Semantics of HoTT
Lecture Notes

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1 Syntactic categories

Consider a Martin-Löf type theory $\mathbb{T}$. By a Martin-Löf type theory, we mean a type theory with the structural rules of Martin-Löf type theory [Hof97]; we are agnostic about which type formers are included in $\mathbb{T}$.

Definition 1.1. The *syntactic category of* $\mathbb{T}$ is the category, denoted $\mathcal{C}[\mathbb{T}]$, consisting of the following.

- The objects are the contexts of $\mathbb{T}$.[1]
- The morphisms are the *context morphisms*. A *context morphism* $f : \Gamma \to \Delta$ consists of terms

$$\begin{aligned}
&\Gamma \vdash f_0 : \Delta_0 \\
&\Gamma \vdash f_1 : \Delta_1[f_0/y_0] \\
&\vdots \\
&\Gamma \vdash f_n : \Delta_n[f_0/y_0][f_1/y_1]\cdots[f_{n-1}/y_{n-1}]
\end{aligned}$$

  where $\Delta = (y_0 : \Delta_0, y_1 : \Delta_1, ..., y_n : \Delta_n)$.[2]
- Given an object/context Γ, the identity morphism $1_\Gamma : \Gamma \to \Gamma$ consists of the terms

$$\begin{aligned}
&\Gamma \vdash x_0 : \Gamma_0 \\
&\Gamma \vdash x_1 : \Gamma_1 \doteq \Gamma_1[x_0/x_0] \\
&\vdots \\
&\Gamma \vdash x_n : \Gamma_n \doteq \Gamma_n[x_0/x_0][x_1/x_1]\cdots[x_{n-1}/x_{n-1}]
\end{aligned}$$

  where $\Gamma = (x_0 : \Gamma_0, x_1 : \Gamma_1, ..., x_n : \Gamma_n)$.

[1] These are given up to judgmental equality in $\mathbb{T}$: i.e., if $\Gamma \doteq \Delta$ as contexts, then $\Gamma = \Delta$ as objects.

[2] These morphisms are given up to judgmental equality in $\mathbb{T}$: i.e., if $f_0 \doteq g_0 : \Delta_0, ..., f_n \doteq g_n : \Delta_n[\delta_0/y_0]\cdots[\delta_{n-1}/y_{n-1}]$, then $f = g$ as morphisms.

- Given morphisms $f : \Gamma \to \Delta$ and $g : \Delta \to \mathrm{E}$, the composition $g \circ f$ is given by the terms

$$\begin{aligned} &\Gamma \vdash g_0[f] : \mathrm{E}_0 \\ &\Gamma \vdash g_1[f] : \mathrm{E}_1 \\ &\vdots \\ &\Gamma \vdash g_m[f] : \mathrm{E}_m \end{aligned}$$

  where $\Delta = (y_0 : \Delta_0, ..., y_n : \Delta_n)$, $\mathrm{E} = (z_0 : \mathrm{E}_0, ..., z_m : \mathrm{E}_m)$ and where by $g_i[f]$ we mean $g_i[f_0/y_0] \cdots [f_n/y_n]$.

Now we show that left unitality, right unitality, and associativity are satisfied.

- Given $f : \Gamma \to \Delta$, we find that $f \circ 1_\Gamma$ consists of terms of the form $\Gamma \vdash f_i[x] : \Delta_i$. But $f_i[x] \doteq f_i[x_0/x_0] \cdots [x_n/x_n]$, so this is $\Gamma \vdash f_i : \Delta_i$. Thus, $f \circ 1_\Gamma = f$.

- Given $f : \Gamma \to \Delta$, we find that $1_\Gamma \circ f$ consists of terms of the form $\Gamma \vdash x_i[f] : \Gamma_i$. But $x_i[f]$ is $x_i[f_0/x_0] \cdots [f_n/x_n]$, so this is $\Gamma \vdash f_i : \Gamma_i$. Thus, $1_\Gamma \circ f = f$.

- Given $f : \Gamma \to \Delta$, $g : \Delta \to \mathrm{E}$, and $h : \mathrm{E} \to \mathrm{Z}$, we find that $h \circ (g \circ f)$ consists of terms of the form $\Gamma \vdash h_i[g[f]] : \mathrm{Z}_i$. But

$$\begin{aligned} h_i[g[f]] &\doteq h_i[g_0[f]/y_0] \cdots [g_m[f]/y_m] \\ &\doteq h_i\left[(g_0[f_0/x_0] \cdots [f_n/x_n])/y_0\right] \cdots \left[(g_n[f_0/x_0] \cdots [f_n/x_n])/y_n\right] \\ &\doteq h_i[g_0/y_0] \cdots [g_n/y_n][f_0/x_0] \cdots [f_n/x_n] \\ &\doteq h_i[g][f]. \end{aligned}$$

  Thus, $h \circ (g \circ f) = (h \circ g) \circ f$.

We think of $\mathcal{C}[\mathbb{T}]$ as the syntax of $\mathbb{T}$, arranged into a category.

Lemma 1.2. The empty context is the terminal object of $\mathcal{C}[\mathbb{T}]$.

