Definition 3.1. A *category with families* consists of the following.

- A category $\mathcal{C}$.
- A presheaf $\mathcal{T} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$.
- A copresheaf $\mathcal{S} : \int \mathcal{T} \to \mathcal{S}et$ where $\int$ denotes the Grothendieck construction. In other words, for every $\Gamma \in \mathcal{C}$ and $A \in \mathcal{T}(\Gamma)$, there is a set $\mathcal{S}(\Gamma, A)$; for every $s : \Delta \to \Gamma$, there is a function $\mathcal{S}(f, A) : \mathcal{S}(\Gamma, A) \to \mathcal{S}(\Delta, \mathcal{T}(f)A)$; and this is functorial.
- For each object Γ of $\mathcal{C}$ and for each $A \in \mathcal{T}(\Gamma)$, there is an object $\pi_\Gamma : \Gamma.A \to \Gamma$ of $\mathcal{C}/\Gamma$ with the following universal property.

$$\hom_{\mathcal{C}/\Gamma}(s, \pi_\Gamma) \cong \mathcal{S}(\mathcal{T}(s)A).$$

Theorem 3.1. *The syntactic category $\mathcal{C}[\mathbb{T}]$ has the structure of a category with families.*

Proof. The underlying category is $\mathcal{C}[\mathbb{T}]$.

For the presheaf $\mathcal{T} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$, we set $\mathcal{T}(\Gamma)$ to be the types of $\mathbb{T}$ in context Γ. Given $s : \Gamma \to \Delta$, we set $\mathcal{T}(\Delta) \to \mathcal{T}(\Gamma)$ to be substitution by s, which we have previously denoted $-[s]$.

For the copresheaf $\mathcal{S} : \int \mathcal{T} \to \mathcal{S}et$, we set $\mathcal{S}(\Gamma, A)$ to be the terms of A in context Γ. Given $s : \Delta \to \Gamma$, the function $\mathcal{S}(f, A) : \mathcal{S}(\Gamma, A) \to \mathcal{S}(\Delta, \mathcal{T}(f)A)$ is also given by substitution by s.

We have objects $\pi_\Gamma : \Gamma.A \to \Gamma$ of $\mathcal{C}/\Gamma$. For the universal property, consider an arbitrary $s : \Delta \to \Gamma$. Then an $f \in \hom_{\mathcal{C}/\Gamma}(s, \pi_\Gamma)$ consists of (many components which must coincide with s and) and one component

$$\Delta \vdash f : A[s],$$

which is exactly an element of $\mathcal{S}(\mathcal{T}(s)A)$. ◻

Exercise 3.1. *Construct a category with families from an arbitrary display structure, and show that when you apply this to the display structure of Theorem 2.1, you obtain the same category with families as above.*

Exercise 3.2. *Show that from any category with families you obtain a display structure. What is the relationship between this construction and the above construction?*

Categories with families from universes

Thm. Consider a category $\mathcal{C}$ with a distinguished morphism $\begin{matrix}\tilde{U}\\ \downarrow \pi_U \\ U\end{matrix}$ such that for any $f : \Gamma \to U$, there exists a pullback.

$$\begin{tikzcd} \Gamma.A \arrow[r] \arrow[d] & \tilde{U} \arrow[d, "\pi_U"] \\ \Gamma \arrow[r] & U \end{tikzcd}.$$

We can construct a category with families by taking

1) the underlying category to be $\mathcal{C}$
2) the presheaf $Ty : \mathcal{C}^{op} \to Set$ to be given by $hom(-, U)$
3) context extension $\pi_\Gamma : \Gamma.A \to \Gamma$ to be $\Gamma.A$.
4) the copresheaf $Tm : \int Ty \to Set$ to be given by

   $Tm(\Gamma, A) := \{\text{sections of } \pi_\Gamma\}$

   $Tm(f, A) :=$ given by the universal property

   $: Tm(\Gamma, A) \to Tm(\Delta, (Ty\, f) A)$

5) $hom_{\mathcal{C}/\Gamma}(s, \pi_\Gamma) \cong Tm(\Delta, Ty(s) A)$

   $\cong hom_{\mathcal{C}/\Delta}(\mathbb{1}, \chi(s\, g))$

comes from the universal property

Ex. We have such a morphism $\pi_U : \tilde{U} \to U$ in groupoids, where U is the (large) groupoid of small groupoids.

What is $\tilde{U}$?

<u>The Grothendieck construction</u>

We consider functors

$$F : \mathcal{G} \to U.$$

Ex. $F(X) : \mathcal{G}/X$

Ex. Given any presheaf $P : \mathcal{G}^{op} \to Set$, postcompose to get $P : \mathcal{G}^{op} \to U$.

