Differential Topology

1. Manifolds

Problem 2

Problem. Let X, Y, Z be manifolds, and let $f : X \to Y$ and $g : Y \to Z$ be smooth maps. Show that $g \circ f : X \to Z$ is smooth. Show that if f and g are diffeomorphisms, then so is $g \circ f$.

Solution.

By definition, we have the following:

For each $x \in X$, there exists a neighborhood V such that the chart $\phi : V \to U$ is a map from X to some open subset of Euclidean space, and there exists a chart $\psi : V' \to U'$ from Y to Euclidean space such that $f(V) \subseteq V'$ (we can restrict V accordingly). As f is smooth, we know that $\psi \circ f \circ \phi^{-1}$ is a smooth map between open subsets in Euclidean space.

We make a similar argument for $g : Y \to Z$. However, note that we can use ψ as the chart of Y as $f(V) \subseteq V'$, and we only care about the intersection of $\operatorname{dom} g$ with the codomain of f. Therefore, as g is smooth, we know that $\chi \circ g \circ \psi^{-1}$ is a smooth mapping between subsets in Euclidean space (where χ is the chart corresponding to $g(f(V))$ in Z.)

$g \circ f$ is smooth if and only if there exists a smooth mapping between subsets of Euclidean space $\chi \circ (g \circ f) \circ \phi^{-1}$, where χ is a chart for Z and ϕ is a chart for X. From above, we know that $(\chi \circ g \circ \psi^{-1}) \circ (\psi \circ f \circ \phi^{-1})$ is a smooth map between open sets in Euclidan space, which simplifies to above.

For the diffeomorphism case, we have that f, g are diffeomorphisms. Therefore, both have smooth inverses, and we can apply the above argument to prove that $(g \circ f)^{-1} : Z \to X$ is smooth. As such, $g \circ f$ is a diffeomorphism, because it has a smooth inverse.

Problem 5

Problem. Is $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ a diffeomorphism?

Solution. No. $f(x) = x^3$ is differentiable and bijective for all $\mathbb{R}$, but its inverse $f^{-1}(x) = x^{1/3}$ is not differentiable at $x = 0$. Therefore, f is not a diffeomorphism.

Problem 6

Problem. Is the union of the x and y axes in $\mathbb{R}^2$ a manifold?

Solution. Yes. We take the coordinate charts:

$$\begin{aligned} V_{x+} &= \{(x,y) \in \mathbb{R}^2 \mid x > 0, y = 0\}, \\ V_{x-} &= \{(x,y) \in \mathbb{R}^2 \mid x < 0, y = 0\}, \\ V_{y+} &= \{(x,y) \in \mathbb{R}^2 \mid x = 0, y > 0\}, \\ V_{y-} &= \{(x,y) \in \mathbb{R}^2 \mid x = 0, y < 0\}. \end{aligned}$$

These cover the union of the x, y axes such that there is a neighborhood for every point such that it is diffeomorphic to an open subset of $\mathbb{R}^n$. We use V_{x+} as an example.

Let $U = (0, \infty) \subseteq \mathbb{R}$. Then, we define $\phi : U \to V_{x+}$ by $\phi(x) = (x, 0)$, so $\phi^{-1}(x, y) = x$. This is a diffeomorphism, and the treatments are similar for the other sets.

Problem 16

Problem. A curve in a manifold X is a smooth map γ from an interval I in $\mathbb{R}$ to X. The *velocity vector* of γ at time $t_0 \in I$, denoted $\frac{d\gamma}{dt}(t_0)$ is the vector $d\gamma_{t_0}(1) \in T_{x_0}X$, where $x_0 = \gamma(t_0)$.

(a) If $X = \mathbb{R}^n$ and $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$, show that

$$\frac{d\gamma}{dt}(t_0) = (\gamma_1'(t_0), \dots, \gamma_n'(t_0)).$$

(b) Show that T_xX is exactly the set of velocity vectors of curves in X passing through x.

Solution.

(a). Note that $\gamma : \mathbb{R} \to \mathbb{R}^n$. Also note that γ is smooth at t_0, so we can consider the function $d\gamma_{t_0} : I \subseteq \mathbb{R} \to \mathbb{R}^n$. This can be represented as a Jacobian of shape $1 \times n$, which looks like

$$\begin{aligned} &\left(\frac{d\gamma_1}{dt}(t_0), \dots, \frac{d\gamma_n}{dt}(t_0)\right) \\ &= \big(\gamma'(t_0), \dots, \gamma_n'(t_0)\big). \end{aligned}$$

(b) We take $x = \gamma(t)$. $\frac{d\gamma}{dt}(t) \in T_xX$. It is clear that every velocity vector is a member of T_xX, so we will show that all $v \in T_xX$ can be represented as velocity vectors.

