Possible exam questions:

1. What advantages does cubical TT offer over HoTT?

Canonicity

Def. A type theory with $\mathbb{N}$ has canonicity if for every term

$$\vdash n : \mathbb{N}$$

we have

$$\vdash 0 \doteq n : \mathbb{N} \quad \text{or} \quad \vdash s^m 0 \doteq n : \mathbb{N} \text{ for some } m.$$

A type theory with $\mathbb{N}$ and some notion of 'homotopy' $=$ has homotopy canonicity if for every term

$$\vdash n : \mathbb{N}$$

we have

$$\vdash 0 = n : \mathbb{N} \quad \text{or} \quad \vdash s^m 0 = n : \mathbb{N} \text{ for some } m.$$

(Meta definition)

Fact. HoTT does not have canonicity. (Univalence blocks computation)
See: Brunerie's number ($\pi_4 S^3$).

Thm. (Kapulkin-Sattler, unpublished) HoTT has homotopy canonicity.

⇝ Look to the model, since univalence holds there.
⇝ But heavily uses classical logic.

New goal: Model with univalence as constructive theorem.
⇝ new type theory

- work ongoing in sSet by van den Berg + collaborators, Gambino + collaborators
- The models on which CTT is based are not Quillen equivalent to spaces
  (Recent unpublished work by Awodey - Cavallo - Coquand - Riehl - Sattler and others tries to rectify this
  Cavallo - Sattler (See Cavallo's HoTTEST Talk)

Cubical model

All cubical models are based on presheaves $\hat{\square}$ where $\square$ is some cubical category.

Def. The cartesian cube category $\square$ is the free finite product category generated by

$$* \underset{1}{\overset{0}{\rightrightarrows}} \mathbb{I}$$

where $*$ is the terminal object.

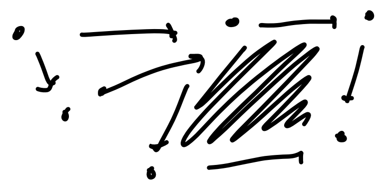
<u>Cubical sets</u> are presheaves $\hat{\square}$

<u>Warning</u>. · There are various versions of $\square$!

<u>Ex</u>. For every $n \in \mathbb{N}$, we have a 'cube' $\mathbb{I}^n$ with projections $\mathbb{I}^{n+1} \to \mathbb{I}^n$, diagonals $\mathbb{I}^n \to \mathbb{I}^{n+1}$, and inclusions $\mathbb{I}^n \to \mathbb{I}^{n+1}$ (generated by 0, 1).

A cubical set is then a collection of

- 0-cells
- 1-cells
- 2-cells
- etc



<u>Fact</u>. There is a geometric realization functor

$$|\cdot| : \hat{\square} \to \mathrm{Top}$$

but this is not a Quillen equivalence.

<u>Thm</u>. This has a model str that models Id-types, Σ-types, Π-types.

$\uparrow$ constructed using constructive logic

Cubical type theory

- We add an explicit interval (in the contexts, not types).
  → layered type theory
- Define a type of paths based on that, prove J-rule and univalence.
- Idea is that $\mathbb{I} \in \widehat{\square}$, but not fibrant.

We let $\mathbb{I}$ be the free de Morgan algebra on a set of variable names.

$0, 1, \wedge, \vee, \neg, \leq,$ st

- $(0, 1, \wedge, \vee, \leq)$ is a bounded distributive lattice
- $\neg$ is an involution
- $\neg, \wedge, \vee$ satisfy De Morgan's law

$$\frac{\Gamma, i : \mathbb{I} \vdash \mathscr{J}}{\Gamma \vdash \mathscr{J}(\varepsilon/i)} \text{ face}$$

$$\frac{\Gamma \vdash \mathscr{J}}{\Gamma, i : \mathbb{I} \vdash \mathscr{J}} \text{ weakening}$$

$$\frac{\Gamma, i : \mathbb{I}, j : \mathbb{I} \vdash \mathscr{J}}{\Gamma, j : \mathbb{I}, i : \mathbb{I} \vdash \mathscr{J}} \text{ exchange}$$

$$\frac{\Gamma, i : \mathbb{I}, j : \mathbb{I} \vdash \mathscr{J}}{\Gamma, i : \mathbb{I} \vdash \mathscr{J}(i/j)} \text{ contraction}$$

$$\frac{\Gamma, i:\mathbb{I} \vdash A \qquad \Gamma \vdash a : A(0/i) \qquad \Gamma \vdash b : A(1/i)}{\Gamma \vdash \mathsf{Path}^i\, A\, a\, b} \qquad \frac{\Gamma, i:\mathbb{I} \vdash A \qquad \Gamma, i:\mathbb{I} \vdash a : A}{\Gamma \vdash \lambda(i:\mathbb{I}).\, a : \mathsf{Path}^i\, A\, a(0/i)\, a(1/i)} \qquad \frac{\Gamma, i:\mathbb{I} \vdash A : \mathscr{U} \qquad \Gamma \vdash a : A(0/i)}{\Gamma \vdash \mathsf{transport}^i\, A\, a : A(1/i)}$$

