Chapter 9 | Models and Theories of Turbulence

9.1 Mathematical Models of Turbulent Flows

In Chapter 7 we introduced the equations of continuity, motion, and thermodynamic energy as mathematical expressions of the conservation of mass, momentum, and heat in an elementary volume of fluid. These are applicable to laminar as well as turbulent flows. In the latter, all the variables and their temporal and spatial derivatives are highly irregular and rapidly varying functions of time and space. This essential property of turbulence makes all the terms in the conservation equations significant, so that no further simplifications of them are feasible, aside from the Boussinesq approximations introduced in Chapter 7.

In particular, the instantaneous equations to the Boussinesq approximations for a thermally stratified turbulent flow, in a rotating frame of reference tied to the earth's surface, are

$$\begin{aligned}
&\frac{\partial \tilde{u}}{\partial x}+\frac{\partial \tilde{v}}{\partial y}+\frac{\partial \tilde{w}}{\partial z}=0 \\
&\frac{\partial \tilde{u}}{\partial t}+\tilde{u}\frac{\partial \tilde{u}}{\partial x}+\tilde{v}\frac{\partial \tilde{u}}{\partial y}+\tilde{w}\frac{\partial \tilde{u}}{\partial z}=f\tilde{v}-\frac{1}{\rho_0}\frac{\partial \tilde{p}_1}{\partial x}+\nu\nabla^2\tilde{u} \\
&\frac{\partial \tilde{v}}{\partial t}+\tilde{u}\frac{\partial \tilde{v}}{\partial x}+\tilde{v}\frac{\partial \tilde{v}}{\partial y}+\tilde{w}\frac{\partial \tilde{v}}{\partial z}=-f\tilde{u}-\frac{1}{\rho_0}\frac{\partial \tilde{p}_1}{\partial y}+\nu\nabla^2\tilde{v} \\
&\frac{\partial \tilde{w}}{\partial t}+\tilde{u}\frac{\partial \tilde{w}}{\partial x}+\tilde{v}\frac{\partial \tilde{w}}{\partial y}+\tilde{w}\frac{\partial \tilde{w}}{\partial z}=\frac{g}{T_0}\tilde{T}_1-\frac{1}{\rho_0}\frac{\partial \tilde{p}_1}{\partial z}+\nu\nabla^2\tilde{w} \\
&\frac{\partial \tilde{\theta}}{\partial t}+\tilde{u}\frac{\partial \tilde{\theta}}{\partial x}+\tilde{v}\frac{\partial \tilde{\theta}}{\partial y}+\tilde{w}\frac{\partial \tilde{\theta}}{\partial z}=\alpha_h\nabla^2\tilde{\theta}
\end{aligned} \tag{9.1}$$

Here, a tilde denotes an instantaneous variable which is the sum of its mean and fluctuating parts.

No general solution to this highly nonlinear system of equations is known and there appears to be no hope of finding one through purely analytical methods. However, approximate numerical solutions through the use of supercomputers have been obtained for a variety of low-Reynolds-number turbulent flows. For large-Reynolds-number flows, such as those encountered in the atmosphere including the PBL, only the Reynolds-averaged equations obtained from a grid-volume averaging (spatial filtering) or an ensemble averaging of the instantaneous equations can be solved. Their numerical solutions including the various finite-difference and finite-element methods constitute the rapidly growing field of computational fluid dynamics.

9.1.1 *Direct numerical simulation*

In this age of superfast computers, it is natural to attempt direct numerical solutions of the instantaneous Navier–Stokes equations for the various turbulent flows of interest using the brute force method of numerical integration on supercomputers or networks of small computers or work stations. This method has been called full-turbulence simulation or direct numerical simulation (DNS); so far it has been limited to only low-Reynolds-number turbulent flows.

Since the instantaneous equations describe all the details of the turbulent fluctuating motion, their direct numerical solution with appropriate boundary conditions should yield every detail of its complex behavior in time and space. Numerical methods are available for solving Equations (9.1) in principle, but there is a serious problem of a numerical model's ability to resolve the whole wide range of scales encountered in turbulent flow. In any direct numerical simulation of turbulence, the flow domain is discretized into three-dimensional grid elements. For an appropriate resolution of small-scale motions, which are mostly responsible for the dissipation of turbulence kinetic energy, the grid size must be less than the microscale η, the characteristic length scale of such motions. For the proper resolution of large energy-containing eddies, the model domain should be larger than the largest scales (l) of interest. Thus, the number of three-dimensional grid elements or grid points at which instantaneous equations are solved at each small time step (this is also determined by the microscale) is of the order of $(l/\eta)^3$. This number for the atmospheric boundary layer can be estimated to fall in the range of 10^{15}–10^{20}. The present supercomputers cannot handle more than 10^9 grid points for direct numerical simulation of turbulence. Thus, so far, DNS has been limited to low-Reynolds-number flows with $l/\eta < 10^3$. Apart from the very large storage requirements, much faster computer speeds will be needed before DNS becomes practically feasible for atmospheric applications.

9.1.2 Large-eddy simulation

A computationally more feasible but less fundamental approach which has been used in micrometeorology and engineering fluid mechanics is large-eddy simulation (LES). It attempts to faithfully simulate only the scales of motion in a certain range of scales between the smallest grid size and the largest dimension of simulated flow domain. The smaller subgrid scales are not resolved, but their important contributions to energy dissipation and minor contributions to turbulent transports are usually parameterized through the use of simpler subgrid scale (SGS) models (Mason, 1994). The origins of the LES technique lie in the early global weather prediction and general circulation models in which the permissible grid spacing could hardly resolve large-scale atmospheric structures.

Large-eddy simulations of turbulent flows, including the PBL, started with the pioneering work of Deardorff (1970a, b, 1972a, b, 1973). The soundness of this modeling approach in simulating neutral and unstable PBLs, even in the presence of moist convection and clouds, has been amply demonstrated by subsequent studies of Deardorff and others (Mason, 1994). A pressing need has been the extension of the LES to the nocturnal stable boundary layer, including the morning and evening transition periods, which are poorly understood. The main difficulty in simulating the stably stratified boundary layer is that the characteristic large-eddy scale (l) becomes small (say, < 10 m) under very stable conditions that are typically encountered in the SBL over land surfaces and most of the energy transfer and other exchange processes are overly influenced or dominated by subgrid scale motions. Therefore, recent LES studies of the SBL have been limited to conditions of mild or moderate stability, strong wind shears, and continuous turbulence across the SBL (Andren, 1995; Ding *et al.*, 2000b; Kosovic and Curry, 2000). Satisfactory large-eddy simulations of the transition from the late afternoon, unstable or convective boundary layer to moderately stable boundary layer in the early evening hours have also been accomplished, especially during the early period of continuous but decaying turbulence. Similar LES studies of very stable boundary layers with frequent episodes of turbulence generation by shear and destruction by buoyancy should become possible in the near future as more powerful next-generation computers become available.