Proof. Let $*$ denote the empty context. A morphism $\Gamma \to *$ consists of components for each component of $*$, that is, nothing. Thus, morphisms $\Gamma \to *$ are unique. $\square$

2 Display map categories

Definition 2.1. Let $\mathcal{C}$ be a category, and consider a subclass $\mathcal{D} \subseteq \mathrm{mor}(\mathcal{C})$. $\mathcal{D}$ is a *display structure* [Tay99] if for every $d : \Gamma \to \Delta$ in $\mathcal{D}$ and every $s : \mathrm{E} \to \Delta$ in $\mathcal{C}$, there is a given pullback $s^*d \in \mathcal{D}$.

We call the elements of $\mathcal{D}$ *display maps*.

In the syntactic category $\mathcal{C}[\mathbb{T}]$, we are often interested in objects of the form $\Gamma, z : A$ for a context Γ and a type A; these are often written as $\Gamma.A$. We are then often interested in morphisms of the form $\pi_\Gamma : \Gamma.A \to \Gamma$ where each component of π_Γ is given by the variable rule. We think of such a morphism as representing the type A in context Γ.

Theorem 2.2. The class of maps of the form $\pi_\Gamma : \Gamma.A \to \Gamma$ forms a display structure in the syntactic category $\mathcal{C}[\mathbb{T}]$.

Proof. Consider π_Γ and s as below, where π_Γ is a display map and s is an arbitrary map.

$$\begin{tikzcd} \Delta.A[s] \arrow[r, "s.A"] \arrow[d, "\pi_\Delta"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d, "\pi_\Gamma"] \\ \Delta \arrow[r, "s"] & \Gamma \end{tikzcd}$$

Let $\Delta.A[s]$ denote the context $\Delta, z : A[s]$, that is more explicitly:

$$\Delta, z : A[s_0/x_0] \cdots [s_n/x_n].$$

Let π_Δ be the projection given by the variable rule at each component. Let $s.A$ denote the morphism consisting of $\Delta, z : A[s] \vdash s_i : \Gamma_i[s_0/x_0] \cdots [s_{i-1}/x_{i-1}]$ for each component Γ_i of Γ and $\Delta, z : A[s] \vdash z : A[s]$. We claim that this makes the square above into a pullback square.

Consider a context Z with maps $f : Z \to \Delta$ and $g : Z \to \Gamma.A$ making the appropriate square commute. Let h denote the composite $f : Z \to \Gamma$. Then all components of g but the last component coincide with h; denote the last component of g by $Z \vdash g_A : A[h]$. We can construct a map $Z \to \Delta.A[s]$ whose components are f_i for each Δ_i in Δ, and whose last component is $Z \vdash g_A : A[h] \doteq A[s][f]$. By construction, the two appropriate triangles commute, and any other $z : Z \to \Delta.A[s]$ making these two triangles commute will coincide with our constructed map. (The intuition being that the components of $Z \to \Delta.A[s]$ must basically coincide with the non-redundant components of f and g.) $\square$

Definition 2.3. Let $\mathcal{C}$ be a category, and consider a subclass $\mathcal{D} \subseteq \mathrm{mor}(\mathcal{C})$. $\mathcal{D}$ is a *class of displays* if $\mathcal{D}$ is stable under pullback.

Lemma 2.4. Any class of displays is closed under isomorphism.

Corollary 2.5 (to Theorem 2.2)**.** Let $\mathcal{D}$ denote the closure under isomorphism of the class of maps of the form $\pi_\Gamma : \Gamma.A \to \Gamma$ in $\mathcal{C}[\mathbb{T}]$. Then $\mathcal{D}$ is a class of displays.

Now suppose that we close the class of maps of the form $\pi_\Gamma : \Gamma.A \to \Gamma$ under composition. This is then the class of maps of the form $\pi_\Gamma : \Gamma, \Delta \to \Gamma$ where Γ and Δ are arbitrary contexts.

Lemma 2.6. Now let $\mathcal{D}$ denote the class of maps of the form $\pi_\Gamma : \Gamma, \Delta \to \Gamma$ in $\mathcal{C}[\mathbb{T}]$. Then

1. $\mathcal{D}$ is closed under composition,
2. $\mathcal{D}$ contains all the maps to the terminal object,
3. every identity is in $\mathcal{D}$

Proof. Consider two composable maps in $\mathcal{D}$. Then they must be of the form $\pi_{\Gamma,\Delta} : \Gamma, \Delta, \mathrm{E} \to \Gamma, \Delta$ and $\pi_\Gamma : \Gamma, \Delta \to \Gamma$. Then their composition can be written as $\pi_\Gamma : \Gamma, \Delta, \mathrm{E} \to \Gamma$. Then $\mathcal{D}$ is closed under composition.

Since any context Γ can be written as $*, \Gamma$ or $\Gamma, *$, the unique map $\pi_* : \Gamma \to *$ and the identity $\pi_\Gamma : \Gamma \to \Gamma$ are in $\mathcal{D}$. □

Definition 2.7. A clan [Joy17] is a category $\mathcal{C}$ with a terminal object $*$ and a distinguished class $\mathcal{D}$ of maps such that

1. $\mathcal{D}$ is closed under isomorphisms,
2. $\mathcal{D}$ contains all isomorphisms,
3. $\mathcal{D}$ is closed under composition,
4. $\mathcal{D}$ is stable under pullbacks, and
5. $\mathcal{D}$ contains all maps to the terminal object.

Note that the first requirement follows from the others.

