1 Second-Order Parabolic Differential Equations

Define the elliptic differential operator L_E by

$$L_E u \equiv \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i,j=1}^{n} b_j(x,t)\frac{\partial u}{\partial x_j} + c(x,t)u. \tag{1}$$

Let t represent a time variable. We would have the second-order linear parabolic equation

$$Lu \equiv -\frac{\partial u}{\partial t} + L_E u = f(x,t) \tag{2}$$

This is the n-dimensional second-order linear parabolic equation. Now, if we set $n = 1$, we would get the 1-dimensional second-order linear parabolic equation.

$$-\frac{\partial u}{\partial t} + a(x,t)\frac{\partial^2 u}{\partial x^2} + b(x,t)\frac{\partial u}{\partial x} + c(x,t)u = f(x,t) \tag{3}$$

This is also a Convection-Diffusion Equation, and the coefficients have their own names:

- $a(x,t)$: diffusion term.
- $b(x,t)$: convection term.
- $c(x,t)$: zero term.
- $f(x,t)$: source term.

2 Black Scholes Equation

Define S_t to be the spot at time t, and the option price to be $C(t, S_t)$ at this spot and time. Black Scholes SDE:

$$dS_t = rS_t dt + \sigma S_t dW_t \tag{4}$$

Black Scholes PDE:

$$\frac{\partial C}{\partial t} + rS_t\frac{\partial C}{\partial S_t} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 C}{\partial S_t^2} - rC = 0 \tag{5}$$

Initial condition for a call option:

$$C(T, S_T) = \max(S_T - K, 0)$$

We have 2 types of boundary condition for a call option.
Dirilet condition:

$$C(t,0) = 0,$$
$$C(t, S_{max}) = S_{max} - Ke^{-r(T-t)}.$$

Neumann condition:

$$\frac{\partial^2 C(t,0)}{\partial S^2} = 0,$$
$$\frac{\partial^2 C(t,S_{max})}{\partial S^2} = 0.$$

3 Finite Difference Discretization

Discretization on *time* with N intervals (backward propagation):

$$\begin{aligned}
\Delta t &= T/N, \\
t_n &= n\Delta t, \quad n = N, (N-1), ..., 2, 1, 0.
\end{aligned}$$

Discretization on *spot* with J intervals:

$$\begin{aligned}
\Delta x &= S_{max}/J, \\
S_j &= j\Delta x, \quad j = 0, 1, 2, ..., J.
\end{aligned}$$

Then we use the abbrevation C_j^n to express the call option price $C(t_n, S_j)$.

3.1 Crank-Nicolson Scheme

Using Crank-Nicolson scheme means we are trying to discretize the Black-Scholes equation at the node $(t_{n+\frac{1}{2}}, S_j)$.
Central approximation for $\frac{\partial C}{\partial t}$ at time $t_{n+\frac{1}{2}}$:

$$\frac{\partial C_j^{n+\frac{1}{2}}}{\partial t} = \frac{C_j^{n+1} - C_j^n}{\Delta t} + O(\Delta t^2)$$

Central approximation for $\frac{\partial C}{\partial S}$ and $\frac{\partial^2 C}{\partial S^2}$ at time $t_{n+\frac{1}{2}}$, spot S_j:

$$\begin{aligned}
\frac{\partial C_j^{n+\frac{1}{2}}}{\partial S} &= \frac{1}{2}\left[\frac{\partial C_j^{n+1}}{\partial S} + \frac{\partial C_j^n}{\partial S}\right] \\
&= \frac{1}{2}\left[\frac{C_{j+1}^{n+1} - C_{j-1}^{n+1}}{\Delta x} + \frac{C_{j+1}^n - C_{j-1}^n}{\Delta x}\right] + O(\Delta x^2), \\
\frac{\partial^2 C_j^{n+\frac{1}{2}}}{\partial S^2} &= \frac{1}{2}\left[\frac{\partial^2 C_j^{n+1}}{\partial S^2} + \frac{\partial^2 C_j^n}{\partial S^2}\right] \\
&= \frac{1}{2}\left[\frac{C_{j+1}^{n+1} - 2C_j^{n+1} + C_{j-1}^{n+1}}{\Delta x^2} + \frac{C_{j+1}^n - 2C_j^n + C_{j-1}^n}{\Delta x^2}\right] + O(\Delta x^2).
\end{aligned}$$

