Almgren Chriss Market Impact Model

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March 27, 2022

Abstract

There are 2 parts. Part 1 is an implementation of modified Almgren-Chriss Market Impact Model. Part 2 is the response to the Optimal Execution.

1 Impact Model

The impact of large trades on market prices is a extensively discussed but scarcely measured phenomenon, of utmost importance to sell-side and buy-side participants. We performed an analysis on a large data set from the TAQ Dataset. We selected 506 stocks and 65 days, ranging from 20070620 to 20070920. We later cleaned the defect stocks data, and narrowed down to 501 stocks and 65 days.

1.1 Data Manipulation

Instead of using raw data, we transformed data into 501 by 55 matrix for each of the input, such as the value between 9:30 AM to 3:30 PM. We performed a 10-day look-back calculation on both the average daily value and the standard deviation of 2-minute mid-quote returns that was already scaled to a day. Since we needed to find a 10-day look-back value, we discarded the days from day 1 to day 10 for all the other inputs.

1.2 Modified Model

The non-linear regression we used for Almgren-Chriss model is

$$h = \eta\sigma\left(\frac{X}{\frac{6}{6.5}V}\right)^{\beta} + < noise >$$

X is the imbalance between 9:30AM and 3:30PM
V is the average daily value
σ is the standard deviation of 2-min returns scaled to 1 day
η is what we want to know
β is what we want to know

We used the above regression model to estimate for η and β. Instead of using volume for Almgren-Chriss model, we choose to use value, calculated using volume-weighted average price and volume. We also modified the imbalance accordingly.

$$\text{value} = \text{volume} \times \text{VWAP}$$

Parameters	Value
η	4.8122
t_η residual	3.2322
t_η residual p value	0.0012
t_η paired	3.6706
t_η paired p value	0.0002
β	0.8351
t_β residual	3.1653
t_β residual p value	0.0016
t_β paired	2.0203
t_β paired p value	0.0434

Table 1: Parameters

1.3 Bootstrapped Results

After the non-linear regression, we used two bootstrap techniques, paired bootstrap and residual bootstrap to estimate the standard error for η and β. We generated 10,000 η and β in both residual and paired bootstrap. Afterwards, we used 2 tailed t-test to calculate the p value. The findings for both bootstraps are in the Table 1.

If we assumed a p-value of 0.05, then, from the table above, we noticed that both η and β were statistically significant in both the paired bootstrap and the residual bootstrap. In other words, the η and the β were valid.

1.4 Residual Analysis

1.4.1 Normality of Residuals

We set out to verify the assumptions for residuals. We performed a normality test, and the p value was 0, which was smaller than assumed alpha 0.05. We rejected the null hypothesis, and concluded that the residuals was not normally distributed. Figure 1 and Figure 2 confirmed our result. They were two plots about the residuals. This was a violation of one of the assumptions of the non-linear regression.

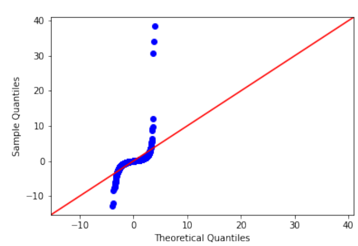

Figure 1: The qqplot

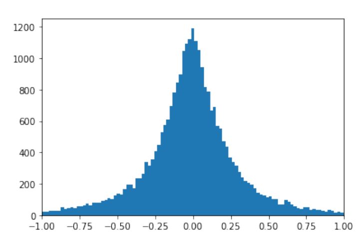

Figure 2: The histogram

1.4.2 Homoskedasticicty of Residuals

We performed a Breusch-Pagan test with p-value of 0.3226. We could not reject the null hypothesis, concluding that the the residuals are homoskedastic. This result satisfied the assumption.

1.4.3 Zero Mean of Residuals

We performed a t-test with p-value of 0.4869. We could not reject the null hypothesis, concluding that the the mean is zero. This result satisfied the assumption.

1.4.4 Independence of Residuals

We performed a Durbin-Watson test. It returned a t-statistic of 2.0426. Since the t-statistic was near to 2 in Durbin-Watson test, we could conclude that the residuals were Independent. This result satisfied the assumption.

