CHAPTER 5
DISTRIBUTED FORCES

CHAPTER OUTLINE

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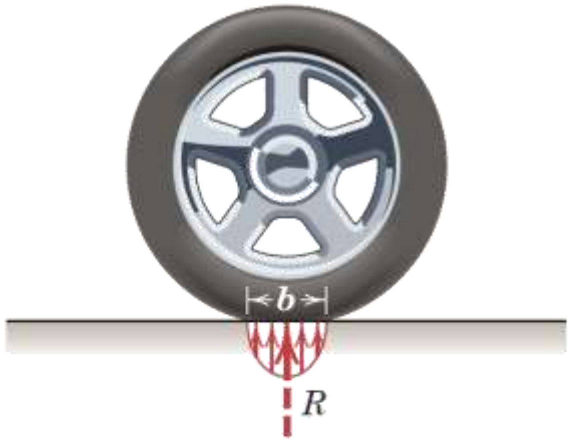
Article 5/1 Introduction

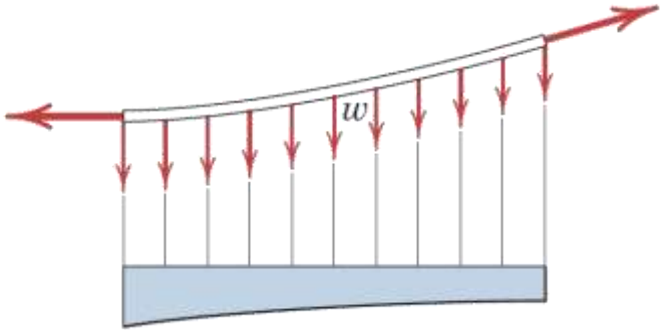
- Concentrated Forces



- Distributed Forces
  - Forces are spread out over a line, area, or volume.
  - Treated as concentrated forces when analyzing their external effects on a body.
  - Occur when a force is applied over a region whose dimensions are not negligible compared with other pertinent dimensions.

Article 5/1 – Categories of Distributed Forces (1 of 3)

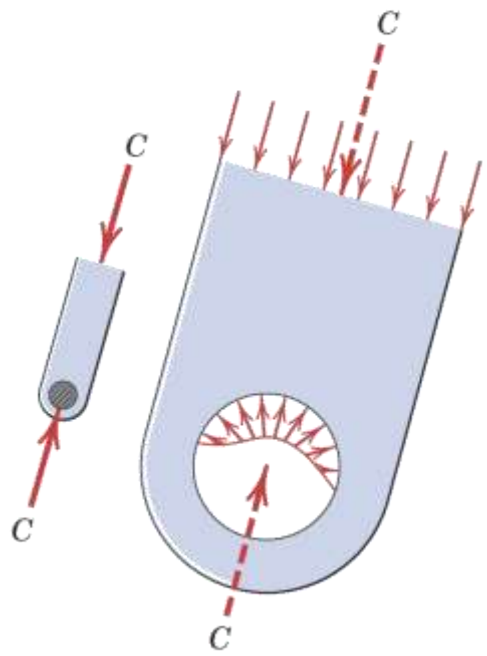
- Line Distribution
  - Illustration

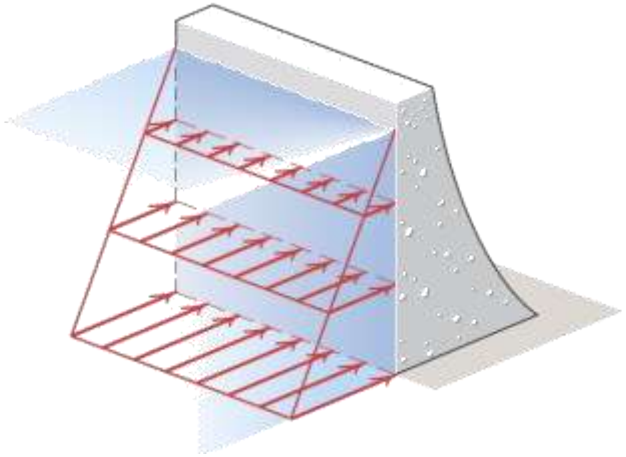


  - Intensity *w* is N/m or lb/ft

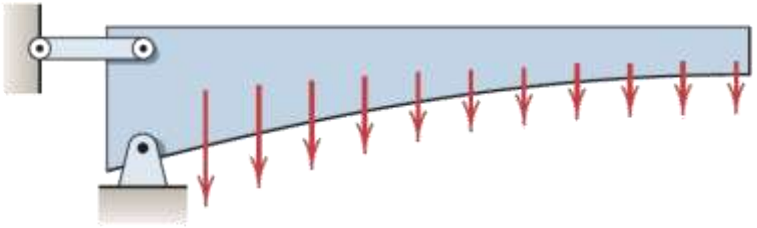
Article 5/1 – Categories of Distributed Forces (2 of 3)

- Area Distribution
  - Illustrations
  - Intensity is termed pressure for fluids and stress for solids.
    - SI unit is the Pascal (Pa).
    - 1 Pa = 1 N/m^2
    - 6895 Pa = 1 lb/in.^{2}
    - Pressure is commonly reported in kilopascals (kPa = 10^3 Pa).
    - Stress is commonly reported in megapascals (MPa = 10^6 Pa).





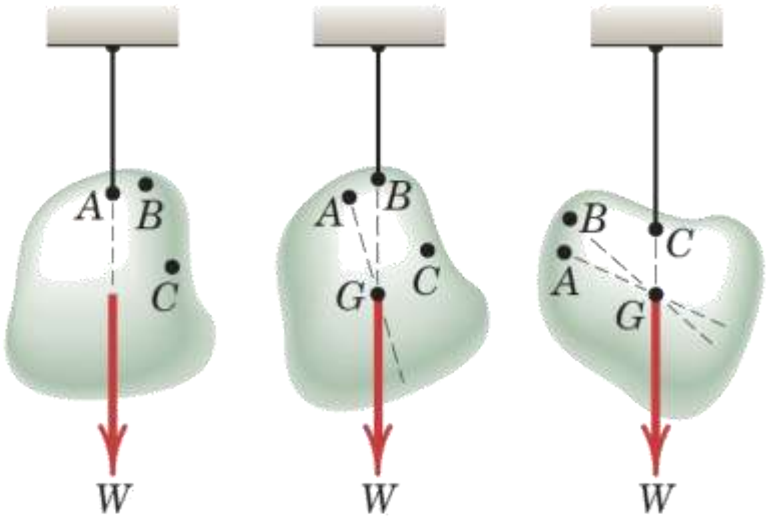
Article 5/1 – Categories of Distributed Forces (3 of 3)



- Volume Distribution (Body Force)
  - Illustration
  - Most common body force is gravitational attraction.
  - Intensity is termed specific weight, $\gamma = \rho g$.
    - ρ = density (mass per unit volume)
    - g = acceleration due to gravity
    - SI units are N/m^3
    - U.S. customary units are lb/ft^3 or $lb/in.^3$

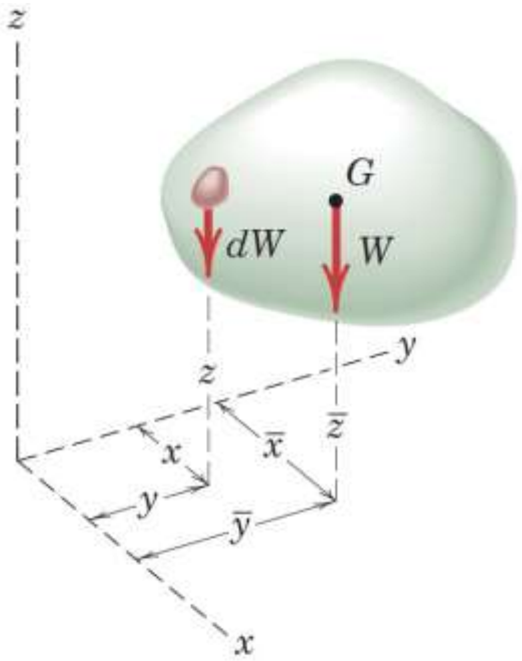
Article 5/2 Center of Mass

- Introduction
- Illustration with Hanging Mass
- Center of Gravity, *G*
- Assumptions



Article 5/2 – Determining the Center of Gravity (1 of 3)

- Principle of Moments
  - Body of Total Weight *W*
  - Center of Gravity Coordinates are $(\bar{x}, \bar{y}, \bar{z})$
  - Differential Element of the Body of Weight *dW*
  - Coordinates of the Element are (x, y, z)
  - The sum of the moments equals the moment of the sum.



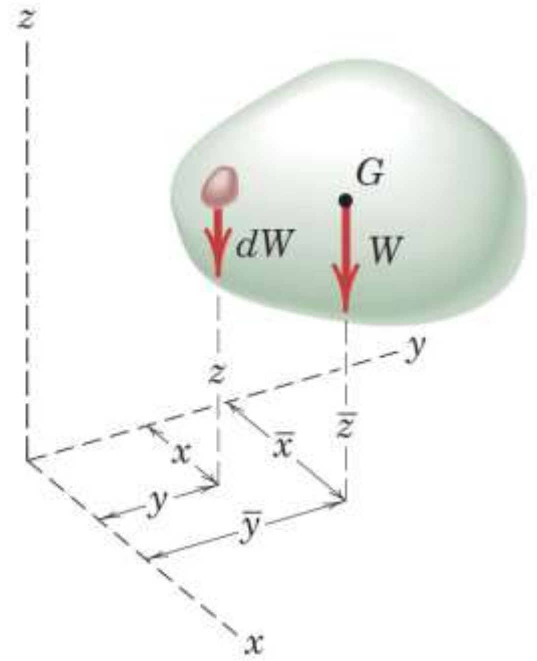
- Equations of Interest

$$\bar{x} = \frac{\int x\, dW}{W} \qquad \bar{y} = \frac{\int y\, dW}{W} \qquad \bar{z} = \frac{\int z\, dW}{W}$$

Article 5/2 – Determining the Center of Gravity (2 of 3)

- Substitute $W = mg$ and $dW = g\ dm$…

$$\boxed{\bar{x} = \frac{\int x\, dm}{m} \qquad \bar{y} = \frac{\int y\, dm}{m} \qquad \bar{z} = \frac{\int z\, dm}{m}}$$



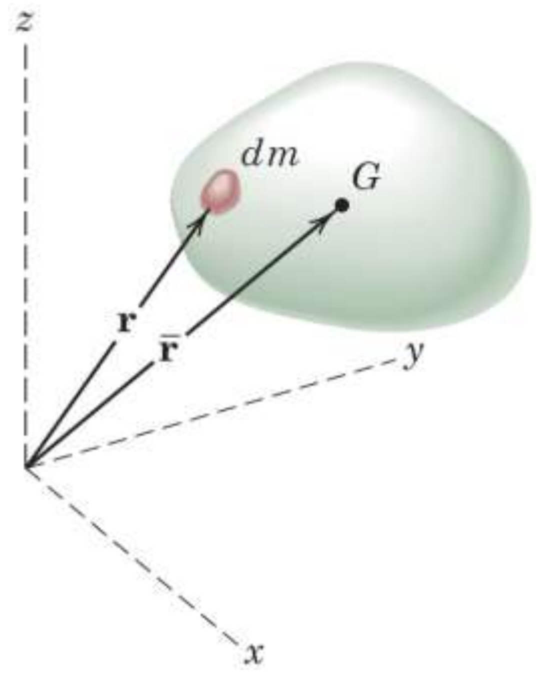
- Substitute $\rho = mV$ and $dm = \rho\ dV$, with ρ variable…

$$\bar{x} = \frac{\int x\rho\, dV}{\int \rho\, dV} \qquad \bar{y} = \frac{\int y\rho\, dV}{\int \rho\, dV} \qquad \bar{z} = \frac{\int z\rho\, dV}{\int \rho\, dV}$$

Article 5/2 – Determining the Center of Gravity (3 of 3)

- Vector Format
  - $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
  - $\bar{\boldsymbol{r}} = \bar{\boldsymbol{x}}\mathbf{i} + \bar{y}\mathbf{j} + \bar{z}\mathbf{k}$

$$\boxed{\bar{\mathbf{r}} = \frac{\int \mathbf{r}\, dm}{m}}$$

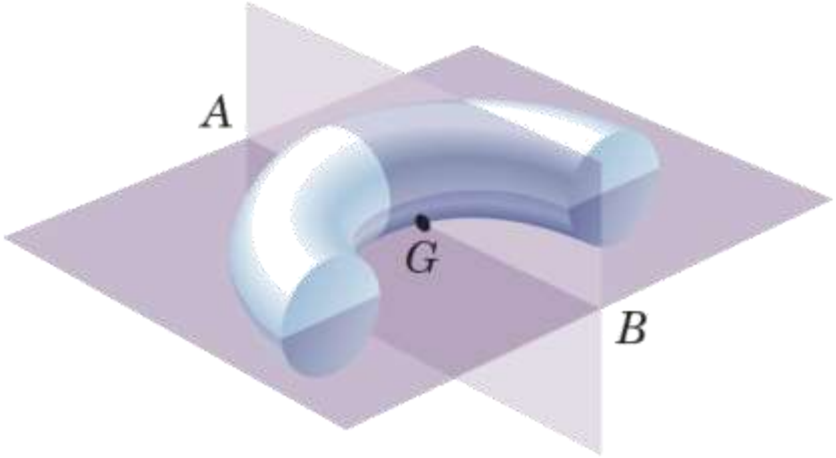


Article 5/2 – Center of Mass versus Center of Gravity

- Center of Mass is…
  - a unique point in the body which is only a function of the distribution of mass.
  - coincident with the center of gravity if the gravitational field is uniform and parallel.
  - still present when the body is removed from a gravitational field.
  - the preferred choice of reference.
  - important in calculating the dynamic response of a body to unbalanced forces.

Article 5/2 – Comments about the Center of Mass

- Tips and Hints
  - Choice of Axes
  - Lines and Planes of Symmetry



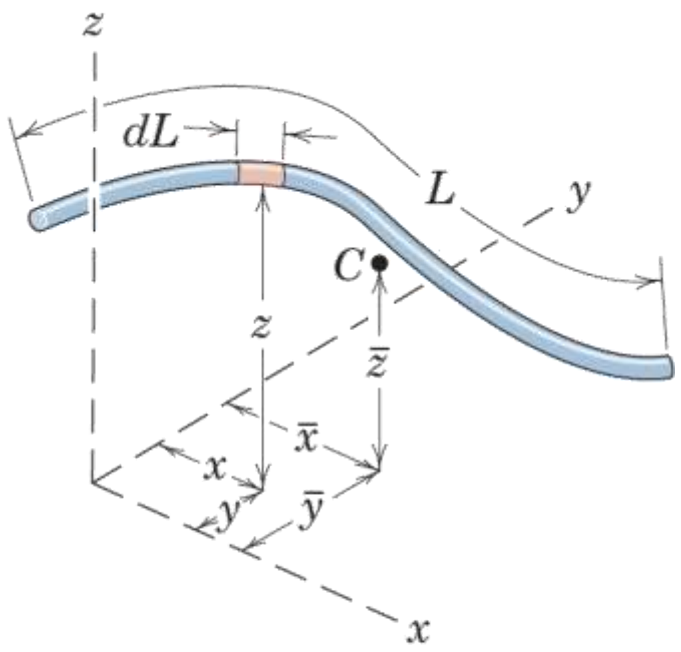
Article 5/3 Centroids of Lines, Areas, and Volumes

- Centroid

  Term used when a center of mass calculation concerns a geometric shape only.

- Centroids of Lines
  - Differential segment $dm = \rho A\, dL$
  - A = cross-sectional area of line
  - Equations of interest if ρ and A are constant…

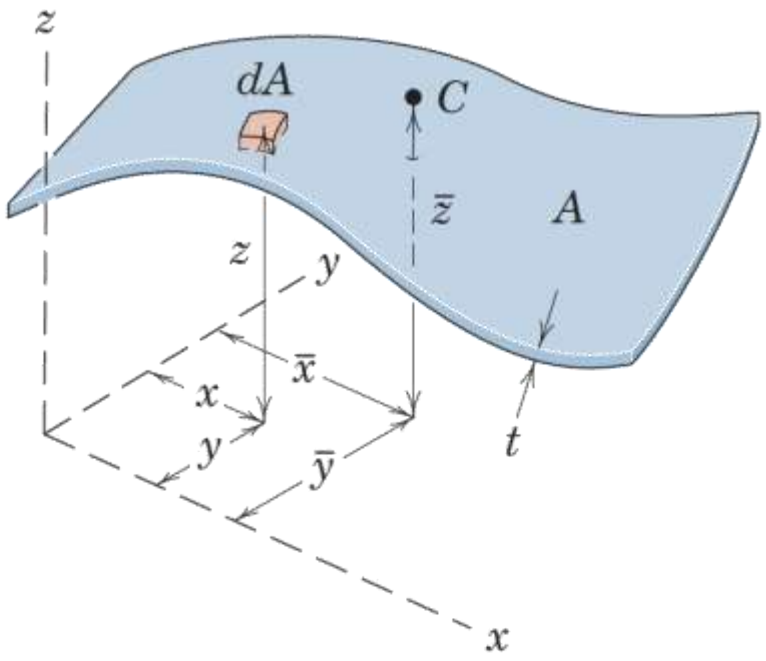
$$\bar{x} = \frac{\int x\, dL}{L} \qquad \bar{y} = \frac{\int y\, dL}{L} \qquad \bar{z} = \frac{\int z\, dL}{L}$$



Article 5/3 – Centroids (cont.)

- Centroids of Areas
  - Differential section $dm = \rho t\, dA$
  - t = thickness of area
  - Equations of interest if ρ and t are constant…

$$\bar{x} = \frac{\int x\, dA}{A} \qquad \bar{y} = \frac{\int y\, dA}{A} \qquad \bar{z} = \frac{\int z\, dA}{A}$$



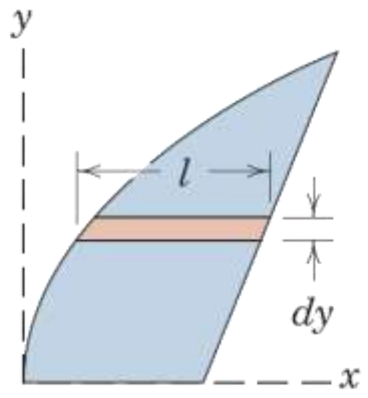
- Centroids of Volumes
  - Differential portion $dm = \rho\, dV$
  - Equations of interest if ρ is constant…

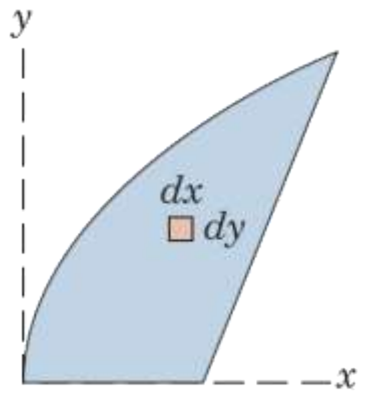
$$\bar{x} = \frac{\int x\, dV}{V} \qquad \bar{y} = \frac{\int y\, dV}{V} \qquad \bar{z} = \frac{\int z\, dV}{V}$$

Article 5/3 – Choice of Element for Integration (1 of 6)

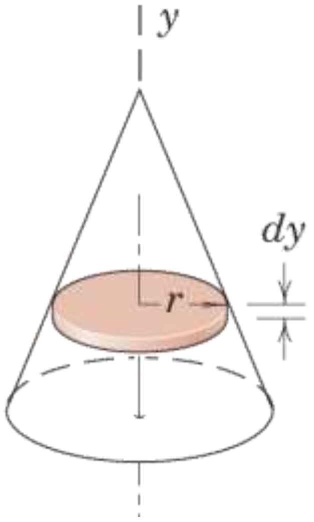
- Order of Element
  - Whenever possible, select a first-order differential element to reduce the number of integrations.

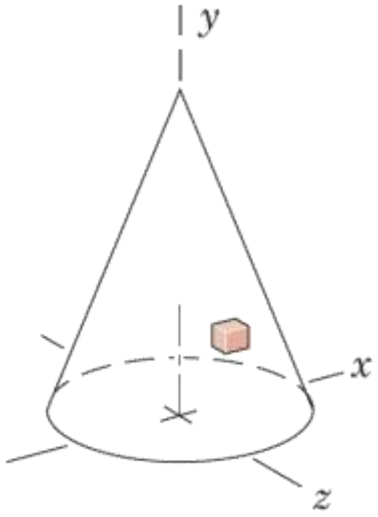
  - Lower: Horizontal Strip, $dA = l\ dy$
  - Higher: Square, $dA = dx\ dy$





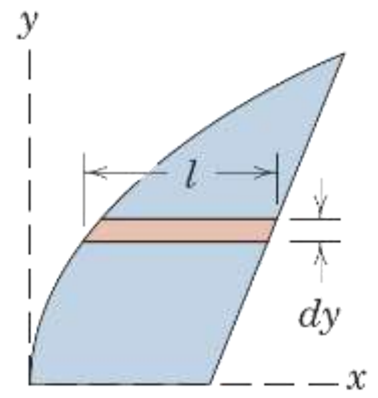
  - Lower: Circular Disk, $dV = \pi r^2\ dy$
  - Higher: Cube, $dV = dx\ dy\ dz$

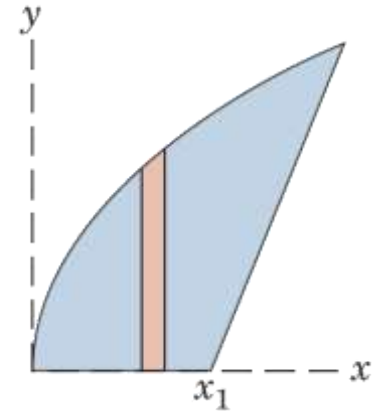




Article 5/3 – Choice of Element for Integration (2 of 6)

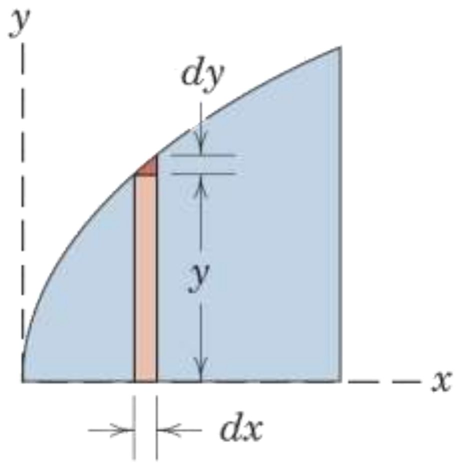
- Continuity
  - Whenever possible, choose an element which can be integrated in one continuous operation to cover the figure.
  - Horizontal strip requires one integration in *dy* because the boundaries for the right and left ends of the strip are continuous in the vertical direction
  - Vertical strip requires two integrations in *dx* because the boundaries for the top and bottom of the strip are not continuous in the horizontal direction.





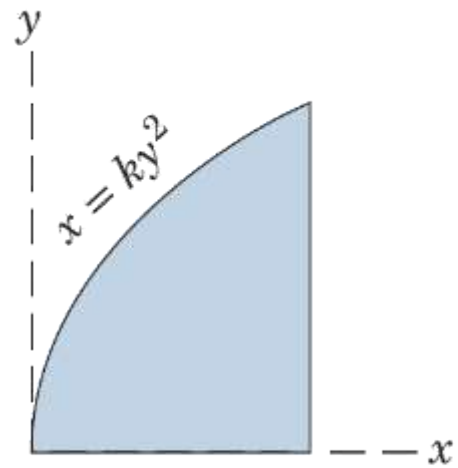
Article 5/3 – Choice of Element for Integration (3 of 6)

- Discarding Higher-Order Terms
  - Higher order terms may always be dropped when compared to lower-order terms.
  - First Order: $dA = y\ dx$
  - Second Order: $dA' = \frac{1}{2}\ dy\ dx$

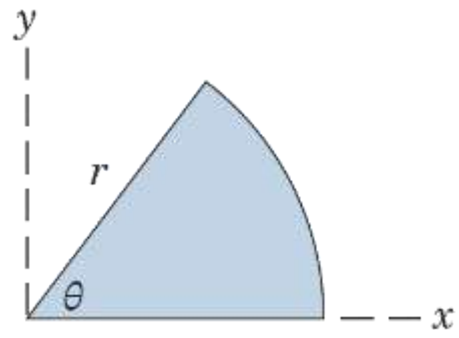


Article 5/3 – Choice of Element for Integration (4 of 6)

- Choice of Coordinates
  - Choose the coordinate system which best matches the boundaries of the figure.
  - Rectangular Coordinates: $x = ky^2$



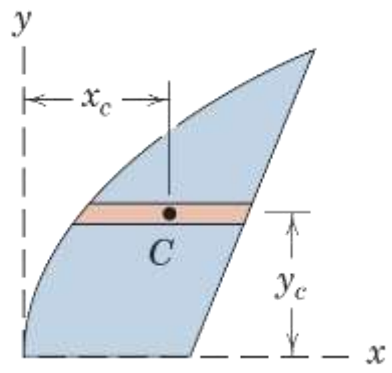
  - Polar Coordinates



Article 5/3 – Choice of Element for Integration (5 of 6)

- Centroidal Coordinate of Element
  - When a first- or second-order differential element is chosen, it is essential to use the coordinate of the centroid of the element for the moment arm in expressing the moment of the differential element.
  - Example with a Horizontal Strip
  - New Equations

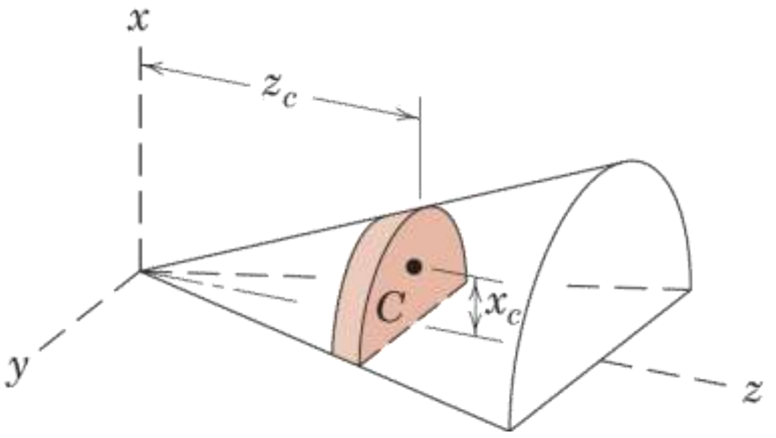
$$\bar{x} = \frac{\int x_c \, dA}{A} \qquad \bar{y} = \frac{\int y_c \, dA}{A} \qquad \bar{z} = \frac{\int z_c \, dA}{A}$$



Article 5/3 – Choice of Element for Integration (6 of 6)

- Centroidal Coordinate of Element (cont.)
  - Example with a Semicircular Slice
  - New Equations

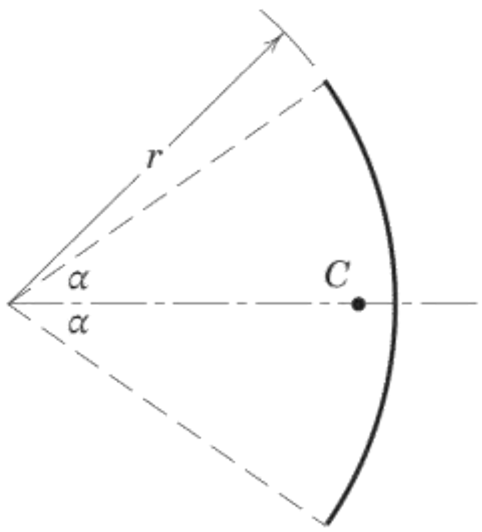
$$\bar{x} = \frac{\int x_c \, dV}{V} \qquad \bar{y} = \frac{\int y_c \, dV}{V} \qquad \bar{z} = \frac{\int z_c \, dV}{V}$$



Article 5/3 – Sample Problem 5/1 (1 of 2)

- Problem Statement

  Locate the centroid of a circular arc as shown in the figure.



Article 5/3 – Sample Problem 5/1 (2 of 2)

- Solution

Choosing the axis of symmetry as the x-axis makes $\bar{y} = 0$. A differential element of arc has the length $dL = r\,d\theta$ expressed in polar coordinates, and the x-coordinate of the element is $r \cos \theta$. ①

Applying the first of Eqs. 5/4 and substituting $L = 2\alpha r$ give

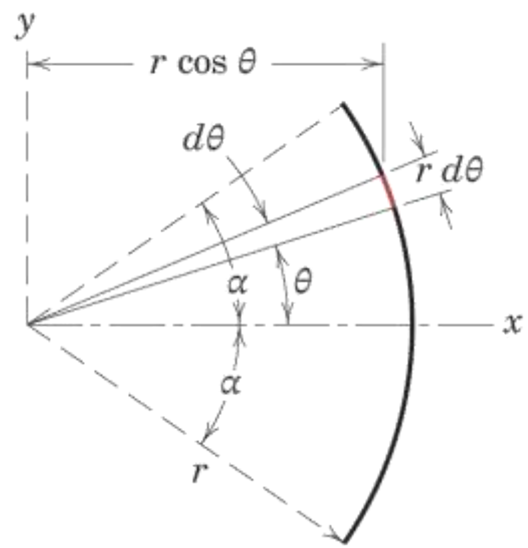
$$\left[L\bar{x} = \int x\,dL\right] \qquad (2\alpha r)\bar{x} = \int_{-\alpha}^{\alpha} (r \cos \theta)\, r\,d\theta$$

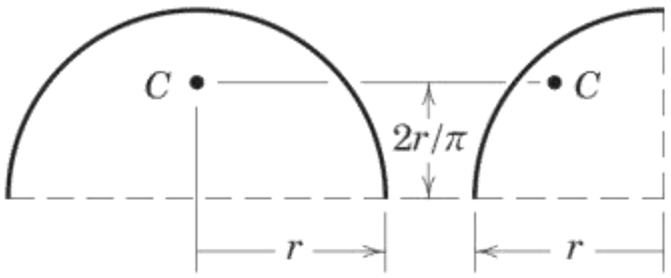
$$2\alpha r\bar{x} = 2r^2 \sin \alpha$$

$$\bar{x} = \frac{r \sin \alpha}{\alpha} \qquad \textit{Ans.}$$

For a semicircular arc $2\alpha = \pi$, which gives $\bar{x} = 2r/\pi$. By symmetry we see immediately that this result also applies to the quarter-circular arc when the measurement is made as shown.

① It should be perfectly evident that polar coordinates are preferable to rectangular coordinates to express the length of a circular arc.

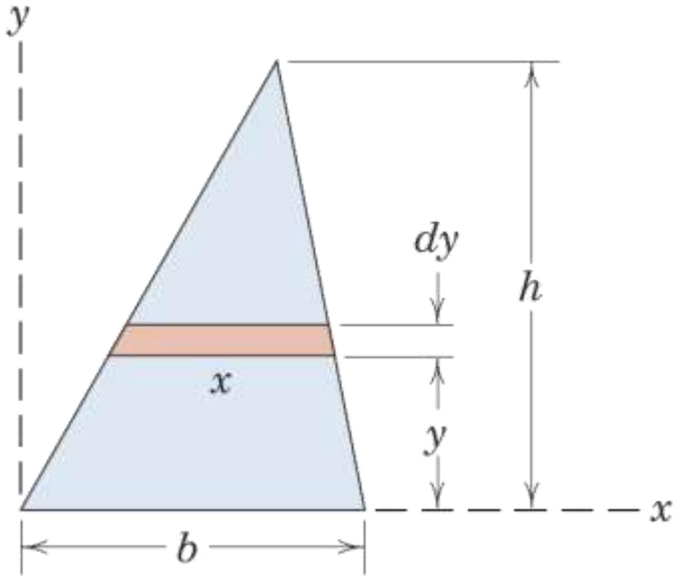




Article 5/3 – Sample Problem 5/2 (1 of 2)

- Problem Statement

Determine the distance *h* from the base of a triangle of altitude *h* to the centroid of its area.



Article 5/3 – Sample Problem 5/2 (2 of 2)

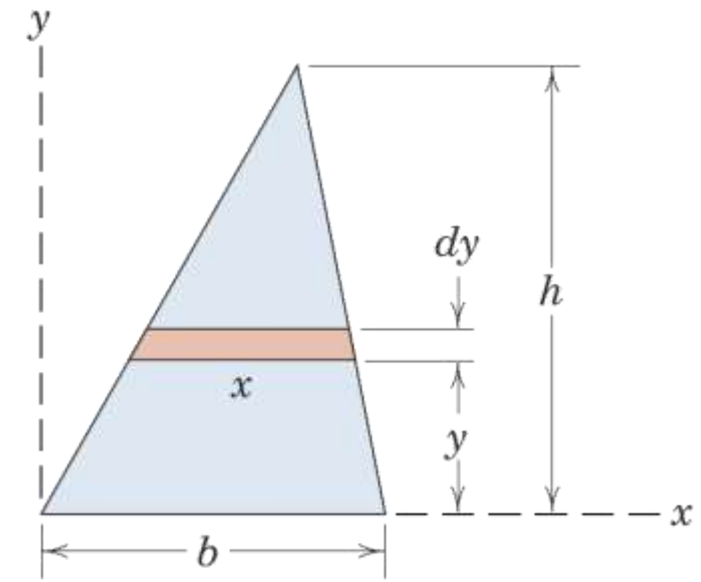
- Solution

The x-axis is taken to coincide with the base. A differential strip of area $dA = x\,dy$ is chosen. ① By similar triangles $x/(h - y) = b/h$. Applying the second of Eqs. 5/5a gives

$$[A\bar{y} = \int y_c\,dA] \qquad \frac{bh}{2}\bar{y} = \int_0^h y\frac{b(h-y)}{h}\,dy = \frac{bh^2}{6}$$

and $$\bar{y} = \frac{h}{3} \qquad \textit{Ans.}$$

This same result holds with respect to either of the other two sides of the triangle considered a new base with corresponding new altitude. Thus, the centroid lies at the intersection of the medians, since the distance of this point from any side is one-third the altitude of the triangle with that side considered the base.

