Type formers in display maps

Fix category $\mathscr{C}$, display maps $\mathscr{D}$.

Contexts $\rightsquigarrow$ objects Γ of $\mathscr{C}$

Types $\rightsquigarrow$ display maps $\Gamma.A \xrightarrow{\pi_\Gamma} \Gamma$ of $\mathscr{C}$

Terms $\rightsquigarrow$ sections of display maps (more generally, context morphisms)

Equality $\rightsquigarrow$ equality

Substitution $\rightsquigarrow$ pullback

Weakening : (also pullback)

$$\begin{tikzcd}
\Delta.\Gamma.A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d, "\pi_\Gamma"] \\
\Delta.\Gamma \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma \arrow[d, "\pi_*"] \\
\Delta \arrow[r] & *
\end{tikzcd}$$

Now interpret rules for type forms as specifying some structure in $(\mathscr{C}, \mathscr{D})$ by the above dictionary.

Unit type

	form	intro	elim	comp
Empty context:	$\dfrac{}{\vdash \mathrm{unit}\ \mathrm{type}}$	$\dfrac{}{\vdash t : \mathrm{unit}}$	$\dfrac{x:\mathrm{unit} \vdash D(x)\ \mathrm{type} \quad \vdash d : D(t)}{x:\mathrm{unit} \vdash J : D(t)}$	$\dfrac{x:\mathrm{unit} \vdash D(x)\ \mathrm{type} \quad \vdash d : D(t)}{\vdash J(x) \equiv d : D(t)}$

display map $\begin{tikzcd} \mathrm{unit} \arrow[d] \\ * \end{tikzcd}$ section $\begin{tikzcd} \mathrm{unit} \arrow[d, shift left] \\ * \arrow[u, shift left, "t"] \end{tikzcd}$

Given solid diagram, $\exists J$

$$\begin{tikzcd} & D \arrow[d] \\ * \arrow[ur, "d"] \arrow[r, "t"'] & \mathrm{unit} \arrow[u, dashed, bend right, "J"'] \end{tikzcd}$$

(Write display maps as $\twoheadrightarrow$ and usually vertically.)

Nonempty context

elim

$$\dfrac{\Gamma, x:\mathrm{unit} \vdash D(x)\ \mathrm{type} \quad \Gamma \vdash d : D(t)}{\Gamma, x:\mathrm{unit} \vdash J : D(t)}$$

comp

$$\dfrac{\Gamma, x:\mathrm{unit} \vdash D(x)\ \mathrm{type} \quad \Gamma \vdash d : D(t)}{\Gamma \vdash J(x) \equiv d : D(t)}$$

Given solid diagram, $\exists J$

$$\begin{tikzcd} & D \arrow[d] \\ \Gamma \arrow[ur, "d"] \arrow[r, "\Gamma.t"'] & \Gamma.\mathrm{unit} \arrow[u, dashed, bend right, "J"'] \end{tikzcd}$$

Def. $(\mathcal{C}, \mathcal{D})$ models the unit type if

- there is a display map $\mathrm{unit} \twoheadrightarrow *$
- with a section $t : * \to \mathrm{unit}$
- such that for every commuting diagram of the form of the solid one below, there is a J as below making the diagram commute

$$\begin{tikzcd} & D \arrow[d] \\ \Gamma \arrow[ur, "d"] \arrow[r, "\Gamma.t"'] & \Gamma.\mathrm{unit} \arrow[u, dashed, bend right, "J"'] \end{tikzcd}$$

Thm. $(\mathcal{C}, \mathcal{D})$ models the unit type if the identity $* \to * \in \mathcal{D}$.

Pf. Take $t := 1_*$. Then $\Gamma.t : \Gamma \cong \Gamma.*$, so we take $\bar{t}$ to be $d(\Gamma.t)^{-1}$.

NB. If $1_* \in \mathcal{D}$, then every iso $\in \mathcal{D}$.

Σ-types

Def. $(\mathcal{C}, \mathcal{D})$ models Σ-types if

- for every display map $D \xrightarrow{\pi_\Gamma} \Gamma$ Δ, there is a display map $\Sigma_\Gamma D \to \cancel{*}\ \Delta$,
- with a morphism $D \xrightarrow{\langle\rangle} \Sigma_\Gamma D$ over Δ
- a morphism $\Sigma_\Gamma D \xrightarrow{\pi_1} \Gamma$ over Δ
- a morphism $\Sigma_\Gamma D \xrightarrow{\pi_2} D$ such that the following commutes

$$\begin{tikzcd} \Sigma_\Gamma D \arrow[r, "\pi_2"] \arrow[dr, "\pi_1"'] & D \arrow[d] \\ & \Gamma \end{tikzcd} \quad (\text{over } \Delta)$$

- such that $\langle\rangle$ and π_2 are inverse.