First we take the case of $X = \mathbb{R}^n$. We want to show there exists a γ such that for all $x, v \in \mathbb{R}^{\mathrm{n}}$ we have $\frac{d\gamma}{dt}(t) = v$.

WLOG fix $t \in I$. When we write $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$, each γ_i is a function $\gamma_i : I \subseteq \mathbb{R} \to \mathbb{R}$, and each γ_i can be selected independently of each other. The only constraint is each γ_i being smooth, which makes γ smooth. We can select each γ_i such that $\gamma_i'(t) = v_i$, and as such we can construct a γ for each $v \in \mathbb{R}^n = T_xX$. This also generalizes for all t.

Now, we relax the constraint on $X = \mathbb{R}^n$. Nevertheless, the same argument applies. $T_x X$ is an n -dimensional vector space, so it is isomorphic to $\mathbb{R}^n$. Let $\phi : T_x X \to \mathbb{R}^n$ be an isomorphism. Then, as for all $v \in T_x X$, one can write $\phi(v)$ as a velocity vector, there is a bijective mapping between the space of velocity vectors and $T_x X$.

2. Regular Values

Problem 3

Problem. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x, y, z) = x^2 + y^2 - z^2$. Find all the critical points and critical values of f. For which $a \in \mathbb{R}$ is $f^{-1}(a)$ a submanifold of $\mathbb{R}^3$? Given $a, b \in \mathbb{R}$, find a criterion for when $f^{-1}(a)$ and $f^{-1}(b)$ are diffeomorphic.

Solution. For some point $a = (a_1, a_2, a_3)$, we have that

$$df_a(h) = 2a_1 h_1 + 2a_2 h_2 - 2a_3 h_3.$$

This is obviously surjective onto $\mathbb{R}$ for all points not $a = (0, 0, 0)$, at which point $df_a(h) = 0$. Therefore, for all $a \neq (0, 0, 0)$, $f^{-1}(a)$ is a submanifold of $\mathbb{R}^3$ by the preimage theorem, as these values of a are regular values.

For $f^{-1}(a), f^{-1}(b)$ to be diffeomorphic, there must be a differentiable and bijective map between the two. We can analyze them some more:

$$f^{-1}(a) = \{(x, y, z) \mid x^2 + y^2 - z^2 = a\}$$
$$f^{-1}(b) = \{(x, y, z) \mid x^2 + y^2 - z^2 = b\}$$

I claim that they are diffeomorphic when a, b are of the same sign, with the mapping given by $\phi(x, y, z) = \left(\sqrt{\frac{b}{a}}x, \sqrt{\frac{b}{a}}y, \sqrt{\frac{b}{a}}z\right)$ from $f^{-1}(a)$ to $f^{-1}(b)$. This is smooth when a, b are of the same sign and nonzero. (This also makes sense - this sign difference corresponds to a shift between a one sheeted and a two sheeted hyperboloid).

Problem 5

Problem. Let $p(z)$ be some nonconstant polynomial with complex coefficients. Show that $p : \mathbb{C} \to \mathbb{C}$ is a submersion except at finitely many points. What are the critical points and critical values?

Solution. As p is nonconstant, $dp(z)$ has at least one zero, and if p is of degree n, then $dp(z)$ has $n - 1$ zeroes. At these zeroes, the differential map $dp_z(h)$ will not be surjective (as they will all go to zero), therefore these are the critical points (and there are finitely many). The critical values are $p(z_0), \ldots, p(z_{n-1})$, where $z_0, \ldots, z_{n-1}$ are the zeroes of the polynomial.

Problem 19

Problem. Find a map $f : \mathbb{R} \to \mathbb{R}$ whose critical values are dense.

Solution. Let q_n be the n-th rational (can enumerate, they're countable). Then, take the map

$$\sum_n q_n \sin\left(\frac{x}{n}\right).$$

This is smooth, and has a critical point at $\frac{\pi}{n}$. In the interval $[0, \pi]$, there is a unique critical value associated with each rational number, and as the rationals are dense, the critical values of this function are dense.