A more detailed description of the large-eddy simulation approach and its applications to the PBL will be given in Chapter 13. Here, it should suffice to say that LES is viewed as the most promising tool for future research in turbulence and is expected to provide a better understanding of the transport phenomena, well beyond the reach of semiempirical theories and most routine observations. Since it requires an investment of huge computer resources (there are also a few other limitations), LES will most likely remain a research tool. It would be very useful, of course, for generating numerical turbulence data for studying large-eddy structures, developing parameterizations for simpler models, and designing physical experiments (Wyngaard and Moeng, 1993).

9.1.3 Ensemble-averaged turbulence models

Direct numerical simulation of turbulent flows is based on the 'primitive' Navier–Stokes equations of motion. The large-eddy simulation utilizes the same equations, but averaged over the finite grid volume. Numerical integrations of both systems yield highly irregular, turbulent flow variables as functions of time and space. Average statistics, such as means, variances, and higher moments, are then calculated from this 'simulated turbulence' data. In most applications, however, only the average statistics are needed and how they come about and other details of turbulent motions are usually of little or no concern. One may then ask, 'Why not use the equations of mean motion to compute the desired average statistics?' This question was first addressed by Osborne Reynolds toward the end of the nineteenth century. He also suggested some simple averaging rules or conditions and derived what came to be known as the Reynolds-averaged equations.

If f and g are two independent variables or functions of variables with mean values $\bar{f}$ and $\bar{g}$, and c is a constant, the Reynolds-averaging conditions are

$$\begin{aligned} &\overline{f+g} = \bar{f} + \bar{g} \\ &\overline{cf} = c\bar{f};\ \overline{\bar{f}g} = \bar{f}\bar{g} \\ &\overline{\partial f/\partial s} = \partial \bar{f}/\partial s;\ \overline{\int f ds} = \int \bar{f} ds \end{aligned} \tag{9.2}$$

where $s = x$, y, z, or t. These conditions are used in deriving the equations for mean variables from those of the instantaneous variables. Since these are strictly satisfied only by ensemble averaging, this type of averaging is generally implied in theory. The time and space averages often used in practice can satisfy the Reynolds-averaging rules only under certain idealized conditions (e.g., stationarity and homogeneity of flow), which were discussed in Chapter 8.

The usual procedure for deriving Reynolds-averaged equations is to substitute in Equation (9.1) $\tilde{u} = U + u$, $\tilde{v} = V + v$, etc., and take their averages using the Reynolds-averaging conditions [Equation (9.2)]. Consider first the continuity equation

$$\partial(U + u)/\partial x + \partial(V + v)/\partial y + \partial(W + w)/\partial z = 0 \tag{9.3}$$

which, after averaging, gives

$$\partial U/\partial x + \partial V/\partial y + \partial W/\partial z = 0 \tag{9.4}$$

Subtracting Equation (9.4) from (9.3), one obtains the continuity equation for the fluctuating motion

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \tag{9.5}$$

Thus, the form of continuity equation remains the same for instantaneous, mean, and fluctuating motions. This is not the case, however, for the equations of conservation of momentum and heat, because of the presence of nonlinear advection terms in those equations.

Let us consider, for example, the advection terms in the instantaneous equation of the conservation of heat

$$\tilde{a}_\theta = \tilde{u}(\partial\tilde{\theta}/\partial x) + \tilde{v}(\partial\tilde{\theta}/\partial y) + \tilde{w}(\partial\tilde{\theta}/\partial z) \tag{9.6a}$$

which, after utilizing the continuity equation, can also be written in the form

$$\tilde{a}_\theta = (\partial/\partial x)(\tilde{u}\tilde{\theta}) + (\partial/\partial y)(\tilde{v}\tilde{\theta}) + (\partial/\partial z)(\tilde{w}\tilde{\theta}) \tag{9.6b}$$

Expressing variables as sums of their mean and fluctuating parts in Equation (9.6b) and averaging, one obtains the Reynolds-averaged advection terms

$$\begin{aligned} A_\theta = {} & \frac{\partial}{\partial x}(U\Theta) + \frac{\partial}{\partial y}(V\Theta) + \frac{\partial}{\partial z}(W\Theta) \\ & + \frac{\partial}{\partial x}(\overline{u\theta}) + \frac{\partial}{\partial y}(\overline{v\theta}) + \frac{\partial}{\partial z}(\overline{w\theta}) \end{aligned} \tag{9.7a}$$

or, after using the mean continuity equation,

$$A_\theta = U\frac{\partial\Theta}{\partial x} + V\frac{\partial\Theta}{\partial y} + W\frac{\partial\Theta}{\partial z} + \frac{\partial}{\partial x}(\overline{u\theta}) + \frac{\partial}{\partial y}(\overline{v\theta}) + \frac{\partial}{\partial z}(\overline{w\theta}) \tag{9.7b}$$

Thus, upon averaging, the nonlinear advection terms yield not only the terms which may be interpreted as advection or transport by mean flow, but also several additional terms involving covariances or turbulent fluxes. These latter terms are the spatial gradients (divergence) of turbulent transports.

Following the above procedure with each component of Equation (9.1), one obtains the Reynolds-averaged equations for the conservation of mass, momentum, and heat:

$$
\begin{aligned}
&\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}=0\\
&\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}+W\frac{\partial U}{\partial z}=fV-\frac{1}{\rho_0}\frac{\partial P_1}{\partial x}+\nu\nabla^2 U\\
&\qquad -\left(\frac{\partial\overline{u^2}}{\partial x}+\frac{\partial\overline{uv}}{\partial y}+\frac{\partial\overline{uw}}{\partial z}\right)\\
&\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial x}+V\frac{\partial V}{\partial y}+W\frac{\partial V}{\partial z}=-fU-\frac{1}{\rho_0}\frac{\partial P_1}{\partial y}+\nu\nabla^2 V\\
&\qquad -\left(\frac{\partial\overline{uv}}{\partial x}+\frac{\partial\overline{v^2}}{\partial y}+\frac{\partial\overline{vw}}{\partial z}\right)\\
&\frac{\partial W}{\partial t}+U\frac{\partial W}{\partial x}+V\frac{\partial W}{\partial y}+W\frac{\partial W}{\partial z}=\frac{g}{T_0}T_1-\frac{1}{\rho_0}\frac{\partial P_1}{\partial z}+\nu\nabla^2 W\\
&\qquad -\left(\frac{\partial\overline{wu}}{\partial x}+\frac{\partial\overline{wv}}{\partial y}+\frac{\partial\overline{w^2}}{\partial z}\right)\\
&\frac{\partial\Theta}{\partial t}+U\frac{\partial\Theta}{\partial x}+V\frac{\partial\Theta}{\partial y}+W\frac{\partial\Theta}{\partial z}=\alpha_h\nabla^2\Theta-\left(\frac{\partial\overline{u\theta}}{\partial x}+\frac{\partial\overline{v\theta}}{\partial y}+\frac{\partial\overline{w\theta}}{\partial z}\right)
\end{aligned}
\tag{9.8}
$$