Theorem 2.8. Let $\mathcal{D}$ denote the closure under isomorphism of morphisms of the form $\pi_\Gamma : \Gamma, \Delta \to \Gamma$ in $\mathcal{C}[\mathbb{T}]$. This is a clan.

Proof. The first requirement holds by construction.

By Lemma 2.6, $\mathcal{D}$ contains all identities. Since it is then closed under isomorphism, it contains all isomorphism.

The closure under isomorphisms of a class that is closed under composition is still closed under composition, so $\mathcal{D}$ is closed under composition by Lemma 2.6.

Consider any $\pi_\Gamma : \Gamma, \Delta \to \Gamma$. We can write this as a composition of the form

$$\Gamma.\Delta_0 ... \Delta_n \xrightarrow{\pi_{\Gamma.\Delta_0 ... \Delta_{n-1}}} \Gamma.\Delta_0 ... \Delta_{n-1} \xrightarrow{\pi_{\Gamma.\Delta_0 ... \Delta_{n-2}}} \dots \xrightarrow{\pi_\Gamma} \Gamma$$

To take a pullback of $\pi_\Gamma : \Gamma, \Delta \to \Gamma$, we can take pullbacks of each of the component maps (which are in $\mathcal{D}$ by Theorem 2.2) and compose. Since $\mathcal{D}$ is closed under composition, the pullback of π_Γ is in $\mathcal{D}$.

$\mathcal{D}$ contains all maps to the terminal object by Lemma 2.6. □

The presence of Σ-types and a unit type allows us to conflate contexts and types.

Theorem 2.9. If $\mathbb{T}$ has Σ-types (with both computation/β and uniqueness/η rules [nLaa]) and a unit type, then the closure under isomorphism of the class of maps of the form $\pi_\Gamma : \Gamma.A \to \Gamma$ is a clan (and indeed, is the same class as in Theorem 2.8).

Proof. It is clear the the class of maps considered here is contained in the class of Theorem 2.8.

Thus, we show that any map of the form $\pi_\Gamma : \Gamma, \Delta \to \Gamma$ is isomorphism to one of the form $\pi_\Gamma : \Gamma, A \to \Gamma$. We let A be the following iterated Σ-type in context Γ.

$$\sum_{x_0:\Delta_0} \sum_{x_1:\Delta_1} \cdots \sum_{x_{n-1}:\Delta_{n-1}} \Delta_n$$

Then we claim that $\Gamma, \Delta \cong \Gamma.A$ and this commutes with the projections to Γ.

Let the morphism $f : \Gamma, \Delta \cong \Gamma.A$ have components given by the variable rule for each component Γ_i in Γ. For the component corresponding to A, we take

$$\Gamma, x_0 : \Delta_0, ..., x_n : \Delta_n \vdash \langle x_0, ..., x_n \rangle : \sum_{x_0:\Delta_0} \sum_{x_1:\Delta_1} \cdots \sum_{x_{n-1}:\Delta_{n-1}} \Delta_n.$$

For the morphism $g : \Gamma.A \to \Gamma, \Delta$, we again let the components corresponding to each Γ_i be given by the variable rule. For the component at a Δ_i, we take

$$\Gamma, y : \sum_{x_0:\Delta_0} \sum_{x_1:\Delta_1} \cdots \sum_{x_{n-1}:\Delta_{n-1}} \Delta_n \vdash \pi_i y : \Delta_i[\pi_0 y/x_0] \cdots [\pi_{i-1} y/x_{i-1}].$$

These morphisms clearly commute with the projections to Γ, since every component of all the morphisms in question at a Γ_i is given by the variable rule.

The fact that f and g are inverse to each other amount to the computation and uniqueness rules for Σ-types. $\square$

3 Categories with families

Definition 3.1. A *category with families* consists of the following.

- A category $\mathcal{C}$.
- A presheaf $\mathcal{T} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$.
- A copresheaf $\mathcal{S} : \int \mathcal{T} \to \mathcal{S}et$ where $\int$ denotes the Grothendieck construction. In other words, for every $\Gamma \in \mathcal{C}$ and $A \in \mathcal{T}(\Gamma)$, there is a set $\mathcal{S}(\Gamma, A)$; for every $s : \Delta \to \Gamma$, there is a function $\mathcal{S}(f, A) : \mathcal{S}(\Gamma, A) \to \mathcal{S}(\Delta, \mathcal{T}(f)A)$; and this is functorial.
- For each object Γ of $\mathcal{C}$ and for each $A \in \mathcal{T}(\Gamma)$, there is an object $\pi_\Gamma : \Gamma.A \to \Gamma$ of $\mathcal{C}/\Gamma$ with the following universal property.

$$\hom_{\mathcal{C}/\Gamma}(s, \pi_\Gamma) \cong \mathcal{S}(\mathcal{T}(s)A).$$

Theorem 3.2. The syntactic category $\mathcal{C}[\mathbb{T}]$ has the structure of a category with families.

Proof. The underlying category is $\mathcal{C}[\mathbb{T}]$.

For the presheaf $\mathcal{T} : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$, we set $\mathcal{T}(\Gamma)$ to be the types of $\mathbb{T}$ in context Γ. Given $s : \Gamma \to \Delta$, we set $\mathcal{T}(\Delta) \to \mathcal{T}(\Gamma)$ to be substitution by s, which we have previously denoted $-[s]$.