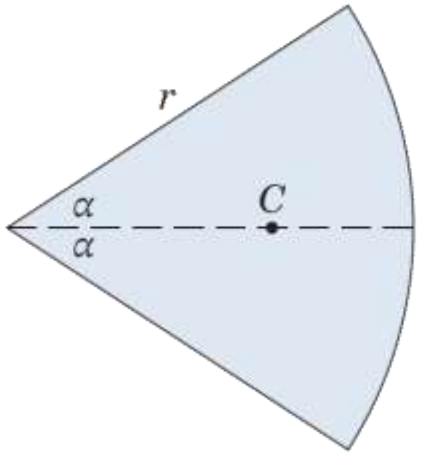


① We save one integration here by using the first-order element of area. Recognize that dA must be expressed in terms of the integration variable y; hence, $x = f(y)$ is required.

Article 5/3 – Sample Problem 5/3 (1 of 3)

- Problem Statement

  Locate the centroid of the area of a circular sector with respect to its vertex.



Article 5/3 – Sample Problem 5/3 (2 of 3)

- Solution I

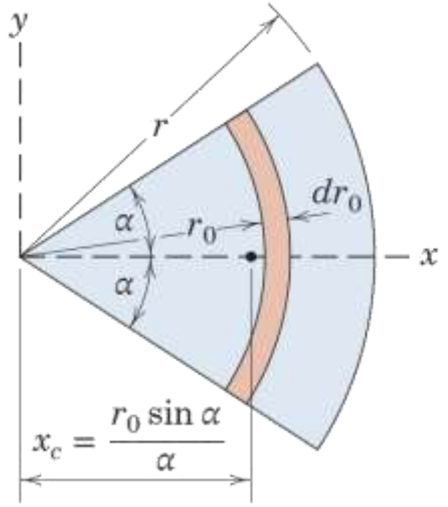
The x-axis is chosen as the axis of symmetry, and $\bar{y}$ is therefore automatically zero. We may cover the area by moving an element in the form of a partial circular ring, as shown in the figure, from the center to the outer periphery. The radius of the ring is r_0 and its thickness is dr_0, so that its area is $dA = 2r_0\alpha\, dr_0$. ①

The x-coordinate to the centroid of the element from Sample Problem 5/1 is $x_c = r_0 \sin\alpha/\alpha$, where r_0 replaces r in the formula. ② Thus, the first of Eqs. 5/5a gives

$$\left[A\bar{x} = \int x_c\, dA\right] \qquad \frac{2\alpha}{2\pi}(\pi r^2)\bar{x} = \int_0^r \left(\frac{r_0 \sin\alpha}{\alpha}\right)(2r_0\alpha\, dr_0)$$

$$r^2\alpha\bar{x} = \frac{2}{3}r^3 \sin\alpha$$

$$\bar{x} = \frac{2}{3}\frac{r\sin\alpha}{\alpha} \qquad \textit{Ans.}$$



① Note carefully that we must distinguish between the variable r_0 and the constant r.

② Be careful not to use r_0 as the centroidal coordinate for the element.

Article 5/3 – Sample Problem 5/3 (3 of 3)

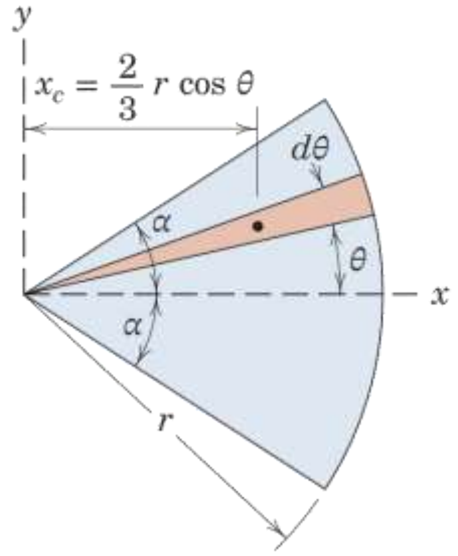
- Solution II

The area may also be covered by swinging a triangle of differential area about the vertex and through the total angle of the sector. This triangle, shown in the illustration, has an area $dA = (r/2)(r\,d\theta)$, where higher-order terms are neglected. From Sample Problem 5/2 the centroid of the triangular element of area is two-thirds of its altitude from its vertex, so that the x-coordinate to the centroid of the element is $x_c = \frac{2}{3}r\cos\theta$. Applying the first of Eqs. 5/5a gives

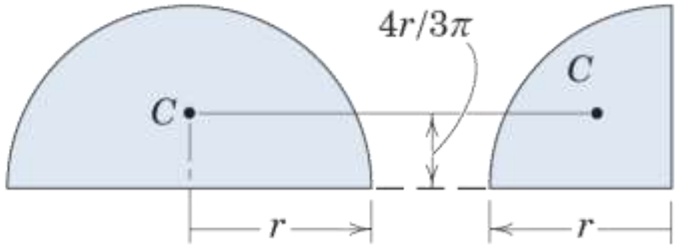
$$\left[A\bar{x} = \int x_c\,dA\right] \qquad (r^2\alpha)\bar{x} = \int_{-\alpha}^{\alpha} (\tfrac{2}{3}r\cos\theta)(\tfrac{1}{2}r^2\,d\theta)$$

$$r^2\alpha\bar{x} = \tfrac{2}{3}r^3\sin\alpha$$

and as before $$\bar{x} = \frac{2}{3}\,\frac{r\sin\alpha}{\alpha} \qquad \textit{Ans.}$$



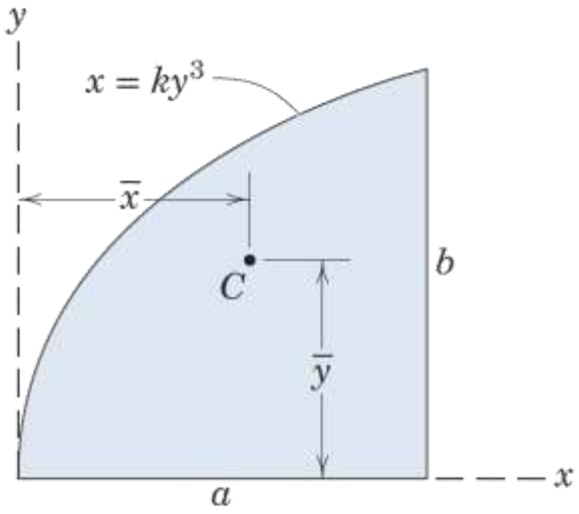
For a semicircular area $2\alpha = \pi$, which gives $\bar{x} = 4r/3\pi$. By symmetry we see immediately that this result also applies to the quarter-circular area where the measurement is made as shown.



It should be noted that, if we had chosen a second-order element $r_0\,dr_0\,d\theta$, one integration with respect to θ would yield the ring with which *Solution I* began. On the other hand, integration with respect to r_0 initially would give the triangular element with which *Solution II* began.

Article 5/3 – Sample Problem 5/4 (1 of 3)

- Problem Statement
  - Locate the centroid of the area under the curve $x = ky^3$ from $x = 0$ to $x = a$.



Article 5/3 – Sample Problem 5/4 (2 of 3)

- Solution I: Vertical Strip

A vertical element of area $dA = y\,dx$ is chosen as shown in the figure. The x-coordinate of the centroid is found from the first of Eqs. 5/5a. Thus,

$$\left[A\bar{x} = \int x_c\,dA\right] \qquad \bar{x}\int_0^a y\,dx = \int_0^a xy\,dx \quad ①$$

Substituting $y = (x/k)^{1/3}$ and $k = a/b^3$ and integrating give

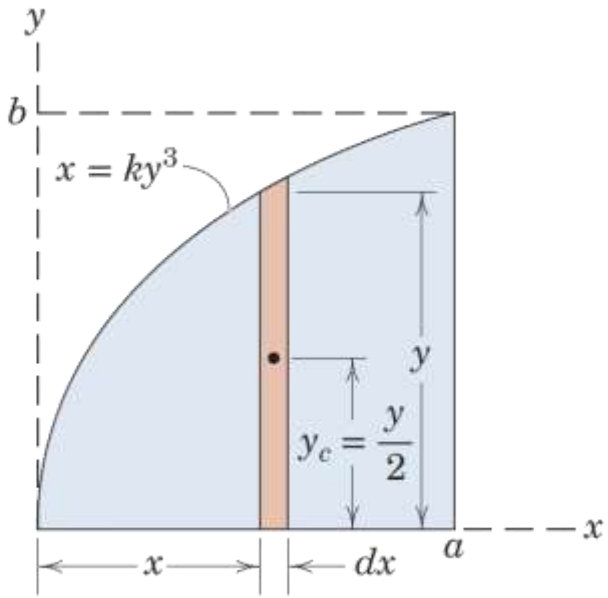
$$\frac{3ab}{4}\bar{x} = \frac{3a^2b}{7} \qquad \bar{x} = \tfrac{4}{7}a \qquad \textit{Ans.}$$

In the solution for $\bar{y}$ from the second of Eqs. 5/5a, the coordinate to the centroid of the rectangular element is $y_c = y/2$, where y is the height of the strip governed by the equation of the curve $x = ky^3$. Thus, the moment principle becomes

$$\left[A\bar{y} = \int y_c\,dA\right] \qquad \frac{3ab}{4}\bar{y} = \int_0^a \left(\frac{y}{2}\right) y\,dx$$

Substituting $y = b(x/a)^{1/3}$ and integrating give

$$\frac{3ab}{4}\bar{y} = \frac{3ab^2}{10} \qquad \bar{y} = \tfrac{2}{5}b \qquad \textit{Ans.}$$



① Note that $x_c = x$ for the vertical element.

Article 5/3 – Sample Problem 5/4 (3 of 3)

- Solution II: Horizontal Strip

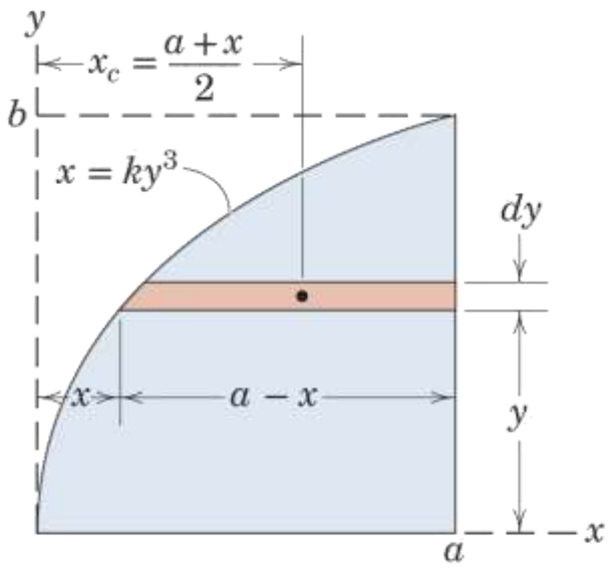
The horizontal element of area shown in the lower figure may be employed in place of the vertical element. The x-coordinate to the centroid of the rectangular element is seen to be $x_c = x + \frac{1}{2}(a - x) = (a + x)/2$, which is simply the average of the coordinates a and x of the ends of the strip. Hence,

$$\left[A\bar{x} = \int x_c \, dA\right] \qquad \bar{x}\int_0^b (a - x)\, dy = \int_0^b \left(\frac{a + x}{2}\right)(a - x)\, dy$$

The value of $\bar{y}$ is found from

$$\left[A\bar{y} = \int y_c \, dA\right] \qquad \bar{y}\int_0^b (a - x)\, dy = \int_0^b y(a - x)\, dy$$

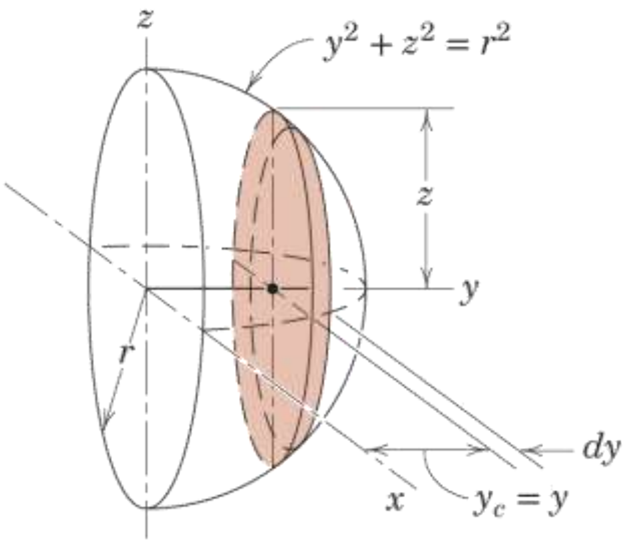
where $y_c = y$ for the horizontal strip. The evaluation of these integrals will check the previous results for $\bar{x}$ and $\bar{y}$.



Article 5/3 – Sample Problem 5/5 (1 of 4)

- Problem Statement

  Locate the centroid of the volume of a hemisphere of radius *r* with respect to its base.



- Solution I: Circular Slice

With the axes chosen as shown in the figure, $\bar{x} = \bar{z} = 0$ by symmetry. The most convenient element is a circular slice of thickness dy parallel to the x-z plane. Since the hemisphere intersects the y-z plane in the circle $y^2 + z^2 = r^2$, the radius of the circular slice is $z = +\sqrt{r^2 - y^2}$. The volume of the elemental slice becomes

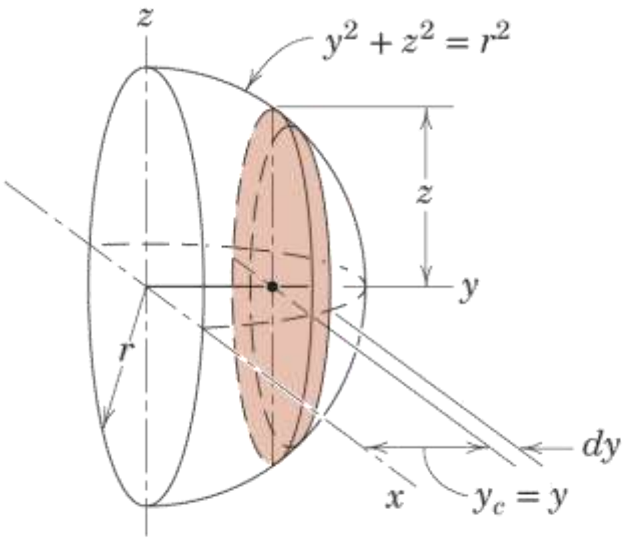
$$dV = \pi(r^2 - y^2)\,dy \quad ①$$

The second of Eqs. 5/6a requires

$$\left[V\bar{y} = \int y_c\,dV\right] \qquad \bar{y}\int_0^r \pi(r^2 - y^2)\,dy = \int_0^r y\pi(r^2 - y^2)\,dy$$

where $y_c = y$. Integrating gives

$$\frac{2}{3}\pi r^3 \bar{y} = \frac{1}{4}\pi r^4 \qquad \bar{y} = \frac{3}{8}r \qquad \textit{Ans.}$$



① Can you identify the higher-order element of volume which is omitted from the expression for dV?

Article 5/3 – Sample Problem 5/5 (3 of 4)

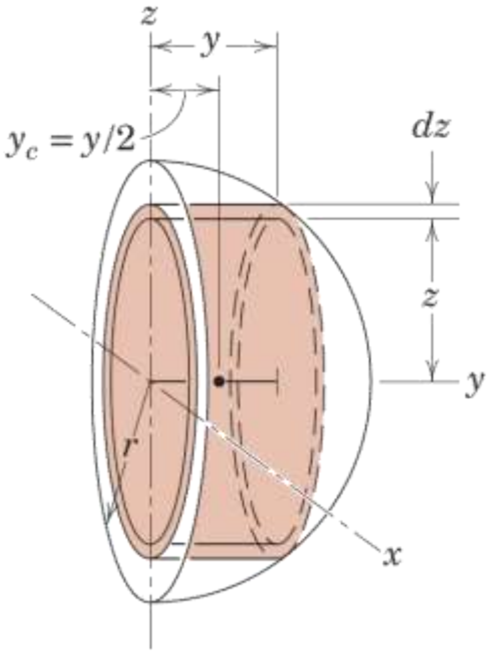
- Solution II: Cylindrical Shell

Alternatively we may use for our differential element a cylindrical shell of length y, radius z, and thickness dz, as shown in the lower figure. By expanding the radius of the shell from zero to r, we cover the entire volume. By symmetry the centroid of the elemental shell lies at its center, so that $y_c = y/2$. The volume of the element is $dV = (2\pi z\, dz)(y)$. Expressing y in terms of z from the equation of the circle gives $y = +\sqrt{r^2 - z^2}$. Using the value of $\frac{2}{3}\pi r^3$ computed in *Solution I* for the volume of the hemisphere and substituting in the second of Eqs. 5/6*a* give us

$$\left[V\bar{y} = \int y_c\, dV\right] \qquad \left(\tfrac{2}{3}\pi r^3\right)\bar{y} = \int_0^r \frac{\sqrt{r^2 - z^2}}{2}\left(2\pi z\sqrt{r^2 - z^2}\right) dz$$

$$= \int_0^r \pi(r^2 z - z^3)\, dz = \frac{\pi r^4}{4}$$

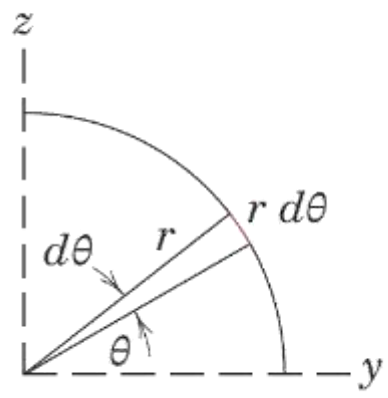
$$\bar{y} = \tfrac{3}{8}r \qquad \textit{Ans.}$$



Article 5/3 – Sample Problem 5/5 (4 of 4)

- Solution III: Use θ as the Variable

As an alternative, we could use the angle θ as our variable with limits of 0 and $\pi/2$. The radius of either element would become $r \sin \theta$, whereas the thickness of the slice in *Solution I* would be $dy = (r\,d\theta) \sin \theta$ and that of the shell in *Solution II* would be $dz = (r\,d\theta) \cos \theta$. The length of the shell would be $y = r \cos \theta$.

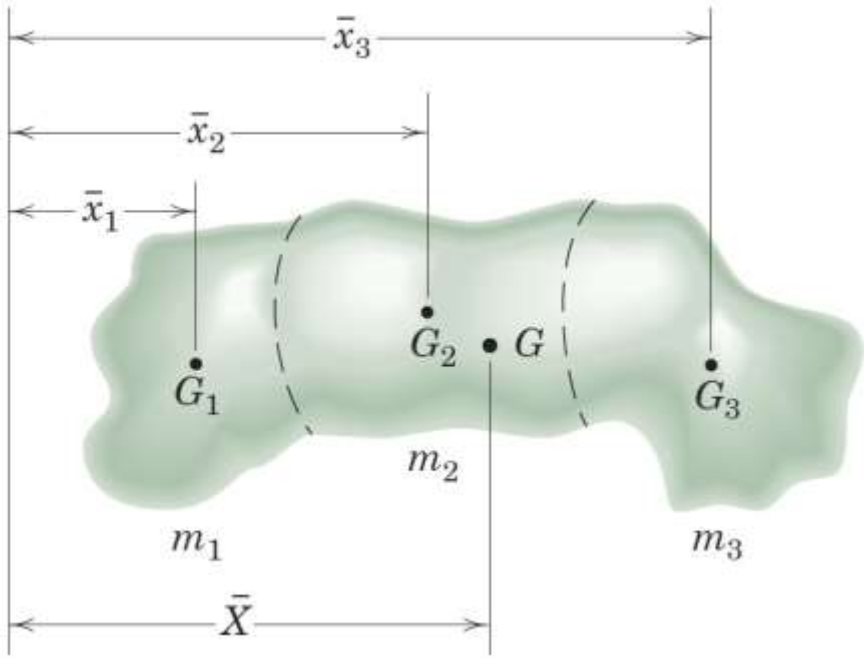


Article 5/4 Composite Bodies and Figures; Approximations

- Overview of Composites

- General Expressions

$$\bar{X} = \frac{\Sigma m\bar{x}}{\Sigma m} \qquad \bar{Y} = \frac{\Sigma m\bar{y}}{\Sigma m} \qquad \bar{Z} = \frac{\Sigma m\bar{z}}{\Sigma m}$$



- Analogous equations exist for lines, areas, and volumes by replacing the *m*’s with *L*’s, *A*’s, and *V*’s, respectively.

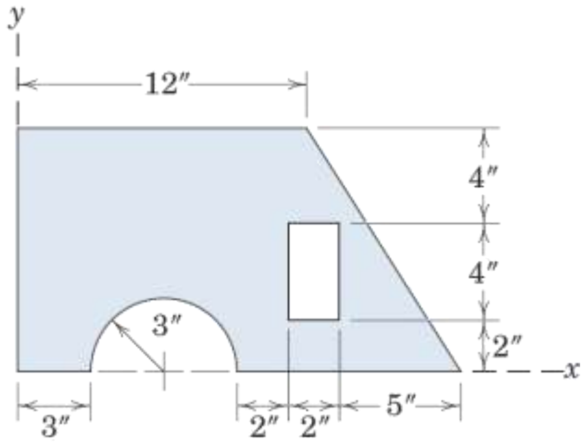
Article 5/4 – Comments about Composite Bodies

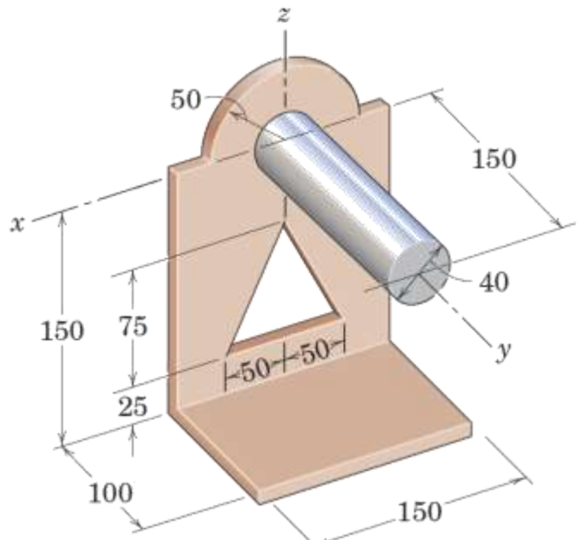
- Identify composite pieces based on Table D/3 and D/4 in Appendix D.

- Treat holes and cutouts as negative quantities. In the upper shape, the rectangular cutout has an area of -8 in.^{2}.

- Measure the centroidal coordinate for every piece from the origin of your reference axes. In the upper shape, the horizontal centroidal coordinate of the rectangular cutout is 12 in., not 1 in.

- Most objects consist of a single type of composite piece geometry (lines, areas, or volumes), but sometimes they are combined. For example, the upper shape consists of areas, while the lower shape consists of areas and volumes. To find the centroid of the lower shape, you must use the mass of the individual pieces.

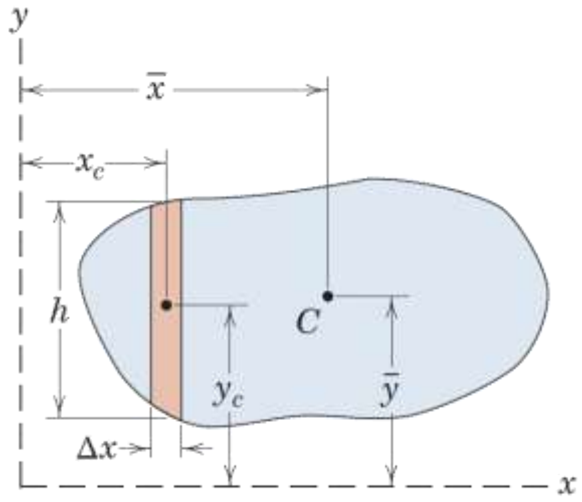




Dimensions in millimeters

Article 5/4 – An Approximation Method

When the boundaries of an area (or volume) are not expressible in terms of simple geometrical shapes or mathematical expressions…

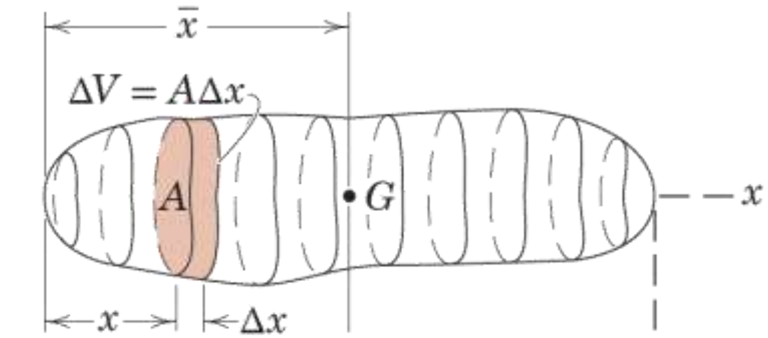


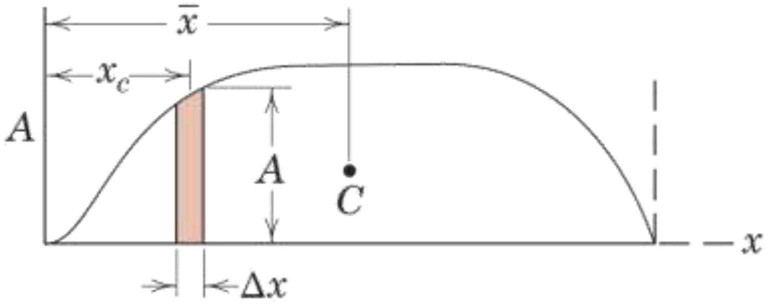
- Divide the body into individual pieces.
- Calculate the area $A = h\,\Delta x$ of each piece.
- Determine the centroidal coordinates, x_c and y_c for each piece.
- Evaluate the expression…

$$\bar{x} = \frac{\Sigma A x_c}{\Sigma A} \qquad \bar{y} = \frac{\Sigma A y_c}{\Sigma A}$$

Article 5/4 – Irregular Volumes

The centroidal location of an irregular volume may be simplified to one of locating the centroid of an area.





- Plot the magnitudes of the cross-sectional areas *A*.
- Calculate the volume $V = A\,\Delta x$ of individual slices.
- Determine the centroidal coordinates x_c of each slice.
- Evaluate the expression…

$$\bar{x} = \frac{\Sigma(A\ \Delta x)x_c}{\Sigma A\ \Delta x} \qquad \text{which equals} \qquad \bar{x} = \frac{\Sigma V x_c}{\Sigma V}$$

Article 5/4 – Centroids of Common Shapes (1 of 6)

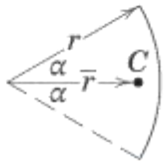
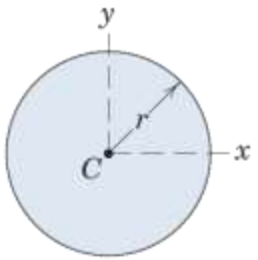
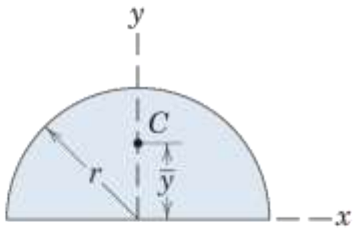
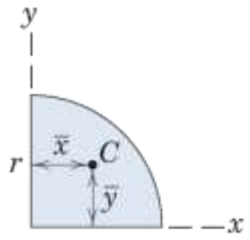
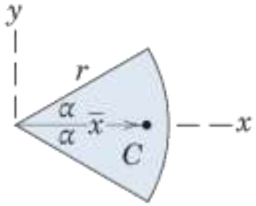
Figure		Centroid
Arc Segment		$\bar{r} = \dfrac{r \sin \alpha}{\alpha}$
Quarter and Semicircular Arcs		$\bar{y} = \dfrac{2r}{\pi}$
Circular Area		—
Semicircular Area		$\bar{y} = \dfrac{4r}{3\pi}$

Figure		Centroid
Quarter-Circular Area		$\bar{x} = \bar{y} = \dfrac{4r}{3\pi}$
Area of Circular Sector		$\bar{x} = \dfrac{2}{3}\dfrac{r \sin \alpha}{\alpha}$

Article 5/4 – Centroids of Common Shapes (2 of 6)

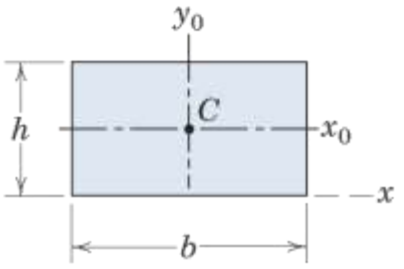
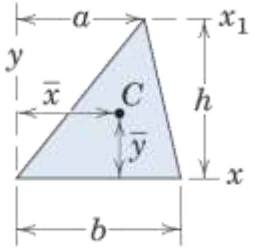
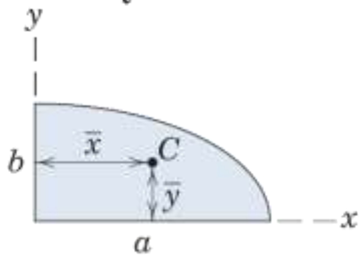
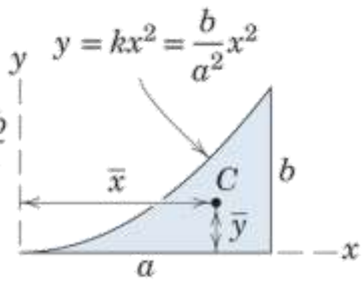
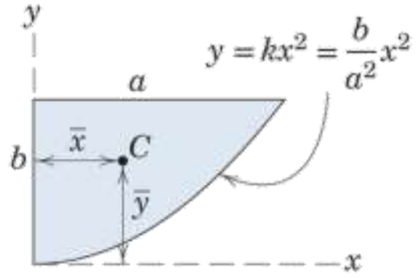
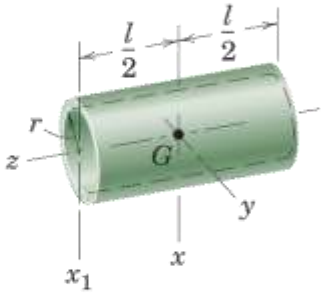
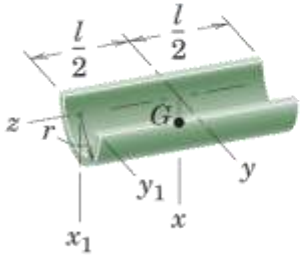
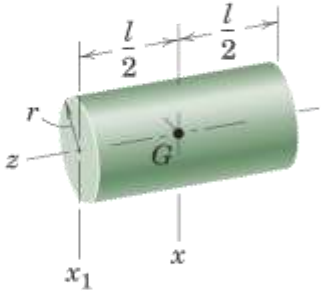
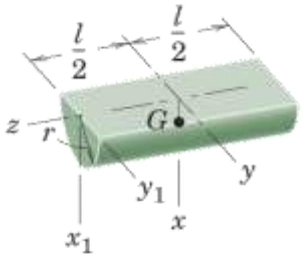
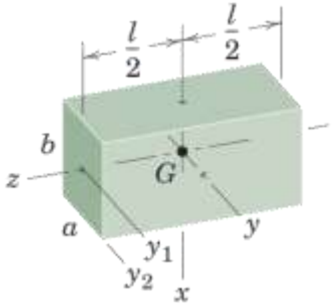
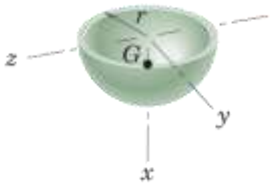
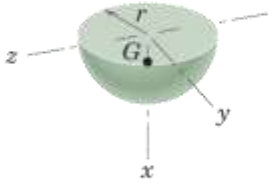
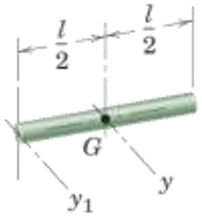
Figure	Centroid
Rectangular Area 	—
Triangular Area 	$\bar{x} = \frac{a+b}{3}$ $\bar{y} = \frac{h}{3}$
Area of Elliptical Quadrant 	$\bar{x} = \frac{4a}{3\pi}$ $\bar{y} = \frac{4b}{3\pi}$

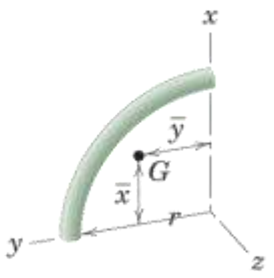
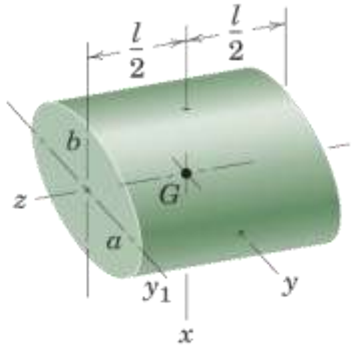
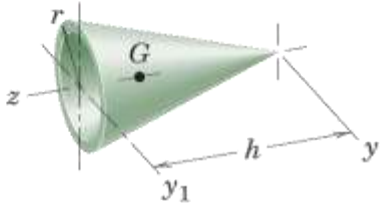
Figure	Centroid
Subparabolic Area Area $A = \frac{ab}{3}$ 	$\bar{x} = \frac{3a}{4}$ $\bar{y} = \frac{3b}{10}$
Parabolic Area Area $A = \frac{2ab}{3}$ 	$\bar{x} = \frac{3a}{8}$ $\bar{y} = \frac{3b}{5}$