(following description as neg type w/ uniqueness)

Thm. $(\mathcal{C}, \mathcal{D})$ models Σ-types if $\mathcal{D}$ is closed under composition.

Pf. Take $\Sigma_\Gamma D := D$ and $\Sigma_\Gamma D \twoheadrightarrow \Delta$ to be the composition of $D \twoheadrightarrow \Gamma \twoheadrightarrow \Delta$.

NB. In $\mathcal{C}(\mathbb{T})$, every object/context Γ has $\Gamma \twoheadrightarrow \Gamma' \twoheadrightarrow \cdots \twoheadrightarrow *$, so with Σ-types, every $\Gamma \to *$ is a display map. We assume this.

Π-types.

Def. $(\mathcal{C}, \mathcal{D})$ models Π-types if

- for every display map $\pi_\Gamma : D \to \Gamma$, $\to \Delta$ there is a display map $\prod_\Gamma D \to \cancel{*}\ \Delta$

given $E \to \Delta$ and
$$\begin{tikzcd} E \arrow[r] \arrow[dr] & \Pi_\Gamma D \arrow[d] \\ & \Delta \end{tikzcd}$$

- such that sections of $\pi_* : \prod_\Gamma D \to \cancel{*}\ \Delta$ are in bijection with sections of $\pi_\Gamma : D \to \Gamma$ (over Δ)

$$\begin{tikzcd} \Gamma . E \arrow[r] \arrow[d] & D \arrow[dl] \\ \Gamma \end{tikzcd}$$

$$\begin{tikzcd} D \arrow[d] & \prod_\Gamma D \arrow[d] \\ \Gamma \arrow[u, dashed, bend left] \arrow[r] & \cancel{*}\ \Delta \arrow[u, dashed, bend left] \end{tikzcd}$$

Thm. $(\mathcal{C}, \mathcal{D})$ models Π-types if $\mathcal{C}$ is locally cartesian closed and $\prod_g f \in \mathcal{D}$ whenever f, g are.

Pf. Being lcc means that for every pair $X \xrightarrow{f} Y \xrightarrow{g} Z$ of composable morphisms, we have an object $\prod_g f$ of $\mathcal{C}/Z$ with the universal property

$$\hom_{\mathcal{C}/Z}\left(z, \prod_g f\right) \cong \hom_{\mathcal{C}/Y}(g^* z, f)$$

Note that this is exactly the bijection required. $\square$

Id-types

Empty context:

form

$$\frac{a,b:A}{\vdash \mathrm{Id}(a,b)\ \mathrm{type}}$$

$$\rightsquigarrow \quad \frac{}{a,b:A \vdash \mathrm{Id}(a,b)\ \mathrm{type}}$$

$$\rightsquigarrow \quad \begin{tikzcd} \mathrm{Id}\,A \arrow[d] \\ A\cdot A \end{tikzcd}$$

intro

$$\frac{a:A}{\vdash r_a : \mathrm{Id}(a,a)}$$

$$\rightsquigarrow \quad \frac{}{a:A \vdash r_a : \mathrm{Id}(a,a)}$$

$$\rightsquigarrow \quad \text{section} \quad \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Id}\,A \arrow[d] \\ A \arrow[r] \arrow[u, dashed, bend left] \arrow[ur, dashed, "r_A"'] & A\cdot A \end{tikzcd}$$

elim + comp

$$\frac{a,b:A,\ p:\mathrm{Id}(a,b) \vdash D\ \mathrm{type} \qquad a:A \vdash d : D(a,a,r_a)}{a,b:A,\ p:\mathrm{Id}(a,b) \vdash J : D(a,b,p) \qquad a:A \vdash J(a) \equiv d : D(a,a,r_a)}$$

$$\rightsquigarrow \quad \begin{tikzcd} & D \arrow[d] \\ A \arrow[ur, "d"] \arrow[r, "r_A"'] & \mathrm{Id}\,A \arrow[u, dashed, bend right, "J"'] \end{tikzcd}$$

