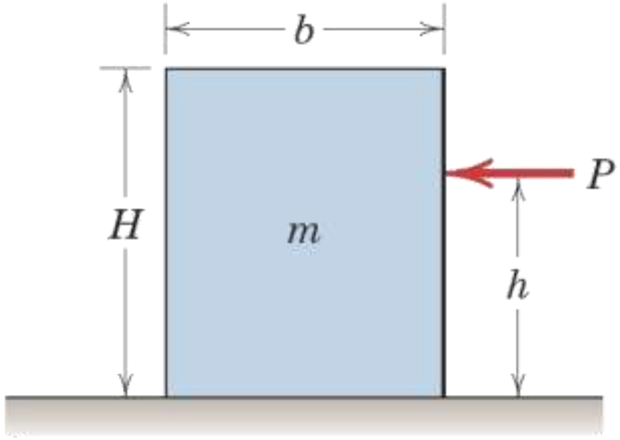




Article 6/3 – Sample Problem 6/4 (1 of 3)

- Problem Statement

  The homogeneous rectangular block of mass m, width b, and height H is placed on the horizontal surface and subjected to a horizontal force P which moves the block along the surface with a constant velocity. The coefficient of kinetic friction between the block and the surface is μ_k. Determine (a) the greatest value which h may have so that the block will slide without tipping over and (b) the location of a point C on the bottom face of the block through which the resultant of the friction and normal forces acts if $h = H/2$.



- Part (*a*) Maximum Value of *h*

With the block on the verge of tipping, we see that the entire reaction between the plane and the block will necessarily be at A. The free-body diagram of the block shows this condition. Since slipping occurs, the friction force is the limiting value $\mu_k N$, and the angle θ becomes $\theta = \tan^{-1} \mu_k$. The resultant of F_k and N passes through a point B through which P must also pass, since three coplanar forces in equilibrium are concurrent. ① Hence, from the geometry of the block

$$\tan\theta = \mu_k = \frac{b/2}{h} \qquad h = \frac{b}{2\mu_k} \qquad \textit{Ans.}$$

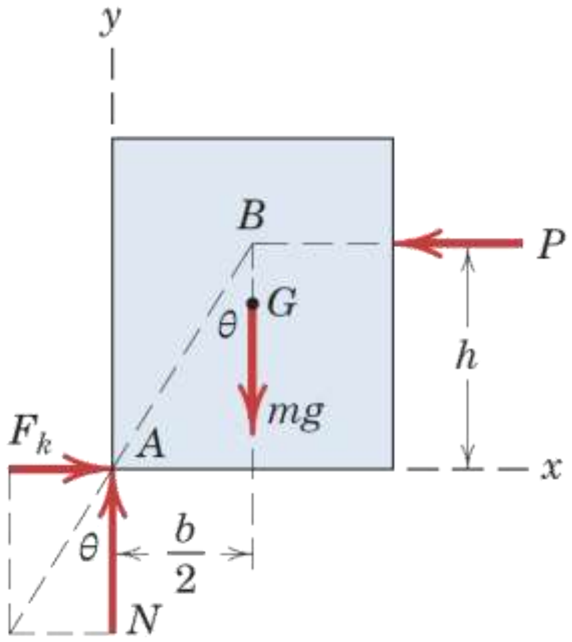
If h were greater than this value, moment equilibrium about A would not be satisfied, and the block would tip over.

Alternatively, we may find h by combining the equilibrium requirements for the x- and y-directions with the moment-equilibrium equation about A. Thus,

$$[\Sigma F_y = 0] \qquad N - mg = 0 \qquad N = mg$$

$$[\Sigma F_x = 0] \qquad F_k - P = 0 \qquad P = F_k = \mu_k N = \mu_k mg$$

$$[\Sigma M_A = 0] \qquad Ph - mg\frac{b}{2} = 0 \qquad h = \frac{mgb}{2P} = \frac{mgb}{2\mu_k mg} = \frac{b}{2\mu_k} \qquad \textit{Ans.}$$



① Recall that the equilibrium equations apply to a body moving with a constant velocity (zero acceleration) just as well as to a body at rest.

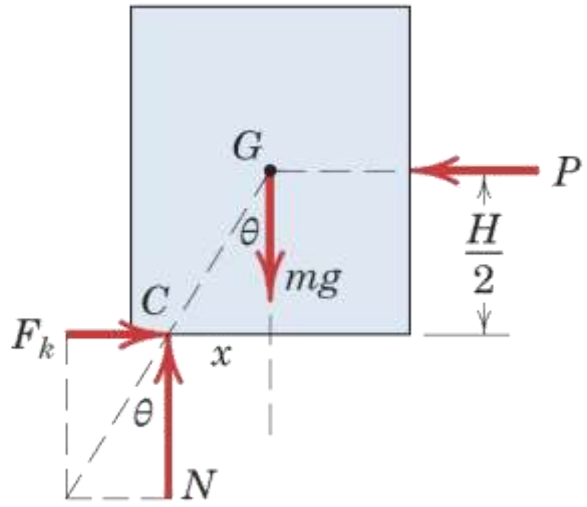
Article 6/3 – Sample Problem 6/4 (3 of 3)

- Part (b) $h = H/2$

With $h = H/2$ we see from the free-body diagram for case (b) that the resultant of F_k and N passes through a point C which is a distance x to the left of the vertical centerline through G. The angle θ is still $\theta = \phi = \tan^{-1} \mu_k$ as long as the block is slipping. Thus, from the geometry of the figure we have

$$\frac{x}{H/2} = \tan\theta = \mu_k \qquad \text{so} \qquad x = \mu_k H/2 \quad ② \qquad \textit{Ans.}$$

If we were to replace μ_k by the static coefficient μ_s, then our solutions would describe the conditions under which the block is (a) on the verge of tipping and (b) on the verge of slipping, both from a rest position.

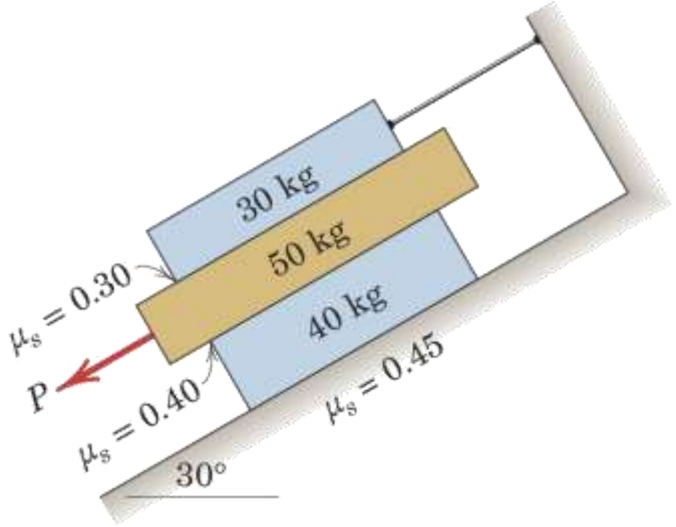


② Alternatively, we could equate the moments about G to zero, which would give us $F(H/2) - Nx = 0$. Thus, with $F_k = \mu_k N$ we get $x = \mu_k H/2$.

Article 6/3 – Sample Problem 6/5 (1 of 4)

- Problem Statement

  The three flat blocks are positioned on the 30° incline as shown, and a force *P* parallel to the incline is applied to the middle block. The upper block is prevented from moving by a wire which attaches it to the fixed support. The coefficient of static friction for each of the three pairs of mating surfaces is shown. Determine the maximum value which *P* may have before any slipping takes place.



Article 6/3 – Sample Problem 6/5 (2 of 4)

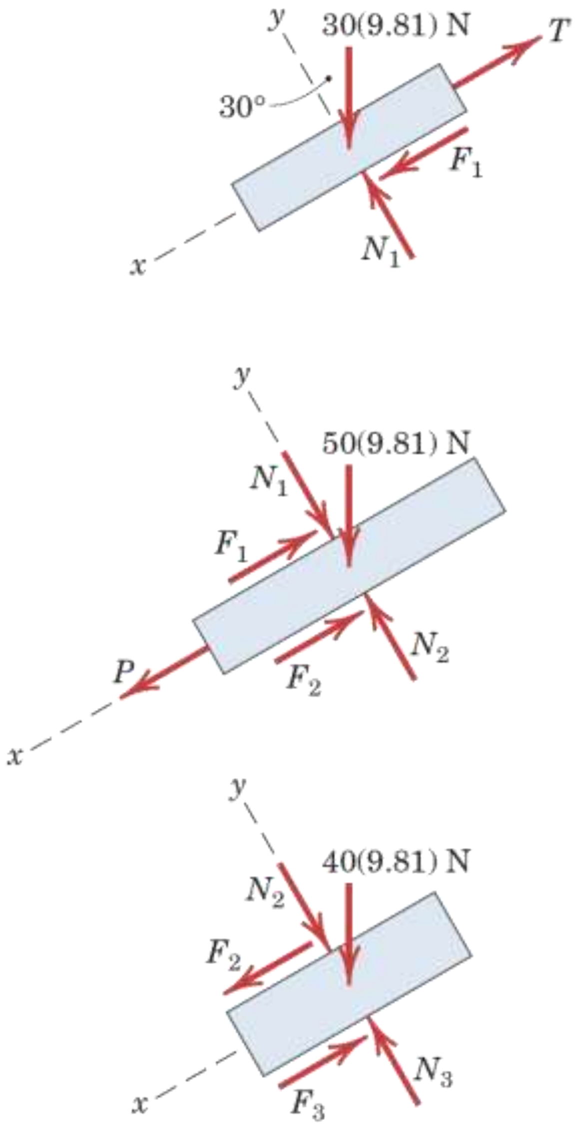
- Free-Body Diagrams

- Normal Force Calculations

The normal forces, which are in the y-direction, may be determined without reference to the friction forces, which are all in the x-direction. Thus,

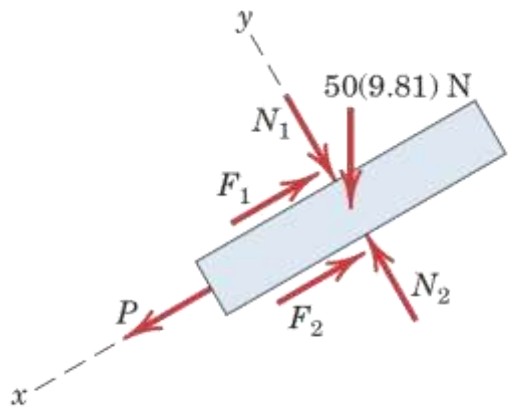
$[\Sigma F_y = 0]$	(30-kg)	$N_1 - 30(9.81)\cos 30° = 0$	$N_1 = 255$ N
	(50-kg)	$N_2 - 50(9.81)\cos 30° - 255 = 0$	$N_2 = 680$ N
	(40-kg)	$N_3 - 40(9.81)\cos 30° - 680 = 0$	$N_3 = 1019$ N

① In the absence of friction the middle block, under the influence of P, would have a greater movement than the 40-kg block, and the friction force F_2 will be in the direction to oppose this motion as shown.



Article 6/3 – Sample Problem 6/5 (3 of 4)

- Assume 50-kg Block Slips

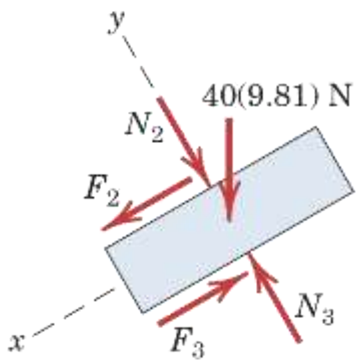


We will assume arbitrarily that only the 50-kg block slips, so that the 40-kg block remains in place. Thus, for impending slippage at both surfaces of the 50-kg block, we have

$$[F_{max} = \mu_s N] \qquad F_1 = 0.30(255) = 76.5\text{ N} \qquad F_2 = 0.40(680) = 272\text{ N}$$

The assumed equilibrium of forces at impending motion for the 50-kg block gives

$$[\Sigma F_x = 0] \qquad P - 76.5 - 272 + 50(9.81)\sin 30° = 0 \qquad P = 103.1\text{ N}$$



Article 6/3 – Sample Problem 6/5 (4 of 4)

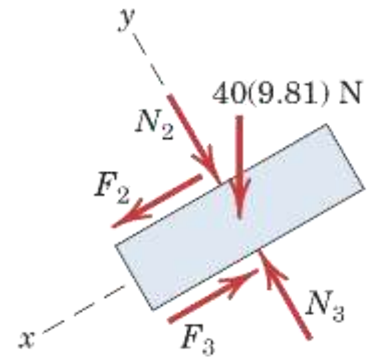
- Check Assumption

We now check on the validity of our initial assumption. For the 40-kg block with $F_2 = 272$ N the friction force F_3 would be given by

$$[\Sigma F_x = 0] \qquad 272 + 40(9.81)\sin 30° - F_3 = 0 \qquad F_3 = 468 \text{ N}$$

But the maximum possible value of F_3 is $F_3 = \mu_s N_3 = 0.45(1019) = 459$ N. Thus, 468 N cannot be supported and our initial assumption was wrong. We conclude, therefore, that slipping occurs first between the 40-kg block and the incline. With the corrected value $F_3 = 459$ N, equilibrium of the 40-kg block for its impending motion requires

$$[\Sigma F_x = 0] \qquad F_2 + 40(9.81)\sin 30° - 459 = 0 \qquad F_2 = 263 \text{ N} \quad ②$$



② We see now that F_2 is less than $\mu_s N_2 = 272$ N.

- Revise Calculation for *P*

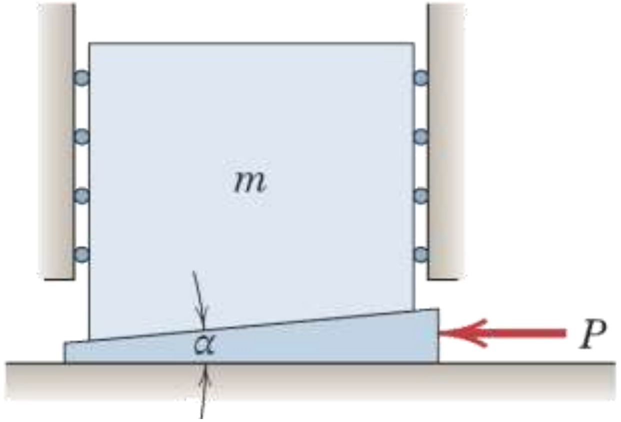
Equilibrium of the 50-kg block gives, finally,

$$[\Sigma F_x = 0] \qquad P + 50(9.81)\sin 30° - 263 - 76.5 = 0$$

$$P = 93.8 \text{ N} \qquad \textit{Ans.}$$

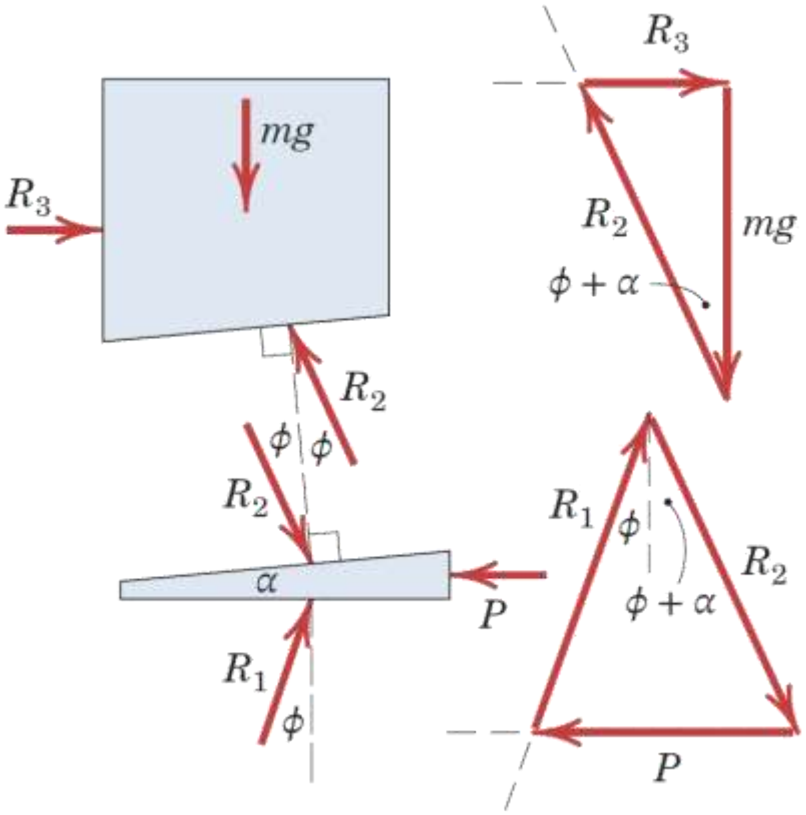
Article 6/4 Wedges

- Overview and Illustration



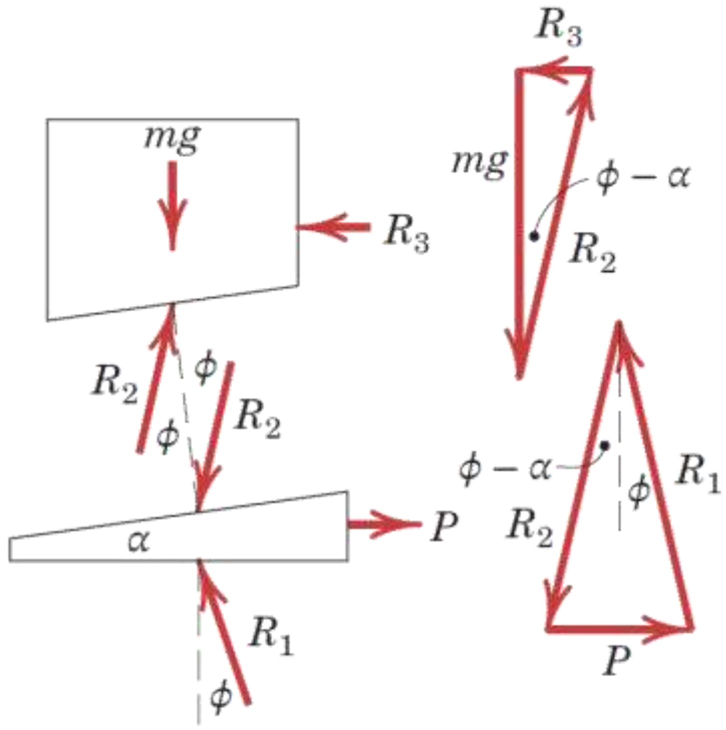
Article 6/4 – Raising a Load

- Free-Body Diagrams
- Wedge Angle, α
- Force Polygons



Article 6/4 – Lowering a Load

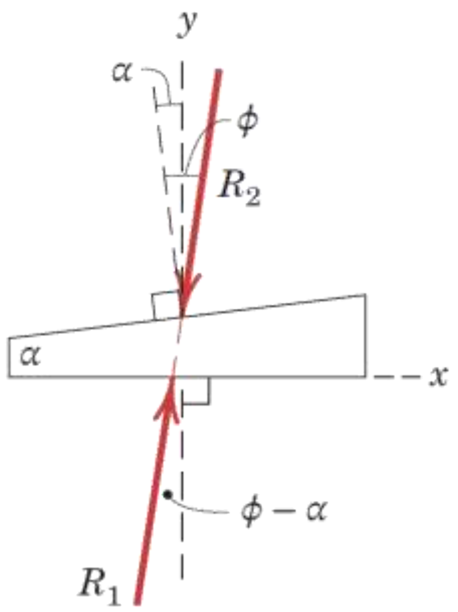
- Free-Body Diagrams
- Wedge Angle, α
- Force Polygons