When these equations are compared with the corresponding instantaneous equations [Equation (9.1)], one finds that most of the terms (except for the turbulent transport terms) are similar and can be interpreted in the same way. However, there are several fundamental differences between these two sets of equations, which forces us to treat them in entirely different ways. While Equation (9.1) deals with instantaneous variables varying rapidly and irregularly in time and space, Equation (9.8) deals with mean variables which are comparatively well behaved and vary only slowly and smoothly. While all the terms in the former equation set may be significant and cannot be ignored *a priori*, the mean flow equations can be greatly simplified by neglecting the molecular diffusion terms outside of possible viscous sublayers and also other terms on the basis of certain boundary layer approximations and considerations of stationarity and horizontal homogeneity, whenever applicable. For example, it is easy to show that for a horizontally homogeneous and stationary PBL the equations of mean flow reduce to

$$\begin{aligned} -f(V - V_g) &= -\partial\overline{uw}/\partial z \\ f(U - U_g) &= -\partial\overline{vw}/\partial z \end{aligned} \tag{9.9}$$

An important consequence of averaging the equations of motion is the appearance of turbulent flux-divergence terms which contain unknown variances and covariances. Note that in Equation (9.8) there are many more unknowns than the number of equations. While the number of extra unknowns may be reduced to one or two in certain simple flow situations, the Reynolds-averaged equations remain essentially unclosed and, hence, unsolvable. This so-called closure problem of turbulence has been a major stumbling block in developing a rigorous and general theory of turbulence. It is a consequence of the nonlinearity of the original (instantaneous) equations of motion. Many semiempirical theories and turbulence closure models have been proposed to get around the closure problem, but none of them has proved to be entirely satisfactory. The simplest gradient-transport theories, hypotheses, or relations, which are still widely used in micrometeorology, form the basis of the so-called first-order closure models; these are based on the equations of mean motion. More sophisticated higher-order closure models are based on the equations of mean motion, turbulence variances, covariances, and even higher moments.

Derivation of the dynamical equations for the various turbulence statistics is outside the scope of this book. It would suffice to say that the fundamental closure problem remains, or even gets worse, as one includes second and higher-order moments in a turbulence closure model. The addition of the turbulence kinetic energy equation to the equations of motion forms an optimal set of dynamical equations on which most frequently used turbulence models in computational fluid dynamics, boundary-layer meteorology, and mesoscale meteorology are based.

9.1.4 Turbulence kinetic energy equation

An approximate form of the turbulence kinetic energy equation for gradually developing atmospheric boundary layer flows is

$$\frac{DE}{Dt} = -\overline{uw}\frac{\partial U}{\partial z} - \overline{vw}\frac{\partial V}{\partial z} + \frac{g}{T_{v0}}\overline{w\theta_v} - \frac{\partial}{\partial z}\left(\overline{we} + \frac{\overline{wp}}{\rho_0}\right) - \varepsilon \tag{9.10}$$

where E and e are mean and fluctuating turbulence kinetic energies per unit mass

$$E = \tfrac{1}{2}(\overline{u^2} + \overline{v^2} + \overline{w^2}) \tag{9.11}$$

$$e = \tfrac{1}{2}(u^2 + v^2 + w^2) \tag{9.12}$$

$\overline{w\theta_v}$ is the virtual heat flux, and ε is the rate of energy dissipation. The left-hand side of Equation (9.10) represents local and advective changes of E, while the various terms on the right-hand side represent shear production, buoyancy production or destruction, turbulent transport including that due to pressure fluctuations, and the rate of dissipation due to viscosity.

The relative importance of shear generation and buoyancy production/destruction terms for the maintenance of turbulence in the atmospheric boundary layer has already been discussed in Section 8.2. Their ratio is used to define the flux Richardson number

$$\mathrm{Rf} = \frac{g\overline{w\theta_v}}{T_{v0}} \bigg/ \left(\overline{uw}\frac{\partial U}{\partial z} + \overline{vw}\frac{\partial V}{\partial z}\right) \tag{9.13}$$

which is related to the gradient Richardson number as

$$\mathrm{Rf} = (K_h/K_m)\mathrm{Ri} \tag{9.14}$$

For the horizontally homogeneous PBL, the left-hand side of Equation (9.10) simplifies to $\partial E/\partial t$, and any imbalance between the various source (e.g., shear production) and sink (e.g., dissipation and buoyancy destruction) terms will determine whether turbulence kinetic energy increases, remains constant, or decreases with time.

In the unstable surface layer, both shear and buoyancy mechanisms act in concert to produce vigorous turbulence, part of which is transported upward to the mixed layer. Due to much reduced wind shears in the convective mixed layer, however, turbulence there is produced primarily by buoyancy. Although turbulence kinetic energy increases with time in response to increasing heat flux and increasing PBL height in the morning hours, $\partial E/\partial t$ is found to be an order of magnitude smaller than the individual production and dissipation terms in the TKE equation. Thus, there is an approximate balance between production, dissipation, and transport terms of that equation.

In the stable boundary layer, on the other hand, the buoyancy acts in opposition to the shear production, resulting in rather weak and, sometimes, decaying turbulence. Based on their relative magnitudes, Richardson (1920) arrived at his well-known criterion of $\mathrm{Rf} < \mathrm{Rf_c} = 1$ for the maintenance of turbulence. However, he did not include the important dissipation term in his theoretical considerations. Subsequent theoretical and experimental studies of turbulence in the SBL have yielded much lower values of $\mathrm{Rf_c} = 0.2$–0.5 (Arya, 1972; Stull, 1988). The criterion is better formulated in terms of the critical flux Richardson number ($\mathrm{Rf_c}$), rather than $\mathrm{Ri_c}$.