The other parameters become:

$$\begin{aligned}
rS &= rj\Delta x, \\
\frac{1}{2}\sigma^2 S^2 &= \frac{1}{2}\sigma^2 j^2 \Delta x^2, \\
C = C_j^{n+\frac{1}{2}} &= \frac{C_j^{n+1} - C_j^{n-1}}{2}.
\end{aligned}$$

The discretized version of Black-Scholes equation is:

$$\begin{aligned}
\frac{C_j^{n+1} - C_j^n}{\Delta t} + rj\Delta x \cdot \frac{1}{2}\left[\frac{C_{j+1}^{n+1} - C_{j-1}^{n+1}}{\Delta x} + \frac{C_{j+1}^n - C_{j-1}^n}{\Delta x}\right] + \\
\frac{1}{2}\sigma^2 j^2 \Delta x^2 \cdot \frac{1}{2}\left[\frac{C_{j+1}^{n+1} - 2C_j^{n+1} + C_{j-1}^{n+1}}{\Delta x^2} + \frac{C_{j+1}^n - 2C_j^n + C_{j-1}^n}{\Delta x^2}\right] - r\frac{C_j^{n+1} - C_j^{n-1}}{2} = 0.
\end{aligned} \tag{6}$$

3.2 Matrix Form of Finite Difference

For the Black-Scholes equation, the Convection-Diffusion terms are specified as below.

- diffusion term: $a_j = a(t_n, S_j) = \frac{1}{2}\sigma^2 j^2 \Delta x^2$
- convection term: $b_j = b(t_n, S_j) = rj\Delta x$
- zero term: $c_j = c(t_n, S_j) = -r$
- source term: $f(t_n, S_j) = 0$

The discretized B-S equation can be transformed to:

$$\begin{cases} -\frac{1}{2\Delta x^2}a_j + \frac{1}{4\Delta x}b_j & \cdot C^n_{j-1} \\ \frac{1}{\Delta t} + \frac{1}{\Delta x^2}a_j - \frac{1}{2}c_j & \cdot C^n_j \\ -\frac{1}{2\Delta x^2}a_j - \frac{1}{4\Delta x}b_j & \cdot C^n_{j+1} \end{cases} = \begin{cases} \frac{1}{2\Delta x^2}a_j - \frac{1}{4\Delta x}b_j & \cdot C^{n+1}_{j-1} \\ \frac{1}{\Delta t} - \frac{1}{\Delta x^2}a_j + \frac{1}{2}c_j & \cdot C^{n+1}_j \\ \frac{1}{2\Delta x^2}a_j + \frac{1}{4\Delta x}b_j & \cdot C^{n+1}_{j+1} \end{cases} \tag{7}$$

Multiplying Δt at both side:

$$\begin{cases} -\frac{a_j\Delta t}{2\Delta x^2} + \frac{b_j\Delta t}{4\Delta x} & \cdot C^n_{j-1} \\ 1 + \frac{a_j\Delta t}{\Delta x^2} - \frac{c_j\Delta t}{2} & \cdot C^n_j \\ -\frac{a_j\Delta t}{2\Delta x^2} - \frac{b_j\Delta t}{4\Delta x} & \cdot C^n_{j+1} \end{cases} = \begin{cases} \frac{a_j\Delta t}{2\Delta x^2} - \frac{b_j\Delta t}{4\Delta x} & \cdot C^{n+1}_{j-1} \\ 1 - \frac{a_j\Delta t}{\Delta x^2} + \frac{c_j\Delta t}{2} & \cdot C^{n+1}_j \\ \frac{a_j\Delta t}{2\Delta x^2} + \frac{b_j\Delta t}{4\Delta x} & \cdot C^{n+1}_{j+1} \end{cases} \tag{8}$$