Article 6/4 – Self-Locking of a Wedge (1 of 2)

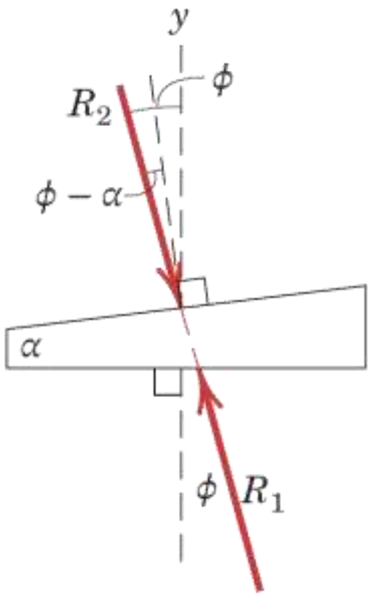
- Explanation of Self-Locking

- Impending Slip at the Upper Surface

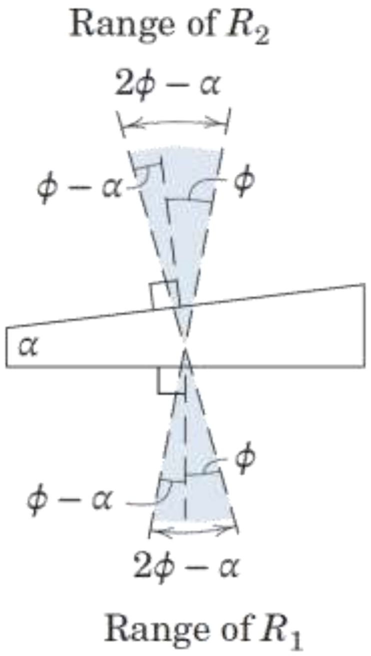


Article 6/4 – Self-Locking of a Wedge (2 of 2)

- Impending Slip at the Lower Surface

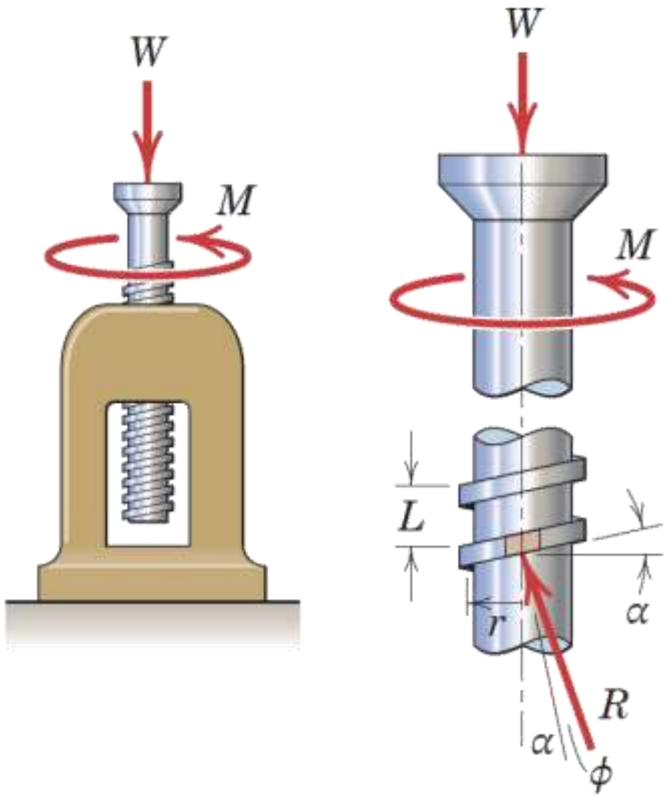


- Range for No Slip



Article 6/5 Screws

- Introduction
- Force Analysis
- Applied Moment, $M = Wr \tan(\alpha + \phi)$
- Lead, L
- Helix Angle, $\tan \alpha = L/2\pi r$

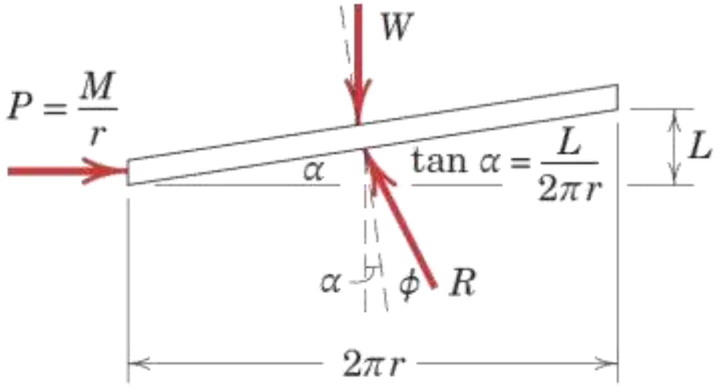


Article 6/5 – Unwrapped Thread View

- Applied Moment, $M = Wr \tan(\alpha + \phi)$

- Helix Angle, $\tan \alpha = L/2\pi r$

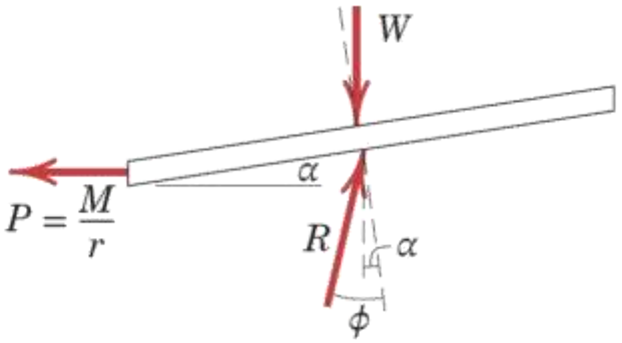
- Equivalent Force, $P = M/r$



(a) To raise load

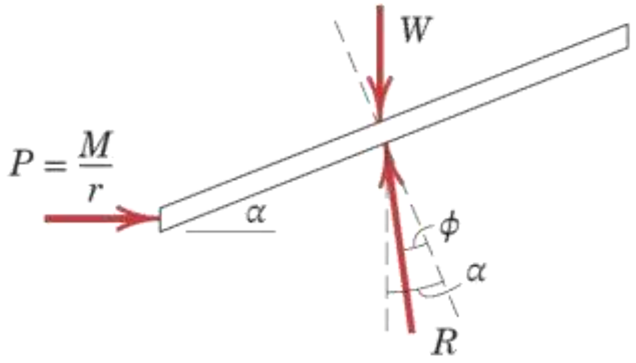
Article 6/5 – Conditions for Unwinding

- Case I: $\alpha < \phi$



(*b*) To lower load (α<ϕ)

- Case II: $\alpha > \phi$

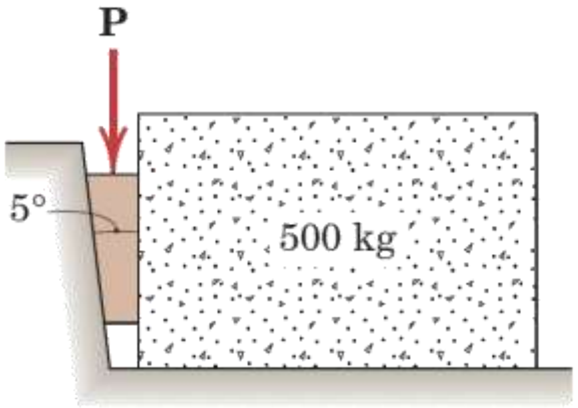


(*c*) To lower load (α>ϕ)

Article 6/5 – Sample Problem 6/6 (1 of 4)

- Problem Statement

  The horizontal position of the 500-kg rectangular block of concrete is adjusted by the 5° wedge under the action of the force **P**. If the coefficient of static friction for both wedge surfaces is 0.30 and if the coefficient of static friction between the block and the horizontal surface is 0.60, determine the least force *P* required to move the block.



Article 6/5 – Sample Problem 6/6 (2 of 4)

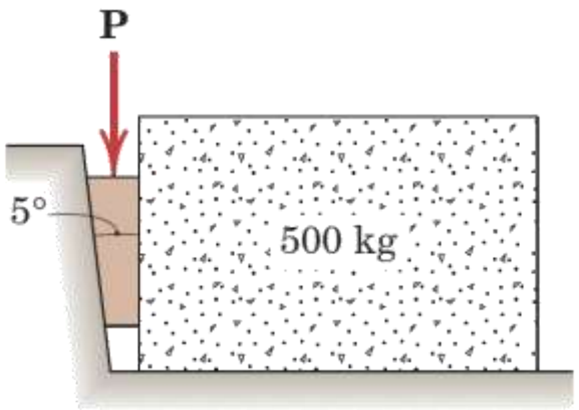
- Solution

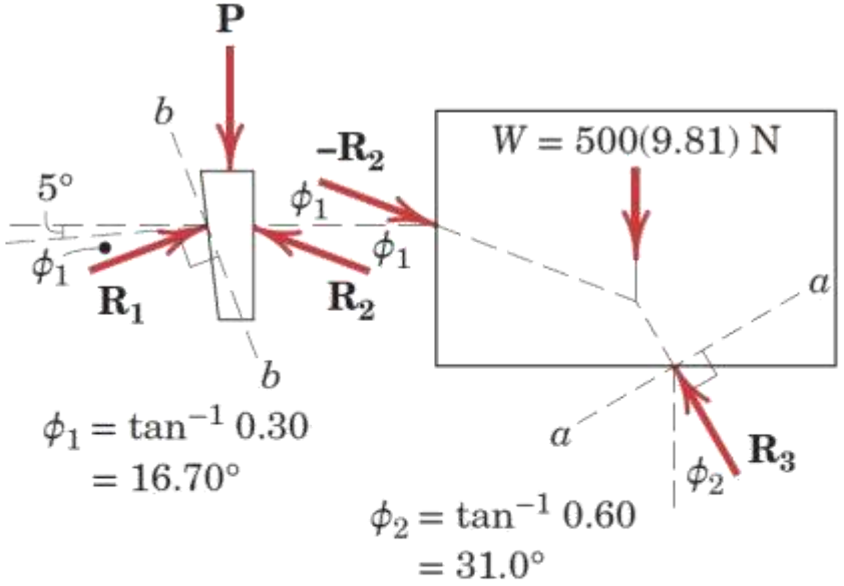
The free-body diagrams of the wedge and the block are drawn with the reactions $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$ inclined with respect to their normals by the amounts of the friction angles for impending motion. ① The friction angle for limiting static friction is given by $\phi = \tan^{-1} \mu$. Each of the two friction angles is computed and shown on the diagram.

We start our vector diagram expressing the equilibrium of the block at a convenient point A and draw the only known vector, the weight $\mathbf{W}$ of the block. Next we add $\mathbf{R}_3$, whose 31.0° inclination from the vertical is now known. The vector $-\mathbf{R}_2$, whose 16.70° inclination from the horizontal is also known, must close the polygon for equilibrium. Thus, point B on the lower polygon is determined by the intersection of the known directions of $\mathbf{R}_3$ and $-\mathbf{R}_2$, and their magnitudes become known.

For the wedge we draw $\mathbf{R}_2$, which is now known, and add $\mathbf{R}_1$, whose direction is known. The directions of $\mathbf{R}_1$ and $\mathbf{P}$ intersect at C, thus giving us the solution for the magnitude of $\mathbf{P}$.

① Be certain to note that the reactions are inclined from their normals in the direction to oppose the motion. Also, we note the equal and opposite reactions $\mathbf{R}_2$ and $-\mathbf{R}_2$.





Article 6/5 – Sample Problem 6/6 (3 of 4)

- Algebraic Solution

The simplest choice of reference axes for calculation purposes is, for the block, in the direction a-a normal to $\mathbf{R}_3$ and, for the wedge, in the direction b-b normal to $\mathbf{R}_1$. ② The angle between $\mathbf{R}_2$ and the a-direction is $16.70° + 31.0° = 47.7°$. Thus, for the block

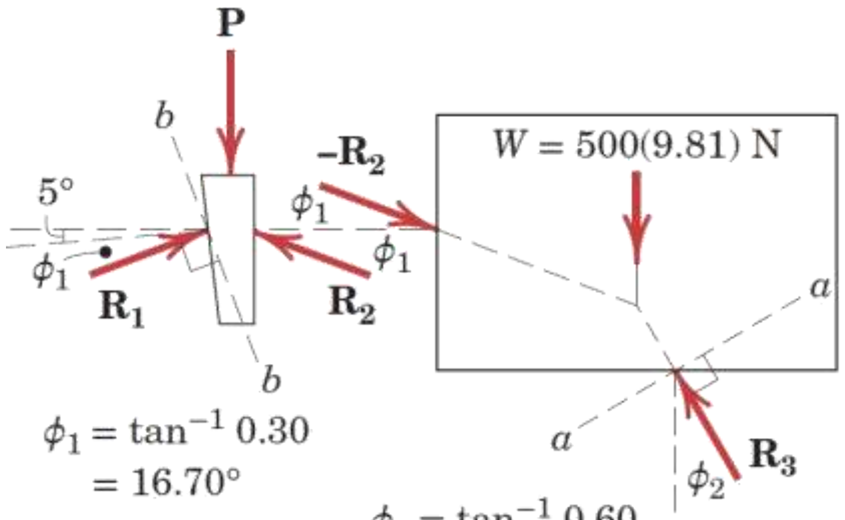
$[\Sigma F_a = 0]$ $\quad 500(9.81) \sin 31.0° - R_2 \cos 47.7° = 0$

$$R_2 = 3750 \text{ N}$$

For the wedge the angle between $\mathbf{R}_2$ and the b-direction is $90° - (2\phi_1 + 5°) = 51.6°$, and the angle between $\mathbf{P}$ and the b-direction is $\phi_1 + 5° = 21.7°$. Thus,

$[\Sigma F_b = 0]$ $\quad 3750 \cos 51.6° - P \cos 21.7° = 0$

$$P = 2500 \text{ N} \qquad \textit{Ans.}$$



$\phi_1 = \tan^{-1} 0.30$
$= 16.70°$

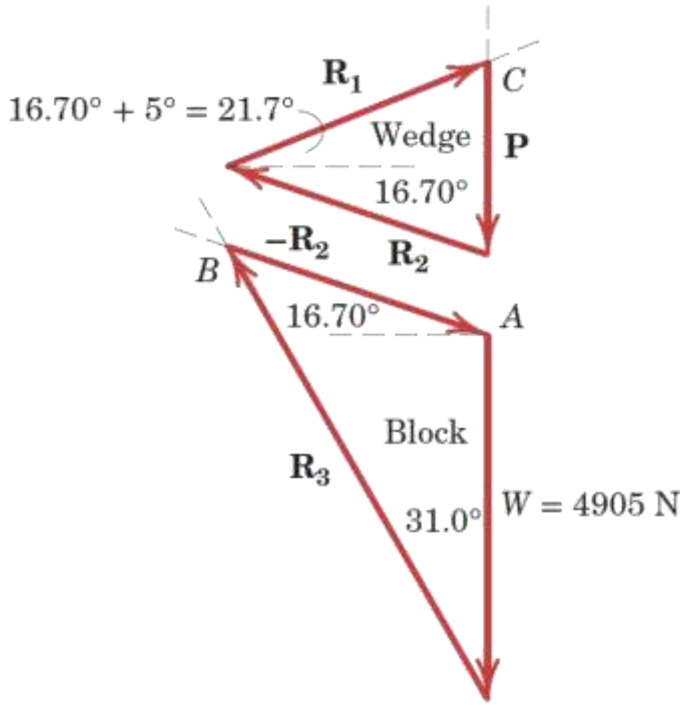
$\phi_2 = \tan^{-1} 0.60$
$= 31.0°$

② It should be evident that we avoid simultaneous equations by eliminating reference to $\mathbf{R}_3$ for the block and $\mathbf{R}_1$ for the wedge.

Article 6/5 – Sample Problem 6/6 (4 of 4)

- Graphical Solution

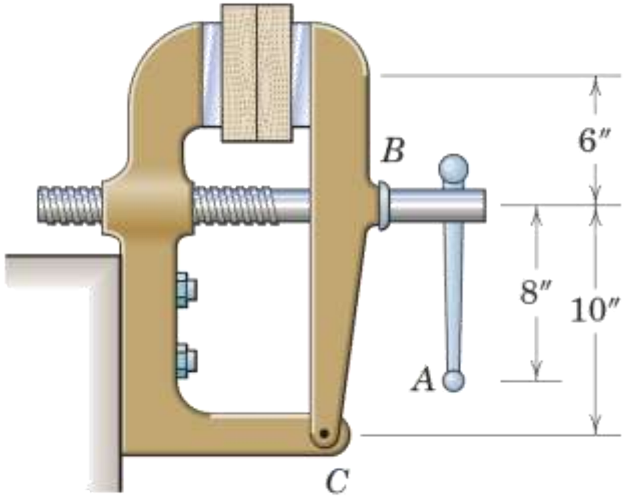
The accuracy of a graphical solution is well within the uncertainty of the friction coefficients and provides a simple and direct result. By laying off the vectors to a reasonable scale following the sequence described, we obtain the magnitudes of **P** and the **R**'s easily by scaling them directly from the diagrams.



Article 6/5 – Sample Problem 6/7 (1 of 4)

- Problem Statement

  The single-threaded screw of the vise has a mean diameter of 1 in. and has 5 square threads per inch. The coefficient of static friction in the threads is 0.20. A 60-lb pull applied normal to the handle at *A* produces a clamping force of 1000 lb between the jaws of the vise. (*a*) Determine the frictional moment M_B, developed at *B*, due to the thrust of the screw against the body of the jaw. (*b*) Determine the force *Q* applied normal to the handle at *A* required to loosen the vise.



Article 6/5 – Sample Problem 6/7 (2 of 4)

- Equilibrium of the Jaw

$[\Sigma M_C = 0] \qquad 1000(16) - 10T = 0 \qquad T = 1600 \text{ lb}$

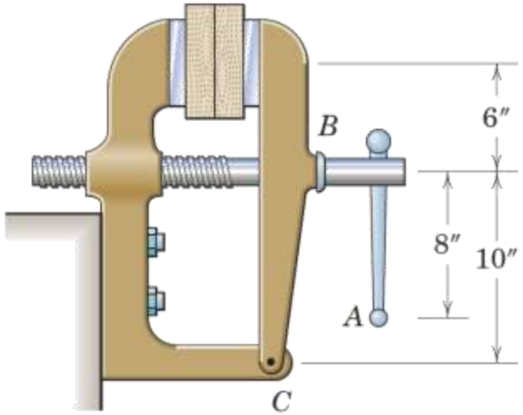
The helix angle α and the friction angle ϕ for the thread are given by

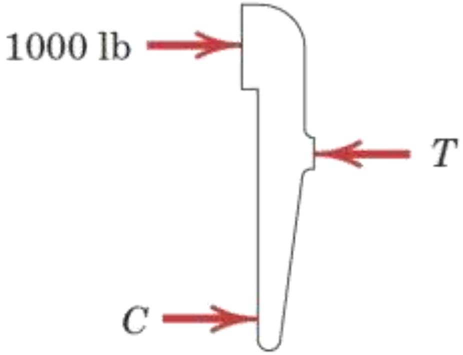
$$\alpha = \tan^{-1}\frac{L}{2\pi r} = \tan^{-1}\frac{1/5}{2\pi(0.5)} = 3.64^\circ \quad ①$$

$$\phi = \tan^{-1}\mu = \tan^{-1}0.20 = 11.31^\circ$$

where the mean radius of the thread is $r = 0.5$ in.