1.5 The Influence of Stock Activity on η and β

1.5.1 Methodology

We split the stocks into 2 halves based on the total value in 65 days. Value was calculated the same as before, a product of volume and VWAP at the corresponding time. The stocks with high value were labeled active, and low value were labeled inactive. The η and β were calculated on each dataset.

1.5.2 Activity Results

We found that the stock activity did influence the parameters η and β. They looked fairly different from one another.

Parameters	Value
Active Stocks	
η	10.5237
β	0.8800
Inactive Stocks	
η	3.2514
β	1.1190

Table 2: Parameter Comparison

Therefore, we could conclude that the active stocks had a higher η and lower β than those of the inactive stocks. The stock activity did have an influence on the parameters.

2 Optimal Execution

This is a response to problem 2.

a

i

The resulting HJB takes the form

$$0 = (\partial_t + \frac{1}{2}\sigma^2\partial_{ss})H - \phi q^2 + \sup_v((v(S - kv)\partial x - bv\partial v - v\partial q)H) \tag{1}$$

subject to

$$H(T, x, S, q) = x + Sq - \alpha q^2$$

The first order condition gives:

$$
\begin{aligned}
\frac{\partial((v(S-kv)\partial x - bv\partial v - v\partial q)H)}{\partial v} &= 0 \\
S\partial_x H - 2vk\partial_x H - b\partial_s H - \partial_q H &= 0 \\
v^\star &= \frac{S\partial_x H - b\partial_s H - \partial_q H}{2k\partial_x H}
\end{aligned}
$$

Substituting the expression for $v^\star$ into Equation (1) gives

$$
0 = (\partial_t + \frac{1}{2}\sigma^2\partial_{ss})H - \phi q^2 + \frac{1}{4k}\frac{(S\partial_x H - b\partial_s H - \partial_q H)^2}{\partial_x H} \tag{2}
$$

We make the ansatz

$$
H(t, x, S, q) = x + Sq + h(t, S, q)
$$

where $h(T, S, q) = -\alpha q^2$.
Equation (2) thus becomes:

$$
0 = (\partial_t + \frac{1}{2}\sigma^2\partial_{ss})h - \phi q^2 + \frac{1}{4k}(b(q + \partial_s h) + \partial_q h)^2 \tag{3}
$$

We observe that Equation (3) doesn't explicitly depend on S. $h(T, S, q)$ is also independent of S. Therefore we can write,

$$
h(t, S, q) = h(t, q)
$$

Equation (3) becomes

$$
0 = (\partial_t h - \phi q^2 + \frac{1}{4k}(b(q + \partial_s h) + \partial_q h)^2
$$

The expression for $v^\star$ also simplifies to

$$
v^\star = -\frac{1}{2k}(\partial_q h + bq)
$$

By separation of variable, let

$$
h(t, q) = h_2(t)q^2
$$

Therefore,

$$
\begin{aligned}
0 &= \partial_t h_2 - \phi + \frac{1}{k}(h_2 + \frac{1}{2}b)^2 \\
h_2(T) &= -\alpha
\end{aligned}
$$

Rewrite $h_2(t) = -\frac{1}{2}b + \chi(t)$ with $\chi(T) = \frac{1}{2}b - \alpha$:

$$
\begin{aligned}
0 &= \partial_t \chi - \phi + \frac{1}{k}\chi^2 \\
0 &= \partial_t \chi k - \phi k + \chi^2 \\
\frac{\partial_t \chi k}{\phi k - \chi^2} &= \frac{1}{k} \\
\int_t^T \frac{\partial_t \chi k}{\phi k - \chi^2} ds &= \int_t^T \frac{1}{k} ds \\
\log(\frac{\sqrt{k\phi} + \chi(T)}{\sqrt{k\phi} - \chi(T)}) - \log(\frac{\sqrt{k\phi} + \chi(t)}{\sqrt{k\phi} - \chi(t)}) &= 2\sqrt{\phi}k(T - t)
\end{aligned}
$$

Rewriting gives

$$\chi(t) = \sqrt{k\phi}\frac{1+\zeta e^{2\gamma(T-t)}}{1-\zeta e^{2\gamma(T-t)}}$$

