Semantics of HoTT
Lecture Notes

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April 17, 2024

1 Syntactic categories

Consider a Martin-Löf type theory $\mathbb{T}$. By a Martin-Löf type theory, we mean a type theory with the structural rules of Martin-Löf type theory [Hof97]; we are agnostic about which type formers are included in $\mathbb{T}$.

Definition 1.1. The *syntactic category of* $\mathbb{T}$ is the category, denoted $\mathcal{C}[\mathbb{T}]$, consisting of the following.

- The objects are the contexts of $\mathbb{T}$.[1]
- The morphisms are the *context morphisms*. A *context morphism* $f : \Gamma \to \Delta$ consists of terms

$$\begin{aligned} &\Gamma \vdash f_0 : \Delta_0 \\ &\Gamma \vdash f_1 : \Delta_1[f_0/y_0] \\ &\vdots \\ &\Gamma \vdash f_n : \Delta_n[f_0/y_0][f_1/y_1]\cdots[f_{n-1}/y_{n-1}] \end{aligned}$$

  where $\Delta = (y_0 : \Delta_0, y_1 : \Delta_1, ..., y_n : \Delta_n)$.[2]
- Given an object/context Γ, the identity morphism $1_\Gamma : \Gamma \to \Gamma$ consists of [fill in the blank]
- Given morphisms $f : \Gamma \to \Delta$ and $g : \Delta \to \mathrm{E}$, the composition $g \circ f$ is given by [fill in the blank]

Now we show that left unitality, right unitality, and associativity are satisfied.

- [fill in the blank]

[1] These are given up to judgmental equality in $\mathbb{T}$: i.e., if $\Gamma \doteq \Delta$ as contexts, then $\Gamma = \Delta$ as objects.

[2] These morphisms are given up to judgmental equality in $\mathbb{T}$: i.e., if $f_0 \doteq g_0 : \Delta_0, ..., f_n \doteq g_n : \Delta_n[\delta_0/y_0]\cdots[\delta_{n-1}/y_{n-1}]$, then $f = g$ as morphisms.

- [fill in the blank]
- [fill in the blank]

We think of $\mathcal{C}[\mathbb{T}]$ as the syntax of $\mathbb{T}$, arranged into a category.

Lemma 1.1. *The empty context is the terminal object of* $\mathcal{C}[\mathbb{T}]$.

Proof. [fill in the blank] □

2 Display map categories

Definition 2.1. Let $\mathcal{C}$ be a category, and consider a subclass $\mathcal{D} \subseteq \mathrm{mor}(\mathcal{C})$. $\mathcal{D}$ is a *display structure* [Tay99] if for every $d : \Gamma \to \Delta$ in $\mathcal{D}$ and every $s : \mathrm{E} \to \Delta$ in $\mathcal{C}$, there is a given pullback $s^*d \in \mathcal{D}$.

We call the elements of $\mathcal{D}$ *display maps.*

In the syntactic category $\mathcal{C}[\mathbb{T}]$, we are often interested in objects of the form $\Gamma, z : A$ for a context Γ and a type A; these are often written as $\Gamma.A$. We are then often interested in morphisms of the form $\pi_\Gamma : \Gamma.A \to \Gamma$ where each component of π_Γ is given by the variable rule. We think of such a morphism as representing the type A in context Γ.

Theorem 2.1. *The class of maps of the form* $\pi_\Gamma : \Gamma.A \to \Gamma$ *form display structure in the syntactic category* $\mathcal{C}[\mathbb{T}]$.

Proof. [fill in the blank] □

Definition 2.2. Let $\mathcal{C}$ be a category, and consider a subclass $\mathcal{D} \subseteq \mathrm{mor}(\mathcal{C})$. $\mathcal{D}$ is a *class of displays* if $\mathcal{D}$ is stable under pullback.

Lemma 2.1. *Any class of displays is closed under isomorphism.*

Proof. [fill in the blank] □

Corollary 2.1 (to Theorem 2.1)**.** *Let* $\mathcal{D}$ *denote the closure under isomorphism of the class of maps of the form* $\pi_\Gamma : \Gamma.A \to \Gamma$ *in* $\mathcal{C}[\mathbb{T}]$. *Then* $\mathcal{D}$ *is a class of displays.*

Proof. [fill in the blank] □

Now suppose that we close the class of maps of the form $\pi_\Gamma : \Gamma.A \to \Gamma$ under composition. This is then the class of maps of the form $\pi_\Gamma : \Gamma, \Delta \to \Gamma$ where Γ and Δ are arbitrary contexts.

Lemma 2.2. *Now let* $\mathcal{D}$ *denote the class of maps of the form* $\pi_\Gamma : \Gamma, \Delta \to \Gamma$ *in* $\mathcal{C}[\mathbb{T}]$. *Then*

1. *$\mathcal{D}$ is closed under composition,*
2. *$\mathcal{D}$ contains all the maps to the terminal object,*

3. *every identity is in $\mathcal{D}$*

Proof. [fill in the blank] $\square$

Definition 2.3. A clan [Joy17] is a category $\mathcal{C}$ with a terminal object $*$ and a distinguished class $\mathcal{D}$ of maps such that

1. $\mathcal{D}$ is closed under isomorphisms,
2. $\mathcal{D}$ contains all isomorphisms,
3. $\mathcal{D}$ is closed under composition,
4. $\mathcal{D}$ is stable under pullbacks, and
5. $\mathcal{D}$ contains all maps to the terminal object.

Note that the first requirement follows from the others.

Theorem 2.2. *Let $\mathcal{D}$ denote the closure under isomorphism of morphisms of the form $\pi_\Gamma : \Gamma, \Delta \to \Gamma$ in $\mathcal{C}[\mathbb{T}]$. This is a clan.*

Proof. [fill in the blank] $\square$

The presence of Σ-types and a unit type allows us to conflate contexts and types.

Theorem 2.3. *If $\mathbb{T}$ has Σ-types (with both computation/β and uniqueness/η rules [nLa]) and a unit type, then the closure under isomorphism of the class of maps of the form $\pi_\Gamma : \Gamma.A \to \Gamma$ constitutes a clan (and indeed, is the same class as in Theorem 2.2).*

Proof. [fill in the blank] $\square$

References

[Hof97] Martin Hofmann. Syntax and semantics of dependent types. *Extensional Constructs in Intensional Type Theory*, pages 13–54, 1997.

[Joy17] André Joyal. Notes on clans and tribes. *arXiv preprint arXiv:1710.10238*, 2017.

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[Tay99] Paul Taylor. *Practical foundations of mathematics*, volume 59. Cambridge University Press, 1999.