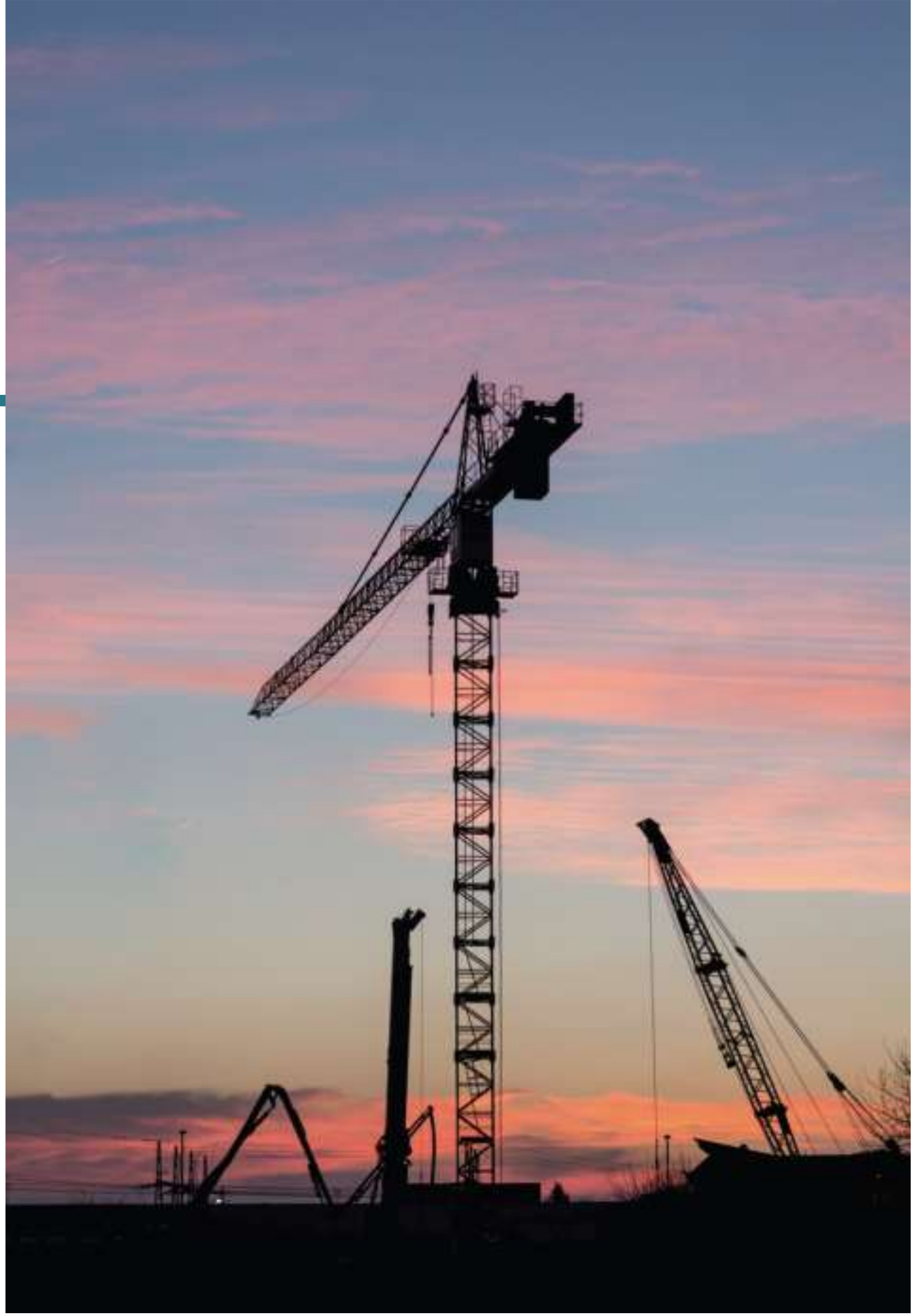
CHAPTER 2
FORCE SYSTEMS

CHAPTER OUTLINE



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Article 2/1 Introduction

- Chapter Purpose

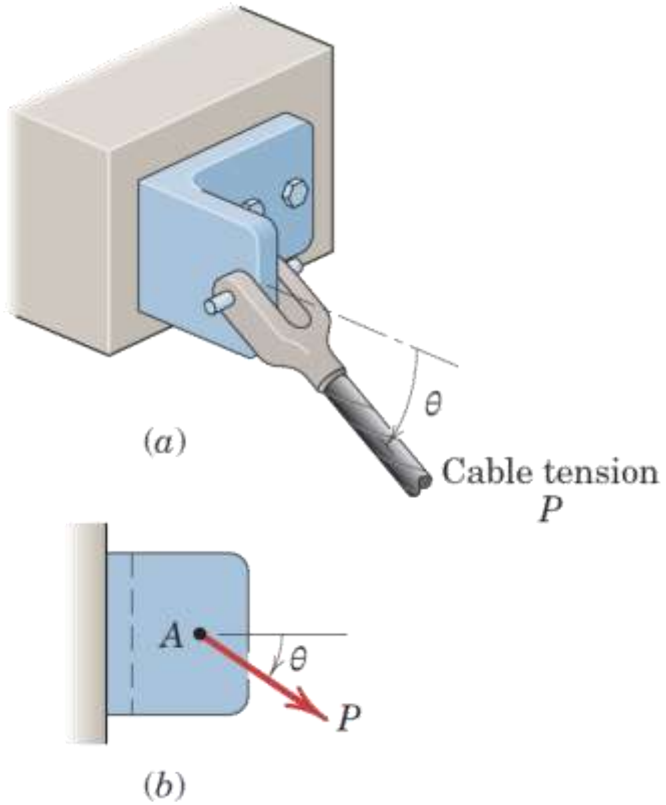
- Importance of Chapter 2 Concepts

Article 2/2 Force

- A Reminder and Illustration

- Vector Quantity

- External and Internal Effects

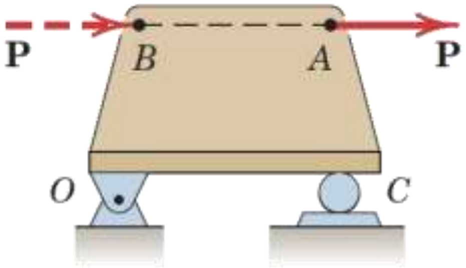


Article 2/2 – Principle of Transmissibility

- Statement of the Principle

  A force may be applied at any point on its given line of action without altering the resultant effects of the force external to the rigid body on which it acts.

- Illustration



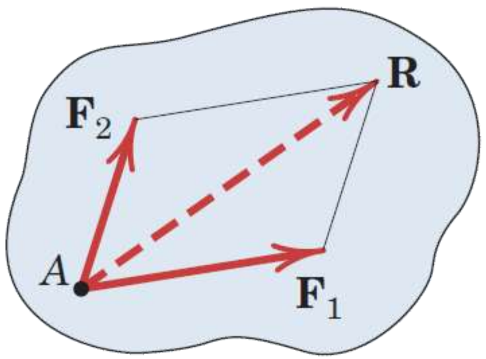
- Sliding Vectors

Article 2/2 – Force Classification

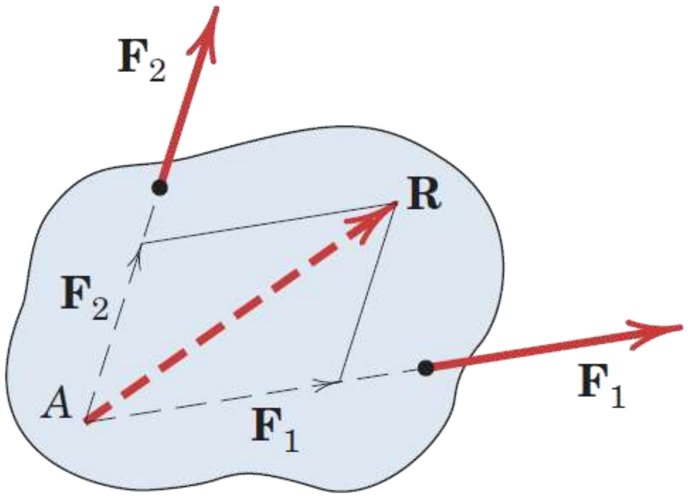
- Contact Force
- Body Force
- Concentrated Force
- Distributed Force
- Action and Reaction Pairs

Article 2/2 – Concurrent Forces (1 of 2)

- Definition and Vector Sum

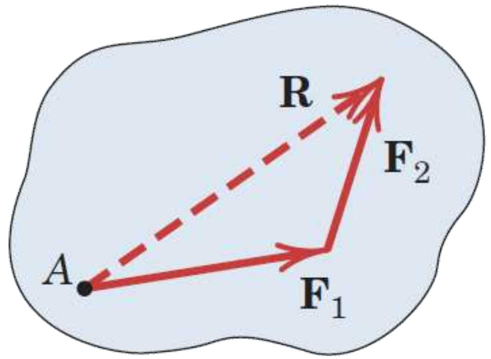


- Application of Transmissibility

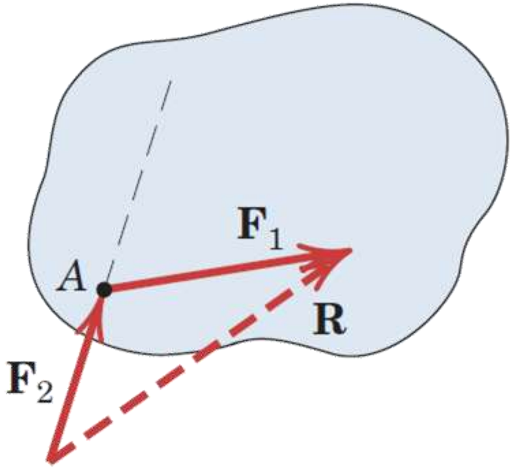


Article 2/2 – Concurrent Forces (2 of 2)

- Parallelogram Law of Vector Addition

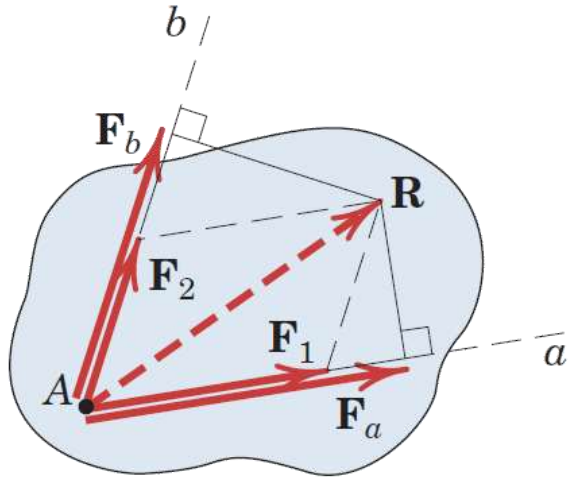


- Order of Addition



Article 2/2 – Vector Components

- Vector Components: $\mathbf{F}_1$ and $\mathbf{F}_2$
  - Obey the Parallelogram Law of Addition
  - Directed along the Axes of Choice

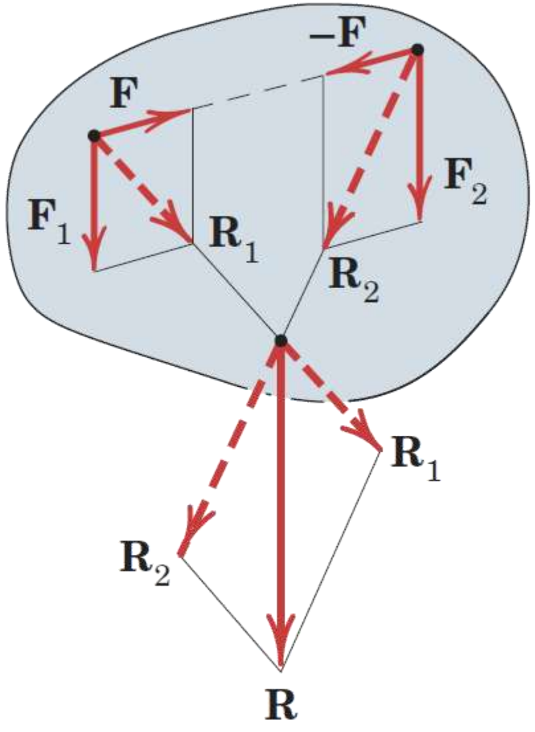


- Vector Projections: $\mathbf{F}_a$ and $\mathbf{F}_b$
  - Do not obey the Parallelogram Law of Addition
  - Directed along the Axes of Choice

Article 2/2 – Special Case of Vector Addition

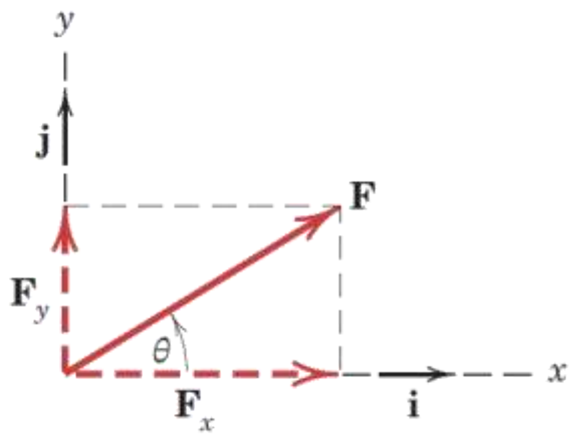
- Parallel Vectors: $\mathbf{F}_1$ and $\mathbf{F}_2$

- Procedure
  - Add Equal, Opposite, Collinear Forces **F** and **-F**
  - Find the Resultants $\mathbf{R}_1$ and $\mathbf{R}_2$
  - Locate the Point of Concurrency for the Resultants
  - Add the Resultants at the Point of Concurrency



2/3 Rectangular Components

- Illustration



- Vector Components: $\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y$

- Scalar Components: $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j}$

- Other Useful Relationships

$$F_x = F\cos\theta \qquad F = \sqrt{F_x^{\,2} + F_y^{\,2}}$$

$$F_y = F\sin\theta \qquad \theta = \tan^{-1}\frac{F_y}{F_x}$$

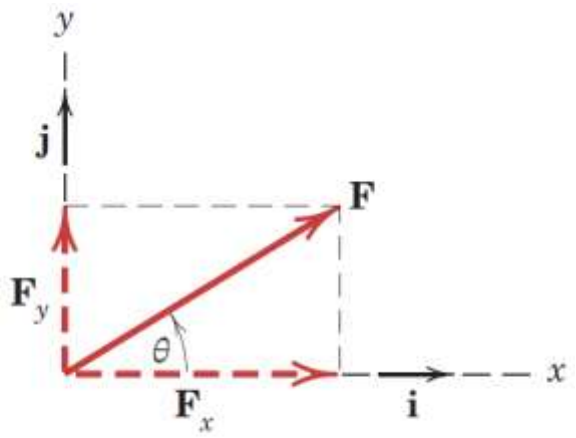
2/3 Conventions for Describing Vector Components

- Vector Magnitude, F
  - Lightface, Italic Font
  - Always Positive

- Scalar Component, F_x
  - Lightface, Italic Font
  - Positive or Negative

- Force Vector Depiction
  - Solid, Red Arrow

- Component Vector Depiction
  - Dashed, Red Arrow



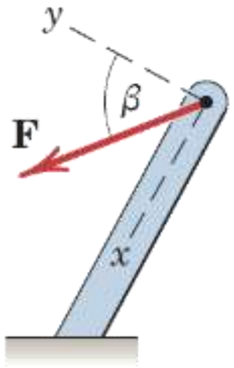
2/3 Determining the Components of a Force (1 of 2)

- It is often the case that…
  - Dimensions or axes are not given in horizontal and vertical directions.
  - Angles are not measured counterclockwise from the x-axis.
  - Coordinates do not originate from the line of action of a force.

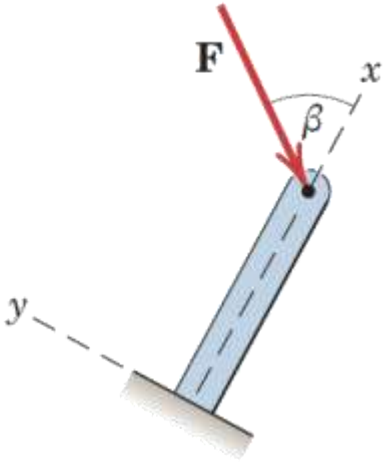
- We still need to be able to find the components of a force vector!

2/3 Determining the Components of a Force (2 of 2)

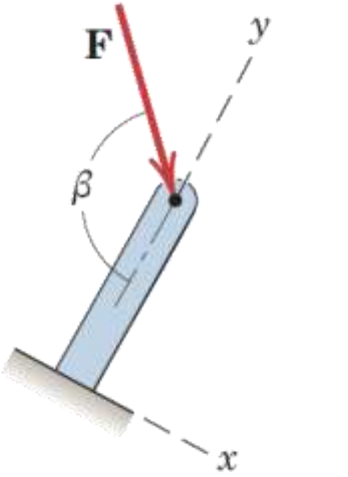
- Some Illustrations



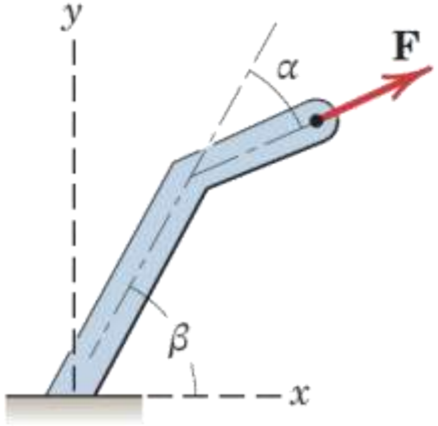
$F_x = F \sin \beta$
$F_y = F \cos \beta$



$F_x = -F \cos \beta$
$F_y = -F \sin \beta$



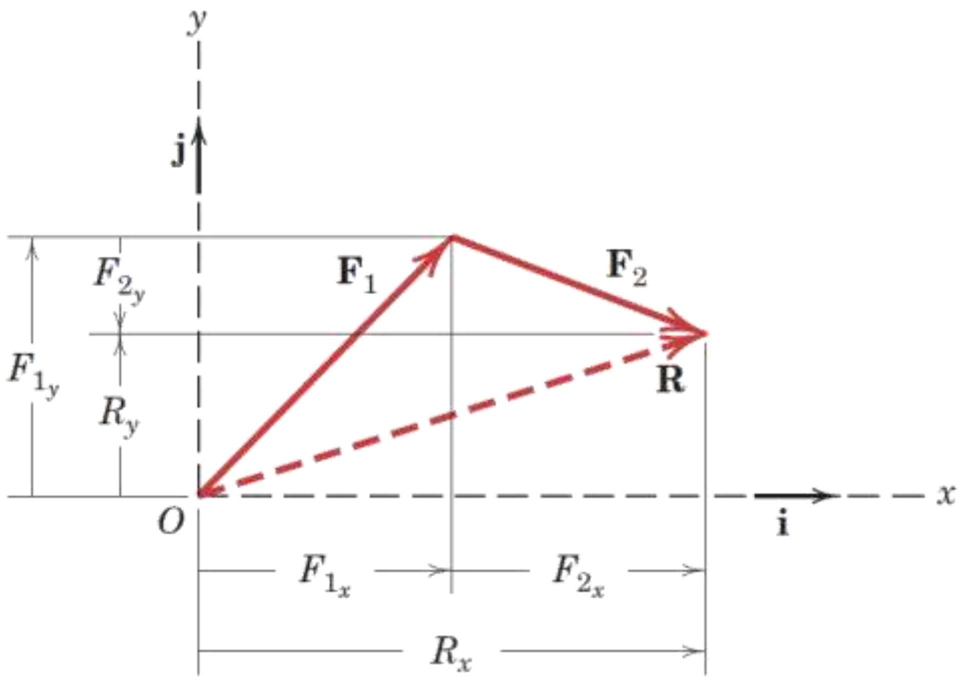
$F_x = F \sin(\pi - \beta)$
$F_y = -F \cos(\pi - \beta)$



$F_x = F \cos(\beta - \alpha)$
$F_y = F \sin(\beta - \alpha)$

2/3 Finding Resultants using Components

- Illustration



- Mathematics

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = (F_{1_x}\mathbf{i} + F_{1_y}\mathbf{j}) + (F_{2_x}\mathbf{i} + F_{2_y}\mathbf{j})$$

$$R_x\mathbf{i} + R_y\mathbf{j} = (F_{1_x} + F_{2_x})\mathbf{i} + (F_{1_y} + F_{2_y})\,\mathbf{j}$$

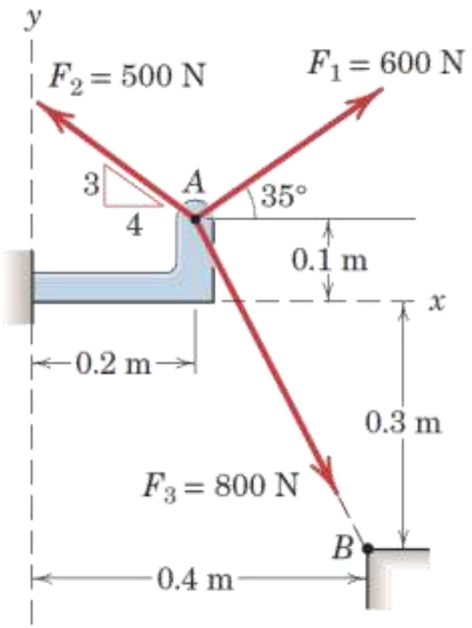
$$R_x = F_{1_x} + F_{2_x} = \Sigma F_x$$

$$R_y = F_{1_y} + F_{2_y} = \Sigma F_y$$

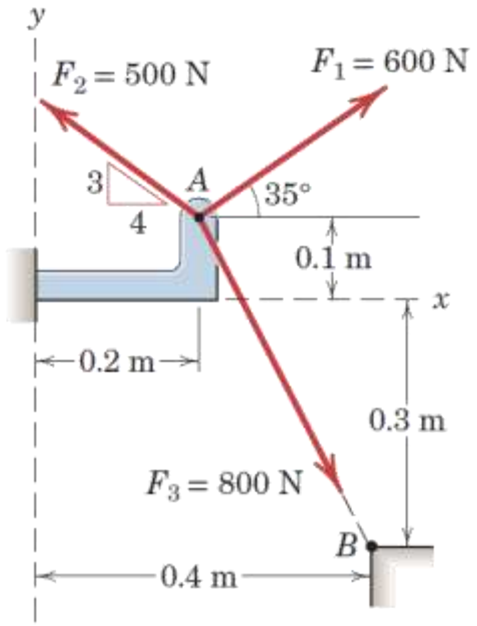
Article 2/3 – Sample Problem 2/1 (1 of 3)

- Problem Statement

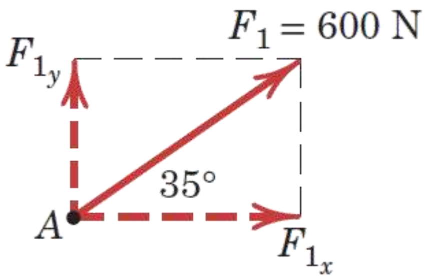
The forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, all of which act on point *A* of the bracket, are specified in three different ways. Determine the *x* and *y* scalar components of each of the three forces.



Article 2/3 – Sample Problem 2/1 (2 of 3)



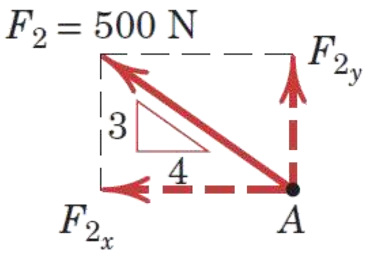
- Scalar Components of $\mathbf{F}_1$



$$F_{1_x} = 600 \cos 35° = 491 \text{ N} \qquad \textit{Ans.}$$

$$F_{1_y} = 600 \sin 35° = 344 \text{ N} \qquad \textit{Ans.}$$

- Scalar Components of $\mathbf{F}_2$

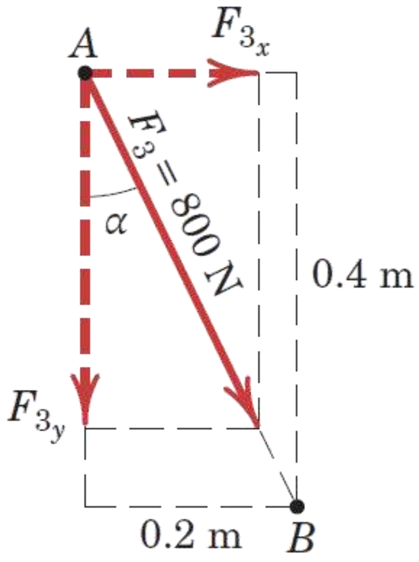


$$F_{2_x} = -500(\tfrac{4}{5}) = -400 \text{ N} \qquad \textit{Ans.}$$

$$F_{2_y} = 500(\tfrac{3}{5}) = 300 \text{ N} \qquad \textit{Ans.}$$

Article 2/3 – Sample Problem 2/1 (3 of 3)

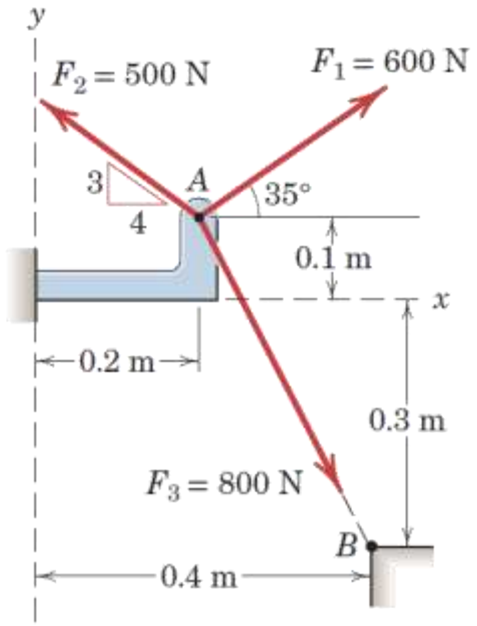
- Scalar Components of $\mathbf{F}_3$



$$\alpha = \tan^{-1}\left[\frac{0.2}{0.4}\right] = 26.6^\circ$$

$$F_{3_x} = F_3 \sin\alpha = 800 \sin 26.6^\circ = 358 \text{ N} \quad ① \qquad Ans.$$

$$F_{3_y} = -F_3 \cos\alpha = -800 \cos 26.6^\circ = -716 \text{ N} \qquad Ans.$$



- Alternative Calculation

$$\mathbf{F}_3 = F_3\mathbf{n}_{AB} = F_3\frac{\overrightarrow{AB}}{\overline{AB}} = 800\left[\frac{0.2\mathbf{i} - 0.4\mathbf{j}}{\sqrt{(0.2)^2 + (-0.4)^2}}\right]$$

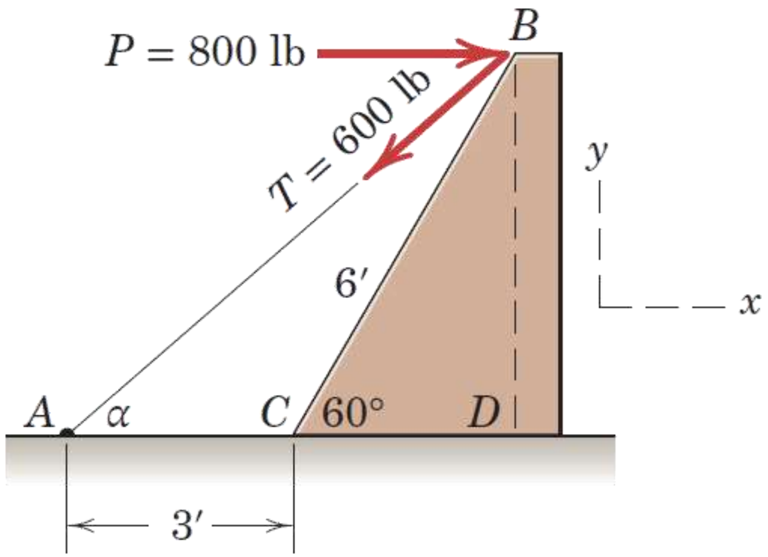
$$= 800\,[0.447\mathbf{i} - 0.894\mathbf{j}]$$

$$= 358\mathbf{i} - 716\mathbf{j} \text{ N}$$

Article 2/3 – Sample Problem 2/2 (1 of 4)

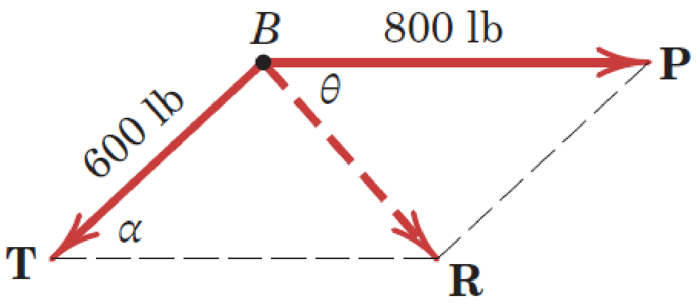
- Problem Statement

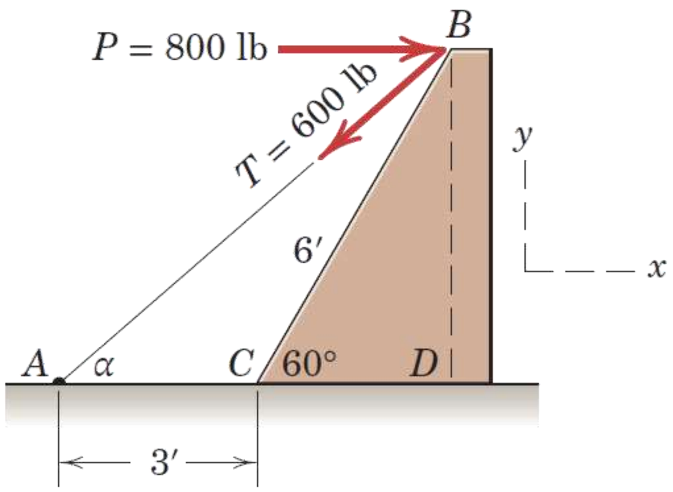
  Combine the two forces **P** and **T**, which act on the fixed structure at *B*, into a single equivalent force **R**.