① Be careful to calculate the helix angle correctly. Its tangent is the lead L (advancement per revolution) divided by the mean circumference $2\pi r$ and not by the diameter $2r$.





Article 6/5 – Sample Problem 6/7 (3 of 4)

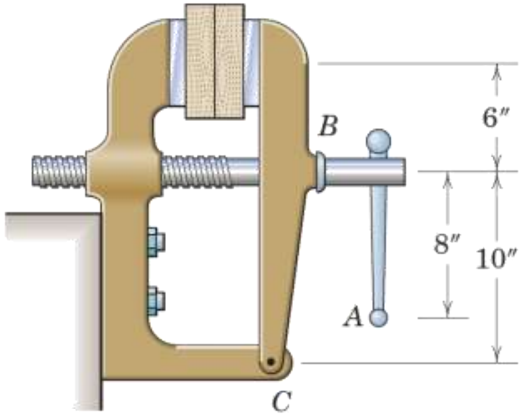
- (*a*) To Tighten

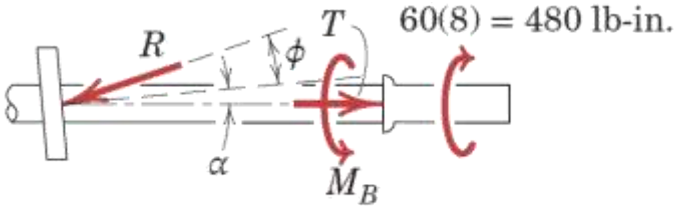
The isolated screw is simulated by the free-body diagram shown where all of the forces acting on the threads of the screw are represented by a single force R inclined at the friction angle ϕ from the normal to the thread. The moment applied about the screw axis is $60(8) = 480$ lb-in. in the clockwise direction as seen from the front of the vise. The frictional moment M_B due to the friction forces acting on the collar at B is in the counterclockwise direction to oppose the impending motion. From Eq. 6/3 with T substituted for W, the net moment acting on the screw is

$$M = Tr \tan(\alpha + \phi)$$

$$480 - M_B = 1600(0.5) \tan(3.64° + 11.31°)$$

$$M_B = 266 \text{ lb-in.} \qquad \textit{Ans.}$$





(*a*) To tighten

Article 6/5 – Sample Problem 6/7 (4 of 4)

- (*b*) To Loosen

The free-body diagram of the screw on the verge of being loosened is shown with R acting at the friction angle from the normal in the direction to counteract the impending motion. ② Also shown is the frictional moment $M_B = 266$ lb-in. acting in the clockwise direction to oppose the motion. The angle between R and the screw axis is now $\phi - \alpha$, and we use Eq. 6/3*a* with the net moment equal to the applied moment M' minus M_B. Thus

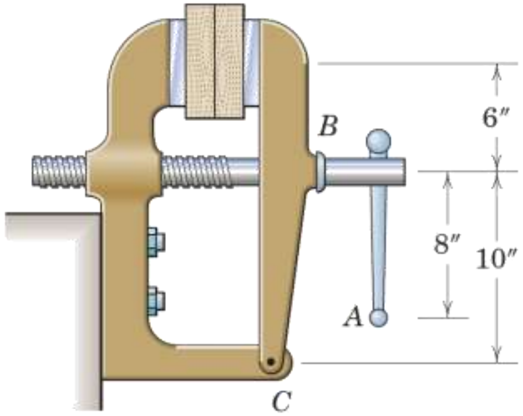
$$M = Tr \tan(\phi - \alpha)$$

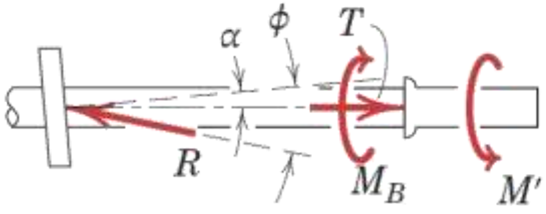
$$M' - 266 = 1600(0.5) \tan(11.31° - 3.64°)$$

$$M' = 374 \text{ lb-in.}$$

Thus, the force on the handle required to loosen the vise is

$$Q = M'/d = 374/8 = 46.8 \text{ lb} \qquad \textit{Ans.}$$





(*b*) To loosen

Article 6/6 Journal Bearings

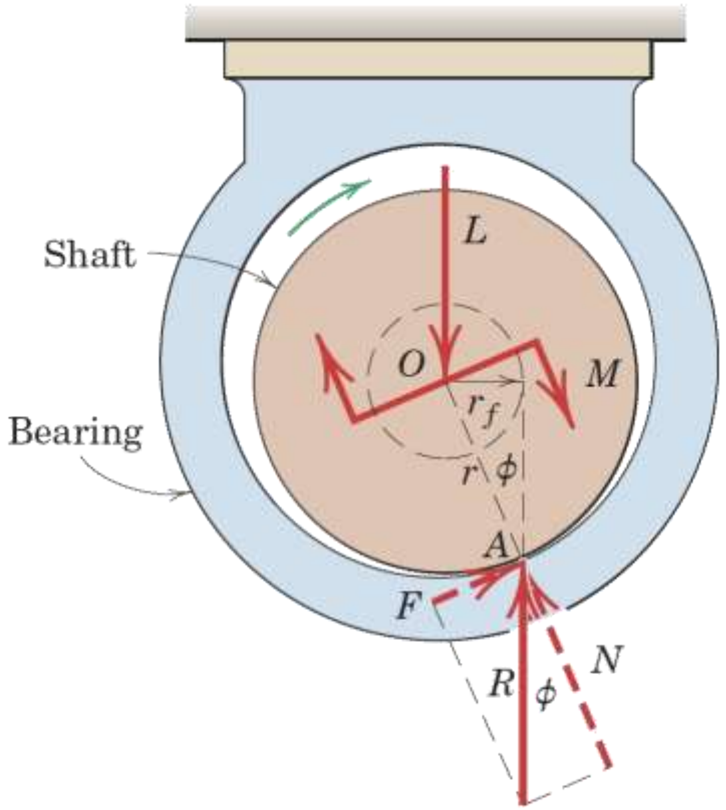
- Purpose

- Illustration

- Load Analysis

- Torque to Maintain Rotation

$$M = Lr \sin\phi = \mu Lr$$



Article 6/7 Thrust Bearings; Disk Friction

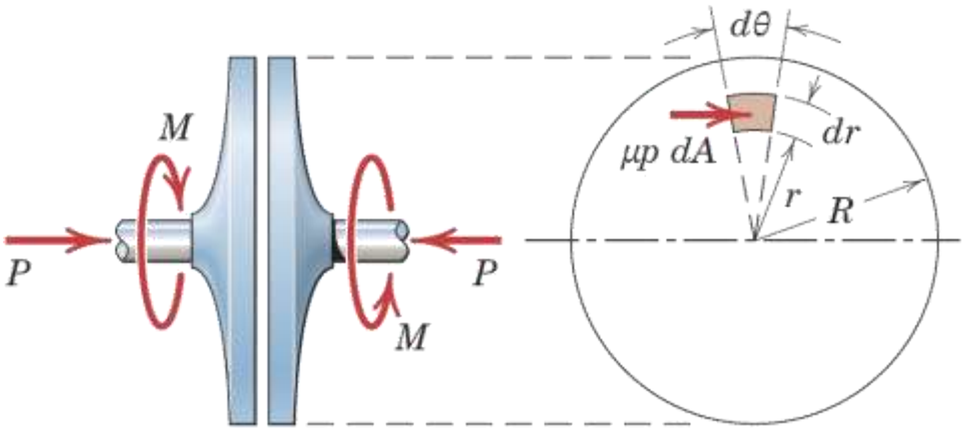
- Purpose

- Example

Westend61 GmbH/Alamy Stock Photo

Article 6/7 – Disk Friction Load Analysis

- Free-Body Diagram
  - Axial Force, P
  - Transmitted Torque, M
  - Pressure Between the Plates, $p = P/\pi R^2$



- Friction Force on a Differential Area

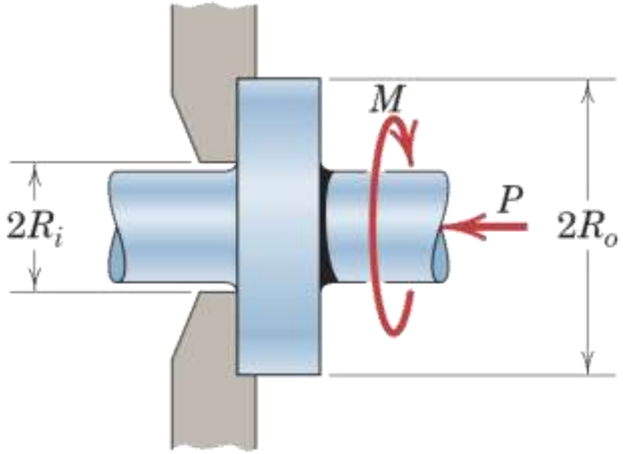
  $dF = \mu p\ dA$

- Integration over the Disk Area

  $M = 2/3\ \mu PR$

Article 6/7 – Collar Bearing Load Analysis

- Example with a Collar Bearing
  - Inside Contact Radius, R_i
  - Outside Contact Radius, R_o



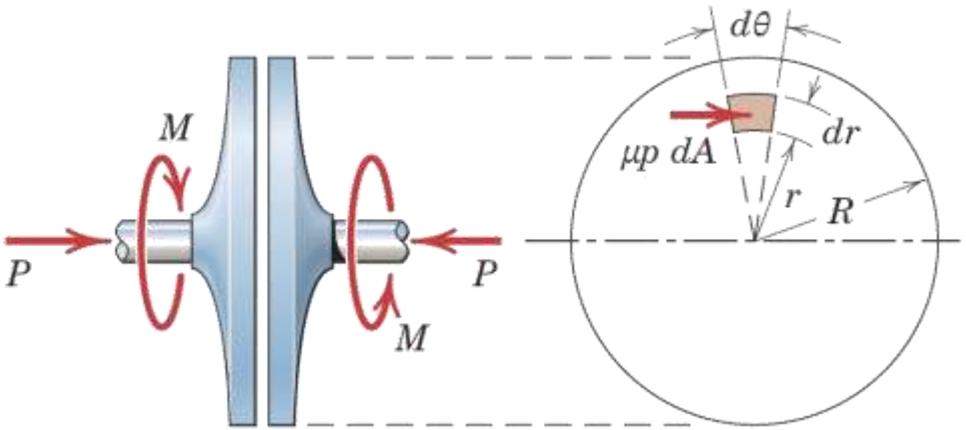
- Integration over the Disk Area

$$M = 2/3\ \mu P(R_o^3 - R_i^3)/(R_o^2 - R_i^2)$$

Article 6/7 – Effect of Wear-In

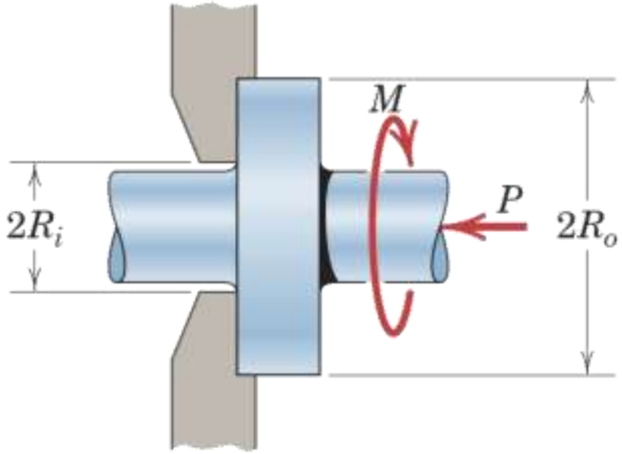
- Solid Disks

  $M = 1/2\ \mu PR$



- Ring Disks

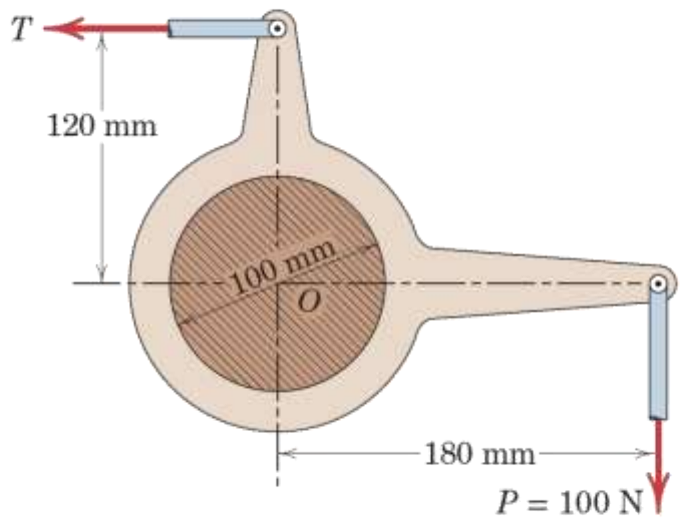
  $M = 1/2\ \mu P(R_o + R_i)$



Article 6/7 – Sample Problem 6/8 (1 of 4)

- Problem Statement

  The bell crank fits over a 100-mm-diameter shaft which is fixed and cannot rotate. The horizontal force *T* is applied to maintain equilibrium of the crank under the action of the vertical force *P* = 100 N. Determine the maximum and minimum values which *T* may have without causing the crank to rotate in either direction. The coefficient of static friction μ between the shaft and the bearing surface of the crank is 0.20.



Article 6/7 – Sample Problem 6/8 (2 of 4)

- Solution

Impending rotation occurs when the reaction R of the fixed shaft on the bell crank makes an angle $\phi = \tan^{-1} \mu$ with the normal to the bearing surface and is, therefore, tangent to the friction circle. Also, equilibrium requires that the three forces acting on the crank be concurrent at point C. These facts are shown in the free-body diagrams for the two cases of impending motion.

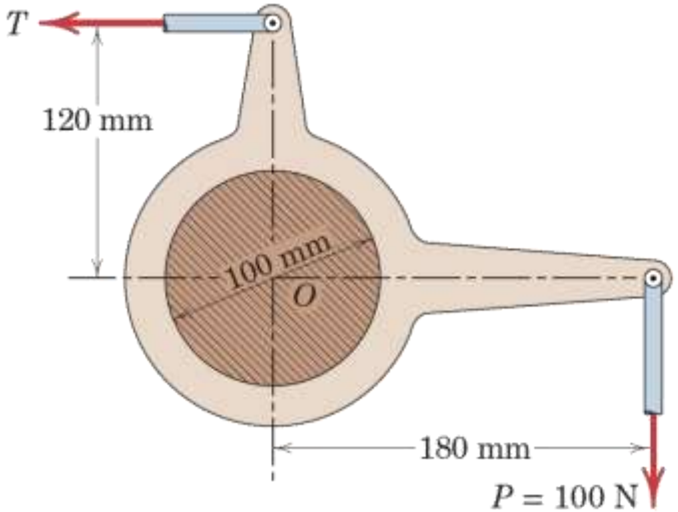
The following calculations are needed:

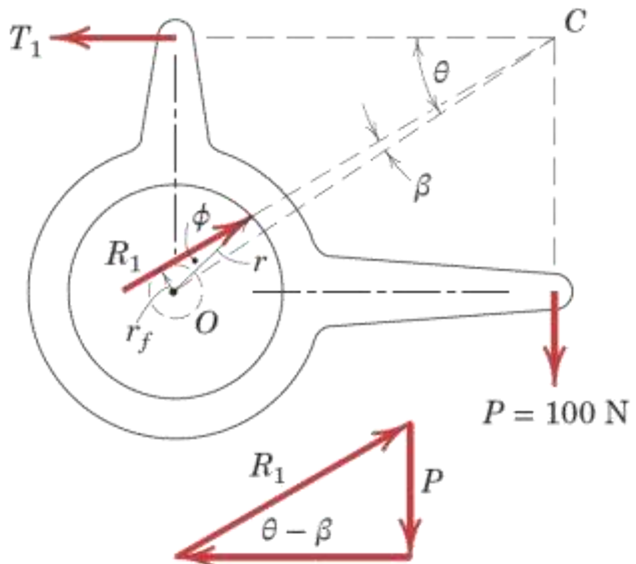
Friction angle $\phi = \tan^{-1} \mu = \tan^{-1} 0.20 = 11.31°$

Radius of friction circle $r_f = r \sin \phi = 50 \sin 11.31° = 9.81$ mm

Angle $\theta = \tan^{-1} \dfrac{120}{180} = 33.7°$

Angle $\beta = \sin^{-1} \dfrac{r_f}{\overline{OC}} = \sin^{-1} \dfrac{9.81}{\sqrt{(120)^2 + (180)^2}} = 2.60°$



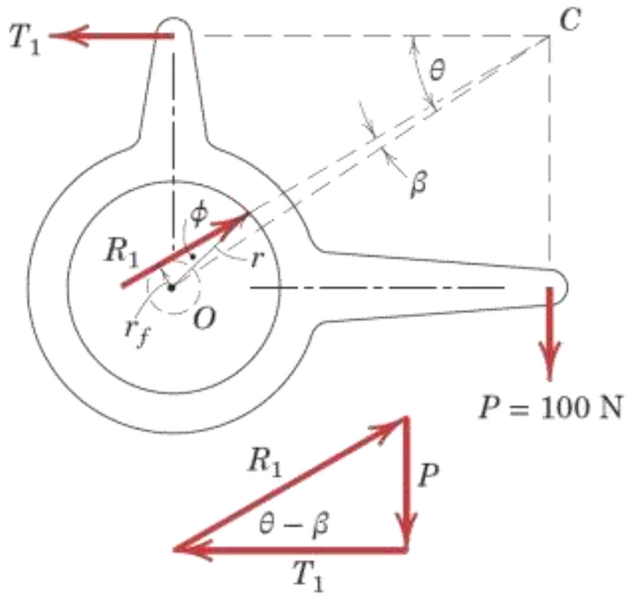


Article 6/7 – Sample Problem 6/8 (3 of 4)

- Impending Counterclockwise Motion

$$T_1 = P \cot(\theta - \beta) = 100 \cot(33.7° - 2.60°)$$

$$T_1 = T_{max} = 165.8 \text{ N} \qquad \textit{Ans.}$$



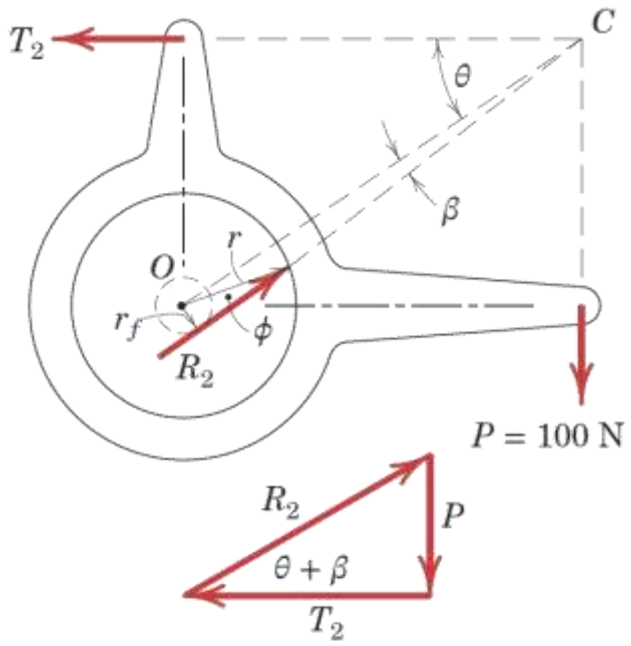
(a) Counterclockwise motion impends

Article 6/7 – Sample Problem 6/8 (4 of 4)