For the copresheaf $\mathcal{S} : \int \mathcal{T} \to \mathcal{S}et$, we set $\mathcal{S}(\Gamma, A)$ to be the terms of A in context Γ. Given $s : \Delta \to \Gamma$, the function $\mathcal{S}(f, A) : \mathcal{S}(\Gamma, A) \to \mathcal{S}(\Delta, \mathcal{T}(f)A)$ is also given by substitution by s.

We have objects $\pi_\Gamma : \Gamma.A \to \Gamma$ of $\mathcal{C}/\Gamma$. For the universal property, consider an arbitrary $s : \Delta \to \Gamma$. Then an $f \in \hom_{\mathcal{C}/\Gamma}(s, \pi_\Gamma)$ consists of (many components which must coincide with s and) and one component

$$\Delta \vdash f : A[s],$$

which is exactly an element of $\mathcal{S}(\mathcal{T}(s)A)$. $\square$

Theorem 3.3. Consider a category $\mathcal{C}$ and a display structure $\mathcal{D}$. Assume that for every object Γ of $\mathcal{C}$, the collection of display maps with codomain Γ form a set[3] and that pullback is functorial.

Then there is a category with families on $\mathcal{C}$.

Proof. We construct the category with families as follows.

- The underlying category is $\mathcal{C}$.
- The presheaf $\mathcal{T}$ is given by sending an object Γ of $\mathcal{C}$ to the set of display maps with domain Γ. The contravariant functorial action is given by pullback: i.e. given $T \in \mathcal{T}(\Gamma)$ and $f : \Delta \to \Gamma$, set $\mathcal{T}(f)T := f^*T$.
- The copresheaf $\mathcal{S}$ is given by setting $\mathcal{S}(\Gamma, T)$ to be the set of sections of T. The functorial action is again given by pullback.
- For each object Γ of $\mathcal{C}$ and $T \in \mathcal{T}(\Gamma)$, we take $\Gamma.T$ to be the domain of the display map T, and take π_Γ to be T itself. Now, under the assignments that we have made, the isomorphism we need to establish says that $\hom_{\mathcal{C}/\Gamma}(s, T)$ is in natural bijection with sections of s^*T. But this is given by the universal property of s^*T, and thus holds (and holds naturally). $\square$

Theorem 3.4. Given a category with families $(\mathcal{C}, \mathcal{T}, \mathcal{S})$, there is a display structure $\mathcal{D}$ on $\mathcal{C}$.

Proof. We take $\mathcal{D}$ to be all morphisms of the form $\pi_\Gamma : \Gamma.A \to \Gamma$.

Let $s : \Delta \to \Gamma$. We want to show that $\pi_\Delta : \Delta.(\mathcal{T}(s)A) \to \Delta$ is a pullback of π_Γ along s. Then we will say that π_Δ is the chosen pullback of π_Γ along s.

First, let ι_x denote the composition of the following bijections (where the first and third are part of the definition of category with families, and the middle is functoriality of $\mathcal{T}$).

$$\hom_{\mathcal{C}/\Delta}(x, \pi_\Delta) \cong \mathcal{S}(\mathcal{T}(x)\mathcal{T}(s)A) \cong \mathcal{S}(\mathcal{T}(sx)A) \cong \hom_{\mathcal{C}/\Gamma}(sx, \pi_\Gamma)$$

[3] In general, they may form a proper class. If this hypothesis is not satisfied, you can prove a version of this theorem by introducing a notion of 'smallness' for display maps.

We need to complete the pullback square. Let $\tilde{s}$ denote

$$\iota_{\pi_\Delta}(1_{\pi_\Delta}) \in \hom_{\mathcal{C}/\Gamma}(s\pi_\Delta, \pi_\Gamma).$$

Now we claim that the square below is a pullback.

To this end, consider a $x : Z \to \Delta$ and $y : Z \to \Gamma.A$.

$$\begin{tikzcd}
Z \arrow[rrr, bend left, "y"] \arrow[rrd, bend right, "x"'] & & \Delta.(\mathcal{T}(s)A) \arrow[r, "\tilde{s}"] \arrow[d, "\pi_\Delta"] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma.A \arrow[d, "\pi_\Gamma"] \\
& & \Delta \arrow[r, "s"] & \Gamma
\end{tikzcd}$$

Then we have a morphism $y : sx \to \pi_\Gamma$ in the slice $\mathcal{C}/\Gamma$, and so $\iota_x^{-1}y$ is a morphism $x \to \pi_\Delta$ in $\mathcal{C}/\Delta$. That is, $\iota_x^{-1}y$ is a morphism $Z \to \Delta.(\mathcal{T}(s)A)$ making the bottom-left triangle commute. Naturality of ι_x ensures that the upper triangle commutes.