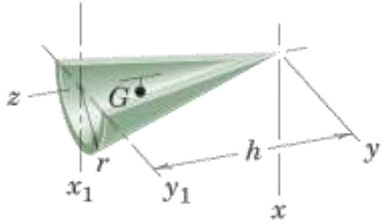
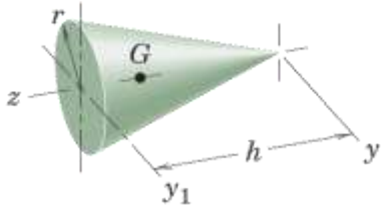
Article 5/4 – Centroids of Common Shapes (3 of 6)

Body		Mass Center
	Circular Cylindrical Shell	—
	Half Cylindrical Shell	$\bar{x} = \frac{2r}{\pi}$
	Circular Cylinder	—
	Semicylinder	$\bar{x} = \frac{4r}{3\pi}$
	Rectangular Parallelepiped	—

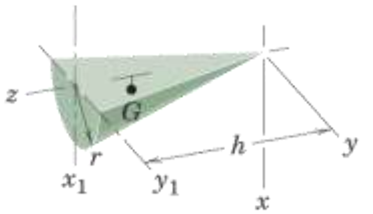
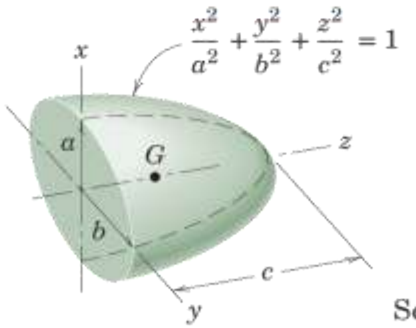
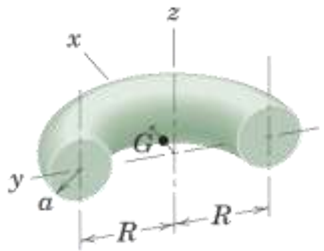
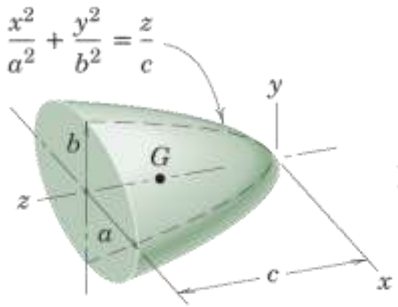
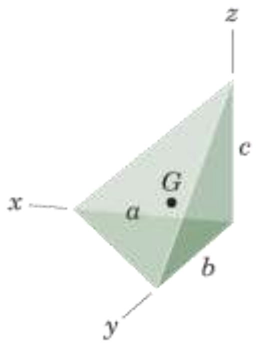
<table>
<tr><th colspan="2">Body</th><th>Mass Center</th></tr>
<tr><td>
</td><td>Spherical Shell</td><td>—</td></tr>
<tr><td>
</td><td>Hemispherical Shell</td><td>$\bar{x} = \frac{r}{2}$</td></tr>
<tr><td>
</td><td>Sphere</td><td>—</td></tr>
<tr><td>
</td><td>Hemisphere</td><td>$\bar{x} = \frac{3r}{8}$</td></tr>
<tr><td>
</td><td>Uniform Slender Rod</td><td>—</td></tr>
</table>

Article 5/4 – Centroids of Common Shapes (5 of 6)

Body		Mass Center
	Quarter-Circular Rod	$\bar{x} = \bar{y}$ $= \dfrac{2r}{\pi}$
	Elliptical Cylinder	—
	Conical Shell	$\bar{z} = \dfrac{2h}{3}$

Body		Mass Center
	Half Conical Shell	$\bar{x} = \dfrac{4r}{3\pi}$ $\bar{z} = \dfrac{2h}{3}$
	Right Circular Cone	$\bar{z} = \dfrac{3h}{4}$

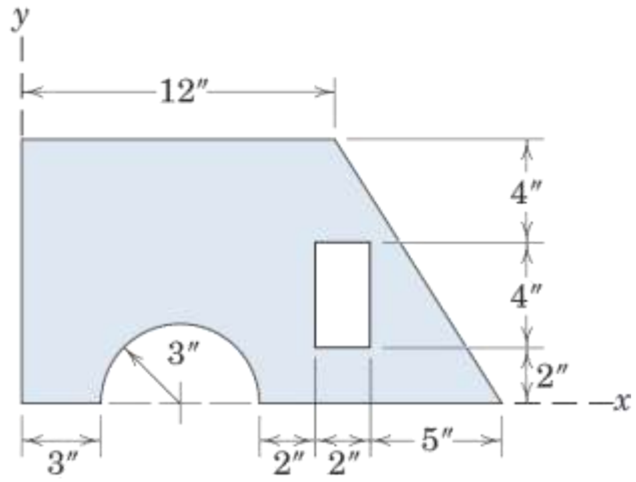
Article 5/4 – Centroids of Common Shapes (6 of 6)

Body	Mass Center
 Half Cone	$\bar{x} = \frac{r}{\pi}$ $\bar{z} = \frac{3h}{4}$
 Semiellipsoid	$\bar{z} = \frac{3c}{8}$
 Half Torus	$\bar{x} = \frac{a^2 + 4R^2}{2\pi R}$
 Elliptic Paraboloid	$\bar{z} = \frac{2c}{3}$
 Rectangular Tetrahedron	$\bar{x} = \frac{a}{4}$ $\bar{y} = \frac{b}{4}$ $\bar{z} = \frac{c}{4}$

Article 5/4 – Sample Problem 5/6 (1 of 3)

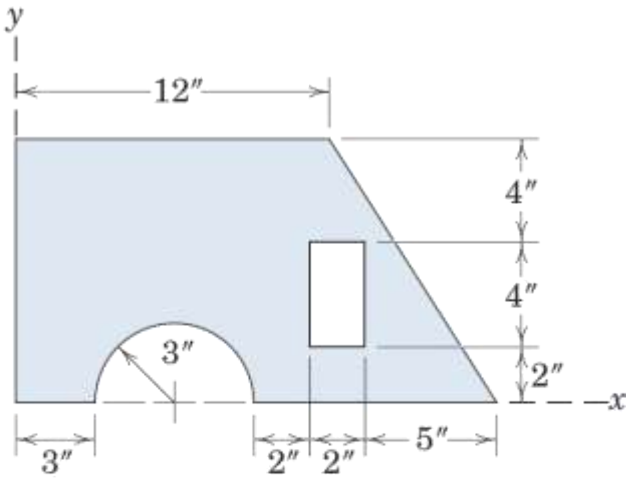
- Problem Statement

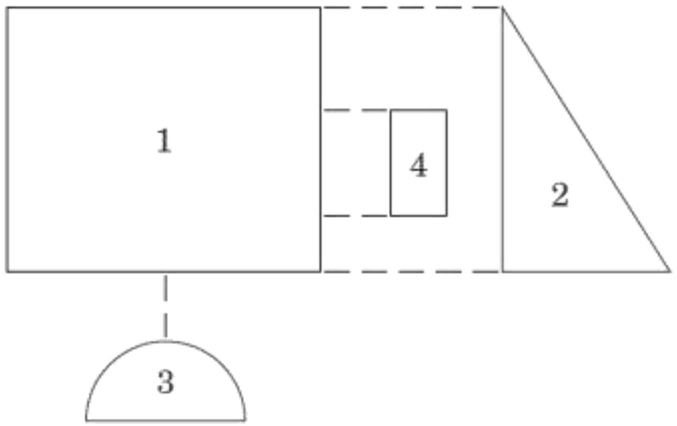
  Locate the centroid of the shaded area.



Article 5/4 – Sample Problem 5/6 (2 of 3)

- Composite Shapes
  - Rectangular Area (1)
  - Triangular Area (2)
  - Semicircular Cutout (3)
  - Square Cutout (4)





Article 5/4 – Sample Problem 5/6 (2 of 3)

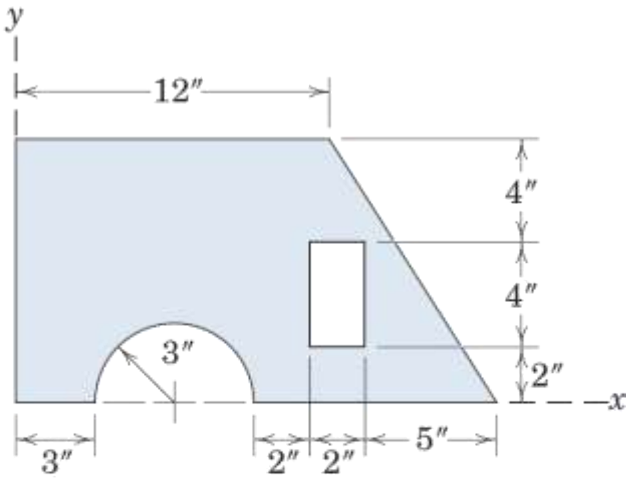
- Tabulated Values

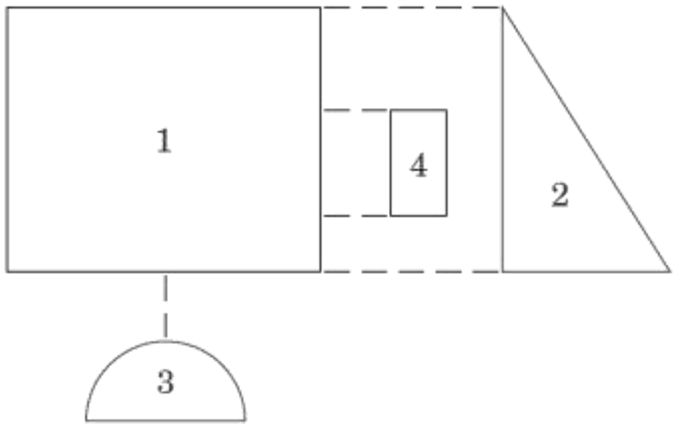
PART	A in.^{2}	$\bar{x}$ in.	$\bar{y}$ in.	$\bar{x}A$ in.^{3}	$\bar{y}A$ in.^{3}
1	120	6	5	720	600
2	30	14	10/3	420	100
3	–14.14	6	1.273	–84.8	–18
4	–8	12	4	–96	–32
TOTALS	127.9			959	650

The area counterparts to Eqs. 5/7 are now applied and yield

$$\left[\bar{X} = \frac{\Sigma A\bar{x}}{\Sigma A}\right] \qquad \bar{X} = \frac{959}{127.9} = 7.50 \text{ in.} \qquad \textit{Ans.}$$

$$\left[\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A}\right] \qquad \bar{Y} = \frac{650}{127.9} = 5.08 \text{ in.} \qquad \textit{Ans.}$$

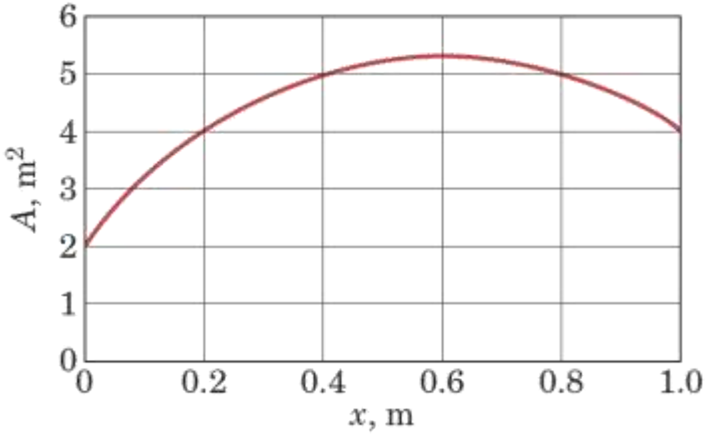




Article 5/4 – Sample Problem 5/7 (1 of 2)

- Problem Statement

  Approximate the x-coordinate of the volume centroid of a body whose length is 1 m and whose cross-sectional area varies with x as shown in the figure.



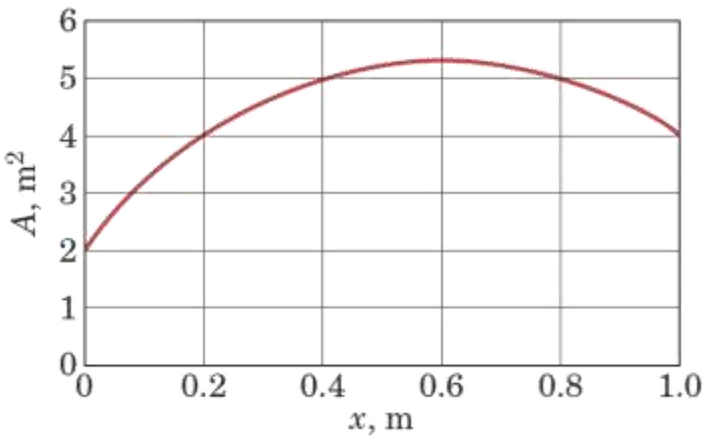
Article 5/4 – Sample Problem 5/7 (2 of 2)

- Solution

The body is divided into five sections. For each section, the average area, volume, and centroid location are determined and entered in the following table:

INTERVAL	A_{av} m^2	Volume V m^3	$\bar{x}$ m	$V\bar{x}$ m^4
0–0.2	3	0.6	0.1	0.060
0.2–0.4	4.5	0.90	0.3	0.270
0.4–0.6	5.2	1.04	0.5	0.520
0.6–0.8	5.2	1.04	0.7	0.728
0.8–1.0	4.5	0.90	0.9	0.810
TOTALS		4.48		2.388

$$\left[\bar{X} = \frac{\Sigma V\bar{x}}{\Sigma V}\right] \qquad \bar{X} = \frac{2.388}{4.48} = 0.533 \text{ m} \quad ① \qquad \textit{Ans.}$$

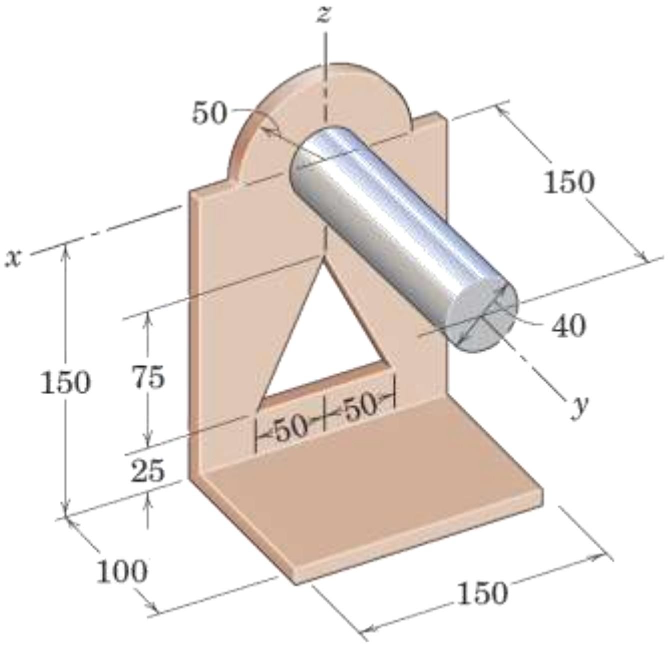


① Note that the shape of the body as a function of y and z does not affect $\bar{X}$.

Article 5/4 – Sample Problem 5/8 (1 of 4)

- Problem Statement

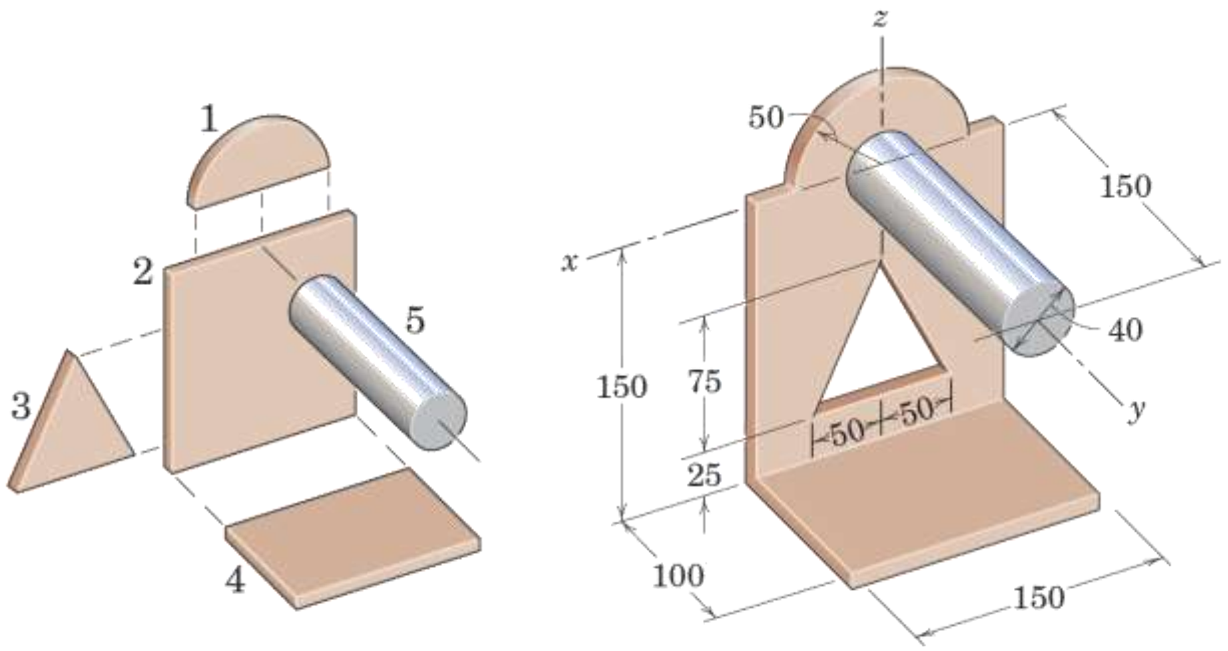
  Locate the center of mass of the bracket-and-shaft combination. The vertical face is made from sheet metal which has a mass of 25 kg/m^2. The material of the horizontal base has a mass of 40 kg/m^2, and the steel shaft has a density of 7.83 Mg/m^3.



Dimensions in millimeters

Article 5/4 – Sample Problem 5/8 (2 of 4)

- Composite Shapes
  - Semicircular Area (1)
  - Square Plate (2)
  - Triangular Cutout (3)
  - Rectangular Flat (4)
  - Cylindrical Shaft (5)



Dimensions in millimeters

Article 5/4 – Sample Problem 5/8 (3 of 4)

- Solution Comments

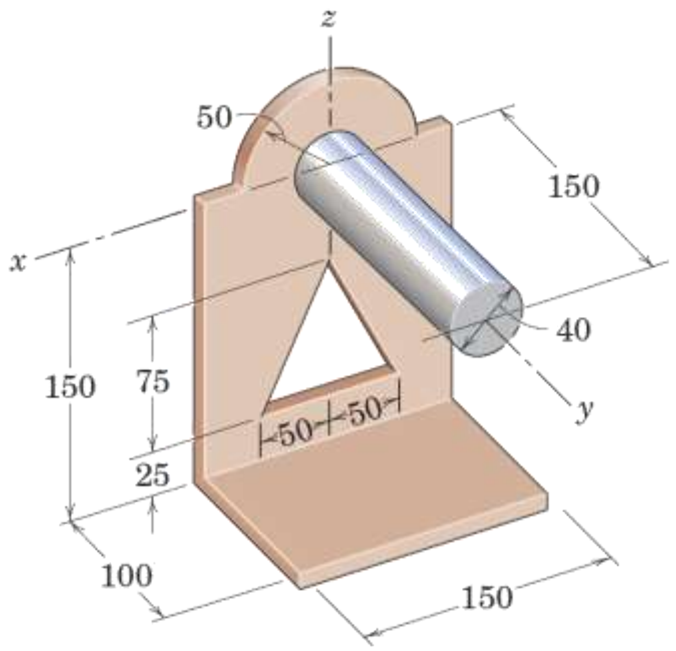
The composite body may be considered to be composed of the five elements shown in the lower portion of the illustration. The triangular part will be taken as a negative mass. For the reference axes indicated, it is clear by symmetry that the x-coordinate of the center of mass is zero.

The mass m of each part is easily calculated and should need no further explanation. For Part 1 we have from Sample Problem 5/3

$$\bar{z} = \frac{4r}{3\pi} = \frac{4(50)}{3\pi} = 21.2 \text{ mm}$$

For Part 3 we see from Sample Problem 5/2 that the centroid of the triangular mass is one-third of its altitude above its base. Measurement from the coordinate axes becomes

$$\bar{z} = -[150 - 25 - \tfrac{1}{3}(75)] = -100 \text{ mm}$$



Dimensions in millimeters

Article 5/4 – Sample Problem 5/8 (4 of 4)

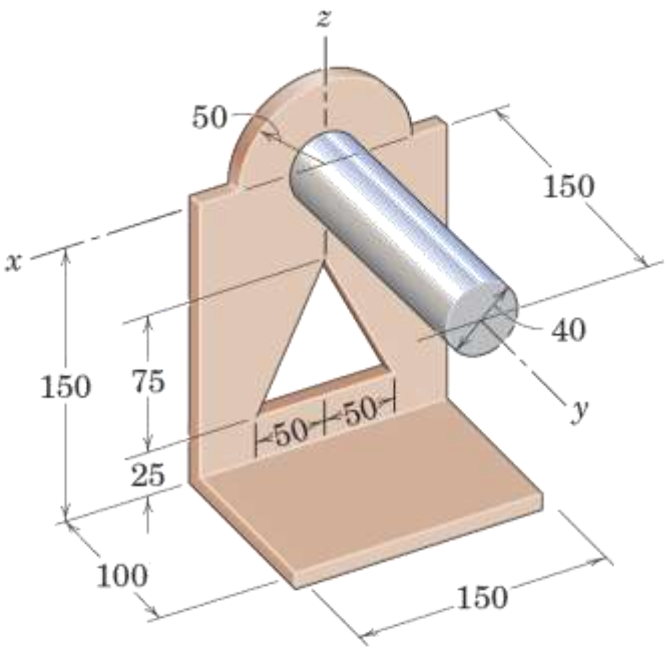
- Tabulated Values

PART	m kg	$\bar{y}$ mm	$\bar{z}$ mm	$m\bar{y}$ kg·mm	$m\bar{z}$ kg·mm
1	0.098	0	21.2	0	2.08
2	0.562	0	–75.0	0	–42.19
3	–0.094	0	–100.0	0	9.38
4	0.600	50.0	–150.0	30.0	–90.00
5	1.476	75.0	0	110.7	0
TOTALS	2.642			140.7	–120.73

Equations 5/7 are now applied and the results are

$$\left[\bar{Y} = \frac{\Sigma m\bar{y}}{\Sigma m}\right] \qquad \bar{Y} = \frac{140.7}{2.642} = 53.3 \text{ mm} \qquad \textit{Ans.}$$

$$\left[\bar{Z} = \frac{\Sigma m\bar{z}}{\Sigma m}\right] \qquad \bar{Z} = \frac{-120.73}{2.642} = -45.7 \text{ mm} \qquad \textit{Ans.}$$

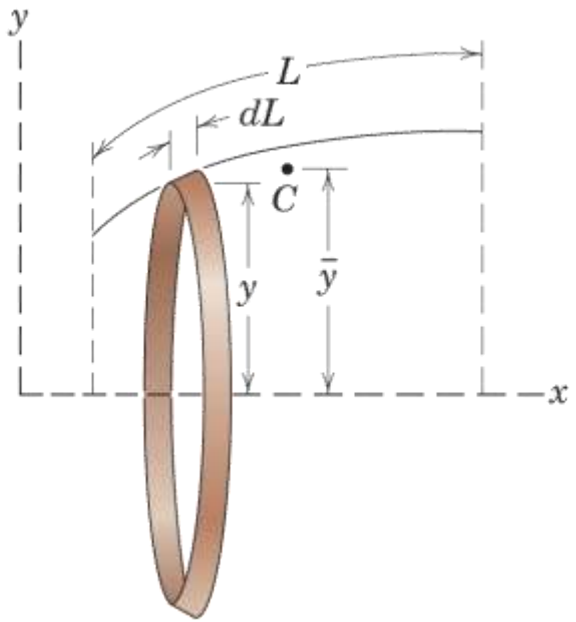


Dimensions in millimeters

Article 5/5 Theorems of Pappus

- Area of a Revolved Line Segment
  - L = length of a revolved line segment
  - $\bar{y}$ = y-coordinate of the centroid C for the line of length L

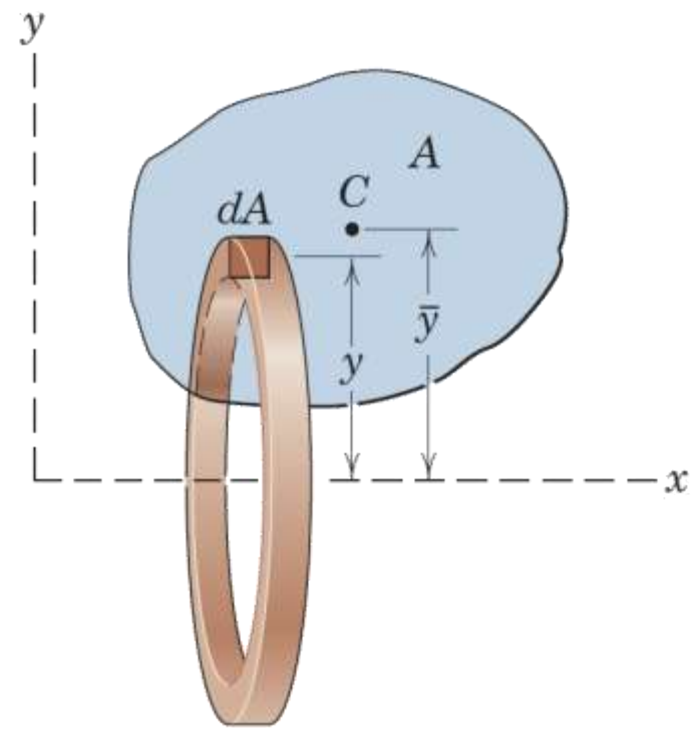
  - Equation of Interest: $A = 2\pi\bar{y}L$



- For a partial revolution through the angle θ…
  - $A = \theta\bar{y}L$ where θ is expressed in radians

Article 5/5 – Theorems of Pappus (cont.)

- Volume of a Solid of Revolution
  - A = area of a revolved section
  - $\bar{y}$ = y-coordinate of the centroid C for the area A
  - Equation of Interest: $V = 2\pi\bar{y}A$

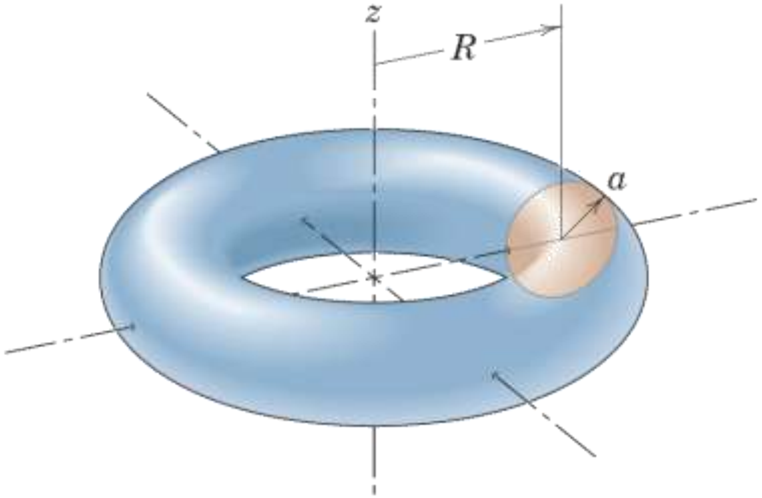


- For a partial revolution through the angle θ…
  - $V = \theta\bar{y}A$ where θ is expressed in radians

Article 5/5 – Sample Problem 5/9 (1 of 2)

- Problem Statement

  Determine the volume *V* and surface area *A* of the complete torus of circular cross section.



Article 5/5 – Sample Problem 5/9 (2 of 2)

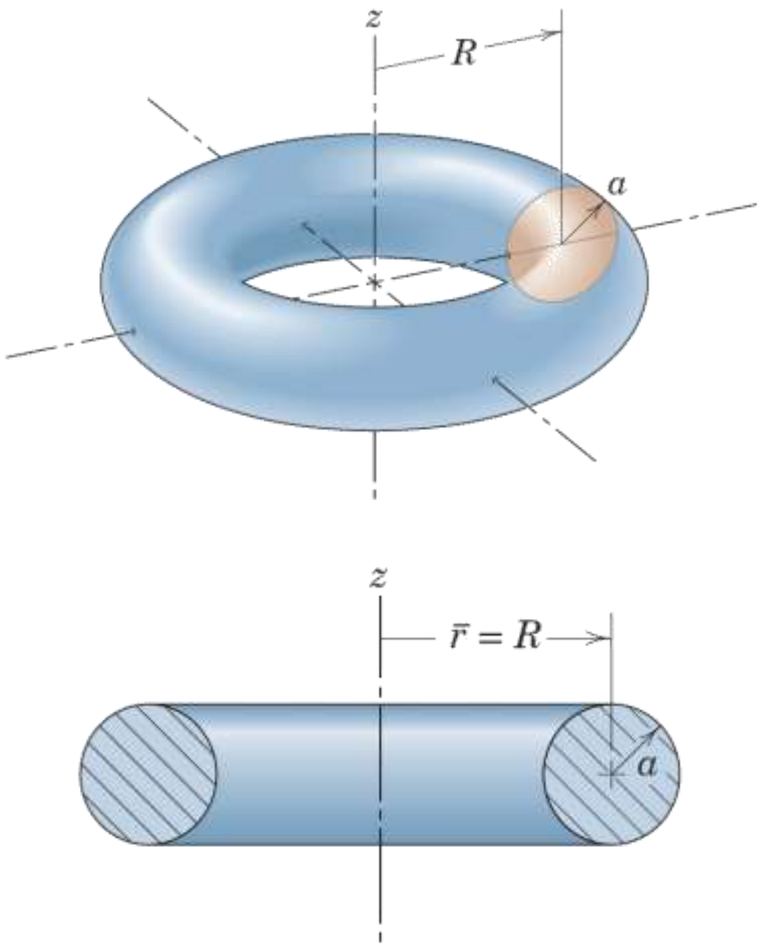
- Solution

$$V = \theta \bar{r} A = 2\pi(R)(\pi a^2) = 2\pi^2 R a^2 \quad ① \qquad \textit{Ans.}$$

Similarly, using Eq. 5/8*a* gives

$$A = \theta \bar{r} L = 2\pi(R)(2\pi a) = 4\pi^2 R a \qquad \textit{Ans.}$$

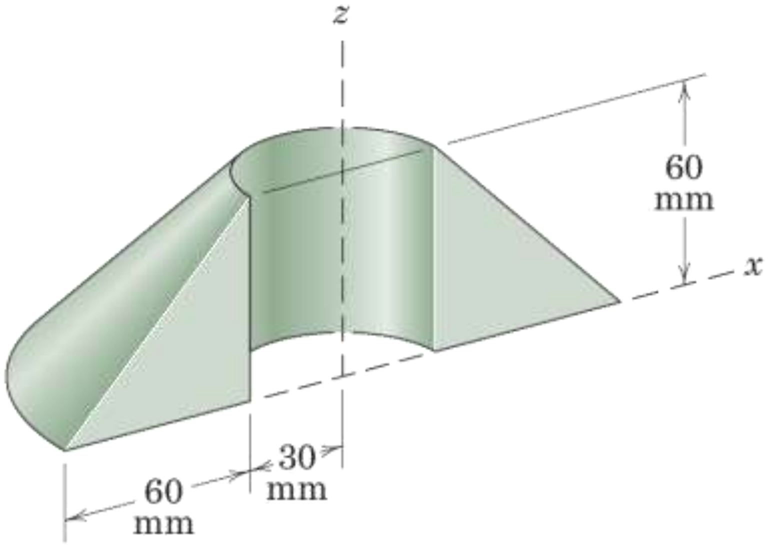
① We note that the angle θ of revolution is 2π for the complete ring. This common but special-case result is given by Eq. 5/9.



Article 5/5 – Sample Problem 5/10 (1 of 2)

- Problem Statement

  Calculate the volume V of the solid generated by revolving the 60-mm right-triangular area through 180° about the z-axis. If this body were constructed of steel, what would be its mass m?