Def. The <u>Grothendieck construction</u> produces a functor

$$\begin{array}{c} \int F \\ \pi_F \big\downarrow \\ \mathcal{G} \end{array}$$

that is the analog of a Σ-type (with projection).

The objects of $\int F$ are pairs (G, X) where $G \in \mathcal{G}$ and $X \in FG$.
The morphisms $(G, X) \to (H, Y)$ consist of $f : G \to H$ and

$g : F(f) X \to Y$.

NB. Notice the similarity between the morphisms of $\int F$ and the characterization of $=$ in Σ-types.

Obs. Functors of the form $\pi_F : \int F \to \mathcal{G}$ have a special property. Consider a fiber $\pi_F^{-1}(G)$. This is $F(G)$.

$$
\begin{tikzcd}
\int F \arrow[d, "\pi_F"'] & & FG \arrow[r, "Ff"] & FH \\
 & & X \arrow[u, phantom, "\vdots"] & F(f)X \arrow[u, phantom, "\vdots"] \\
\mathcal{G} & & G \arrow[r, "f"] & H
\end{tikzcd}
$$

Since there is a functor $F(G) \to F(H)$ for any $f : G \to H$, we similarly get a functor between fibers.

Furthermore, given $f : G \to H$ and $X \in \int F$ such that $\pi_F X = G$, ('above' G) there is an object $F(f)X$ above H and a canonical morphism $(f, id) : X \to F(f)X$.

Def. An <u>isofibration</u> is a functor $F : \mathcal{C} \to \mathcal{D}$ such that for any $f : G \to H$ in $\mathcal{D}$ and X above G, there is a morphism $\tilde{f} : X \to f^* X$ above G.

Thm. The Grothendieck construction on functors $F : \mathcal{G} \to \mathcal{U}$ always produces an isofibration. This underlies an equivalence* of categories.

Now, we consider

$$U \xrightarrow{1_U} U$$

to get $\tilde{U} := \int 1_U \xrightarrow{\pi} U$.

This gives us a universal "classifying" isofibration. Indeed, every other one is a (2-)pullback

$$\begin{tikzcd} \int A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, "\pi"] \\ \Gamma \arrow[r, "A"'] & U \end{tikzcd}.$$

<u>Cor</u>. There is a category with families whose category of contexts is Grpd, whose types are functors $\mathcal{G} \to U$, and where context extension is given by the Grothendieck construction.

<u>Why CwF and not display maps?</u>

Functors $\mathcal{G} \to U$ are easier to work with than isofibrations.

<u>Strict substitution</u>. The rules of type theory ensure substitution is <u>strictly</u> functorial.

$$\text{E.g.} \quad A[x/x] \doteq A, \quad A[f][g] \doteq A[f[g]].$$

In a category with display maps, substitution is modeled as pullback.

$$\begin{tikzcd} A[x/x] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\ \Gamma \arrow[r, "1"] & \Gamma \end{tikzcd} \qquad \begin{tikzcd} A[f][g] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A[f] \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d] \\ E \arrow[r, "g"] & \Delta \arrow[r, "f"] & \Gamma \end{tikzcd}$$

$\cong A[f[g]]$

but this is only pseudofunctorial in general.

By moving to an equivalent environment where structure is possible/easier to express, we resolve this issue.

Now

$A[x/x]$ is $\Gamma \xrightarrow{1} \Gamma \xrightarrow{A} U$

$A[f][g]$ and $A[f[g]]$ are

$$E \xrightarrow{g} \Delta \xrightarrow{f} \Gamma \xrightarrow{A} U \;.$$

Id types and fibrations

- In an isofibration $F : \mathcal{C} \to \mathcal{D}$, given $f : X \to Y$ in $\mathcal{D}$, get a functor $FX \to FY$.

  This is transport, so isofibrations do act like dependent types.

- The identity type in this model is given by

  $\hom(-,-) : A.A \cong A \times A \longrightarrow Set \hookrightarrow U$ $\qquad x:A,\ y:A \vdash Id(x,y)$

- Reflexivity is given by the identity morphisms.

Σ-types.

- Given a $B : A \to U$, take $\int B$.

$$\frac{x : A \vdash B(x)}{\vdash \sum_{x:A} B(x)}$$

Π-types

- Given $B : A \to U$, need groupoid that represents the collection of terms of B, i.e. sections of $\begin{matrix} A.B \\ \downarrow \\ A \end{matrix}$.

Indeed, remembering that Gpd is a 2-category, the set of sections has the structure of groupoid.