

Type formers in display maps

Fix category \mathcal{C} , display maps \mathcal{D} .

Contexts \leadsto objects Γ of \mathcal{C}

Types \leadsto display maps $\Gamma.A \xrightarrow{\pi_\Gamma} \Gamma$ of \mathcal{C}

Terms \leadsto sections of display maps (more generally, context morphisms)

Equality \leadsto equality

Substitution \leadsto pullback

Weakening: (also pullback)

$$\begin{array}{ccc} \Delta.\Gamma.A & \longrightarrow & \Gamma.A \\ \downarrow \lrcorner & & \downarrow \pi_\Gamma \\ \Delta.\Gamma & \longrightarrow & \Gamma \\ \downarrow \lrcorner & & \downarrow \pi_\Delta \\ \Delta & \longrightarrow & * \end{array}$$

Now interpret rules for type forms as specifying some structure in $(\mathcal{C}, \mathcal{D})$ by the above dictionary.

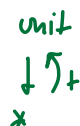
Unit type

	form	intro	elim	comp
Empty context :	$\frac{}{\vdash \text{unit type}}$	$\frac{}{\vdash !:\text{unit}}$	$\frac{x:\text{unit} \vdash D(x) \text{ type} \quad \vdash d:D(!)}{x:\text{unit} \vdash !:D(!)}$	$\frac{x:\text{unit} \vdash D(x) \text{ type} \quad \vdash d:D(!)}{\vdash !!(x) \equiv d:D(!)}$

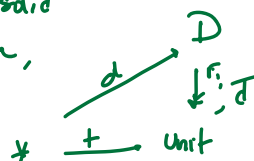
display
map



section



Given solid
diagram,
 $\exists !$

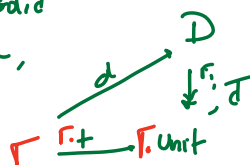


(Write display
maps as \rightarrow
and usually
vertically.)

Nonempty context

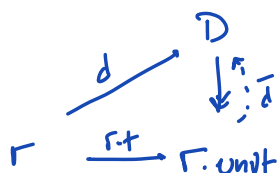
	elim	comp
	$\frac{\Gamma, x:\text{unit} \vdash D(x) \text{ type} \quad \Gamma \vdash d:D(!)}{\Gamma, x:\text{unit} \vdash !:D(!)}$	$\frac{\Gamma, x:\text{unit} \vdash D(x) \text{ type} \quad \Gamma \vdash d:D(!)}{\Gamma \vdash !!(x) \equiv d:D(!)}$

Given solid
diagram,
 $\exists !$



Def. $(\mathcal{C}, \mathcal{D})$ models the unit type if

- there is a display map $\text{unit} \rightarrow *$
- with a section $! : * \rightarrow \text{unit}$
- such that for every commuting diagram of the form of the solid one below, there is a $!$ as below making the diagram commute



Thm. $(\mathcal{C}, \mathcal{D})$ models the unit type if the identity $* \rightarrow * \in \mathcal{D}$.

Pf. Take $! := !_*$. Then $\Gamma. ! : \Gamma \cong \Gamma.*$, so we take $\bar{!}$ to be $d(\Gamma.!)^{-1}$.

NB. If $!_* \in \mathcal{D}$, then every iso $\in \mathcal{D}$.

Σ -types

Def. $(\mathcal{C}, \mathcal{D})$ models Σ -types if

- for every display map $D \xrightarrow{\pi_1} \Gamma$, there is a display map $\Sigma_r D \xrightarrow{\pi_2} \Delta$,
 $\Sigma_r D \rightarrow \Delta$
- with a morphism $D \hookrightarrow \Sigma_r D$ over Δ
- a morphism $\Sigma_r D \xrightarrow{\pi_1} \Gamma$ over Δ
- a morphism $\Sigma_r D \xrightarrow{\pi_2} D$ such that the following commutes

$$\begin{array}{ccc} \Sigma_r D & \xrightarrow{\pi_2} & D \\ \pi_1 \searrow & & \downarrow \\ & & \Gamma \end{array} \quad (\text{over } \Delta)$$

- such that \hookrightarrow and π_2 are inverse.