Article 5/5 – Sample Problem 5/10 (2 of 2)

- Solution

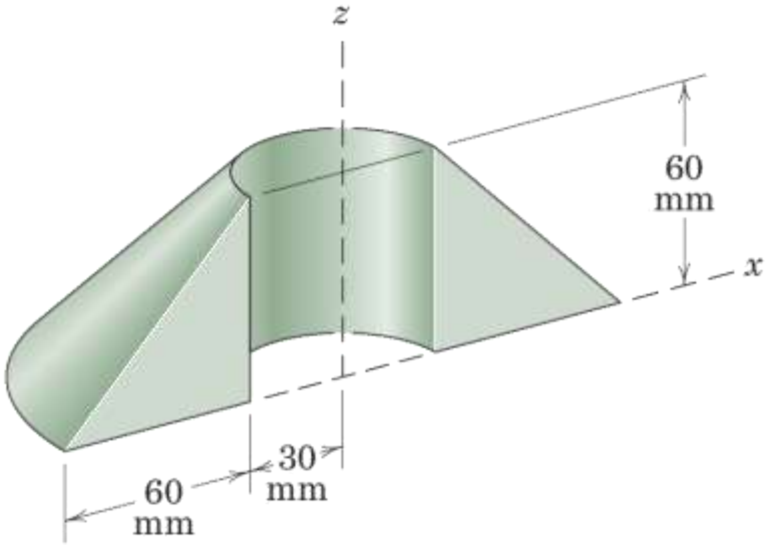
$$V = \theta \bar{r} A = \pi[30 + \tfrac{1}{3}(60)][\tfrac{1}{2}(60)(60)] = 2.83(10^5)\ \text{mm}^3 \quad ① \qquad Ans.$$

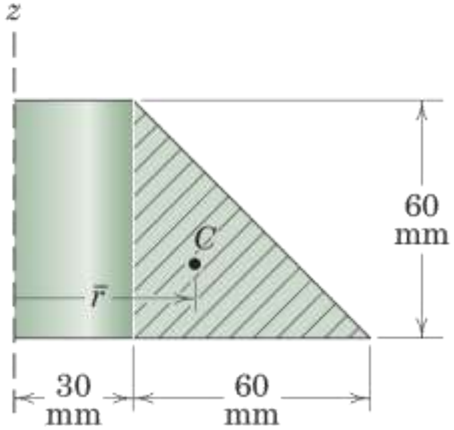
The mass of the body is then

$$m = \rho V = \left[7830\ \frac{\text{kg}}{\text{m}^3}\right][2.83(10^5)\text{mm}^3]\left[\frac{1\ \text{m}}{1000\ \text{mm}}\right]^3$$

$$= 2.21\ \text{kg} \qquad Ans.$$

① Note that θ must be in radians.





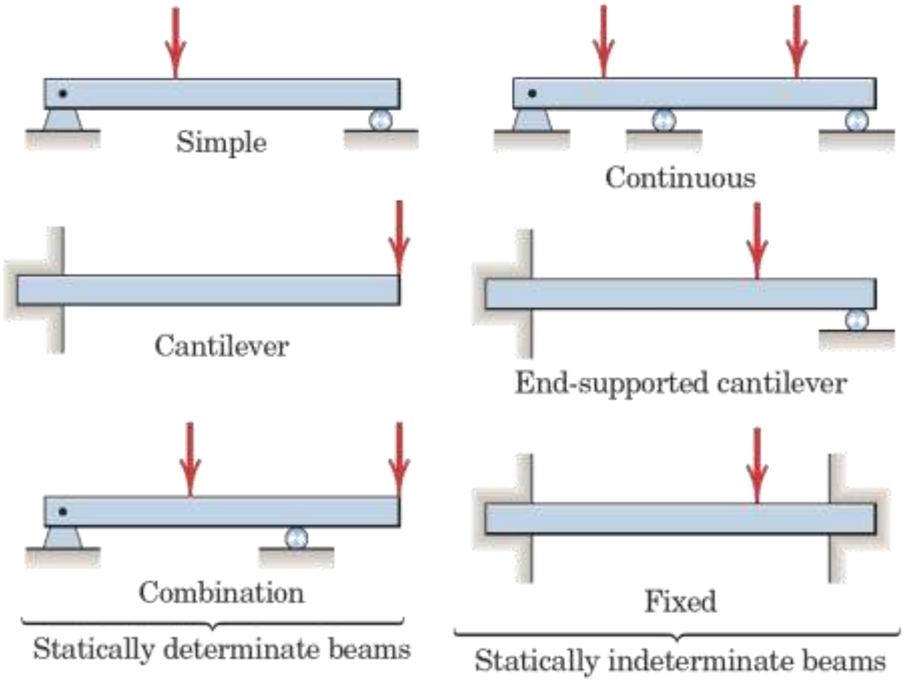
Article 5/6 Beams – External Effects

- Introduction

- Types of Beams
  - Determinate

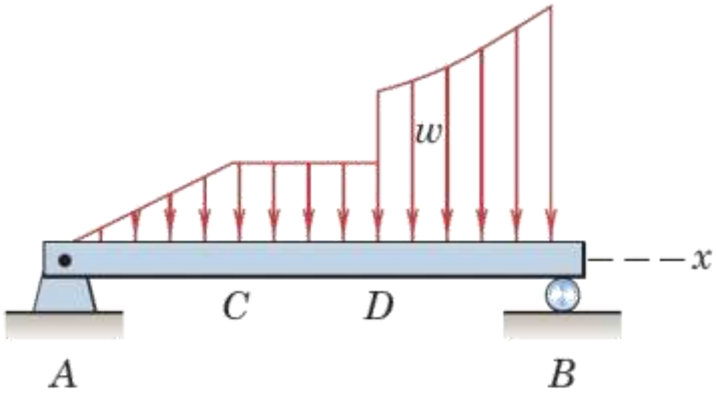
  - Indeterminate

- Types of Supports



Article 5/6 – Distributed Loads

- Illustration



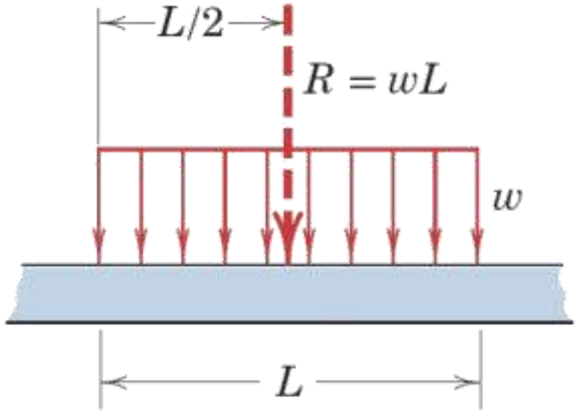
- Intensity of a Distributed Load, *w*
  - Units are Force/Length (N/m or lb/ft)
  - Various Types, e.g., Constant, Linear, Functional

- Question: How do we analyze the external reactions on a beam subjected to various distributed loads?

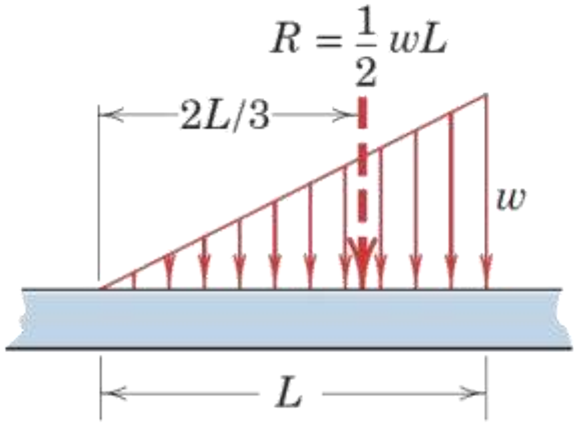
- Answer: We must replace the distributed load by its equivalent concentrated force which acts at the appropriate location.

Article 5/6 – Types of Distributed Loads (1 of 2)

- Uniform Loads

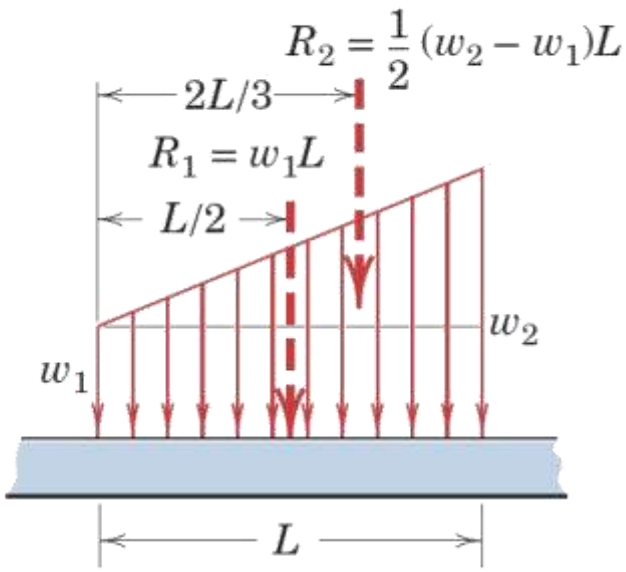


- Linear Loads



Article 5/6 – Types of Distributed Loads (2 of 2)

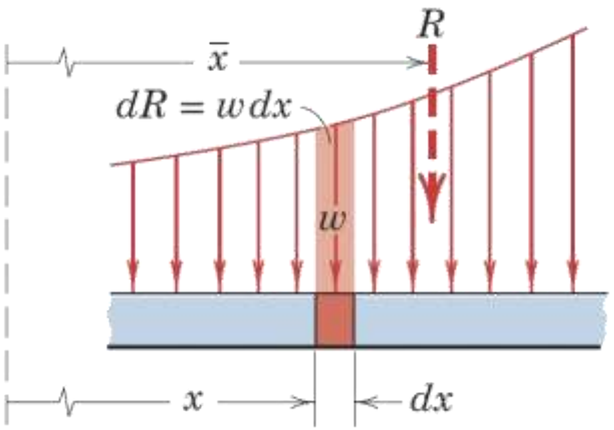
- Trapezoidal Loads



- Functional Loads

$$R = \int w \, dx$$

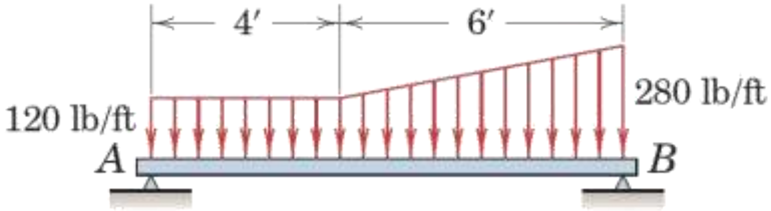
$$\bar{x} = \frac{\int xw \, dx}{R}$$



Article 5/6 – Sample Problem 5/11 (1 of 2)

- Problem Statement

  Determine the equivalent concentrated load(s) and external reactions for the simply supported beam which is subjected to the distributed load shown.



Article 5/6 – Sample Problem 5/11 (2 of 2)

- Solution

The area associated with the load distribution is divided into the rectangular and triangular areas shown. The concentrated-load values are determined by computing the areas, and these loads are located at the centroids of the respective areas. ①

Once the concentrated loads are determined, they are placed on the free-body diagram of the beam along with the external reactions at A and B. Using principles of equilibrium, we have

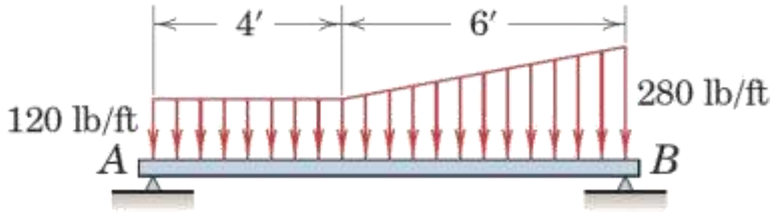
$[\Sigma M_A = 0]$ $\quad 1200(5) + 480(8) - R_B(10) = 0$

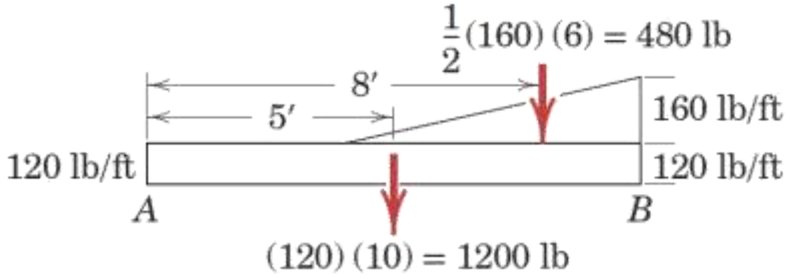
$$R_B = 984 \text{ lb} \qquad \textit{Ans.}$$

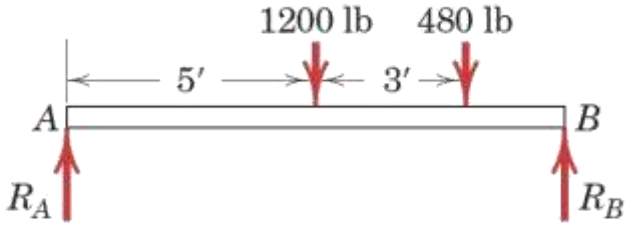
$[\Sigma M_B = 0]$ $\quad R_A(10) - 1200(5) - 480(2) = 0$

$$R_A = 696 \text{ lb} \qquad \textit{Ans.}$$

① Note that it is usually unnecessary to reduce a given distributed load to a *single* concentrated load.



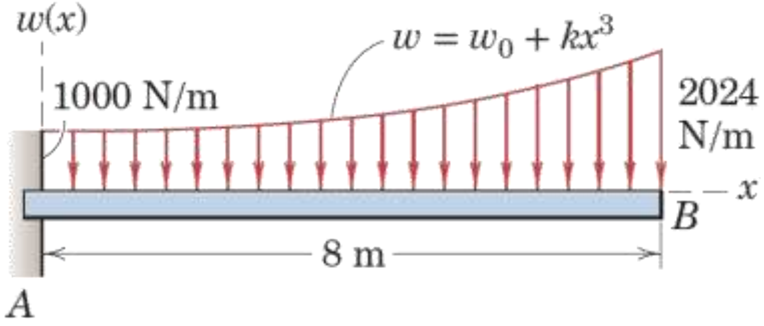




Article 5/6 – Sample Problem 5/12 (1 of 2)

- Problem Statement

  Determine the reaction at the support *A* of the loaded cantilever beam.



Article 5/6 – Sample Problem 5/12 (2 of 2)

- Solution

The constants in the load distribution are found to be $w_0 =$ 1000 N/m and $k = 2$ N/m^4. ① The load R is then

$$R = \int w\,dx = \int_0^8 (1000 + 2x^3)\,dx = \left(1000x + \frac{x^4}{2}\right)\Big|_0^8 = 10\,050 \text{ N}$$

The x-coordinate of the centroid of the area is found by ②

$$\bar{x} = \frac{\int xw\,dx}{R} = \frac{1}{10\,050}\int_0^8 x(1000 + 2x^3)\,dx$$

$$= \frac{1}{10\,050}\left(500x^2 + \tfrac{2}{5}x^5\right)\Big|_0^8 = 4.49 \text{ m}$$

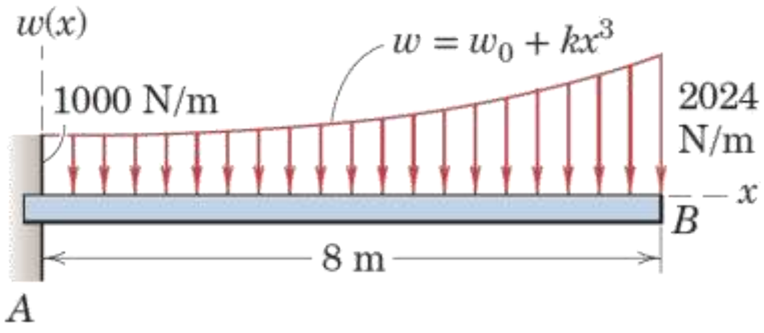
From the free-body diagram of the beam, we have

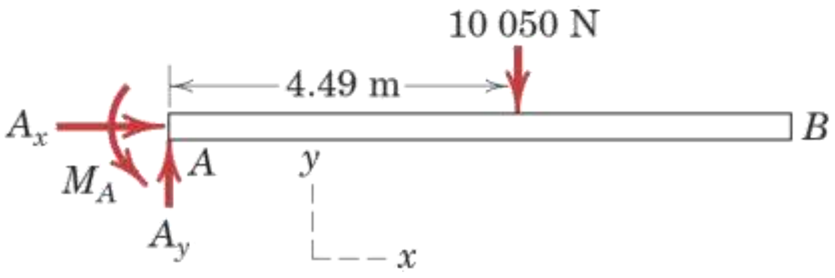
$[\Sigma M_A = 0]$ $\qquad M_A - (10\,050)(4.49) = 0$

$M_A = 45\,100$ N·m $\qquad$ *Ans.*

$[\Sigma F_y = 0]$ $\qquad A_y = 10\,050$ N $\qquad$ *Ans.*

Note that $A_x = 0$ by inspection.



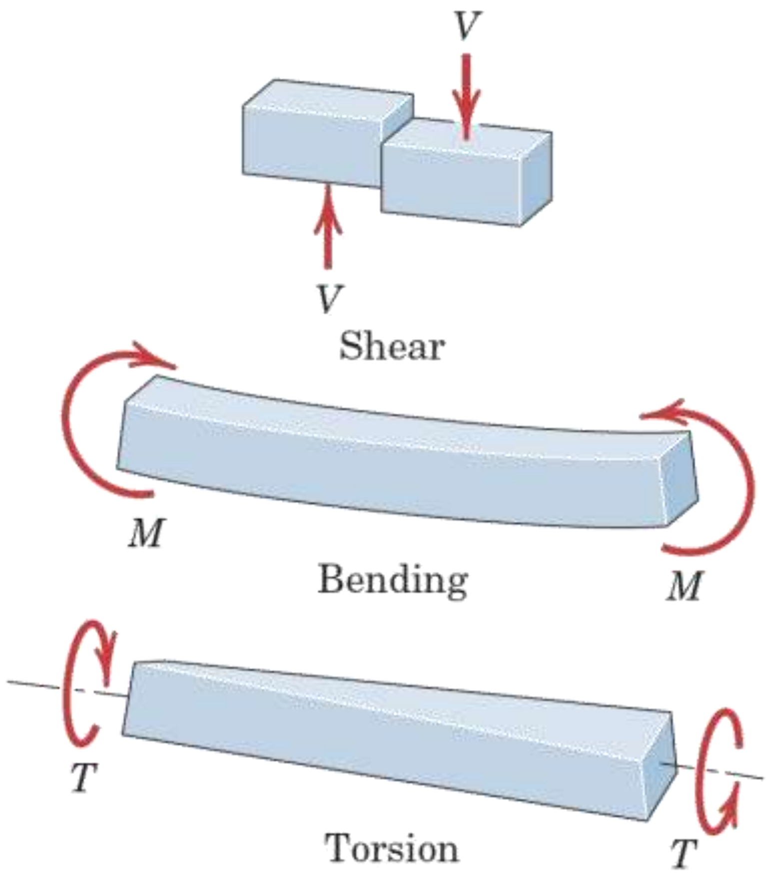


① Use caution with the units of the constants w_0 and k.

② The student should recognize that the calculation of R and its location $\bar{x}$ is simply an application of centroids as treated in Art. 5/3.

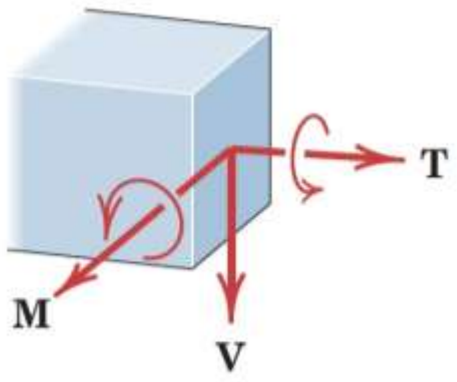
Article 5/7 Beams – Internal Effects

- Internal Loads in Beams
  - Shear Force
  - Bending Moment
  - Torsional Moment



Article 5/7 – Internal Loads (cont.)

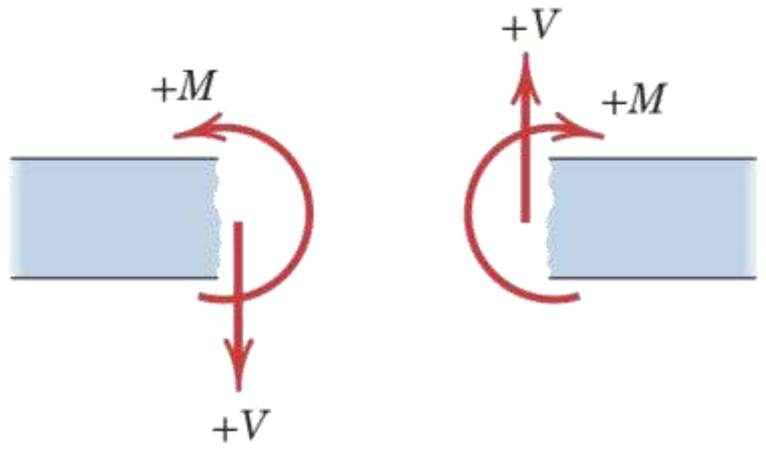
- Combined Loading Effect



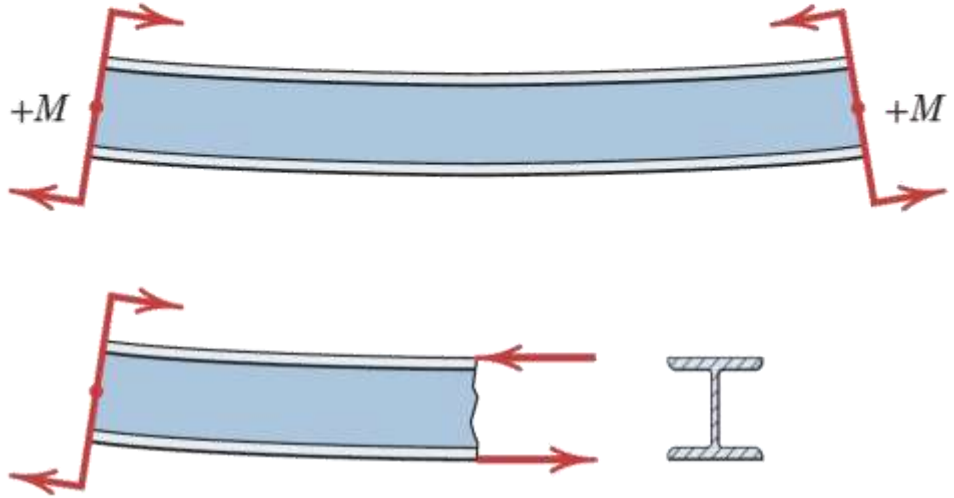
Combined loading

Article 5/7 – Shear and Bending Explored

- Sign Convention for *V* and *M*

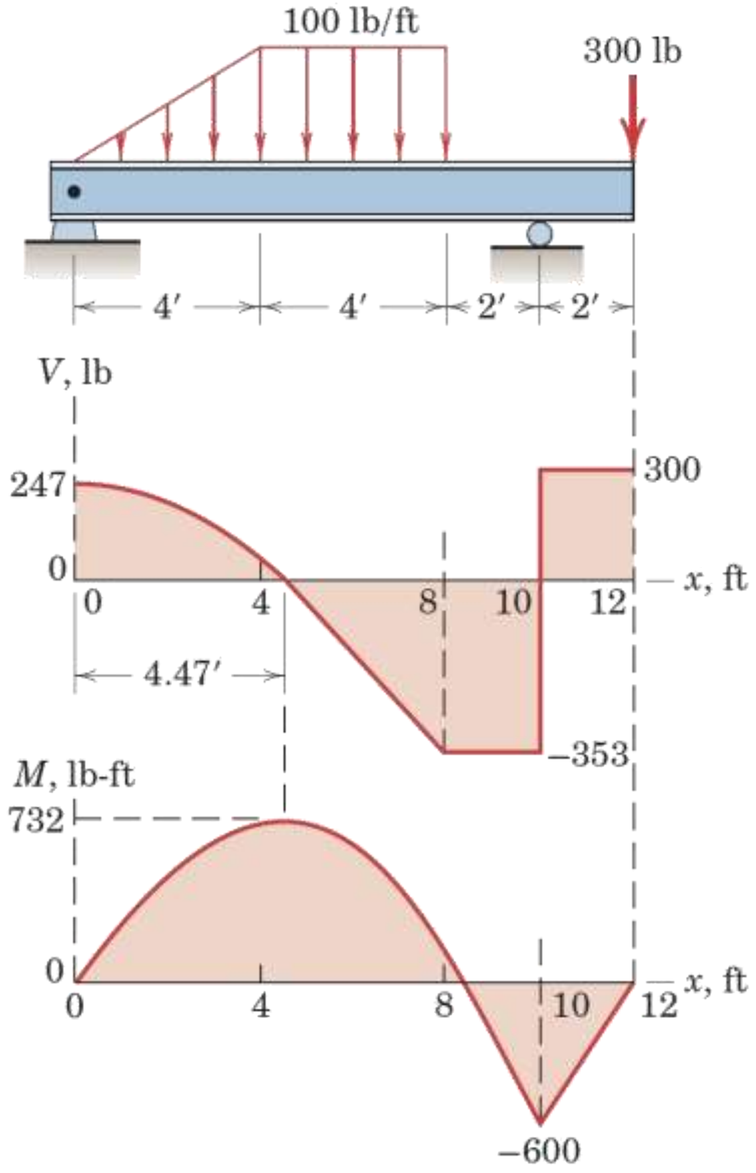


- Visualization of the Bending Moment



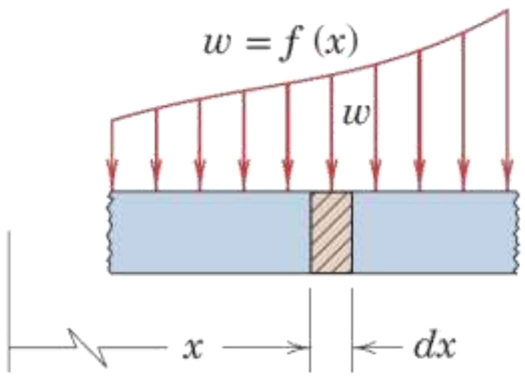
Article 5/7 – Shear-Force and Bending-Moment Diagrams

- Overview
- Solution Steps
  1. Determine the external reactions which act on the beam.
  2. Isolate a portion of the beam, either to the right or the left of an arbitrary transverse section, and construct a free-body diagram of the section. Sections cannot be taken at a discontinuity in the loading diagram, e.g., at a reaction or concentrated force, or at a discontinuity in a distributed load. Loads in the section should be shown in a positive sign convention.
  3. Apply the equations of equilibrium to the isolated portion to determine expressions for the shear force *V* and the bending moment *M* which act in that portion of the beam.
  4. Plot the resulting expressions over the section of the beam for which they are valid.
  5. Continue to the next section until the entire beam has been analyzed.

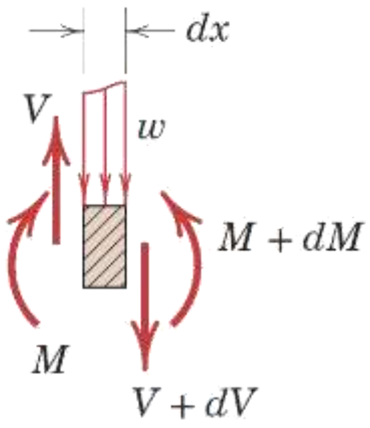


Article 5/7 – General Loading, Shear, and Moment Relationships

- Situation of Interest



- Free-Body Diagram of a Segment

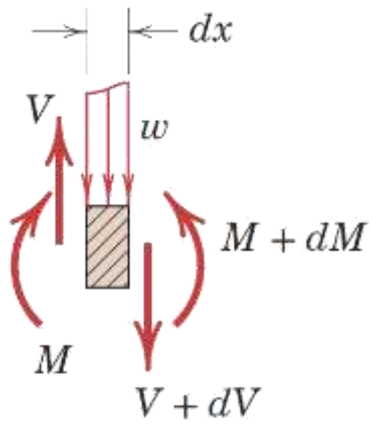


- Solution Process
  - Vertical Equilibrium yields *V*
  - Moment Equilibrium yields *M*

Article 5/7 – Relationships between Shear and Loading (1 of 3)

- Vertical Equilibrium (positive up)

$$V - w\,dx - (V + dV) = 0 \quad \rightarrow \quad dV = -w\,dx$$



- Result 1: Indefinite integral of both sides.

$$\int dV = -\int w\,dx \quad \rightarrow \quad V = C_1 - \int w\,dx$$

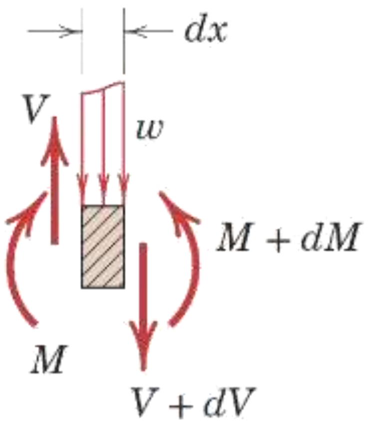
  - The expression for the internal shear force in a region of the beam as a function of *x* is found by integrating the equation for the distributed loading within that region of the beam and adding an appropriate constant. The integral expression must be multiplied by a negative to obtain the correct relationship.

  - The constant C_1 is most often determined as the value of the shear force at the beginning of the section of the beam where the distributed loading is defined.

Article 5/7 – Relationships between Shear and Loading (2 of 3)

- Vertical Equilibrium

$$dV = -w\,dx$$



- Result 2: Definite integral of both sides.

$$\int_{V_1}^{V_2} dV = -\int_{x_1}^{x_2} w\,dx \;\rightarrow\; V_2 - V_1 = \Delta V_{1-2} = -\int_{x_1}^{x_2} w\,dx$$

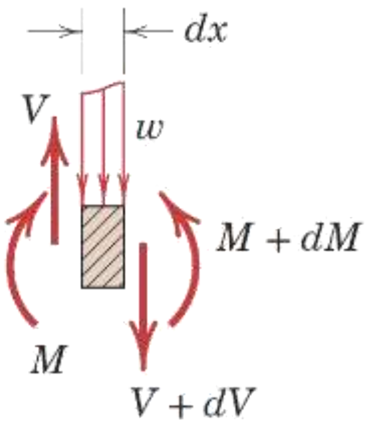
  - The change in the value of the internal shear force between two points on a beam is the negative of the area under the distributed loading curve between those same two points.

  - Note that V_1 is the value of the shear force at the coordinate x_1, and V_2 is the value of the shear force at the coordinate x_2.

Article 5/7 – Relationships between Shear and Loading (3 of 3)

- Vertical Equilibrium

$$dV = -w\,dx$$



- Result 3: Divide both sides by *dx*.

$$w = -\frac{dV}{dx}$$

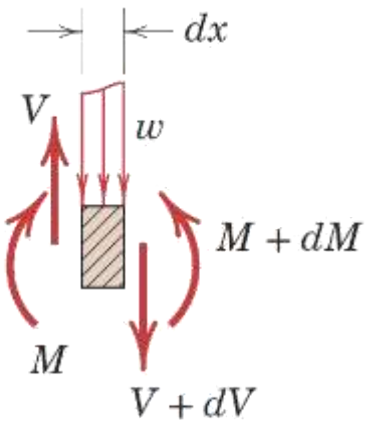
  - The slope of the internal shear-force diagram at any location on a beam is equal to the negative intensity of the distributed loading at that point.
  - If there is no distributed loading at a location in the beam, then the slope is zero at that location meaning the shear force is temporarily constant.

Article 5/7 – Relationships between Moment and Loading (1 of 3)

- Moment Equilibrium (about Left Side, CCW +)

$$-M + \left(M + dM\right) - (V + dV)dx - w\,dx\frac{dx}{2} = 0$$

$$dM = V\ dx$$



- Result 4: Indefinite integral of both sides.

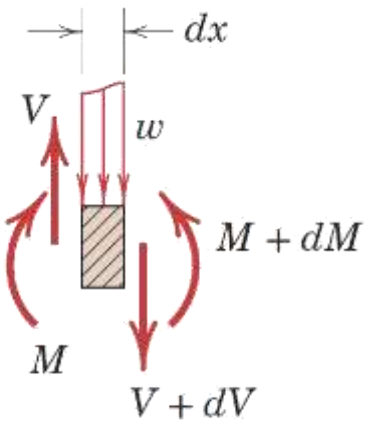
$$\int dM = \int V\ dx \quad \rightarrow \quad M = C_2 + \int V\ dx$$

  - The expression for the internal bending moment in a region of the beam is found by integrating the equation for the internal shear force within that region of the beam and adding an appropriate constant.
  - The constant C_2 is most often determined as the value of the bending moment at the beginning of the section of the beam where the distributed loading is defined.
  - Note that the expression for V was determined from the first integral of the loading function, so the bending moment is the second integral of the loading function.

Article 5/7 – Relationships between Moment and Loading (2 of 3)

- Moment Equilibrium (about Left Side, CCW +)

$$dM = V\ dx$$



- Result 5: Definite integral of both sides.

$$\int_{M_1}^{M_2} dM = \int_{x_1}^{x_2} V\ dx \quad \rightarrow \quad M_2 - M_1 = \Delta M_{1-2} = \int_{x_1}^{x_2} V\ dx$$

  - The change in the value of the internal bending moment between two points on a beam is the area under the shear force diagram between those same two points.

  - Note that M_1 is the value of the bending moment at the coordinate x_1, and M_2 is the value of the bending moment at the coordinate x_2.