$$\rightsquigarrow \quad \begin{tikzcd} A \arrow[r, "d"] \arrow[d, "r_A"'] & D \arrow[d] \\ \mathrm{Id}A \arrow[r, equal] \arrow[ur, dashed, "J"] & \mathrm{Id}A \end{tikzcd}$$

$$\rightsquigarrow \quad \begin{tikzcd} A \arrow[rr] \arrow[dr] \arrow[d, "r_A"'] & & E \arrow[d, "\in \mathcal{D}"] \\ \mathrm{Id}A \arrow[r, equal] \arrow[rr, bend right] \arrow[ur, dashed] & \bullet \arrow[ur] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[u, phantom] & B \\ & \mathrm{Id}A \arrow[ur] & \end{tikzcd}$$

Def. Say that $l \boxslash r$ if for every commuting square

$$\begin{tikzcd} \cdot \arrow[r, "x"] \arrow[d, "l"'] & \cdot \arrow[d, "r"] \\ \cdot \arrow[r, "y"'] \arrow[ur, dashed, "s"] & \cdot \end{tikzcd}$$

$\exists\, s$ making the diagram commute.

Write $\mathcal{L} \boxslash \mathcal{R}$ if $\forall\, l \in \mathcal{L}, r \in \mathcal{R}$, $l \boxslash r$.

Write $\mathcal{L}^{\boxslash} := \{ r \mid l \boxslash r \ \forall\, l \in \mathcal{L} \}$, ${}^{\boxslash}\mathcal{R} := \{ l \mid l \boxslash r \ \forall\, r \in \mathcal{R} \}$.

Ex. $\{r_A \mid A \in \mathcal{C}\} \boxslash \mathcal{D}$

Ex. In a model category $(\mathcal{C}, \mathcal{W}, \mathcal{F})$, we have
$$\mathcal{C} \cap \mathcal{W} \boxslash \mathcal{F}.$$

Def. A weak factorization system $(\mathcal{L}, \mathcal{R})$ on $\mathcal{C}$ consists of

- classes of morphisms $\mathcal{L}, \mathcal{R}$ such that
  $\mathcal{L}^{\boxslash} = \mathcal{R}$, $\mathcal{L} = {}^{\boxslash}\mathcal{R}$
- a <u>factorization</u>: a function that takes any $f : X \to Y$ to

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "Lf"'] & & Y \\ & Mf \arrow[ur, "Rf"'] & \end{tikzcd}$$

$$\mathcal{L} \ni Lf, \qquad Rf \in \mathcal{R}$$

Thm. Consider a wfs $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$ with finite limits. Let $\mathcal{C}_f$ denote the full subcategory spanned by objects X s.t. $X \to * \in \mathcal{R}$. Then

0. The wfs restricts to one $(\mathcal{L}_f, \mathcal{R}_f)$ on $\mathcal{C}_f$.
1. $(\mathcal{C}_f, \mathcal{R}_f)$ is a clan. (HW).
2. $(\mathcal{C}_f, \mathcal{R}_f)$ satisfies the rules above (Id-types in empty context).

Pf. (2) Obtain r_A, Id_A via the factorization.

$$\begin{tikzcd} A \arrow[r, "r_A"] \arrow[rr, bend right, "\Delta"'] & \mathrm{Id}\,A \arrow[r, two heads, "\pi_{A.A}"] & A \times A \end{tikzcd}$$

Then $r_A \boxslash \mathcal{R}_f$, so it satisfies elim + comp.

Ex. In Top, take $\mathcal{L} :=$ {maps w/ ho ext prop $\wedge$ ho eq}

$\mathcal{R} :=$ {maps w/ ho lifting prop}

Get

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "c"'] & & A \times A \\ & A^I \arrow[ur, "ev"'] & \end{tikzcd}$$

A^I $=$ $\mathrm{Id}A$

not lcc

Ex. In sSet, take $\mathcal{L} :=$ {closure of $D^i \hookrightarrow D^{i+1}$}

$\mathcal{R} :=$ {Kan fibrations}

$\mathcal{C}_f :=$ full subcategory on Kan complexes

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "c"'] & & A \times A \\ & A^I \arrow[ur, "ev"'] & \end{tikzcd}$$

lcc

Ex. In Grpd, take $\mathcal{R} :=$ {iso fibrations}

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, "c"'] & & A \times A \\ & A^{\cong} \arrow[ur, "ev"'] & \end{tikzcd}$$

not lcc, but enough