Define α, β, γ as:

$$\begin{aligned} \alpha_j &= \frac{a_j\Delta t}{2\Delta x^2} - \frac{b_j\Delta t}{4\Delta x}, \\ \beta_j &= -\frac{a_j\Delta t}{\Delta x^2} + \frac{c_j\Delta t}{2}, \\ \gamma_j &= \frac{a_j\Delta t}{2\Delta x^2} + \frac{b_j\Delta t}{4\Delta x}. \end{aligned}$$

Then become:

$$-\alpha_j C^n_{j-1} + (1-\beta_j)C^n_j - \gamma_j C^n_{j+1} = \alpha_j C^{n+1}_{j-1} + (1+\beta_j)C^{n+1}_j + \gamma_j C^{n+1}_{j+1}. \tag{9}$$

Matrix form:

$$\underbrace{\begin{bmatrix} -\alpha_1 & 1-\beta_1 & -\gamma_1 & & & \\ & -\alpha_2 & 1-\beta_2 & -\gamma_2 & & \\ & & \dots & \dots & \dots & \\ & & & -\alpha_{J-1} & 1-\beta_{J-1} & -\gamma_{J-1} \end{bmatrix}}_{(J-1)\times(J+1)} \underbrace{\begin{bmatrix} C^n_0 \\ C^n_1 \\ C^n_2 \\ \vdots \\ C^n_{J-1} \\ C^n_J \end{bmatrix}}_{(J+1)\times 1} = \underbrace{\begin{bmatrix} \alpha_1 & 1+\beta_1 & \gamma_1 & & & \\ & \alpha_2 & 1+\beta_2 & \gamma_2 & & \\ & & \dots & \dots & \dots & \\ & & & \alpha_{J-1} & 1+\beta_{J-1} & \gamma_{J-1} \end{bmatrix}}_{(J-1)\times(J+1)} \underbrace{\begin{bmatrix} C^{n+1}_0 \\ C^{n+1}_1 \\ C^{n+1}_2 \\ \vdots \\ C^{n+1}_{J-1} \\ C^{n+1}_J \end{bmatrix}}_{(J+1)\times 1}, \tag{10}$$

Under different boundary conditions, the matrix form of equation systems are different.

3.2.1 Dirilet Boundary

At the new time step t_n, the boundary node values are known:

$$C(t_n, S_0) = 0,$$
$$C(t_n, S_{J-1}) = S_{max} - Ke^{-r(T-t_n)},$$

where $t_n = n\Delta t$, and $n = N, (N-1), ..., 2, 1, 0$ (back propagation).

The matrix form equation systems (10) can be changed to:

$$\underbrace{\begin{bmatrix} 1-\beta_1 & -\gamma_1 & & \\ -\alpha_2 & 1-\beta_2 & -\gamma_2 & \\ & ... & ... & ... \\ & & -\alpha_{J-1} & 1-\beta_{J-1} \end{bmatrix}}_{(J-1)\times(J-1)} \underbrace{\begin{bmatrix} C_1^n \\ C_2^n \\ \vdots \\ C_{J-1}^n \end{bmatrix}}_{(J-1)\times 1} + \underbrace{\begin{bmatrix} -\alpha_1 C_0^n \\ 0 \\ \vdots \\ -\gamma_{J-1} C_J^n \end{bmatrix}}_{(J-1)\times 1} =$$
$$\underbrace{\begin{bmatrix} 1+\beta_1 & \gamma_1 & & \\ \alpha_2 & 1+\beta_2 & \gamma_2 & \\ & ... & ... & ... \\ & & \alpha_{J-1} & 1+\beta_{J-1} \end{bmatrix}}_{(J-1)\times(J-1)} \underbrace{\begin{bmatrix} C_1^{n+1} \\ C_2^{n+1} \\ \vdots \\ C_{J-1}^{n+1} \end{bmatrix}}_{(J-1)\times 1} + \underbrace{\begin{bmatrix} \alpha_1 C_0^{n+1} \\ 0 \\ \vdots \\ \gamma_{J-1} C_J^{n+1} \end{bmatrix}}_{(J-1)\times 1}, \tag{11}$$