Article 2/3 – Sample Problem 2/2 (2 of 4)

- Graphical Solution (Scaled Drawing)





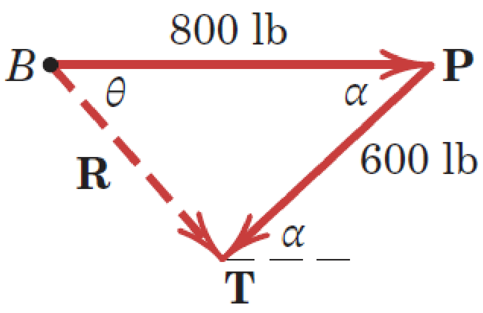
$$\tan \alpha = \frac{\overline{BD}}{\overline{AD}} = \frac{6 \sin 60^\circ}{3 + 6 \cos 60^\circ} = 0.866 \qquad \alpha = 40.9^\circ$$

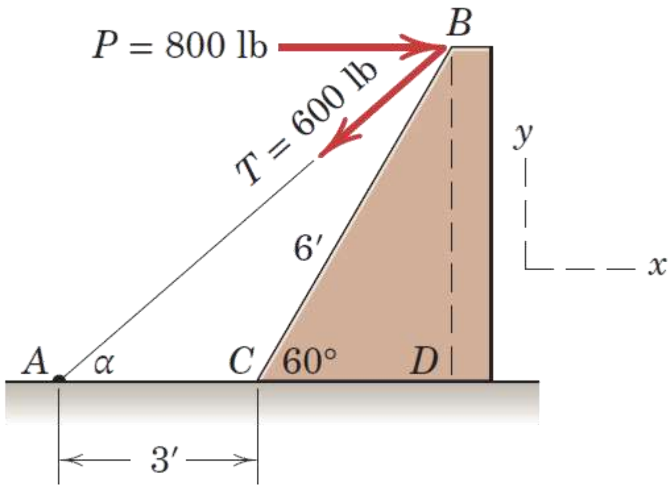
Measurement of the length R and direction θ of the resultant force $\mathbf{R}$ yields the approximate results

$$R = 525 \text{ lb} \qquad \theta = 49^\circ \qquad \textit{Ans.}$$

Article 2/3 – Sample Problem 2/2 (3 of 4)

- Geometric Solution





$$R^2 = (600)^2 + (800)^2 - 2(600)(800)\cos 40.9° = 274{,}300$$

$$R = 524 \text{ lb} \qquad \textit{Ans.}$$

From the law of sines, we may determine the angle θ which orients **R**. Thus,

$$\frac{600}{\sin\theta} = \frac{524}{\sin 40.9°} \qquad \sin\theta = 0.750 \qquad \theta = 48.6° \qquad \textit{Ans.}$$

Article 2/3 – Sample Problem 2/2 (4 of 4)

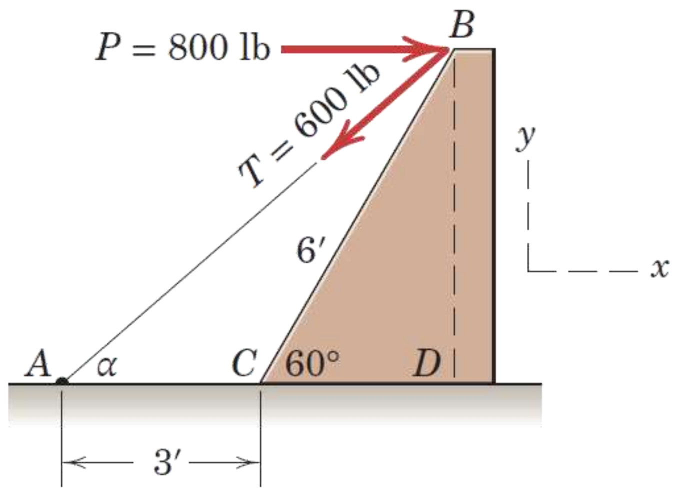
- Algebraic Solution

$$R_x = \Sigma F_x = 800 - 600 \cos 40.9° = 346 \text{ lb}$$

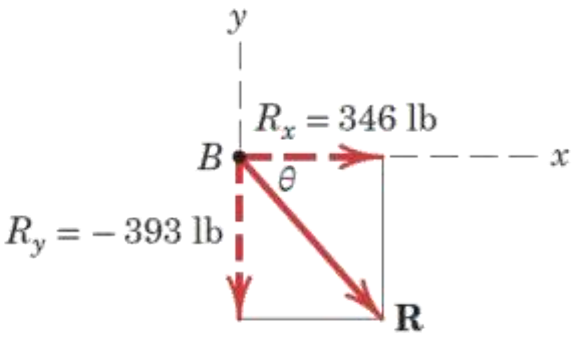
$$R_y = \Sigma F_y = -600 \sin 40.9° = -393 \text{ lb}$$

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(346)^2 + (-393)^2} = 524 \text{ lb} \qquad \textit{Ans.}$$

$$\theta = \tan^{-1} \frac{|R_y|}{|R_x|} = \tan^{-1} \frac{393}{346} = 48.6° \qquad \textit{Ans.}$$



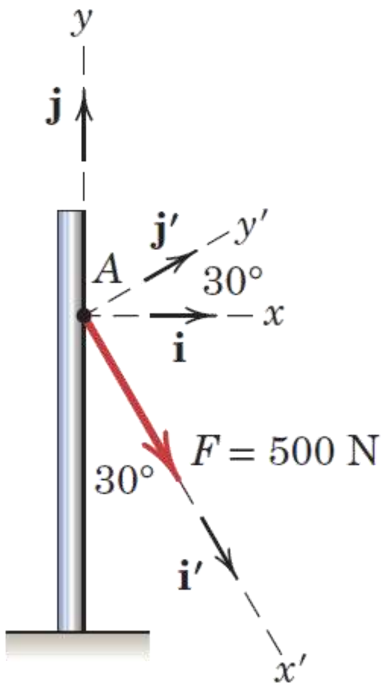
- Vector Representation



Article 2/3 – Sample Problem 2/3 (1 of 3)

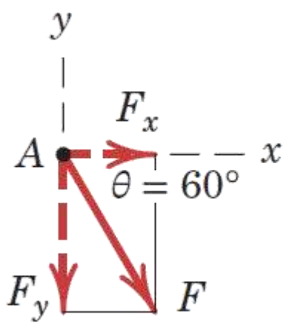
- Problem Statement

  The 500-N force **F** is applied to the vertical pole as shown. (1) Write **F** in terms of the unit vectors **i** and **j** and identify both its vector and scalar components. (2) Determine the scalar components of the force vector **F** along the *x*′- and *y*′-axes. (3) Determine the scalar components of **F** along the *x*- and *y*′-axes.



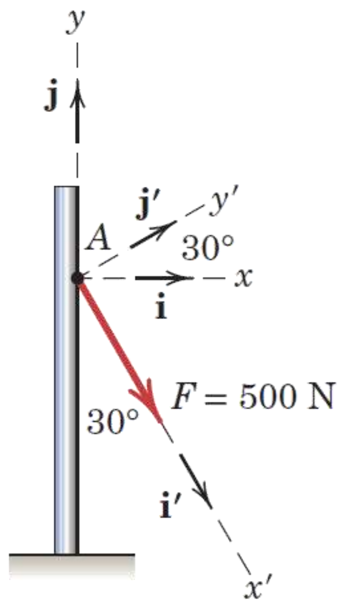
Article 2/3 – Sample Problem 2/3 (2 of 3)

- Part 1 Solution

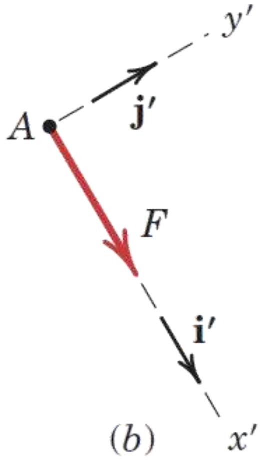


$$\mathbf{F} = (F\cos\theta)\mathbf{i} - (F\sin\theta)\mathbf{j}$$
$$= (500\cos 60°)\mathbf{i} - (500\sin 60°)\mathbf{j}$$
$$= (250\mathbf{i} - 433\mathbf{j})\text{ N} \qquad \textit{Ans.}$$

The scalar components are $F_x = 250$ N and $F_y = -433$ N. The vector components are $\mathbf{F}_x = 250\mathbf{i}$ N and $\mathbf{F}_y = -433\mathbf{j}$ N.



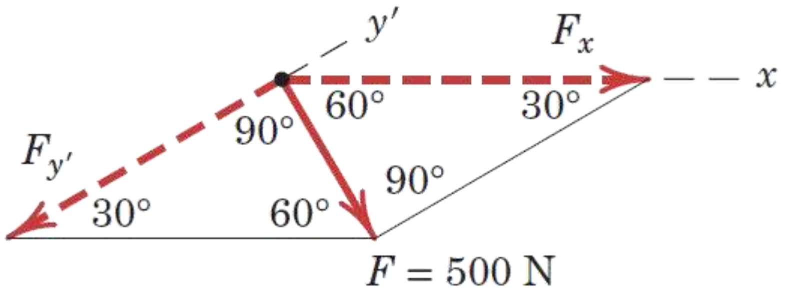
- Part 2 Solution



$$F_{x'} = 500\text{ N} \qquad F_{y'} = 0 \qquad \textit{Ans.}$$

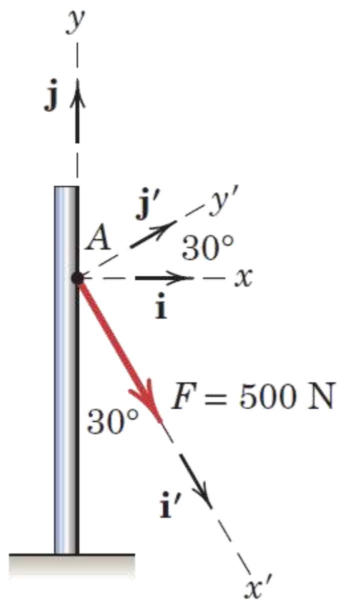
Article 2/3 – Sample Problem 2/3 (3 of 3)

- Part 3 Solution



$$\frac{|F_x|}{\sin 90°} = \frac{500}{\sin 30°} \qquad |F_x| = 1000 \text{ N} \quad ①$$

$$\frac{|F_{y'}|}{\sin 60°} = \frac{500}{\sin 30°} \qquad |F_{y'}| = 866 \text{ N}$$



The required scalar components are then

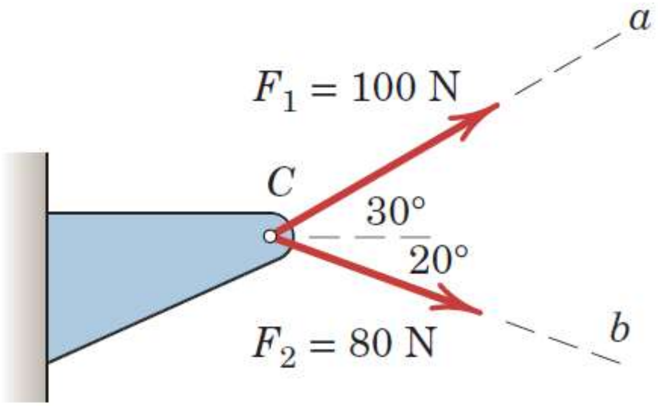
$$F_x = 1000 \text{ N} \qquad F_{y'} = -866 \text{ N} \qquad \textit{Ans.}$$

① Obtain F_x and $F_{y'}$ graphically and compare your results with the calculated values.

Article 2/3 – Sample Problem 2/4 (1 of 2)

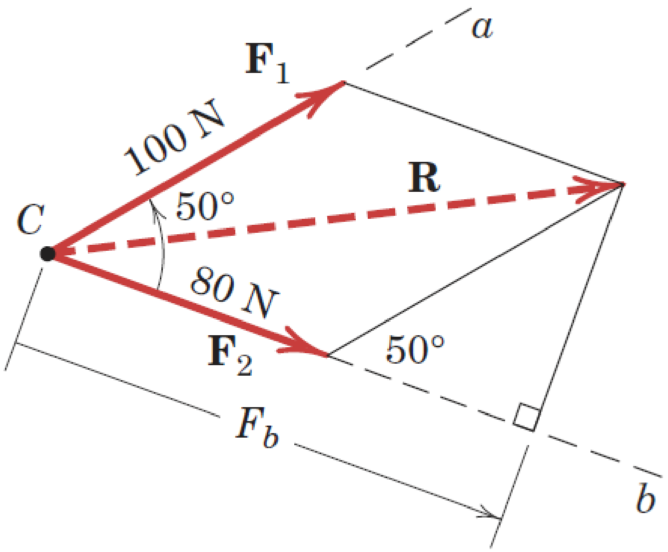
- Problem Statement

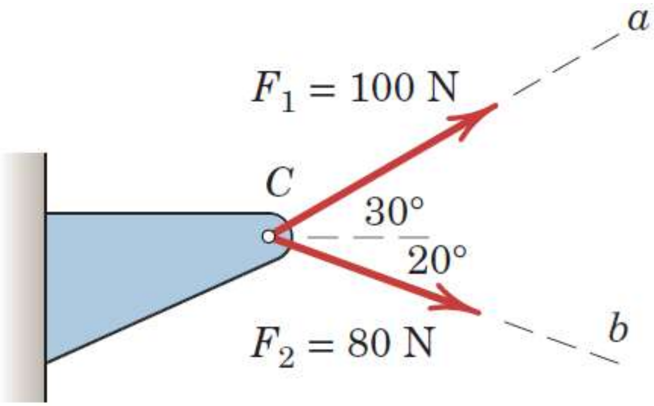
  Forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on the bracket as shown. Determine the projection F_b of their resultant $\mathbf{R}$ onto the b-axis.



Article 2/3 – Sample Problem 2/4 (1 of 2)

- Solution





$$R^2 = (80)^2 + (100)^2 - 2(80)(100)\cos 130° \qquad R = 163.4\text{ N}$$

The figure also shows the orthogonal projection $\mathbf{F}_b$ of $\mathbf{R}$ onto the b-axis. Its length is

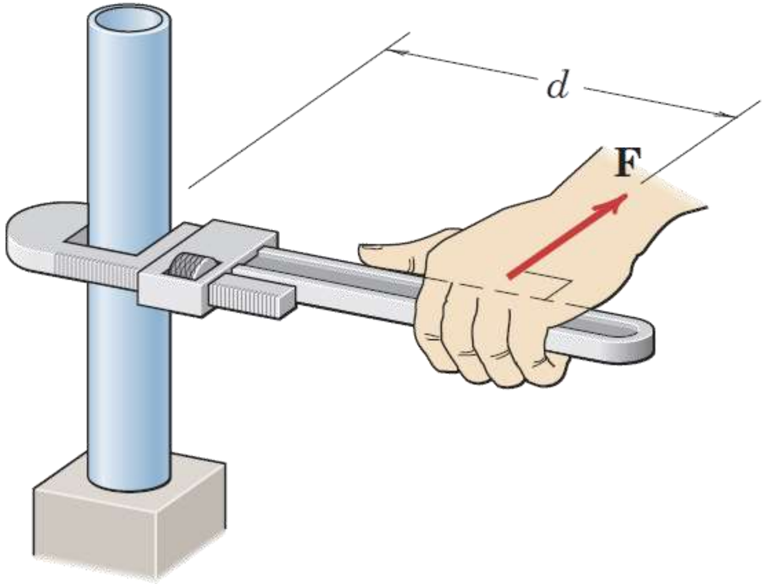
$$F_b = 80 + 100\cos 50° = 144.3\text{ N} \qquad \textit{Ans.}$$

Article 2/4 Moment

- A moment is the tendency of a force to rotate a body about an axis.

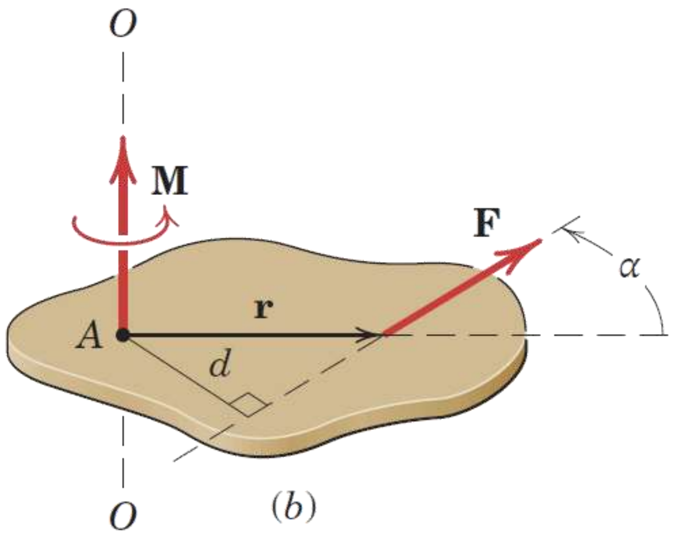
- Illustration: Pipe Wrench

- Things to Note
  - Direction and Orientation of the Force
  - Axis of Rotation
  - Direction of Rotation
  - Effective Length, *d*



Article 2/4 – Moment about a Point (1 of 3)

- Scalar Development
  - Moment Arm, *d*
  - Moment Vector, **M**
  - Axis of Rotation, *O*-*O*
  - Moment Magnitude, $M = Fd$
  - Direction of Rotation
  - Units



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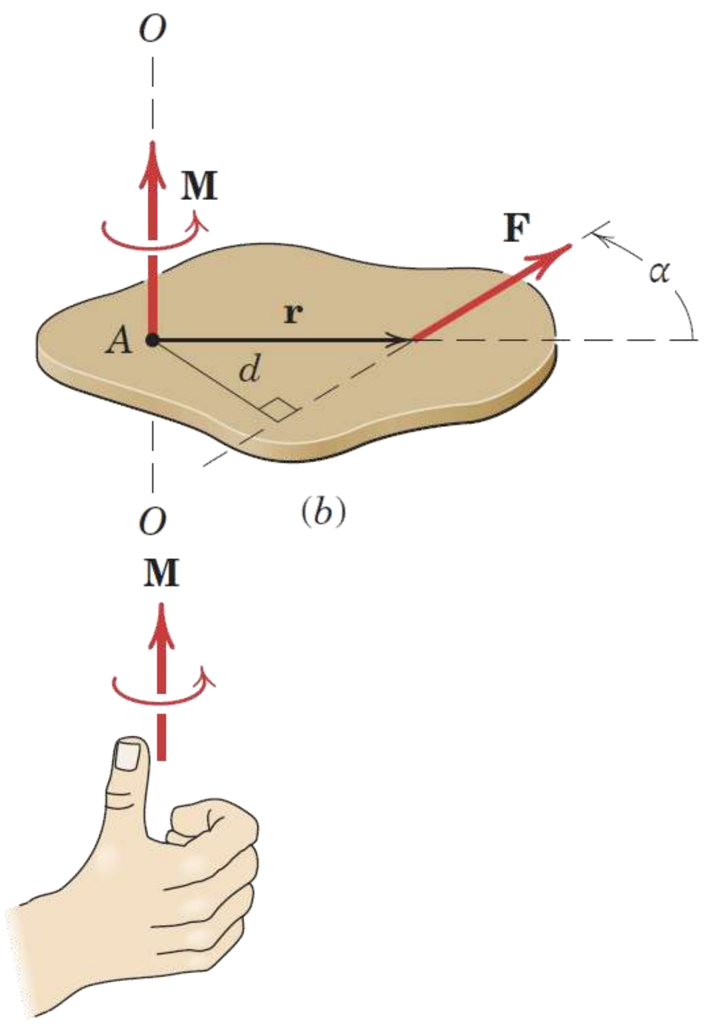
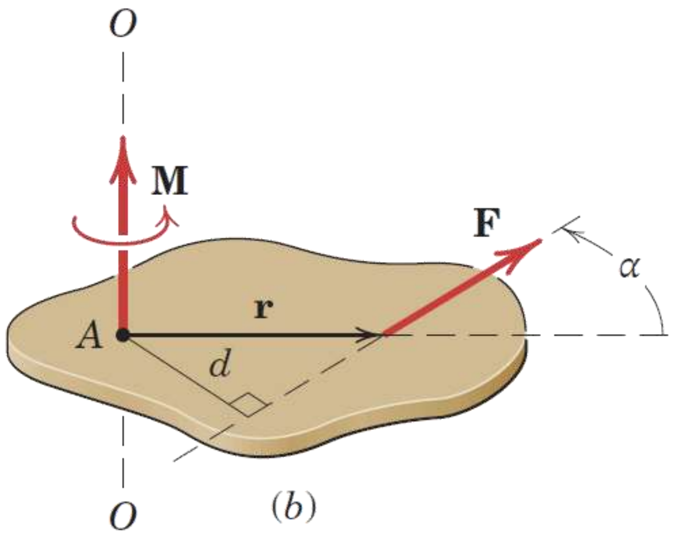
Article 2/4 – Moment about a Point (2 of 3)

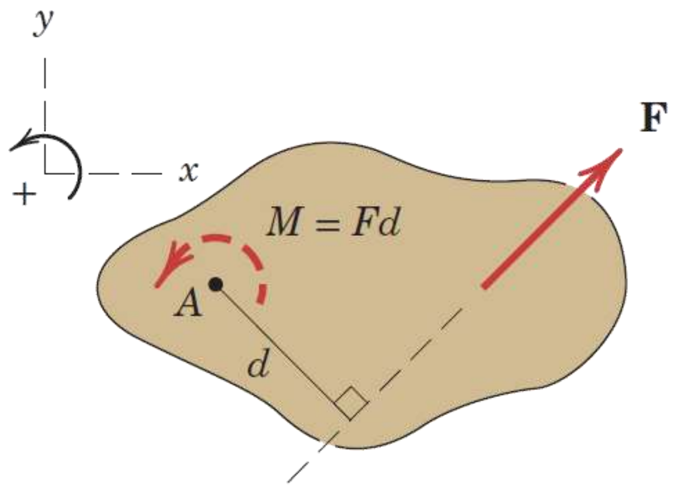
- The Right-Hand Rule
  1. Position your right hand such that your fingers point in the same direction as the force.
  2. Orient your hand such that the point you are computing the moment about is on the same side as your palm. From the figure at right, your hand is positioned such that the moment arm *d* intersects the middle of your palm.
  3. Close your fingers to make a fist and extend your thumb straight up. From the figure at right, imagine closing your fist around line *O*-*O*, and your thumb would point in the direction of the moment vector. Curling your fingers about this line would represent the rotation of the moment about the axis.



Article 2/4 – Moment about a Point (3 of 3)

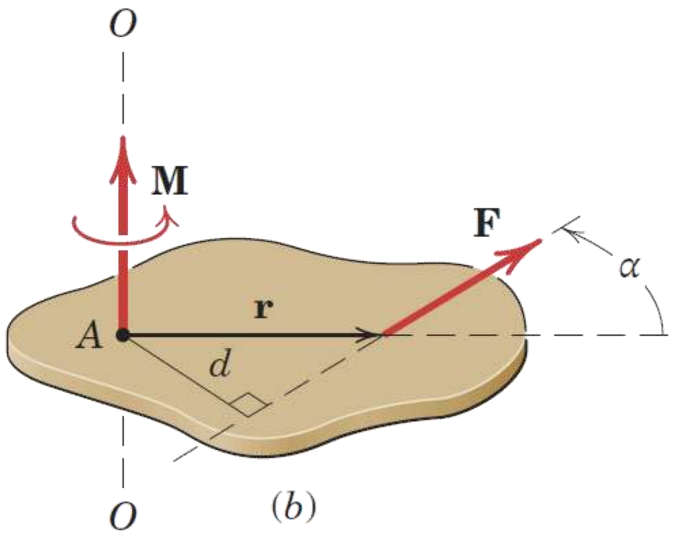
- Planar View
- Sign Conventions
  - Counterclockwise, CCW
  - Clockwise, CW
  - User-Defined
- Two-Dimensional Representation





Article 2/4 – The Cross Product

- Vector Expression for Moments
  - Position Vector, **r**
  - Moment Vector, **M = r** × **F**
- Advantages of the Cross Product
- Disadvantages of the Cross Product



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Article 2/4 – Varignon’s Theorem (1 of 2)

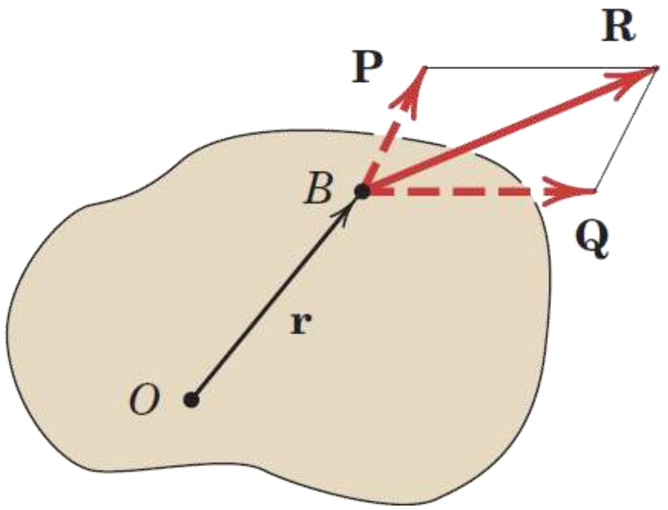
- The Theorem Stated

  The moment of a force about any point is equal to the sum of the moments of the components of the force about the same point.