9.2 Gradient-transport Theories

In order to close the set of equations given by Equation (9.8) or its simplified version for a given flow situation, the variances and covariances must either be specified in terms of other variables or additional equations must be developed. In the latter approach, the closure problem is only shifted to a higher level in the hierarchy of equations that can be developed. We will not discuss here the so-called higher-order closure schemes or models that have been developed in recent years and that have been found to have their own limitations and problems. The older and more widely used approach has been based on the assumed (hypothetical) analogy between molecular and turbulent transfers. It is called the gradient-transport approach, because turbulent transports or fluxes are sought to be related to the appropriate gradients of mean variables (velocity, temperature, etc.). Several different hypotheses have been used in developing such relationships.

9.2.1 Eddy viscosity (diffusivity) hypothesis

In analogy with Newton's law of molecular viscosity [see Equation (7.1), Chapter 7], J. Boussinesq in 1877 proposed that the turbulent shear stress in the direction of flow may be expressed as

$$\tau = \rho K_{\mathrm{m}}(\partial U/\partial z) \tag{9.15}$$

in which K_{m} is called the eddy exchange coefficient of momentum or simply the eddy viscosity, which is analogous to the molecular kinematic viscosity ν. In analogy with the more general constitutive relations [Equation (7.2)], one can also generalize Equation (9.15) to express the various Reynolds stress components in terms of mean gradients. In particular, when the mean gradients in the x and y directions can be neglected in comparison to those in the z direction (the usual boundary layer approximation), we have the simpler eddy viscosity relations for the vertical fluxes of momentum

$$\begin{aligned} \overline{uw} &= -K_{\mathrm{m}}(\partial U/\partial z) \\ \overline{vw} &= -K_{\mathrm{m}}(\partial V/\partial z) \end{aligned} \tag{9.16}$$

Similar relations have been proposed for the turbulent fluxes of heat, water vapor, and other transferable constituents (e.g., pollutants), which are analogous to Fourier's and Fick's laws of molecular diffusion of heat and mass.

Those frequently used in micrometeorology are the relations for the vertical fluxes of heat ($\overline{\theta w}$) and water vapor ($\overline{qw}$)

$$\begin{aligned} \overline{\theta w} &= -K_{\mathrm{h}}(\partial \Theta / \partial z) \\ \overline{qw} &= -K_{\mathrm{w}}(\partial Q / \partial z) \end{aligned} \tag{9.17}$$

in which K_{h} and K_{w} are called the eddy exchange coefficients or eddy diffusivities of heat and water vapor, respectively, and Q and q denote the mean and fluctuating parts of specific humidity.

It should be recognized that the above gradient-transport relations are not the expressions of any sound physical laws in the same sense that their molecular counterparts are. These are not based on any rigorous theory, but only on an intuitive assumption of similarity or analogy between molecular and turbulent transfers. Under ordinary circumstances, one would expect heat to flow from warmer to colder regions, roughly in proportion to the temperature gradient. Similarly, momentum and mass transfers may be expected to be proportional to and down the mean gradients. However, these expectations are not always borne out by experimental data in turbulent flows, including the atmospheric boundary layer.

The analogy between molecular and turbulent transfers has subsequently been found to be very weak and qualitative only. Eddy diffusivities, as determined from their defining relations, Equations (9.16) and (9.17), are usually several orders of magnitude larger than their molecular counterparts, indicating the dominance of turbulent mixing over molecular exchanges. More importantly, eddy diffusivities cannot simply be regarded as fluid properties; these are actually turbulence or flow properties, which can vary widely from one flow to another and from one region to another in the same flow. Eddy diffusivities show no apparent dependence on molecular properties, such as mass density, temperature, etc., and have nothing in common with molecular diffusivities, except for the same dimensions.

There are other limitations of the K theory which we have not mentioned so far. The basic notion of down-gradient transport implied in the theory may be questioned. There are practical situations when turbulent fluxes are in no way related to the local gradients. For example, in a convective mixed layer the potential temperature gradient becomes near zero or slightly positive, while the heat is transported upward in significant amounts. This would imply infinite or even negative values of K_{h}, indicating that K theory becomes invalid in this case. Even in other situations, the specification of eddy diffusivities in a rational manner is not always easy, but quite difficult.

In spite of the above limitations of the implied analogy between molecular and turbulent diffusion, Equations (9.16) and (9.17) need not be restrictive, since they replace only one set of unknowns (fluxes) for another (eddy diffusivities). Some restrictions are imposed, however, when one assumes that eddy diffusivities depend on the coordinates and flow parameters in some definite manner. This, then, constitutes a semiempirical theory that is based on a hypothesis and is subject to experimental verification. The simplest assumption, which Boussinesq proposed originally, is that eddy diffusivities are constants for the whole flow. It turns out that this assumption works well in free turbulent flows such as jets, wakes, and mixing layers, away from any boundaries, and is often used in the free atmosphere. But when applied to boundary layers and channel flows, it leads to incorrect results. In general, the assumption of constant eddy diffusivity is not applicable near a rigid surface. But here other reasonable hypotheses can be made regarding the variation of eddy diffusivity with distance from the surface. For example, a linear distribution of K_m in the neutral surface layer works quite well. Suggested modifications of the K distribution in thermally stratified conditions are usually based on other theoretical considerations and empirical data, which will be discussed later. In any case, the K theory is quite useful and is widely used in micrometeorology.

9.2.2 Mixing-length hypothesis

In an attempt to specify eddy viscosity as a function of geometry and flow parameters, L. Prandtl in 1925 further extended the molecular analogy by ascribing a hypothetical mechanism for turbulent mixing. According to the kinetic theory of gases, momentum and other properties are transferred when molecules collide with each other. The theory leads to an expression of molecular viscosity as a product of mean molecular velocity and the mean free path length (the average distance traveled by molecules before collision). Prandtl hypothesized a similar mechanism of transfer in turbulent flows by assuming that eddies or 'blobs' of fluid (analogous to molecules) break away from the main body of the fluid and travel a certain distance, called the mixing length (analogous to free path length), before they mix suddenly with the new environment. If the velocity, temperature, and other properties of a blob or parcel are different from those of the environment with which the parcel mixes, fluctuations in these properties would be expected to occur as a result of the exchanges of momentum, heat, etc. If such eddy motions occur more or less randomly in all directions, it can be easily shown that net (average) exchanges of momentum, heat, etc., will occur only in the direction of decreasing velocity, temperature, etc.

In order to illustrate the above mechanism for the generation of turbulent fluctuations and their covariances (fluxes), we consider the usual case of increasing mean velocity with height in the lower atmosphere (Figure 9.1).