That is, for any $z : x \to \pi_\Delta$, naturality produces the following commutative diagram where z^* denotes precomposition.

$$\begin{tikzcd}
\hom_{\mathcal{C}/\Delta}(\pi_\Delta, \pi_\Delta) \arrow[r, "\iota_{\pi_\Delta}"] \arrow[d, "z^*"] & \hom_{\mathcal{C}/\Gamma}(s\pi_\Delta, \pi_\Gamma) \arrow[d, "z^*"] \\
\hom_{\mathcal{C}/\Delta}(x, \pi_\Delta) \arrow[r, "\iota_x"] & \hom_{\mathcal{C}/\Gamma}(sx, \pi_\Gamma)
\end{tikzcd}$$

It tells us that $z^*\iota_{\pi_\Delta}1_{\pi_\Delta} = \iota_x z^* 1_{\pi_\Delta}$. But $z^*\iota_{\pi_\Delta}1_{\pi_\Delta} = z^*\tilde{s} = \tilde{s}z$ and $\iota_x z^* 1_{\pi_\Delta} = \iota_x z$. Thus, $\tilde{s}z = \iota_x z$. When $z = \iota_x^{-1}y$, we find that $\tilde{s}(\iota_x^{-1}y) = y$ and the upper triangle commutes.

To show that this is unique, consider another $z : Z \to \Delta.(\mathcal{T}(s)A)$ making the diagram commute. If $\iota_x z = y$, then $z = \iota_x^{-1}y$ since ι_x is a bijection. But by our above calculation, $\iota_x z = \tilde{s}z$, and thus $\iota_x z = y$. $\square$

Exercise 3.5 (Open ended)**.** What is the relationship between the two above constructions?

4 Semantic universes

Definition 4.1. Consider a category $\mathcal{C}$. Say that a morphism $\pi_U : \tilde{U} \to U$ is a *universe* if for any $A : \Gamma \to U$ in $\mathcal{C}$, there exists a chosen pullback, which will be denoted as in the following.

$$\begin{tikzcd}
\Gamma.A \arrow[r] \arrow[d, "\pi_\Gamma"] \arrow[dr, phantom, "\lrcorner", very near start] & \tilde{U} \arrow[d, "\pi_U"] \\
\Gamma \arrow[r, "A"] & U
\end{tikzcd}$$

Theorem 4.2. Consider a category $\mathcal{C}$ with a universe $\pi_U : \tilde{U} \to U$. Let $\mathcal{D}$ denote the class of all pullbacks of π_U. Then $\mathcal{D}$ is a display structure.

Proof. We need to show that there exist chosen pullbacks of any $\pi_Gamma : \Gamma.A \to \Gamma$ along any $f : \Delta \to \Gamma$. We let the chosen pullback be $\Gamma_Delta : \Delta.(Af) \to \Delta$, i.e., the chosen pullback of π_U along Af. By the pullback-pasting law this is a pullback. $\square$

Theorem 4.3. Consider a category $\mathcal{C}$ with a universe $\pi_U : \tilde{U} \to U$. Then $\mathcal{C}$ has the structure of a category with families.

Proof. We need to show that there exist chosen pullbacks of any $\pi_Gamma : \Gamma.A \to \Gamma$ along any $f : \Delta \to \Gamma$. We let the chosen pullback be $\Gamma_Delta : \Delta.(Af) \to \Delta$, i.e., the chosen pullback of π_U along Af. By the pullback-pasting law this is a pullback. We construct the category with families as follows.

- The underlying category is $\mathcal{C}$.
- We set the presheaf $\mathcal{T} := \hom(-, U)$.
- For each object Γ of $\mathcal{C}$ and $A \in \mathcal{T}(\Gamma)$, we take $\pi_\Gamma : \Gamma.A \to \Gamma$ to be the specified pullback of π_U along A.
- The copresheaf $\mathcal{S}$ is given by setting $\mathcal{S}(\Gamma, T)$ to be the set of sections of π_Γ. The functorial action is given by pullback.
- Now, under the assignments that we have made, the isomorphism we need to establish says that $\hom_{\mathcal{C}/\Gamma}(s, \pi_\Gamma)$ is in natural bijection with sections of $s^*(\pi_\Gamma)$. But this is given by the universal property of $s^*(\pi_\Gamma)$ (and is a nice diagram chase to sketch out). $\square$

Exercise 4.4 (Open ended)**.** Here, we have constructed display structures and categories with families out of categories with universes. Above, we constructed display structures from categories with families and vice versa. What is the relationship between all these constructions? Notice that there is something of a mismatch when these constructions are applied to the following example.

Example 4.5. Consider the (1-)category $\mathcal{G}$ of groupoids; this is closed under pullback. Let U denote a groupoid whose objects are small groupoids and whose isomorphisms are functors which are isomorphisms. Let $\pi_U : \tilde{U} \to U$ denote the Grothendieck construction of the identity $U \to U$. Then this is a universe in $\mathcal{G}$, and thus $\mathcal{G}$ has the structure of a display structure (where display maps are pullbacks of π_U, equivalently 'small' isofibrations) and a category with families (where types are functors into U).

See [nLac] and [nLab] for details about isofibrations and the Grothendieck construction.

Exercise 4.6. Show that every pullback of π_U is an isofibration. Show that every isofibration with small fibers is a pullback of π_U. You should use the Grothendieck construction.

5 Type formers in display maps

In this section, fix a category $\mathcal{C}$ with display maps $\mathcal{D}$. We will only make minimal assumptions in order to see why the other assumptions are natural. We assume that $\mathcal{C}$ has a terminal object and that $\mathcal{D}$ is stable under pullback.