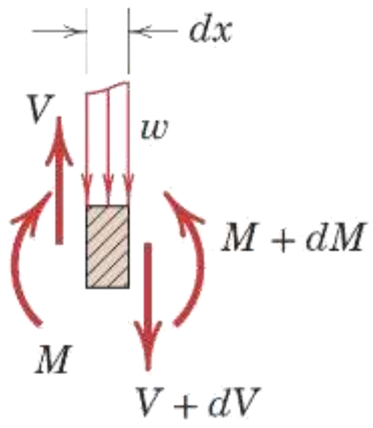
Article 5/7 – Relationships between Moment and Loading (3 of 3)

- Moment Equilibrium (about Left Side, CCW +)

$$dM = V\ dx$$

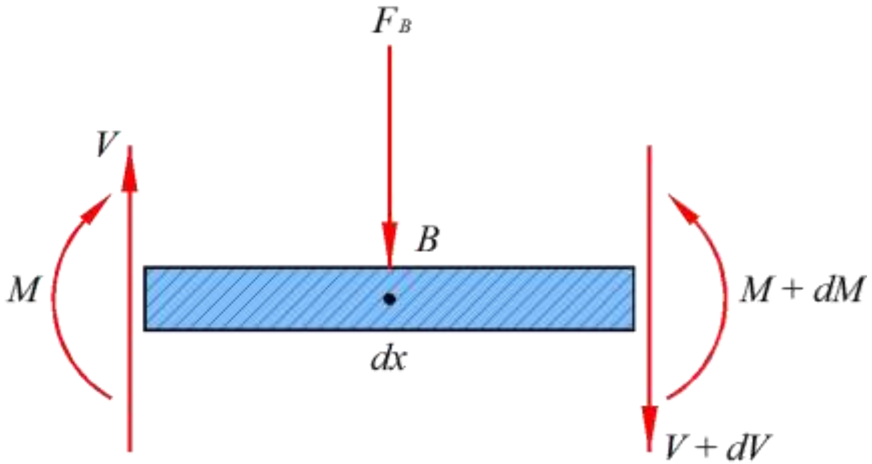
- Result 6: Divide both sides by *dx*.

$$V = \frac{dM}{dx}$$



  - The slope of the internal bending moment diagram at any location on a beam is equal to the value of the internal shear force at that point.
  - If there is no shear force at a location in the beam, then the slope is zero at that location meaning the bending moment is temporarily constant.
  - When V passes through zero and is continuous function of x with $dV/dx \neq 0$, the bending moment M will be a maximum or a minimum. Critical values of M also occur when V crosses the zero axis discontinuously which occurs at a concentrated load.

Article 5/7 – Effect of a Concentrated Force (1 of 2)



- Situation of Interest

- Vertical Equilibrium (positive up)

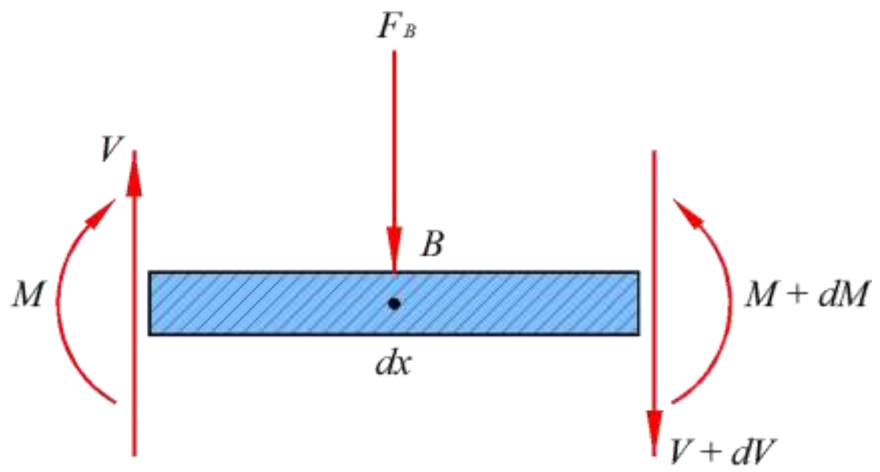
$$V - (V + dV) - F_B = 0 \quad \rightarrow \quad dV = -F_B$$

- Result 7
  - An externally applied, concentrated force creates a change in the internal shear force that is consistent with both the magnitude and direction of the concentrated force.
  - A downward force creates an instantaneous *drop* in the internal shear force and an upward force creates an instantaneous *rise* in the internal shear force. The drop or rise is equal to the magnitude of the concentrated force.

Article 5/7 – Effect of a Concentrated Force (2 of 2)

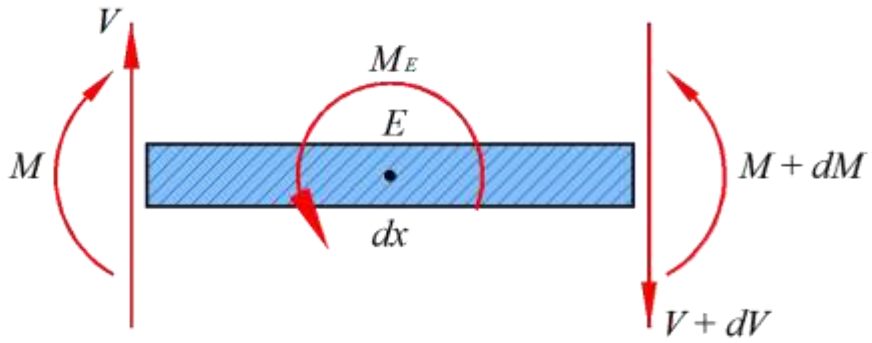
- Moment Equilibrium (at *B*, CCW+, *dx*→0)

$$-M - V\frac{dx}{2} - (V + dV)\frac{dx}{2} + (M + dM) = 0 \quad \rightarrow \quad dM = 0$$



- Result 8
  - An externally applied, concentrated force has no effect on the value of the internal bending moment at the point of application.
  - An externally applied, concentrated force will create a sudden change in the slope of the bending-moment diagram at the point of application.

Article 5/7 – Effect of a Concentrated Moment (1 of 2)



- Situation of Interest

- Vertical Equilibrium (positive up)

$$V-(V+dV)=0 \quad \rightarrow \quad dV=0$$

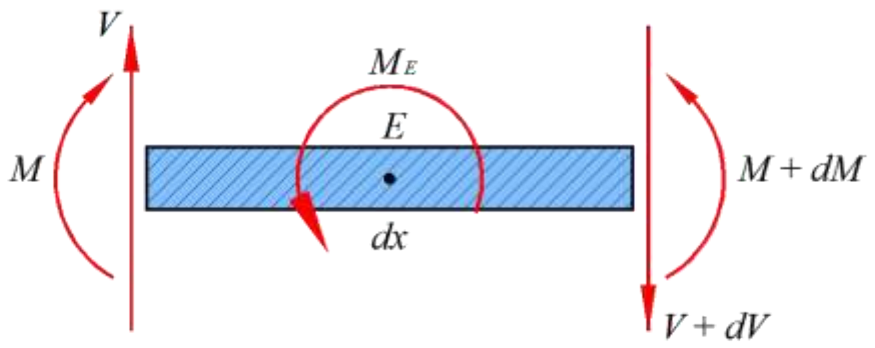
- Result 9
  - An externally applied, concentrated moment has no effect on the value of the internal shear force at the point of application for the moment.

Article 5/7 – Effect of a Concentrated Moment (2 of 2)

- Moment Equilibrium (at *B*, CCW+, *dx*→0)

$$-M - V\frac{dx}{2} - (V + dV)\frac{dx}{2} + (M + dM) + M_E = 0$$

$$dM = -M_E$$

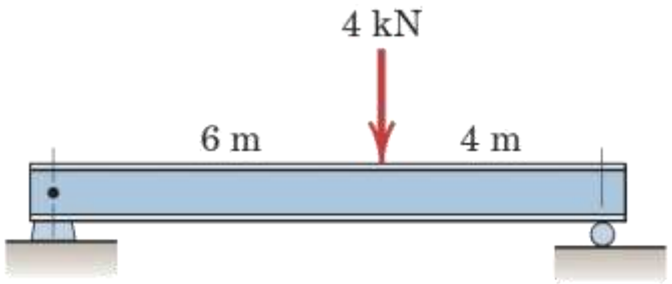


- Result 10
  - An externally applied, concentrated moment creates a change in the internal bending moment that is consistent with the magnitude of the concentrated moment, but is inconsistent with the direction of the concentrated moment.
  - An externally applied *counterclockwise* moment creates an instantaneous *drop* in the internal bending moment and an externally applied *clockwise* moment creates an instantaneous *rise* in the internal bending moment. The *drop* or *rise* is equal to the magnitude of the concentrated moment.

Article 5/7 – Sample Problem 5/13 (1 of 4)

- Problem Statement

  Determine the shear and moment distributions produced in the simple beam by the 4-kN concentrated load.

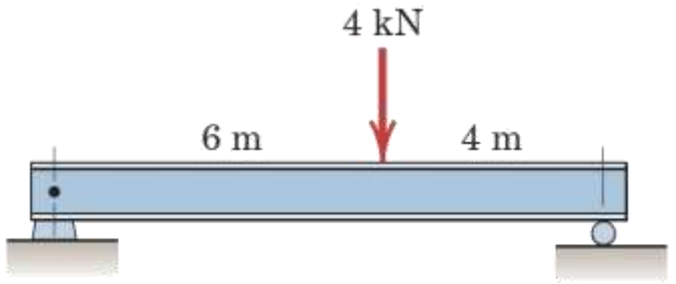


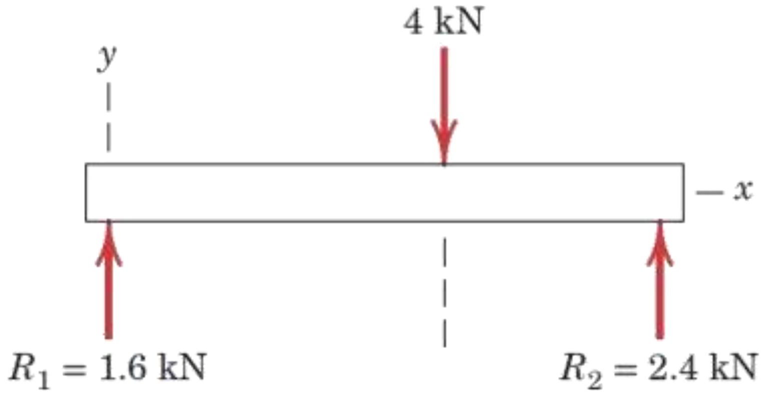
Article 5/7 – Sample Problem 5/13 (2 of 4)

- Support Reactions

From the free-body diagram of the entire beam we find the support reactions, which are

$$R_1 = 1.6 \text{ kN} \qquad R_2 = 2.4 \text{ kN}$$





Article 5/7 – Sample Problem 5/13 (3 of 4)

- Beam Segment Equilibrium

A section of the beam of length x is next isolated with its free-body diagram on which we show the shear V and the bending moment M in their positive directions. Equilibrium gives

$[\Sigma F_y = 0]$ $\quad 1.6 - V = 0 \quad V = 1.6\text{ kN}$

$[\Sigma M_{R_1} = 0]$ $\quad M - 1.6x = 0 \quad M = 1.6x$

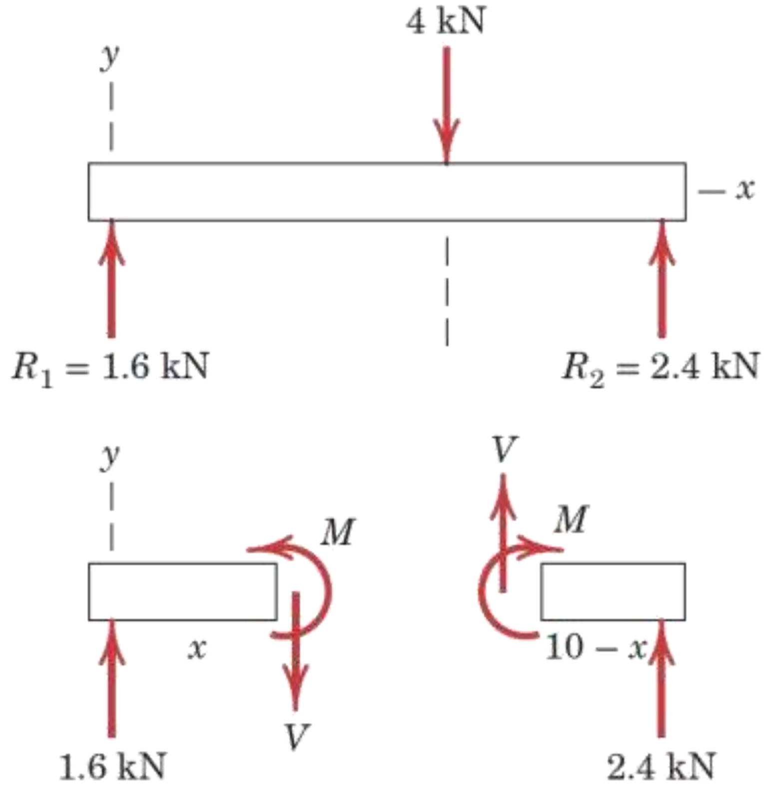
These values of V and M apply to all sections of the beam to the left of the 4-kN load. ①

A section of the beam to the right of the 4-kN load is next isolated with its free-body diagram on which V and M are shown in their positive directions. Equilibrium requires

$[\Sigma F_y = 0]$ $\quad V + 2.4 = 0 \quad V = -2.4\text{ kN}$

$[\Sigma M_{R_2} = 0]$ $\quad -(2.4)(10 - x) + M = 0 \quad M = 2.4(10 - x)$

These results apply only to sections of the beam to the right of the 4-kN load.

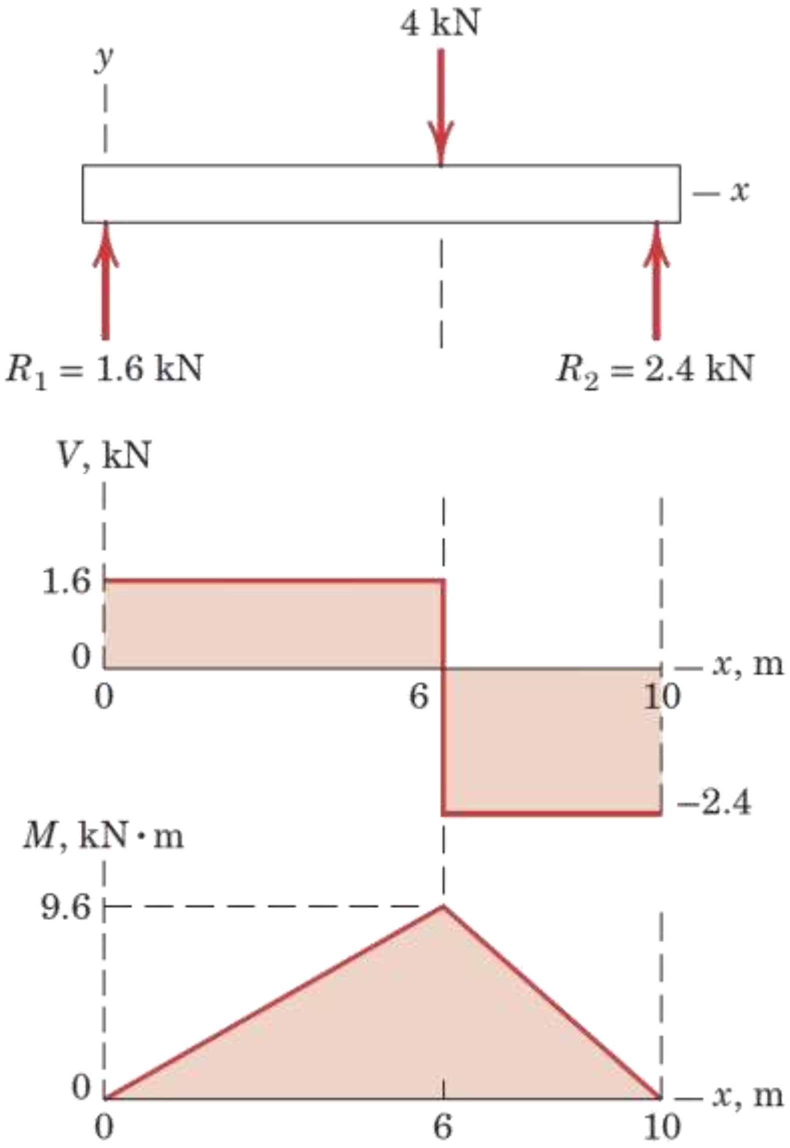


① We must be careful not to take our section at a concentrated load (such as $x = 6$ m) since the shear and moment relations involve discontinuities at such positions.

Article 5/7 – Sample Problem 5/13 (4 of 4)

- *V* and *M* Diagrams

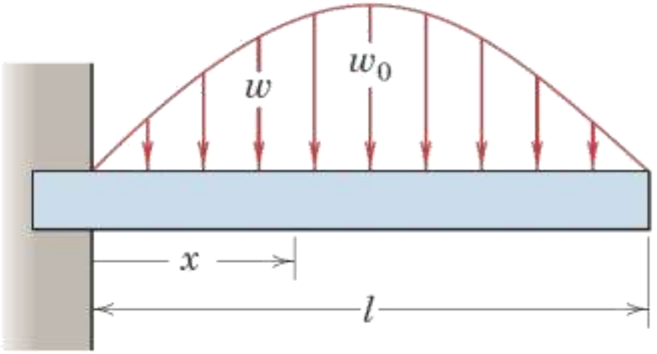
The values of V and M are plotted as shown. The maximum bending moment occurs where the shear changes direction. As we move in the positive x-direction starting with $x = 0$, we see that the moment M is merely the accumulated area under the shear diagram.



Article 5/7 – Sample Problem 5/14 (1 of 5)

- Problem Statement

  The cantilever beam is subjected to the load intensity (force per unit length) which varies as $w = w_0 \sin(\pi x/l)$. Determine the shear force V and bending moment M as functions of the ratio x/l.



Article 5/7 – Sample Problem 5/14 (2 of 5)

- Support Reactions

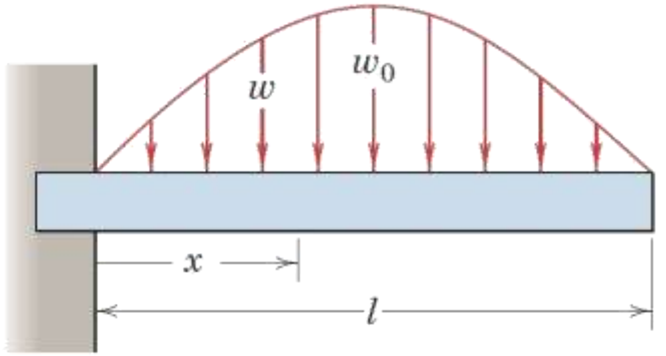
The free-body diagram of the entire beam is drawn first so that the shear force V_0 and bending moment M_0 which act at the supported end at $x = 0$ can be computed. By convention V_0 and M_0 are shown in their positive mathematical senses. A summation of vertical forces for equilibrium gives

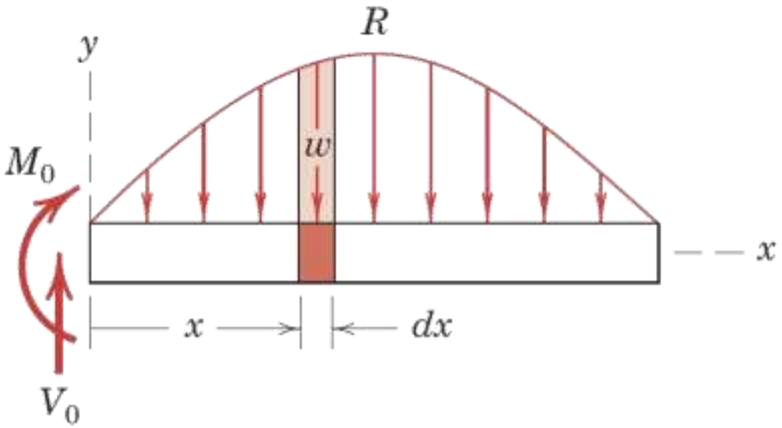
$$[\Sigma F_y = 0] \qquad V_0 - \int_0^l w\,dx = 0 \qquad V_0 = \int_0^l w_0 \sin\frac{\pi x}{l}\,dx = \frac{2w_0 l}{\pi}$$

A summation of moments about the left end at $x = 0$ for equilibrium gives ①

$$[\Sigma M = 0] \qquad -M_0 - \int_0^l x(w\,dx) = 0 \qquad M_0 = -\int_0^l w_0 x \sin\frac{\pi x}{l}\,dx$$

$$M_0 = \frac{-w_0 l^2}{\pi^2}\left[\sin\frac{\pi x}{l} - \frac{\pi x}{l}\cos\frac{\pi x}{l}\right]_0^l = -\frac{w_0 l^2}{\pi}$$





① In this case of symmetry, it is clear that the resultant $R = V_0 = 2w_0 l/\pi$ of the load distribution acts at midspan, so that the moment requirement is simply $M_0 = -Rl/2 = -w_0 l^2/\pi$. The minus sign tells us that physically the bending moment at $x = 0$ is opposite to that represented on the free-body diagram.

Article 5/7 – Sample Problem 5/14 (3 of 5)

- Shear-Force Expression

From a free-body diagram of an arbitrary section of length x, integration of Eq. 5/10 permits us to find the shear force internal to the beam. Thus,

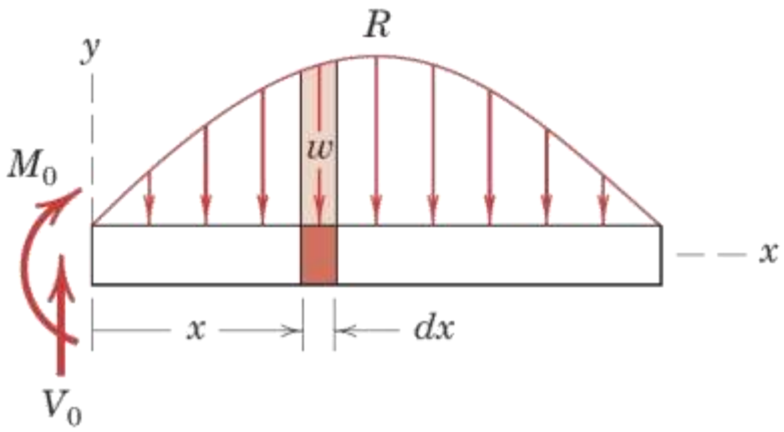
$$[dV = -w\,dx] \qquad \int_{V_0}^{V} dV = -\int_0^x w_0 \sin\frac{\pi x}{l}\,dx \quad ②$$

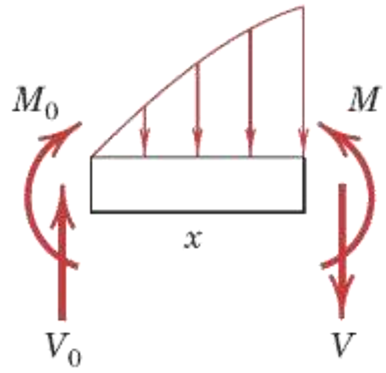
$$V - V_0 = \left[\frac{w_0 l}{\pi}\cos\frac{\pi x}{l}\right]_0^x \qquad V - \frac{2w_0 l}{\pi} = \frac{w_0 l}{\pi}\left(\cos\frac{\pi x}{l} - 1\right)$$

or in dimensionless form

$$\frac{V}{w_0 l} = \frac{1}{\pi}\left(1 + \cos\frac{\pi x}{l}\right) \qquad \textit{Ans.}$$

② The free-body diagram serves to remind us that the integration limits for V as well as for x must be accounted for. We see that the expression for V is positive, so that the shear force is as represented on the free-body diagram.





Article 5/7 – Sample Problem 5/14 (4 of 5)

- Bending-Moment Expression

The bending moment is obtained by integration of Eq. 5/11, which gives

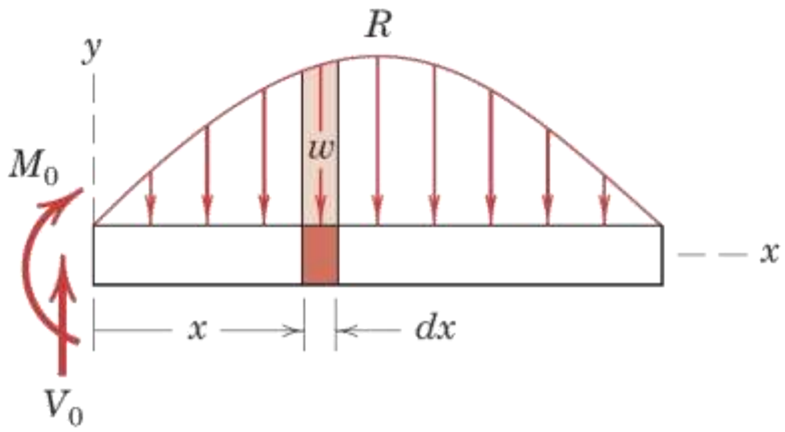
$$[dM = V\,dx] \qquad \int_{M_0}^{M} dM = \int_0^x \frac{w_0 l}{\pi}\left(1 + \cos\frac{\pi x}{l}\right) dx$$

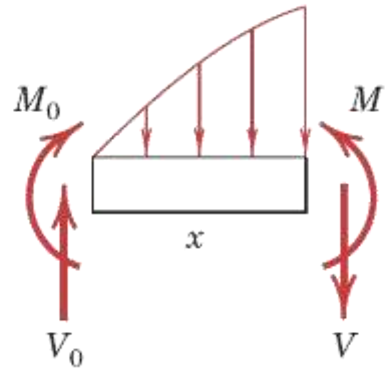
$$M - M_0 = \frac{w_0 l}{\pi}\left[x + \frac{l}{\pi}\sin\frac{\pi x}{l}\right]_0^x$$

$$M = -\frac{w_0 l^2}{\pi} + \frac{w_0 l}{\pi}\left[x + \frac{l}{\pi}\sin\frac{\pi x}{l} - 0\right]$$

or in dimensionless form

$$\frac{M}{w_0 l^2} = \frac{1}{\pi}\left(\frac{x}{l} - 1 + \frac{1}{\pi}\sin\frac{\pi x}{l}\right) \qquad \textit{Ans.}$$

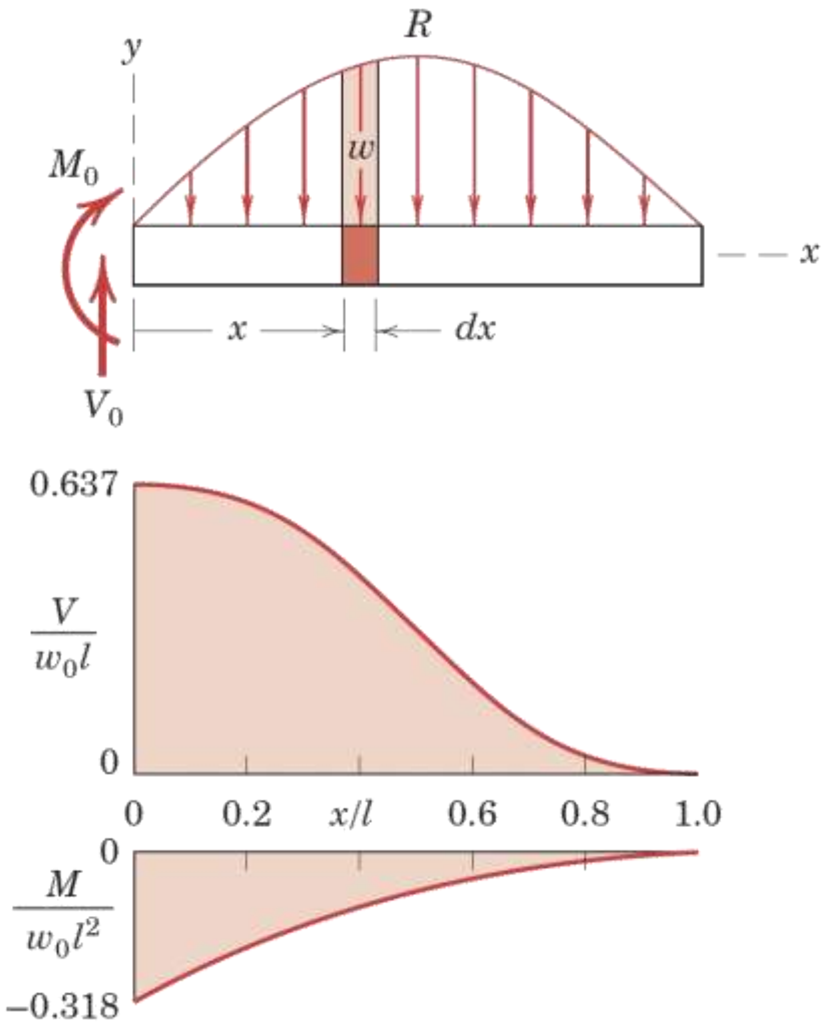




Article 5/7 – Sample Problem 5/14 (5 of 5)

- *V* and *M* Diagrams

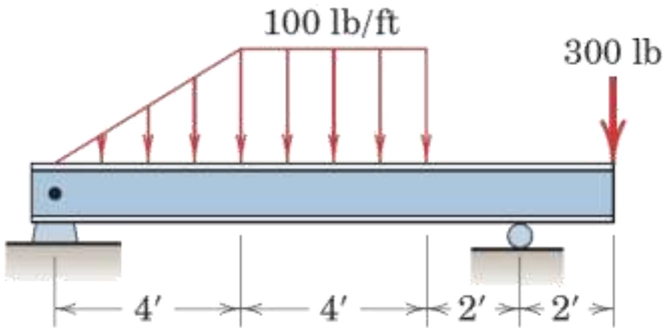
The variations of $V/w_0 l$ and $M/w_0 l^2$ with x/l are shown in the bottom figures. The negative values of $M/w_0 l^2$ indicate that physically the bending moment is in the direction opposite to that shown.



Article 5/7 – Sample Problem 5/15 (1 of 6)

- Problem Statement

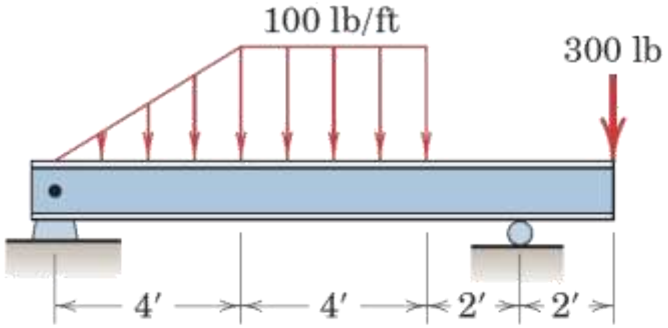
  Draw the shear-force and bending-moment diagrams for the loaded beam and determine the maximum moment M and its location x from the left end.

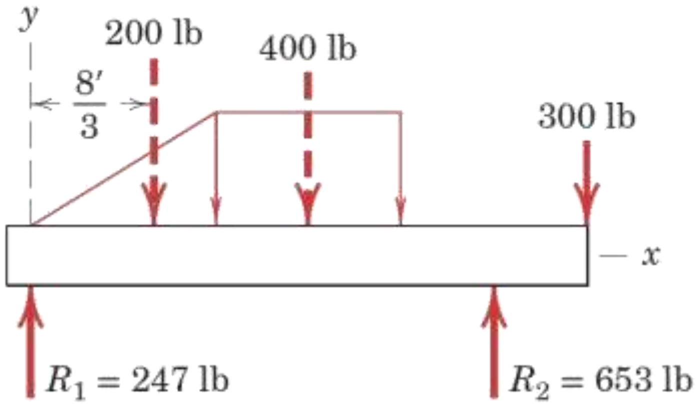


Article 5/7 – Sample Problem 5/15 (2 of 6)

- Support Reactions

  The support reactions are most easily obtained by considering the resultants of the distributed loads as shown on the free-body diagram of the beam as a whole.