- Impending Clockwise Motion

$$T_2 = P \cot(\theta + \beta) = 100 \cot(33.7^\circ + 2.60^\circ)$$

$$T_2 = T_{\min} = 136.2 \text{ N} \qquad \textit{Ans.}$$



(*b*) Clockwise motion impends

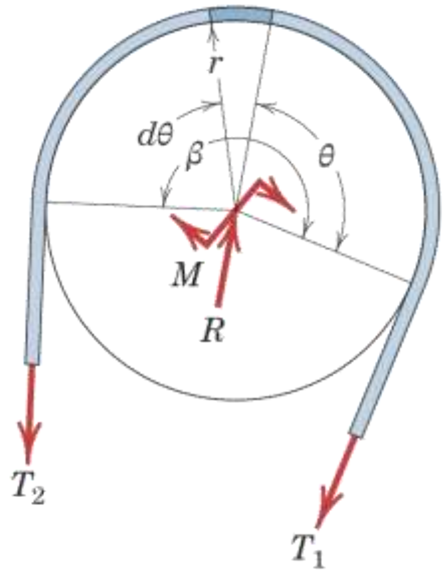
Article 6/8 Flexible Belts

- Introduction

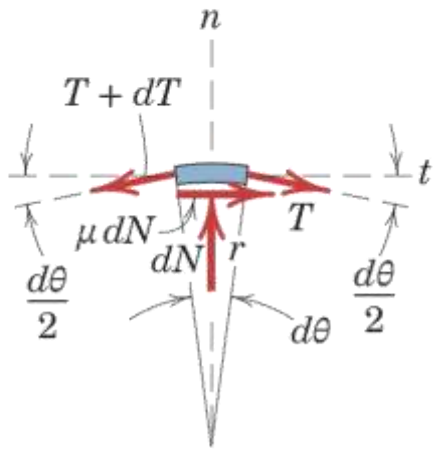
- Illustration

Article 6/8 – Derivation (1 of 2)

- Situation of Interest



- Free-Body Diagram of a Belt Segment

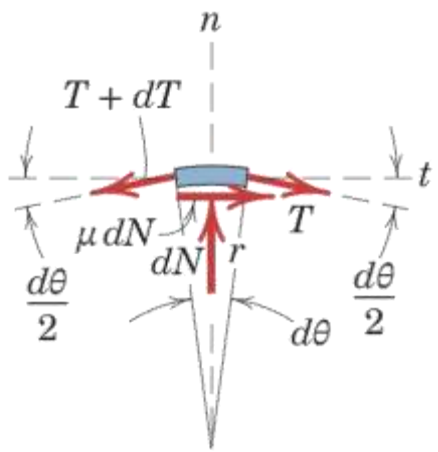


Article 6/8 – Derivation (2 of 2)

- Equilibrium Conditions

$$T\cos\frac{d\theta}{2} + \mu\, dN = (T + dT)\cos\frac{d\theta}{2} \qquad dN = (T + dT)\sin\frac{d\theta}{2} + T\sin\frac{d\theta}{2}$$

$$\mu\, dN = dT \qquad dN = T\, d\theta$$



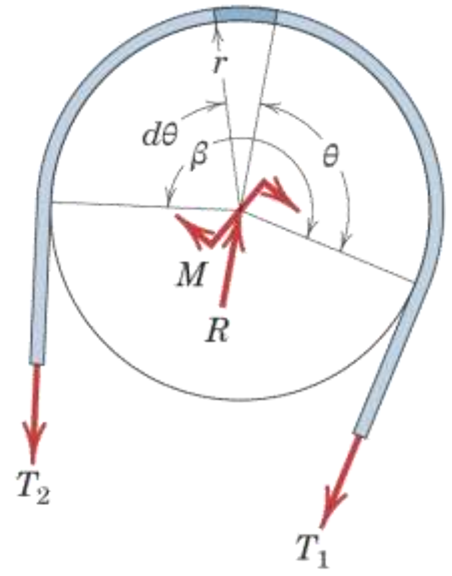
- Combine and Integrate

$$dT/T = \mu\, d\theta \quad \rightarrow \quad T_2 = T_1 e^{\mu\beta}$$

Article 6/8 – The Belt Friction Equation

- Comments about the Equation: $T_2 = T_1 e^{\mu\beta}$

  - T_2 = the larger of the two belt tensions (it points in the direction the belt will tend to move in along the surface).
  - T_1 = the smaller of the two belt tensions (it opposes the motion of the belt along the surface).
  - μ = the coefficient of friction between the belt and the surface.
  - β = the total angle of belt contact with the surface. It must be expressed in radians.

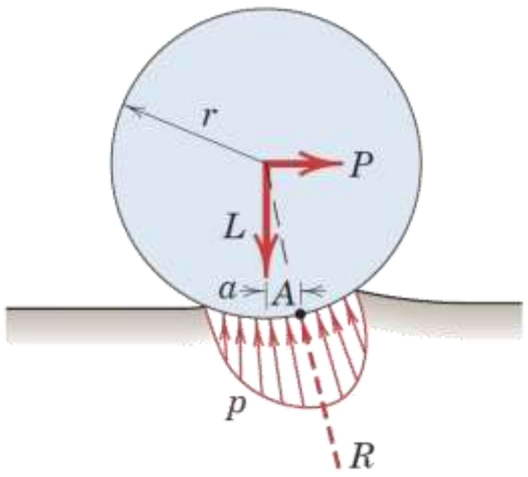


Article 6/9 Rolling Resistance

- Introduction

- Illustration

- Load Analysis
  - Load on the Wheel, L
  - Force to Maintain Speed, P
  - Distribution of Pressure over the Contact Area, p
  - Resultant Pressure-Force Distribution, R
  - Location of Resultant Pressure Force, A
  - Forward Shift of the Pressure Center from the Wheel Center, a

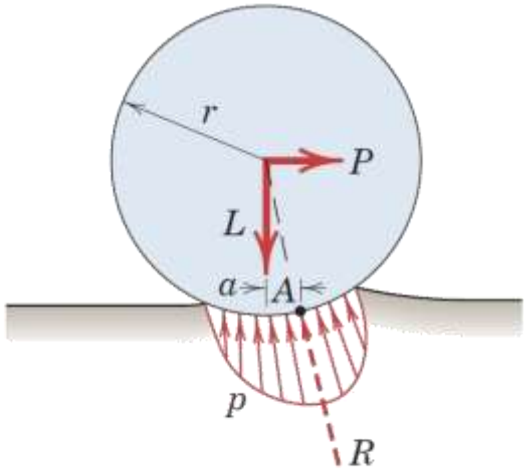


Article 6/9 – Load Analysis

- Moment Sum about the Pressure Center

  $P = a/rL = \mu_r L$

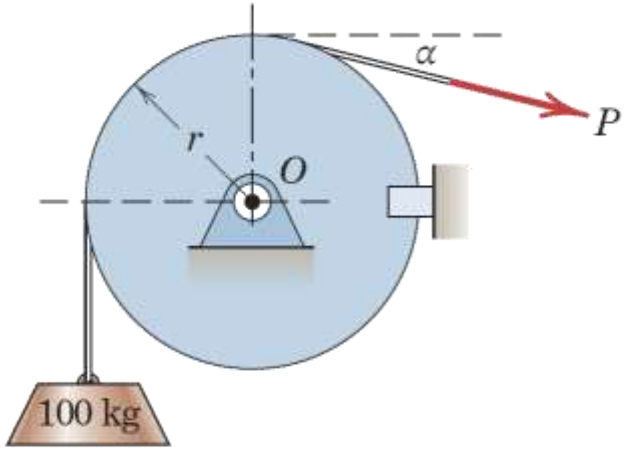
- Coefficient of Rolling Resistance, μ_r



Article 6/9 – Sample Problem 6/9 (1 of 3)

- Problem Statement

  A flexible cable which supports the 100-kg load is passed over a fixed circular drum and subjected to a force P to maintain equilibrium. The coefficient of static friction μ between the cable and the fixed drum is 0.30. (a) For $\alpha = 0$, determine the maximum and minimum values which P may have in order not to raise or lower the load. (b) For $P = 500$ N, determine the minimum value which the angle α may have before the load begins to slip.



Article 6/9 – Sample Problem 6/9 (2 of 3)

- Part (a) $\alpha = 0$

With $\alpha = 0$ the angle of contact is $\beta = \pi/2$ rad. ① For impending upward motion of the load, $T_2 = P_{max}$, $T_1 = 981$ N, and we have

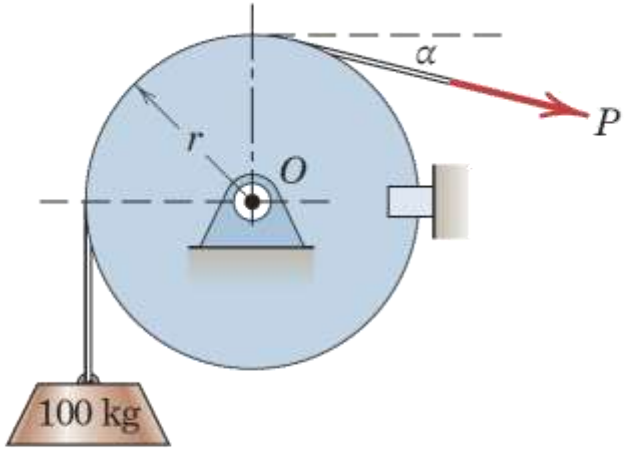
$$P_{max}/981 = e^{0.30(\pi/2)} \qquad P_{max} = 981(1.602) = 1572 \text{ N} \quad ② \quad \textit{Ans.}$$

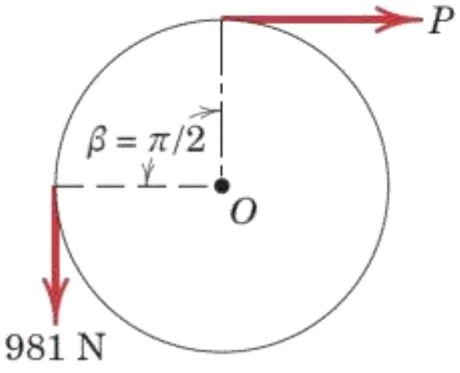
For impending downward motion of the load, $T_2 = 981$ N and $T_1 = P_{min}$. Thus,

$$981/P_{min} = e^{0.30(\pi/2)} \qquad P_{min} = 981/1.602 = 612 \text{ N} \qquad \textit{Ans.}$$

① We are careful to note that β must be expressed in radians.

② In our derivation of Eq. 6/7 be certain to note that $T_2 > T_1$.





(a) $\alpha = 0$

Article 6/9 – Sample Problem 6/9 (3 of 3)

- Part (b) P = 500 N

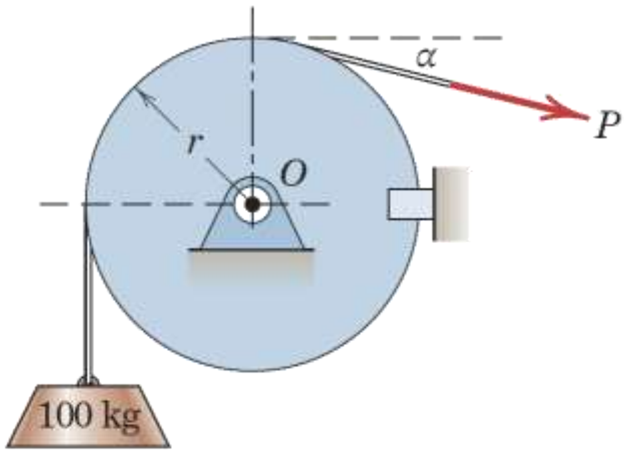
With $T_2 = 981$ N and $T_1 = P = 500$ N, Eq. 6/7 gives us

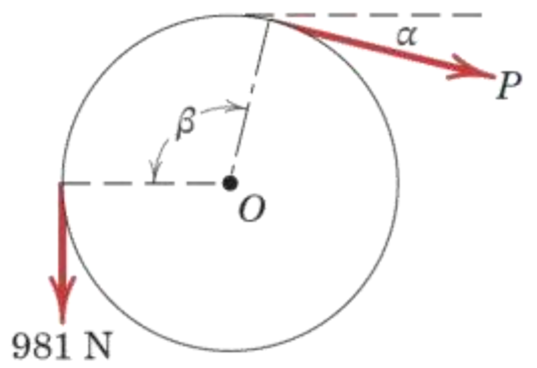
$$981/500 = e^{0.30\beta} \qquad 0.30\beta = \ln(981/500) = 0.674$$

$$\beta = 2.25 \text{ rad} \qquad \text{or} \qquad \beta = 2.25\left(\frac{360}{2\pi}\right) = 128.7^\circ$$

$$\alpha = 128.7^\circ - 90^\circ = 38.7^\circ \quad ③ \qquad \textit{Ans.}$$

③ As was noted in the derivation of Eq. 6/7, the radius of the drum does not enter into the calculations. It is only the angle of contact and the coefficient of friction which determine the limiting conditions for impending motion of the flexible cable over the curved surface.



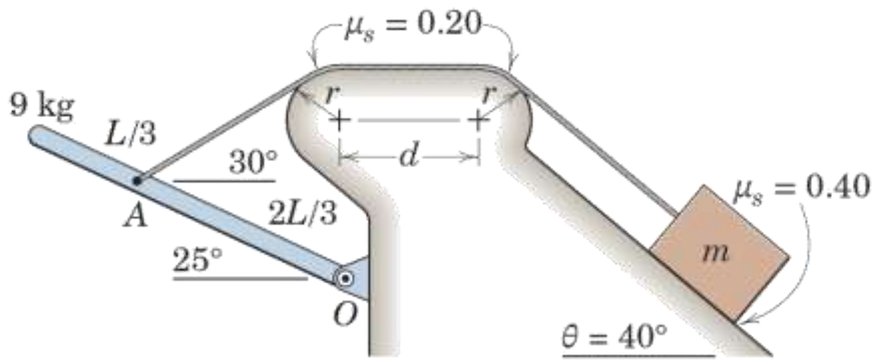


(b) P = 500 N

Article 6/9 – Sample Problem 6/10 (1 of 4)

- Problem Statement

  Determine the range of mass *m* over which the system is in static equilibrium. The coefficient of static friction between the cord and the upper curved surface is 0.20, while that between the block and the incline is 0.40. Neglect friction at the pivot *O*.

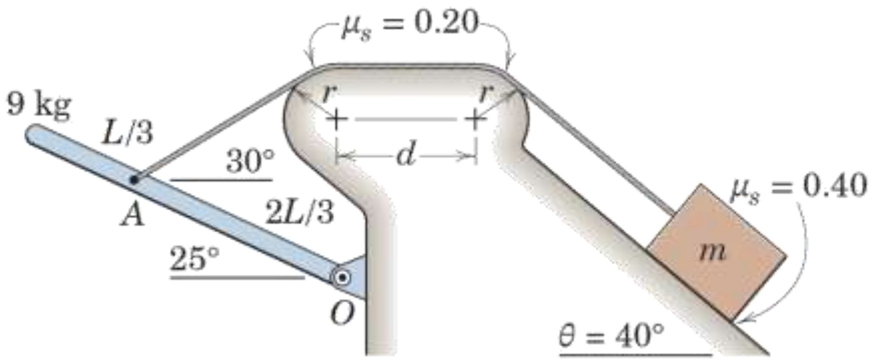


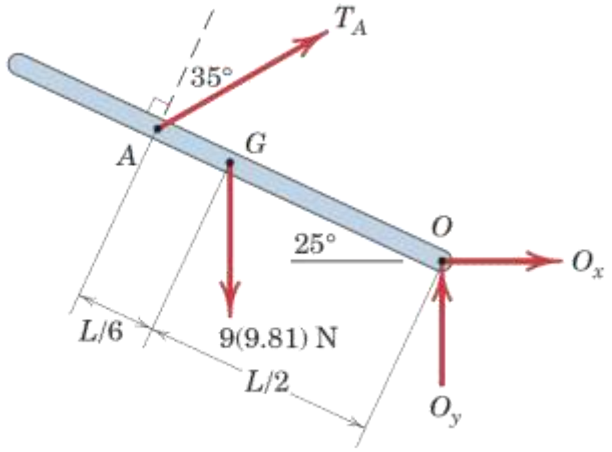
Article 6/9 – Sample Problem 6/10 (2 of 4)

- Equilibrium of Slender Bar

$$[\Sigma M_O = 0] \qquad -T_A\left(\frac{2L}{3}\cos 35^\circ\right) + 9(9.81)\left(\frac{L}{2}\cos 25^\circ\right) = 0$$

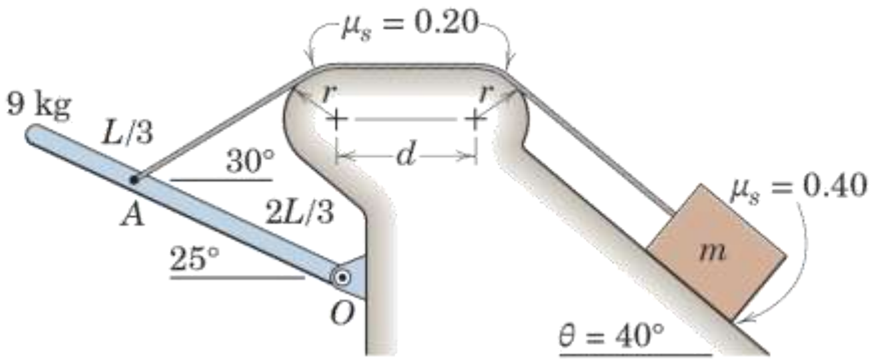
$$T_A = 73.3 \text{ N}$$





Article 6/9 – Sample Problem 6/10 (3 of 4)

- Case I: Motion Impends Up the Incline



The tension $T_A = 73.3$ N is the larger of the two tensions associated with the rough rounded surface. From Eq. 6/7 we have

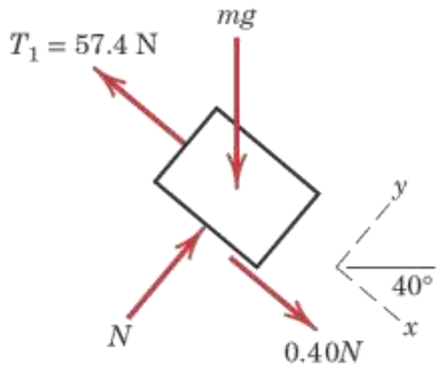
$[T_2 = T_1 e^{\mu_s \beta}]$ $\qquad 73.3 = T_1 e^{0.20[30° + 40°]\pi/180°}$ $\qquad T_1 = 57.4$ N ①

From the FBD of the block for Case I:

$[\Sigma F_y = 0]$ $\qquad N - mg \cos 40° = 0 \qquad N = 0.766mg$

$[\Sigma F_x = 0]$ $\qquad -57.4 + mg \sin 40° + 0.40(0.766mg) = 0$

$$mg = 60.5 \text{ N} \qquad m = 6.16 \text{ kg}$$

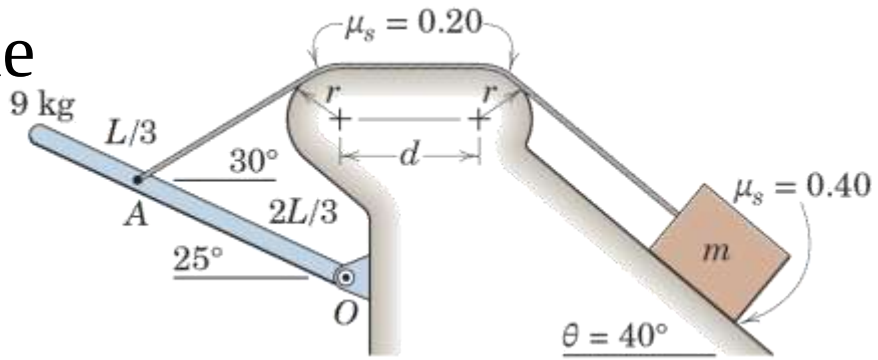


Case I

① Only the total angular contact enters Eq. 6/7 (as β). So our results are independent of the quantities r and d.