We explain how to interpret various type formers in a class of display maps. The rules that comprise a type former stipulate some structure involving contexts, types, and terms. Thus, we will interpret these rules as stipulating some structure in such a category with display maps involving objects (contexts), display maps (types), and morphisms (terms). We interpret substitution by pullback and weakening also by certain pullbacks.

We ask that all operations are stable under pullback because in the type theory they are stable under substitution. In cases where it is possible to ask for functoriality of this stability, we do, since that is the case in type theory.

Theorem 5.1. Consider a contexts Γ, Δ and Γ, Δ' in our type theory $\mathbb{T}$. In the syntactic category $\mathcal{C}[\mathbb{T}]$, the context Γ, Δ, Δ' is the pullback of the projections $\Gamma, \Delta \to \Gamma$ and $\Gamma, \Delta' \to \Gamma$.

Proof. [fill in the blank] $\square$

5.1 The unit type

Consider the rules for the unit type. Interpreted semantically, they say that there exists a unit type, i.e. a display map $\mathtt{unit} \to *$ (by formation), together with a section $\mathtt{tt} : * \to \mathtt{unit}$ (introduction). The elimination rule tells us that given any object Γ and display map $\pi_{\Gamma.\mathtt{unit}} : D \to \Gamma.\mathtt{unit}$ together with a morphism $d : \Gamma \to D$ making the solid diagram below commute, there is a section $\overline{d}$ of $\pi_{\Gamma.\mathtt{unit}}$. Here, $\Gamma.\mathtt{tt}$ is the weakening of $\mathtt{tt}$ by Γ, so it is the product of $\mathtt{tt}$ by Γ. The computation rule tells us that this section makes the diagram commute.

$$\begin{tikzcd}
 & \Gamma.\mathtt{unit}.D \arrow[d, "\pi_{\Gamma.\mathtt{unit}}"] \\
\Gamma \arrow[ur, "d"] \arrow[r, "\Gamma.\mathtt{tt}"] & \Gamma.\mathtt{unit} \arrow[u, dotted, bend left, "\overline{d}"]
\end{tikzcd}$$

Moreover, this needs to be stable under substitution in Γ. That is, given a morphism $s : \Delta \to \Gamma$, pulling back the above diagram along s should produce the corresponding diagram in context Δ.

Definition 5.2. A category with display maps $(\mathcal{C}, \mathcal{D})$ *models Σ-types* if

- there is a display map $\mathtt{unit} \to *$
- with a section $\mathtt{tt} : * \to \mathtt{unit}$ such that
- for any diagram of the form of the following solid diagram,

$$\begin{tikzcd}
 & \Gamma.\mathtt{unit}.D \arrow[d, "\pi_{\Gamma.\mathtt{unit}}"] \\
\Gamma \arrow[ur, "d"] \arrow[r, "\Gamma.\mathtt{tt}"] & \Gamma.\mathtt{unit} \arrow[u, dotted, bend left, "\overline{d}"]
\end{tikzcd}$$

there is a dotted arrow as above making the diagram commute

- and given any morphism $s : \Delta \to \Gamma$, the above operation is stable under pullback: that is, $\overline{s^*d} = s^*\overline{d}$.

Theorem 5.3. Suppose that for the terminal object $*$, the identity $1_* : * \to *$ is a display map. Then $(\mathcal{C}, \mathcal{D})$ gives an interpretation of the unit type.

Proof. [fill in the blank] □

We want our category with display maps to model terminal objects, so we often assume the hypothesis of the preceding theorem. But then we find the following.

Lemma 5.4. The identity $1_* : * \to *$ is a display map if and only if all isomorphisms are display maps.

Proof. [fill in the blank] □

6 Dependent sum types

Consider the rules for the Σ-types. The formation rule tells us that for any judgment of the form $\Gamma, A \vdash B$ TYPE, there is a judgment of the form $\Gamma \vdash \Sigma_A B$ TYPE. Thus, semantically we interpret this as saying that for any pair of display maps $\Gamma.A.B \xrightarrow{\pi_{\Gamma.A}} \Gamma.A \xrightarrow{\pi_\Gamma} \Gamma$, there is a display map $\pi_\Gamma : \Sigma_A B \to \Gamma$.

The introduction rule gives us a judgment of the form $\Gamma, x : A, y : B \vdash \langle x, y \rangle : \Sigma_A B$. We interpret this as a context morphism $\langle\rangle : \Gamma.A.B \to \Sigma_A B$ over Γ: that is, the following diagram must commute.

$$\begin{tikzcd}
\Gamma.A.B \arrow[r, "\langle\rangle"] \arrow[d, "\pi_{\Gamma.A}"] & \Sigma_A B \arrow[dd, "\pi_\Gamma"] \\
\Gamma.A \arrow[dr, "\pi_\Gamma"] & \\
 & \Gamma
\end{tikzcd}$$

Now, we follow the negative presentation of Σ-types, including both the computation and uniqueness rules (see [nLaa]). When one includes the uniqueness rule, this is often called a *strong* Σ*-type*.