- The Theorem Illustrated – Vectors

$$\mathbf{R} = \mathbf{P} + \mathbf{Q}$$

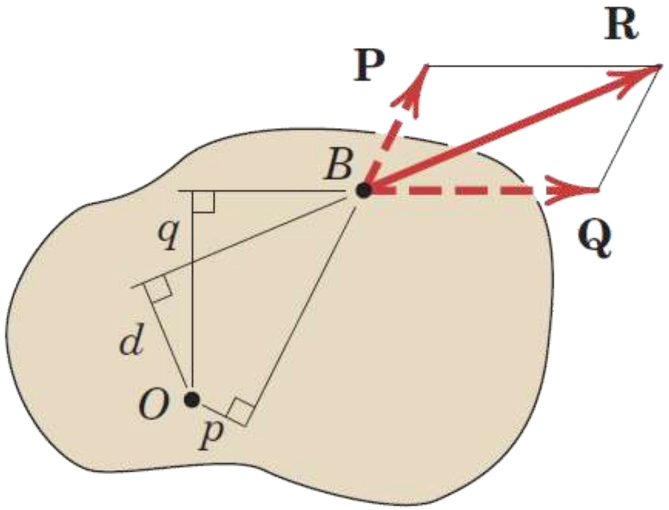
$$\mathbf{M}_O = \mathbf{r} \times \mathbf{R} = \mathbf{r} \times (\mathbf{P} + \mathbf{Q}) = \mathbf{r} \times \mathbf{P} + \mathbf{r} \times \mathbf{Q}$$



Article 2/4 – Varignon's Theorem (1 of 2)

- The Theorem Illustrated – Scalars

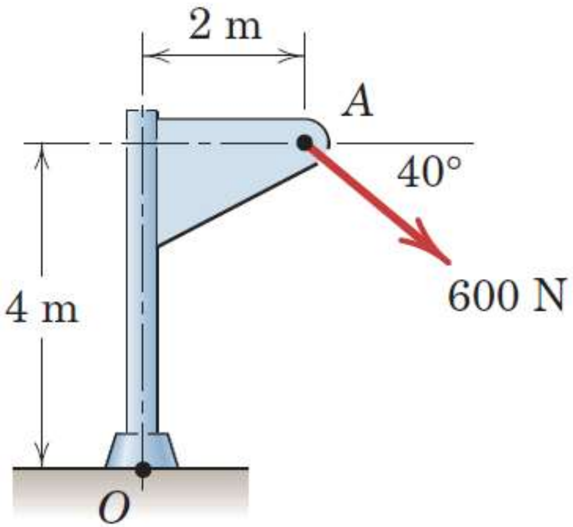
$M_O = Rd = -pP + qQ$ (Assumes CW +)



Article 2/4 – Sample Problem 2/5 (1 of 5)

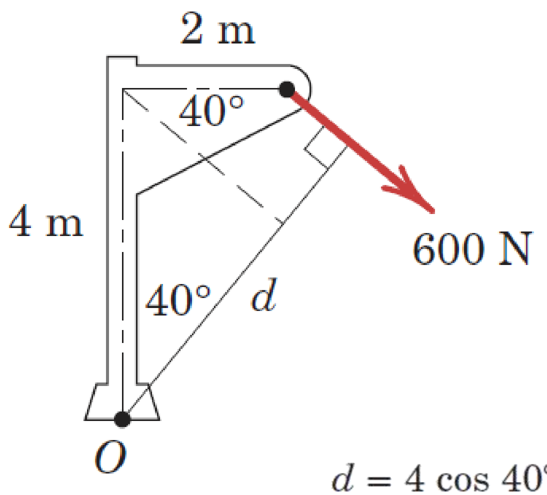
- Problem Statement

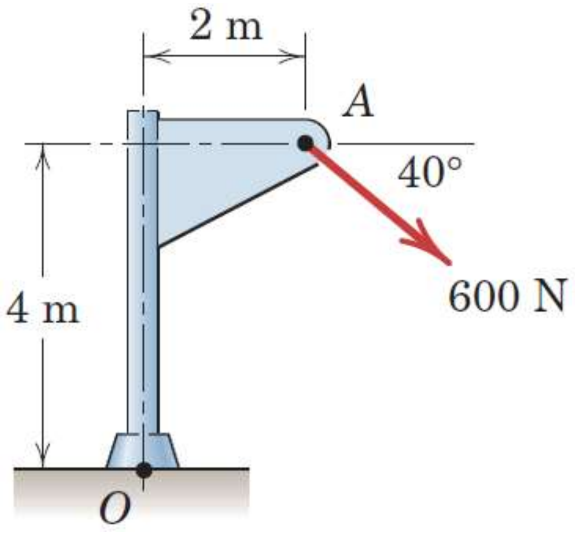
  Calculate the magnitude of the moment about the base point *O* of the 600-N force in five different ways.



Article 2/4 – Sample Problem 2/5 (2 of 5)

- Method 1: Use the Moment Arm (CW is +)





$$d = 4 \cos 40° + 2 \sin 40° = 4.35 \text{ m}$$

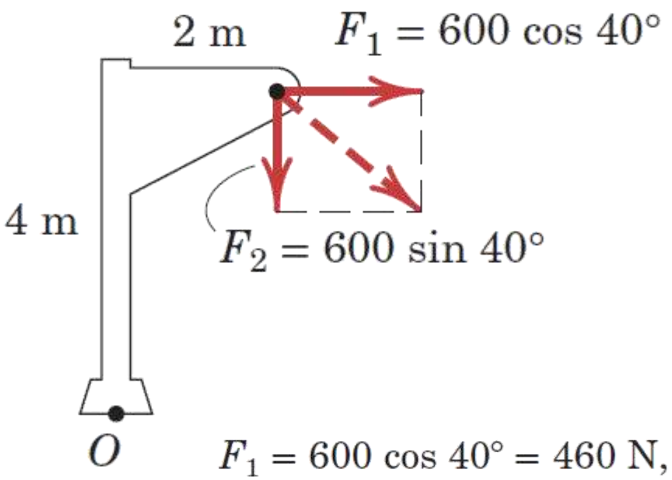
By $M = Fd$ the moment is clockwise and has the magnitude ①

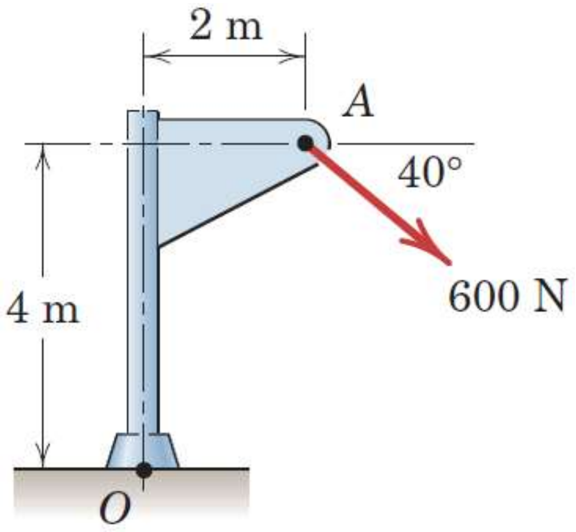
$$M_O = 600(4.35) = 2610 \text{ N}\cdot\text{m} \qquad \textit{Ans.}$$

① The required geometry here and in similar problems should not cause difficulty if the sketch is carefully drawn.

Article 2/4 – Sample Problem 2/5 (3 of 5)

- Method 2: Use Components at *A* (CW is +)





$F_1 = 600 \cos 40° = 460 \text{ N}, \qquad F_2 = 600 \sin 40° = 386 \text{ N}$

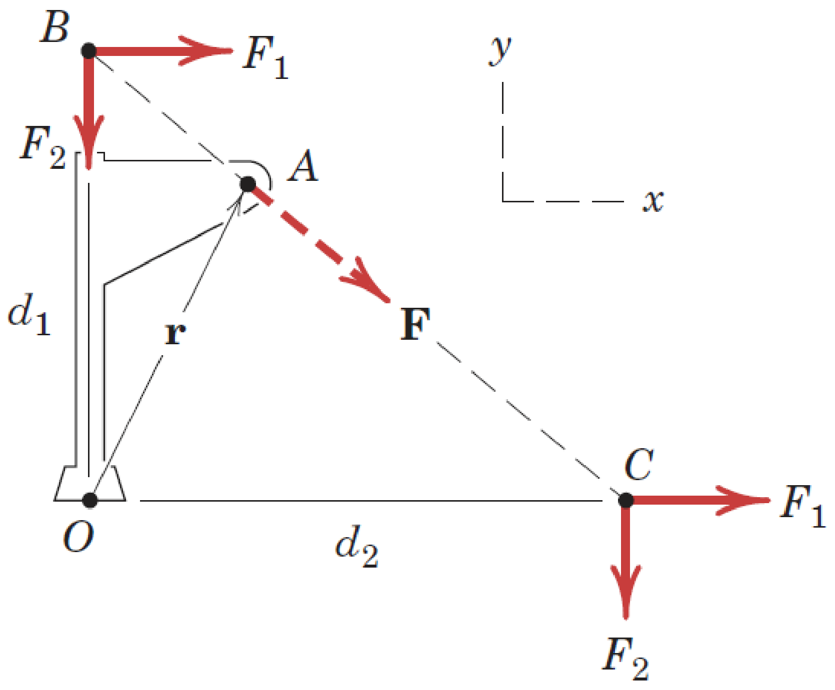
By Varignon's theorem, the moment becomes

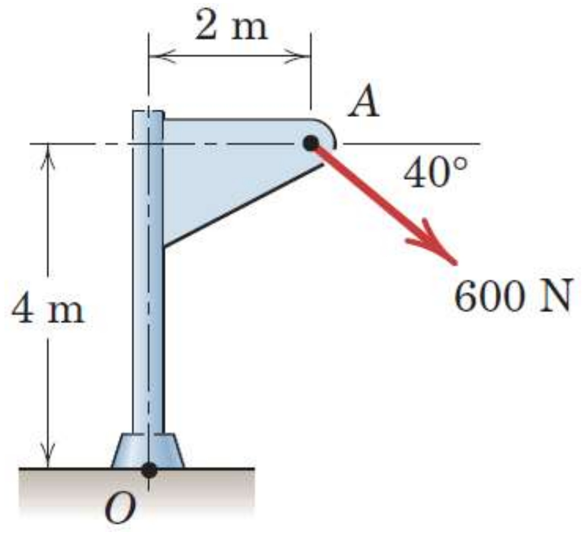
$$M_O = 460(4) + 386(2) = 2610 \text{ N}\cdot\text{m} \quad ② \qquad \textit{Ans.}$$

② This procedure is frequently the shortest approach.

Article 2/4 – Sample Problem 2/5 (4 of 5)

- Methods 3 and 4: Alternative Moment Arms





③ The fact that points B and C are not on the body proper should not cause concern, as the mathematical calculation of the moment of a force does not require that the force be on the body.

$d_1 = 4 + 2 \tan 40° = 5.68$ m

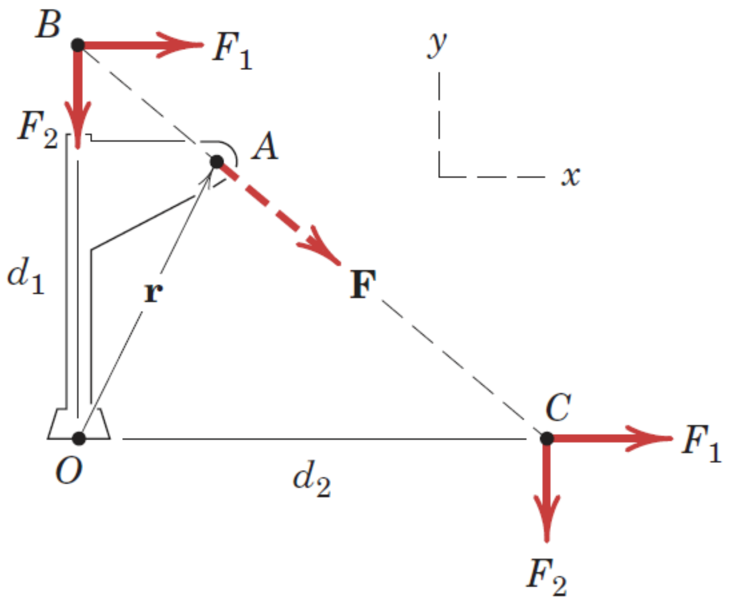
$M_O = 460(5.68) = 2610$ N·m *Ans.*

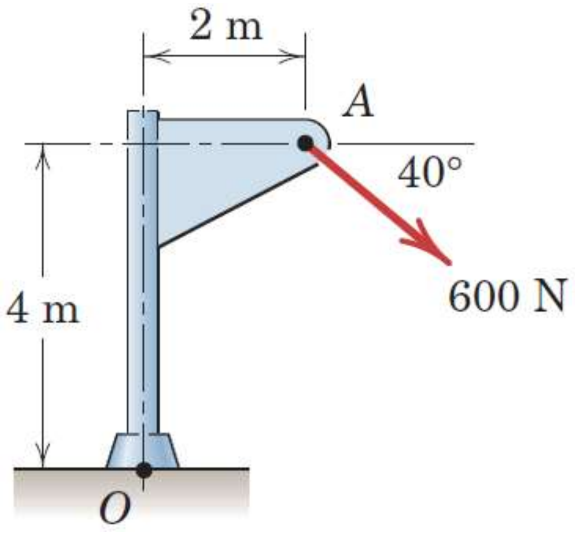
$d_2 = 2 + 4 \cot 40° = 6.77$ m

$M_O = 386(6.77) = 2610$ N·m *Ans.*

Article 2/4 – Sample Problem 2/5 (5 of 5)

- Method 5: Vector Approach





$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = (2\mathbf{i} + 4\mathbf{j}) \times 600(\mathbf{i}\cos 40° - \mathbf{j}\sin 40°) \quad ④$$

$$= -2610\mathbf{k}\ \text{N}\cdot\text{m}$$

④ Alternative choices for the position vector $\mathbf{r}$ are $\mathbf{r} = d_1\mathbf{j} = 5.68\mathbf{j}$ m and $\mathbf{r} = d_2\mathbf{i} = 6.77\mathbf{i}$ m.

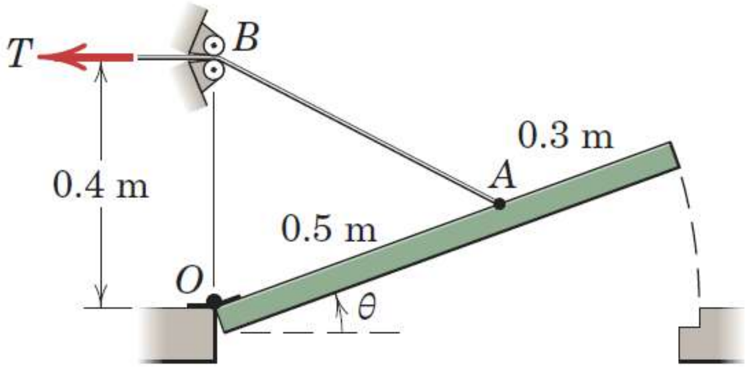
The minus sign indicates that the vector is in the negative z-direction. The magnitude of the vector expression is

$$M_O = 2610\ \text{N}\cdot\text{m} \qquad \textit{Ans.}$$

Article 2/4 – Sample Problem 2/6 (1 of 4)

- Problem Statement

  The trap door *OA* is raised by the cable *AB*, which passes over the small frictionless guide pulleys at *B*. The tension everywhere in the cable is *T*, and this tension applied at *A* causes a moment M_O about the hinge at *O*. Plot the quantity M_O/T as a function of the door elevation angle θ over the range $0 \leq \theta \leq 90°$ and note minimum and maximum values. What is the physical significance of this ratio?



Article 2/4 – Sample Problem 2/6 (2 of 4)

- Tension Vector

$$\mathbf{n}_{AB} = \frac{\mathbf{r}_{AB}}{r_{AB}} = \frac{\mathbf{r}_{OB} - \mathbf{r}_{OA}}{r_{AB}} \quad ①$$

Using the x-y coordinates of our figure, we can write

$$\mathbf{r}_{OB} = 0.4\mathbf{j}\text{ m} \qquad \mathbf{r}_{OA} = 0.5(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})\text{ m} \quad ②$$

$$\mathbf{r}_{AB} = \mathbf{r}_{OB} - \mathbf{r}_{OA} = 0.4\mathbf{j} - (0.5)(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$$

$$= -0.5\cos\theta\mathbf{i} + (0.4 - 0.5\sin\theta)\mathbf{j}\text{ m}$$

So $$r_{AB} = \sqrt{(0.5\cos\theta)^2 + (0.4 - 0.5\sin\theta)^2}$$

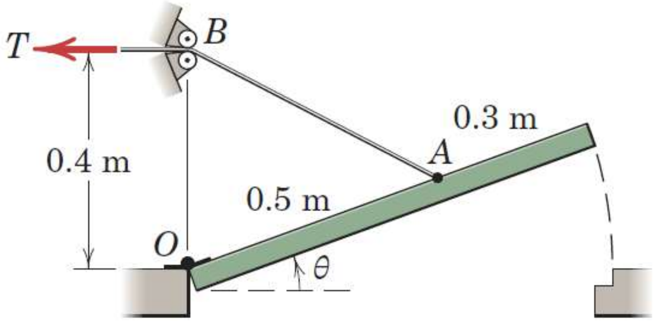
$$= \sqrt{0.41 - 0.4\sin\theta}\text{ m}$$

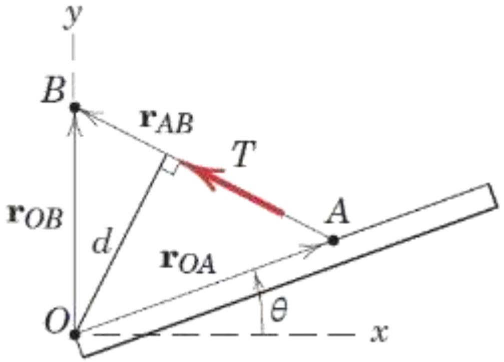
The desired unit vector is

$$\mathbf{n}_{AB} = \frac{\mathbf{r}_{AB}}{r_{AB}} = \frac{-0.5\cos\theta\mathbf{i} + (0.4 - 0.5\sin\theta)\mathbf{j}}{\sqrt{0.41 - 0.4\sin\theta}}$$

Our tension vector can now be written as

$$\mathbf{T} = T\mathbf{n}_{AB} = T\left[\frac{-0.5\cos\theta\mathbf{i} + (0.4 - 0.5\sin\theta)\mathbf{j}}{\sqrt{0.41 - 0.4\sin\theta}}\right]$$





① Recall that any unit vector can be written as a vector divided by its magnitude. In this case the vector in the numerator is a position vector.

② Recall that any vector may be written as a magnitude times an "aiming" unit vector.

Article 2/4 – Sample Problem 2/6 (3 of 4)

- Moment Vector

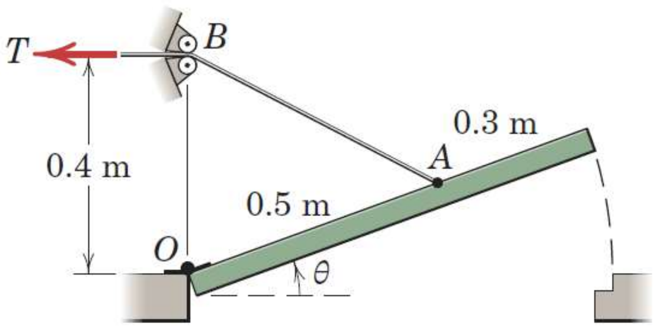
The moment of **T** about point O, as a vector, is $\mathbf{M}_O = \mathbf{r}_{OB} \times \mathbf{T}$, where $\mathbf{r}_{OB} =$ $0.4\mathbf{j}$ m, or ③

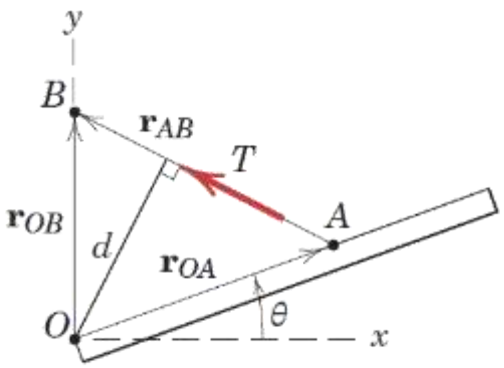
$$\mathbf{M}_O = 0.4\mathbf{j} \times T\left[\frac{-0.5\cos\theta\mathbf{i} + (0.4 - 0.5\sin\theta)\mathbf{j}}{\sqrt{0.41 - 0.4\sin\theta}}\right]$$

$$= \frac{0.2T\cos\theta}{\sqrt{0.41 - 0.4\sin\theta}}\mathbf{k}$$

The magnitude of $\mathbf{M}_O$ is

$$M_O = \frac{0.2T\cos\theta}{\sqrt{0.41 - 0.4\sin\theta}}$$



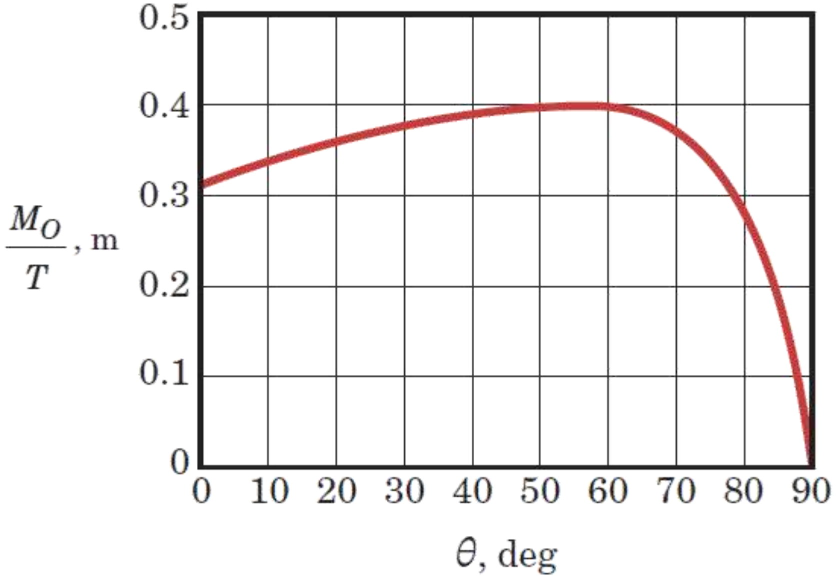


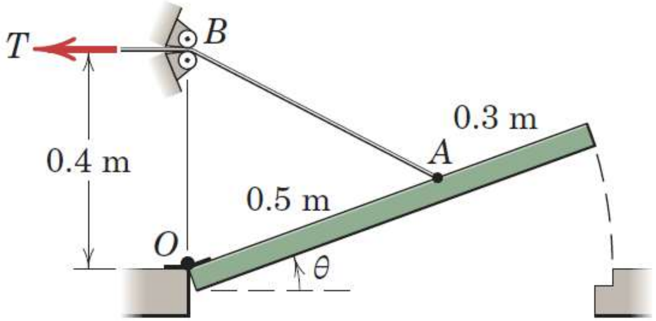
③ In the expression $\mathbf{M} = \mathbf{r} \times \mathbf{F}$, the position vector **r** runs *from* the moment center *to* any point on the line of action of **F**. Here, $\mathbf{r}_{OB}$ is more convenient than $\mathbf{r}_{OA}$.

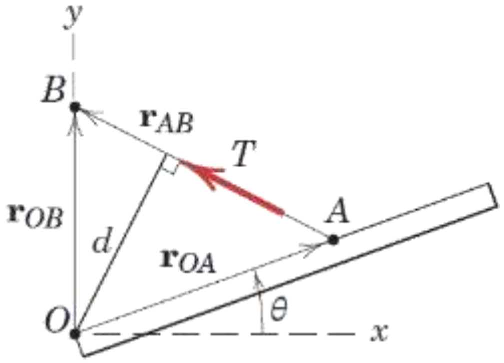
- Desired Expression and Plot

$$\frac{M_O}{T} = \frac{0.2 \cos \theta}{\sqrt{0.41 - 0.4 \sin \theta}} \qquad \textit{Ans.}$$

which is plotted in the accompanying graph. The expression M_O/T is the moment arm d (in meters) which runs from O to the line of action of $\mathbf{T}$. It has a maximum value of 0.4 m at $\theta = 53.1°$ (at which point $\mathbf{T}$ is horizontal) and a minimum value of 0 at $\theta = 90°$ (at which point $\mathbf{T}$ is vertical). The expression is valid even if T varies.







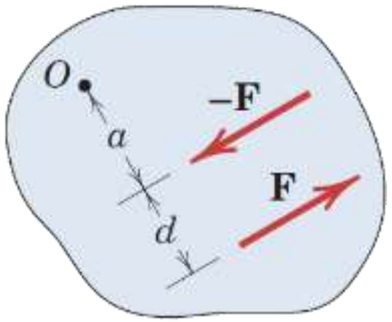
Article 2/5 Couple

- Definition

  The moment produced by two equal, opposite, and noncollinear forces is called a couple.