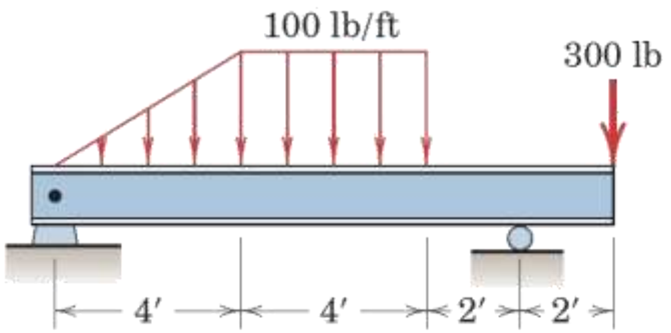


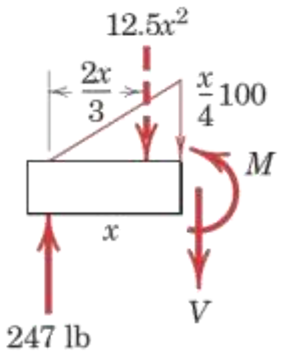
Article 5/7 – Sample Problem 5/15 (3 of 6)

- First Beam Segment: $0 < x < 4$ ft

$$[\Sigma F_y = 0] \qquad V = 247 - 12.5x^2$$

$$[\Sigma M = 0] \qquad M + (12.5x^2)\frac{x}{3} - 247x = 0 \qquad M = 247x - 4.17x^3$$





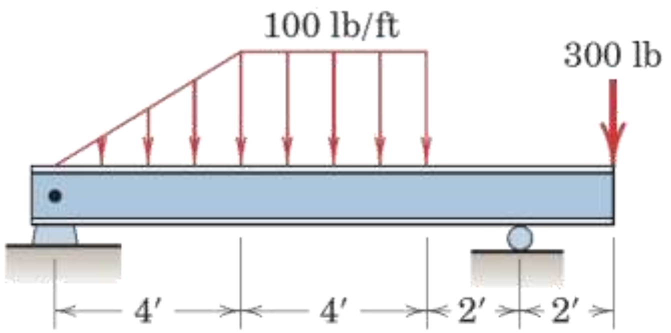
Article 5/7 – Sample Problem 5/15 (4 of 6)

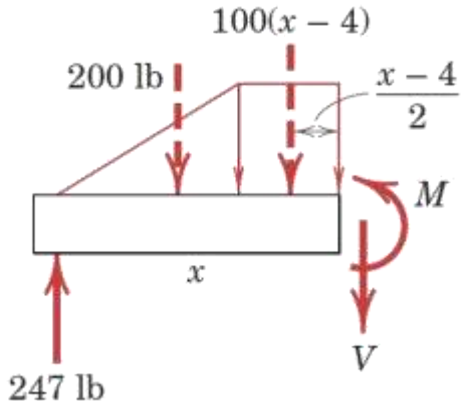
- Second Beam Segment: $4 < x < 8$ ft

$[\Sigma F_y = 0] \qquad V + 100(x-4) + 200 - 247 = 0 \qquad V = 447 - 100x$

$[\Sigma M = 0] \qquad M + 100(x-4)\dfrac{x-4}{2} + 200[x - \tfrac{2}{3}(4)] - 247x = 0$

$$M = -267 + 447x - 50x^2$$



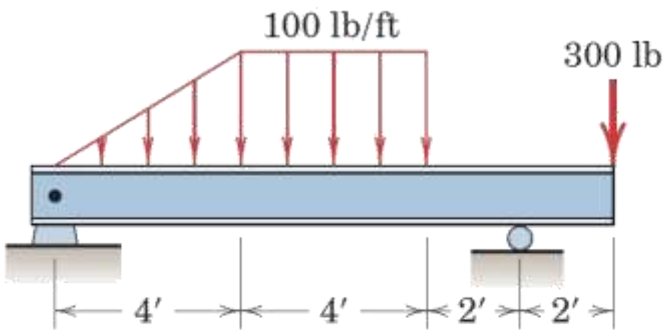


Article 5/7 – Sample Problem 5/15 (5 of 6)

- Third Beam Segment: $8 < x < 10$ ft

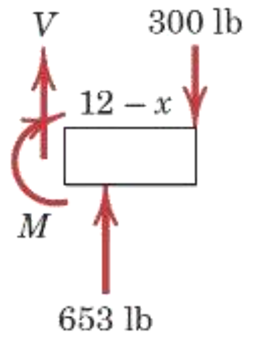
$[\Sigma F_y = 0]$ $V + 653 - 300 = 0$ $V = -353$ lb

$[\Sigma M = 0]$ $-M + 653(12 - x - 2) - 300(12 - x) = 0$ $M = 2930 - 353x$



- Fourth Beam Segment: $10 < x < 12$ ft

The last interval may be analyzed by inspection. The shear is constant at +300 lb, and the moment follows a straight-line relation beginning with zero at the right end of the beam.



Article 5/7 – Sample Problem 5/15 (6 of 6)

- *V* and *M* Diagrams

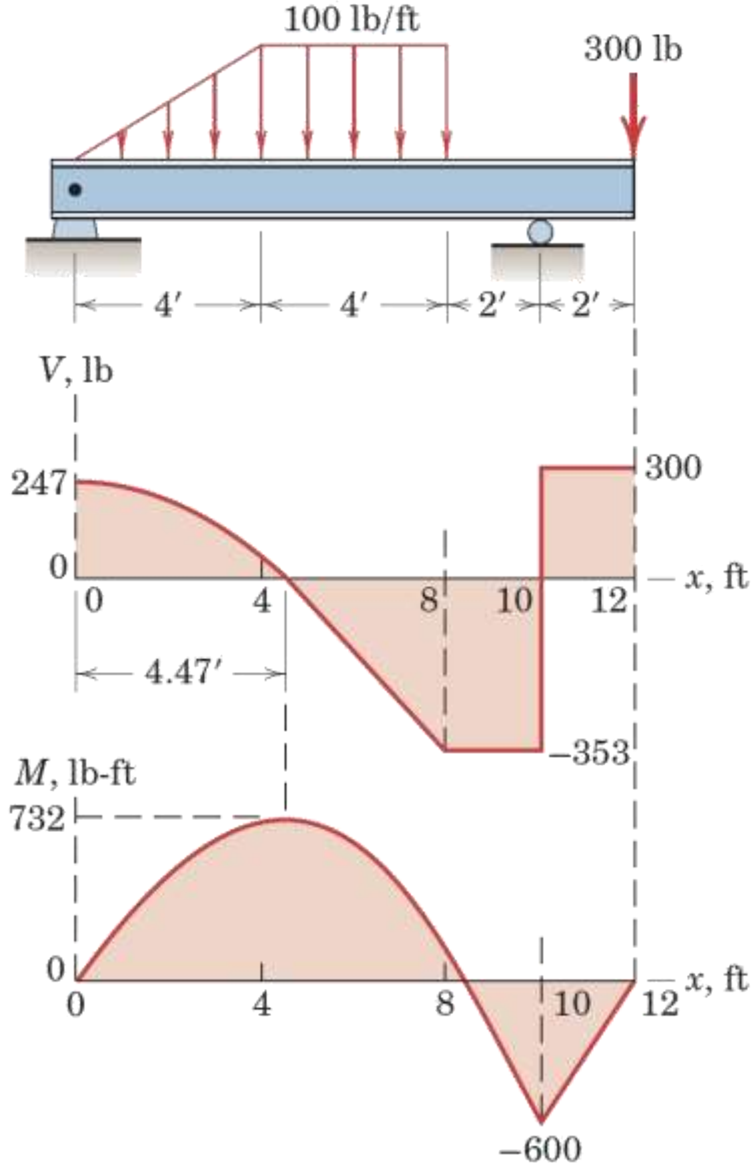
The maximum moment occurs at $x = 4.47$ ft, where the shear curve crosses the zero axis, and the magnitude of M is obtained for this value of x by substitution into the expression for M for the second interval. The maximum moment is

$$M = 732 \text{ lb-ft} \qquad \textit{Ans.}$$

As before, note that the change in moment M up to any section equals the area under the shear diagram up to that section. For instance, for $x < 4$ ft,

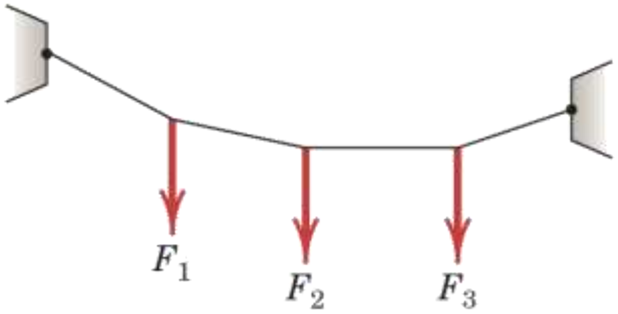
$$\left[\Delta M = \int V\,dx\right] \qquad M - 0 = \int_0^x (247 - 12.5x^2)\,dx$$

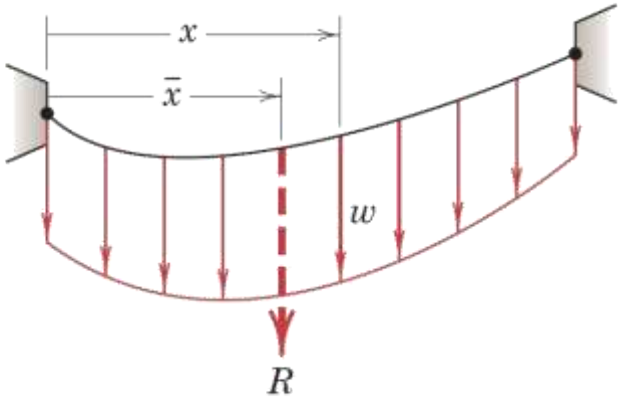
and, as above, $$M = 247x - 4.17x^3$$



Article 5/8 Flexible Cables

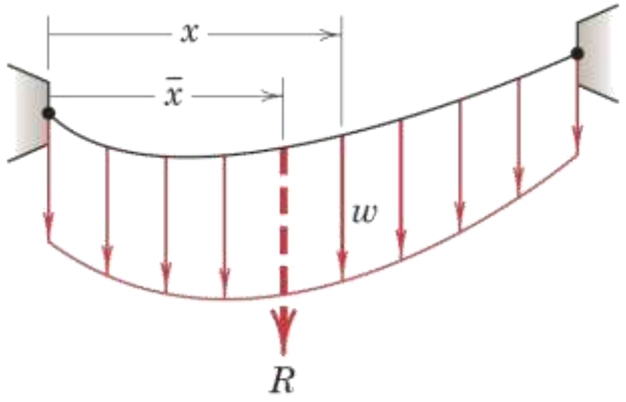
- Introduction
- Examples
- Parameters of Importance
  - Tension
  - Span
  - Sag
  - Length
  - Load Intensity



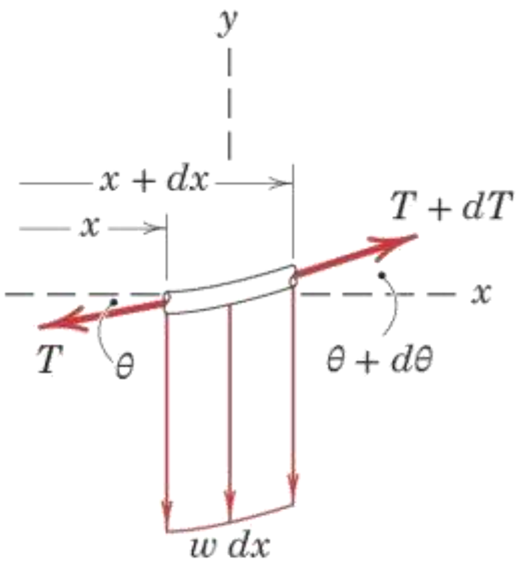


Article 5/8 – General Relationships (1 of 3)

- Variable and Continuous Load, *w*



- Resultant, $R = \int w \, dx$



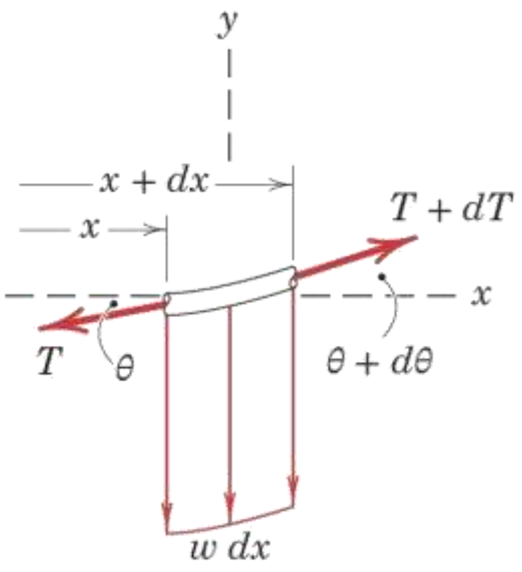
- Location, $R\bar{x} = \int xw \, dx$

Article 5/8 – General Relationships (2 of 3)

- Equilibrium of a Cable Segment

$$\Sigma F_x = 0: \quad (T + dT)\cos(\theta + d\theta) - T\cos\theta = 0$$

$$\Sigma F_y = 0: \quad (T + dT)\sin(\theta + d\theta) - T\sin\theta - w\,dx = 0$$



- Trigonometric and Small-Angle Substitutions

$$\sin(\theta + d\theta) = \sin\theta\cos d\theta + \cos\theta\sin d\theta \qquad \sin d\theta \cong d\theta$$

$$\cos(\theta + d\theta) = \cos\theta\cos d\theta - \sin\theta\sin d\theta \qquad \cos d\theta \cong 1$$

- Result

$$T\cos\theta\ d\theta + dT\sin\theta = w\ dx$$

$$-T\sin\theta\ d\theta + dT\cos\theta = 0$$

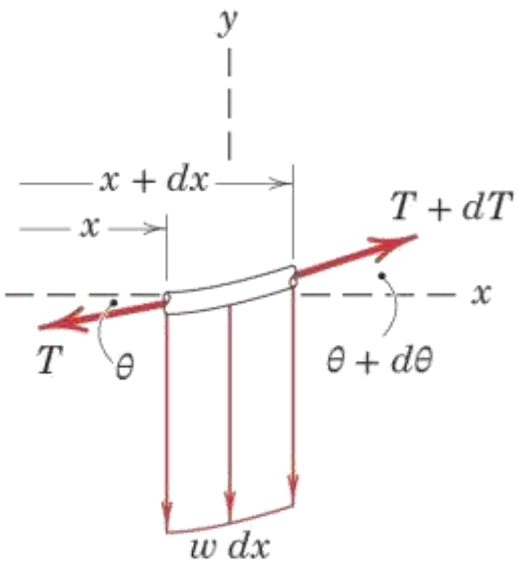
Article 5/8 – General Relationships (3 of 3)

- Rewrite Equations

$$d\left(T\sin\theta\right) = w\,dx$$

$$d\left(T\cos\theta\right) = 0$$

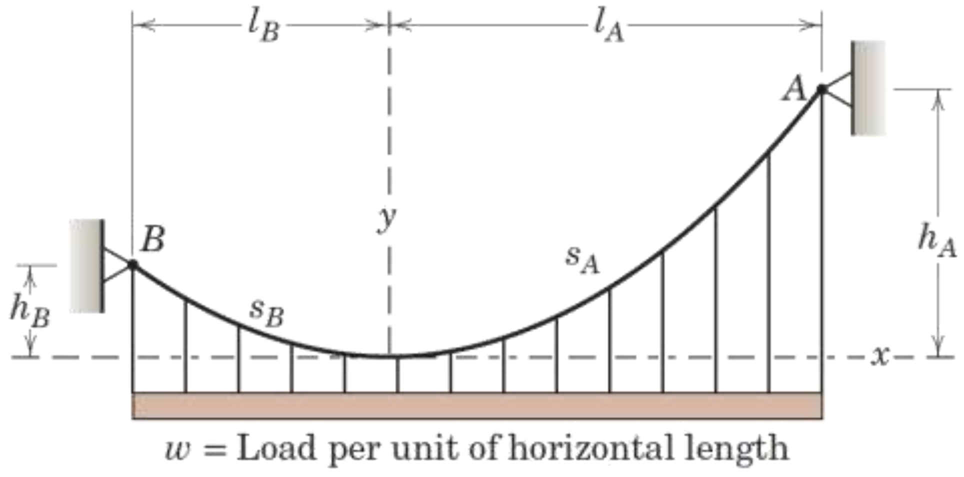
- Result
  - Horizontal component of the cable tension remains unchanged, so we introduce the variable $T_0 = T\cos\theta$ to represent it.
  - Substitute $T = T_0/\cos\theta$ into the first equation to obtain the differential equation for the flexible cable.
  - Note: $\tan\theta = dy/dx$

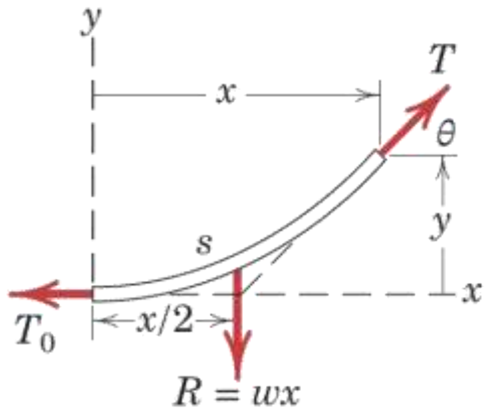


$$\boxed{\frac{d^2y}{dx^2} = \frac{w}{T_0}}$$

Article 5/8 – Parabolic Cables (1 of 3)

- Characteristics
  - Cable Weight is Negligible
  - Vertical Loading, *w*, is Constant
  - Span, *l*
  - Sag, *h*
  - Length, *s*
  - Shape is a Parabolic Arch
  - Coordinate Origin at Lowest Point
- Free-Body Diagram of a Cable Segment





Article 5/8 – Parabolic Cables (2 of 3)

- Integrate General Relationship Twice…

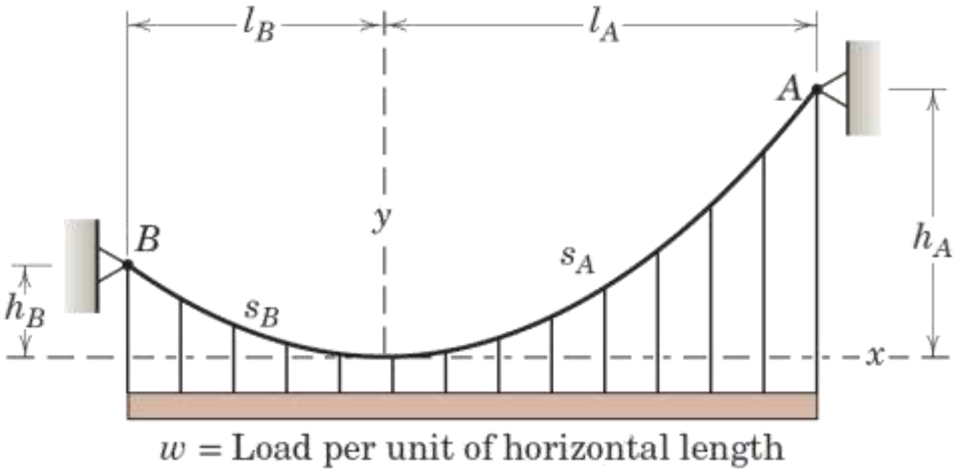
$$\boxed{y = \frac{wx^2}{2T_0}}$$

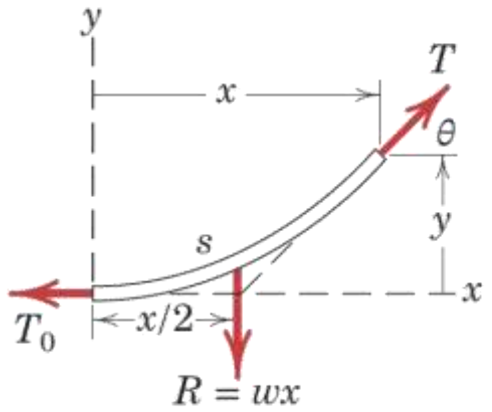
- Equations of Interest

$$T_0 = \frac{wl_A^{\,2}}{2h_A} = \frac{wl_B^{\,2}}{2h_B} \qquad \text{and} \qquad T = w\sqrt{x^2 + \left(\frac{l_A^{\,2}}{2h_A}\right)^2}$$

$$T_A = wl_A\sqrt{1+\left(\frac{l_A^{\,2}}{2h_A}\right)^2} \qquad \text{and} \qquad T_B = wl_B\sqrt{1+\left(\frac{l_B^{\,2}}{2h_B}\right)^2}$$

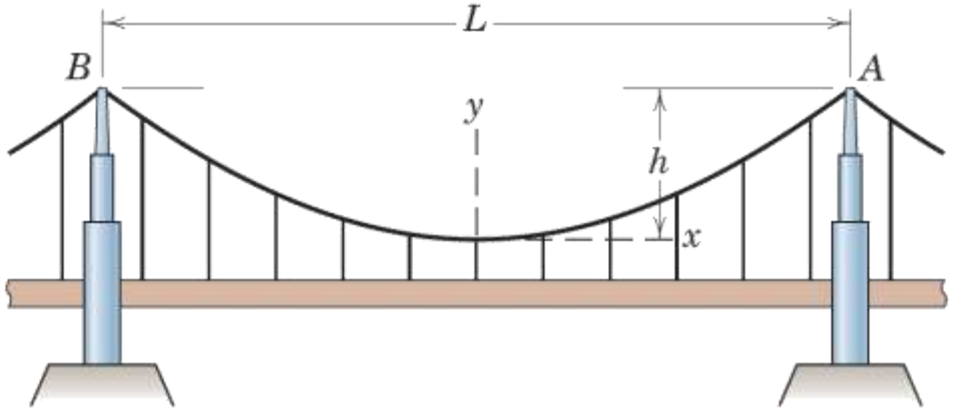
$$s_A = \int_0^{l_A}\sqrt{1+\left(\frac{wx}{T_0}\right)^2}\,dx \qquad \text{and} \qquad s_B = \int_0^{l_B}\sqrt{1+\left(\frac{wx}{T_0}\right)^2}\,dx$$





Article 5/8 – Parabolic Cables (3 of 3)

- Suspension Bridge
  - Total Span, $L = 2l_A$
  - Sag, $h = h_A$
  - Total Length, $S = 2s_A$



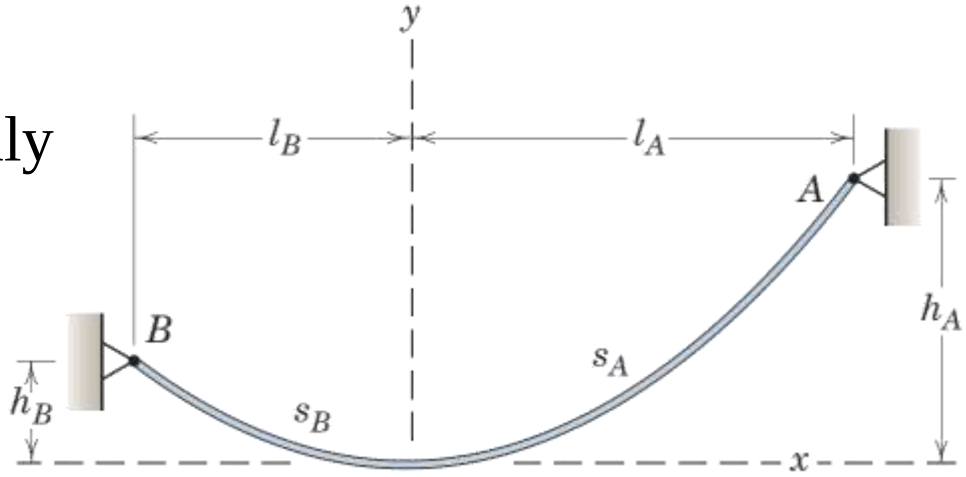
- Equations of Interest

$$T_{\max} = \frac{wL}{2}\sqrt{1 + (L/4h)^2}$$

$$S = L\left[1 + \frac{8}{3}\left(\frac{h}{L}\right)^2 - \frac{32}{5}\left(\frac{h}{L}\right)^4 + \cdots\right]$$

Article 5/8 – Catenary Cables (1 of 3)

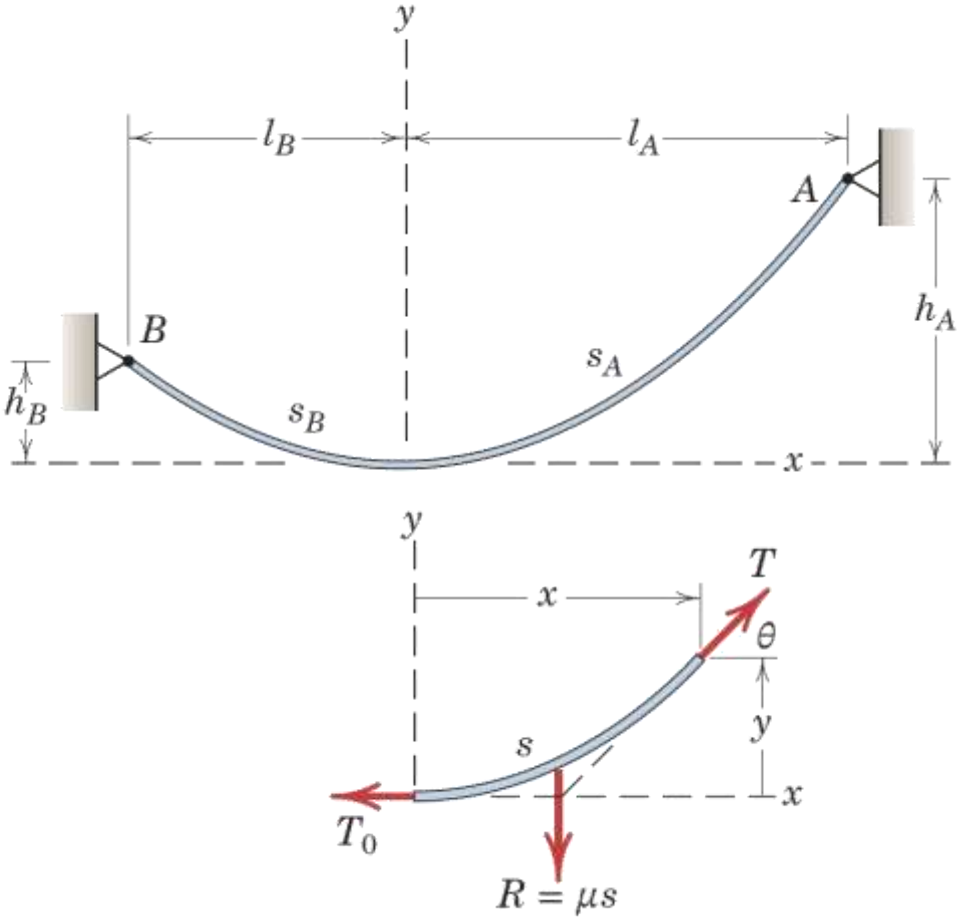
- Characteristics
  - Cable Hangs Under Action of its Weight Only
  - Cable Weight per Unit Length, μ
  - Span, l
  - Sag, h
  - Length, s
  - Shape is a Catenary
  - Coordinate Origin at Lowest Point



Article 5/8 – Catenary Cables (2 of 3)

- Free-Body Diagram of a Cable Segment
  - Resultant, $R = \mu s$
  - From Before, $w\,dx = \mu\,ds$
  - Substitute into the General Expression…

$$\boxed{\frac{d^2y}{dx^2} = \frac{w}{T_0}} \longrightarrow \boxed{\frac{d^2y}{dx^2} = \frac{\mu}{T_0}\frac{ds}{dx}}$$



Article 5/8 – Catenary Cables (3 of 3)

- Equations of Interest

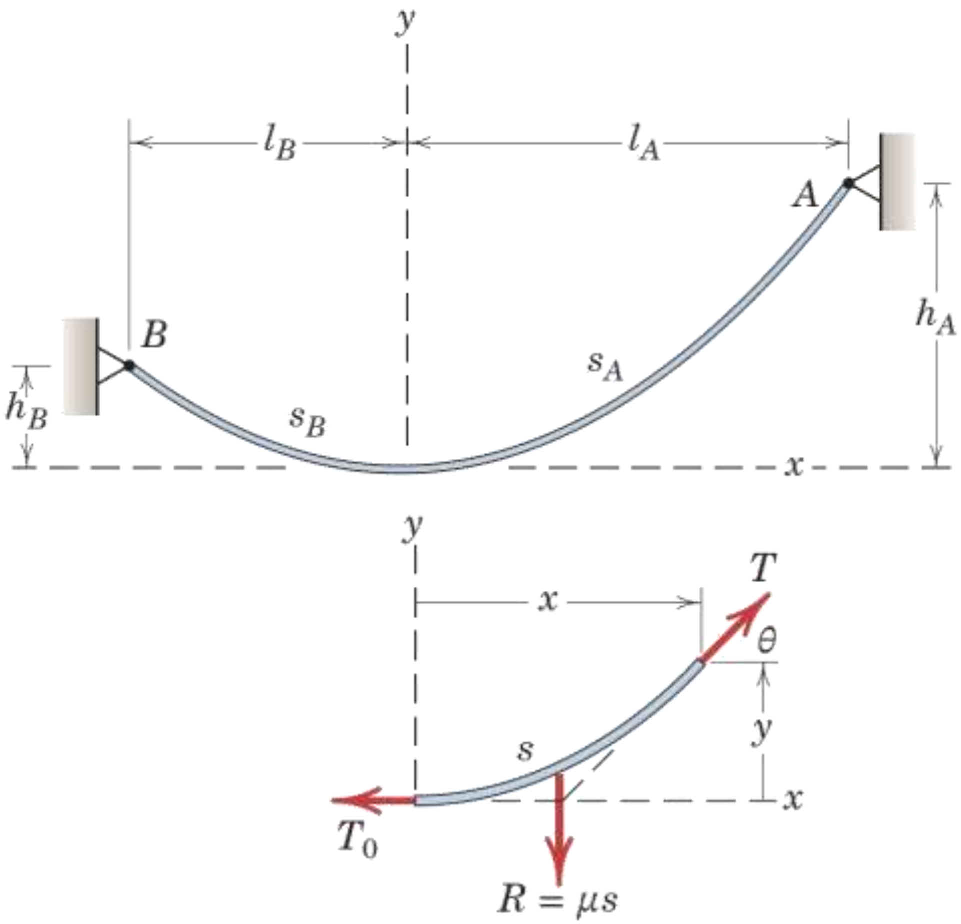
$$y = \frac{T_0}{\mu}\left(\cosh\frac{\mu x}{T_0} - 1\right)$$

$$s = \frac{T_0}{\mu}\sinh\frac{\mu x}{T_0}$$

$$T = T_0 \cosh\frac{\mu x}{T_0}$$

$$T = T_0 + \mu y$$

$$T^2 = \mu^2 s^2 + T_0^{\,2}$$

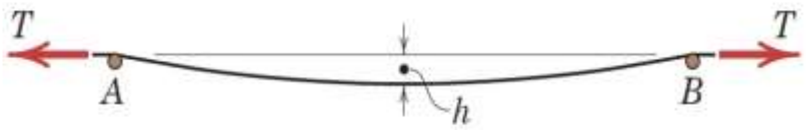


- Note: Catenary cable problems generally require numerical solutions!

Article 5/8 – Sample Problem 5/16 (1 of 2)

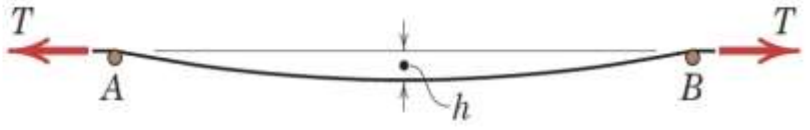
- Problem Statement

  A 100-ft length of surveyor's tape weighs 0.6 lb. When the tape is stretched between two points on the same level by a tension of 10 lb at each end, calculate the sag *h* in the middle.



Article 5/8 – Sample Problem 5/16 (2 of 2)

- Solution



The weight per unit length is $\mu = 0.6/100 = 0.006$ lb/ft. The total length is $2s = 100$ or $s = 50$ ft.

$[T^2 = \mu^2 s^2 + T_0^{\,2}]$ $\qquad 10^2 = (0.006)^2(50)^2 + T_0^{\,2}$

$$T_0 = 9.995 \text{ lb} \quad ①$$

$[T = T_0 + \mu y]$ $\qquad 10 = 9.995 + 0.006h$

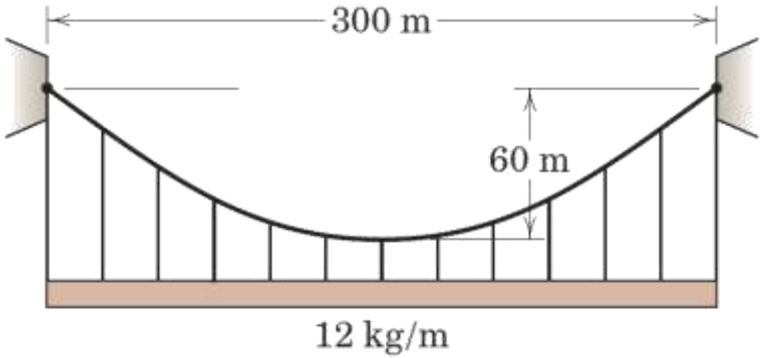
$$h = 0.750 \text{ ft} \quad \text{or} \quad 9.00 \text{ in.} \qquad \textit{Ans.}$$

① An extra significant figure is displayed here for clarity.

Article 5/8 – Sample Problem 5/17 (1 of 3)

- Problem Statement

  The light cable supports a mass of 12 kg per meter of horizontal length and is suspended between the two points on the same level 300 m apart. If the sag is 60 m, find the tension at midlength, the maximum tension, and the total length of the cable.



Article 5/8 – Sample Problem 5/17 (2 of 3)

- Midlength Tension

With a uniform horizontal distribution of load, the solution of part (*b*) of Art. 5/8 applies, and we have a parabolic shape for the cable. For $h = 60$ m, $L = 300$ m, and $w = 12(9.81)(10^{-3})$ kN/m, the relation following Eq. 5/14 with $l_A = L/2$ gives for the midlength tension

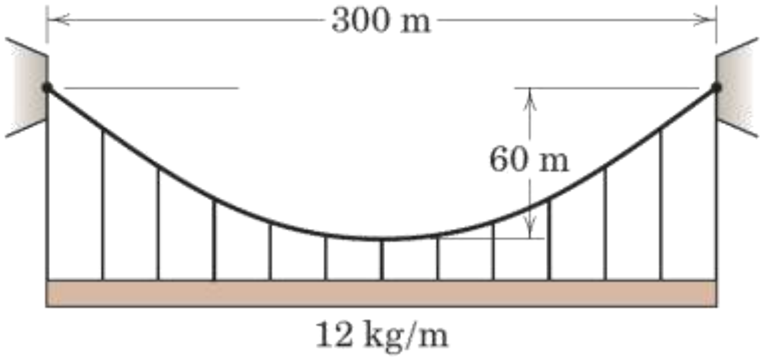
$$\left[T_0 = \frac{wL^2}{8h}\right] \qquad T_0 = \frac{0.1177(300)^2}{8(60)} = 22.1 \text{ kN} \qquad \textit{Ans.}$$

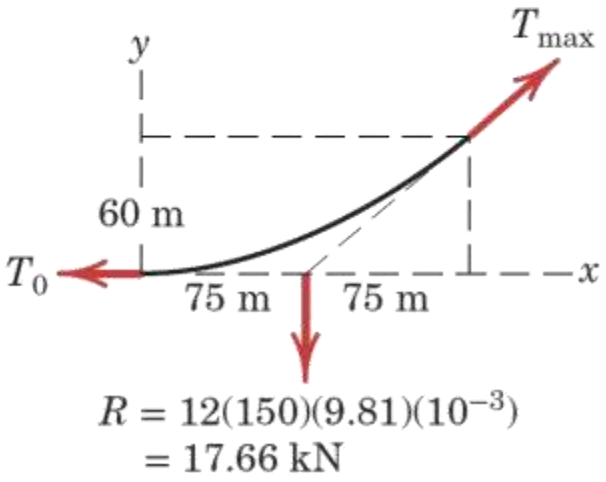
- Maximum Tension

The maximum tension occurs at the supports and is given by Eq. 5/15*b*. Thus,

$$\left[T_{\max} = \frac{wL}{2}\sqrt{1 + \left(\frac{L}{4h}\right)^2}\right]$$

$$T_{\max} = \frac{12(9.81)(10^{-3})(300)}{2}\sqrt{1 + \left(\frac{300}{4(60)}\right)^2} = 28.3 \text{ kN} \quad ① \quad \textit{Ans.}$$





① *Suggestion:* Check the value of $T_{\max}$ directly from the free-body diagram of the right-hand half of the cable, from which a force polygon may be drawn.