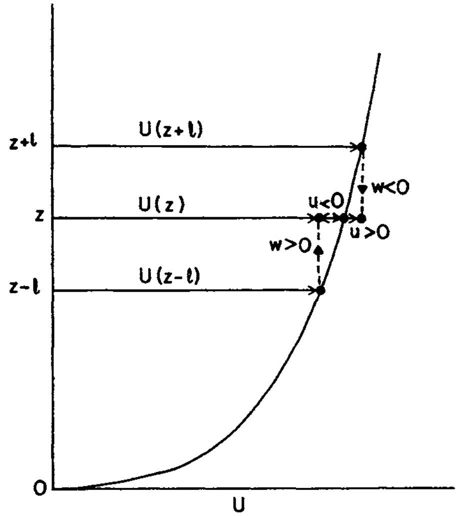

Figure 9.1 Schematic of mean velocity in the lower atmosphere and expected correlations between the longitudinal and vertical velocity fluctuations due to fluid parcels coming from above or below mixing with the surrounding fluid.

The longitudinal velocity fluctuations at a level z may be assumed to occur as a result of mixing with the environment, at this level, of fluid parcels arriving from different levels above and below. For example, a parcel arriving from below (level $z - l$) will give rise to a negative fluctuation at level z of magnitude

$$u = U(z - l) - U(z) \sim -l(\partial U/\partial z) \tag{9.18}$$

associated with its positive vertical velocity (fluctuation) w. The last approximation is based on the assumption of a linear velocity profile over the mixing length l, which is considered here a fluctuating quantity with positive values for the upward motions and negative for the downward motions of the parcel. Considering the action of many parcels arriving at z and taking the average, we obtain an expression for the momentum flux

$$\overline{uw} = -\overline{lw}(\partial U/\partial z) \tag{9.19}$$

which is not very helpful, because there is no way of measuring l. Another mixing-length expression for $\overline{uw}$ was obtained by Prandtl, after assuming that in a turbulent flow the velocity fluctuations in all directions are of the same order of magnitude and are related to each other, so that

$$w \sim -u \cong l(\partial U/\partial z) \tag{9.20}$$

From Equations (9.18) and (9.20) one obtains

$$\overline{uw} \sim -\overline{l^2}(\partial U/\partial z)^2$$

or Prandtl's original mixing-length relation

$$\overline{uw} = -l_m^2 |\partial U/\partial z| (\partial U/\partial z) \tag{9.21}$$

in which l_m is a mean mixing length, which is proportional to the root-mean-square value of the fluctuating length l, and the absolute value of the velocity gradient is introduced to ensure that the momentum flux is down the gradient.

Note that Equations (9.20) and (9.21) are derived for the unidirectional mean flow in the x direction. These can be generalized to the PBL flow with U and V profiles as

$$w \sim l|\partial \mathbf{V}/\partial z| \tag{9.22}$$

$$\overline{uw} = -l_m^2 |\partial \mathbf{V}/\partial z| (\partial U/\partial z) \tag{9.23}$$

$$\overline{vw} = -l_m^2 |\partial \mathbf{V}/\partial z| (\partial V/\partial z) \tag{9.24}$$

Further, extensions of the mixing-length hypothesis to the vertical transfers of heat and water vapor lead to the following analogous relations for their fluxes:

$$\overline{\theta w} = -l_m l_h |\partial \mathbf{V}/\partial z| (\partial \Theta/\partial z) \tag{9.25}$$

$$\overline{qw} = -l_m l_w |\partial \mathbf{V}/\partial z| (\partial Q/\partial z) \tag{9.26}$$

where l_h and l_w are the mixing lengths for heat and water vapor transfers, which might differ from that of momentum.

Equations (9.23)–(9.26) constitute closure relations if the various mixing lengths can be prescribed as functions of the flow geometry and, possibly, other flow properties. If large eddies are mainly responsible for momentum and other exchanges in a turbulent flow, it can be argued that l_m should be directly related to the characteristic large-eddy length scale. Some investigators make no distinction between the two, although, strictly speaking, a proportionality coefficient of the order of one is usually involved. In free turbulent flows and in the outer parts of boundary layer and channel flows, where large-eddy size may not have significant spatial variations, the assumption of constant mixing length ($l_m = l_0$) is found to be reasonable. In surface or wall layers, the large-eddy size, at least normal to the surface, varies roughly in proportion to the distance from the surface, so that mixing length is also expected to be proportional to the distance ($l_m \sim z$). In the atmospheric boundary layer, the mixing-length distribution is further expected to depend on thermal stratification and boundary layer thickness, and will be considered later.

The mixing-length hypothesis may be used to specify eddy viscosity. From a comparison of Equations (9.16) and (9.23), it is obvious that

$$K_{\mathrm{m}} = l_{\mathrm{m}}^2 |\partial \mathbf{V}/\partial z| \tag{9.27}$$

An alternative specification is obtained from a comparison of Equations (9.16) and (9.19)

$$K_{\mathrm{m}} = c_{\mathrm{m}} l_{\mathrm{m}} \sigma_{\mathrm{w}} \tag{9.28}$$

where c_{m} is a constant. Other parametric relations of eddy viscosity involving the product of a characteristic length scale and a turbulence velocity scale have been proposed in the literature. Similar relations are given for K_{h} and K_{w}; e.g.,

$$K_{\mathrm{h}} = l_{\mathrm{m}} l_{\mathrm{h}} |\partial \mathbf{V}/\partial z| \tag{9.29}$$

or, alternatively,

$$K_{\mathrm{h}} = c_{\mathrm{h}} l_{\mathrm{h}} \sigma_{\mathrm{w}} \tag{9.30}$$

Example Problem 1

In the neutral surface layer, eddy viscosity and mixing length can be expressed as $K_{\mathrm{m}} = kzu_*$ and $l_{\mathrm{m}} = kz$. Derive an expression for wind distribution, assuming that the friction velocity $u_* = (-\overline{uw})^{1/2}$ is independent of height in this constant-flux, neutral surface layer.

Solution

For the neutral surface layer, the eddy viscosity relation (9.16) can be expressed as

$$u_*^2 = K_{\mathrm{m}} \frac{\partial U}{\partial z} = kzu_* \frac{\partial U}{\partial z}$$

so that,

$$\frac{\partial U}{\partial z} = \frac{u_*}{kz}$$

whose integration with respect to z yields

$$U = \frac{u_*}{k} \ln z + A$$

where A is a constant of integration. In order to determine A we use the lower boundary condition at $z = z_0$, $U = 0$ (here, one cannot use the boundary condition right at the surface, i.e., at $z = 0$, for various reasons). This boundary condition gives $A = -(u_*/k)\ln z_0$ and the wind profile as

$$U = \frac{u_*}{k}\ln(z/z_0)$$

which is the well known logarithmic wind profile law, in which z_0 is called the roughness length.