3. Transversality

Problem 6

Problem. Prove that the fixed-point in the Brouwer fixed-point theorem need not be an interior point.

Solution. Consider the rotation of $\mathbb{B}^n$ by $\pi/2$ clockwise. This has two fixed points on $\partial\mathbb{B}$ -- the north and south poles.

This is a counterexample, unsure if the question asks to prove that for all n, there exists $f : \mathbb{B}^n \to \mathbb{B}^n$ such that it has a fixed point on the boundary?

Problem 7

Problem. Show that the map from the solid torus to itself need not have a fixed-point. What goes wrong in the proof of the Brouwer fixed-point theorem if we adapt it to the torus?

Solution. The solid torus can be rotated along its central axis -- these rotations do not have fixed points yet are maps from the torus to itself. To specify, we parametrize the torus as thus:

$$\begin{aligned}\mathbb{T} = \{(x, y, z) \in \mathbb{R}^3 \mid\, & x = (1 + r\cos\theta)\cos\phi \\ & y = (1 + r\cos\theta)\sin\phi \\ & z = r\sin\theta \\ & r \in \left[-\frac{1}{2}, \frac{1}{2}\right], \theta \in [0, 2\pi), \phi \in [0, 2\pi)\}.\end{aligned}$$

Let $f : \mathbb{T} \to \mathbb{T}$ such that $f(r, \theta, \phi) = (r, \theta, \phi + \pi/2)$. This has no fixed points.

The proof of the Brouwer fixed-point theorem fails when applied to the solid torus because $x \in \partial\mathbb{T}$ does not mean that $g(x) = x$! as there are multiple points where the ray $tx + (1 - t)f(x)$ can intersect the boundary for some choices of f and x. This happens when the ray passes through the hole in the torus. And if g is not the identity on $\partial\mathbb{T}$, then the proof by contradiction fails (one cannot construct a retraction).

Problem 8

Problem. Prove the Brouwer fixed-point theorem for continuous functions.

Solution. Let $f : \mathbb{B}^n \to \mathbb{B}^n$ be a continuous function. We will use a proof by contradiction to show it has at least one fixed point.

Assume f has no fixed points. Let $\epsilon = \min |f(x) - x|$. As f has no fixed points, then $\epsilon > 0$. By Stone-Weierstrass, there exists a polynomial P such that

$$|f(x) - P(x)| < \epsilon/2$$

for all x. Using the Reverse Triangle Inequality, we get

$$\begin{aligned} |f(x) - x - (P(x) - x)| &< \epsilon/2, \\ \Big||f(x) - x| - |P(x) - x|\Big| &< \epsilon/2. \end{aligned}$$

Therefore,

$$|f(x) - x| - \epsilon/2 < |P(x) - x| < |f(x) - x| + \epsilon/2.$$

Observe that it is impossible for $P(x) - x$ to have a zero, as its norm has a hard positive lower bound. However, this is a contradiction -- by the Brouwer fixed-point theorem for smooth functions, $P(x)$ should have a fixed point and therefore $P(x) - x$ should have a zero for some x. Ergo, it is impossible for f to have no fixed-points, and we have proven the Brouwer fixed-point theorem for continuous functions.

4. Mod 2 Intersections

Problem 1

Problem. Prove that there exists $z \in \mathbb{C}$ such that

$$z^7 + \cos(|z|^2)(1 + 93z^4) = 0.$$

Solution. Let $f(z) = z^7 + \cos(|z|^2)(1 + 93z^4)$. Observe that $f|_{\mathbb{R}}$ is smooth on the interval $(-\infty, 0)$. We will prove that there exists an $x \in (-\infty, 0)$ such that $f(x) = 0$, and as such prove the problem statement (as $x \in \mathbb{C}$.)

As f is smooth on $(-\infty, 0)$, the intermediate value theorem applies, so to prove the existence of a zero on this interval we must find two values $a, b \in (-\infty, 0)$ such that one is negative and the other positive.

Let $b = -1$. $f(-1) = (-1)^7 + \cos(1)(1 + 93) > 0$. Let $a = -\sqrt{\frac{\pi}{2}}$. $f(a) = (-\sqrt{\frac{\pi}{2}})^7 + 0 < 0$. By IVT, there is some value on the interval $x \in (a, b)$ such that $f(x) = 0$.