Article 5/8 – Sample Problem 5/17 (3 of 3)

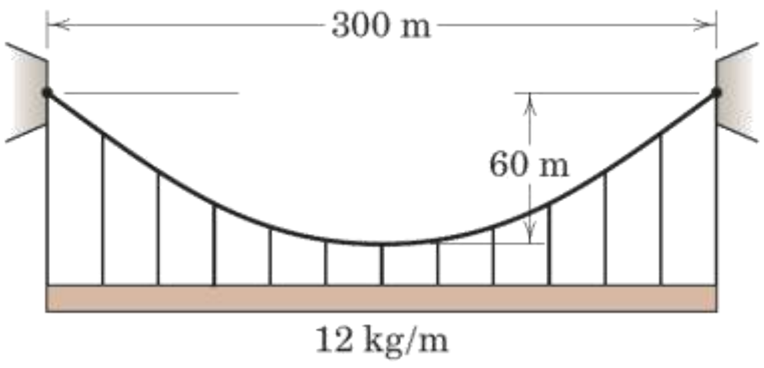
- Cable Length

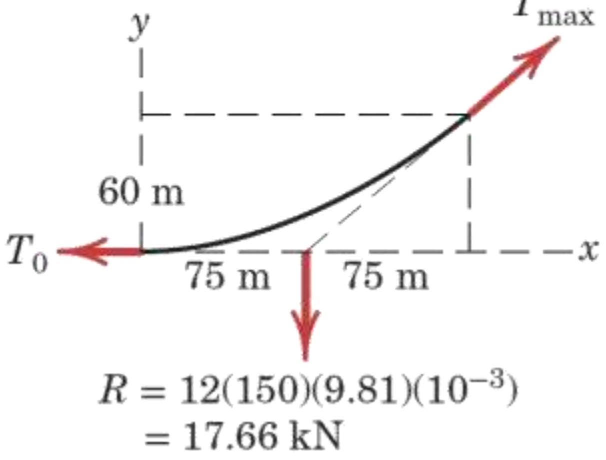
The sag-to-span ratio is 60/300 = 1/5 < 1/4. Therefore, the series expression developed in Eq. 5/16*a* is convergent, and we may write for the total length

$$S = 300\left[1 + \frac{8}{3}\left(\frac{1}{5}\right)^2 - \frac{32}{5}\left(\frac{1}{5}\right)^4 + \cdots\right]$$

$$= 300[1 + 0.1067 - 0.01024 + \cdots]$$

$$= 329 \text{ m} \qquad \textit{Ans.}$$

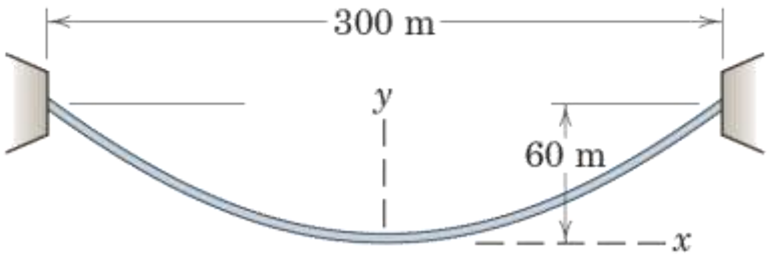




Article 5/8 – Sample Problem 5/18 (1 of 3)

- Problem Statement

  Replace the cable of Sample Problem 5/17, which is loaded uniformly along the horizontal, by a cable which has a mass of 12 kg per meter of its own length and supports its own weight only. The cable is suspended between two points on the same level 300 m apart and has a sag of 60 m. Find the tension at midlength, the maximum tension, and the total length of the cable.



Article 5/8 – Sample Problem 5/18 (2 of 3)

- Midlength Tension

With a load distributed uniformly along the length of the cable, the solution of part (*c*) of Art. 5/8 applies, and we have a catenary shape of the cable. Equations 5/20 and 5/21 for the cable length and tension both involve the minimum tension T_0 at midlength, which must be found from Eq. 5/19. Thus, for $x = 150$ m, $y = 60$ m, and $\mu = 12(9.81)(10^{-3}) = 0.1177$ kN/m, we have

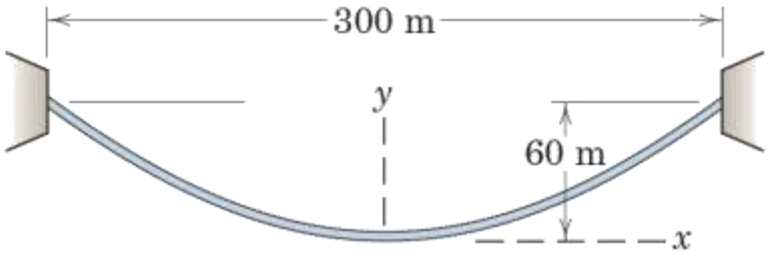
$$60 = \frac{T_0}{0.1177}\left[\cosh\frac{(0.1177)(150)}{T_0} - 1\right]$$

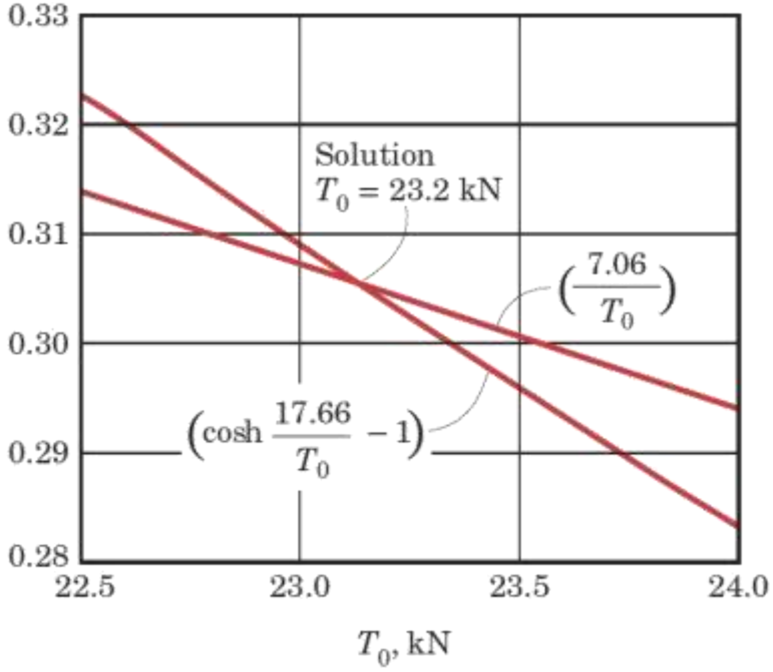
or

$$\frac{7.06}{T_0} = \cosh\frac{17.66}{T_0} - 1$$

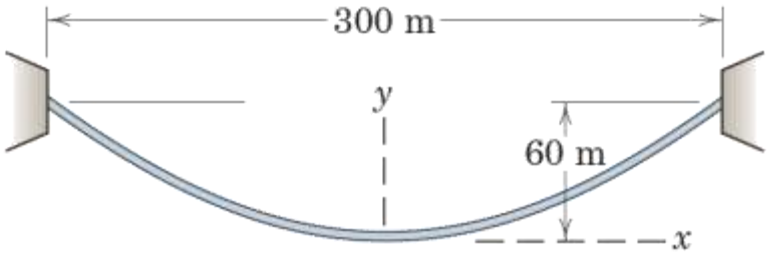
This equation can be solved graphically. We compute the expression on each side of the equals sign and plot it as a function of T_0. The intersection of the two curves establishes the equality and determines the correct value of T_0. This plot is shown in the figure accompanying this problem and yields the solution

$$T_0 = 23.2 \text{ kN}$$





Article 5/8 – Sample Problem 5/18 (3 of 3)



- Maximum Tension

The maximum tension occurs for maximum y and from Eq. 5/22 is

$$T_{\max} = 23.2 + (0.1177)(60) = 30.2 \text{ kN} \qquad \textit{Ans.}$$

- Cable Length

From Eq. 5/20 the total length of the cable becomes

$$2s = 2\frac{23.2}{0.1177}\sinh\frac{(0.1177)(150)}{23.2} = 330 \text{ m} \quad ① \qquad \textit{Ans.}$$

① Note that the solution of Sample Problem 5/17 for the parabolic cable gives a very close approximation to the values for the catenary even though we have a fairly large sag. The approximation is even better for smaller sag-to-span ratios.

Article 5/9 Fluid Statics

- Introduction

- Important Concepts
  - A fluid is any continuous substance which, when at rest, is unable to support a shear force. Fluids are either gaseous or liquid.

  - A shear force is any force which is tangent to the surface on which it acts.

  - Fluids at rest can only exert normal forces on a bounding surface.

  - The statics of fluids is called *hydrostatics* when the fluid is a liquid and *aerostatics* when the fluid is a gas.

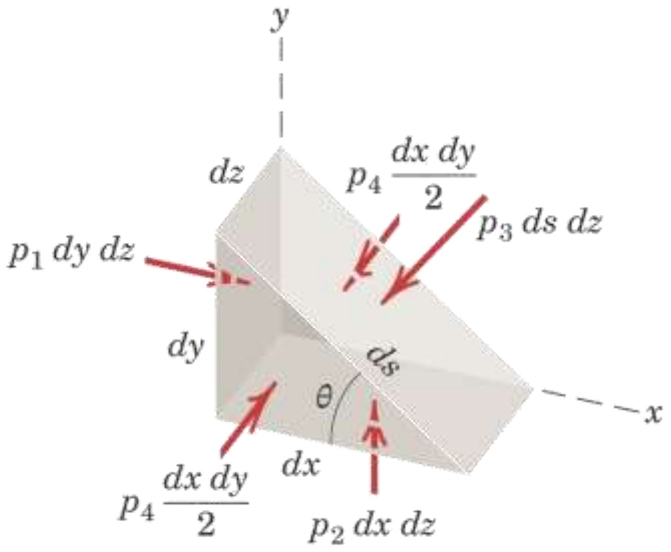
Article 5/9 – Fluid Pressure at a Point (1 of 3)

- Pascal's Law
  - The pressure at any given point in a fluid is the same in all directions.

- Equilibrium of a Fluid Prism

$$\Sigma F_x = 0: \quad p_1 dy\ dz - p_3 ds\ dz \sin\theta = 0$$

$$\Sigma F_y = 0: \quad p_2 dx\ dz - p_3 ds\ dz \cos\theta = 0$$



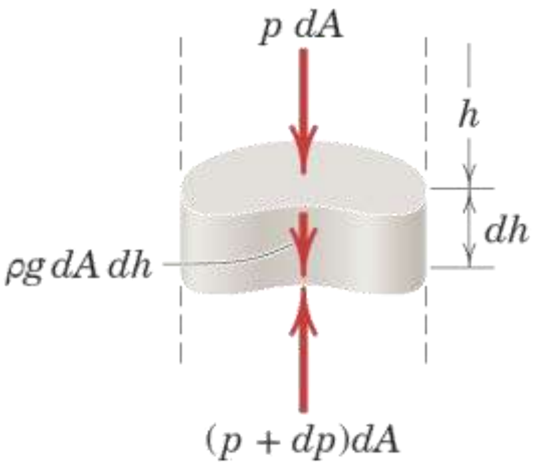
- Since $ds \sin\theta = dy$ and $ds \cos\theta = dx$…

$$p_1 = p_2 = p_3 = p$$

- Rotating the element 90° would show the same thing for p_4.

Article 5/9 – Fluid Pressure at a Point (2 of 3)

- Effect of Depth on Pressure in a Fluid at Rest



- Vertical Equilibrium of a Fluid Slug, +*h* is Down

$$\Sigma F_h = 0: \quad \rho g \; dh \; dA + p \; dA - \left(p + dp\right) dA = 0$$

$$dp = \rho g \; dh$$

- Generalized Result: $p = p_0 + \rho g h$
  - p = the pressure at some depth h in a fluid
  - p_0 = the pressure on the surface of the fluid
  - ρ = the density of the fluid, constant if the fluid is incompressible
  - g = the acceleration of gravity, constant if the fluid depth is small
  - h = the depth below the surface of the fluid

Article 5/9 – Atmospheric, Absolute, and Gage Pressure

- Atmospheric Pressure
  - Represented by the weight of a column of air which extends from the surface of the earth to the top of the atmosphere.
  - Standard value is 14.7 lb/in.^{2} or 101.3 kPa.
  - In many problems, p_0 is atmospheric pressure.

- Gage Pressure
  - Is a measurement of pressure which is made relative to atmospheric pressure.
  - Gage pressure in fluid statics is computed as ρgh.

- Absolute Pressure
  - Is the sum of atmospheric and gage pressure.
  - In fluid statics, absolute pressure is computed as $p_0 + \rho gh$.

Article 5/9 – Units of Pressure

- Pressure is a force divided by an area.

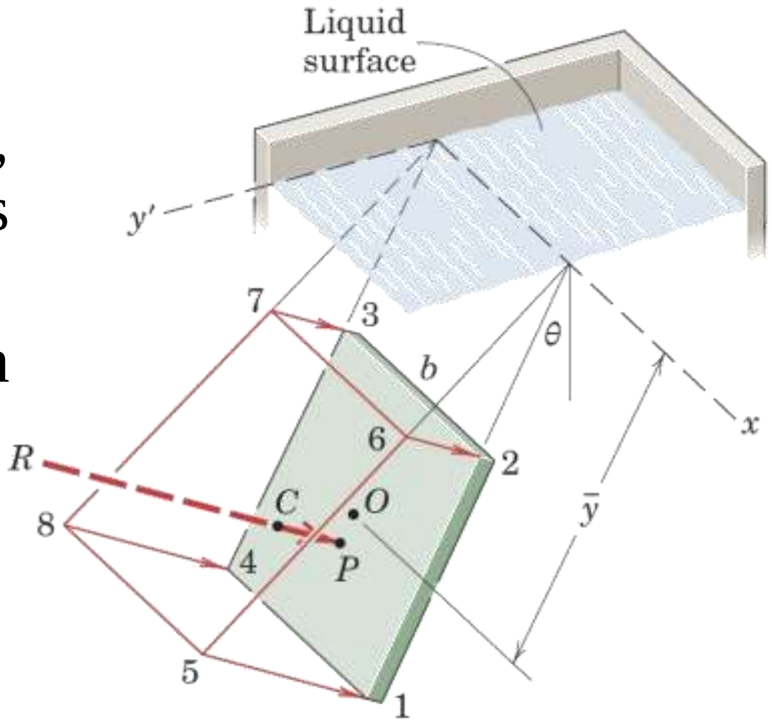
- SI Units
  - The most common unit is the Pascal (Pa), which is a newton per square meter.
  - $Pa = N/m^2$
  - kilopascal (kPa) = 10^3 Pa

- U.S. Units
  - The most common unit is a pound per square inch (psi or $lb/in.^2$) or a pound per square foot (psf or lb/ft^2).

- Conversion: $1\ lb/in.^2 = 6895\ Pa$

Article 5/9 – Hydrostatic Pressure on Submerged Rectangular Surfaces (1 of 4)

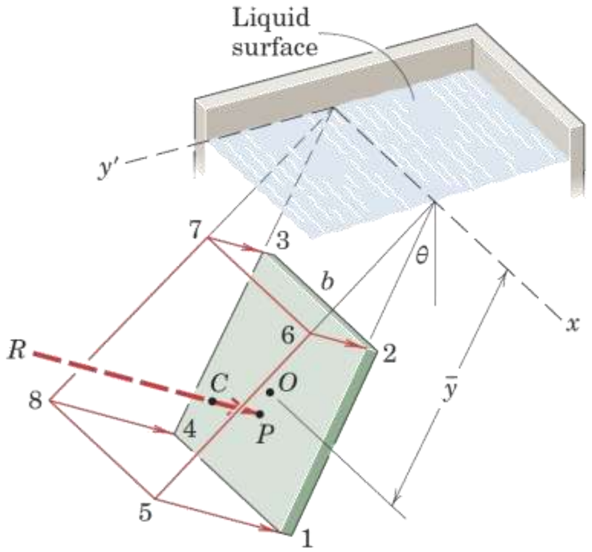
- Pressure
  - Varies linearly with depth according to ρgh, where the pressure at the liquid surface is atmospheric.
  - The pressure forms a trapezoidal prism which acts over the surface of the submerged plate.

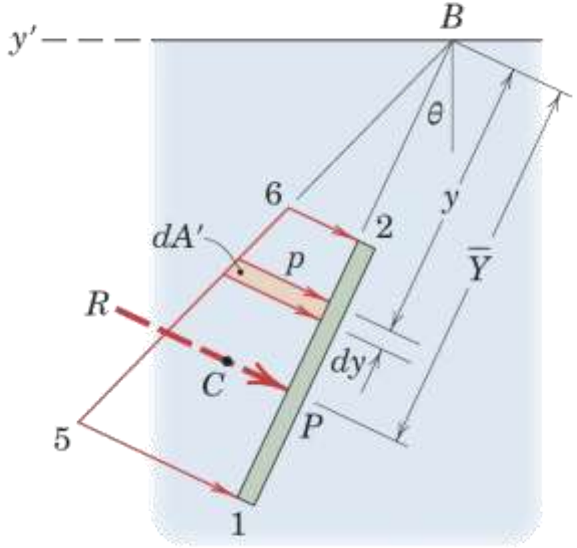


- Resultant R
  - Equal to the volume of the pressure prism.
  - Acts at the center of pressure P, which is the centroid of the pressure prism, not the centroid of the plate.

Article 5/9 – Hydrostatic Pressure on Submerged Rectangular Surfaces (2 of 4)

- Calculation of R – Method 1
  - $R = bA'$
    - b = plate width normal to the plane of the figure.
    - A' = geometrical area defined by the trapezoidal distribution of pressure, 1-2-6-5.
    - Acts at the center of pressure P.
- Calculation of R – Method 2
  - $R = p_{av}A = \rho g\bar{h}A$
    - p_{av} = average pressure acting on the plate, ½ $(p_1 + p_2)$, or the pressure which exists at the average depth, measured to the centroid O of the plate.
    - A = area of the plate.
    - $\bar{h} = \bar{y}\cos\theta$, is the average depth of the plate, which is measured to the plate centroid O.
    - Acts at the center of pressure P.

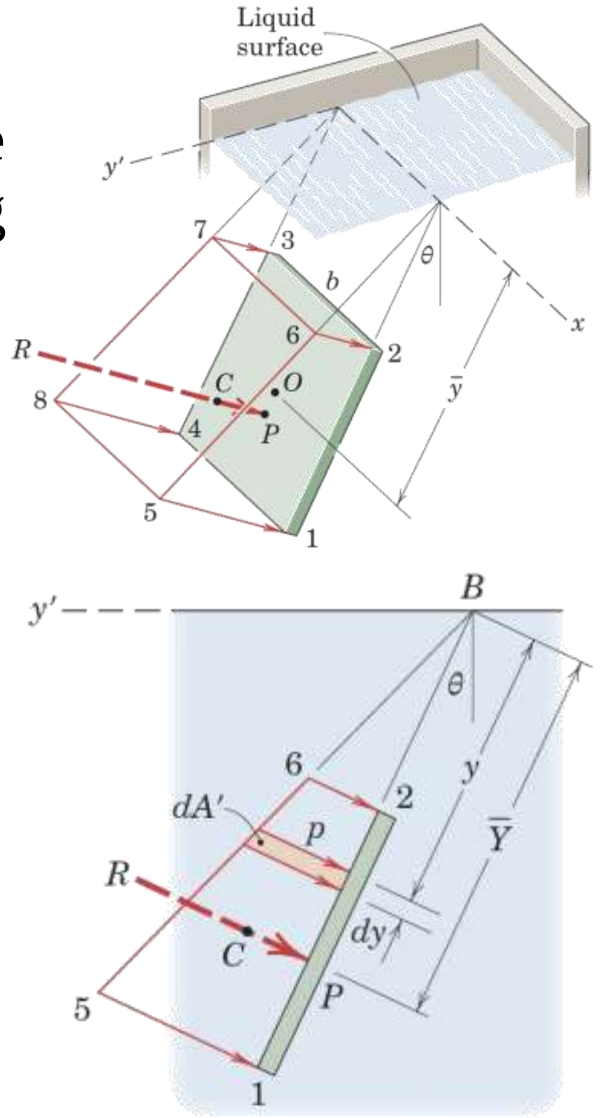




Article 5/9 – Hydrostatic Pressure on Submerged Rectangular Surfaces (3 of 4)

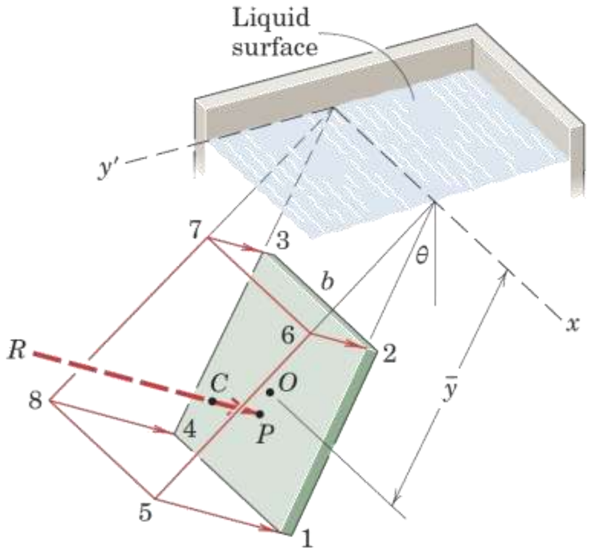
- Center of Pressure Location
  - The center of pressure P is located at the centroid of the trapezoidal pressure prism, which is easily found using composite techniques or evaluating the integral…

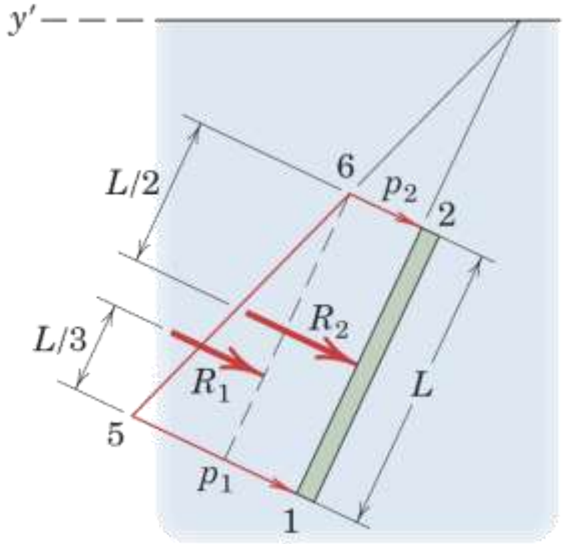
$$\overline{Y} = \frac{\int y\, dA'}{\int dA'}$$



Article 5/9 – Hydrostatic Pressure on Submerged Rectangular Surfaces (4 of 4)

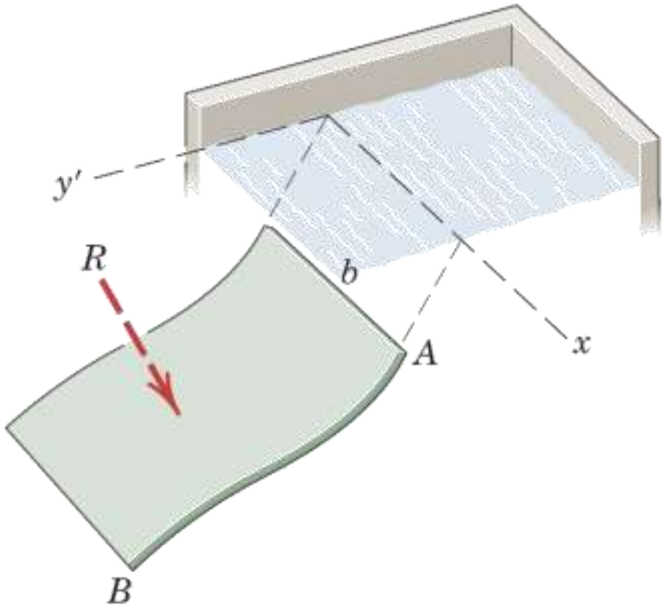
- Calculation of R – Method 3
  - $R_1 = \frac{1}{2}(p_1 - p_2)A$
  - Acts at the centroid of the triangular portion of the pressure prism.
  - $A = bL$ is the area of the plate
  - $R_2 = p_2 A$
  - Acts at the centroid of the rectangular portion of the pressure prism.





Article 5/9 – Hydrostatic Pressure on Submerged Cylindrical Surfaces (1 of 2)

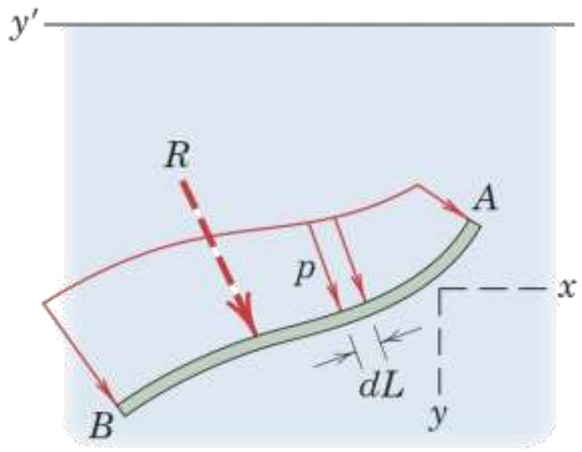
- Illustration



- Two-Dimensional Representation

- Calculation Procedure – Method 1

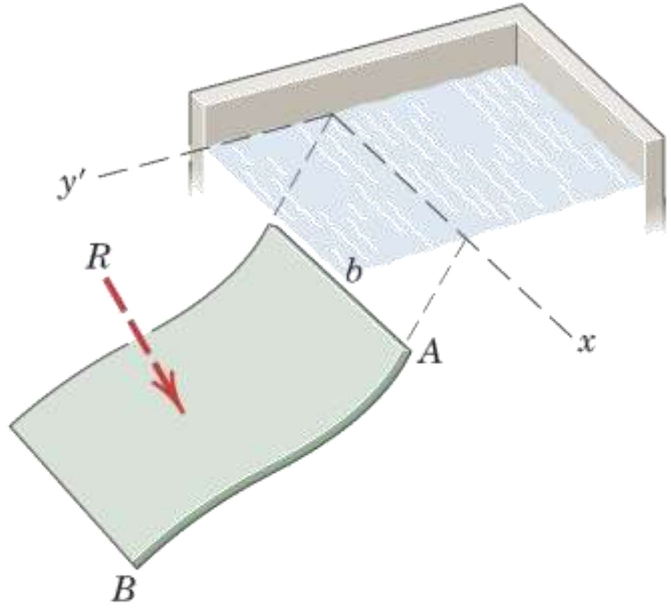
$$R_x = b\int (p\,dL)_x = b\int p\,dy \qquad \text{and} \qquad R_y = b\int (p\,dL)_y = b\int p\,dx$$

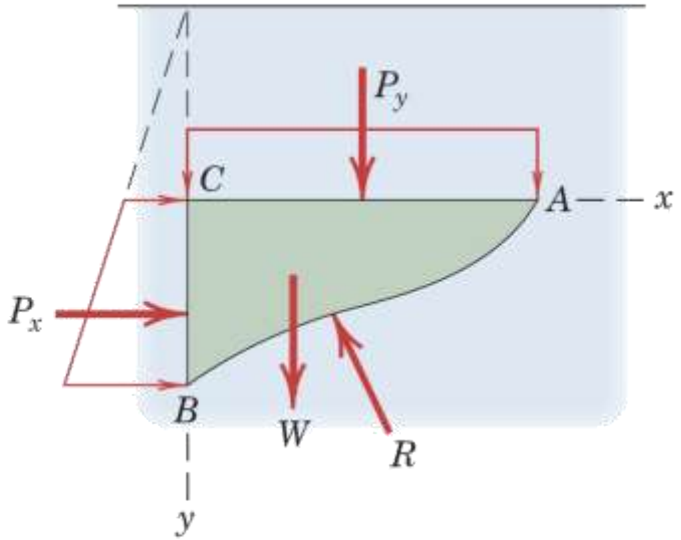


- Moment Equation will give Location

Article 5/9 – Hydrostatic Pressure on Submerged Cylindrical Surfaces (2 of 2)

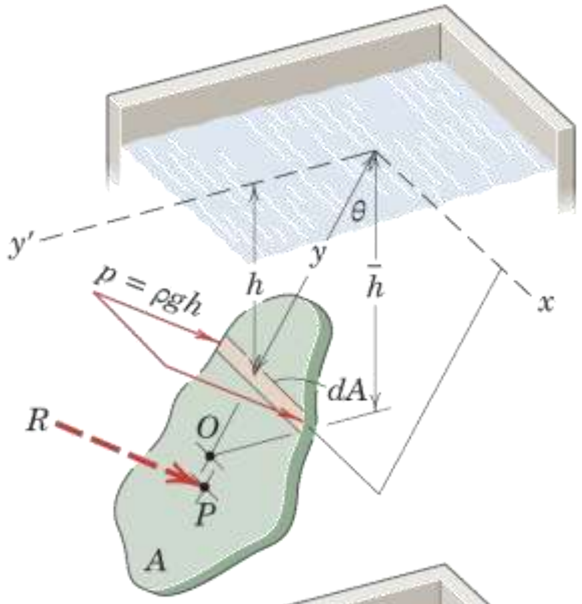
- Calculation Procedure – Method 2
  - P_x = pressure along *CB* and can be found using the techniques established for flat rectangular plates.
  - P_y = pressure along *AC* and can be found using the techniques established for flat rectangular plates.
  - *W* = weight of the fluid block and is the area *ABC* times the plate width *b*. The weight passes through the centroid of area *ABC*.

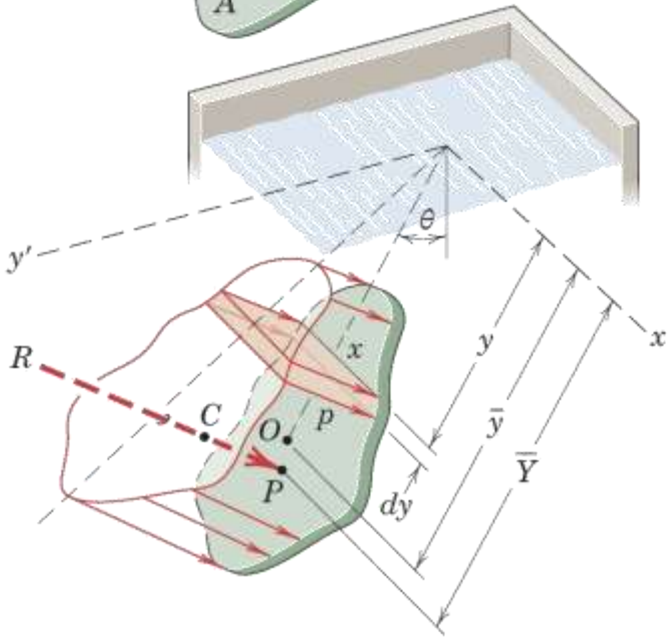




Article 5/9 – Hydrostatic Pressure on Submerged Irregular Flat Surfaces (1 of 2)

- Illustration
- Resultant
  - $R = \int \rho g h x \, dy$
  - $R = \rho g \bar{h} A$
  - A = area of the plate.
  - $\bar{h} = \bar{y} \cos\theta$, is the average depth of the plate, which is measured to the plate centroid O.
- Location
  - Acts at the center of pressure P which is the centroid of the irregular pressure volume.