The mixing-length relation (9.21) leads to the same result as

$$u_*^2 = l_{\mathrm{m}}^2\left(\frac{\partial U}{\partial z}\right)^2 = k^2 z^2\left(\frac{\partial U}{\partial z}\right)^2$$

or,

$$\frac{\partial U}{\partial z} = \frac{u_*}{kz}$$

9.3 Dimensional Analysis and Similarity Theories

Dimensional analysis and similarity considerations are extensively used in micrometeorology, as well as in other areas of science and engineering. Therefore, these analytical methods are briefly discussed in this section, while their applications in micrometeorology will be given in later chapters.

9.3.1 Dimensional analysis

Dimensional analysis is a simple but powerful method of investigating a variety of scientific phenomena and establishing useful relationships between the various quantities or parameters, based on their dimensions. One can define a set of fundamental dimensions, such as length [L], time [T], mass [M], etc., and express the dimensions of all the quantities involved in terms of these fundamental dimensions. A representation of the dimensions of a quantity or a parameter in terms of fundamental dimensions constitutes a dimensional formula, e.g., the dimensional formula for fluid viscosity is $[\mu] = [\mathrm{ML}^{-1}\mathrm{T}^{-1}]$. If the exponents in the dimensional formula are all zero, the parameter under consideration is dimensionless. One can form dimensionless parameters from appropriate combinations of dimensional quantities; e.g., the Reynolds number

$\mathrm{Re} = VL\rho/\mu$ is a dimensionless combination of fluid velocity V, the characteristic length scale L, density ρ, and viscosity μ.

Dimensionless groups or parameters are of special significance in any dimensional analysis in which the main objective is to seek certain functional relationships between the various dimensionless parameters. There are several reasons for considering dimensionless groups instead of dimensional quantities or variables. First, mathematical expressions of fundamental physical laws are dimensionally homogeneous (i.e., all the terms in an expression or equation have the same dimensions) and can be written in dimensionless forms simply by an appropriate choice of scales for normalizing the various quantities. Second, dimensionless relations represented in mathematical or graphical form are independent of the system of units used and they facilitate comparisons between data obtained by different investigators at different locations and times. Third, and, perhaps, the most important reason for working with dimensionless parameters is that nondimensionalization always reduces the number of parameters that are involved in a functional relationship. This follows from the well-known Buckingham Pi theorem, which states that if m quantities $(Q_1, Q_2, \ldots, Q_m)$, involving n fundamental dimensions, form a dimensionally homogeneous equation, the relationship can always be expressed in terms of $m - n$ independent dimensionless groups $(\Pi_1, \Pi_2, \ldots, \Pi_{m-n})$ made of the original m quantities. Thus, the dimensional functional relationship

$$f(Q_1, Q_2, \ldots, Q_m) = 0 \tag{9.31}$$

is equivalent to the dimensionless relation

$$F(\Pi_1, \Pi_2, \ldots, \Pi_{m-n}) = 0 \tag{9.32}$$

or, alternatively,

$$\Pi_1 = F_1(\Pi_2, \Pi_3, \ldots, \Pi_{m-n}) \tag{9.33}$$

In particular, when only one dimensionless group can be formed out of all the quantities, i.e., when $m - n = 1$, that group must be a constant, since it cannot be a function of any other parameters. If there are two Π-groups, one must be a unique function of the other, and so on. Dimensional analysis does not give actual forms of the functions F, F_1, etc., or values of any dimensionless constants that might result from the analysis. This must be obtained by other means, such as further theoretical considerations and experimental observations. It is common practice to follow dimensional analysis by a systematic experimental study of the phenomenon to be investigated.

9.3.2 Similarity theory

The Buckingham Pi theorem and dimensional analysis discussed above are merely mathematical formalisms and do not deal with the physics of the problem. The actual formulation of a similarity theory involves several steps, some of which require physical intuition, other theoretical considerations, prior observational information, and possibly new experiments designed to test the theory. The five steps involved in developing and testing a similarity theory are:

(1) Define the scope of the theory with all the restrictive assumptions clearly stated.
(2) Select an optimal set of relevant independent variables on which one or more variable of interest may depend. This constitutes a similarity hypothesis about the functional dependence between the various variables. Only one dependent variable is considered at a time in such a functional relationship, but one can have several functional relationships (e.g., one for each dependent variable).
(3) Perform dimensional analysis after determining the number of possible independent dimensionless groups and organize the variables into dimensionless groups.
(4) Express functional relationships between dimensionless groups, one of which should contain the dependent variable. These constitute the similarity relations or similarity theory predictions (one for each dependent variable).
(5) Gather relevant data from previous experiments that satisfy the restrictive assumptions of the similarity theory, or perform a new experiment to test the initial similarity hypothesis and similarity theory predictions. Experimental data will tell us whether the original similarity hypothesis is correct or not. If the theory is verified by experimental data, the latter can also be used to determine empirical forms of the various similarity functions by appropriate curve fitting through data plots.

The ultimate result of this five-step procedure is a set of empirical equations or fitted curves through plots of experimental data, all involving dimensionless similarity parameters. For a successful similarity theory, verified by experiments, the empirical similarity relations are expected to be universal and can be used at other locations with different surface characteristics and under different meteorological conditions.

In the first step, certain restrictive assumptions are made in order to reduce the number of independent variables involved in a similarity hypothesis, so that one can reduce the number of dimensionless parameters to a minimum that is consistent with the physics. The fewer the dimensionless parameters, the more powerful are the similarity theory predictions, and the easier it is to verify them and to determine empirical similarity relations or constants from carefully conducted experiments and observations. For example, the most commonly used simplifying assumptions in all the proposed PBL similarity theories are: (1) mean flow is steady or stationary; (2) mean flow is horizontally homogeneous, implying a flat and homogeneous surface; (3) viscosity and other molecular diffusivities are not relevant in the bulk of the PBL flow outside molecular sublayers; and (4) near-surface canopy variables can be ignored in the formulation of a similarity theory for the horizontally homogeneous part of the PBL. Additional assumptions are sometimes made depending on the type of the PBL (e.g., for a barotropic PBL, geostrophic winds are independent of height) and its stability regime (e.g., under convective conditions, shear effects and the Coriolis parameter are ignored). In surface-layer similarity theories, the Coriolis parameter, geostrophic winds and shears, and the PBL height are all considered irrelevant.