The elimination rules give us judgments $\Gamma, x : \Sigma_A B \vdash \pi_1 x : A$ and $\Gamma, x : \Sigma_A B \vdash \pi_2 x : B(\pi_1 x)$. Semantically, this corresponds to a morphism $\pi : \Gamma.\Sigma_A B \to \Gamma.A.B$ over Γ: that is, the following diagram must commute.

$$\begin{tikzcd}
\Sigma_A B \arrow[r, "\pi"] \arrow[dd, "\pi_\Gamma"'] & \Gamma.A.B \arrow[d, "\pi_{\Gamma.A}"] \\
 & \Gamma.A \arrow[dl, "\pi_\Gamma"] \\
\Gamma &
\end{tikzcd}$$

The computation rule tells us that $\pi\langle\rangle = 1_{\Gamma.A.B}$ and uniqueness tells us that $\langle\rangle\pi = 1_{\Gamma.\Sigma_A B}$.

Moreover, we ask that this structure is stable under pullback (up to isomorphism). Thus, for any $s : \Delta \to \Gamma$, we must have $s^*(\Sigma_A B) \cong \Sigma_{s^*A}(s^*B)$.

Definition 6.1. A category with display maps $(\mathcal{C}, \mathcal{D})$ *models strong* Σ*-types* if

- for any pair of display maps $\Gamma.A.B \xrightarrow{\pi_{\Gamma.A}} \Gamma.A \xrightarrow{\pi_\Gamma} \Gamma$, there is a display map $\pi_\Gamma : \Sigma_A B \to \Gamma$
- together with an isomorphism $\langle\rangle : \Gamma.A.B \to \Sigma_A B$ making the following diagram commute

$$\begin{tikzcd} \Gamma.A.B \arrow[r, "\langle\rangle"] \arrow[d, "\pi_{\Gamma.A}"] & \Sigma_A B \arrow[dd, "\pi_\Gamma"] \\ \Gamma.A \arrow[dr, "\pi_\Gamma"] & \\ & \Gamma \end{tikzcd}$$

- such that for any $s : \Delta \to \Gamma$, we have an isomorphism $s^*(\Sigma_A B) \cong \Sigma_{s^*A}(s^*B)$ commuting with $\langle\rangle$, where by $\Sigma_{s^*A}(s^*B)$, we mean the domain of the display map obtained from $s^*(\Gamma.A.B) \xrightarrow{s^*(\Gamma.A)} s^*(\Gamma.A) \xrightarrow{\pi_\Delta} \Delta$.

Theorem 6.2. If $\mathcal{D}$ is closed under composition, then $(\mathcal{C}, \mathcal{D})$ models strong Σ-types (and thus it also models Σ-types without the uniqueness rule).

Proof. [fill in the blank] $\square$

Theorem 6.3. If $\mathcal{D}$ is closed under isomorphism and $(\mathcal{C}, \mathcal{D})$ models strong Σ-types, then $\mathcal{D}$ is closed under composition.

Proof. [fill in the blank] $\square$

7 Dependent product types

Consider the rules for the Π-types. The formation rule, which has the same form as the one for Σ-types, tells us that for any judgment of the form $\Gamma, A \vdash B$ TYPE, there is a judgment of the form $\Gamma \vdash \Pi_A B$ TYPE. Thus, semantically we interpret this as saying that for any pair of display maps $\Gamma.A.B \xrightarrow{d} \Gamma.A \xrightarrow{e} \Gamma$, there is a display map $\pi(d, e) : \Pi_A B \to \Gamma$.

We ask that this is stable under pullback/substitution. Thus, we ask that for any morphism $s : \Delta \to \Gamma$, we have an isomorphism $i_s : s^*\pi(d, e) \cong \pi(s^*d, s^*e)$ in $\mathcal{C}/\Delta$.

The introduction rule tells us that given a judgement of the form $\Gamma, x : A \vdash b : B$, we obtain a judgment of the form $\Gamma \vdash \lambda x.b : \Pi_A B$. Semantically, this says that given a section of d, we get a section of π, as illustrated below.

$$
\begin{tikzcd}
\Gamma.A.B \arrow[d, "d"] & \Pi_A B \arrow[d, "\pi(d,e)"'] \\
\Gamma.A \arrow[u, bend left] \arrow[r, "e"'] & \Gamma \arrow[u, bend right, dotted]
\end{tikzcd}
$$

That is, we get a function $\lambda : \hom_{\mathcal{C}/\Gamma.A}(1, d) \to \hom_{\mathcal{C}/\Gamma}(1, \pi(d, e))$.

The elimination rule tells us that given a judgment of the form $\Gamma \vdash f : \Pi_A B$, we get $\Gamma, x : A \vdash fx : B$. Semantically, this corresponds to a function in the other direction: $\epsilon : \hom_{\mathcal{C}/\Gamma}(1, \pi(d, e)) \to \hom_{\mathcal{C}/\Gamma.A}(1, d)$.