- Illustration and Derivation (Scalars)

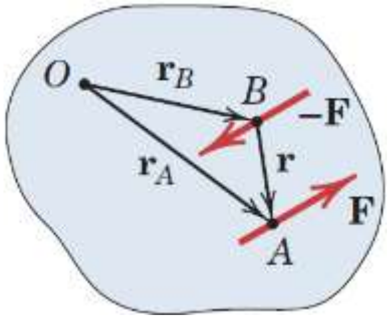
$$M_O = F(a + d) - Fa = Fd$$



- Irrelevance of the Moment Center *O*

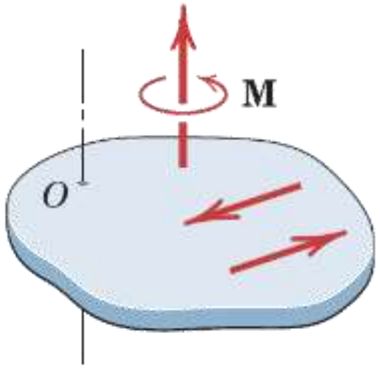
Article 2/5 – Vector Algebra Method

- Illustration and Derivation (Vectors)



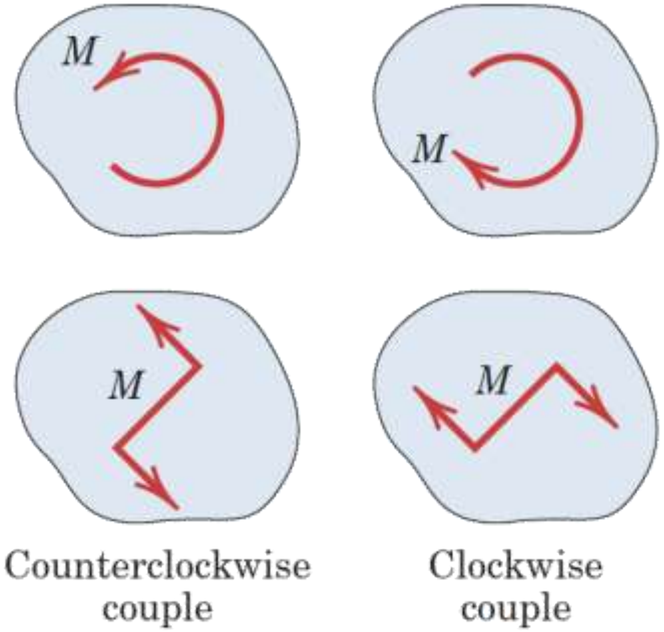
$$\mathbf{M}_O = \mathbf{r}_A \times \mathbf{F} + \mathbf{r}_B \times (-\mathbf{F}) = (\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F} = \mathbf{r} \times \mathbf{F}$$

- The Couple is a Free Vector



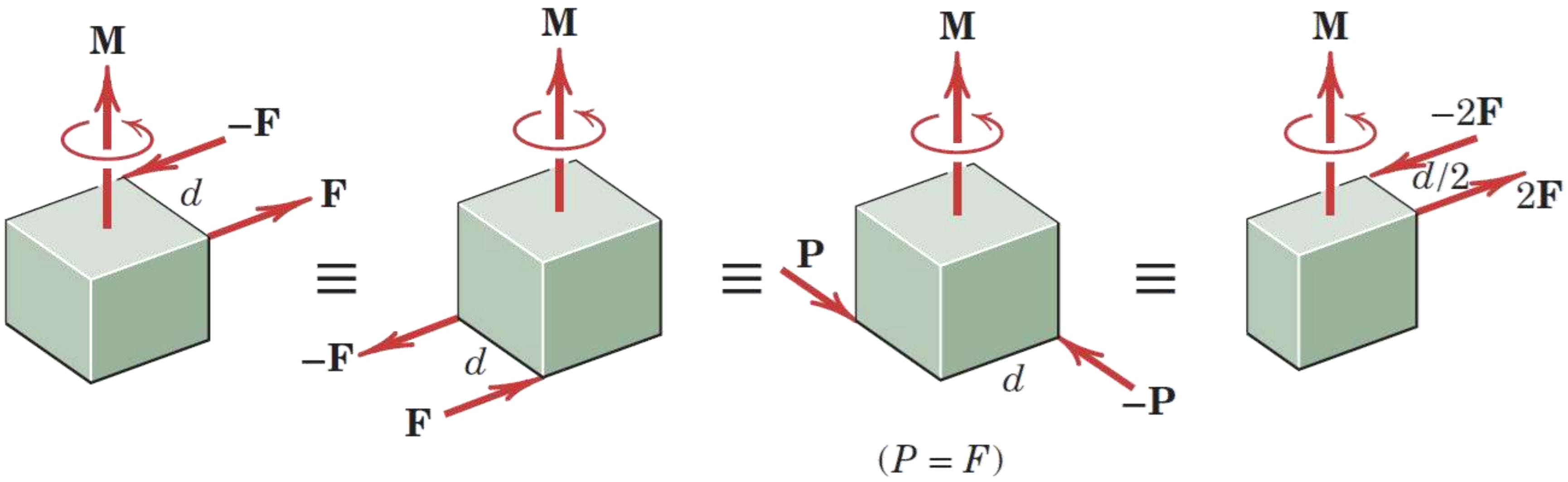
Article 2/5 – Alternative Representations of Couples

- Two-Dimensional Representations



Article 2/5 – Equivalent Couples

- Illustration

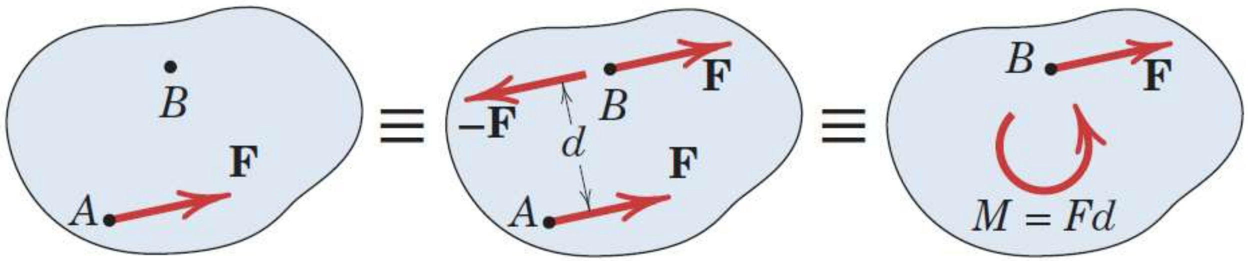


Article 2/5 – Force-Couple Systems (1 of 2)

- Principle of the Force-Couple System

  Any force which acts at a particular location on a body can be replaced by an equivalent force which acts at a different location and a couple.

- Illustration of the Process



Article 2/5 – Force-Couple Systems (2 of 2)

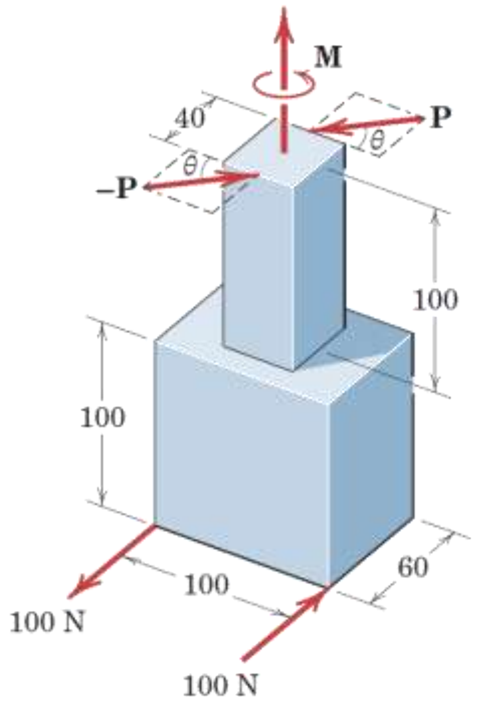
- Steps to Create a Force-Couple System
    1. Write the force as a vector.
    2. Compute the moment or couple which the force creates about the point.
    3. Redraw the force acting at the new location.
    4. Sketch the couple acting at the new location.

- Important Reminder
    - The force-couple system has the same effect on the body which the original force had. It is simply a different way to visualize the effect of the force acting at a new location.

Article 2/5 – Sample Problem 2/7 (1 of 2)

- Problem Statement

  The rigid structural member is subjected to a couple consisting of the two 100-N forces. Replace this couple by an equivalent couple consisting of the two forces **P** and −**P**, each of which has a magnitude of 400 N. Determine the proper angle θ.



Dimensions in millimeters

Article 2/5 – Sample Problem 2/7 (2 of 2)

- Solution

$[M = Fd]$ $\quad\quad M = 100(0.1) = 10\ \text{N}\cdot\text{m}$

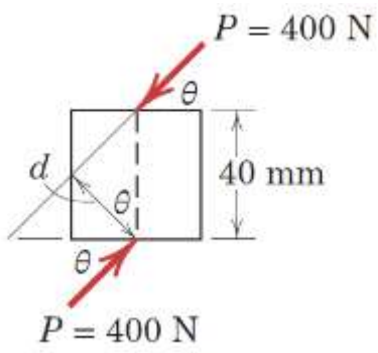
The forces **P** and **–P** produce a counterclockwise couple

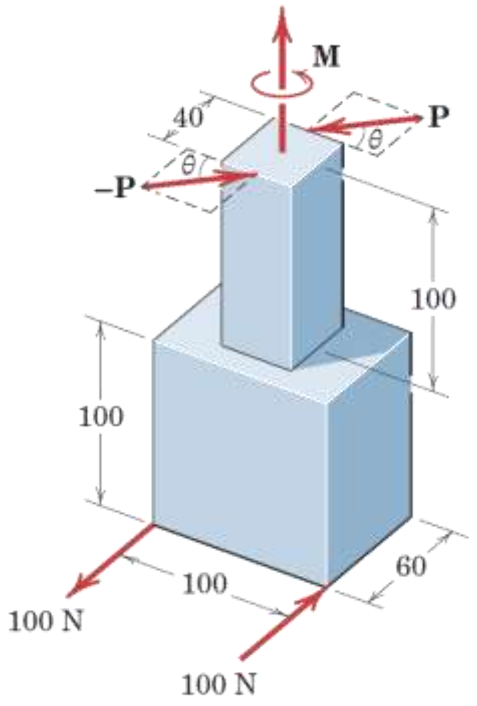
$$M = 400(0.040)\cos\theta$$

Equating the two expressions gives ①

$$10 = (400)(0.040)\cos\theta$$

$$\theta = \cos^{-1}\frac{10}{16} = 51.3^\circ \qquad \textit{Ans.}$$





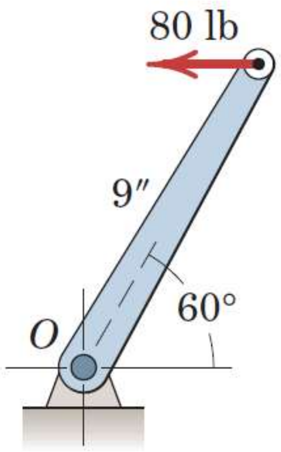
Dimensions in millimeters

① Since the two equal couples are parallel free vectors, the only dimensions which are relevant are those which give the perpendicular distances between the forces of the couples.

Article 2/5 – Sample Problem 2/8 (1 of 2)

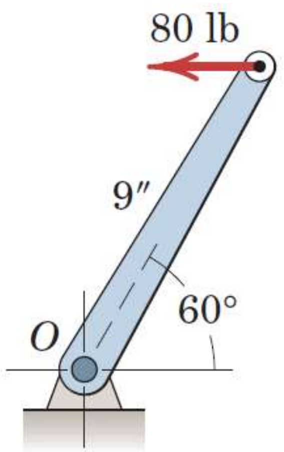
- Problem Statement

  Replace the horizontal 80-lb force acting on the lever by an equivalent system consisting of a force at *O* and a couple.



Article 2/5 – Sample Problem 2/8 (2 of 2)

- Solution

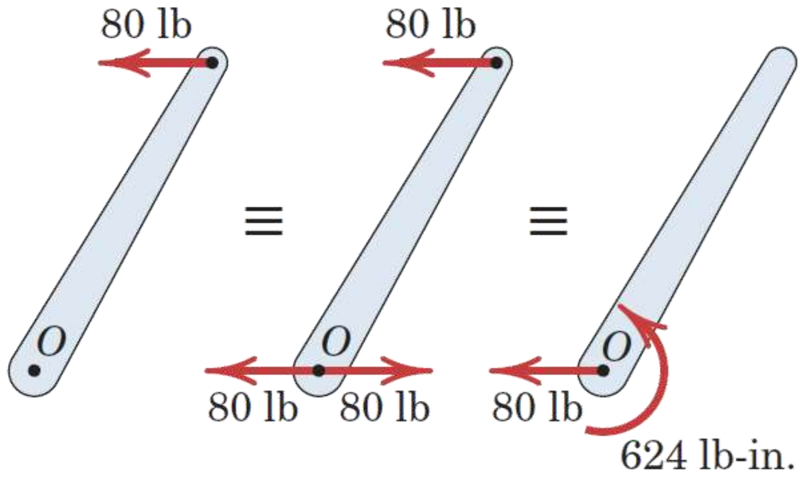


[$M = Fd$] $\quad M = 80(9 \sin 60°) = 624$ lb-in. *Ans.*

Thus, the original force is equivalent to the 80-lb force at O and the 624-lb-in. couple as shown in the third of the three equivalent figures. ①

HELPFUL HINT

① The reverse of this problem is often encountered, namely, the replacement of a force and a couple by a single force. Proceeding in reverse is the same as replacing the couple by two forces, one of which is equal and opposite to the 80-lb force at O. The moment arm to the second force would be $M/F = 624/80 = 7.79$ in., which is 9 sin 60°, thus determining the line of action of the single resultant force of 80 lb.



Article 2/6 Resultants

- Definition

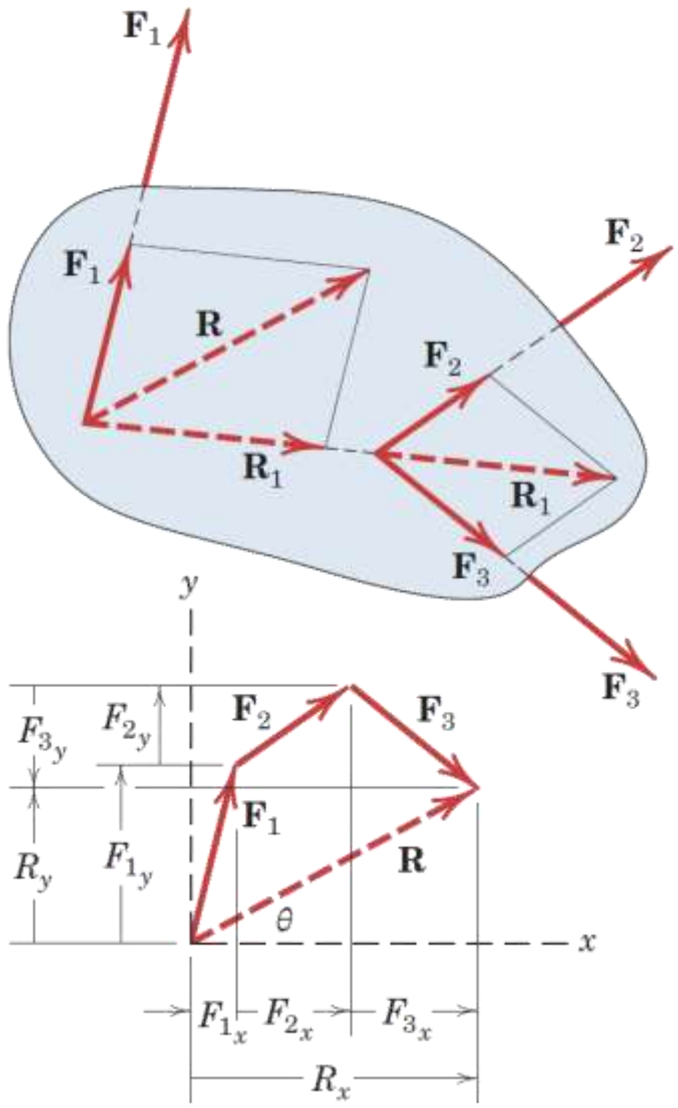
  The *resultant* of a system of forces is the simplest force combination which can replace the original forces without altering the external effect on the rigid body to which the forces are applied.

- Equilibrium Condition

- Nonequilibrium Condition

Article 2/6 – Planar Force System

- Illustration



- Equations of Interest

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots = \Sigma\mathbf{F}$$

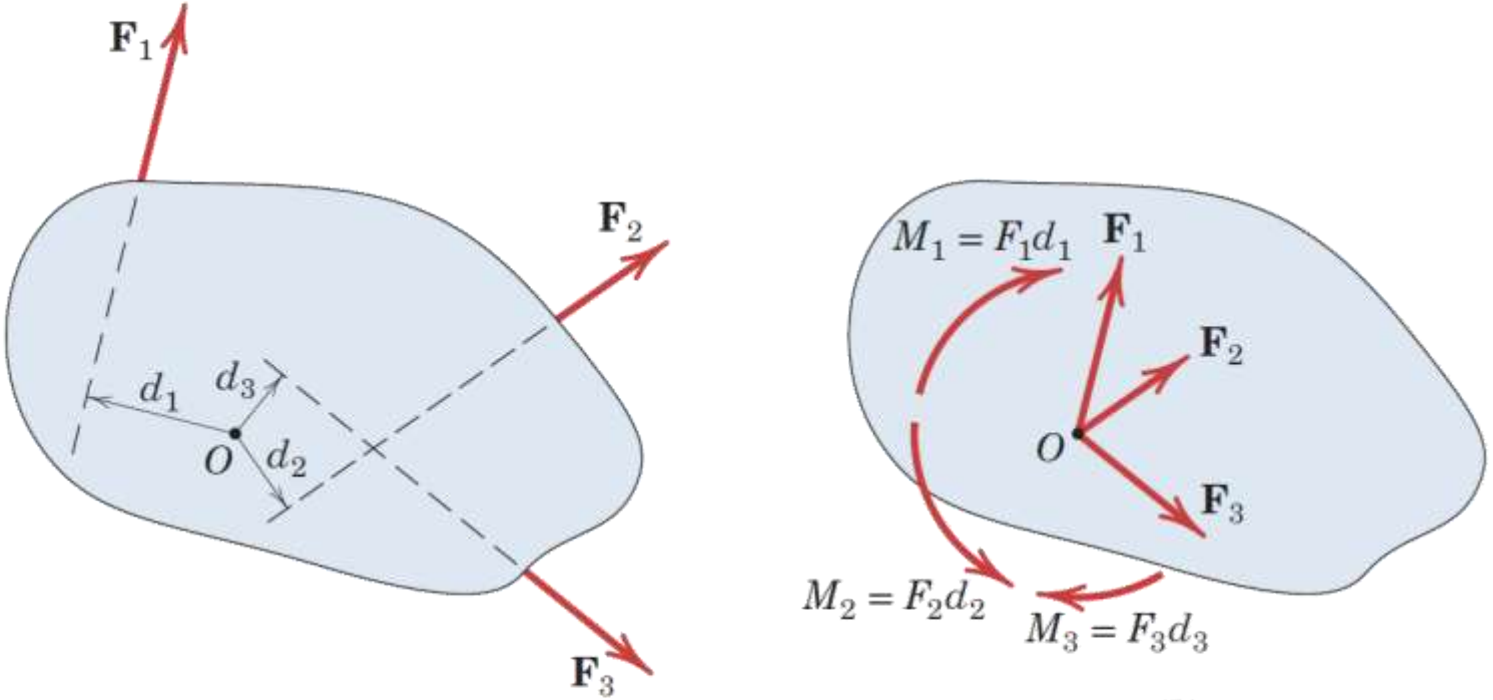
$$R_x = \Sigma F_x \qquad R_y = \Sigma F_y \qquad R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$$

$$\theta = \tan^{-1}\frac{R_y}{R_x} = \tan^{-1}\frac{\Sigma F_y}{\Sigma F_x}$$

Article 2/6 – Algebraic Method (1 of 3)

- Finding the Resultant and Line of Action

  1. Choose a convenient reference point and move all forces to that point. This process is depicted for a three-force system in Figs. (*a*) and (*b*) below where M_1, M_2, and M_3 are the couples resulting from the transfer of forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ from their respective original lines of action to lines of action through point O.

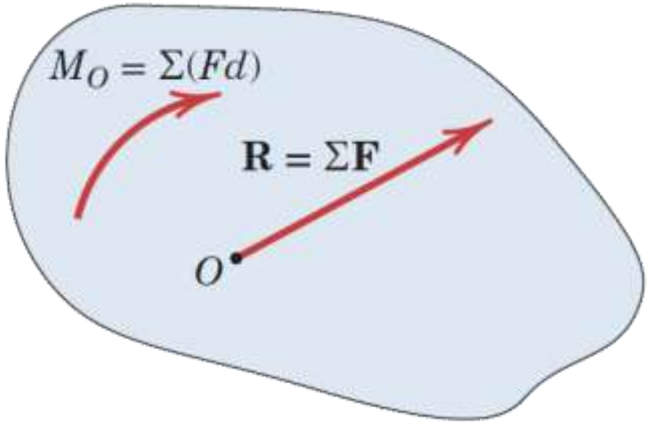


(*a*) (*b*)

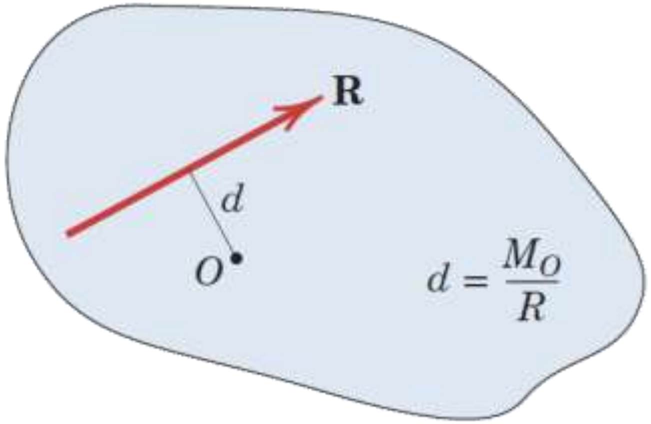
Article 2/6 – Algebraic Method (2 of 3)

- Finding the Resultant and Line of Action (cont.)

  2. Add all forces at O to form the resultant force **R**, and add all couples to form the resultant couple M_O. We now have the single force–couple system, as shown below in Fig. (*c*).
  3. Find the line of action of **R** by requiring **R** to have a moment of M_O about point O. Note that the force system in Fig. (*d*) is equivalent to the initial force system from Fig. (a) and that $\Sigma(Fd)$ in Fig. (*a*) is equal to Rd in Fig. (*d*).



(*c*)



(*d*)

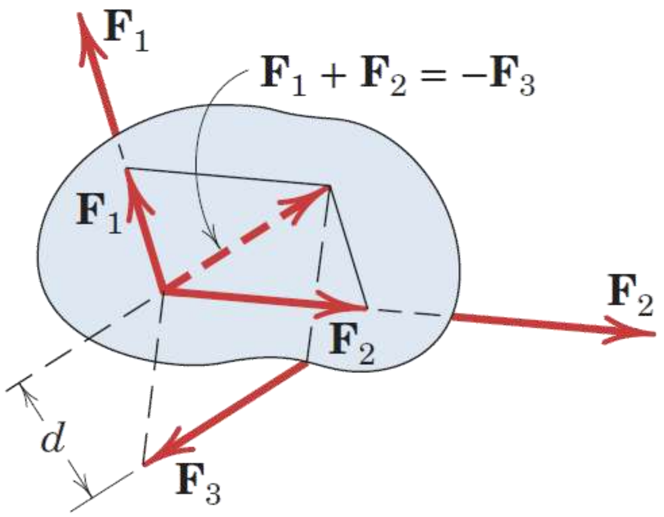
Article 2/6 – Algebraic Method (3 of 3)

- Equations of Interest
- Concurrent Force System
- Parallel Force System
- Zero-Resultant Force System

$$\mathbf{R} = \Sigma \mathbf{F}$$

$$M_O = \Sigma M = \Sigma(Fd)$$

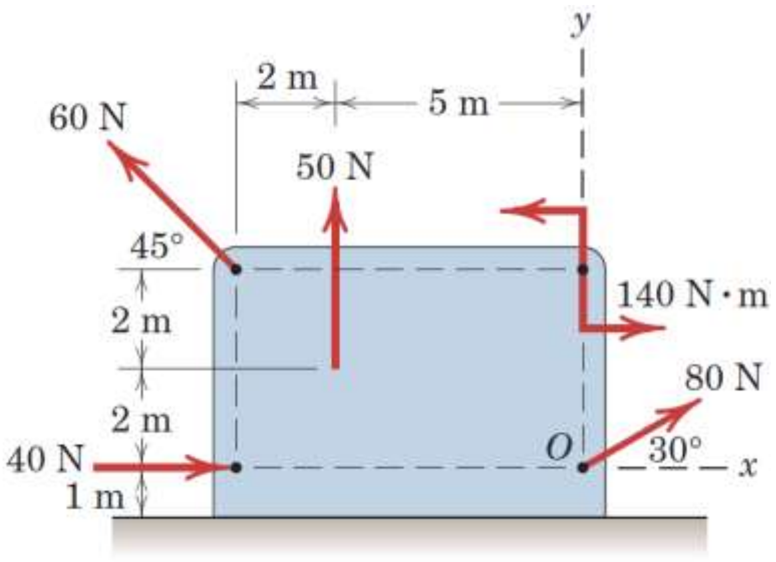
$$Rd = M_O$$



Article 2/6 – Sample Problem 2/9 (1 of 4)

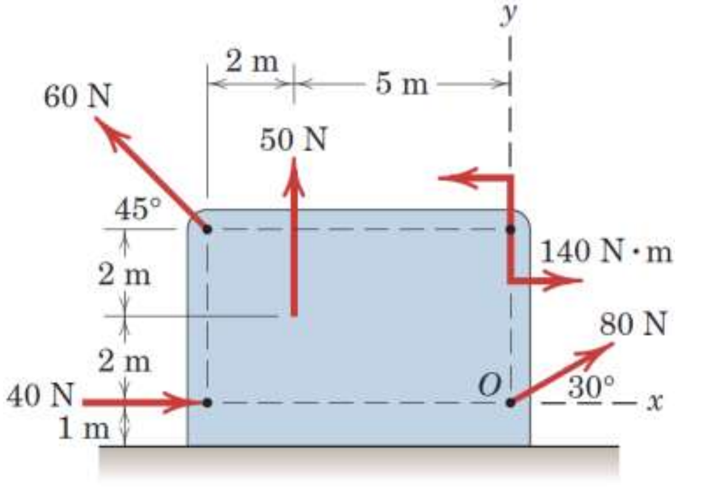
- Problem Statement

  Determine the resultant of the four forces and one couple which act on the plate shown.



Article 2/6 – Sample Problem 2/9 (2 of 4)

- Equivalent Force-Couple System



$[R_x = \Sigma F_x]$ $\quad R_x = 40 + 80 \cos 30° - 60 \cos 45° = 66.9 \text{ N}$

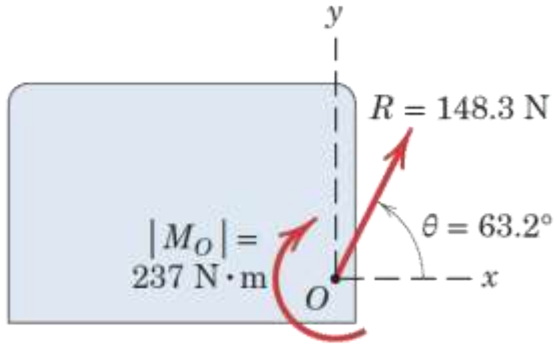
$[R_y = \Sigma F_y]$ $\quad R_y = 50 + 80 \sin 30° + 60 \cos 45° = 132.4 \text{ N}$

$[R = \sqrt{R_x^{\,2} + R_y^{\,2}}]$ $\quad R = \sqrt{(66.9)^2 + (132.4)^2} = 148.3 \text{ N}$ *Ans.*

$\left[\theta = \tan^{-1} \dfrac{R_y}{R_x}\right]$ $\quad \theta = \tan^{-1} \dfrac{132.4}{66.9} = 63.2°$ *Ans.*

$[M_O = \Sigma(Fd)]$ $\quad M_O = 140 - 50(5) + 60 \cos 45°(4) - 60 \sin 45°(7)$ ①

$= -237 \text{ N·m}$



① We note that the choice of point O as a moment center eliminates any moments due to the two forces which pass through O. Had the clockwise sign convention been adopted, M_O would have been +237 N·m, with the plus sign indicating a sense which agrees with the sign convention. Either sign convention, of course, leads to the conclusion of a clockwise moment M_O.