The second step, involving the selection of independent variables in the formulation of a similarity hypothesis, is the most crucial step in the development of a successful similarity theory. On the one hand, one cannot ignore any of the important variables or parameters on which the dependent variable really depends, because this might lead to a completely wrong or unphysical relationship. On the other hand, if unnecessary and irrelevant variables are included in the original similarity hypothesis, they will unnecessarily complicate the analysis and make empirical determination of the various functional relationships extremely difficult, if not impossible. In principle, experimental data should tell us if there are irrelevant variables or similarity parameters, which can be dropped from the similarity theory without any significant loss of generality. It is always desirable to keep the number of independent variables to a minimum, consistent with physics. Sometimes, it may become necessary to break down the domain of the problem or phenomenon under investigation into several small subdomains, so that simpler similarity hypotheses can be formulated for each of them separately. For example, the atmospheric boundary layer is usually divided into a surface layer and an outer layer or a mixed layer for dimensional analysis and similarity considerations.

The third step is straightforward, once the number of independent dimensionless groups is determined by the Buckingham's Pi theorem. But, there is always some flexibility and choice in formulating dimensionless parameters. For this, it is often convenient to first determine the appropriate scales of length, velocity, etc., from the independent variables. Then, dimensionless parameters can be determined merely by inspection. If there are more than one length or velocity scales, their ratio forms a dimensionless similarity parameter.

The similarity relations or predictions in the fourth step are simply expressions of dimensionless groups containing dependent variables as unspecified functions of other dimensionless (similarity) parameters. If there is no similarity parameter that can be formed by independent variables only, the dependent parameter (II-group) must simply be a constant.

Finally, a thorough experimental verification of the similarity hypothesis and resulting similarity relations is necessary before a proposed similarity theory becomes widely accepted and successful. For this, experiments are conducted at different locations, under more or less idealized conditions implied in the theory. However, the whole wide range of parameters should be covered. Sometimes, experimental data can verify some of the similarity relations, but not others, in which case the theory is considered only partially successful with applicability to only specified variables.

9.3.3 Examples of similarity theories

In order to illustrate the method and usefulness of dimensional analysis and similarity, let us consider the possible relationship for the mean potential temperature gradient ($\partial\Theta/\partial z$) as a function of the height (z) above a uniform heated surface, the surface heat flux (H_0), the buoyancy parameter (g/T_0) which appears in the expressions for static stability and buoyant acceleration, and the relevant fluid properties (ρ and c_p) in the near-surface layer when free convection dominates any mechanical mixing (this latter condition permits dropping of all shear-related parameters from consideration). To establish a functional relationship in the dimensional form

$$f(\partial\Theta/\partial z, H_0, g/T_0, z, \rho, c_p) = 0 \tag{9.34}$$

would require extensive observations of temperature as a function of height and surface heat flux at different times and locations (to represent different types of surfaces and radiative regimes). If the relationship is to be further generalized to other fluids, laboratory experiments using different fluids will also be necessary. Considerable simplification can be achieved, however, if ρ and c_p are combined with H_0 into what may be called kinematic heat flux $H_0/\rho c_p$, so that Equation (9.34) can be written as

$$F(\partial\Theta/\partial z, H_0/\rho c_p, g/T_0, z) = 0 \tag{9.35}$$

If we now use the method of dimensional analysis, realizing that only one dimensionless group can be formed from the above quantities, we obtain

$$(\partial\Theta/\partial z)(H_0/\rho c_p)^{-2/3}(g/T_0)^{1/3}z^{4/3} = C \tag{9.36}$$

in which the left-hand side is the dimensionless group that is predicted to be a constant. The value of the constant C can be determined from only one carefully conducted experiment, although a thorough experimental verification of the above relationship might require more extensive observations.

The desired dimensionless group from a given number of quantities can often be formed merely by inspection. A more formal and general approach would be to write and solve a system of algebraic equations for the exponents of various quantities involved in the dimensionless group. For example, the dimensionless group formed out of all the parameters in Equation (9.35) may be assumed as

$$\Pi_1 = (\partial\Theta/\partial z)(H_0/\rho c_p)^a (g/T_0)^b z^c \tag{9.37}$$

in which we have arbitrarily assigned a value of unity to one of the indices (here, the exponent of $\partial\Theta/\partial z$), because any arbitrary power of a dimensionless quantity is also dimensionless. Writing Equation (9.37) in terms of our chosen fundamental dimensions (length, time, and temperature), we have

$$[L^0T^0K^0] = [KL^{-1}][KLT^{-1}]^a[LT^{-2}K^{-1}]^b[L]^c$$

from which we obtain the algebraic equations

$$\begin{aligned} 0 &= -1 + a + b + c \\ 0 &= -a - 2b \\ 0 &= 1 + a - b \end{aligned} \tag{9.38}$$

whose solution gives $a = -2/3$, $b = 1/3$, and $c = 4/3$. Substituting these values in Equation (9.37) and equating the only dimensionless group to a constant, then, yields Equation (9.36).

Another approach is to first formulate the characteristic scales of length, velocity, etc., from combinations of independent variables and then use these scales to normalize the dependent variables. In the case of multiple scales, their ratios form the independent dimensionless groups. In the above example of temperature distribution over a heated surface, with $\partial\Theta/\partial z$ as the dependent variable and the remaining quantities as independent variables, the following scales can be formulated out of the latter:

$$\begin{aligned}
&\text{length:} && z \\
&\text{temperature:} && \theta_f = \left(\frac{H_0}{\rho c_p}\right)^{2/3} \left(\frac{g}{T_0}\right)^{-1/3} z^{-1/3} \\
&\text{velocity:} && u_f = \left(\frac{H_0}{\rho c_p} \frac{g}{T_0} z\right)^{1/3}
\end{aligned} \tag{9.39}$$

Then, the appropriate dimensionless group involving the dependent variable is $(z/\theta_f)(\partial\Theta/\partial z)$, which must be a constant, since no other independent dimensionless groups can be formed from independent variables. This procedure also leads to Equation (9.36); it is found to be more convenient to use when a host of dependent variables are functions of the same set of independent variables. For example, standard deviations of temperature and vertical velocity fluctuations in the free convective surface layer are given by

$$\begin{aligned}
\sigma_\theta/\theta_f &= C_\theta \\
\sigma_w/u_f &= C_w
\end{aligned} \tag{9.40}$$

or, after substituting from Equation (9.39),

$$\begin{aligned}
\sigma_\theta &= C_\theta (H_0/\rho c_p)^{2/3} (g/T_0)^{-1/3} z^{-1/3} \\
\sigma_w &= C_w [(H_0/\rho c_p)(g/T_0) z]^{1/3}
\end{aligned} \tag{9.41}$$

Equations (9.41) have proved to be quite useful for the atmospheric surface layer under daytime unstable conditions and are supported by many observations (Monin and Yaglom, 1971, Chapter 5; Wyngaard, 1973) from which $c_\theta \cong 1.3$ and $C_w \cong 1.4$.