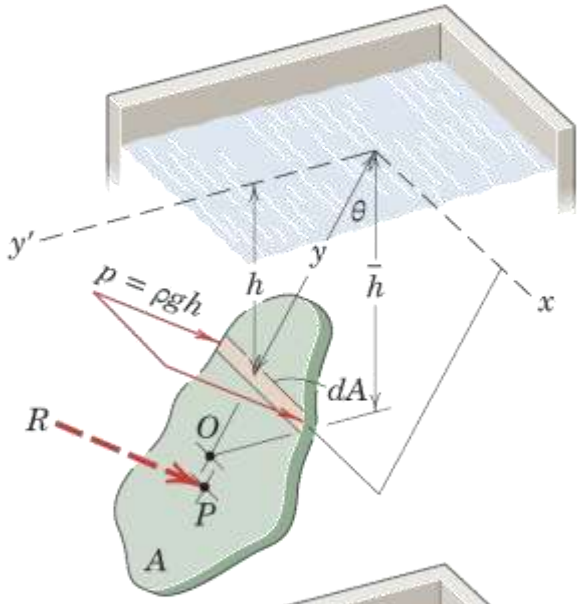


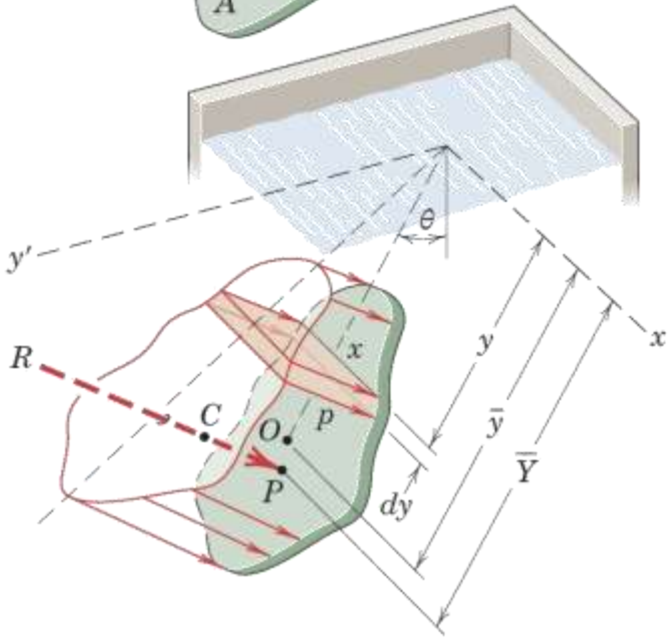
Article 5/9 – Hydrostatic Pressure on Submerged Irregular Flat Surfaces (2 of 2)

- Center of Pressure
  - The center of pressure P is located at the centroid of the trapezoidal pressure prism, which is easily found using composite techniques or evaluating the integral…

$$\bar{Y} = \frac{\int y(px\,dy)}{\int px\,dy}$$

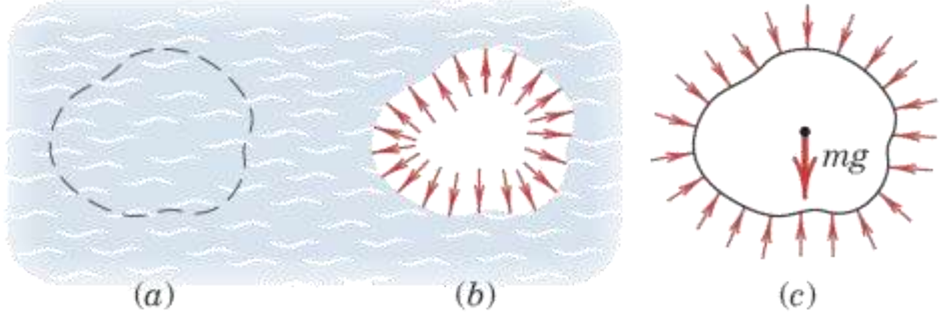
  - Note that the pressure $p = \rho g h = \rho g y \cos\theta$.
  - Also note that the center of pressure P and the centroid of the plate O are not the same.





Article 5/9 – Buoyancy (1 of 4)

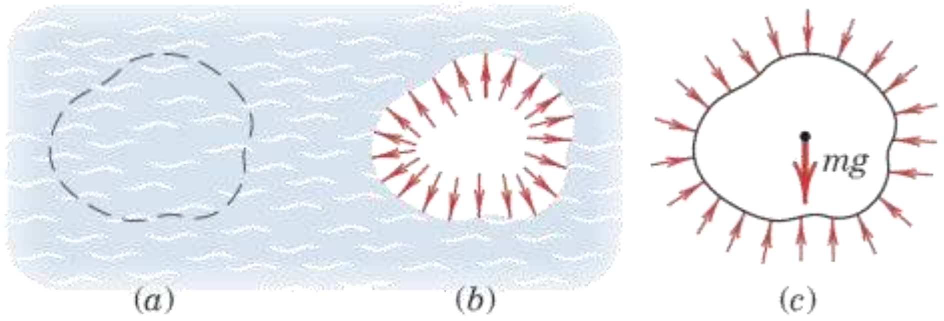
- Illustration



- Description
  - Fluid Portion (*a*)
  - Replaced Fluid Portion Force Distribution (*b*)
  - Free-Body Diagram of Fluid Portion (*c*)
  - Equilibrium Requirements

Article 5/9 – Buoyancy (2 of 4)

- Buoyant Force, $F = \rho g V$
  - ρ = density of the fluid
  - g = acceleration of gravity
  - V = volume of the displaced fluid

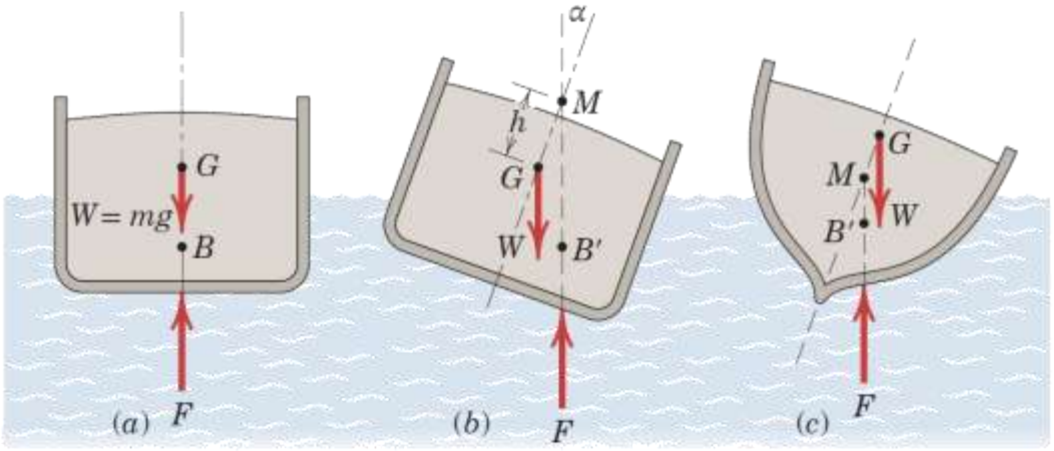


- The buoyant force is equal, opposite, and collinear with the weight of the fluid portion.

- For objects at rest in a fluid, the buoyant force will be equal in magnitude to the weight of the fluid displaced by the object.

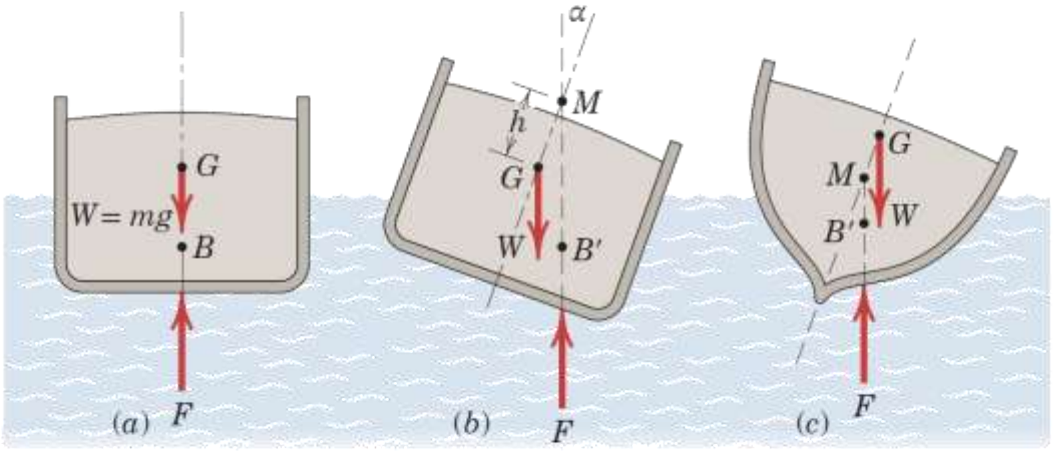
Article 5/9 – Buoyancy (3 of 4)

- Stability of a Floating Object
  - *B* = center of buoyancy, which is the centroid of displaced fluid volume.
  - The buoyant force *F* will pass through the center of buoyancy, and is equal and opposite to the weight of the ship, as shown in Figure (*a*).
  - If the ship lists through an angle α, the shape of the displaced volume will change, and the center of buoyancy will shift to B'.



Article 5/9 – Buoyancy (4 of 4)

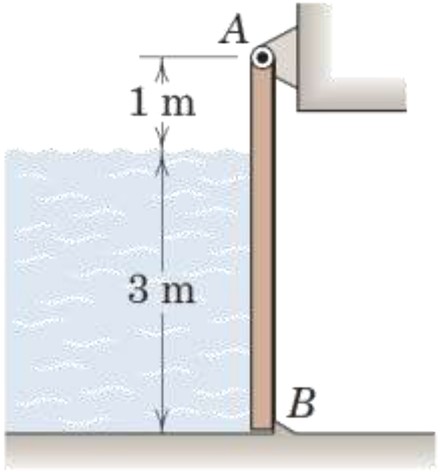
- Stability of a Floating Object (cont.)
  - Metacenter *M* is the intersection of a vertical line through the center of buoyancy and the ship centerline.
  - Metacentric height *h* is the distance of the metacenter from the mass center of the ship. For most hull shapes *h* is roughly constant for lists up to 20°.
  - If *M* is above *G*, the buoyant force creates a restoring moment against listing.
  - If *M* is below *G*, the buoyant force will cause the list to increase.



Article 5/9 – Sample Problem 5/19 (1 of 2)

- Problem Statement

  A rectangular plate, shown in vertical section *AB*, is 4 m high and 6 m wide (normal to the plane of the paper) and blocks the end of a fresh-water channel 3 m deep. The plate is hinged about a horizontal axis along its upper edge through *A* and is restrained from opening by the fixed ridge *B* which bears horizontally against the lower edge of the plate. Find the force *B* exerted on the plate by the ridge.



Article 5/9 – Sample Problem 5/19 (2 of 2)

- Solution

The free-body diagram of the plate is shown in section and includes the vertical and horizontal components of the force at A, the unspecified weight $W = mg$ of the plate, the unknown horizontal force B, and the resultant R of the triangular distribution of pressure against the vertical face.

The density of fresh water is $\rho = 1.000\ \text{Mg/m}^3$ so that the average pressure is

$[p_{av} = \rho g\bar{h}]$ $\qquad p_{av} = 1.000(9.81)(\frac{3}{2}) = 14.72\ \text{kPa}$ ①

The resultant R of the pressure forces against the plate becomes

$[R = p_{av} A]$ $\qquad R = (14.72)(3)(6) = 265\ \text{kN}$

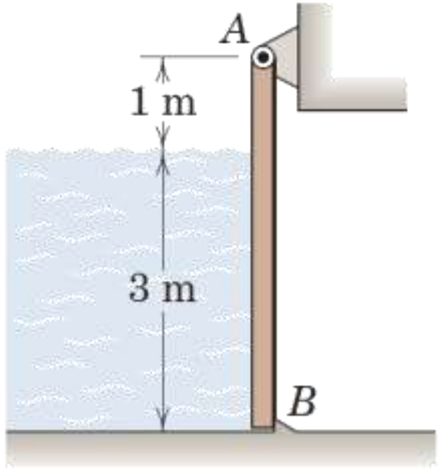
This force acts through the centroid of the triangular distribution of pressure, which is 1 m above the bottom of the plate. A zero moment summation about A establishes the unknown force B. Thus,

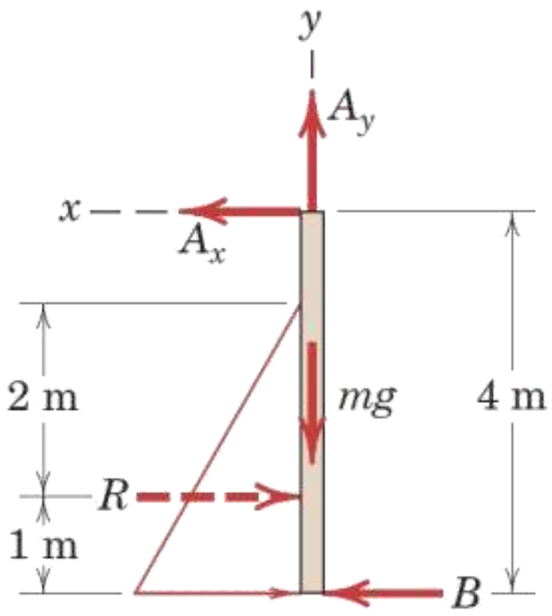
$[\Sigma M_A = 0]$ $\qquad 3(265) - 4B = 0 \qquad B = 198.7\ \text{kN}$ $\qquad$ *Ans.*

① Note that the units of pressure ρgh are

$$\left(10^3\frac{\text{kg}}{\text{m}^3}\right)\left(\frac{\text{m}}{\text{s}^2}\right)(\text{m}) = \left(10^3\frac{\text{kg}\cdot\text{m}}{\text{s}^2}\right)\left(\frac{1}{\text{m}^2}\right)$$

$$= \text{kN/m}^2 = \text{kPa}.$$

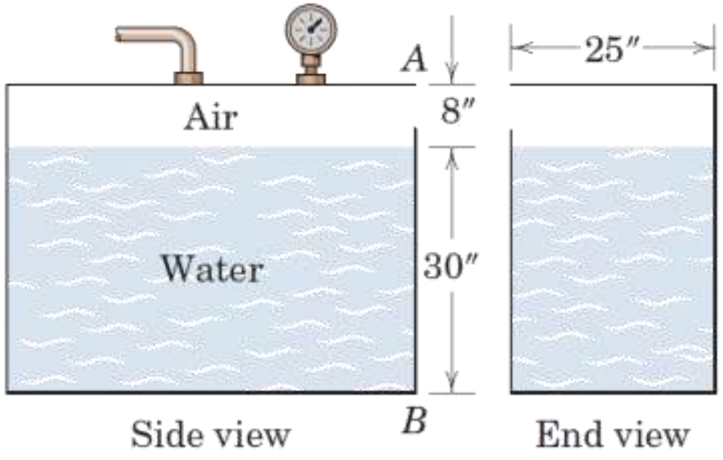




Article 5/9 – Sample Problem 5/20 (1 of 2)

- Problem Statement

  The air space in the closed fresh-water tank is maintained at a pressure of 0.80 lb/in.^{2} (above atmospheric). Determine the resultant force R exerted by the air and water on the end of the tank.



Article 5/9 – Sample Problem 5/20 (2 of 2)

- Solution

The pressure distribution on the end surface is shown, where $p_0 = 0.80$ lb/in.^{2} The specific weight of fresh water is $\mu = \rho g = 62.4/1728 = 0.0361$ lb/in.^{3} so that the increment of pressure Δp due to the water is

$$\Delta p = \mu \, \Delta h = 0.0361(30) = 1.083 \text{ lb/in.}^2$$

The resultant forces R_1 and R_2 due to the rectangular and triangular distributions of pressure, respectively, are ①

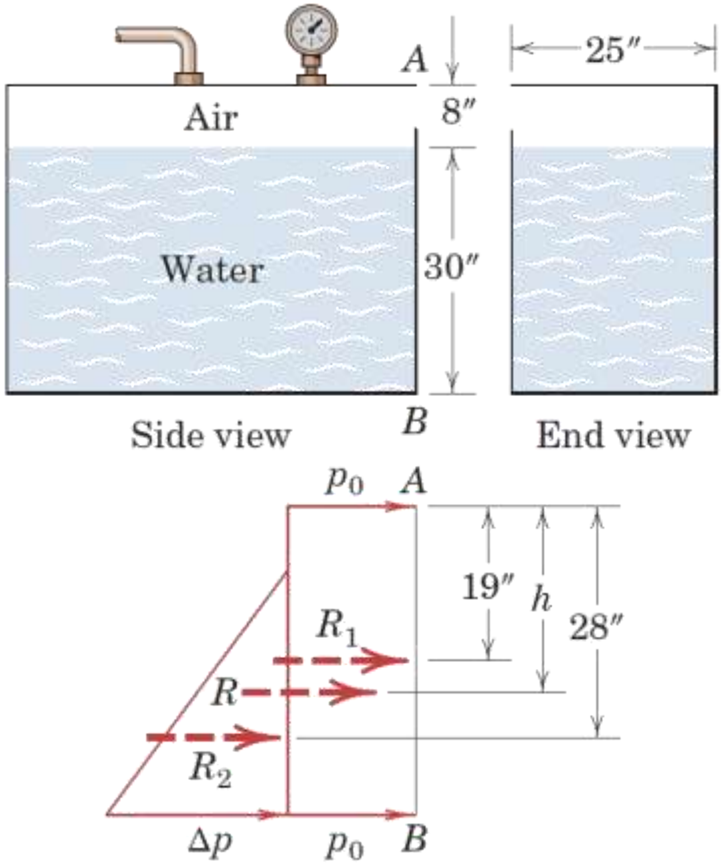
$$R_1 = p_0 A_1 = 0.80(38)(25) = 760 \text{ lb}$$

$$R_2 = \Delta p_{av} A_2 = \frac{1.083}{2}(30)(25) = 406 \text{ lb}$$

The resultant is then $R = R_1 + R_2 = 760 + 406 = 1166$ lb. *Ans.*

We locate R by applying the moment principle about A noting that R_1 acts through the center of the 38-in. depth and that R_2 acts through the centroid of the triangular pressure distribution 20 in. below the surface of the water and $20 + 8 = 28$ in. below A. Thus,

$[Rh = \Sigma M_A]$ $\quad 1166h = 760(19) + 406(28) \quad h = 22.1$ in. *Ans.*

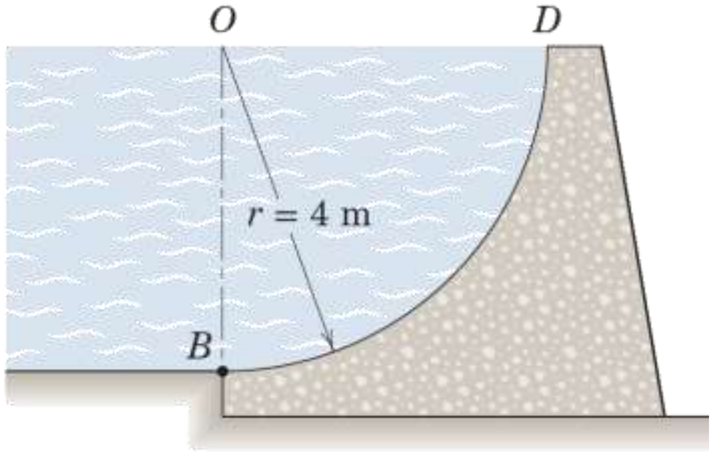


① Dividing the pressure distribution into these two parts is decidedly the simplest way in which to make the calculation.

Article 5/9 – Sample Problem 5/21 (1 of 3)

- Problem Statement

  Determine completely the resultant force R exerted on the cylindrical dam surface by the water. The density of fresh water is 1.000 Mg/m^3, and the dam has a length b, normal to the paper, of 30 m.



Article 5/9 – Sample Problem 5/21 (2 of 3)

- Solution I

The circular block of water BDO is isolated and its free-body diagram is drawn. The force P_x is

$$P_x = \rho g \bar{h} A = \frac{\rho g r}{2} br = \frac{(1.000)(9.81)(4)}{2}(30)(4) = 2350 \text{ kN} \quad ①$$

The weight W of the water passes through the mass center G of the quarter-circular section and is

$$mg = \rho g V = (1.000)(9.81)\frac{\pi(4)^2}{4}(30) = 3700 \text{ kN}$$

Equilibrium of the section of water requires

$[\Sigma F_x = 0]$ $\qquad R_x = P_x = 2350 \text{ kN}$

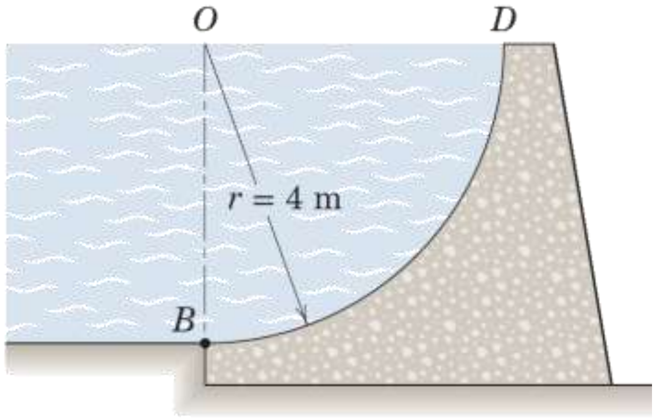
$[\Sigma F_y = 0]$ $\qquad R_y = mg = 3700 \text{ kN}$

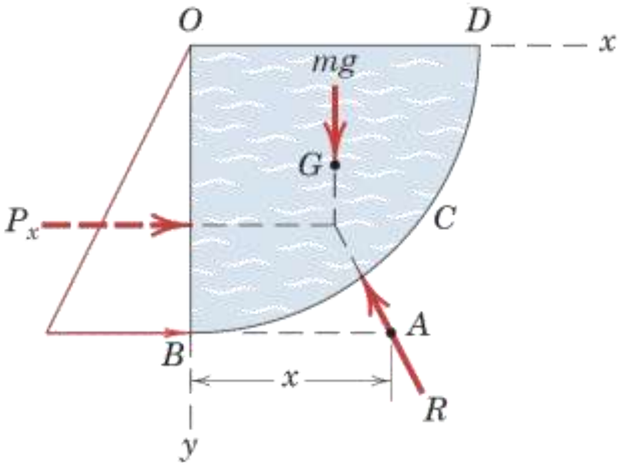
The resultant force R exerted by the fluid on the dam is equal and opposite to that shown acting on the fluid and is

$[R = \sqrt{R_x^{\,2} + R_y^{\,2}}]$ $\qquad R = \sqrt{(2350)^2 + (3700)^2} = 4380 \text{ kN}$ $\qquad$ *Ans.*

The x-coordinate of the point A through which R passes may be found from the principle of moments. Using B as a moment center gives

$$P_x \frac{r}{3} + mg\frac{4r}{3\pi} - R_y x = 0, \quad x = \frac{2350\left(\frac{4}{3}\right) + 3700\left(\frac{16}{3\pi}\right)}{3700} = 2.55 \text{ m} \quad \textit{Ans.}$$





① See note ① in Sample Problem 5/19 if there is any question about the units for $\rho g \bar{h}$.

Article 5/9 – Sample Problem 5/21 (3 of 3)

- Alternative Solution

The force acting on the dam surface may be obtained by a direct integration of the components ②

$$dR_x = p\,dA\cos\theta \qquad \text{and} \qquad dR_y = p\,dA\sin\theta$$

where $p = \rho gh = \rho gr\sin\theta$ and $dA = b(r\,d\theta)$. Thus,

$$R_x = \int_0^{\pi/2} \rho gr^2 b\sin\theta\cos\theta\,d\theta = -\rho gr^2 b\left[\frac{\cos 2\theta}{4}\right]_0^{\pi/2} = \tfrac{1}{2}\rho gr^2 b$$

$$R_y = \int_0^{\pi/2} \rho gr^2 b\sin^2\theta\,d\theta = \rho gr^2 b\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = \tfrac{1}{4}\pi\rho gr^2 b$$

Thus, $R = \sqrt{R_x^{\,2} + R_y^{\,2}} = \frac{1}{2}\rho gr^2 b\sqrt{1+\pi^2/4}$. Substituting the numerical values gives

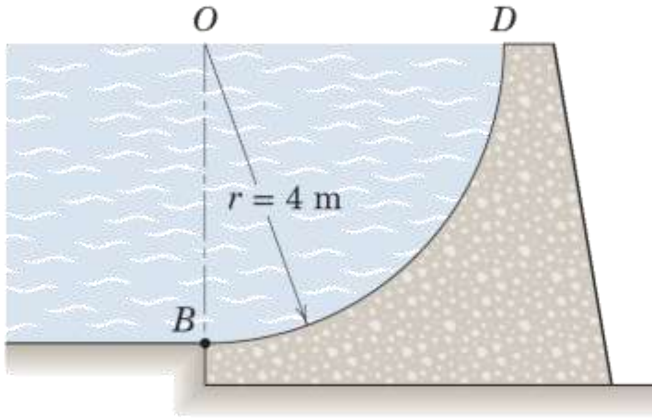
$$R = \tfrac{1}{2}(1.000)(9.81)(4^2)(30)\sqrt{1+\pi^2/4} = 4380 \text{ kN} \qquad \textit{Ans.}$$

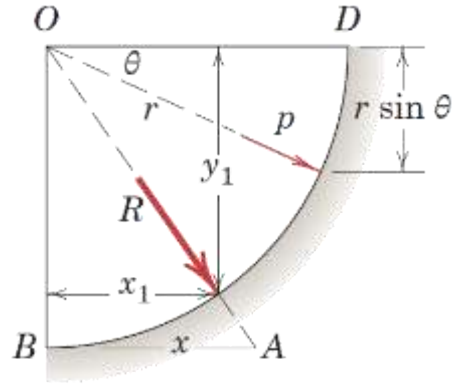
Since dR always passes through point O, we see that R also passes through O and, therefore, the moments of R_x and R_y about O must cancel. So we write $R_x y_1 = R_y x_1$, which gives us

$$x_1/y_1 = R_x/R_y = (\tfrac{1}{2}\rho gr^2 b)/(\tfrac{1}{4}\pi\rho gr^2 b) = 2/\pi$$

By similar triangles we see that

$$x/r = x_1/y_1 = 2/\pi \qquad \text{and} \qquad x = 2r/\pi = 2(4)/\pi = 2.55 \text{ m} \qquad \textit{Ans.}$$



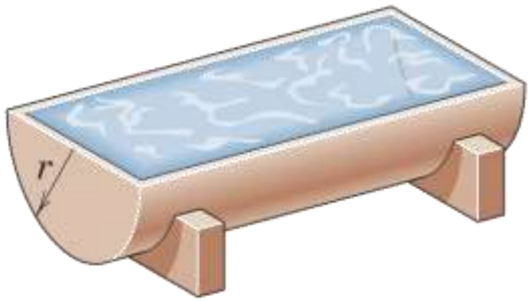


② This approach by integration is feasible here mainly because of the simple geometry of the circular arc.

Article 5/9 – Sample Problem 5/22 (1 of 3)

- Problem Statement

  Determine the resultant force R exerted on the semicircular end of the water tank shown in the figure if the tank is filled to capacity. Express the result in terms of the radius r and the water density ρ.



Article 5/9 – Sample Problem 5/22 (2 of 3)

- Solution I

We will obtain R first by a direct integration. With a horizontal strip of area $dA = 2x\,dy$ acted on by the pressure $p = \rho g y$, the increment of the resultant force is $dR = p\,dA$ so that

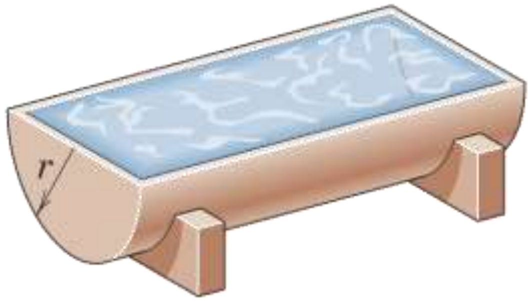
$$R = \int p\,dA = \int \rho g y(2x\,dy) = 2\rho g \int_0^r y\sqrt{r^2 - y^2}\,dy$$

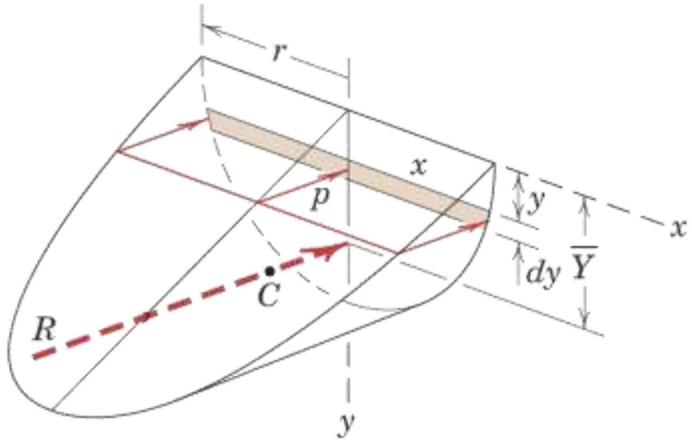
Integrating gives $\qquad R = \frac{2}{3}\rho g r^3 \qquad$ *Ans.*

The location of R is determined by using the principle of moments. Taking moments about the x-axis gives

$$\left[R\bar{Y} = \int y\,dR\right] \qquad \frac{2}{3}\rho g r^3 \bar{Y} = 2\rho g \int_0^r y^2\sqrt{r^2 - y^2}\,dy$$

Integrating gives $\qquad \frac{2}{3}\rho g r^3 \bar{Y} = \frac{\rho g r^4}{4}\frac{\pi}{2} \qquad$ and $\qquad \bar{Y} = \frac{3\pi r}{16} \qquad$ *Ans.*





Article 5/9 – Sample Problem 5/22 (3 of 3)

- Solution II

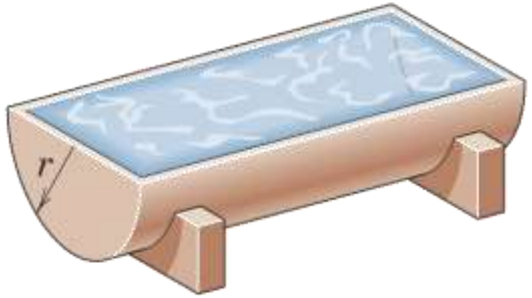
We may use Eq. 5/25 directly to find R, where the average pressure is $\rho g\bar{h}$ and $\bar{h}$ is the coordinate to the centroid of the area over which the pressure acts. For a semicircular area, $\bar{h} = 4r/(3\pi)$.

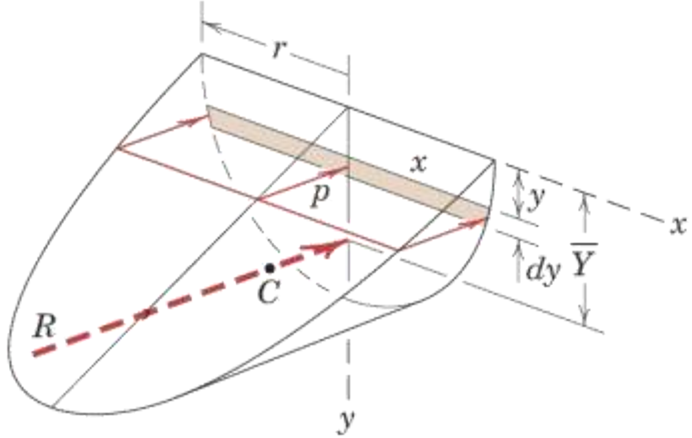
$[R = \rho g\bar{h}A]$ $\qquad R = \rho g \frac{4r}{3\pi}\frac{\pi r^2}{2} = \frac{2}{3}\rho g r^3$ $\qquad$ *Ans.*

which is the volume of the pressure-area figure.

The resultant R acts through the centroid C of the volume defined by the pressure-area figure. ① Calculation of the centroidal distance $\bar{Y}$ involves the same integral obtained in *Solution I*.

① Be very careful not to make the mistake of assuming that R passes through the centroid of the area over which the pressure acts.

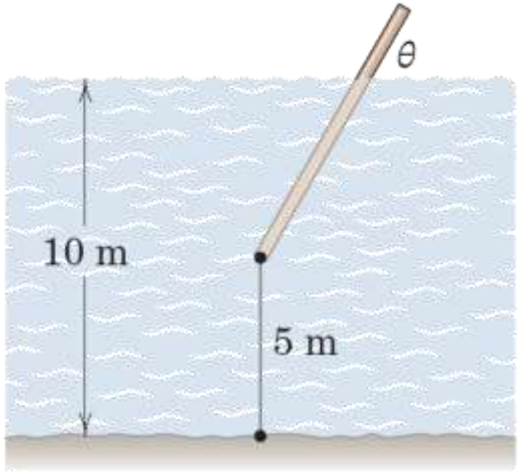




Article 5/9 – Sample Problem 5/23 (1 of 2)

- Problem Statement

  A buoy in the form of a uniform 8-m pole 0.2 m in diameter has a mass of 200 kg and is secured at its lower end to the bottom of a fresh-water lake with 5 m of cable. If the depth of the water is 10 m, calculate the angle θ made by the pole with the horizontal.



Article 5/9 – Sample Problem 5/23 (2 of 2)

- Solution

The free-body diagram of the buoy shows its weight acting through G, the vertical tension T in the anchor cable, and the buoyancy force B which passes through centroid C of the submerged portion of the buoy. Let x be the distance from G to the waterline. The density of fresh water is $\rho = 10^3$ kg/m^3, so that the buoyancy force is

$[B = \rho g V]$ $\quad\quad B = 10^3(9.81)\pi(0.1)^2(4 + x)$ N

Moment equilibrium, $\Sigma M_A = 0$, about A gives

$$200(9.81)(4 \cos \theta) - [10^3(9.81)\pi(0.1)^2(4 + x)]\frac{4 + x}{2} \cos \theta = 0$$

Thus, $\quad x = 3.14$ m $\quad$ and $\quad \theta = \sin^{-1}\left(\frac{5}{4 + 3.14}\right) = 44.5°$ $\quad$ *Ans.*