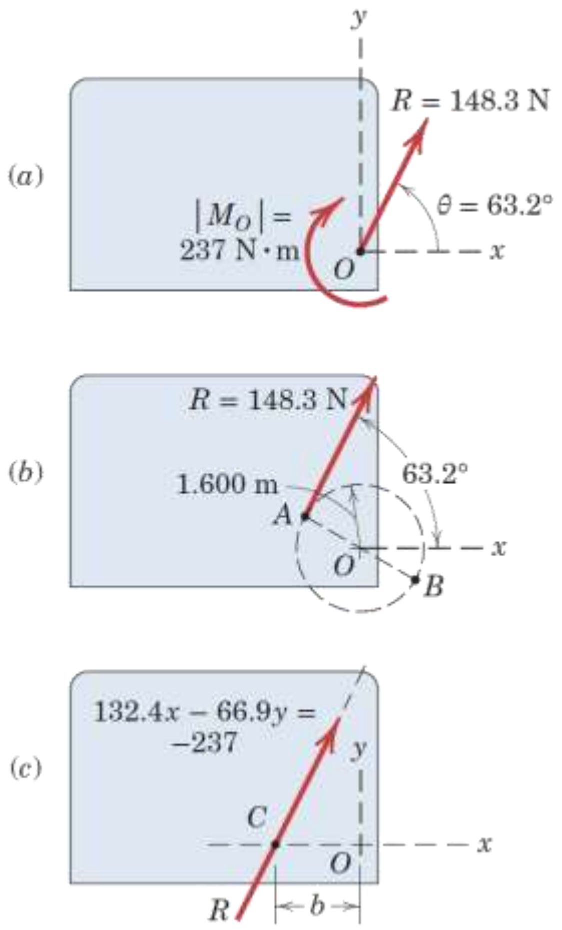
Article 2/6 – Sample Problem 2/9 (3 of 4)

- Line of Action for the Resultant

$[Rd = |M_O|]$ $\qquad 148.3d = 237 \qquad d = 1.600\text{ m} \qquad$ *Ans.*

- Alternative Solution (Point *C* on *x*-axis)

$$R_y b = |M_O| \qquad \text{and} \qquad b = \frac{237}{132.4} = 1.792\text{ m}$$



Article 2/6 – Sample Problem 2/9 (4 of 4)

- Vector Approach for the Line of Action

$$\mathbf{r} \times \mathbf{R} = \mathbf{M}_O$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ is a position vector running from point O to any point on the line of action of $\mathbf{R}$. Substituting the vector expressions for $\mathbf{r}$, $\mathbf{R}$, and $\mathbf{M}_O$ and carrying out the cross product result in

$$(x\mathbf{i} + y\mathbf{j}) \times (66.9\mathbf{i} + 132.4\mathbf{j}) = -237\mathbf{k}$$

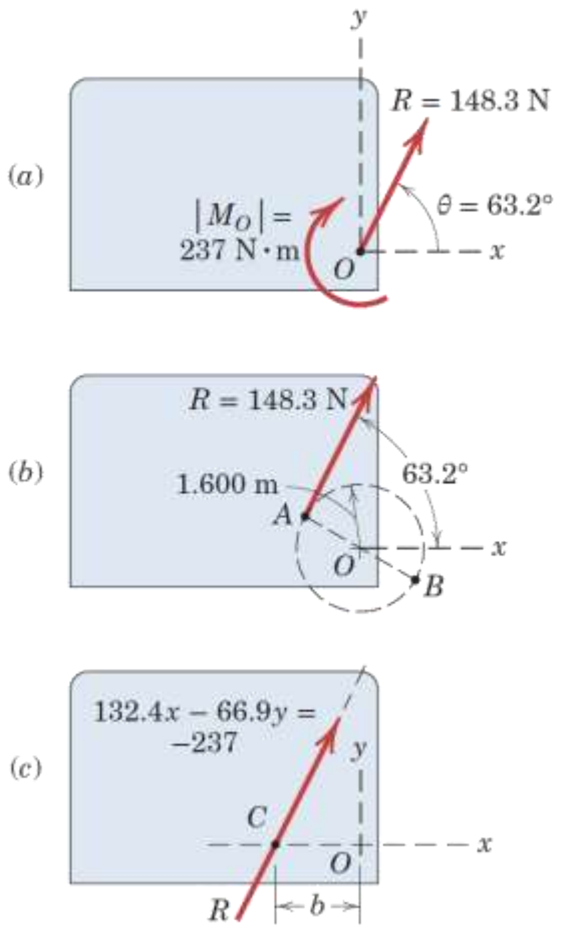
$$(132.4x - 66.9y)\mathbf{k} = -237\mathbf{k}$$

Thus, the desired line of action, Fig. c, is given by

$$132.4x - 66.9y = -237$$

By setting $y = 0$, we obtain $x = -1.792$ m, which agrees with our earlier calculation of the distance b. ②

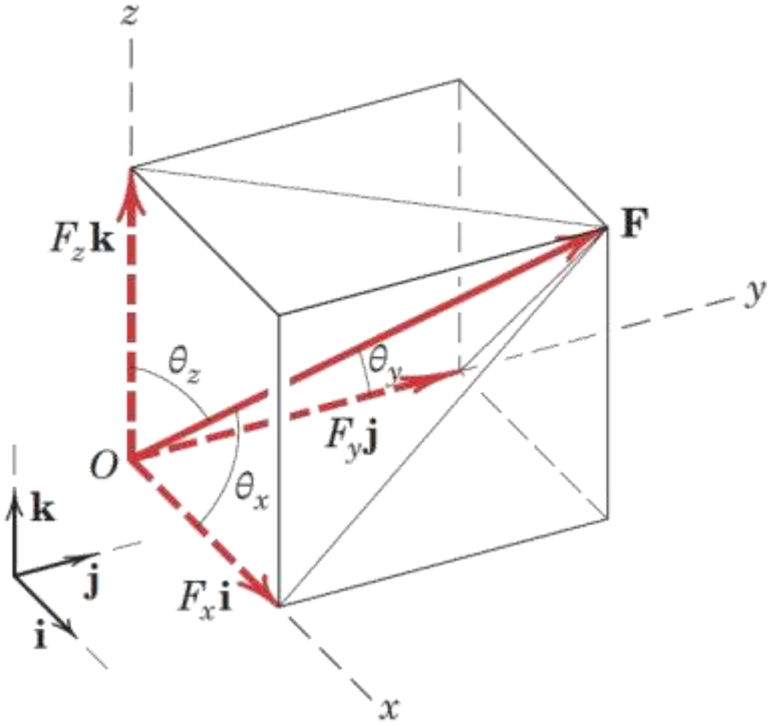
② Note that the vector approach yields sign information automatically, whereas the scalar approach is more physically oriented. You should master both methods.



Article 2/7 Rectangular Components (3D)

- Illustration and Equations of Interest

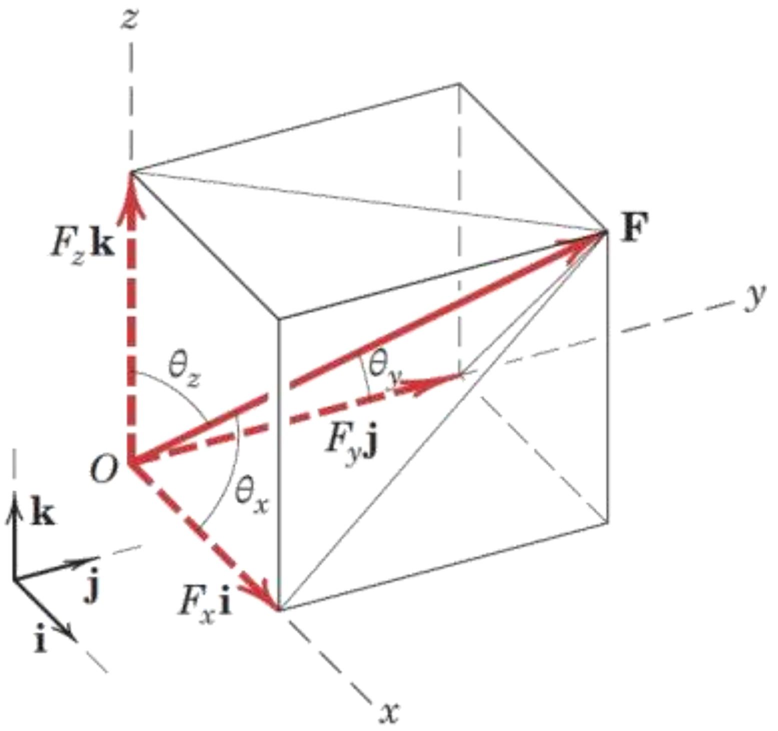
$F_x = F \cos \theta_x$	$F = \sqrt{F_x{}^2 + F_y{}^2 + F_z{}^2}$
$F_y = F \cos \theta_y$	$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$
$F_z = F \cos \theta_z$	$\mathbf{F} = F(\mathbf{i} \cos \theta_x + \mathbf{j} \cos \theta_y + \mathbf{k} \cos \theta_z)$



Article 2/7 – Rectangular Components (cont.)

- Magnitude and Direction Format
  - $\mathbf{F} = F\mathbf{n}_f$ where $\mathbf{n}_f$ is a unit vector in the direction of $\mathbf{F}$.
  - $\mathbf{n}_f = \cos\theta_x\mathbf{i} + \cos\theta_y\mathbf{j} + \cos\theta_z\mathbf{k}$

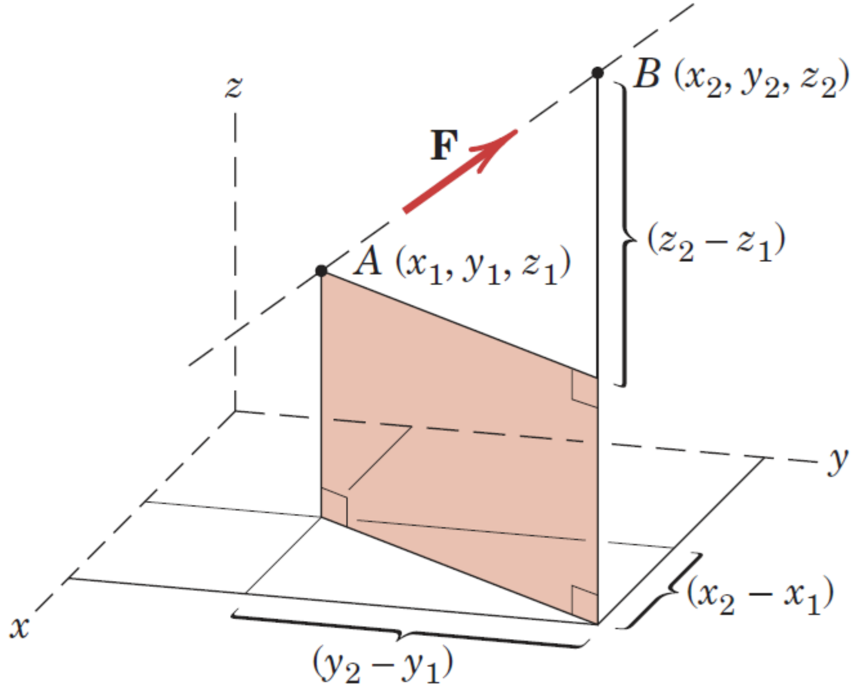
- Direction Cosine Format
  - $\mathbf{F} = F\mathbf{n}_f = F(l\mathbf{i} + m\mathbf{j} + n\mathbf{k})$
  - $l = \cos\theta_x$
  - $m = \cos\theta_y$
  - $n = \cos\theta_z$
  - $l^2 + m^2 + n^2 = 1$



Article 2/7 – Writing Vector Components (1 of 2)

- Specification by two points on the line of action of the force.

$$\mathbf{F} = F\mathbf{n}_F = F\frac{\overrightarrow{AB}}{AB} = F\frac{(x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$$



Article 2/7 – Writing Vector Components (2 of 2)

- Specification by two angles which orient the line of action of the force.

  - Horizontal and Vertical Components

    $$F_{xy} = F \cos \phi$$

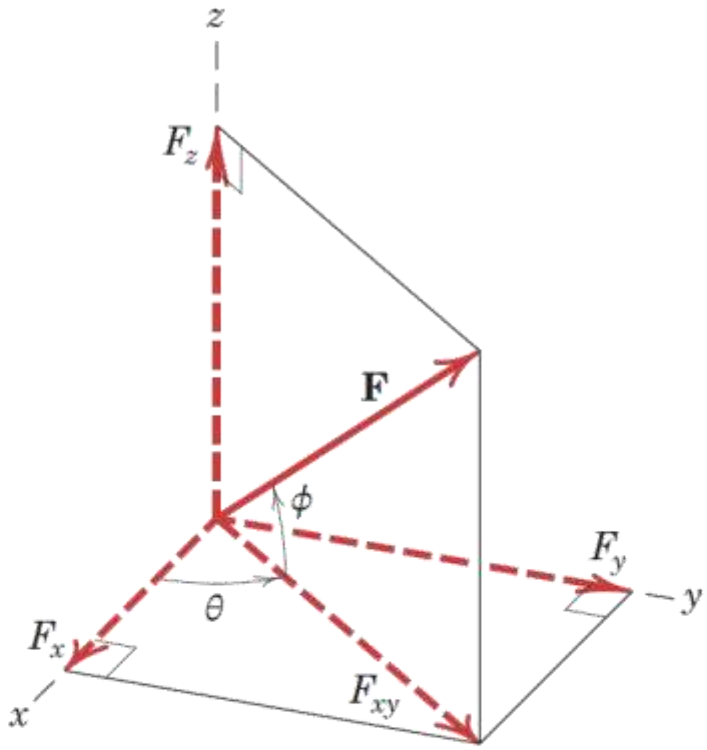
    $$F_z = F \sin \phi$$

  - x- and y-Components

    $$F_x = F_{xy} \cos \theta = F \cos \phi \cos \theta$$

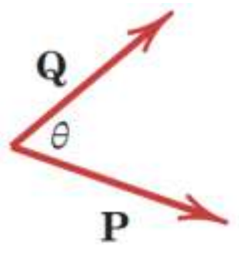
    $$F_y = F_{xy} \sin \theta = F \cos \phi \sin \theta$$

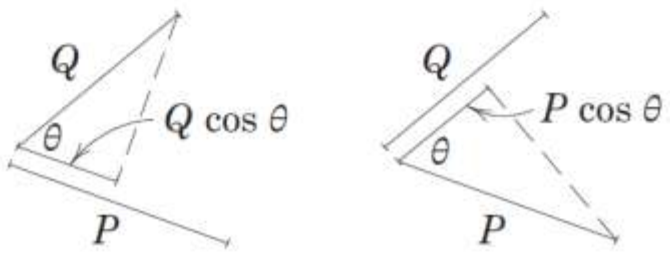
  - Other Combinations of Angles



Article 2/7 – The Dot Product

- Definitions and Illustration
  - $\mathbf{P}\cdot\mathbf{Q} = \mathbf{Q}\cdot\mathbf{P} = PQ\cos\theta$
  - $\theta = \cos^{-1}(\mathbf{P}\cdot\mathbf{Q}\,/PQ)$





- Mathematical Features of the Dot Product

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$$

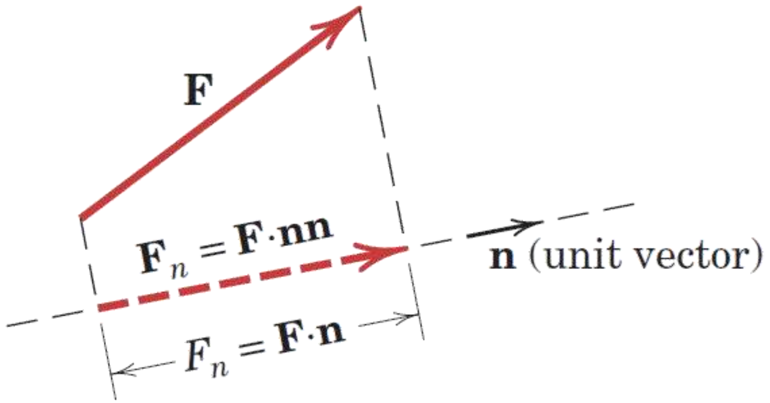
$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{i} = \mathbf{i}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{j} = 0$$

$$\begin{aligned}\mathbf{P}\cdot\mathbf{Q} &= (P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k})\cdot(Q_x\mathbf{i} + Q_y\mathbf{j} + Q_z\mathbf{k}) \\ &= P_xQ_x + P_yQ_y + P_zQ_z\end{aligned}$$

$$\mathbf{P}\cdot\mathbf{P} = P_x^{\,2} + P_y^{\,2} + P_z^{\,2}$$

Article 2/7 – Finding Projections of Forces onto Lines

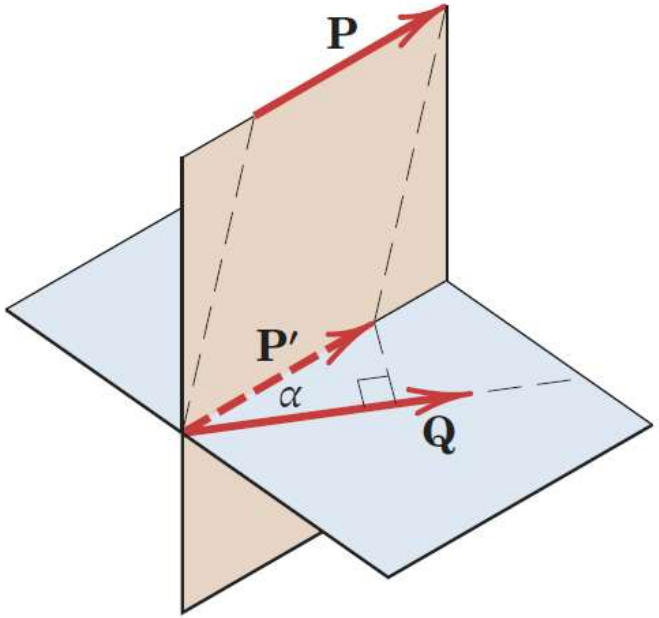
- Scalar Projection of a Force onto a line, F_n
  1. Write the force as a vector.
  2. Write a unit vector in the direction of the line.
  3. Dot the force vector with the unit vector.



- Vector Projection of a Force, $\mathbf{F}_n$
  1. Write the scalar projection of the force onto the line.
  2. Multiply the scalar projection by the unit vector for the line.

Article 2/7 – Finding the Angle between Two Vectors

- Dot Product Reminder
  - $\mathbf{P}\cdot\mathbf{Q} = \mathbf{Q}\cdot\mathbf{P} = PQ\cos\alpha$
  - $\alpha = \cos^{-1}(\mathbf{P}\cdot\mathbf{Q}\ / PQ)$



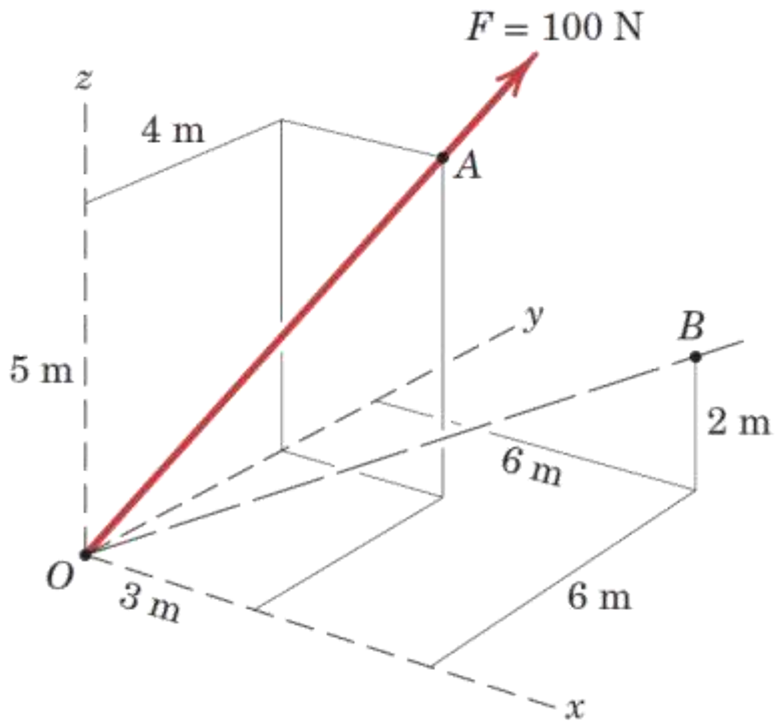
- Solution Steps
  1. Write each vector.
  2. Take a dot product between the vectors.
  3. Divide the dot product by the product of the magnitudes of the vectors.
  4. Take the inverse cosine of this ratio.

- This process is more easily carried out with unit vectors.

Article 2/7 – Sample Problem 2/10 (1 of 4)

- Problem Statement

A force **F** with a magnitude of 100 N is applied at the origin *O* of the axes *x*-*y*-*z* as shown. The line of action of **F** passes through a point *A* whose coordinates are 3 m, 4 m, and 5 m. Determine (*a*) the *x*, *y*, and *z* scalar components of **F**, (*b*) the projection F_{xy} of **F** on the *x*-*y* plane, and (*c*) the projection F_{OB} of **F** along the line *OB*.



Article 2/7 – Sample Problem 2/10 (2 of 4)

- Scalar Components of *F*

$$\mathbf{F} = F\mathbf{n}_{OA} = F\frac{\overrightarrow{OA}}{\overline{OA}} = 100\left[\frac{3\mathbf{i}+4\mathbf{j}+5\mathbf{k}}{\sqrt{3^2+4^2+5^2}}\right]$$

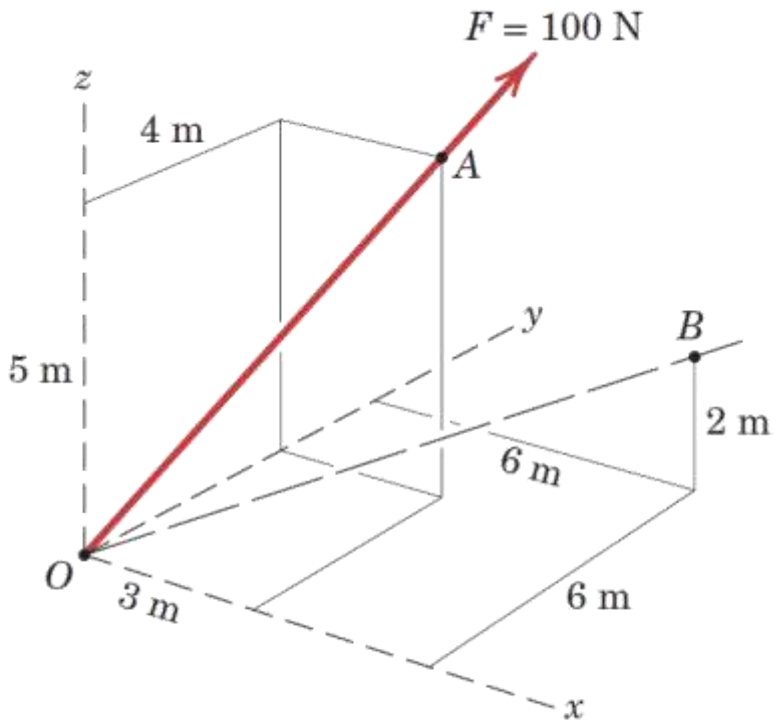
$$= 100[0.424\mathbf{i} + 0.566\mathbf{j} + 0.707\mathbf{k}]$$

$$= 42.4\mathbf{i} + 56.6\mathbf{j} + 70.7\mathbf{k}\ \text{N}$$

The desired scalar components are thus

$$F_x = 42.4\ \text{N} \qquad F_y = 56.6\ \text{N} \qquad F_z = 70.7\ \text{N} \quad ① \qquad \textit{Ans.}$$

① In this example all scalar components are positive. Be prepared for the case where a direction cosine, and hence the scalar component, is negative.