5. Oriented Intersections

Problem 5

Problem. Prove that the Mobius strip is nonorientable.

Solution. We consider the Mobius strip M with the following embedding in Euclidean space:
$M(t, \theta) = \left(\left(1 - t\sin\frac{\theta}{2}\right)\cos\theta, \left(1 - t\sin\frac{\theta}{2}\right)\sin\theta, t\cos\frac{\theta}{2}\right)$.
such that $-\frac{1}{2} \leq t \leq \frac{1}{2}$ and $0 \leq \theta \leq 2\pi$. We then explicitly calculate the function of the normal vector to the centerline, which is $n(\theta) = (\cos\theta, \sin\theta, \cos\frac{\theta}{2})$. The normal vector should be equal at $\theta = 0, \theta = 2\pi$, but is instead $(1, 0, 1)$ and $(1, 0, -1)$, which are not equal. Therefore, the Mobius strip is not orientable.

Problem 6

Problem. Prove that if X is nonorientable, than $X \times Y$ is nonorientable for all Y.

Solution. Assume $X \times Y$ is orientable. We know there is no smooth choice of orientation for $T_x X$. Let y be some fixed point in Y. We know that $X \times \{y\}$ is diffeomorphic to X, but the tangent space $T_{(x,y)} X \times \{y\}$ is also isomorphic to $T_x X$ (as y is a fixed point). Therefore, this space is nonorientable, and expanding this over all y gives you that $X \times Y$ is not orientable.

Problem 9

Problem. Prove that there exists some $z \in \mathbb{C}$ such that $z^2 = e^{-|z|^2}$.

We can use the intermediate value theorem to prove that there is a zero of $f(x) = x^2 - e^{-|x|^2}$ over the reals. At $x = 0$, $f(0) = -1$. As $\lim_{x\to\infty} f(x) = \infty$, there exists some large number R such that for all $x > R$, $f(x) > 0$. Therefore, there is a real number satisfying this, which is also a complex number, so this has been proven.

6. The Lefschetz Fixed-Point Theorem

Problem 7

Problem. Compute $\chi(O(n))$, where $O(n)$ is the orthogonal group.

Solution. Left multiplication by some matrix $A \neq I$ in $O(n)$ has no fixed points. Left multiplication by A is homotopic to left multiplication by I. Therefore, the global Lefschetz number of left multiplication by A is zero, and as such $\chi(O(n)) = 0$.

More rigorously: let $M \in O(n)$, and let $f(M) = AM$, for $A \in O(n)/\{I\}$. A homotopy between it and the identity map is given by $f_t(M) = (A^T)^t AM$. f has no fixed points, so the Euler characteristic is 0.

Problem 8

Problem. More generally, explain how to compute $\chi(X)$, where X is a compact Lie group.

Solution. The Euler characteristic is zero. Consider left multiplication by some element $x \in X$ not the identity. This is homotopic to the identity, as the inversion map is smooth on Lie groups. Left multiplication by some element not the identity will have no fixed points (the identity element is unique in groups). Therefore, the Euler Characteristic will be zero as a result of the global Lefschetz number of the map being zero.

7. Vector Fields

Problem 2.

Problem. Let $\mathbf{v}$ be the vector field on $\mathbb{R}^2$ defined by $\mathbf{v}(x, y) = (x, y)$. Show that the family of diffeomorphisms $h_t : \mathbb{R}^2 \to \mathbb{R}^2$ is tangent to $\mathbf{v}$. That is, for any z, the curve $t \mapsto h_t(z)$ is tangent to $\mathbf{v}$. Compare $\operatorname{ind}_0(\mathbf{v})$ and $L_0(h_t)$.

Solution. We have that $\mathbf{v}(z) = z$, and we wish to show that $h_t'(z) = \mathbf{v}(h_t'(z))$. Well, w.r.t t, $h_t'(z) = z$, so this is shown. $\operatorname{ind}_0(\mathbf{v}) = L_0(h_t)$ by Proposition 3.2 (the curve is tangent at time 0, and the maps have no fixed points other than the origin) and both are equal to 1.

Problem 4

Problem. We mentioned that the torus admits a nowhere vanishing vector field. Describe such a vector field.

Solution. For a torus embedded in three dimensions, $\mathbf{v}(x, y, z) = (-\frac{y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}, 0)$. Each vector will have norm 1. It's the vector field generated by "combing" the torus counterclockwise.