The above similarity theory was originally proposed by Obukhov and is more commonly known as the local free convection similarity theory. Local free convection similarity and scaling are not found to be applicable to horizontal velocity fluctuations, so that σ_u and σ_v are not simply proportional to u_f. Instead, they are found to be strongly influenced by large-eddy motions (convective updrafts and downdrafts) which extend through the whole depth of the CBL. Since the PBL height is ignored in the local free convection similarity theory, its applicability is limited to vertical velocity and temperature fluctuations.

If the boundary layer height (h) is also added to the list in Equation (9.35), we would have, according to the pi theorem, two independent dimensionless groups and the predicted functional relationship would be

$$\frac{\partial \Theta}{\partial z}\left(\frac{H_0}{\rho c_p}\right)^{-2/3}\left(\frac{g}{T_0}\right)^{1/3} z^{4/3} = F(z/h) \tag{9.42}$$

Similarly, σ_w/u_f and σ_θ/θ_f would be predicted to be functions of z/h. However, subsequent analysis of experimental data might indicate that the dependence of these on z/h is very weak or nonexistent and, hence, the irrelevance of h in the original hypothesis. The inclusion of h would be quite justified and even necessary, however, if we were to investigate the turbulence structure of the mixed layer, or σ_u and σ_v even in the surface layer. For example, the mixed-layer similarity hypothesis proposed by Deardorff (1970b) states that turbulence structure in the convective mixed layer depends on z, g/T_0, $H_0/\rho c_p$, and h. Then, the relevant mixed-layer similarity scales are:

$$\begin{aligned} &\text{length:} && h \\ &\text{temperature:} && T_* = \left(\frac{H_0}{\rho c_p}\right)^{2/3}\left(\frac{gh}{T_0}\right)^{-1/3} \\ &\text{velocity:} && W_* = \left(\frac{H_0}{\rho c_p}\frac{g}{T_0}h\right)^{1/3} \end{aligned} \tag{9.43}$$

The corresponding similarity predictions are that the dimensionless structure parameters σ_u/W_*, σ_w/W_*, etc., must be some unique function of z/h. This mixed-layer similarity theory has proved to be very useful in describing turbulence and diffusion in the convective boundary layer (Arya, 1999).

As the number of independent variables and parameters is increased in a similarity hypothesis, not only the number of independent dimensionless groups increases, but also the possible combinations of variables in forming such groups become large. The possibility of experimentally determining their functional relationships becomes increasingly remote as the number of dimensionless groups increases beyond two or three. A generalized PBL similarity theory will have to include all the possible factors influencing the PBL under the whole range of conditions encountered and, hence, might be too unwieldy for practical use. Simpler PBL similarity theories and scaling will be described in Chapter 13, while the surface layer similarity theory/scaling will be discussed in more detail in Chapters 10–12.

Example Problem 2

In a convective boundary layer during the midday period when $T_0 = 300$ K, $H_0 = 500$ W m^{-2}, and $h = 1500$ m, calculate and compare the standard deviations of vertical velocity and temperature fluctuations at the 10 m and 100 m height levels.

Solution

Using the local free-convection similarity relations (9.41) with $C_\theta = 1.3$, $C_w = 1.4$, and $\rho c_p = 1200$ J K^{-1} m^{-3}, the computed values of σ_w and σ_θ at the two heights in the convective surface layer ($z < 0.1h = 150$ m) are as follows:

Height, z (m)	σ_w (m s^{-1})	σ_θ (K)
10	0.72	1.05
100	1.55	0.49

Note that σ_w increases and σ_θ decreases by a factor of $10^{1/3} \cong 2.15$ as the height increases from 10 to 100 m.

9.4 Applications

The theories and models of turbulence discussed in this chapter are widely used in micrometeorology. Some of the specific applications are as follows:

- Calculating the mean structure (e.g., vertical profiles of mean velocity and temperature) of the PBL.
- Calculating the turbulence structure (e.g., profiles of fluxes, variances of fluctuations, and scales of turbulence) of the PBL.
- Providing plausible theoretical explanations for turbulent exchange and mixing processes in the PBL.
- Providing suitable frameworks for analyzing and comparing micrometeorological data from different sites.
- Suggesting simple methods of estimating turbulent fluxes from mean profile observations.

Problems and Exercises

1. Compare and contrast the instantaneous and the Reynolds-averaged equations of motion for the PBL.

2.

(a) What boundary layer approximations are often used for simplifying the Reynolds-averaged equations of motion for the PBL?

(b) What are the other simplifying assumptions commonly used in micrometeorology and the conditions in which they may not be valid?

3. Show, step by step, how Equation (9.8) is reduced to Equation (9.9) for a horizontally homogeneous and stationary PBL.

4.

(a) Write down the mean thermodynamic energy equation for a horizontally homogeneous PBL and discuss the rationale for retaining the time-tendency term in the same.

(b) Describe a method for estimating the diurnal variation of surface heat flux from hourly soundings of temperature in the PBL.

5. Show that with the assumption of a constant eddy viscosity the equations of mean motion in the PBL become similar to those for the laminar Ekman layer and, hence, discuss some of the limitations of the above assumption.

6. If the surface layer turbulence under neutral stability conditions is characterized by a large-eddy length scale l, which is proportional to height, and a velocity scale u_*, which is independent of height, suggest the plausible expressions of mixing length and eddy viscosity in this layer.

7. A similarity hypothesis proposed by von Karman states that mixing length in a turbulent shear flow depends only on the first and second spatial derivatives of mean velocity ($\partial U/\partial z$, $\partial^2 U/\partial z^2$) in the direction of shear. Suggest an expression for mixing length which is consistent with the above similarity hypothesis.

8.

(a) In a neutral, barotropic PBL, ageostrophic velocity components ($U - U_g$ and $V - V_g$) may be assumed to depend only on the height z above the surface, the friction velocity scale u_*, and the Coriolis parameter f. On the basis of dimensional analysis suggest the appropriate similarity relations for velocity distribution and the PBL height h.

(b) If the PBL height in Problem 8(a) above was determined by a low-level inversion, with its base at z_i, what should be the corresponding form of similarity relations?

9.

(a) In a convective boundary layer during the midday period when $T_0 = 310$ K, $H_0 = 300$ W m^{-2}, and $h = 1000$ m, calculate and compare the standard deviations of velocity and temperature fluctuations at the heights of 10, 50, and 100 m.

(b) Show that, in the convective surface layer, the local free convection similarity relations can also be expressed in terms of mixed-layer similarity scales as

$$\sigma_\theta / T_* = C_\theta (z/h)^{-1/3}; \; \sigma_w / W_* = C_w (z/h)^{1/3}$$