Equality types of various types

bool: $(\mathrm{true} = \mathrm{true}) \simeq (\mathrm{false} = \mathrm{false}) \simeq \mathbb{1}$ (Similar for $\mathbb{N}$, other inductive types)

Σ: $(s = t) \simeq \left(\sum_{p : \pi_1 s = \pi_1 t} \mathrm{tr}_p \, \pi_2 s = \pi_2 t \right)$ (Similar for $\times$)

Π. funext: $(f = g) \simeq (f \sim g)$

What about other types?

=-types: For $p, q : a = b$, maybe want

$$(p = q) \simeq \mathbb{1}$$

so that all 'higher paths are trivial'.

- Called **UIP**: uniqueness of identity proofs
- Equivalent to **Axiom K**
- Validated by interpretation into logic, sets.

U: For A, B, want

$$(A = B) \simeq (A \simeq B)$$

Called <u>univalence axiom</u> (UA).

Validated by interpretation into spaces.

<u>Thm</u>. UA implies funext.

- UIP $\wedge$ funext $\not\Rightarrow \bot$
- UA + UIP $\Rightarrow \bot$

We choose UA.

<u>Def</u>. For types $A, B : U$, have

$$\text{id-to-equiv} : A = B \to A \simeq B.$$

<u>Axiom</u>. The univalence axiom asserts

$$ua : \text{isEquiv}(\text{id-to-equiv}).$$

<u>NB</u>: Now the fact that isEquiv is a proposition is very nice.

<u>NB</u>: In earlier lectures we had a lot of small facts about equivalences that were cited by Rijke.

e.g. For $x:A \vdash Bx, Cx$ if $Bx \simeq Cx$ for all x, then $\sum_{x:A} Bx \simeq \sum_{x:A} Cx$.

This is provable without UA, but it is obvious with the univalence axiom, since everything respects equality.

<u>Homotopy content</u>.

Before UA, everything was interpretable in Set. $\longrightarrow$ Now, it is not.

The universe cannot be a set.

Intuition: if everything else is a set, then
U has terms: every set
U has paths: every equivalence of sets (isomorphisms)
→ so U is the groupoid of sets

<u>Ex</u>. Consider $\text{bool} : U$. We have
$$\text{id}, \text{not} : (\text{bool} \simeq \text{bool}) = (\text{bool} = \text{bool})$$

If there was
$$\alpha : \mathrm{id} = \mathrm{not},$$
then $\alpha\,\mathrm{true} : (\mathrm{true} = \mathrm{false}) \simeq \emptyset$,
so by contradiction $\mathrm{id} \neq \mathrm{not}$.
Thus, $\mathrm{bool} = \mathrm{bool}$ is not a prop, so U is not a set.

Univalence for logic and sets

Thm. The univalence axiom implies
$$(P =_{\mathrm{Prop}} Q) \simeq (P \leftrightarrow Q).$$

Lem. $(P \leftrightarrow Q)$ is a proposition.

Cor. Prop is a set.

Def. $\mathrm{Set} := \sum_{S:\mathrm{Type}} \mathrm{isSet}(S)$

Thm. The univalence axiom implies
$$(A =_{\mathrm{Set}} B) \simeq (A \cong B).$$

Pf. We have
$$(A =_{\mathrm{Set}} B) \simeq (A =_{U} B) \simeq (A \simeq B)$$

Now,

$$\text{isEquiv}\, f \rightleftarrows \text{isqEquiv}\, f$$

$$\text{isqEquiv}\, f :\equiv \sum_{g:B\to A} \underbrace{gf = \text{id}}_{\substack{\text{prop} \\ \underbrace{\prod_x gfx = x}_{\text{prop}}}} \times \underbrace{fg = \text{id}}_{\substack{\text{prop} \\ \underbrace{\prod_y fgy = y}_{\text{prop}}}}$$

Given such g, g', we have

$$g = gfg' = g'$$

so isqEquiv f is a prop.

By univalence for props, we find

$$\text{isEquiv}\, f \simeq \text{isqEquiv}\, f,$$

so $(A \simeq B) \simeq A \cong B)$ and thus $(A = B) \simeq (A \cong B)$. $\square$

Lem. $A \cong B$ is a set.

Pf. $A \cong B :\equiv \sum_{f:A\to B} \underbrace{\text{isEquiv}\, f}_{\text{prop}}$

so it suffices to show $A \to B$ is a set. Consider $f, g : A \to B$.

Then

$$f = g \simeq \underbrace{\prod_{x:A} \underbrace{fx = gx}_{\text{Prop}}}_{\text{Prop}}$$

$\square$

<u>Cor.</u> Set is a groupoid.

<u>Groups</u>

<u>Def.</u> $\mathrm{Grp} := \sum_{G:\mathrm{Set}} \sum_{e:G} \sum_{m:G \to G \to G} \sum_{i:G \to G} \prod_{x:G} (m(e,x) = x) \times (m(x,e) = x)$
$\times \prod_{x,y,z:G} ((xy)z = x(yz))$
$\times \prod_{x:G} (m(ix,x) = e) \times (m(x,ix) = e)$.

<u>Q.</u> Why do we ask G to be a set?

<u>Thm.</u> The univalence axiom implies

$$(G =_{\mathrm{Grp}} H) \simeq (G \cong H)$$

<u>Pf.</u> sketch $(G =_{\mathrm{Grp}} H)$ is equivalent to a Σ-type of equalities.

The first component is in $(G =_{Set} H) \overset{E}{\simeq} (G \underset{Set}{\cong} H)$.

Transporting e along $p : G = H$ is the same as $(Ep)e$.

$$tr_p\, e = (Ep)e$$

by path induction, so

$$(tr_p e_G = e_H) \simeq ((Ep)e_G = e_H)$$

So the beginning of this Σ-type is

$$\sum_{i : G \underset{Set}{\cong} H} \; \sum_{p : ie_G = e_H} \; \sum_{q : i m_G i^{-1} = m_H} \cdots$$

$\square$

Lem. $G \cong H$ is a set.

Cor. Grp is a groupoid.

Fact. We have the same univalence principle for any algebraic structure on a set.

Moral: univalence allows us to do mathematics up to the appropriate notion of sameness in a type (in these examples).

→ 'Structure identity principle' (Aczel, Coquand)

→ 'identity of indiscernables' (Leibniz)