The computation rule (β-rule) tells us that $\epsilon\lambda = 1$, and uniqueness (η-rule) tells us that $\lambda\epsilon = 1$. Thus, we have an isomorphism

$$\lambda : \hom_{\mathcal{C}/\Gamma.A}(1, d) \to \hom_{\mathcal{C}/\Gamma}(1, \pi(d, e)).$$

We also ask that this operation is stable under substitution/pullback and that substitution/pullback is functorial. That is, we ask that the following diagrams commute for any $s : \Delta \to \Gamma$ and $t : \mathrm{E} \to \Delta$.

$$
\begin{tikzcd}
\hom_{\mathcal{C}/\Gamma.A}(1, d) \arrow[r, "\lambda"] \arrow[dd, "s^*"] & \hom_{\mathcal{C}/\Gamma}(1, \pi(d, e)) \arrow[d, "s^*"] \\
 & \hom_{\mathcal{C}/\Delta}(1, s^*\pi(d, e)) \arrow[d, "i"] \\
\hom_{\mathcal{C}/s^*(\Delta.A)}(1, s^*d) \arrow[r, "\lambda"] & \hom_{\mathcal{C}/\Delta}(1, \pi(s^*d, s^*e))
\end{tikzcd}
$$

$$
\begin{tikzcd}
\hom_{\mathcal{C}/\Gamma.A}(1, d) \arrow[rr, "\lambda"] \arrow[d, "\cong"] & & \hom_{\mathcal{C}/\Gamma}(1, \pi(d, e)) \arrow[d, "\cong"] \\
\hom_{\mathcal{C}/\Gamma.A}(1, 1_\Gamma^* d) \arrow[r, "\lambda"] & \hom_{\mathcal{C}/\Gamma}(1, \pi(1_\Gamma^* d, 1_\Gamma^* e)) \arrow[r, "i^{-1}"] & \hom_{\mathcal{C}/\Delta}(1, 1_\Gamma^* \pi(d, e))
\end{tikzcd}
$$

$$
\begin{tikzcd}
\hom_{\mathcal{C}/(st)^*(\Gamma.A)}(1, (st)^*d) \arrow[r, "\lambda"] \arrow[d, "\cong"] & \hom_{\mathcal{C}/\mathrm{E}}(1, \pi((st)^*d, (st)^*e)) \arrow[r, "i^{-1}"] & \hom_{\mathcal{C}/\Gamma}(1, (st)^*\pi(d, e)) \arrow[d, "\cong"] \\
\hom_{\mathcal{C}/t^*s^*(\Gamma.A)}(1, t^*s^*d) \arrow[r, "\lambda"] & \hom_{\mathcal{C}/\mathrm{E}}(1, \pi(t^*s^*d, t^*s^*e)) \arrow[r, "i^{-1}"] & \hom_{\mathcal{C}/\Gamma}(1, t^*s^*\pi(d, e))
\end{tikzcd}
$$

However, notice that the second diagram above is an instance of the first.

Definition 7.1. Say that $(\mathcal{C}, \mathcal{D})$ *models* Π-*types* if

- for any pair of display maps $\Gamma.A.B \xrightarrow{d} \Gamma.A \xrightarrow{e} \Gamma$, there is a display map $\pi(d, e) : \Pi_A B \to \Gamma$
- together with an isomorphism $i_s : s^*\pi(d, e) \cong \pi(s^*d, s^*e)$ in $\mathcal{C}/\Delta$ for any morphism $s : \Delta \to \Gamma$ and

- a bijection

$$\lambda : \hom_{\mathcal{C}/\Gamma.A}(1, d) \to \hom_{\mathcal{C}/\Gamma}(1, \pi(d, e))$$

- making the following diagrams commute for any $s : \Delta \to \Gamma$, $t : \mathrm{E} \to \Delta$.

$$\begin{tikzcd}
\hom_{\mathcal{C}/\Gamma.A}(1, d) \arrow[r, "\lambda"] \arrow[dd, "s^*"] & \hom_{\mathcal{C}/\Gamma}(1, \pi(d, e)) \arrow[d, "s^*"] \\
& \hom_{\mathcal{C}/\Delta}(1, s^*\pi(d, e)) \arrow[d, "i_*"] \\
\hom_{\mathcal{C}/s^*(\Delta.A)}(1, s^*d) \arrow[r, "\lambda"] & \hom_{\mathcal{C}/\Delta}(1, \pi(s^*d, s^*e))
\end{tikzcd}$$

$$\begin{tikzcd}
\hom_{\mathcal{C}/(st)^*(\Gamma.A)}(1, (st)^*d) \arrow[r, "\lambda"] \arrow[d, "\cong"] & \hom_{\mathcal{C}/\mathrm{E}}(1, \pi((st)^*d, (st)^*e)) \arrow[r, "i^{-1}"] & \hom_{\mathcal{C}/\Gamma}(1, (st)^*\pi(d, e)) \arrow[d, "\cong"] \\
\hom_{\mathcal{C}/t^*s^*(\Gamma.A)}(1, t^*s^*d) \arrow[r, "\lambda"] & \hom_{\mathcal{C}/\mathrm{E}}(1, \pi(t^*s^*d, t^*s^*e)) \arrow[r, "i^{-1}"] & \hom_{\mathcal{C}/\Gamma}(1, t^*s^*\pi(d, e))
\end{tikzcd}$$

Definition 7.2. Say that $(\mathcal{C}, \mathcal{D})$ is *closed under local exponentials* if for every composable pair of display maps $\Gamma.A.B \xrightarrow{d} \Gamma.A \xrightarrow{e} \Gamma$, there is a display map $\pi(d, e) : \Pi_A B \to \Gamma$ together with the following universal property.

$$\hom_{\mathcal{C}/\Gamma.A}(e^*s, d) \cong \hom_{\mathcal{C}/\Gamma}(s, \pi(d, e))$$

Theorem 7.3. We have that $(\mathcal{C}, \mathcal{D})$ is closed under local exponentials if and only if it models Π-types.

Proof. [fill in the blank] □

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