NONSTANDARD METHODS IN RAMSEY THEORY

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4.1. **Introduction.** To define what it means for a number to be an *infintesimal* or *infinitely large*, we will construct an ordered field called the *hyperreals* $^*\mathbb{R}$ which contains infinitesimal and infinitely large numbers while preserving essential properties of the real numbers $\mathbb{R}$. To do so, we will make a system in which *sequences* of numbers can represent infinitely small and infinitely large numbers.

Definition 4.1. *Equivalence* of two real-valued sequences $r = \langle r_1, r_2, r_3, \ldots \rangle$ and $s = \langle s_1, s_2, s_3, \ldots \rangle$ exists if and only if the *agreement set* of r and s

$$E_{rs} = \{n : r_n = s_n\}$$

is *large*.

"Largeness" as a concept will be rigorously defined with filters later in this section, but its relevant properties are as follows:

- $\mathbb{N}$ is large, while $\emptyset$ is not large.
- If A and B are large sets, and $A \cap B \subseteq C$, then C is large.
- For any subset A of $\mathbb{N}$, either A or its complement A^c (the set of elements in $\mathbb{N}$ not in A) is large.

Remark 4.2. These definitions imply that either the even naturals $\{2, 4, 6, \ldots\}$ or the odd naturals $\{1, 3, 5, \ldots\}$ are large, but not both. Such arbitrary choices are inevitable in this definition of largeness.

4.2. **Filters.**

Definition 4.3. A *filter* over some nonempty set I is a nonempty collection $\mathcal{F} \subseteq \mathcal{P}(I)$ such that:

- if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
- if $A \in \mathcal{F}$ and $A \subseteq B \subseteq I$, then $B \in \mathcal{F}$.

Equivalently, $A \cap B \in \mathcal{F}$ if and only if $A, B \in \mathcal{F}$.

A *proper filter* is a filter $\mathcal{F}$ which does not contain the empty set $\emptyset$. An *ultrafilter* is a proper filter $\mathcal{U}$ such that for any $A \subseteq I$, either A or its complement A^c is a member of $\mathcal{U}$. A *principal ultrafilter* is an ultrafilter which contains a finite set, and a *nonprincipal ultrafilter* only contains cofinite sets (a set is cofinite if its complement is finite).

Date: July 4, 2022.

Definition 4.4. The *filter generated by* $\mathcal{H}$ (denoted $\mathcal{F}^{\mathcal{H}}$) for some $\mathcal{H} \subseteq \mathcal{P}(I)$ is the collection

$$\mathcal{F}^{\mathcal{H}} = \{A \subseteq I : A \subseteq B_1 \cap \cdots \cap B_n \text{ for some } n \in \mathbb{N} \text{ and some } B_i \in \mathcal{H},\ 1 \leq i \leq n\}.$$

When $\mathcal{H}$ has a single member B, we call $\mathcal{F}^{\mathcal{H}}$ the *principal filter generated by* B. When $B = \{i\}, i \in I$ —that is, B has a single element—$\mathcal{F}^{\mathcal{H}}$ is the *principal ultrafilter generated by* i, denoted by $\mathcal{F}^i$.

Remark 4.5. An ultrafilter is principal (contains a finite set) if and only if it contains a single element set.

Proof. It is trivial in the forward direction. For the opposite, let $A \in \mathcal{U}$, where $\mathcal{U}$ is some ultrafilter and A is a finite set. For some element $a \in A$, either $\{i\} \in \mathcal{U}$ or $\{i\}^c \in \mathcal{U}$, by the definition of the ultrafilter. The former proves the claim, while the latter implies $A - i \in \mathcal{U}$, where $A - i$ is the set of all elements of A not including i. By repeating the same process on $A - i$, we recursively prove that the existence of a finite set in $\mathcal{U}$ implies the existence of a single element set, as one can continue this process until A has two elements, and either i or $A - i$ is a member of $\mathcal{U}$. ■

Definition 4.6. A collection $\mathcal{H} \subseteq \mathcal{P}(I)$ has the *finite intersection property* (abbreviated as FIP) if the intersection of every nonempty finite subcollection of $\mathcal{H}$ is nonempty. That is,

$$B_1 \cap \cdots \cap B_n \neq \emptyset$$

for all $n \in \mathbb{N}$ and all $B_1, \ldots, B_n \in \mathcal{H}$.

While this treatment of filters is enough for us to begin our construction of the hyperreals, we also need to state the axiom of choice in the form of Zorn's lemma.

Zorn's Lemma. *If $(P, \leq)$ is a partially ordered set in which every linearly ordered subset (or "chain") has an upper bound in P, then P contains a $\leq -$maximal element.*

While specific discussion of the axiom of choice and its equivalents (namely: Zorn's lemma) is beyond the scope of this paper, this enables us to use the property of set inclusion $\subseteq$ as a partial ordering property. Any partial ordering property (denoted by $\leq$) must satisfy:

(1) Reflexivity: For some $a, a \leq a$.
(2) Antisymmetry: If $a \leq b$ and $b \leq a$ then $a = b$.
(3) Transitivity: If $a \leq b$ and $b \leq c$ then $a \leq c$.