Article 6/9 – Sample Problem 6/10 (4 of 4)

- Case II: Motion Impends Down the Incline



The value $T_A = 73.3$ N is unchanged, but now this is the smaller of the two tensions in Eq. 6/7.

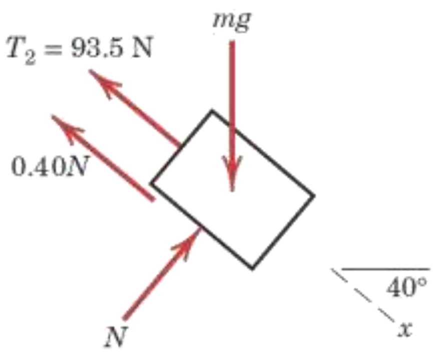
$[T_2 = T_1 e^{\mu_s \beta}]$ $\quad\quad T_2 = 73.3e^{0.20[30° + 40°]\pi/180°}$ $\quad\quad T_2 = 93.5$ N

Considering the FBD of the block for Case II, we see that the normal force N is unchanged from Case I.

$[\Sigma F_x = 0]$ $\quad\quad -93.5 - 0.4(0.766mg) + mg\sin 40° = 0$

$$mg = 278 \text{ N} \qquad m = 28.3 \text{ kg}$$

So the requested range is $6.16 \leq m \leq 28.3$ kg. ② *Ans.*



Case II

② Re-solve the entire problem if the ramp angle θ were changed to 20°, with all other given information remaining constant. Be alert for a surprising result!

CHAPTER 7
VIRTUAL WORK

CHAPTER OUTLINE

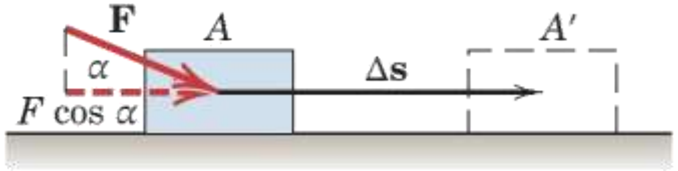
BuildPix/(c)Construction Photography/Photoshoot

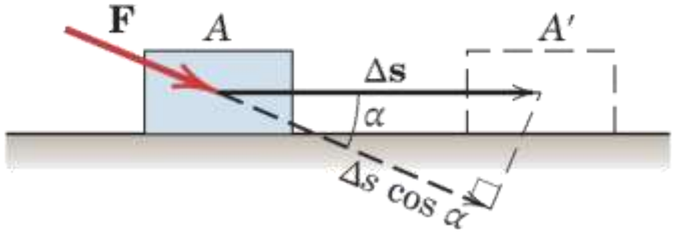
Article 7/1 Introduction

- Introduction

Article 7/2 Work

- Work of a Force
  - Illustration



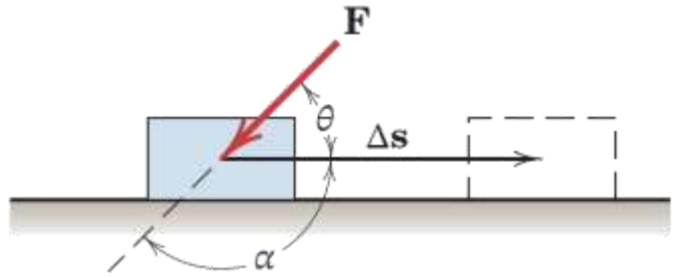


  - Calculation
    - $U = (F \cos \alpha)\Delta s = F(\Delta s \cos \alpha)$
  - Scalar Quantity

Article 7/2 – Sign Convention

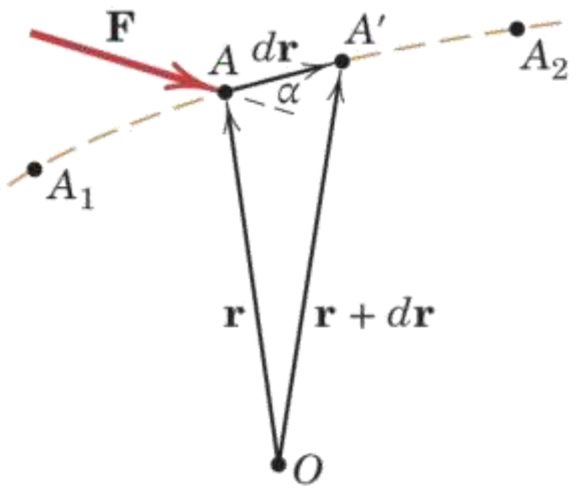
- Positive Work
  - Working component of *F* is in the same direction as the displacement.

- Negative Work
  - Working component of *F* is in the opposite direction as the displacement.

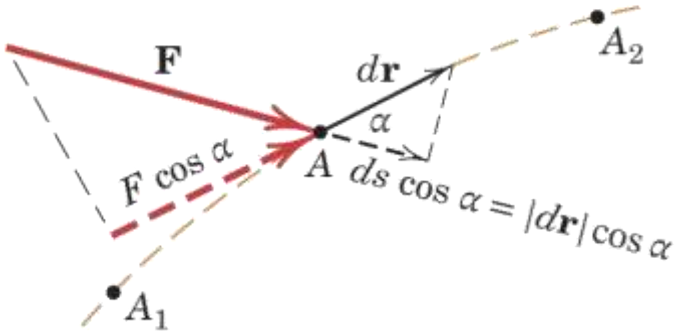


Article 7/2 – General Situation

- General Illustration of Work by a Force

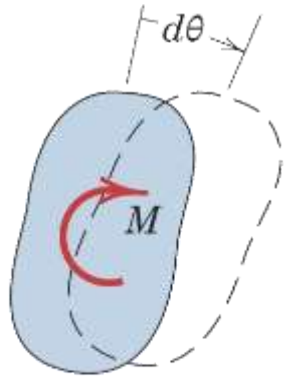


- General Expression for Work by a Force
  - $dU = \mathbf{F} \cdot d\mathbf{r}$

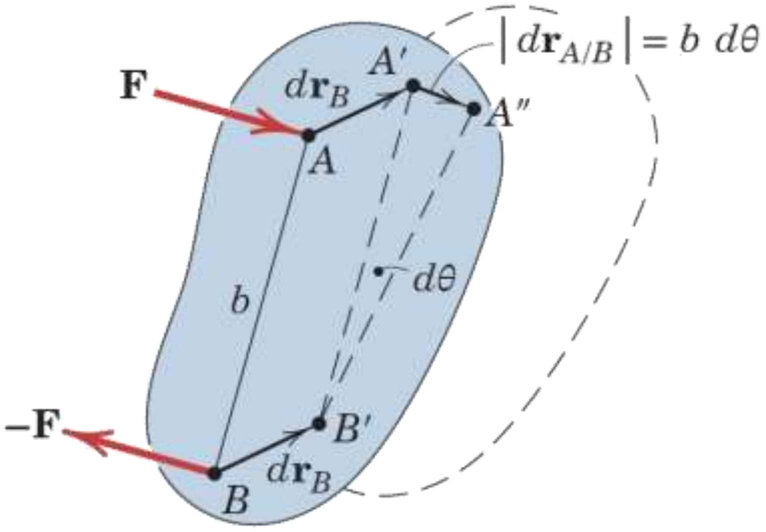


Article 7/2 – Work of a Couple

- General Illustration of Work by a Couple



- General Expression for Work by a Couple
  - $dU = M\, d\theta$



- Sign Convention
  - Positive if couple acts in the same sense as rotation.
  - Negative if couple acts in the opposite sense as rotation.

Article 7/2 – Dimensions and Units of Work

- Dimensions
  - (Force) × (Distance)

- SI Units
  - Joule (J)
  - J = N·m

- U.S. Customary Units
  - foot-pound (ft-lb)
  - Not to be confused with moment which is pound-foot (lb-ft) and a vector quantity, whereas work is a scalar quantity.

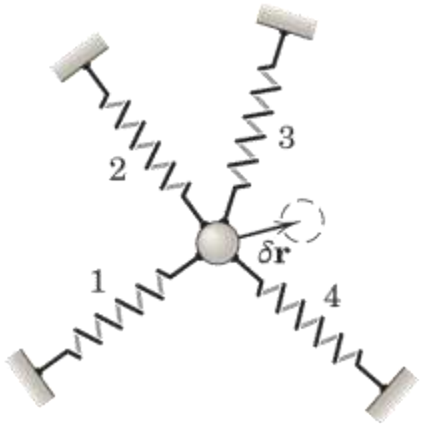
Article 7/2 – Virtual Work

- Virtual Displacement
  - Does not actually exist, but is assumed to exist.
  - Infinitesimal in size.
  - Consistent with the constrains of the system.
  - Instantaneous in time.

- Virtual Work done by a Force and Couple
  - $\delta U = \mathbf{F} \cdot d\mathbf{r}$ or $\delta U = F\, \delta s \cos \alpha$
  - $\delta U = M\, \delta\theta$

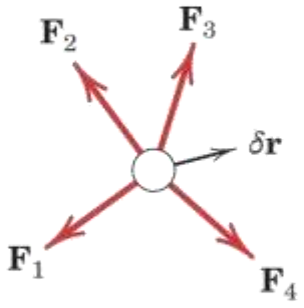
Article 7/3 Equilibrium

- Equilibrium of a Particle



- Virtual Work done on the Particle

$$\delta U = \mathbf{F}_1 \cdot \delta\mathbf{r} + \mathbf{F}_2 \cdot \delta\mathbf{r} + \mathbf{F}_3 \cdot \delta\mathbf{r} + \cdots = \Sigma\mathbf{F} \cdot \delta\mathbf{r}$$



- Scalar Expression of Virtual Work

$$\begin{aligned}\delta U = \Sigma\mathbf{F} \cdot \delta\mathbf{r} &= (\mathbf{i}\,\Sigma F_x + \mathbf{j}\,\Sigma F_y + \mathbf{k}\,\Sigma F_z) \cdot (\mathbf{i}\,\delta x + \mathbf{j}\,\delta y + \mathbf{k}\,\delta z) \\ &= \Sigma F_x\,\delta x + \Sigma F_y\,\delta y + \Sigma F_z\,\delta z = 0\end{aligned}$$

- Result

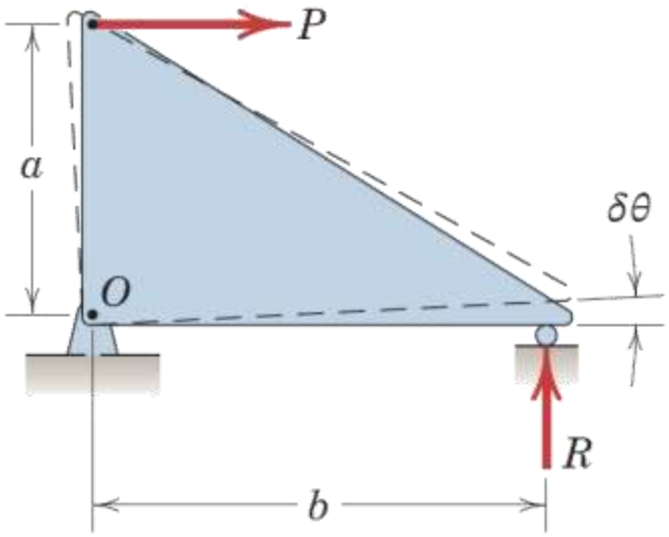
Article 7/3 – Equilibrium of a Rigid Body

- Equilibrium of a Rigid Body

- Determine the Roller Force, *R*

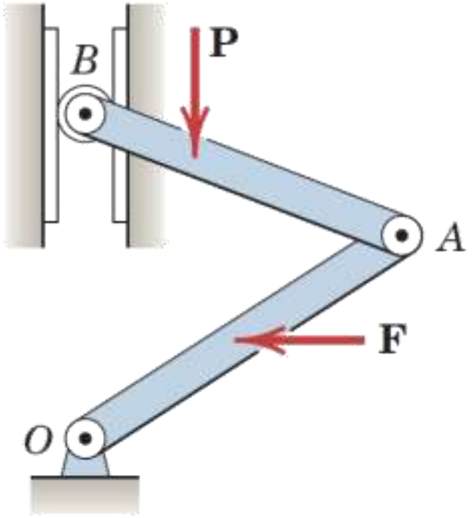
- Virtual Work Calculation
  - $\delta U = 0 = Rb\ \delta\theta - Pa\ \delta\theta$

- Result
  - $Pa - Rb = 0$
  - Equivalent to Moment Summation about Hinge *O*



Article 7/3 – Equilibrium of Ideal Systems of Rigid Bodies

- Ideal System
  - Friction is Neglected
  - Connections do not Absorb Energy



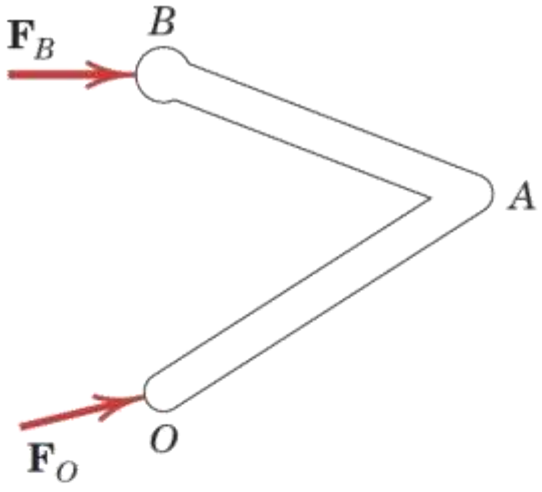
(a) Active forces

- Example

- Active Forces
  - Perform Virtual Work

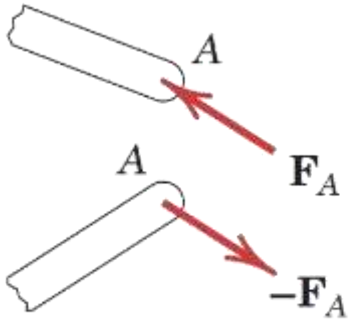
Article 7/3 – Other Types of Forces to Consider

- Reactive Forces
  - Act at Fixed Locations
  - Do not Perform Virtual Work



(b) Reactive forces

- Internal Forces
  - Occur in Equal and Opposite Pairs
  - Net Virtual Work is Zero



(c) Internal forces

Article 7/3 – Principle of Virtual Work (1 of 2)

- The Principle Stated
  - The virtual work done by external active forces on an ideal mechanical system in equilibrium is zero for any and all virtual displacements consistent with the constraints.
- Mathematical Expression
  - $\delta U = 0$

Article 7/3 – Principle of Virtual Work (2 of 2)

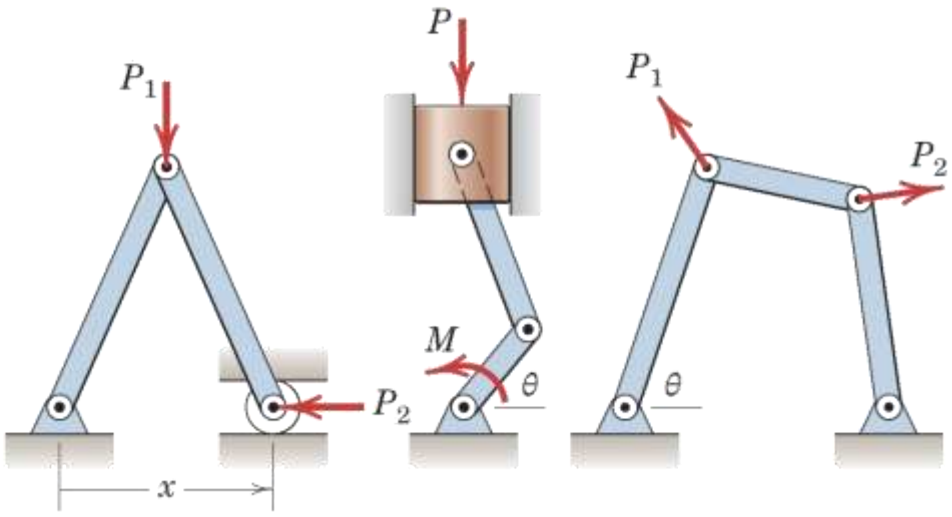
- Advantages of Virtual Work
  - It is not necessary to dismember ideal systems to establish the relations between active forces.
  - The relations between active forces may be determined directly without reference to the reactive forces.

- General Comments
  - Method of virtual work requires internal friction forces do negligible work, or else the work by friction must be considered.
  - Construct an active-force diagram which shows only the *active forces* which act on the body. The reactive forces and internal forces do not perform any virtual work and do not need to be included.

Article 7/3 – Degrees of Freedom (1 of 2)

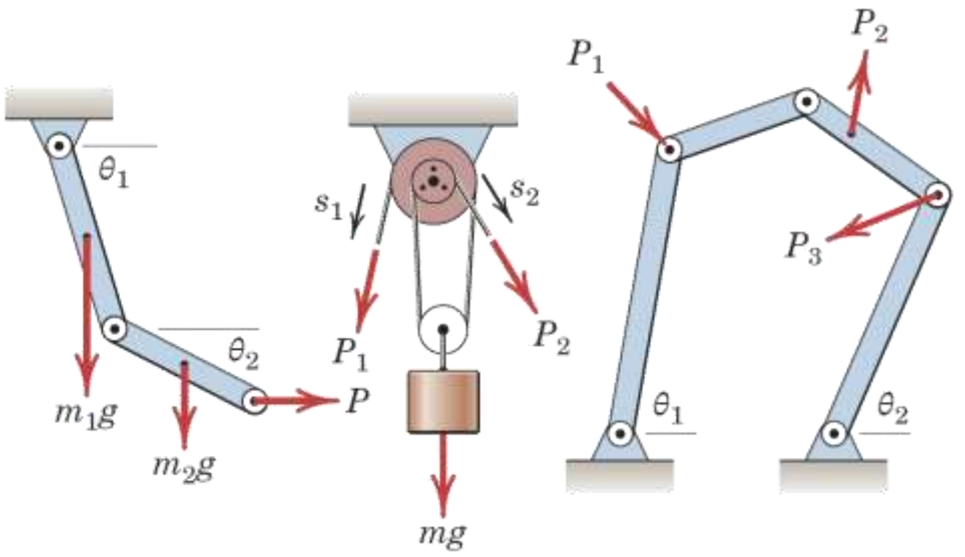
- Definition
  - The number of degrees of freedom of a mechanical system is the number of independent coordinates needed to specify completely the configuration of the system.
  - The principle of virtual work may be applied as many times as there are degrees of freedom.
  - With each application, allow only one independent coordinate to change at a time while holding the others constant.