(following description as neg type w/ uniqueness)

Thm. $(\mathcal{C}, \mathcal{D})$ models Σ -types if \mathcal{D} is closed under composition.

Pf. Take $\Sigma_r D := D$ and $\Sigma_r D \rightarrow \Delta$ to be the composition of $D \rightarrow \Gamma \rightarrow \Delta$.

NB. In $\mathcal{C}(\Pi)$, every object/context Γ has $\Gamma \rightarrow \Gamma' \rightarrow \dots \rightarrow *$, so with Σ -types,

every $\Gamma \rightarrow *$ is a display map. We assume this.

Π -types.

Def. $(\mathcal{C}, \mathcal{P})$ models Π -types if

- for every display map $\pi_r: D \rightarrow \Gamma$, there is a display map $\Pi_r D \rightarrow \Delta$

- such that sections of $\pi_*: \Pi_r D \rightarrow \Delta$ are in bijection with sections of $\pi_r: D \rightarrow \Gamma$ (over Δ)

$\Gamma.E \rightarrow D$



Thm. $(\mathcal{C}, \mathcal{P})$ models Π -types if \mathcal{C} is locally cartesian closed and $\Pi f \in \mathcal{P}$ whenever f, g are.

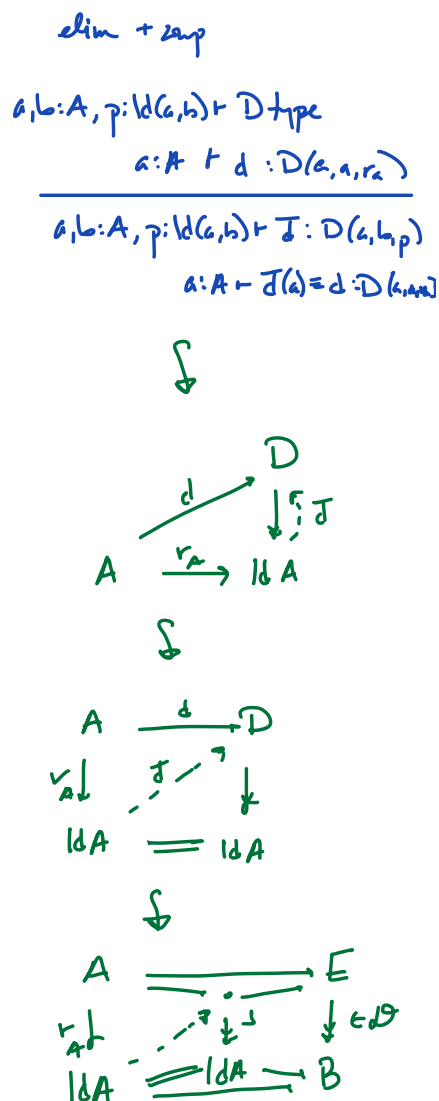
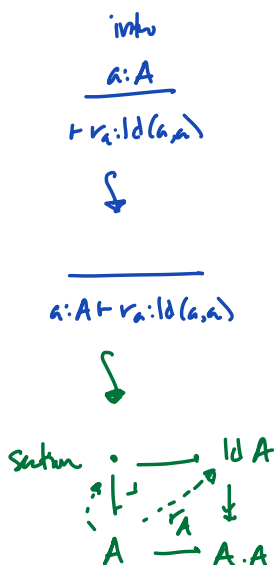
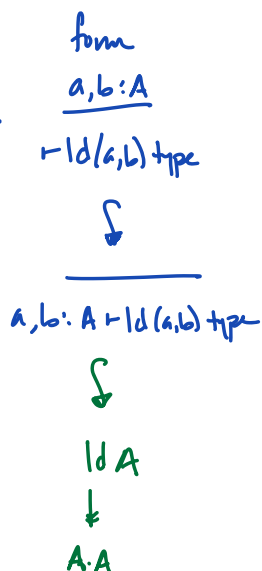
Pf. Being lcc means that for every pair $x \xrightarrow{f} y \xleftarrow{g} z$ of composable morphisms, we have an object $\Pi_g f$ of \mathcal{C}/Z with the universal property

$$\text{hom}_{\mathcal{C}/Z} (z, \Pi_g f) \cong \text{hom}_{\mathcal{C}/Y} (g^* z, f)$$

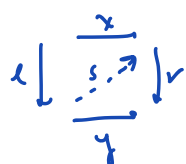
Note that this is exactly the bijection required. □

Id - types

Empty context:



Def. Say that $\mathcal{L} \boxplus \mathcal{R}$ if for every commuting square



\exists s making the diagram commute.