Problem 7

Problem. Let f be a real function on a manifold X in $\mathbb{R}^N$. For each $x \in X$, $df_x : T_xX \to \mathbb{R}$ is a linear functional on T_xX. Thus there exists some $\mathbf{v}(x) \in T_xX$ such that $df_x(w) = \mathbf{v}(x) \cdot w$. This vector field $\mathbf{v}$ is called the gradient field of f, and we write $\mathbf{v} = \operatorname{grad}(f)$. Show that if $X = \mathbb{R}^n$, then

$$\operatorname{grad}(f) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

Solution. If $X = \mathbb{R}^n$, then $T_x X = \mathbb{R}^n$, so we're dealing with a linear map from $\mathbb{R}^n$ to $\mathbb{R}$, which is simply a Jacobian. E.g,

$$\begin{aligned} df_x(w) &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} w_i \\ &= \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right) \cdot w. \end{aligned}$$

Problem 12

Problem. Show that any Lie group of positive dimension has a nowhere vanishing vector field. Conclude that all Lie groups have Euler characteristic 0, and therefore that it is not possible to put a Lie group structure on $\mathbb{S}^n$ where n is even.

Solution. Let G be a Lie group. Pick $g \in G$ not the identity -- then left multiplication by g is a smooth map. The vector field $\mathbf{v}(x) = g$ is invariant and smooth and admitted onto G. Therefore, Lie groups have Euler characteristic zero, and as $\mathbb{S}^n$ has nonzero Euler characteristic for even n then they cannot admit a Lie group structure.

8. Introduction to Morse Theory

Problem 1

Problem. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = x^3 - 3xy^2$. Show that f has a critical point at $(0, 0)$. Is it nondegenerate? Draw some level sets, including the level set at 0.

Solution. The Jacobian is:

$$\mathbf{J}(f) = (3x^2 - 3y^2 \quad -6xy)$$

which is 0 at $(0, 0)$. Therefore, f has a critical point at $(0, 0)$. The Hessian is:

$$H(f) = \begin{pmatrix} 6x & -6y \\ -6y & -6x \end{pmatrix},$$

which is also 0 at $(0, 0)$. Therefore, this zero is degenerate.

Problem 2

Problem. Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ be Morse functions. Show that $f + g : X \times Y \to \mathbb{R}$ defined by $(f + g)(x, y) = f(x) + g(y)$ is a Morse function on $X \times Y$. What are its critical points?

Solution. The Hessian of $f + g$ is as follows:

$$\mathbf{H}(f+g) = \begin{bmatrix} \mathbf{H}(f) & 0 \\ \\ 0 & \mathbf{H}(g). \end{bmatrix}$$

Any derivatives of the form $\frac{\partial^2 f}{\partial x_i \partial y_j}$ or $\frac{\partial^2 f}{\partial y_i \partial x_j}$ will be 0 by definition. So, as the Hessian is a block diagonal matrix of $\mathbf{H}(f)$ and $\mathbf{H}(g)$, both of which have discriminant nonzero, then $\mathbf{H}(f+g)$ also has discriminant nonzero, which means it's invertible and nonsingular. Therefore, $f+g$ is Morse.

The critical points of $f+g$ exist when $d(f+g)_{(x,y)} = 0$. As this is just when the Jacobian is zero, each partial first derivative of $f+g$ must be zero. This is equivalent to (x, y) being a critical point of $f+g$ when x is a critical point of f and y is a critical point of g. All are nondegenerate.

Problem 4

Problem. Define $f : \mathbb{CP}^n \to \mathbb{R}$ by

$$f(z_0, \ldots, z_n) = \frac{\sum_{j=0}^n j|z_j|^2}{\sum_{j=0}^n |z_j|^2}.$$

Show that this is actually a well defined function on $\mathbb{CP}^n$. What are its critical points? Show that f is a Morse function, and find the indices of all the critical points. What do you think the sublevel sets look like, up to homotopy equivalence?

Solution. f would be undefined when $\sum_{j=0}^n |z_j|^2 = 0$, which would only occur if $z_0, \ldots, z_n = 0$. This is not in the domain of f, as $\mathbb{CP}^n$ does not contain 0. Therefore, f is well defined.