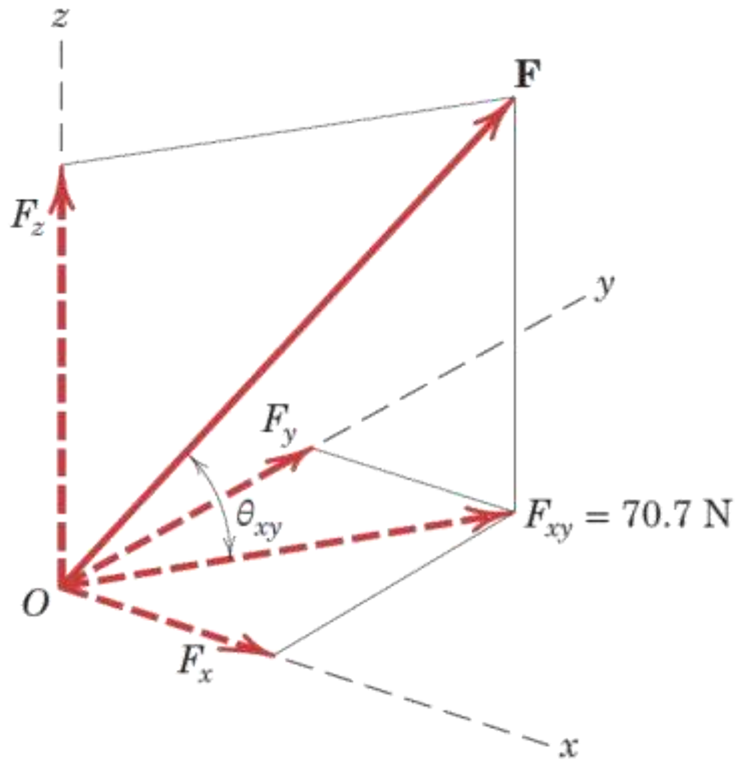


Article 2/7 – Sample Problem 2/10 (3 of 4)

- Projection of *F* into the *x*-*y* Plane

$$\cos\theta_{xy} = \frac{\sqrt{3^2+4^2}}{\sqrt{3^2+4^2+5^2}} = 0.707$$

so that $F_{xy} = F\cos\theta_{xy} = 100(0.707) = 70.7\text{ N}$ *Ans.*



Article 2/7 – Sample Problem 2/10 (4 of 4)

- Projection of *F* onto Line *OB*

$$\mathbf{n}_{OB} = \frac{\overrightarrow{OB}}{\overline{OB}} = \frac{6\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}}{\sqrt{6^2 + 6^2 + 2^2}} = 0.688\mathbf{i} + 0.688\mathbf{j} + 0.229\mathbf{k}$$

The scalar projection of $\mathbf{F}$ on OB is

$$F_{OB} = \mathbf{F}\cdot\mathbf{n}_{OB} = (42.4\mathbf{i} + 56.6\mathbf{j} + 70.7\mathbf{k})\cdot(0.688\mathbf{i} + 0.688\mathbf{j} + 0.229\mathbf{k}) \quad ②$$

$$= (42.4)(0.688) + (56.6)(0.688) + (70.7)(0.229)$$

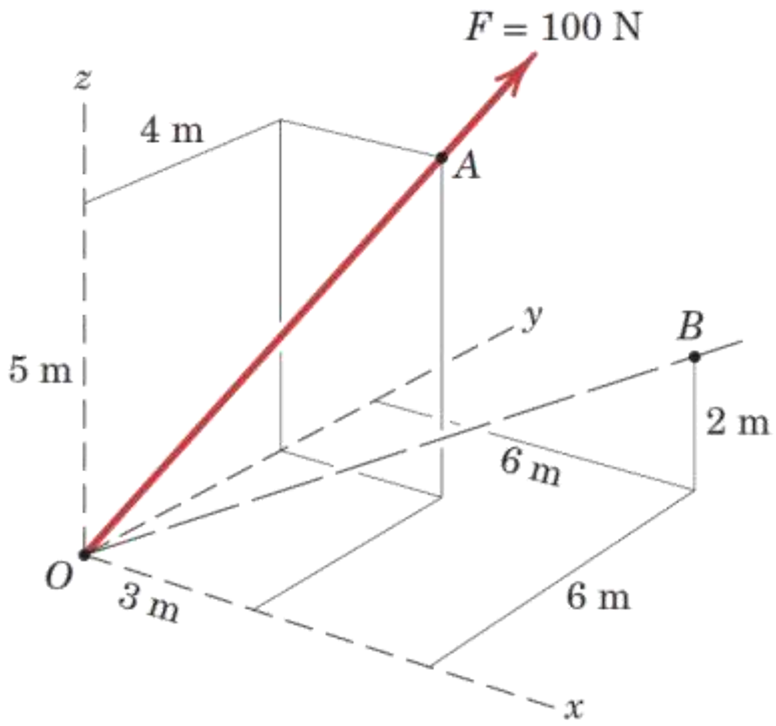
$$= 84.4 \text{ N} \qquad \textit{Ans.}$$

If we wish to express the projection as a vector, we write

$$\mathbf{F}_{OB} = \mathbf{F}\cdot\mathbf{n}_{OB}\mathbf{n}_{OB}$$

$$= 84.4(0.688\mathbf{i} + 0.688\mathbf{j} + 0.229\mathbf{k})$$

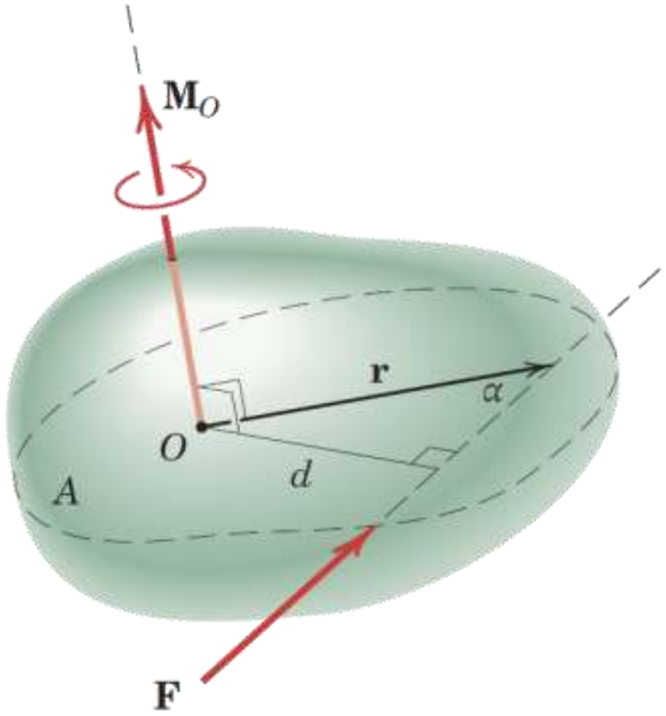
$$= 58.1\mathbf{i} + 58.1\mathbf{j} + 19.35\mathbf{k} \text{ N}$$



② The dot product automatically finds the projection or scalar component of $\mathbf{F}$ along line OB as shown.

Article 2/8 Moment and Couple (3D)

- Moments in Three Dimensions
  - Operate Identically to Moments in Two Dimensions
  - More Complicated to Visualize

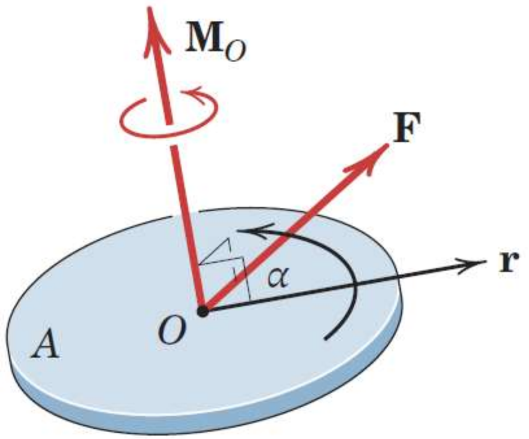


- Scalar Approach: $M_O = Fd$
  - More Difficult to Accomplish
  - Lacks Sign Information

- Vector Approach: $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$
  - Easy to Compute
  - Sign Information is Included Automatically

Article 2/8 – Right-Hand Rule Reminder

- Direction and Sense of the Moment
  - Established by Right-Hand Rule
  - Perpendicular to the Plane which Contains **r** and **F**
  - Cross Product Order is Essential



Article 2/8 – Cross Products (1 of 2)

- Definitions and Illustration

$$\mathbf{P} \times \mathbf{Q} = (P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k}) \times (Q_x\mathbf{i} + Q_y\mathbf{j} + Q_z\mathbf{k})$$
$$= (P_yQ_z - P_zQ_y)\mathbf{i} + (P_zQ_x - P_xQ_z)\mathbf{j} + (P_xQ_y - P_yQ_x)\mathbf{k}$$

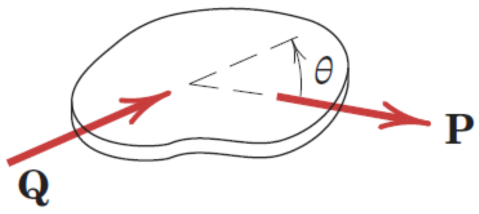
$$|\mathbf{P} \times \mathbf{Q}| = PQ \sin\theta$$

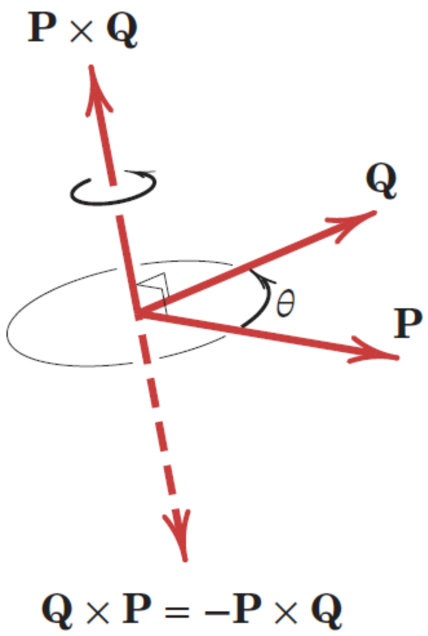
- Mathematical Features of the Cross Product

Distributive law $\quad \mathbf{P} \times (\mathbf{Q} + \mathbf{R}) = \mathbf{P} \times \mathbf{Q} + \mathbf{P} \times \mathbf{R}$

From the definition of the cross product, using a *right-handed coordinate system*, we get

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$
$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$
$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$$





Article 2/8 – Cross Products (2 of 2)

- Calculation via Determinant

$$\mathbf{P} \times \mathbf{Q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}$$

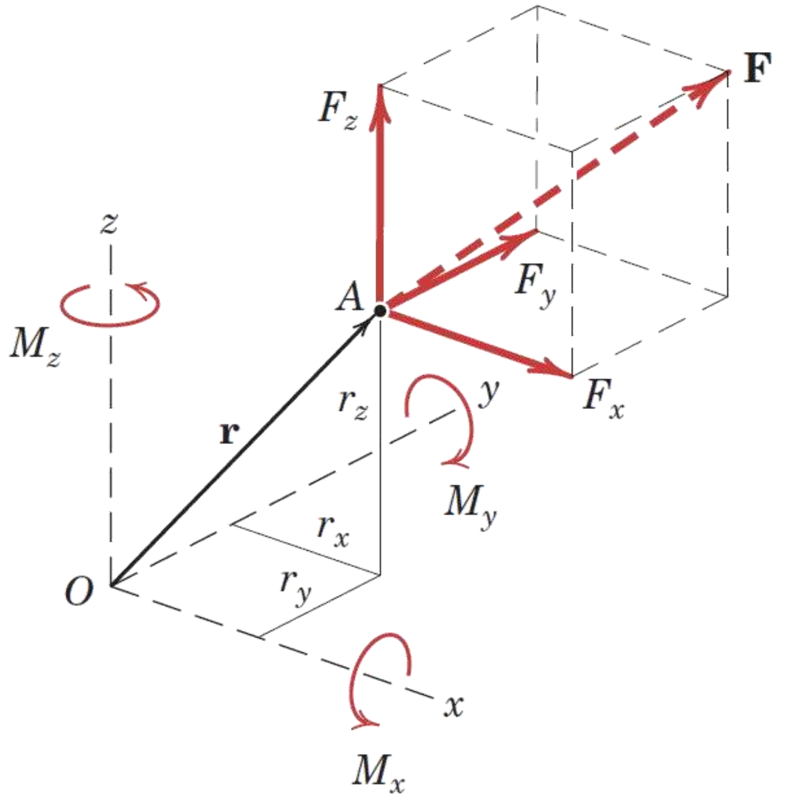
$$\begin{aligned} \mathbf{P} \times \mathbf{Q} &= (P_x\mathbf{i} + P_y\mathbf{j} + P_z\mathbf{k}) \times (Q_x\mathbf{i} + Q_y\mathbf{j} + Q_z\mathbf{k}) \\ &= (P_yQ_z - P_zQ_y)\mathbf{i} + (P_zQ_x - P_xQ_z)\mathbf{j} + (P_xQ_y - P_yQ_x)\mathbf{k} \end{aligned}$$

Article 2/8 – Moment made by a General Force

- Vector Components of $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$

$$\boxed{\mathbf{M}_O = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}}$$

$$\mathbf{M}_O = (r_y F_z - r_z F_y)\mathbf{i} + (r_z F_x - r_x F_z)\mathbf{j} + (r_x F_y - r_y F_x)\mathbf{k}$$

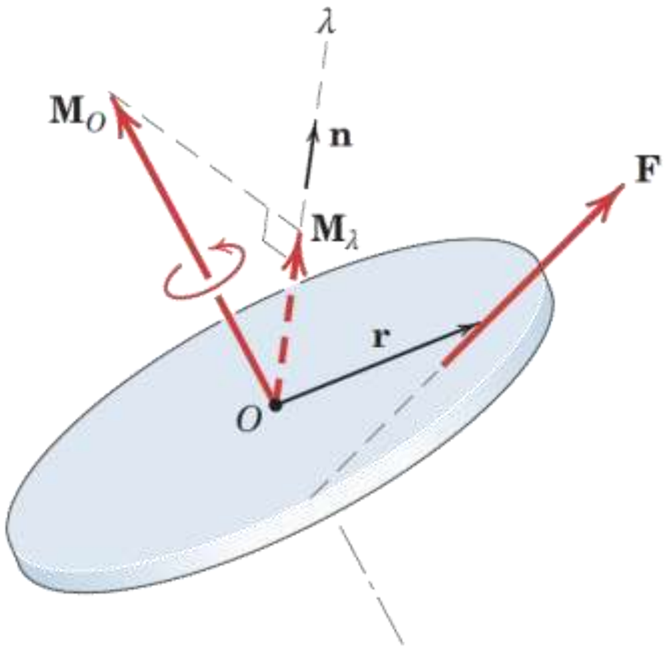


- Scalar Components

$$M_x = r_y F_z - r_z F_y \qquad M_y = r_z F_x - r_x F_z \qquad M_z = r_x F_y - r_y F_x$$

Article 2/8 – Moment about an Arbitrary Axis

- Illustration

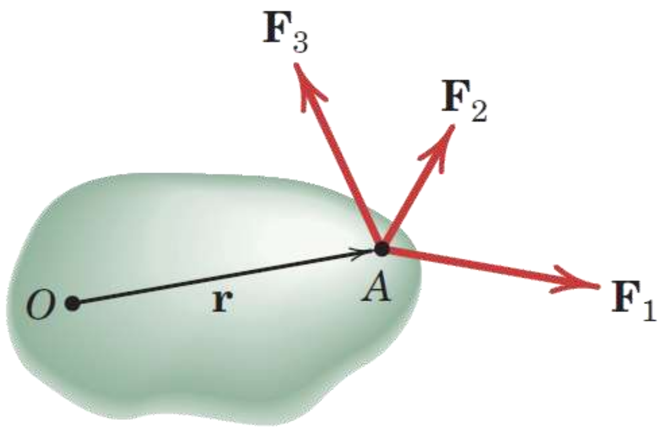


- Scalar Expression of the Moment about an Axis, M_λ
  1. Write the force as a vector.
  2. Write a position vector from any point on the axis to any point on the line of action of the force.
  3. Compute the moment of the force about the point.
  4. Write a unit vector in the direction of the axis.

- Vector Expression of the Moment about an Axis, $\mathbf{M}_\lambda$
  1. Write the scalar expression of the moment about the axis.
  2. Multiply the scalar expression of the moment about the axis by the unit vector in the direction of the axis.

Article 2/8 – Varignon's Theorem in Three Dimensions

- Illustration

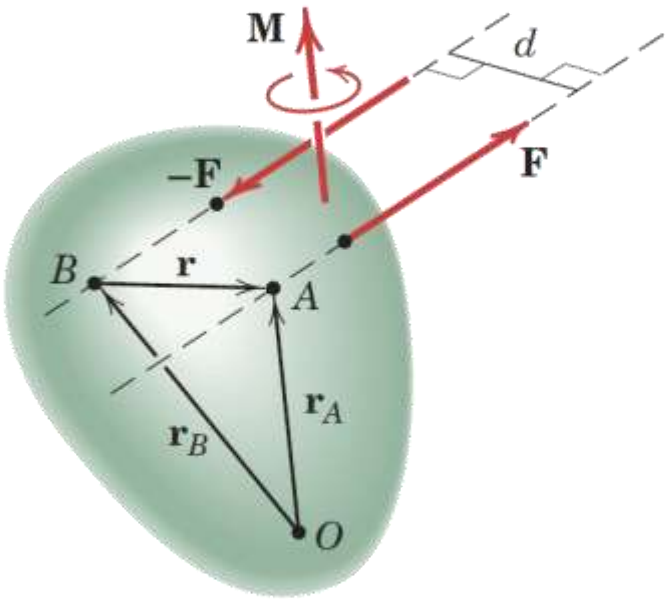


- Mathematics

$$\boxed{\mathbf{M}_O = \Sigma(\mathbf{r} \times \mathbf{F}) = \mathbf{r} \times \mathbf{R}}$$

Article 2/8 – Couples in Three Dimensions (1 of 3)

- Illustration



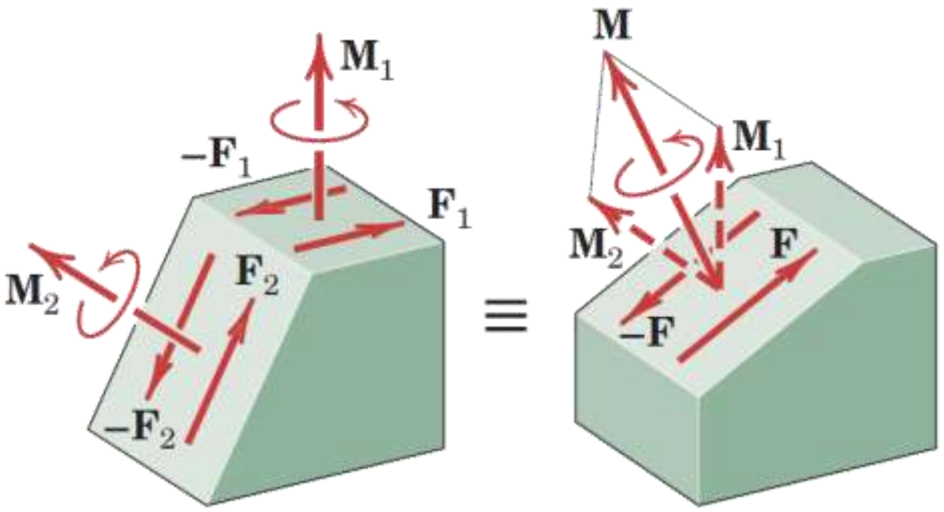
- Mathematics

$$\mathbf{M}_O = \mathbf{r}_A \times \mathbf{F} + \mathbf{r}_B \times (-\mathbf{F}) = (\mathbf{r}_A - \mathbf{r}_B) \times \mathbf{F} = \mathbf{r} \times \mathbf{F}$$

- Comments about Couples
  - Couples are free vectors.
  - You can simply compute the moment of one of the forces about any point on the line of action of the other force.
  - Couple vectors obey all of the corresponding mathematical rules which govern vector quantities.

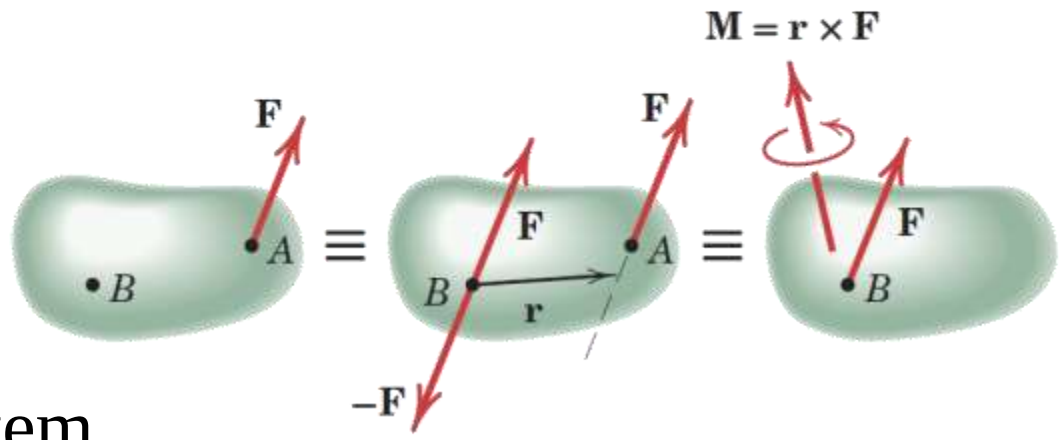
Article 2/8 – Couples in Three Dimensions (2 of 3)

- Adding Couples
  - Couples add with the Parallelogram Rule of Vector Addition



Article 2/8 – Force-Couple Systems

- Illustration of the Process

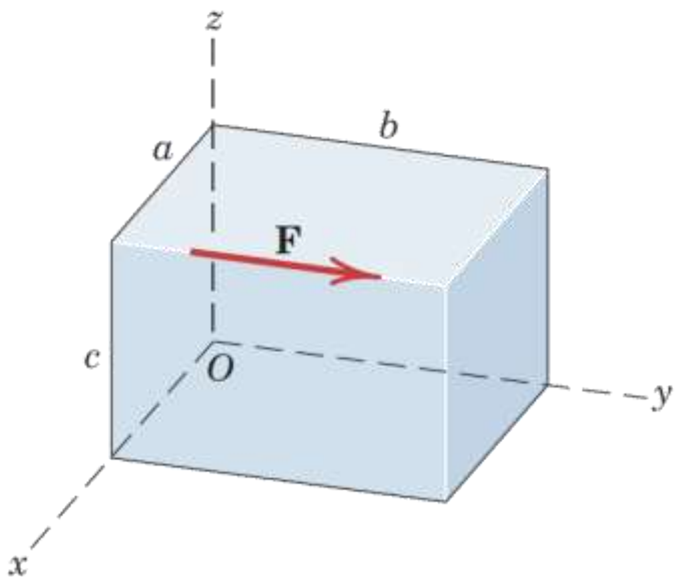


- Steps to Create a Force-Couple System
  1. Write the force as a vector.
  2. Compute the moment or couple which the force creates about the point.
  3. Redraw the force acting at the new location.
  4. Sketch the couple acting at the new location.
- Important Reminder
  - As with two-dimensional force-couple systems, the force-couple system has the same effect on the body which the original force had. It is simply a different way to visualize the effect of the force acting at a new location.

Article 2/8 – Sample Problem 2/11 (1 of 2)

- Problem Statement

  Determine the moment of force **F** about point O (a) by inspection and (b) by the formal cross-product definition $\mathbf{M}_O = \mathbf{r} \times \mathbf{F}$.



Article 2/8 – Sample Problem 2/11 (2 of 2)

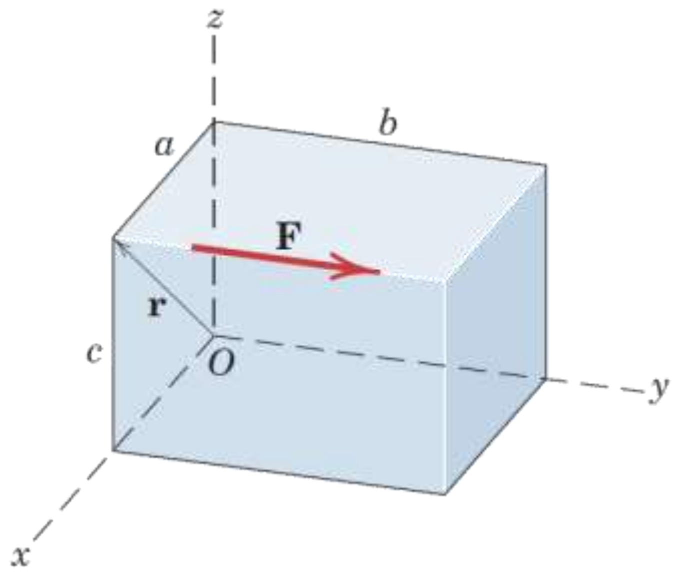
- Solution by Inspection

$$\mathbf{M}_O = -cF\mathbf{i} + aF\mathbf{k} = F(-c\mathbf{i} + a\mathbf{k}) \qquad \textit{Ans.}$$

- Solution by Cross Product

$$\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = (a\mathbf{i} + c\mathbf{k}) \times F\mathbf{j} = aF\mathbf{k} - cF\mathbf{i} \qquad ①$$

$$= F(-c\mathbf{i} + a\mathbf{k}) \qquad \textit{Ans.}$$

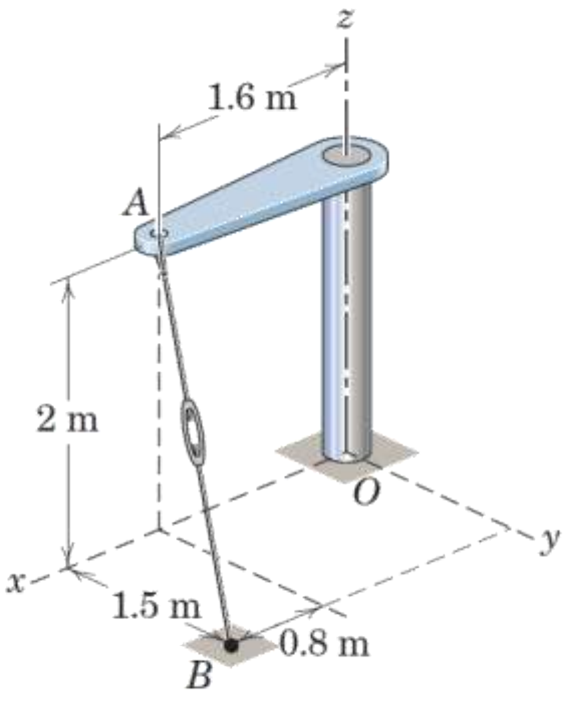


① Again we stress that $\mathbf{r}$ runs *from* the moment center *to* the line of action of $\mathbf{F}$. Another permissible, but less convenient, position vector is $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

Article 2/8 – Sample Problem 2/12 (1 of 2)

- Problem Statement

  The turnbuckle is tightened until the tension in cable *AB* is 2.4 kN. Determine the moment about point *O* of the cable force acting on point *A* and the magnitude of this moment.



Article 2/8 – Sample Problem 2/12 (2 of 2)

- Tension Vector

$$\mathbf{T} = T\mathbf{n}_{AB} = 2.4\left[\frac{0.8\mathbf{i} + 1.5\mathbf{j} - 2\mathbf{k}}{\sqrt{0.8^2 + 1.5^2 + 2^2}}\right]$$

$$= 0.731\mathbf{i} + 1.371\mathbf{j} - 1.829\mathbf{k} \text{ kN}$$

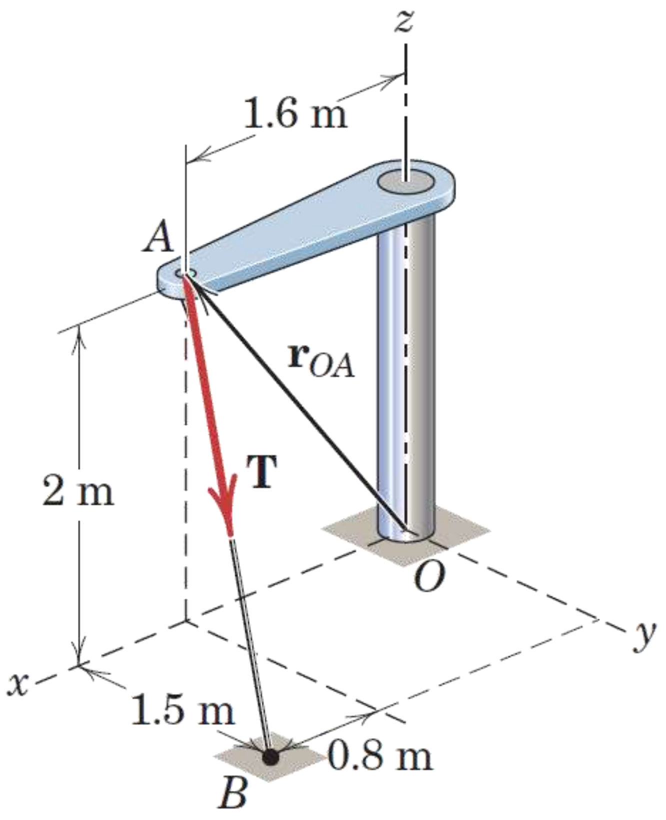
- Moment about Point O

$$\mathbf{M}_O = \mathbf{r}_{OA} \times \mathbf{T} = (1.6\mathbf{i} + 2\mathbf{k}) \times (0.731\mathbf{i} + 1.371\mathbf{j} - 1.829\mathbf{k})$$

$$= -2.74\mathbf{i} + 4.39\mathbf{j} + 2.19\mathbf{k} \text{ kN}\cdot\text{m} \quad ① \qquad \textit{Ans.}$$

- Magnitude of the Moment

$$M_O = \sqrt{2.74^2 + 4.39^2 + 2.19^2} = 5.62 \text{ kN}\cdot\text{m} \qquad \textit{Ans.}$$



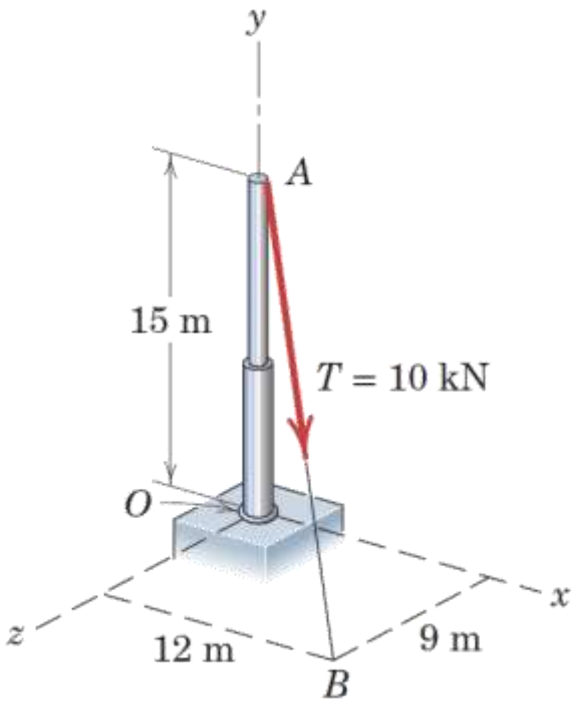
HELPFUL HINT

① The student should verify by inspection the signs of the moment components.