$$\frac{\Gamma \vdash p : \mathsf{Path}^i\, A\, a\, b \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash p\, r : A(r/i)} \qquad \frac{\Gamma, i:\mathbb{I} \vdash A \qquad \Gamma, i:\mathbb{I} \vdash a : A \qquad \Gamma \vdash r : \mathbb{I}}{\Gamma \vdash (\lambda(i:\mathbb{I}).\, a)\, r = a(r/i) : A(r/i)}\,\beta$$

$$\frac{\Gamma \vdash p : \mathsf{Path}^i\, A\, a\, b}{\Gamma \vdash (\lambda(j:\mathbb{I}).\, p\, j) = p : \mathsf{Path}^i\, A\, a\, b}\,\eta \qquad \frac{\Gamma \vdash p : \mathsf{Path}^i\, A\, a\, b}{\Gamma \vdash p\, 0 = a : A(0/i)} \qquad \frac{\Gamma \vdash p : \mathsf{Path}^i\, A\, a\, b}{\Gamma \vdash p\, 1 = b : A(1/i)}$$

$\mathrm{Path}^i\ A\ a b$ is like a Π-type, except that we can fix the 'endpoints' $a\ b$. See extension types in Riehl-Shulman.

<u>Lem</u>. Given a term $\Gamma \vdash a : A$, we have $\Gamma \vdash r_a : \mathrm{Path}^i\, A\, a\, a$.

<u>Pf</u>. r_a is derived by

$$\frac{\dfrac{\vdash A}{i:\mathbb{I} \vdash A} \qquad \dfrac{}{i:\mathbb{I} \vdash a : A}\,\text{weakening}}{\vdash \lambda(i:\mathbb{I}).\, a : \mathrm{Path}^i\, A\, a\, a}$$

<u>Lem</u>. Functions are functorial, i.e., given $\Gamma \vdash f : A \to B$ and $\Gamma \vdash p : \mathrm{Path}^i A\, a\, b$, we get $\Gamma \vdash \mathrm{ap}\, f\, p : \mathrm{Path}^i B\, fa\, fb$.

Moreover, $\mathrm{ap}\ 1_A\ p \doteq p$

$\mathrm{ap}\ g\ (\mathrm{ap}\ f\ p) \doteq \mathrm{ap}\ (g \circ f)\ p$.

Pf. We have

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash p : \mathrm{Path}^i\, A\ a\ b}{\Gamma, i : \mathbb{I} \vdash p\,i : A} \qquad \Gamma \vdash f : A \to B}{\Gamma, i : \mathbb{I} \vdash f(p\,i) : B}}{\lambda(i : \mathbb{I})\, f(p\,i) : \mathrm{Path}^i\, B\ fa\ fb}$$

The equalities are given by the β-equality for the Path type.

Lem. Functional extensionality is provable, in the sense that

$$\left(\prod_{x : A} \mathrm{Path}^i\, B\ fx\ gx \right) \longrightarrow \left| \mathrm{Path}^i\, (A \to B)\ f\ g \right).$$

Pf.

$$\dfrac{\dfrac{\dfrac{\vdash \pi : \prod_{x:A} \mathrm{Path}^i\, B\ fx\ gx}{a : A \vdash \pi a : \mathrm{Path}^i\, B\ fa\ ga}}{\dfrac{a : A, i : \mathbb{I} \vdash \pi\, a\, i : B}{i : \mathbb{I} \vdash \lambda a.\, \pi\, a\, i : A \to B}}}{\vdash (\lambda i : \mathbb{I})\, \lambda a.\pi a i : \mathrm{Path}^i\, (A \to B)\ f\ g}$$

Just as types are modeled by Kan complexes in the standard model, here they are modelled by fibrant cubical types.

Composition operations:

$$
\begin{tikzcd}
a \arrow[r, dashed] & d \\
b \arrow[u, "p"] \arrow[r, "q"'] & c \arrow[u, "r"']
\end{tikzcd}
\qquad i \uparrow, \quad j \rightarrow
$$

$$
\left.\begin{array}{l}
p : \mathrm{Path}^i\, A\; a\; b \\
q : \mathrm{Path}^j\, A\; b\; c \\
r : \mathrm{Path}^i\, A\; a\; b
\end{array}\right\}
\quad \mathrm{comp}^i\, A \begin{bmatrix} (j=0) \mapsto p \\ (j=1) \mapsto r \end{bmatrix} q
$$

Gluing operations:

$$
\begin{tikzcd}
B_0 \arrow[rr, dashed] & & B_1 \\
A(0/j) \arrow[u, "e_0", "\simeq"'] \arrow[rr, dash, "A"'] & & A(1/j) \arrow[u, "e_1"', "\simeq"]
\end{tikzcd}
$$

$$
\left.\begin{array}{l}
i : \mathbb{I} \vdash A \text{ type} \\
\vdash e_0 : A(0/j) \simeq B_0 \\
\vdash e_1 : A(1/j) \simeq B_1
\end{array}\right\}
\quad \mathrm{Glue} \begin{bmatrix} (j=0) \mapsto (B_0, e_0) \\ (j=1) \mapsto (B_1, e_1) \end{bmatrix} A
$$

Theorem. Univalence.

Pf. Given $A \simeq B$, use gluing to get a path:

$$
\begin{tikzcd}
A \arrow[r, dashed] & B \\
A \arrow[u, "\simeq"] \arrow[r] & A \arrow[u, "\simeq"']
\end{tikzcd}
$$