Therefore,

$$\begin{aligned} v^\star &= -\frac{1}{2k}(\partial q(-\frac{1}{2}b + \sqrt{k\phi}\frac{1+\zeta e^{2\gamma(T-t)}}{1-\zeta e^{2\gamma(T-t)}}q^2) + bq) \\ &= -\sqrt{\frac{\phi}{k}}\frac{1+\zeta e^{2\gamma(T-t)}}{1-\zeta e^{2\gamma(T-t)}}Q_t^{v^\star} \end{aligned}$$

Note that as $dQ_t^v = -v_t dt$,

$$\begin{aligned} dQ_t^{v^\star} &= \sqrt{\frac{\phi}{k}}\frac{1+\zeta e^{2\gamma(T-t)}}{1-\zeta e^{2\gamma(T-t)}}Q_t^{v^\star}dt \\ Q_t^{v^\star} &= q_0 \cdot e^{\int_0^t \sqrt{\frac{\phi}{k}\frac{1+\zeta e^{2\gamma(T-t)}}{1-\zeta e^{2\gamma(T-t)}}}ds} \\ &= q_0 \cdot e^{\log(e^{-\gamma(T-s)}-\zeta e^{\gamma(T-s)})\Big|_0^t} \\ &= q_0 \cdot \frac{\zeta e^{\gamma(T-t)} - e^{-\gamma(T-t)}}{\zeta e^{\gamma T} - e^{-\gamma T}} \end{aligned}$$

ii

As, $\lim \phi \to 0$, the expression for $v^\star$ simplifies to

$$v_t^\star = \frac{-2qh_2}{2k}$$

where $\frac{dh_2}{h_2^2} = -\frac{1}{k} + C$. Given the terminal condition, it can be easily calculated that

$$C = -\frac{1}{\alpha} - \frac{T}{K}$$

Substituting gives

$$h_2 = -\frac{\alpha k}{\alpha(T-t) - k}$$

Therefore,

$$v_t^\star = ((T-t) + \frac{k}{\alpha})^{-1}Q_t^{v^\star}, \lim \phi \to 0$$

Similarly, the expression for $Q_t^{v^\star}$ can be simplified to

$$Q_t^{v^\star} = 1 - \frac{t}{T + \frac{k}{\alpha}}, \lim \phi \to 0$$

Note that as ϕ approaches 0, the running penalty in the model disappears, and the expressions for $Q_t^{v^\star}$ and $v_t^\star$ are therefore identical to the results of the scenario with only terminal penalty.

b

i

The resulting HJB takes the form

$$0 = \partial_t H + \mu\partial_s H + \frac{1}{2}\sigma^2\partial_{ss}H + \sup_v((S - kv)v - v(\partial_q H))$$

subject to

$$H(T,S,q) = q(S - \alpha q)$$

The first order condition gives:

$$\begin{aligned}
\frac{\partial((S-kv)v - v(\partial_q H))}{\partial v} &= 0 \\
S - 2kv - \partial_q H &= 0 \\
v^\star &= \frac{S - \partial_q H}{2k}
\end{aligned}$$

ii

Substituting the expression for $v^\star$ into the HJB gives,

$$0 = \partial_t H + \mu \partial_s H + \frac{1}{2}\sigma^2 \partial_{ss} H + \frac{(S - \partial_q H)^2}{4k}$$

We make the ansatz

$$H(t,x,S,q) = Sq + h(t,S,q)$$

where $h(T,S,q) = -\alpha q^2$.
The HJB equation can therefor be written as,

$$0 = \partial_t h + \mu(q + \partial_s h) + \frac{1}{2}\sigma^2 \partial_{ss} H + \frac{(S - \partial_q h)^2}{4k}$$

where $h(T,S,q) = -\alpha q^2$.
Similar to the previous section, we can make the simplification $h(t,S,q) = h(t,q)$.
We rewrite

$$h(t,q) = h_0(t) + h_1(t)q + h_2(t)q^2,$$

where at terminal time:

$$\begin{aligned}
h_0(T) &= 0 \\
h_0(T) &= 0 \\
h_2(T) &= -\alpha
\end{aligned}$$