Rearrange the matrix form equation, and moving the known terms C_0^n, C_J^n to right-hand side, we can further get:

$$\underbrace{\begin{bmatrix} 1-\beta_1 & -\gamma_1 & & \\ -\alpha_2 & 1-\beta_2 & -\gamma_2 & \\ & ... & ... & ... \\ & & -\alpha_{J-1} & 1-\beta_{J-1} \end{bmatrix}}_{(J-1)\times(J-1)} \underbrace{\begin{bmatrix} C_1^n \\ C_2^n \\ \vdots \\ C_{J-1}^n \end{bmatrix}}_{(J-1)\times 1} = \underbrace{\begin{bmatrix} 1+\beta_1 & \gamma_1 & & \\ \alpha_2 & 1+\beta_2 & \gamma_2 & \\ & ... & ... & ... \\ & & \alpha_{J-1} & 1+\beta_{J-1} \end{bmatrix}}_{(J-1)\times(J-1)} \underbrace{\begin{bmatrix} C_1^{n+1} \\ C_2^{n+1} \\ \vdots \\ C_{J-1}^{n+1} \end{bmatrix}}_{(J-1)\times 1} + \underbrace{\begin{bmatrix} \alpha_1 C_0^{n+1} + \alpha_1 C_0^n \\ 0 \\ \vdots \\ \gamma_{J-1} C_J^{n+1} + \gamma_{J-1} C_J^n \end{bmatrix}}_{(J-1)\times 1}. \tag{12}$$

3.2.2 Neumann Boundary

At the new time step t_n, the boundary node values are known:

$$\frac{\partial^2 C(t_n, S_1)}{\partial S^2} = 0 \quad \Rightarrow \quad C_0^n - 2C_1^n + C_2^n = 0$$
$$\frac{\partial^2 C(t_n, S_{J-1})}{\partial S^2} = 0 \quad \Rightarrow \quad C_{J-2}^n - 2C_{J-1}^n + C_J^n = 0$$

where $t_n = n\Delta t$, and $n = N, (N-1), ..., 2, 1, 0$ (back propagation).

Then replacing the C_0^n and C_J^n in the matrix form equation systems (10), we get:

$$\underbrace{\begin{bmatrix} -2\alpha_1+1-\beta_1 & \alpha_1-\gamma_1 & & \\ -\alpha_2 & 1-\beta_2 & -\gamma_2 & \\ & ... & ... & ... \\ & & -\alpha_{J-1}+\gamma_{J-1} & 1-\beta_{J-1}-2\gamma_{J-1} \end{bmatrix}}_{(J-1)\times(J-1)} \underbrace{\begin{bmatrix} C_1^n \\ C_2^n \\ \vdots \\ C_{J-1}^n \end{bmatrix}}_{(J-1)\times 1} = \underbrace{\begin{bmatrix} 2\alpha_1+1+\beta_1 & -\alpha_1+\gamma_1 & & \\ \alpha_2 & 1+\beta_2 & \gamma_2 & \\ & ... & ... & ... \\ & & \alpha_{J-1}-\gamma_{J-1} & 1+\beta_{J-1}+2\gamma_{J-1} \end{bmatrix}}_{(J-1)\times(J-1)} \underbrace{\begin{bmatrix} C_1^{n+1} \\ C_2^{n+1} \\ \vdots \\ C_{J-1}^{n+1} \end{bmatrix}}_{(J-1)\times 1}. \tag{13}$$