Higher categories in HoTT

First: approaches to higher categories in classical mathematics

- Joyal: $\Delta^{op} \to \mathcal{S}et$
- Grothendieck: complicated structure on sets

  ⋮

- Rezk: $\Delta^{op} \to \mathcal{S}paces$

  $\Delta^{op} \to \Delta^{op} \to \mathcal{S}et$

in $\mathcal{S}et$

We can replicate • (all classical mathematics) in h-sets, but not interesting from a homotopical perspective.

→ See: Lean

Also, this is a lot of data to handle in a proof assistant.

→ See: lack of higher categories in Lean

We want to take advantage of the homotopy in HoTT.

E.g., univalence principles (recall univalence for univalent categories)

So, most approaches replicate •.

Basic problem:

- Rezk's approach starts with a (strict) functor (of 1-categories) $\Delta^{op} \to \mathcal{S}paces$.

- But in HoTT, we don't have a 1-category of spaces. (Morally, we have an $(\infty,1)$-category, but we're trying to define that.)

<u>Solutions</u>:

- Two-level type theory (Annenkov, Capriotti, Kraus, Sattler)
  - Two equality types: one for homotopy, one for strict equality

- RS type theory (Riehl, Shulman)
  - Add a layer of 'shapes' representing Δ^{op} in 'context'. Then put a layer of HoTT on top (i.e., $\mathcal{S}paces$).

- Others based on extending the equality type former to a directed equality type former (North, see also Nuyts, Warren).

Complete Segal Spaces (Rezk)

in classical mathematics

Def. A *complete Segal space* is a simplicial space P (functor

$P : \Delta^{op} \to \mathcal{S}paces$)

$$
\begin{tikzcd}
S_{2,0} & S_{2,1} \arrow[l, shift left] \arrow[l] \arrow[l, shift right] & S_{2,2} \arrow[l, shift left] \arrow[l] \arrow[l, shift right] & \\
S_{1,0} \arrow[u, shift left] \arrow[u] \arrow[u, shift right] & S_{1,1} \arrow[l, shift left] \arrow[l] \arrow[l, shift right] \arrow[u, shift left] \arrow[u] \arrow[u, shift right] & S_{1,2} \arrow[l, shift left] \arrow[l] \arrow[l, shift right] \arrow[u, shift left] \arrow[u] \arrow[u, shift right] & \cdots \\
\underbrace{S_{0,0}} \arrow[u, shift left] \arrow[u, shift right] & \underbrace{S_{0,1}} \arrow[l, shift left] \arrow[l, shift right] \arrow[u, shift left] \arrow[u, shift right] & \underbrace{S_{0,2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[u, shift left] \arrow[u, shift right] &
\end{tikzcd}
$$

s.t. Kan complexes

s.t.

1. P is Reedy fibrant
   - ⇝ For each n, $P_n \to M_n P$ is a fibration.
   - ⇝ **Def.** Let Δ_- denote the subcategory of Δ consisting of all objects of Δ and morphisms $n \to m$ such that $n \leq m$.

     Let $n // \Delta^{op}_-$ denote the full subcategory of n / Δ^{op}_- spanning by all objects except for

id_n.

The n^{th} matching object M_nP is
$$\lim_{i \in n /\!/ \underline{\Delta}^{op}} P(\mathrm{cod}\, i).$$

Ex. The 0^{th} matching object M_0P is
$$\lim_{i \in 0 /\!/ \underline{\Delta}^{op}} P_i \cong \lim_{\emptyset} P_i \cong *.$$

So "Reedy fibrant @ 0" means P_0 is a Kan complex.

Ex. The 1^{st} matching object M_1P is
$$\lim_{i \in 1 /\!/ \underline{\Delta}^{op}} P_i \cong P_0 \times P_0$$

So "Reedy fibrant @ 1" means $P_1 \xrightarrow{\pi} P_0 \times P_0$ is a Kan fib.
Notice that we have a pullback

$$\begin{tikzcd}
P_1 \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d] \\
P_0 \times P_0 \arrow[r, "P_1"] & U
\end{tikzcd}$$

where the fibers of π are given by f, so asking in 'the internal language'
$$x : P_0 \times P_0 \vdash P_1(x) : U$$

or

$$x : P_0,\ y : P_0 \vdash P_1(x, y) : U$$

corresponds to having a Reedy fibrant P_1.

Ex. The 2^{nd} matching object M_2P is

$$\lim_{i \in \partial\Delta^2/\Delta^{op}} P_i \cong \lim \begin{pmatrix} P_1 & P_1 & P_1 \\ P_0 & P_0 & P_0 \end{pmatrix}.$$

Being "Reedy fibrant @ 2" means $P_2 \to$ is a Kan fibration.
Type theoretically,

$$x : P_0,\ y : P_0,\ z : P_0,\ f : P_1(x,y),\ g : P_1(y,z),\ h : P_1(x,z)$$

Ex. Compare with the definition of univalent category.
Start with

- $P_0 : U$
- $x, y : P_0 \vdash P_1(x,y) : U$
- Can encode composition as

$$x,y,z : P_0,\ f : P_1(x,y),\ g : P_1(y,z),\ h : P_1(x,z) \vdash$$
$$\vdash P_2 : U$$

where P_2 is a proposition saying whether $g \circ f = h$.

In fact, the interpretation of univalent categories into Kan complexes is exactly 'truncated' complete Segal spaces.

See: <u>Univalence principle</u> by Ahrens - N - Shulman - Tsementzis
where we extend theory of univalent categories to 'any' 'finite height' categorical str.
↑ i.e., n-categories, not ∞-categories.

(using 2LTT.)

2. The <u>Segal map</u>

$$P_n \longrightarrow \underbrace{P_1 \times_{P_0} P_1 \cdots \times_{P_0} P_1}_{n \text{ times}}$$

is a homotopy equivalence.

(Says that P_n is 'the space of composites of n maps'.

3. P is <u>complete</u> (cf. univalent)

$$s_0 : X_0 \longrightarrow X_{eq}$$ subspace of X_1 that are 'isos/equivs'

is a homotopy equivalence

<u>Thm</u>. In 2LTT, we can replicate this (actually a modification based on semisimplicial sets).

<u>RS Type Theory</u>

Like cubical TT.

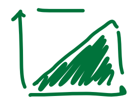
Layers:

1. (cubes) Generated by powers of 2, where 2 is similar to the interval in cubical TT <u>except without</u> $\neg$

   Ex. $x : 2, y : 2$

2. (Topes) 'Subsets' of cubes capturing in particular simplicial sets.

   $x : 2, y : 2 \vdash (x \leq y)\ \text{tope}$



3. MLTT

   Ex. $x : 2, y : 2 \mid x \leq y \vdash A : U$

So this is a tope-indexed MLTT.

They then define Segal types and Rezk types corresponding to Segal spaces and complete Segal spaces in the semantics.