Article 2/8 – Sample Problem 2/13 (1 of 4)

- Problem Statement

  A tension **T** of magnitude 10 kN is applied to the cable attached to the top *A* of the rigid mast and secured to the ground at *B*. Determine the moment M_z of **T** about the *z*-axis passing through the base *O*.

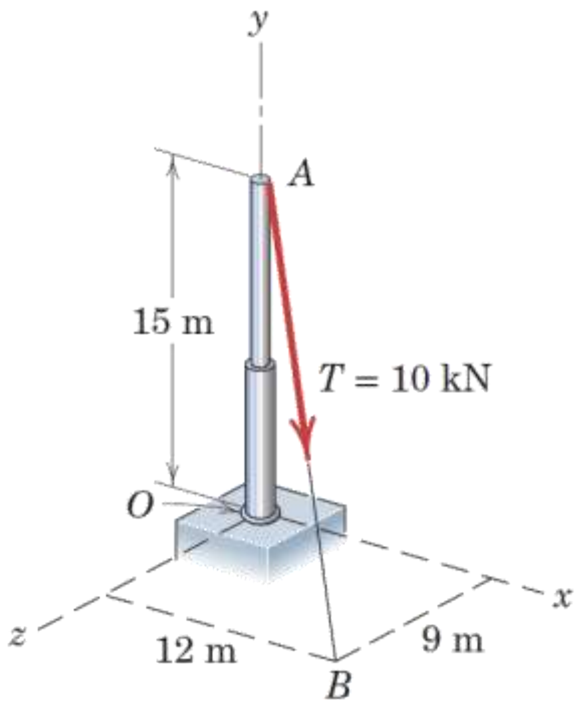


Article 2/8 – Sample Problem 2/13 (2 of 4)

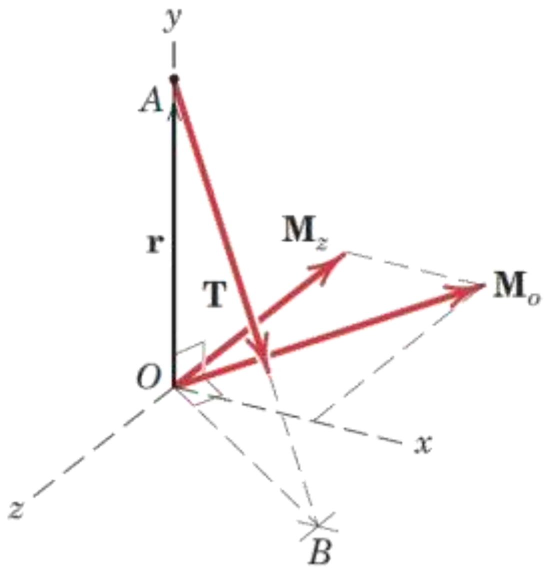
- Tension Vector

$$\mathbf{T} = T\mathbf{n}_{AB} = 10\left[\frac{12\mathbf{i} - 15\mathbf{j} + 9\mathbf{k}}{\sqrt{(12)^2 + (-15)^2 + (9)^2}}\right]$$

$$= 10(0.566\mathbf{i} - 0.707\mathbf{j} + 0.424\mathbf{k})\ \text{kN}$$



- Solution 1 – Cross Product



$[\mathbf{M}_O = \mathbf{r} \times \mathbf{F}]$ $\qquad \mathbf{M}_O = 15\mathbf{j} \times 10(0.566\mathbf{i} - 0.707\mathbf{j} + 0.424\mathbf{k})$

$$= 150(-0.566\mathbf{k} + 0.424\mathbf{i})\ \text{kN}\cdot\text{m}$$

The value M_z of the desired moment is the scalar component of $\mathbf{M}_O$ in the z-direction or $M_z = \mathbf{M}_O \cdot \mathbf{k}$. Therefore,

$$M_z = 150(-0.566\mathbf{k} + 0.424\mathbf{i}) \cdot \mathbf{k} = -84.9\ \text{kN}\cdot\text{m} \qquad \textit{Ans.}$$

Article 2/8 – Sample Problem 2/13 (3 of 4)

- Solution 2 – Two Scalar Components

$M_z = T_{xy}d$, where d is the perpendicular distance from T_{xy} to O. ③ The cosine of the angle between T and T_{xy} is $\sqrt{15^2 + 12^2}/\sqrt{15^2 + 12^2 + 9^2} =$ 0.906, and therefore,

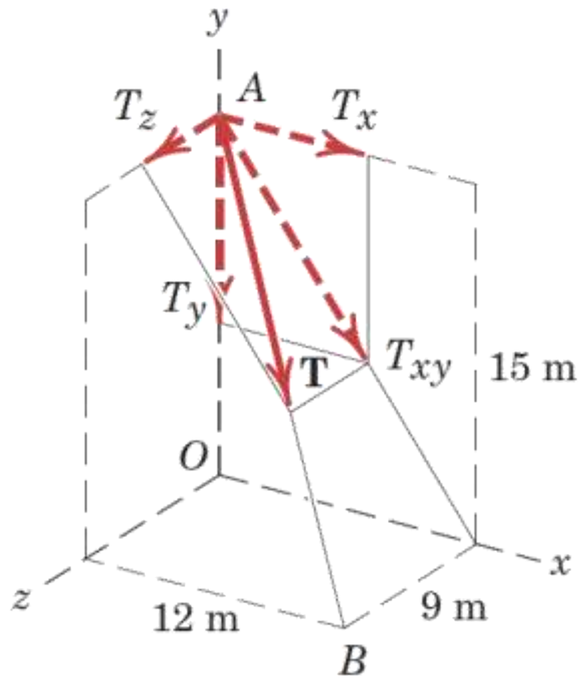
$$T_{xy} = 10(0.906) = 9.06 \text{ kN}$$

The moment arm d equals $\overline{OA}$ multiplied by the sine of the angle between T_{xy} and OA, or

$$d = 15 \frac{12}{\sqrt{12^2 + 15^2}} = 9.37 \text{ m}$$

Hence, the moment of **T** about the z-axis has the magnitude

$$M_z = 9.06(9.37) = 84.9 \text{ kN·m} \qquad \textit{Ans.}$$



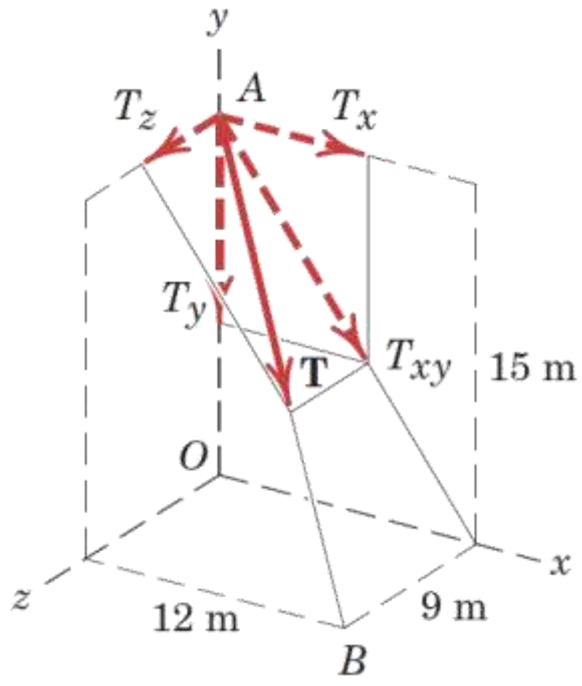
Article 2/8 – Sample Problem 2/13 (4 of 4)

- Solution 3 – Three Scalar Components

By inspection, only T_x makes a moment about point O. The y-component Intersects point O and the z-component is parallel to the z-axis.

From before, $T_x = 10(0.566) = 5.66$ kN.

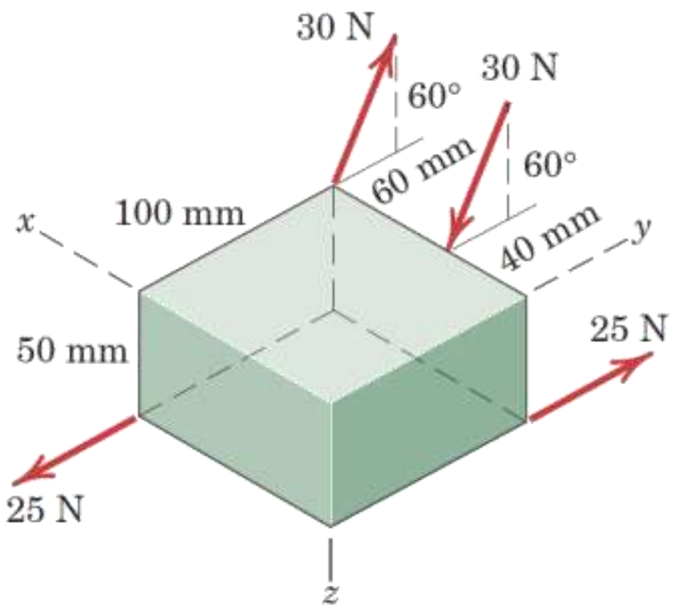
Therefore… $M_z = 5.66(15) = 84.9\ \text{kN}\cdot\text{m}$ *Ans.*



Article 2/8 – Sample Problem 2/14 (1 of 2)

- Problem Statement

  Determine the magnitude and direction of the couple **M** which will replace the two given couples and still produce the same external effect on the block. Specify the two forces **F** and -**F**, applied in the two faces of the block parallel to the *y*-*z* plane, which may replace the four given forces. The 30-N forces act parallel to the *y*-*z* plane.



Article 2/8 – Sample Problem 2/14 (2 of 2)

- Solution

The couple due to the 30-N forces has the magnitude $M_1 = 30(0.06) = 1.80$ N·m. The direction of $\mathbf{M}_1$ is normal to the plane defined by the two forces, and the sense, shown in the figure, is established by the right-hand convention. The couple due to the 25-N forces has the magnitude $M_2 = 25(0.10) = 2.50$ N·m with the direction and sense shown in the same figure. The two couple vectors combine to give the components

$$M_y = 1.80 \sin 60° = 1.559 \text{ N·m}$$

$$M_z = -2.50 + 1.80 \cos 60° = -1.600 \text{ N·m}$$

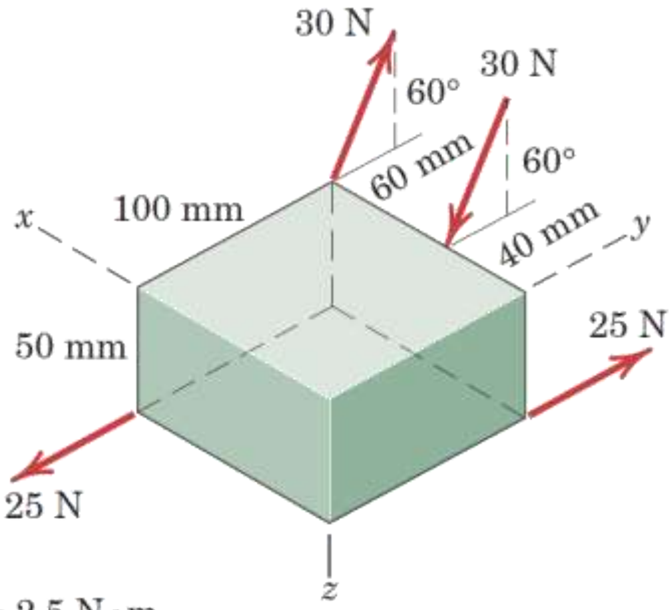
Thus, $M = \sqrt{(1.559)^2 + (-1.600)^2} = 2.23$ N·m ① *Ans.*

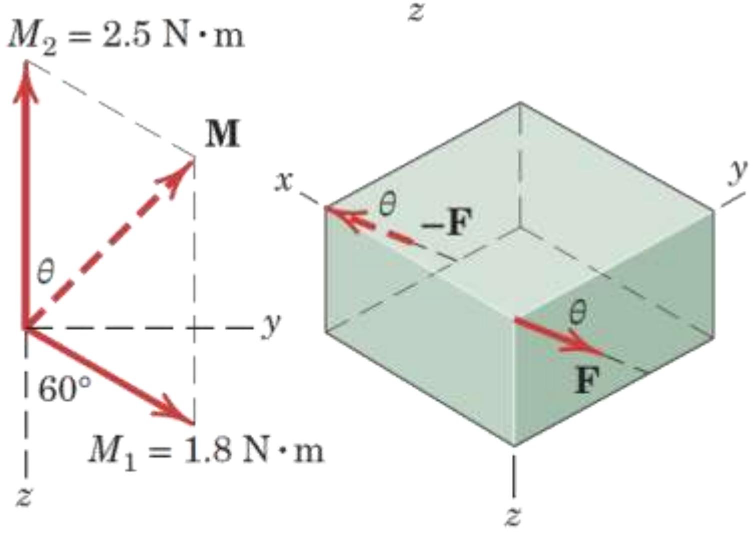
with $\theta = \tan^{-1} \dfrac{1.559}{1.600} = \tan^{-1} 0.974 = 44.3°$ *Ans.*

The forces **F** and –**F** lie in a plane normal to the couple **M**, and their moment arm as seen from the right-hand figure is 100 mm. Thus, each force has the magnitude

$[M = Fd]$ $F = \dfrac{2.23}{0.10} = 22.3$ N *Ans.*

and the direction $\theta = 44.3°$.

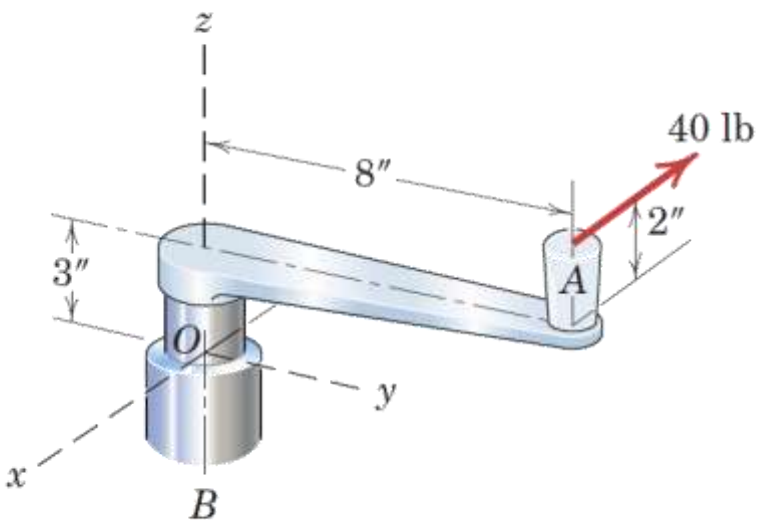




Article 2/8 – Sample Problem 2/15 (1 of 2)

- Problem Statement

  A force of 40 lb is applied at *A* to the handle of the control lever which is attached to the fixed shaft *OB*. In determining the effect of the force on the shaft at a cross section such as that at *O*, we may replace the force by an equivalent force at *O* and a couple. Describe this couple as a vector **M**.



Article 2/8 – Sample Problem 2/15 (2 of 2)

- Solution

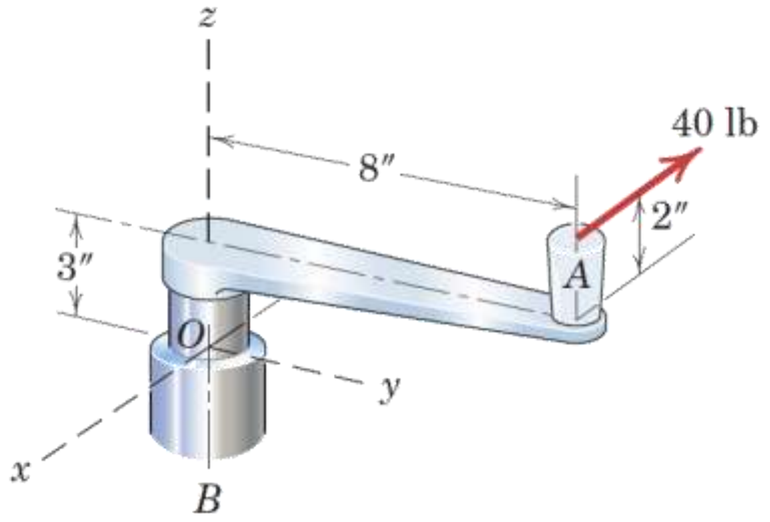
$$\mathbf{M} = (8\mathbf{j} + 5\mathbf{k}) \times (-40\mathbf{i}) = -200\mathbf{j} + 320\mathbf{k} \text{ lb-in.}$$

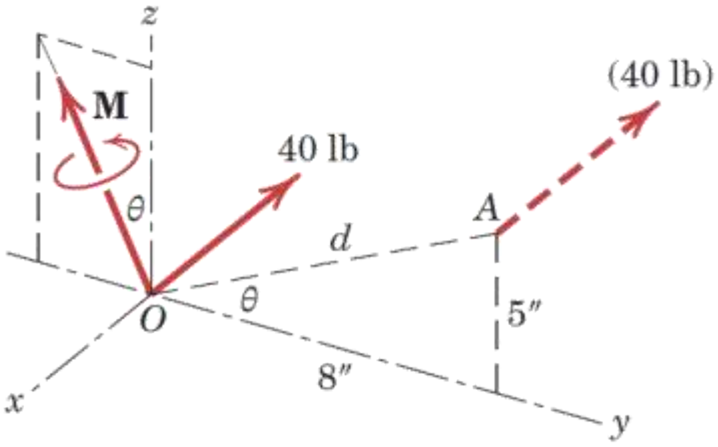
Alternatively we see that moving the 40-lb force through a distance $d = \sqrt{5^2 + 8^2} = 9.43$ in. to a parallel position through O requires the addition of a couple **M** whose magnitude is

$$M = Fd = 40(9.43) = 377 \text{ lb-in.} \qquad \textit{Ans.}$$

The couple vector is perpendicular to the plane in which the force is shifted, and its sense is that of the moment of the given force about O. The direction of **M** in the y-z plane is given by

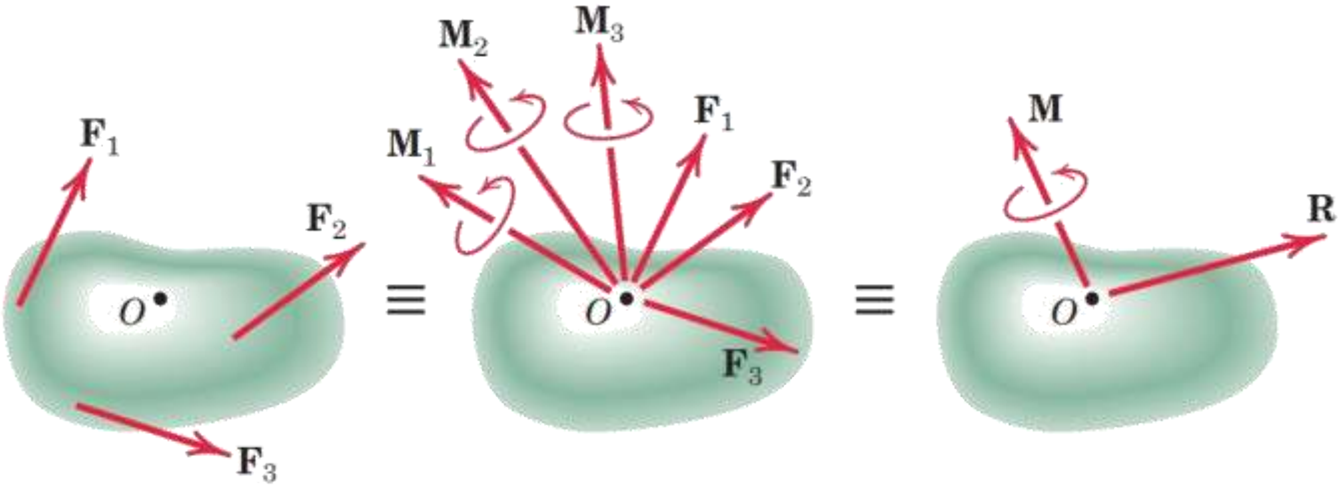
$$\theta = \tan^{-1}\frac{5}{8} = 32.0° \qquad \textit{Ans.}$$





Article 2/9 Resultants (3D)

- Illustration



- Equations of Interest

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots = \Sigma\mathbf{F}$$
$$\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 + \cdots = \Sigma(\mathbf{r}\times\mathbf{F})$$

$$R_x = \Sigma F_x \qquad R_y = \Sigma F_y \qquad R_z = \Sigma F_z$$
$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2 + (\Sigma F_z)^2}$$
$$\mathbf{M}_x = \Sigma(\mathbf{r}\times\mathbf{F})_x \qquad \mathbf{M}_y = \Sigma(\mathbf{r}\times\mathbf{F})_y \qquad \mathbf{M}_z = \Sigma(\mathbf{r}\times\mathbf{F})_z$$
$$M = \sqrt{M_x{}^2 + M_y{}^2 + M_z{}^2}$$

Article 2/9 – Types of Force Systems (1 of 2)

- Concurrent Forces
  - Because the forces are concurrent, there are no moments about the point of concurrency.
  - $\mathbf{R} = \Sigma \mathbf{F}$
- Coplanar Forces
  - Article 2/6 was devoted to this force system.

Article 2/9 – Types of Force Systems (2 of 2)

- Parallel Forces not in the Same Plane
  - Because the forces are parallel, the moment they produce about any point will be perpendicular to the line of action of the resultant.

- Calculation Steps
  1. Find the resultant, $\mathbf{R} = \Sigma\mathbf{F}$
  2. Find the couple at the point, $\mathbf{M}_O = \Sigma\mathbf{M}_O$ (from all forces or applied couples)
  3. Write a position vector $\mathbf{r}$ from the force-couple reference point to any point on the line of action of the resultant $\mathbf{R}$. Typically, the point will be specified in one of the three coordinate-axis planes.
     a. For a point in the x-y plane $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$
     b. For a point in the x-z plane, $\mathbf{r} = x\mathbf{i} + z\mathbf{k}$
     c. For a point in the y-z plane, $\mathbf{r} = y\mathbf{j} + z\mathbf{k}$
  4. Solve the equation $\mathbf{r} \times \mathbf{R} = \mathbf{M}_O$

Article 2/9 – Wrench Resultants (1 of 4)

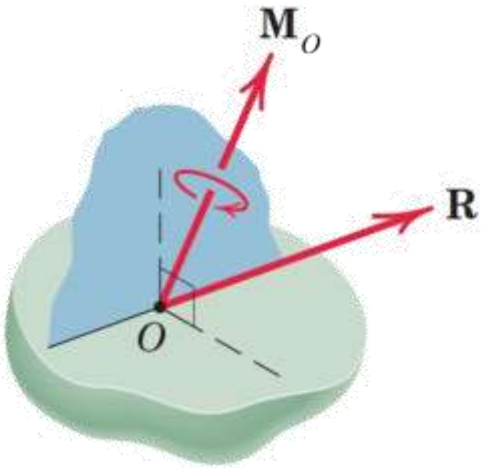
- Occurrence and Illustration
  - Wrenches occur for a system of forces which are not parallel or concurrent. In this case, the resultant couple vector, $\mathbf{M}_O$, will have a component that is parallel to the resultant force **R**. The resultant **R** is not able to produce this component of the moment regardless of its position relative to the force-couple system reference point.

  - The simplified force system will consist of two pieces.
    - A resultant **R** which equals the vector sum of all forces, and is positioned such that it can produce the part of the resultant couple vector which is perpendicular to its line of action.
    - A wrench moment **M** which is equal to the part of the resultant couple vector which is parallel to the line of action of the resultant **R**.