Noting that

$$\begin{aligned}
\partial_t h &= h_0' + h_1' q + h_2' q^2 \\
\partial_q h &= h_1 + 2q h_2
\end{aligned}$$

The HJB expression thus becomes:

$$\begin{aligned}
h_0' + h_1' q + h_2' q^2 + \mu q + \frac{(h_1 + 2q h_2)^2}{4k} &= 0 \\
(h_2' + \frac{(h_2)^2}{k})q^2 + (h_1' + \mu + \frac{h_1 h_2}{k})q + (h_0' + \frac{h_1^2}{4k}) &= 0
\end{aligned}$$

As this holds for $\forall t \geq 0$, $\forall q \geq 0$,

$$\begin{aligned}
h_2' + \frac{(h_2)^2}{k} &= 0 \\
h_1' + \mu + \frac{h_1 h_2}{k} &= 0 \\
h_2' + \frac{h_2^2}{4k} &= 0
\end{aligned}$$

Solving the third ODE gives,

$$\frac{dh_2}{h_2^2} = -\frac{1}{k} + C$$

As $h_2(T) = -\alpha$,

$$h_2(t) = \frac{\alpha k}{\alpha(t-T) - k} \tag{4}$$

Solving the second ODE gives,

$$h_1' - \frac{1}{T - t + \frac{k}{\alpha}} h_1 = -\mu$$

Let $p(t) = e^{-\int \frac{1}{T-t+\frac{k}{\alpha}} ds} = t - T - \frac{k}{\alpha}$,

$$\begin{aligned} p(t)h_1' - \frac{p(t)}{T - t + \frac{k}{\alpha}} h_1 &= -\mu p(t) \\ d(p(t)h_1) &= \mu(T - t + \frac{k}{\alpha})dt \end{aligned}$$

As $h_1(T) = 0$,

$$h_1(t) = \frac{\frac{\mu}{2}(T-t)((T-t) + \frac{2k}{\alpha})}{T - t + \frac{k}{\alpha}} \tag{5}$$

Combining Equation (4) and (5) gives the expression for $v^\star$

$$\begin{aligned} v^\star &= \frac{S - \partial_q H}{2k} \\ &= -\frac{h_1 + 2qh_2}{2k} \\ &= \frac{Q_t^{v^\star}}{T - t + \frac{k}{\alpha}} - \frac{1}{4k}\mu(T-t)\frac{T - t + \frac{2k}{\alpha}}{T - t + \frac{k}{\alpha}} \end{aligned}$$

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From above we have

$$\begin{aligned} v^\star &= -\frac{dQ_t^{v^\star}}{dt} \\ &= \frac{Q_t^{v^\star}}{T - t + \frac{k}{\alpha}} - \frac{1}{4k}\mu(T-t)\frac{T - t + \frac{2k}{\alpha}}{T - t + \frac{k}{\alpha}} \end{aligned}$$

Hence,

$$Q_t^{v^\star} = -\int_0^t \frac{Q_t^{v^\star}}{T - s + \frac{k}{\alpha}} - \frac{\mu(T-s)}{4k}\frac{T - s + \frac{2k}{\alpha}}{T - s + \frac{k}{\alpha}} ds$$

Denote

$$Q_\infty = \lim_{\alpha\to\infty} Q_t^{v^\star} = -\int_0^t \frac{Q_\infty}{T-s} - \frac{\mu(T-s)}{4k} ds$$

We find

$$Q'_\infty = \frac{Q_\infty}{T-t} - \frac{\mu(T-t)}{4k}$$

$$Q'_\infty - \frac{Q_\infty}{T-t} = -\frac{\mu(T-t)}{4k}$$

Multiplying $e^{\int \frac{1}{T-s} ds} = \frac{1}{T-t}$ to both sides gives

$$d(e^{\int \frac{1}{T-s} ds} Q_\infty) = -\frac{\mu dt}{4k}$$
$$\frac{Q_\infty}{T-t} = \frac{\mu}{4k} + C$$

where it can be easily calculated that $C = \frac{q_0}{T}$.

Therefore, $\lim_{\alpha\to\infty} Q_t^{v^\star} = (T-t)(\frac{\mu t}{4k} + \frac{q_0}{T})$.

Reference

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