- One-Degree-of-Freedom Systems



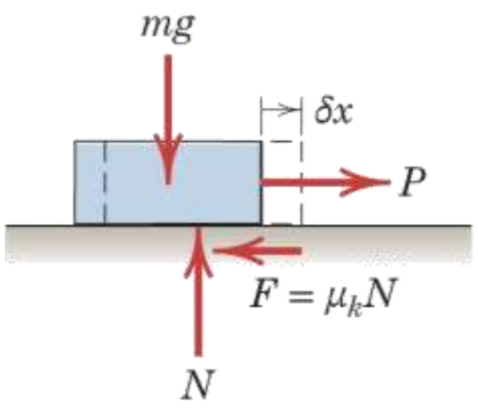
Article 7/3 – Degrees of Freedom (2 of 2)

- Two-Degree-of-Freedom Systems



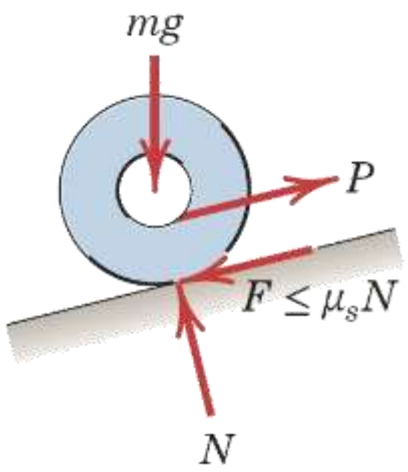
Article 7/3 – Systems with Friction (1 of 2)

- Effect of Friction
  - Dissipates Positive Work by External Forces
  - Negative Work which cannot be Regained



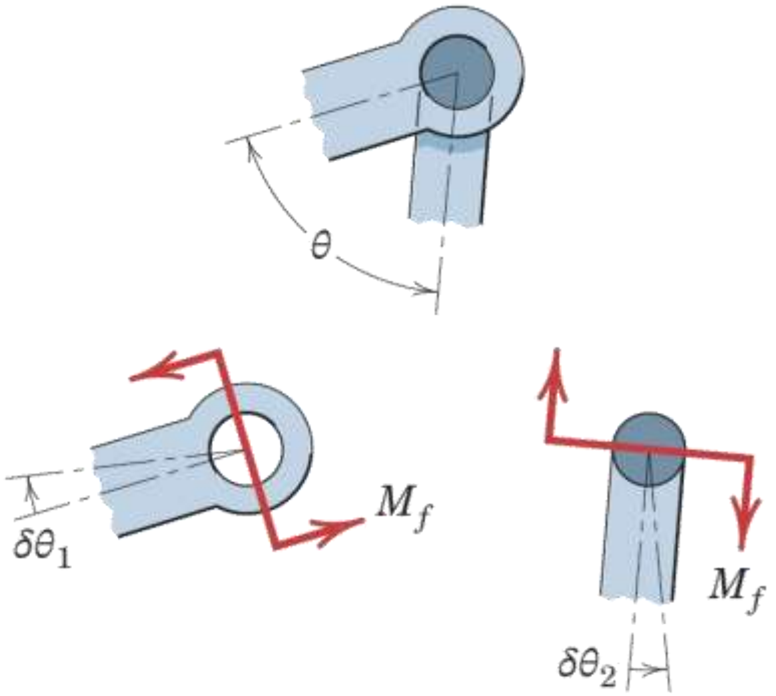
- Illustration with Sliding Block

- Situation of a Rolling Wheel



Article 7/3 – Systems with Friction (2 of 2)

- Friction at a Pinned Connection



Article 7/3 – Mechanical Efficiency (1 of 3)

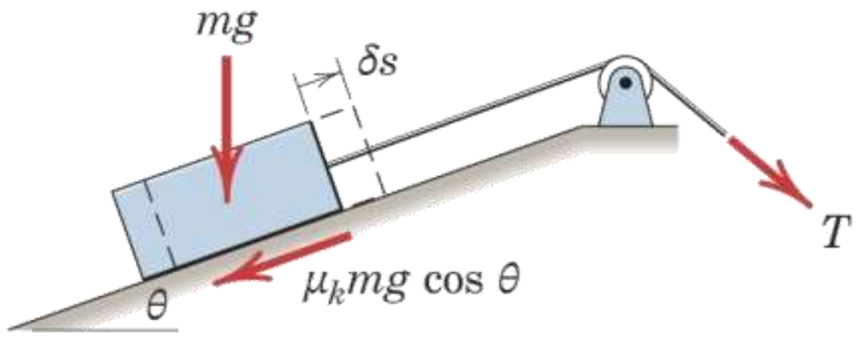
- Efficiency Defined
  - Efficiency is the ratio of the output work from a machine and the input work to the machine.

- Mathematical Expression

$$e = \frac{\text{output work}}{\text{input work}}$$

Article 7/3 – Mechanical Efficiency (2 of 3)

- Example 1: Block Moving up an Incline

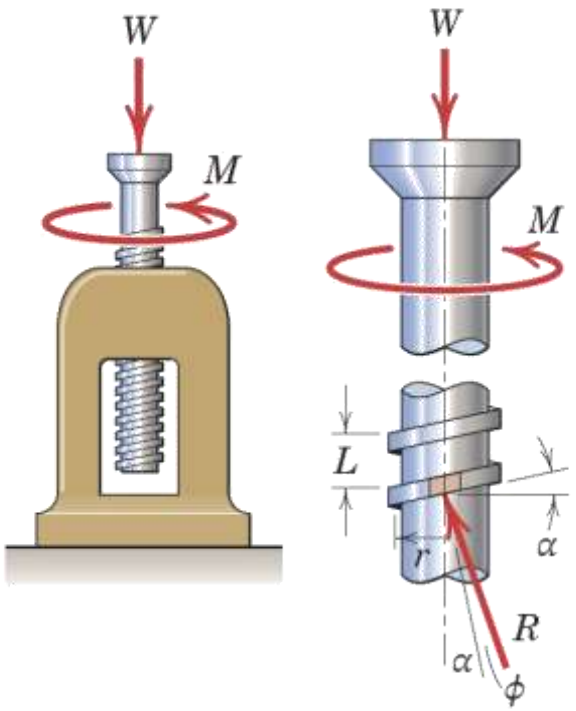


- Efficiency

$$e = \frac{mg\,\delta s\,\sin\theta}{mg(\sin\theta + \mu_k\cos\theta)\,\delta s} = \frac{1}{1 + \mu_k\cot\theta}$$

Article 7/3 – Mechanical Efficiency (3 of 3)

- Example 2: Screw Jack



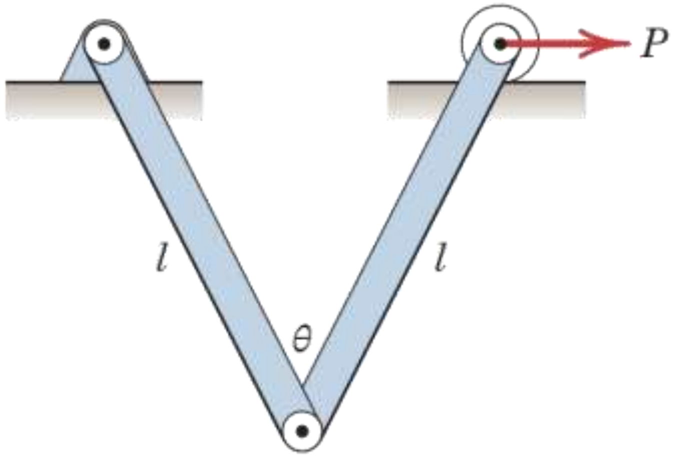
- Efficiency

$$e = \frac{Wr\,\delta\theta \tan\alpha}{Wr\,\delta\theta \tan(\alpha+\phi)} = \frac{\tan\alpha}{\tan(\alpha+\phi)}$$

Article 7/3 – Sample Problem 7/1 (1 of 3)

- Problem Statement

  Each of the two uniform hinged bars has a mass m and a length l, and is supported and loaded as shown. For a given force P determine the angle θ for equilibrium.



Article 7/3 – Sample Problem 7/1 (2 of 3)

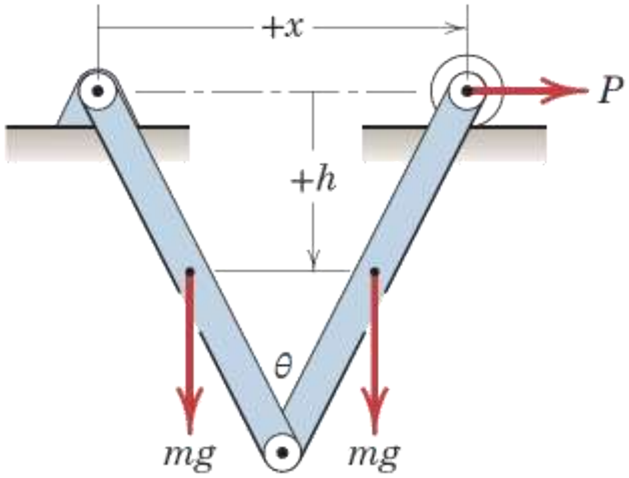
- Solution

The principle of virtual work requires that the total work of all external active forces be zero for any virtual displacement consistent with the constraints. Thus, for a movement δx the virtual work becomes

$[\delta U = 0]$ $$P\,\delta x + 2mg\,\delta h = 0$$ ①

We now express each of these virtual displacements in terms of the variable θ, the required quantity. Hence,

$$x = 2l \sin\frac{\theta}{2} \qquad \text{and} \qquad \delta x = l\cos\frac{\theta}{2}\,\delta\theta$$



① Note carefully that with x positive to the right δx is also positive to the right in the direction of P, so that the virtual work is $P(+\delta x)$. With h positive down δh is also mathematically positive down in the direction of mg, so that the correct mathematical expression for the work is $mg(+\delta h)$. When we express δh in terms of $\delta\theta$ in the next step, δh will have a negative sign, thus bringing our mathematical expression into agreement with the physical observation that the weight mg does negative work as each center of mass moves upward with an increase in x and θ.

Article 7/3 – Sample Problem 7/1 (3 of 3)

- Solution (cont.)

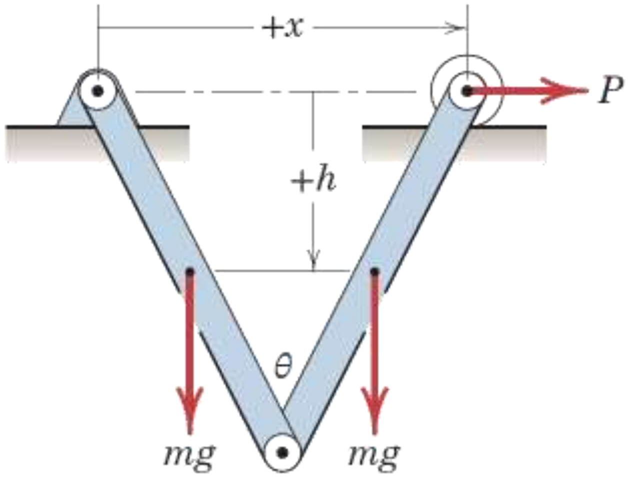
$$h = \frac{l}{2}\cos\frac{\theta}{2} \qquad \text{and} \qquad \delta h = -\frac{l}{4}\sin\frac{\theta}{2}\,\delta\theta \quad ②$$

Substitution into the equation of virtual work gives us

$$Pl\cos\frac{\theta}{2}\,\delta\theta - 2mg\frac{l}{4}\sin\frac{\theta}{2}\,\delta\theta = 0$$

from which we get

$$\tan\frac{\theta}{2} = \frac{2P}{mg} \qquad \text{or} \qquad \theta = 2\tan^{-1}\frac{2P}{mg} \qquad \textit{Ans.}$$

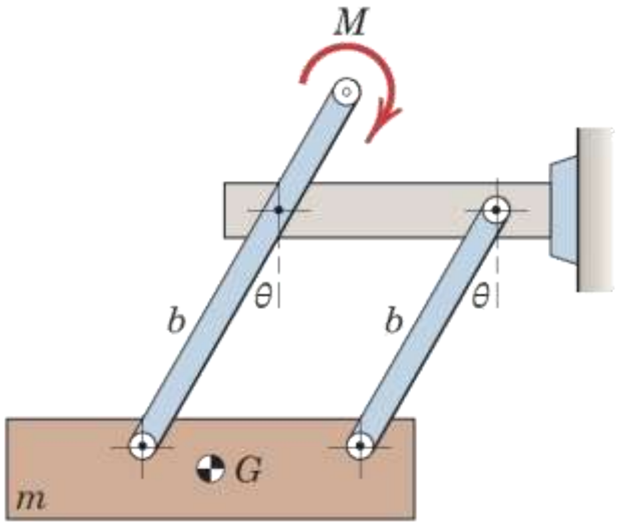


② We obtain δh and δx with the same mathematical rules of differentiation with which we may obtain dh and dx.

Article 7/3 – Sample Problem 7/2 (1 of 3)

- Problem Statement

  The mass m is brought to an equilibrium position by the application of the couple M to the end of one of the two parallel links which are hinged as shown. The links have negligible mass, and all friction is assumed to be absent. Determine the expression for the equilibrium angle θ assumed by the links with the vertical for a given value of M. Consider the alternative of a solution by force and moment equilibrium.



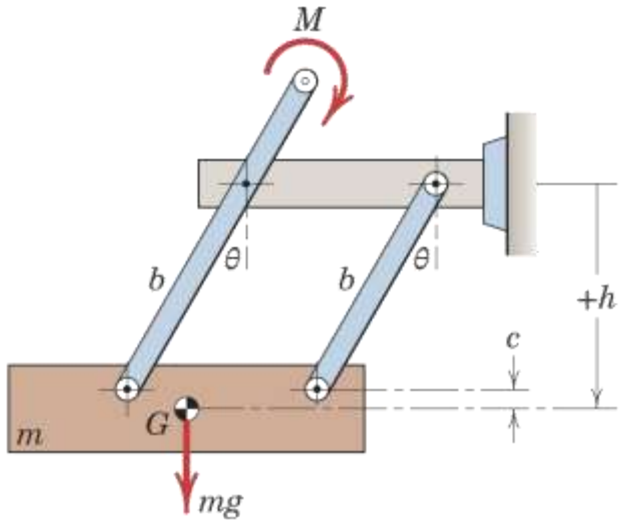
Article 7/3 – Sample Problem 7/2 (2 of 3)

- Solution

The vertical position of the center of mass G is designated by the distance h below the fixed horizontal reference line and is $h = b \cos\theta + c$. The work done by mg during a movement δh in the direction of mg is

$$
\begin{aligned}
+mg\,\delta h &= mg\,\delta(b\cos\theta + c) \\
&= mg(-b\sin\theta\,\delta\theta + 0) \\
&= -mgb\sin\theta\,\delta\theta
\end{aligned}
$$

The minus sign shows that the work is negative for a positive value of $\delta\theta$. ① The constant c drops out since its variation is zero.



① Again, as in Sample Problem 7/1, we are consistent mathematically with our definition of work, and we see that the algebraic sign of the resulting expression agrees with the physical change.

Article 7/3 – Sample Problem 7/2 (3 of 3)

- Solution (cont.)

With θ measured positive in the clockwise sense, $\delta\theta$ is also positive clockwise. Thus, the work done by the clockwise couple M is $+M\ \delta\theta$. Substitution into the virtual-work equation gives us

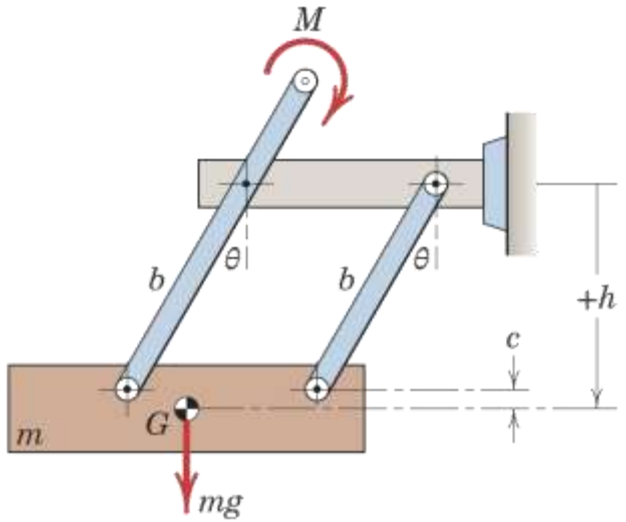
$[\delta U = 0]$ $$M\ \delta\theta + mg\ \delta h = 0$$

which yields

$$M\ \delta\theta = mgb \sin\theta\ \delta\theta$$

$$\theta = \sin^{-1}\frac{M}{mgb} \qquad \textit{Ans.}$$

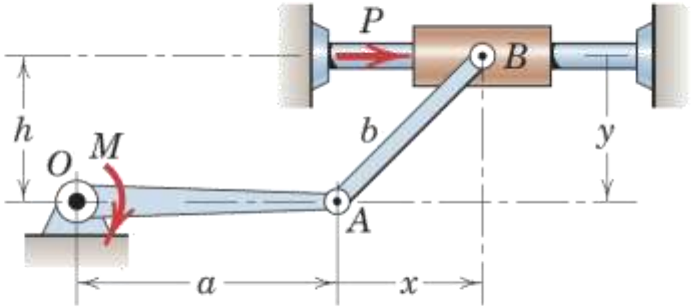
Inasmuch as $\sin\theta$ cannot exceed unity, we see that for equilibrium, M is limited to values that do not exceed mgb.



Article 7/3 – Sample Problem 7/3 (1 of 3)

- Problem Statement

  For link *OA* in the horizontal position shown, determine the force *P* on the sliding collar which will prevent *OA* from rotating under the action of the couple *M*. Neglect the mass of the moving parts.



Article 7/3 – Sample Problem 7/3 (2 of 3)

- Solution

We will give the crank OA a small clockwise angular movement $\delta\theta$ as our virtual displacement and determine the resulting virtual work done by M and P. From the horizontal position of the crank, the angular movement gives a downward displacement of A equal to

$$\delta y = a\,\delta\theta \quad ①$$

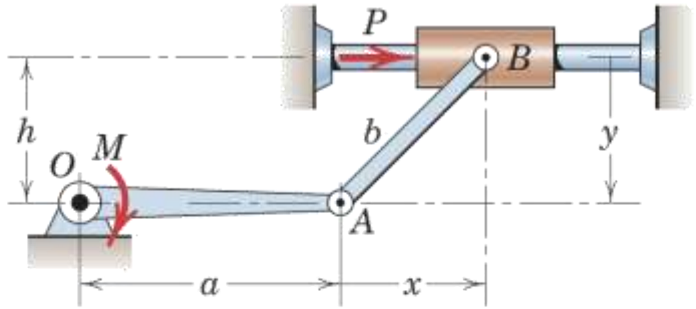
where $\delta\theta$ is, of course, expressed in radians.