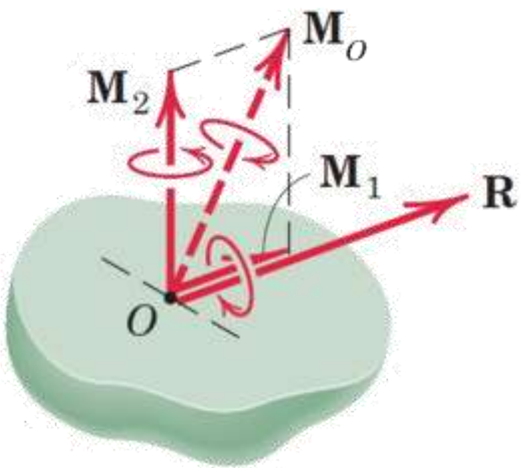
Positive wrench   Negative wrench

Article 2/9 – Wrench Resultants (2 of 4)

- General Force-Couple System

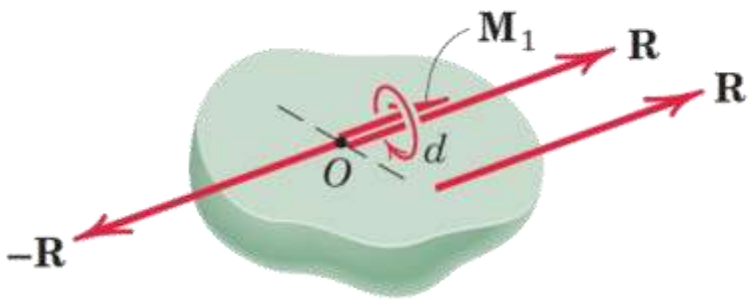


- Components of the Resultant Couple Vector
  - $\mathbf{M}_1$ is Parallel to $\mathbf{R}$
  - $\mathbf{M}_2$ is Perpendicular to $\mathbf{R}$

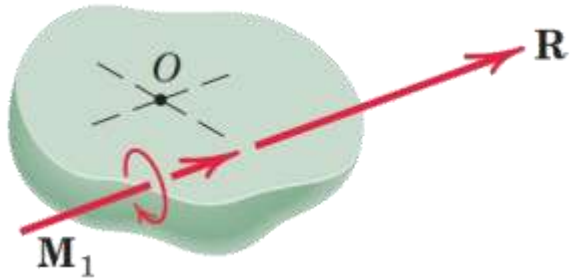


Article 2/9 – Wrench Resultants (3 of 4)

- Relocation of the Resultant by a Couple
  - **R** is moved a distance *d* from the reference point.
  - $Rd = M_2$



- Final Wrench Resultant
  - Force Resultant is Preserved
  - Moment Resultant is Preserved
    - $\mathbf{M}_1$ is Simply Added
    - **R** will Produce $\mathbf{M}_2$
    - $\mathbf{M}_O = \mathbf{M}_1 + \mathbf{M}_2$



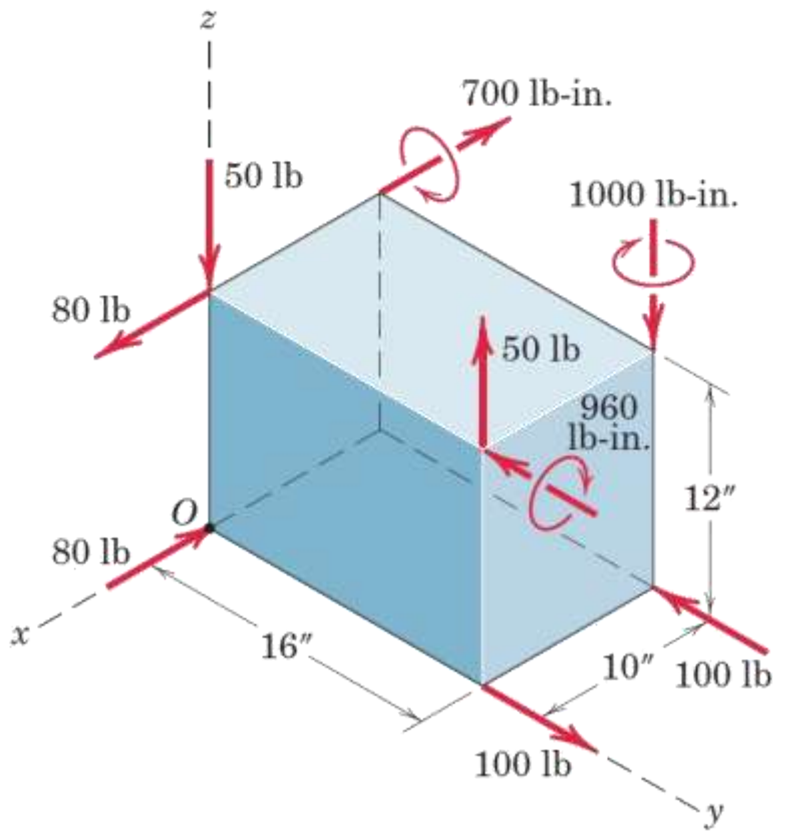
Article 2/9 – Wrench Resultants (4 of 4)

- Calculation Steps
  1. Find the resultant, $\mathbf{R} = \Sigma\mathbf{F}$
  2. Find the couple at the point, $\mathbf{M}_O = \Sigma\mathbf{M}_O$ (from all forces or applied couples)
  3. Find the wrench moment, $\mathbf{M}_1$
     a. Write a unit vector $\mathbf{n}$ in the direction of the resultant $\mathbf{R}$, $\mathbf{n} = \mathbf{R}/R$
     b. Take a Dot Product to find the scalar portion of $\mathbf{M}_O$ in the direction of $\mathbf{R}$, $M_1 = \mathbf{M}_O \cdot \mathbf{n}$
     c. The algebraic sign of M_1 will tell you if the wrench is in the positive or negative sense.
     d. Write the wrench-moment vector, $\mathbf{M}_1 = M_1\mathbf{n}$
  4. Write a position vector $\mathbf{r}$ from the force-couple reference point to any point on the line of action of the resultant $\mathbf{R}$. Typically, the point will be specified in one of the three coordinate-axis planes.
     a. For a point in the x-y plane $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$
     b. For a point in the x-z plane, $\mathbf{r} = x\mathbf{i} + z\mathbf{k}$
     c. For a point in the y-z plane, $\mathbf{r} = y\mathbf{j} + z\mathbf{k}$
  5. Solve the equation $\mathbf{r} \times \mathbf{R} + \mathbf{M}_1 = \mathbf{M}_O$

Article 2/9 – Sample Problem 2/16 (1 of 2)

- Problem Statement

  Determine the resultant of the force and couple system which acts on the rectangular solid.



Article 2/9 – Sample Problem 2/16 (2 of 2)

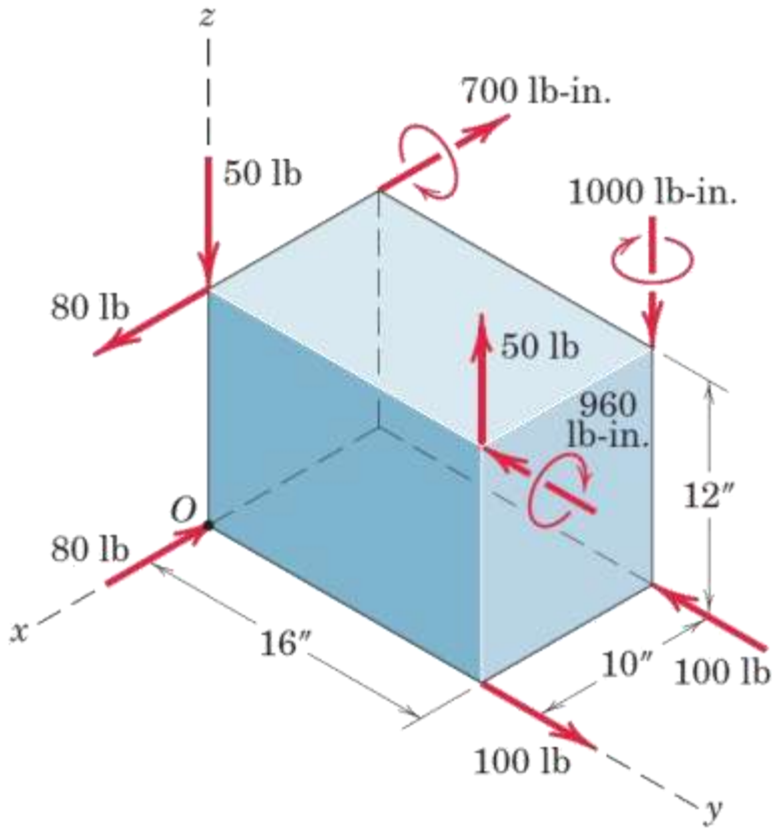
- Solution

$$\mathbf{R} = \Sigma\mathbf{F} = (80 - 80)\mathbf{i} + (100 - 100)\mathbf{j} + (50 - 50)\mathbf{k} = \mathbf{0} \text{ lb} \quad ①$$

The sum of the moments about O is

$$\mathbf{M}_O = [50(16) - 700]\mathbf{i} + [80(12) - 960]\mathbf{j} + [100(10) - 1000]\mathbf{k} \text{ lb-in.}$$

$$= 100\mathbf{i} \text{ lb-in.} \quad ②$$



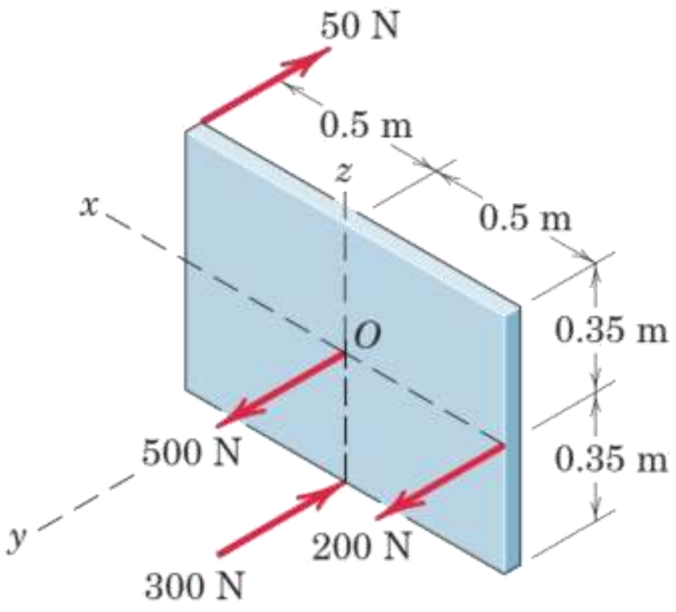
① Since the force summation is zero, we conclude that the resultant, if it exists, must be a couple.

② The moments associated with the force pairs are easily obtained by using the $M = Fd$ rule and assigning the unit-vector direction by inspection. In many three-dimensional problems, this may be simpler than the $\mathbf{M} = \mathbf{r} \times \mathbf{F}$ approach.

Article 2/9 – Sample Problem 2/17 (1 of 2)

- Problem Statement

  Determine the resultant of the system of parallel forces which act on the plate. Solve with a vector approach.



Article 2/9 – Sample Problem 2/17 (2 of 2)

- Solution

$$\mathbf{R} = \Sigma\mathbf{F} = (200 + 500 - 300 - 50)\mathbf{j} = 350\mathbf{j}\ \text{N}$$

$$\mathbf{M}_O = [50(0.35) - 300(0.35)]\mathbf{i} + [-50(0.50) - 200(0.50)]\mathbf{k}$$

$$= -87.5\mathbf{i} - 125\mathbf{k}\ \text{N}\cdot\text{m}$$

The placement of **R** so that it alone represents the above force–couple system is determined by the principle of moments in vector form

$$\mathbf{r} \times \mathbf{R} = \mathbf{M}_O$$

$$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times 350\mathbf{j} = -87.5\mathbf{i} - 125\mathbf{k}$$

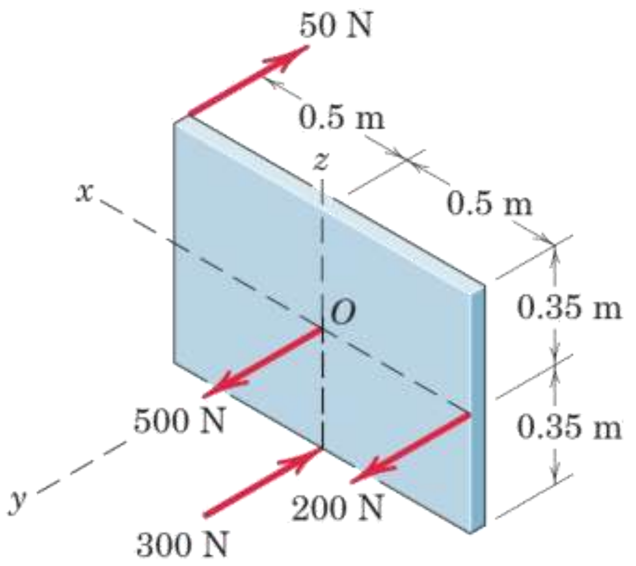
$$350x\mathbf{k} - 350z\mathbf{i} = -87.5\mathbf{i} - 125\mathbf{k}$$

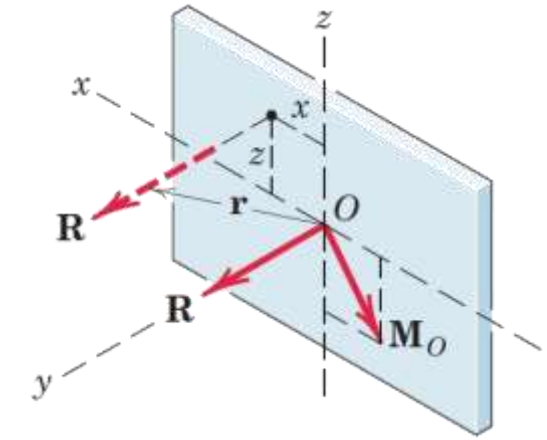
From the one vector equation we may obtain the two scalar equations

$$350x = -125 \qquad \text{and} \qquad -350z = -87.5$$

Hence, $x = -0.357$ m and $z = 0.250$ m are the coordinates through which the line of action of **R** must pass. The value of y may, of course, be any value, as permitted by the principle of transmissibility. Thus, as expected, the variable y drops out of the above vector analysis. ①

① You should also carry out a scalar solution to this problem.

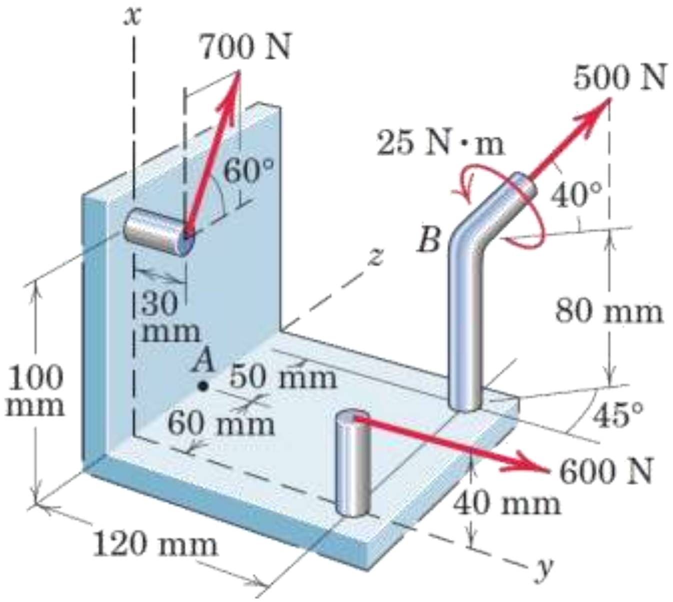




Article 2/9 – Sample Problem 2/18 (1 of 4)

- Problem Statement

  Replace the two forces and the negative wrench by a single force **R** applied at *A* and the corresponding couple **M**.



Article 2/9 – Sample Problem 2/18 (2 of 4)

- Force Resultant

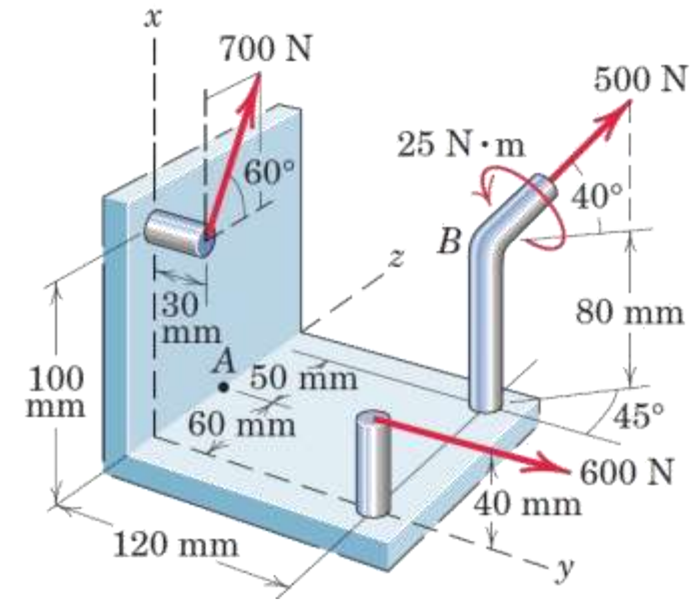
$[R_x = \Sigma F_x]$ $\quad R_x = 500 \sin 40° + 700 \sin 60° = 928 \text{ N}$

$[R_y = \Sigma F_y]$ $\quad R_y = 600 + 500 \cos 40° \cos 45° = 871 \text{ N}$

$[R_z = \Sigma F_z]$ $\quad R_z = 700 \cos 60° + 500 \cos 40° \sin 45° = 621 \text{ N}$

Thus, $\quad \mathbf{R} = 928\mathbf{i} + 871\mathbf{j} + 621\mathbf{k} \text{ N}$

and $\quad R = \sqrt{(928)^2 + (871)^2 + (621)^2} = 1416 \text{ N}$ *Ans.*



- Couple from the 500-N Force

$$[\mathbf{M} = \mathbf{r} \times \mathbf{F}] \quad \mathbf{M}_{500} = (0.08\mathbf{i} + 0.12\mathbf{j} + 0.05\mathbf{k}) \times 500(\mathbf{i} \sin 40° + \mathbf{j} \cos 40° \cos 45° + \mathbf{k} \cos 40° \sin 45°) \quad ①$$

where $\mathbf{r}$ is the vector from A to B.

The term-by-term, or determinant, expansion gives

$$\mathbf{M}_{500} = 18.95\mathbf{i} - 5.59\mathbf{j} - 16.90\mathbf{k} \text{ N·m}$$

① *Suggestion:* Check the cross-product results by evaluating the moments about A of the components of the 500-N force directly from the sketch.

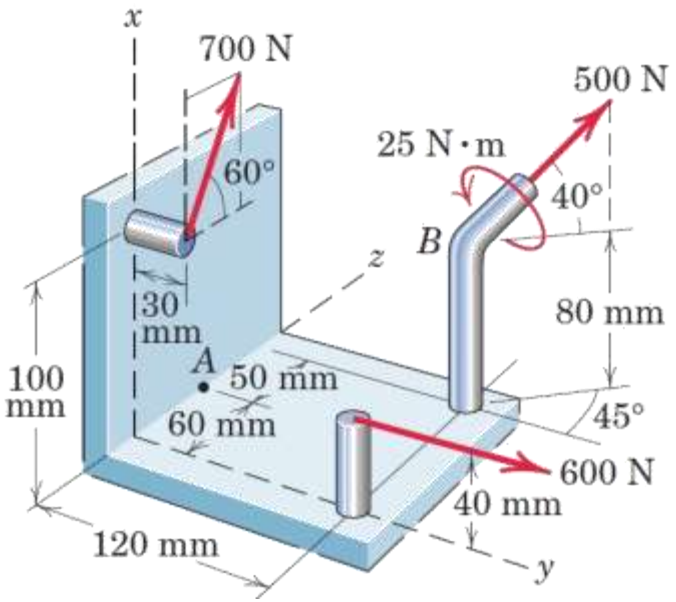
Article 2/9 – Sample Problem 2/18 (3 of 4)

- Couple from the 600-N Force

$$\mathbf{M}_{600} = (600)(0.060)\mathbf{i} + (600)(0.040)\mathbf{k}$$
$$= 36.0\mathbf{i} + 24.0\mathbf{k}\ \text{N}\cdot\text{m}$$

- Couple from the 700-N Force

$$\mathbf{M}_{700} = (700\cos 60^\circ)(0.030)\mathbf{i} - [(700\sin 60^\circ)(0.060) + (700\cos 60^\circ)(0.100)]\mathbf{j} - (700\sin 60^\circ)(0.030)\mathbf{k}$$
$$= 10.5\mathbf{i} - 71.4\mathbf{j} - 18.19\mathbf{k}\ \text{N}\cdot\text{m}$$



② For the 600-N and 700-N forces it is easier to obtain the components of their moments about the coordinate directions through A by inspection of the figure than it is to set up the cross-product relations.

Article 2/9 – Sample Problem 2/18 (4 of 4)

- Couple from the Wrench Moment

$$\mathbf{M}' = 25.0(-\mathbf{i}\sin 40° - \mathbf{j}\cos 40° \cos 45° - \mathbf{k}\cos 40° \sin 45°)$$

$$= -16.07\mathbf{i} - 13.54\mathbf{j} - 13.54\mathbf{k}\ \text{N}\cdot\text{m}$$

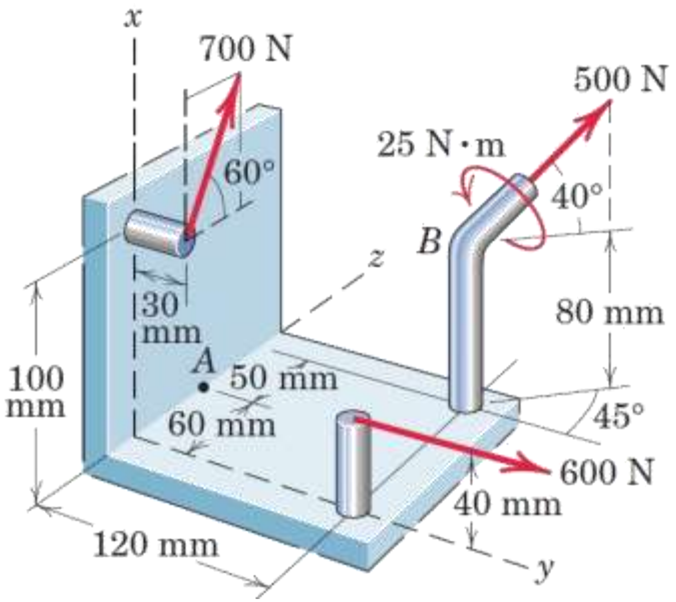
- Resultant Couple at *A*

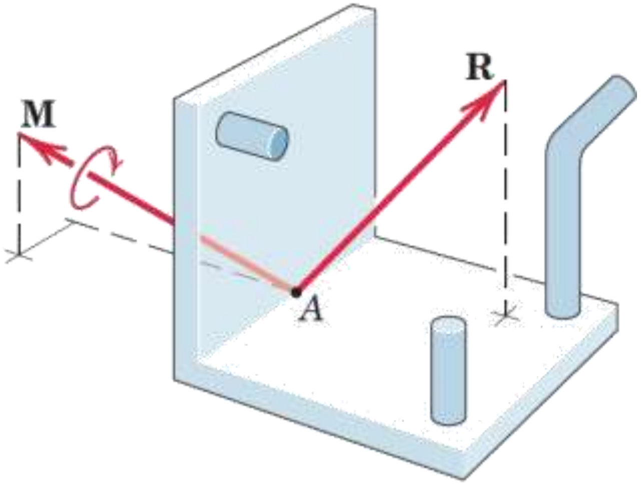
$$\mathbf{M} = 49.4\mathbf{i} - 90.5\mathbf{j} - 24.6\mathbf{k}\ \text{N}\cdot\text{m} \quad ④$$

and $\quad M = \sqrt{(49.4)^2 + (90.5)^2 + (24.6)^2} = 106.0\ \text{N}\cdot\text{m}$ *Ans.*

③ The 25-N·m couple vector of the *wrench* points in the direction opposite to that of the 500-N force, and we must resolve it into its x-, y-, and z-components to be added to the other couple-vector components.

④ Although the resultant couple vector **M** in the sketch of the resultants is shown through *A*, we recognize that a couple vector is a free vector and therefore has no specified line of action.

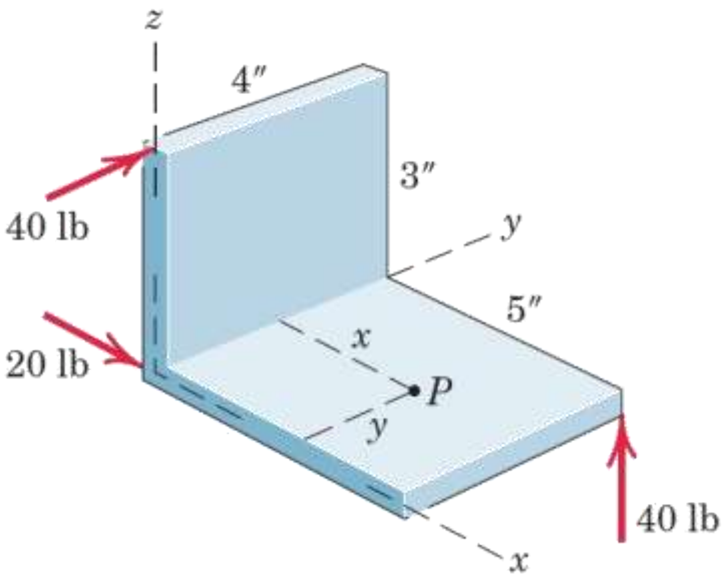




Article 2/9 – Sample Problem 2/19 (1 of 3)

- Problem Statement

  Determine the wrench resultant of the three forces acting on the bracket. Calculate the coordinates of the point *P* in the *x*-*y* plane through which the resultant force of the wrench acts. Also find the magnitude of the couple **M** of the wrench.



Article 2/9 – Sample Problem 2/19 (2 of 3)

- Resultant Force

$$\mathbf{R} = 20\mathbf{i} + 40\mathbf{j} + 40\mathbf{k} \text{ lb} \qquad R = \sqrt{(20)^2 + (40)^2 + (40)^2} = 60 \text{ lb}$$

and its direction cosines are

$$\cos\theta_x = 20/60 = 1/3 \qquad \cos\theta_y = 40/60 = 2/3 \qquad \cos\theta_z = 40/60 = 2/3$$

- Moment about P

The moment of the wrench couple must equal the sum of the moments of the given forces about point P through which $\mathbf{R}$ passes. The moments about P of the three forces are

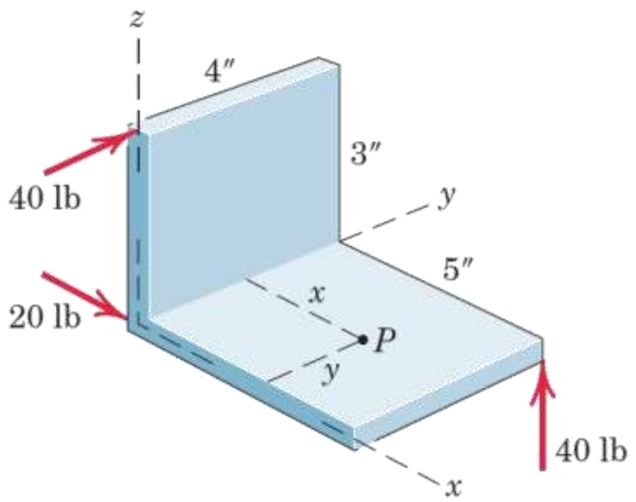
$$(\mathbf{M})_{R_x} = 20y\mathbf{k} \text{ lb-in.}$$

$$(\mathbf{M})_{R_y} = -40(3)\mathbf{i} - 40x\mathbf{k} \text{ lb-in.}$$

$$(\mathbf{M})_{R_z} = 40(4 - y)\mathbf{i} - 40(5 - x)\mathbf{j} \text{ lb-in.}$$

and the total moment is

$$\mathbf{M} = (40 - 40y)\mathbf{i} + (-200 + 40x)\mathbf{j} + (-40x + 20y)\mathbf{k} \text{ lb-in.}$$



Article 2/9 – Sample Problem 2/19 (3 of 3)

- Final Solution

The direction cosines of **M** are

$$\cos\theta_x = (40 - 40y)/M$$

$$\cos\theta_y = (-200 + 40x)/M$$

$$\cos\theta_z = (-40x + 20y)/M$$

where M is the magnitude of **M**. Equating the direction cosines of **R** and **M** gives

$$40 - 40y = \frac{M}{3}$$

$$-200 + 40x = \frac{2M}{3}$$

$$-40x + 20y = \frac{2M}{3}$$

Solution of the three equations gives

$M = -120$ lb-in. $\quad x = 3$ in. $\quad y = 2$ in. $\qquad$ *Ans.*