As the subset $\subseteq$ relation satisfies all three properties, it can be considered a partial ordering.

Theorem 4.7. *Any collection of subsets of I that has the finite intersection property can be extended to an ultrafilter on I.*

Proof. Let $\mathcal{H}$ be a collection of subsets of I that has the FIP. Then, the filter $\mathcal{F}^{\mathcal{H}}$ is proper. Let P be the collection of all proper filters on I that include $\mathcal{F}^{\mathcal{H}}$, partially ordered by set inclusion $\subseteq$. Then, every linearly ordered subset of P has an upper bound in P, so Zorn's lemma implies P has a maximal element, which is a maximal filter on I and therefore an ultrafilter. ■

Corollary 4.8. *Any infinite set has a nonprincipal ultrafilter on it.*

Proof. If I is infinite, the cofinite filter $\mathcal{F}^{co}$ is proper and has the finite intersection property, and so is included in an ultrafilter $\mathcal{F}$. But for any $i \in I$ we have $I - \{i\} \in \mathcal{F}^{co} \subseteq \mathcal{F}$, so $\{i\} \notin \mathcal{F}$, whereas $\{i\} \in \mathcal{F}^i$. Hence $\mathcal{F} \neq \mathcal{F}^i$. Therefore, $\mathcal{F}$ is nonprincipal. **REWRITE THIS WITH LEMMAS** ■

This is the key insight we need to begin our construction of the hyperreals.

4.3. **Construction of the Hyperreals.** Let $\mathbb{R}^{\mathbb{N}}$ be the set of all sequences of real numbers. A member $r \in \mathbb{R}^{\mathbb{N}}$ has the form $r = \langle r_n : n \in \mathbb{N} \rangle$, and we denote this by $r = \langle r_n \rangle$. Addition and multiplication in $\mathbb{R}^{\mathbb{N}}$ are defined as follows, for $r, s \in \mathbb{R}^{\mathbb{N}}$:

$$r \oplus s = \langle r_n + s_n : n \in \mathbb{N} \rangle.$$
$$r \otimes s = \langle r_n \cdot s_n : n \in \mathbb{N} \rangle.$$

Definition 4.9. Let $r, s \in \mathbb{R}^{\mathbb{N}}$ be equivalent by the relation $\equiv$ such that

$$\langle r_n \rangle \equiv \langle s_n \rangle \text{ if and only if } \{n \in \mathbb{N} : r_n = s_n\} \in \mathcal{U}$$

where $\mathcal{U}$ is some nonprincipal ultrafilter on $\mathbb{N}$. We call this relation *almost everywhere agreement*, and it can be said that these two sequences agree on a "large" set.

Remark 4.10. Note that $\equiv$ is an equivalence relation on $\mathbb{R}^{\mathbb{N}}$.

Typically, we set the agreement set $\{n \in \mathbb{N} : r_n = s_n\}$ as $[[r = s]]$, and the equivalence class of some $r \in \mathbb{R}^{\mathbb{N}}$ will be denoted by $[r]$. From this, we can generate the quotient set of $\mathbb{R}^{\mathbb{N}}$ by $\equiv$ as

$${}^*\mathbb{R} = \{[r] : r \in \mathbb{R}^{\mathbb{N}}\}.$$

Define, for $r, s \in \mathbb{R}^{\mathbb{N}}$,

$$[r] + [s] = [r \oplus s],$$
$$[r] \cdot [s] = [r \otimes s],$$
$$[r] < [s] \iff [[r < s]] \in \mathcal{U} \iff \{n \in \mathbb{N} : r_n < s_n\} \in \mathcal{U}.$$

This structure ${}^*\mathbb{R}$ represents the hyperreals (with these properties is a well-ordered field), and this construction has some interesting properties. Take the sequences

$$a = \langle 1, 2, 3, \ldots \rangle,$$
$$b = \langle 2, 2, 3, \ldots \rangle.$$

We can say that a and b are equivalent in the hyperreals, as they agree at a *large* number of instances. However, the sequences

$$c = \langle 1, \frac{1}{2}, \frac{1}{3}, \ldots \rangle$$
$$d = \langle 0, 0, 0, \ldots \rangle$$

are *not* equivalent. Even though they are both approximated to 0 (as we will be determined later), because their agreement set $[[c = d]]$ is *empty* they are not equivalent. While *largeness* as a concept is still yet to be rigorously defined, we have enough of a basis to start utilizing the key property of the hyperreals.

4.4. **Star Maps and the Transfer Principle.**