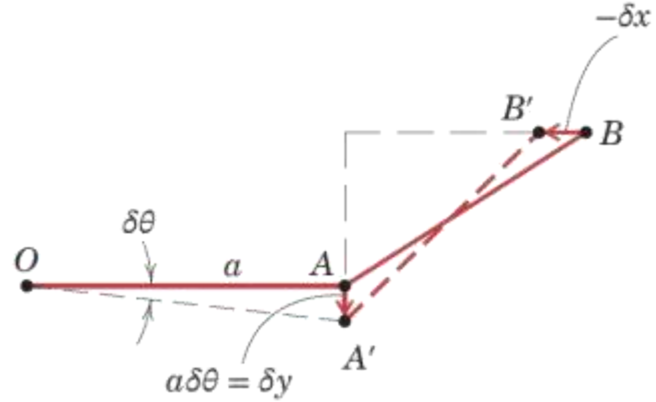
From the right triangle for which link AB is the constant hypotenuse we may write

$$b^2 = x^2 + y^2$$

We now take the differential of the equation and get

$$0 = 2x\,\delta x + 2y\,\delta y \qquad \text{or} \qquad \delta x = -\frac{y}{x}\,\delta y \quad ②$$





① Note that the displacement $a\,\delta\theta$ of point A would no longer equal δy if the crank OA were not in a horizontal position.

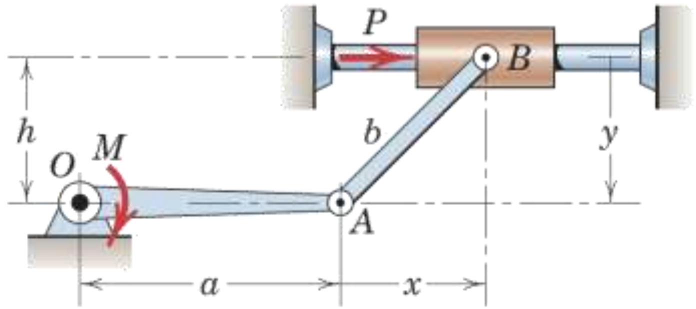
② The length b is constant so that $\delta b = 0$. Notice the negative sign, which merely tells us that if one change is positive, the other must be negative.

Article 7/3 – Sample Problem 7/3 (3 of 3)

- Solution (cont.)

Thus,

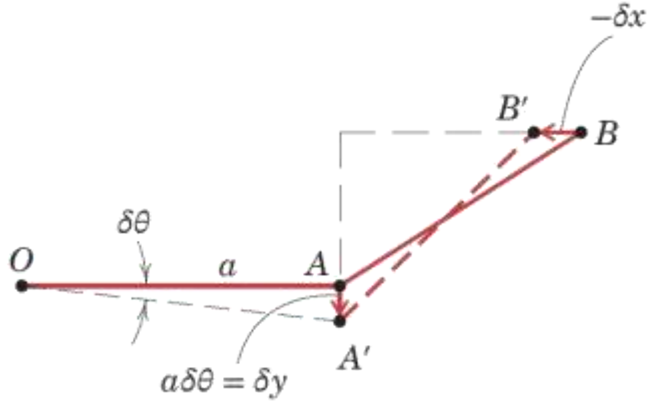
$$\delta x = -\frac{y}{x} a \, \delta\theta$$



and the virtual-work equation becomes

$[\delta U = 0]$ $\qquad M\,\delta\theta + P\,\delta x = 0 \qquad M\,\delta\theta + P\left(-\frac{y}{x} a\,\delta\theta\right) = 0$ ③

$$P = \frac{Mx}{ya} = \frac{Mx}{ha}$$ *Ans.*

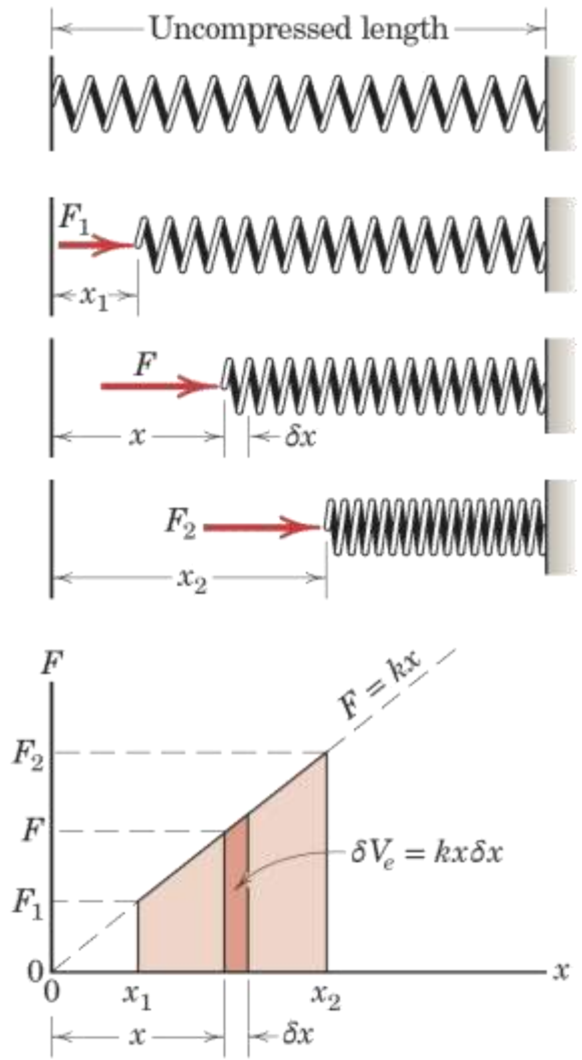


③ We could just as well use a counterclockwise virtual displacement for the crank, which would merely reverse the signs of all terms.

Article 7/4 Potential Energy and Stability

- Elastic Potential Energy

- Illustration
  - Linear Spring Stiffness, k
  - Spring Force, $F = kx$
  - Work done on the Spring, $dU = F\,dx$
  - Potential Energy, $V_e = \frac{1}{2} kx^2$
  - Change in Potential Energy, $\Delta V_e = \frac{1}{2} k(x_2^2 - x_1^2)$

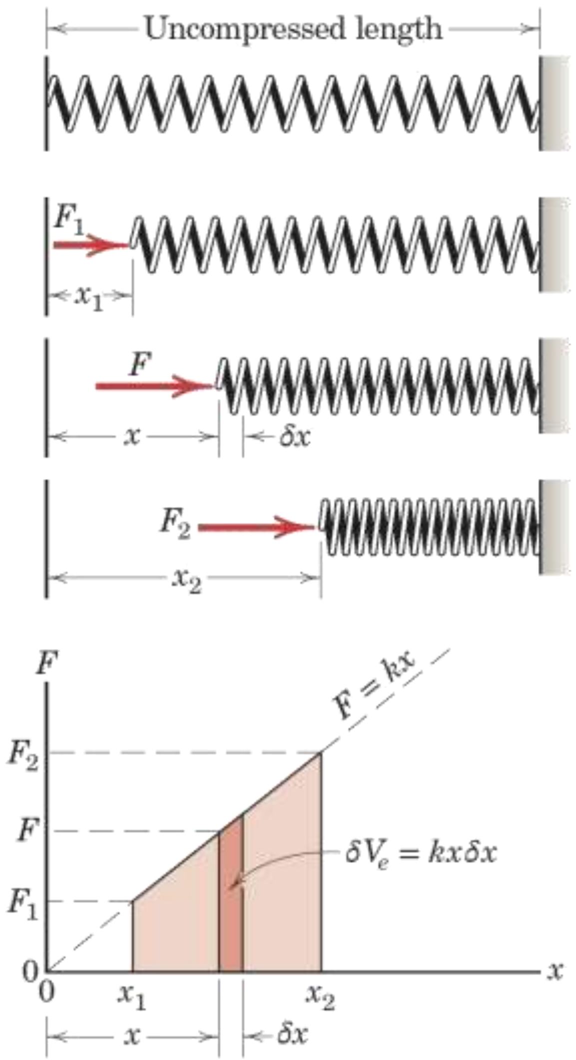


Article 7/4 – Elastic Potential Energy (cont.)

- Virtual Displacements and Potential Energy
  - Virtual Displacement, δx
  - Virtual Change in Potential Energy, $\delta V_e = F\ \delta x$

- Sign Convention
  - If the spring is **compressed**, the virtual change in elastic potential energy is **negative**.
  - If the spring is **stretched**, the virtual change in elastic potential energy is **positive**.

- The work done on the body is the negative of the potential energy change of the spring.



Article 7/4 – Torsional Springs

- A torsional spring resists rotation of a shaft or another element.

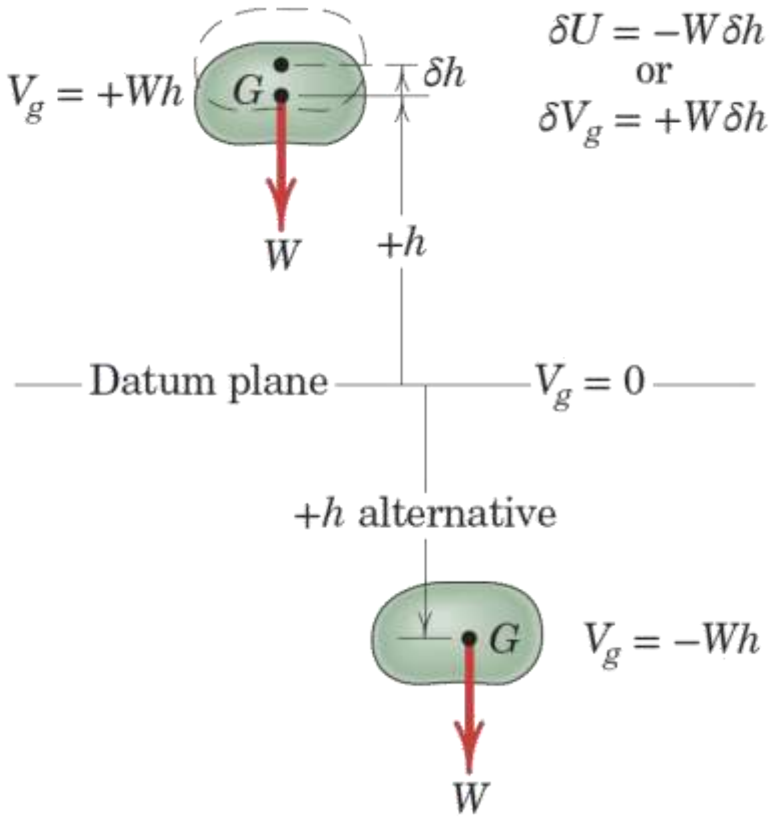
- Torsional Stiffness, k_T
  - Units of torque per radian twist (N·m/rad or lb-ft/rad)

- Elastic Potential Energy, $V_e = \frac{1}{2} k_T \theta^2$

Article 7/4 – Gravitational Potential Energy (1 of 3)

- Illustration & Definition

  The gravitational potential energy V_g is defined as the work done on the body by a force equal and opposite to the weight in bringing the body to the position under consideration from some arbitrary datum plane where the potential energy is defined to be zero.

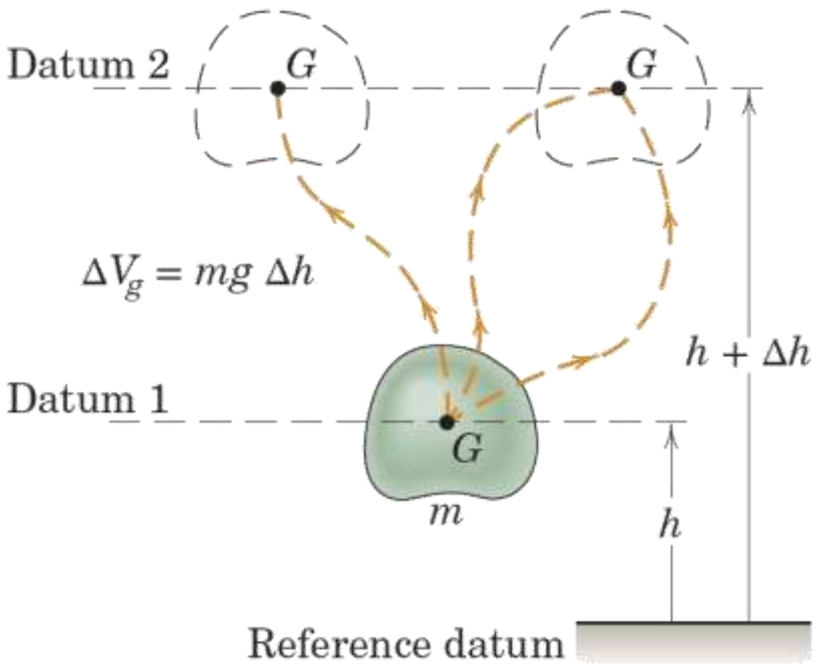


- Potential Energy, $V_g = mgh$
  - Height h is measured from the datum.
  - Above the datum is considered positive.
  - Below the datum is considered negative.

- Change in Potential Energy, $\Delta V_g = mg\,\Delta h$

Article 7/4 – Gravitational Potential Energy (2 of 2)

- Path Irrelevance



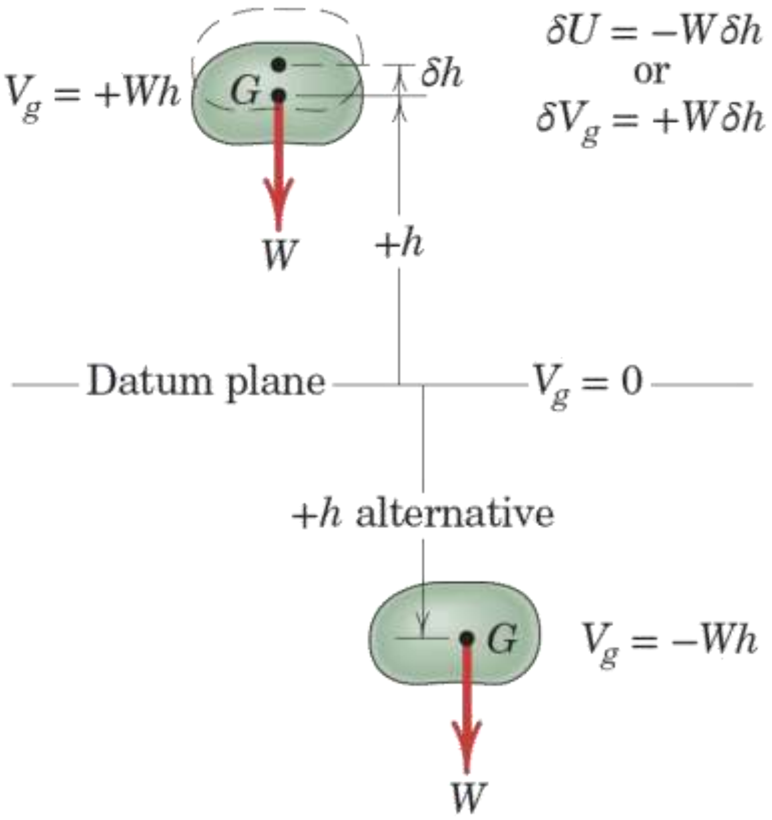
Article 7/4 – Gravitational Potential Energy (3 of 3)

- Virtual Displacements and Potential Energy
  - Virtual Displacement, δh
  - Virtual Change in Potential Energy, $\delta V_g = mg\,\delta h$

- Sign Convention
  - If the virtual displacement is **upward**, the sign of the virtual change in gravitational potential energy is **positive**.

  - If the virtual displacement is **downward**, the sign of the virtual change in gravitational potential energy is **negative**.

- The work done on the body is the negative of the potential energy change of the body.



Article 7/4 – The Principle of Virtual Work Restated

- The Equation Restated

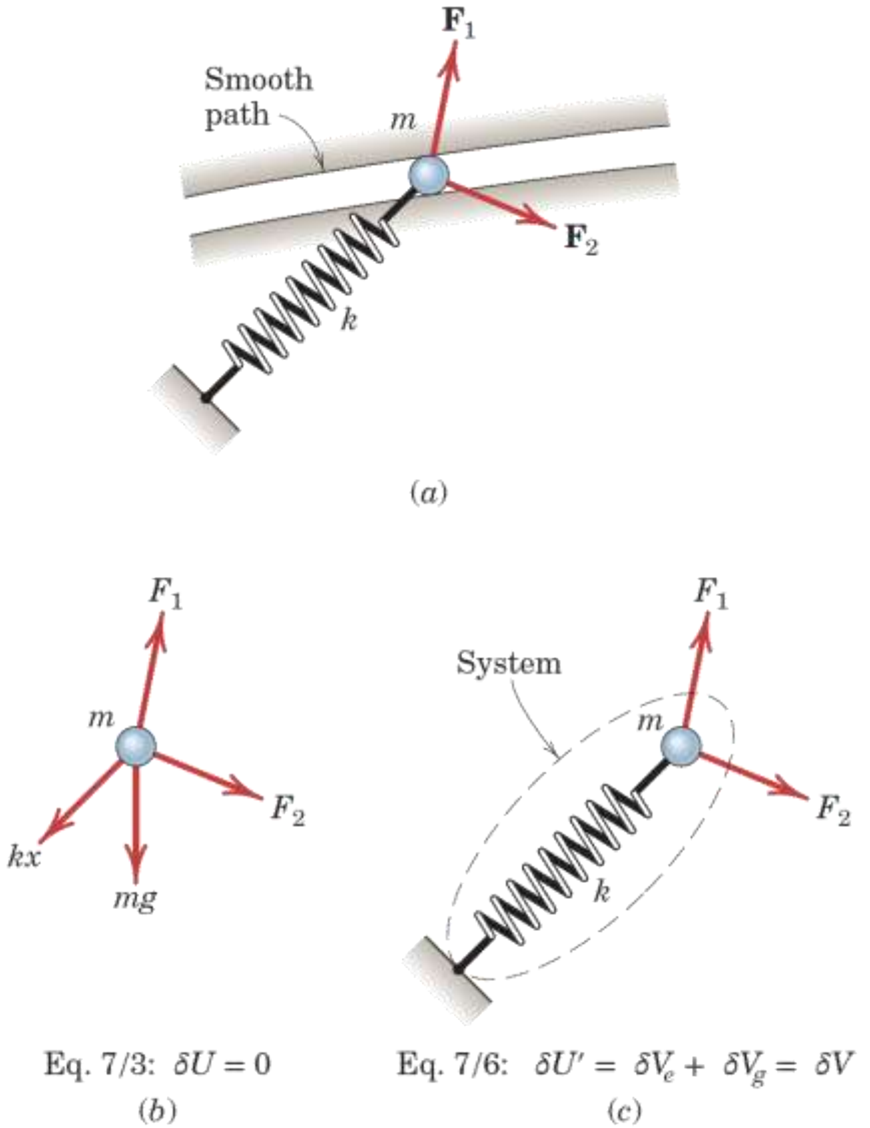
  The virtual work done by all external active forces (other than the gravitational and spring forces accounted for in the potential energy terms) on a mechanical system in equilibrium equals the corresponding change in the total elastic and gravitational potential energy of the system for any and all virtual displacements consistent with the constraints.

- Mathematical Expression: $\delta U' - (\delta V_e + \delta V_g) = 0$ or $\delta U' = \delta V$

  - $\delta U'$ = sum of the work done by all active forces other than springs or weight.
  - $-\delta V_e$ = work done by the spring forces.
  - $-\delta V_g$ = work done by the weight forces.
  - δV = change in total elastic and gravitational potential energy of the system.
  - The equation above is also called the *energy equation*.