Write $\mathcal{L} \boxplus \mathcal{R}$ if $\forall \ell \in \mathcal{L}, r \in \mathcal{R}, \ell \boxplus r$.

Write $\mathcal{L}^\boxplus := \{r \mid \ell \boxplus r \forall \ell \in \mathcal{L}\}, \mathcal{R}^\boxplus := \{\ell \mid \ell \boxplus r \forall r \in \mathcal{R}\}$.

Ex. $\{r_A \mid A \in \mathcal{C}\} \models \sigma$

Ex. In a model category $(\mathcal{C}, \mathcal{W}, \mathcal{F})$, we have
 $\mathcal{C} \cap \mathcal{W} \models \mathcal{F}$.

Def. A weak factorization system $(\mathcal{L}, \mathcal{R})$ on \mathcal{C} consists of

- classes of morphisms \mathcal{L}, \mathcal{R} such that

$$\mathcal{L}^\perp = \mathcal{R}, \mathcal{L} = \mathcal{R}^\perp$$

- a factorization: a function that takes any $f: X \rightarrow Y$ to

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \scriptstyle Lf & \nearrow \scriptstyle Rf \\ & Mf & \\ \scriptstyle L \nearrow & & \scriptstyle R \searrow \\ & \scriptstyle R & \end{array}$$

Thm. Consider a wfs $(\mathcal{L}, \mathcal{R})$ on a category \mathcal{C} with finite limits. Let \mathcal{C}_\perp denote the full subcategory spanned by objects X s.t. $X \rightarrow * \in \mathcal{R}$. Then

1. The wfs restricts to one $(\mathcal{L}_\perp, \mathcal{R}_\perp)$ on \mathcal{C}_\perp .

1. $(\mathcal{C}_\perp, \mathcal{R}_\perp)$ is a clan. (HW).

2. $(\mathcal{C}_\perp, \mathcal{R}_\perp)$ satisfies the wfs above (Id-types in empty context).

Pf. (2) Obtain r_A, l_A via the factorization.

$$\begin{array}{ccccc} A & \xrightarrow{r_A} & l_A & \xrightarrow{\pi_{A,A}} & A \times A \\ & & \searrow & \nearrow & \\ & & \scriptstyle \Delta & & \end{array}$$

Then $r_A \in \mathcal{R}_\perp$, so it satisfies elim + comp.

Ex. In \mathbf{Top} , take $\mathcal{L} := \{\text{maps w/ the ext prop \wedge the eq}\}$

$\mathcal{R} := \{\text{maps w/ the lifting prop}\}$

Get

$$\begin{array}{ccc} A & \xrightarrow{\quad} & A \times A \\ & \searrow c \quad \nearrow ev & \\ & A^{\pm} & \\ & \vdots & \\ & Id_A & \end{array}$$

not loc

Ex. In \mathbf{sSet} , take $\mathcal{L} := \{\text{closure of } D^i \hookrightarrow D^{i+1}\}$

$\mathcal{R} := \{\text{Kan fibrations}\}$

$\mathcal{L}_f := \text{full subcategory on Kan complexes}$

$$\begin{array}{ccc} A & \xrightarrow{\quad} & A \times A \\ & \searrow c \quad \nearrow ev & \\ & A^{\pm} & \end{array}$$

loc

Ex. In \mathbf{Grpd} , take $\mathcal{R} := \{\text{isofibrations}\}$

$$\begin{array}{ccc} A & \xrightarrow{\quad} & A \times A \\ & \searrow c \quad \nearrow ev & \\ & A^{\cong} & \end{array}$$

not

loc,

but
enough