The critical points of f occur when $z_1, \ldots, z_n = 0$ and for any value of z_0. To show this, we take advantage of the properties of complex projective space to set $\sum_{j=0}^n |z_j|^2 = 1$ without loss of generality. This is possible because each element in complex projective space is equivalent to its normalized version. Then, f is simplified to

$$f(z_0, \ldots, z_n) = \sum_{j=0}^n j|z_j|^2.$$

The gradient of this function must be zero. To find the gradient, we substitute $z_j = x_j + iy_j$, and find that

$$f(x_0, y_0, \ldots, x_n, y_n) = \sum_{j=0}^n j(x_j^2 + y_j^2).$$

So,

$$\frac{\partial f}{\partial x_j} = 2jx_j, \quad \frac{\partial f}{\partial y_j} = 2jy_j.$$

These are only simultaneously zero when $x_j, y_j = 0$ for all $j \neq 0$. At $j = 0$, these derivatives are zero for all values of x_0, y_0. Therefore, the critical points of this function are of the form $(z_0, 0 \ldots, 0)$, and are all equivalent to $(1, 0, \ldots, 0)$ in projective space.

Note that this is possible because $\mathbb{CP}^n$ is a manifold contained in Euclidean space of dimension $2n+2$, and we can consider this a function from $\mathbb{R}^{2n+2}$ when writing the Hessian.

Any element of the Hessian of the form $\frac{\partial^2 f}{\partial x_i \partial x_j}$ (where $i \neq j$, and $i, j \in [0, 2n+1]$ ranging over the real analogue of $z_0, \ldots, z_n$) will be 0. The only non-zero elements of the Hessian will be of the form $\frac{\partial^2 f}{\partial x_j^2} = 2j$ on the diagonal.

However, we run into an issue. $\frac{\partial^2 f}{\partial x_0^2} = 0$, $\frac{\partial^2 f}{\partial x_1^2} = 0$. f is invariant under changes to z_0 for all z_0. Therefore, as this term occurs on the Hessian's diagonal, the Hessian will have zero eigenvalues, and therefore be singular.

However, because the Hessian is invariant under z_0, we have exactly the same information by only considering the domain to be over $\mathbb{CP}^{n-1}$, in which case f is Morse because it has only nonzero real eigenvalues.

(Note: considering f not under the equivalence class of $\sum_{j=0}^n |z_j|^2 = 1$ gives the same effect. The extra condition imposed ends up being $\sum_{j=0}^n (k-j)(x_j^2 + y_j^2)$, for some $k \in [0, n]$, and this is impossible to satisfy unless all $x_j, y_j = 0$).

The only critical point, $z_1, \ldots, z_n = 0$, has index 0, so it is a minimum.

Don't know how to describe the sublevel sets. Seems like the max norm of z_j would be $\sqrt{\frac{a}{j}}$, and the tradeoff between the norms of any two z_i, z_j would be linear, so this would be some convex polyhedra.

9. Morse Homology

Problem 1

Problem. Suppose we have a chain complex

$$0 \to A \to 0$$

where A is an abelian group. What is its homology?

Solution. We have that $f_1 : A \to 0$, and $f_2 : 0 \to A$. So, $\ker f_1 = A$, and $\operatorname{im} f_2 = 0$. Therefore, the homology of this chain complex is A for $i = 1$, and 0 for $n = 0$, and trivial for all else.

Problem 2

Problem. A chain complex of abelian groups is said to be an *exact sequence* if all of its homology groups are trivial.
(a) Show that if we have an exact sequence of the form $0 \to A \to B \to 0$, then $A \equiv B$.
(b) Show that if $A \oplus C \equiv B$, then there is an exact sequence $0 \to A \to B \to C \to 0$.
(c) Show that if $0 \to A \to B \to C \to 0$ is an exact sequence, then it need not be the case that $B \equiv A \oplus C$.

Solution.

(a) If we have $f_1 : B \to 0$, and $f_2 : A \to B$ both homeomorphisms, and the sequence is exact, then $\frac{kerf_1}{imf_2}$ has to be the trivial group, which means that $kerf_1 = imf_2$. $kerf_1 = B$, and $imf_2 = A$, which implies that $A \equiv B$.

(b) For the sequence to be exact, $ker(B \to C) = \mathrm{im}(A \to B) \implies \ker(B \to C) = A$, which is possible given that $A \oplus C = B$.

(c) Example: $A \equiv B \equiv C$.

Problem 3

Problem.