Article 7/4 – Active Force Diagrams

- Illustration
  - (*a*) Body of Interest
  - (*b*) Use of Equation 7/3
  - (*c*) Use of Equation 7/6



Article 7/4 – Stability of Equilibrium (1 of 2)

- Introduction
  - Consider a mechanical system where movement is accompanied by changes in gravitational and elastic potential energies and where no work is done on the system by nonpotential forces ($\delta U' = 0$).
  - $\delta U' - (\delta V_e + \delta V_g) = 0$ or $(\delta V_e + \delta V_g) = 0$ or $\delta (V_e + V_g) = 0$ or $\delta V = 0$
  - The total potential energy V of the system is constant or has a stationary value.
- One-Degree-of-Freedom System Implication
  - Potential energy is a function of some variable, x, and its derivatives.
  - Then…$dV/dx = 0$ for equilibrium.

Article 7/4 – Stability of Equilibrium (2 of 2)

- Conditions on Equilibrium

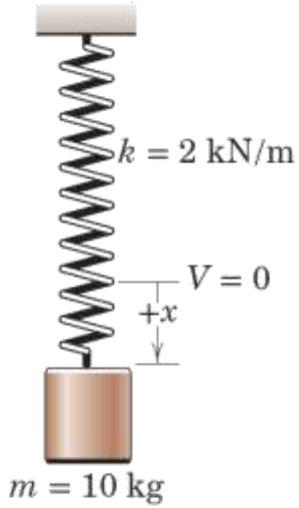
  - Stable: Total Potential Energy is a Minimum
  - Unstable: Total Potential Energy is a Maximum
  - Neutral: Total Potential Energy is a Constant
  - Mathematical Summary

Equilibrium	$\dfrac{dV}{dx} = 0$
Stable	$\dfrac{d^2V}{dx^2} > 0$
Unstable	$\dfrac{d^2V}{dx^2} < 0$

Article 7/4 – Sample Problem 7/4 (1 of 3)

- Problem Statement

  The 10-kg cylinder is suspended by the spring, which has a stiffness of 2 kN/m. Plot the potential energy *V* of the system and show that it is minimum at the equilibrium position.



Article 7/4 – Sample Problem 7/4 (2 of 3)

- Solution

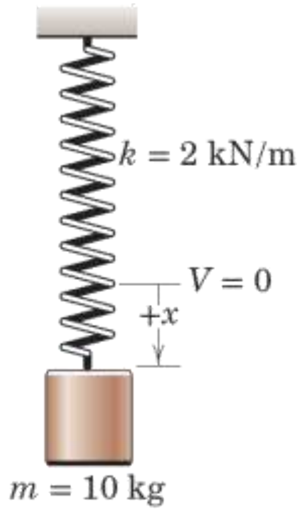
(Although the equilibrium position in this simple problem is clearly where the force in the spring equals the weight mg, we will proceed as though this fact were unknown in order to illustrate the energy relationships in the simplest way.) We choose the datum plane for zero potential energy at the position where the spring is unextended. ①

The elastic potential energy for an arbitrary position x is $V_e = \frac{1}{2}kx^2$ and the gravitational potential energy is $-mgx$, so that the total potential energy is

$$[V = V_e + V_g] \qquad V = \frac{1}{2}kx^2 - mgx$$

Equilibrium occurs where

$$\left[\frac{dV}{dx} = 0\right] \qquad \frac{dV}{dx} = kx - mg = 0 \qquad x = mg/k$$



① The choice is arbitrary but simplifies the algebra.

Article 7/4 – Sample Problem 7/4 (3 of 3)

- Solution (cont.)

Although we know in this simple case that the equilibrium is stable, we prove it by evaluating the sign of the second derivative of V at the equilibrium position. Thus, $d^2V/dx^2 = k$, which is positive, proving that the equilibrium is stable.

Substituting numerical values gives

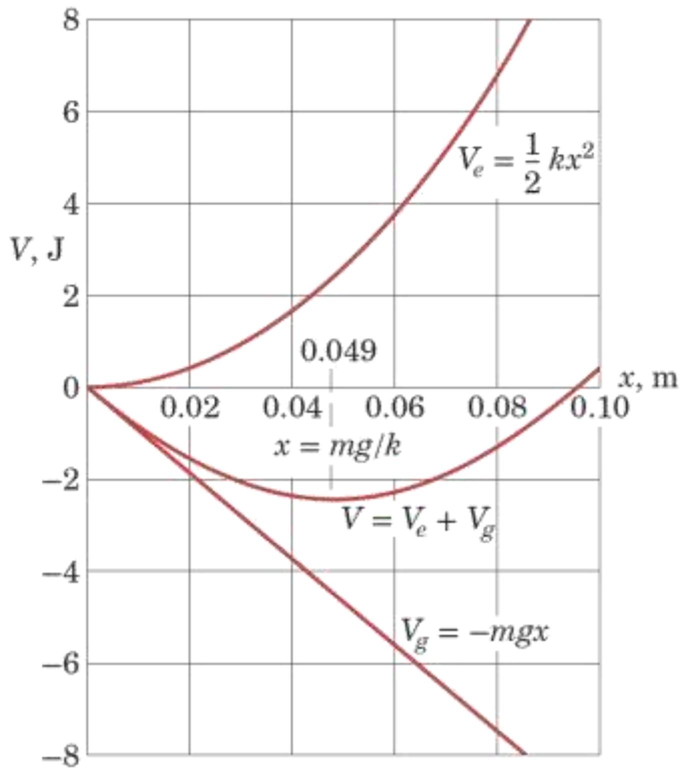
$$V = \frac{1}{2}(2000)x^2 - 10(9.81)x$$

expressed in joules, and the equilibrium value of x is

$$x = 10(9.81)/2000 = 0.0490 \text{ m} \qquad \text{or} \qquad 49.0 \text{ mm} \qquad \textit{Ans.}$$

We calculate V for various values of x and plot V versus x as shown. The minimum value of V occurs at $x = 0.0490$ m where $dV/dx = 0$ and d^2V/dx^2 is positive. ②

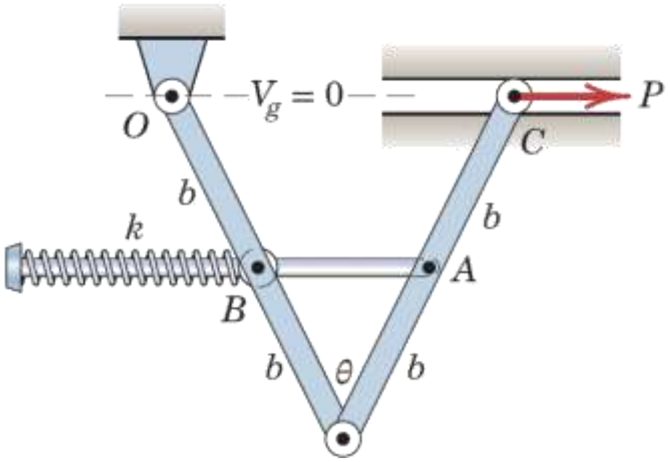
② We could have chosen different datum planes for V_e and V_g without affecting our conclusions. Such a change would merely shift the separate curves for V_e and V_g up or down but would not affect the position of the minimum value of V.



Article 7/4 – Sample Problem 7/5 (1 of 3)

- Problem Statement

  The two uniform links, each of mass m, are in the vertical plane and are connected and constrained as shown. As the angle θ between the links increases with the application of the horizontal force P, the light rod, which is connected at A and passes through a pivoted collar at B, compresses the spring of stiffness k. If the spring is uncompressed in the position where $\theta = 0$, determine the force P which will produce equilibrium at the angle θ.



Article 7/4 – Sample Problem 7/5 (2 of 3)

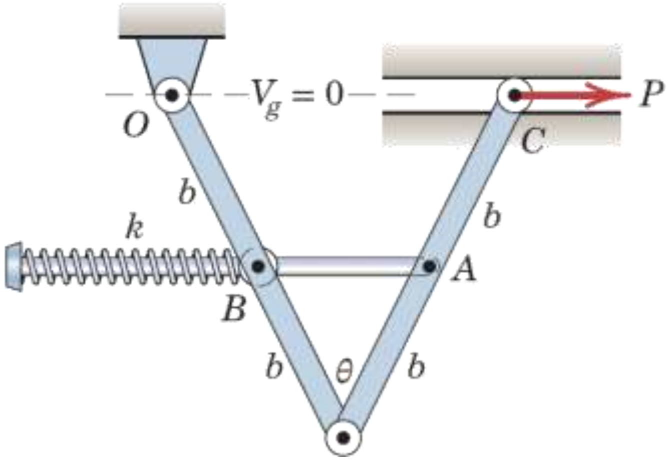
- Solution

The given sketch serves as the active-force diagram of the system. The compression x of the spring is the distance which A has moved away from B, which is $x = 2b \sin \theta/2$. Thus, the elastic potential energy of the spring is

$$[V_e = \tfrac{1}{2}kx^2] \qquad V_e = \tfrac{1}{2}k\left(2b \sin \frac{\theta}{2}\right)^2 = 2kb^2 \sin^2 \frac{\theta}{2}$$

With the datum for zero gravitational potential energy taken through the support at O for convenience, the expression for V_g becomes

$$[V_g = mgh] \qquad V_g = 2mg\left(-b \cos \frac{\theta}{2}\right)$$



Article 7/4 – Sample Problem 7/5 (2 of 3)

- Solution (cont.)

The distance between O and C is $4b \sin \theta/2$, so that the virtual work done by P is

$$\delta U' = P\,\delta\left(4b \sin \frac{\theta}{2}\right) = 2Pb \cos \frac{\theta}{2}\,\delta\theta$$

The virtual-work equation now gives

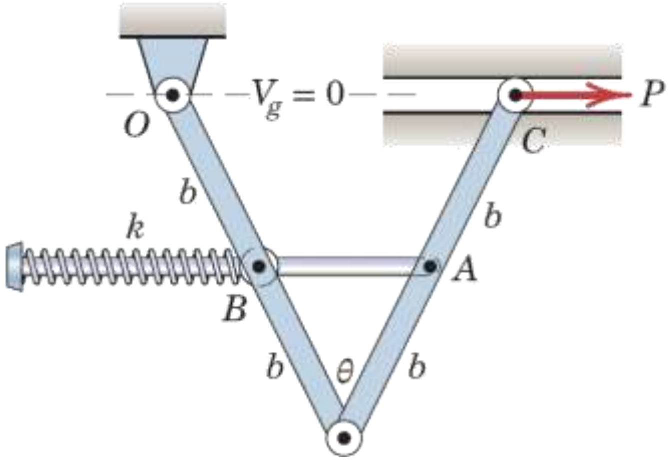
$[\delta U' = \delta V_e + \delta V_g]$

$$2Pb \cos \frac{\theta}{2}\,\delta\theta = \delta\left(2kb^2 \sin^2 \frac{\theta}{2}\right) + \delta\left(-2mgb \cos \frac{\theta}{2}\right)$$

$$= 2kb^2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}\,\delta\theta + mgb \sin \frac{\theta}{2}\,\delta\theta$$

Simplifying gives finally

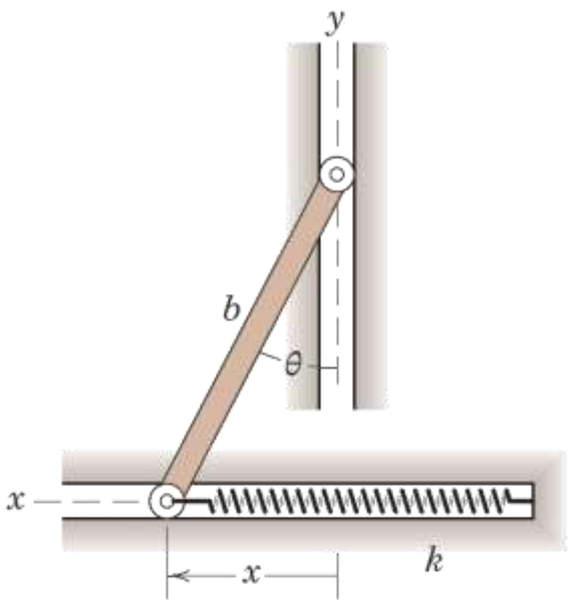
$$P = kb \sin \frac{\theta}{2} + \tfrac{1}{2} mg \tan \frac{\theta}{2} \qquad \textit{Ans.}$$



Article 7/4 – Sample Problem 7/6 (1 of 4)

- Problem Statement

  The ends of the uniform bar of mass *m* slide freely in the horizontal and vertical guides. Examine the stability conditions for the positions of equilibrium. The spring of stiffness *k* is undeformed when $x = 0$.



Article 7/4 – Sample Problem 7/6 (2 of 4)

- Solution

The system consists of the spring and the bar. Since there are no external active forces, the given sketch serves as the active-force diagram. ① We will take the x-axis as the datum for zero gravitational potential energy. In the displaced position the elastic and gravitational potential energies are

$$V_e = \frac{1}{2}kx^2 = \frac{1}{2}kb^2 \sin^2\theta \qquad \text{and} \qquad V_g = mg\,\frac{b}{2}\cos\theta$$

The total potential energy is then

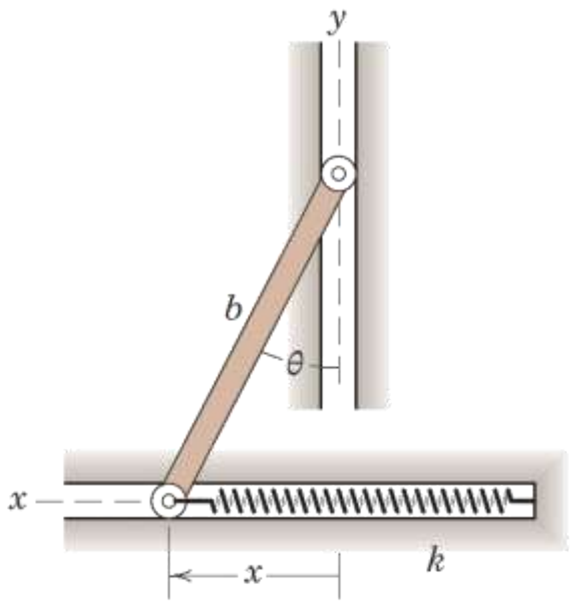
$$V = V_e + V_g = \frac{1}{2}kb^2 \sin^2\theta + \frac{1}{2}mgb\cos\theta$$

Equilibrium occurs for $dV/d\theta = 0$ so that

$$\frac{dV}{d\theta} = kb^2 \sin\theta\cos\theta - \frac{1}{2}mgb\sin\theta = (kb^2\cos\theta - \frac{1}{2}mgb)\sin\theta = 0$$

The two solutions to this equation are given by

$$\sin\theta = 0 \qquad \text{and} \qquad \cos\theta = \frac{mg}{2kb} \quad ②$$



① With no external active forces there is no $\delta U'$ term, and $\delta V = 0$ is equivalent to $dV/d\theta = 0$.

② Be careful not to overlook the solution $\theta = 0$ given by $\sin\theta = 0$.

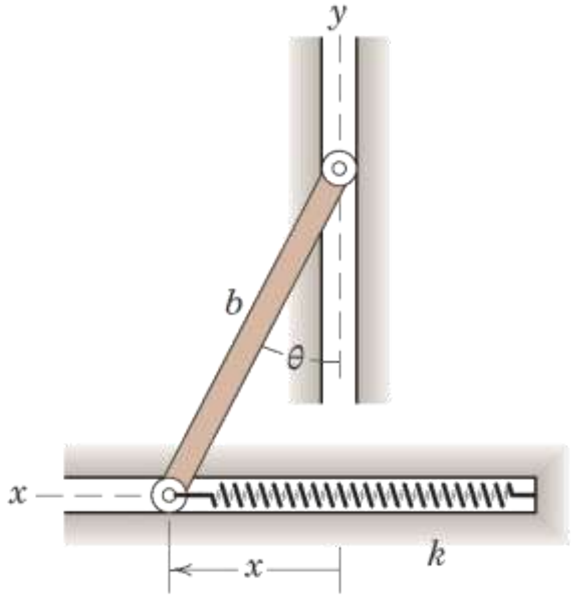
Article 7/4 – Sample Problem 7/6 (3 of 4)

- Stability Equation

We now determine the stability by examining the sign of the second derivative of V for each of the two equilibrium positions. The second derivative is

$$\frac{d^2V}{d\theta^2} = kb^2(\cos^2\theta - \sin^2\theta) - \tfrac{1}{2}mgb\cos\theta$$

$$= kb^2(2\cos^2\theta - 1) - \tfrac{1}{2}mgb\cos\theta$$



- Stability Solution I

$$\sin\theta = 0, \theta = 0$$

$$\frac{d^2V}{d\theta^2} = kb^2(2-1) - \tfrac{1}{2}mgb = kb^2\left(1 - \frac{mg}{2kb}\right)$$

$$= \text{positive (stable)} \qquad \text{if } k > mg/2b$$

$$= \text{negative (unstable)} \qquad \text{if } k < mg/2b \qquad \textit{Ans.}$$

Thus, if the spring is sufficiently stiff, the bar will return to the vertical position even though there is no force in the spring at that position. ③

③ We might not have anticipated this result without the mathematical analysis of the stability.

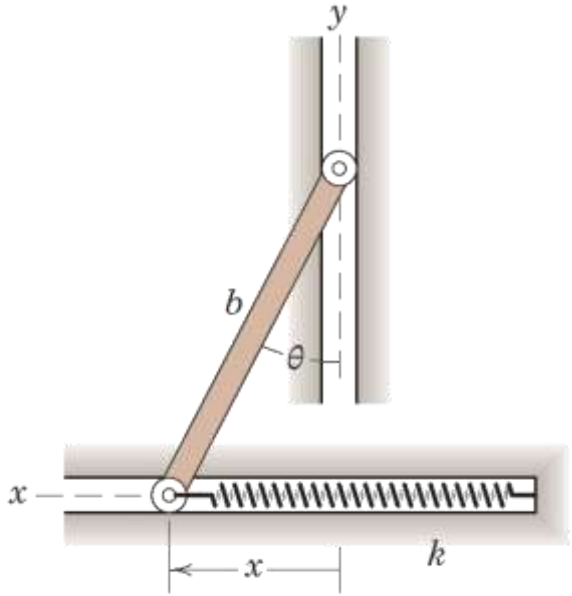
Article 7/4 – Sample Problem 7/6 (3 of 4)

- Stability Solution II

$$\cos\theta = \frac{mg}{2kb}, \theta = \cos^{-1}\frac{mg}{2kb}$$

$$\frac{d^2V}{d\theta^2} = kb^2\left[2\left(\frac{mg}{2kb}\right)^2 - 1\right] - \frac{1}{2}mgb\left(\frac{mg}{2kb}\right) = kb^2\left[\left(\frac{mg}{2kb}\right)^2 - 1\right] \quad \textit{Ans.}$$

Since the cosine must be less than unity, we see that this solution is limited to the case where $k > mg/2b$, which makes the second derivative of V negative. Thus, equilibrium for Solution II is never stable. ④ If $k < mg/2b$, we no longer have Solution II since the spring will be too weak to maintain equilibrium at a value of θ between 0 and 90°.



④ Again, without the benefit of the mathematical analysis of the stability we might have supposed erroneously that the bar could come to rest in a stable equilibrium position for some value of θ between 0 and 90°.