D. $X_{t+\tau} = \phi_{t+\tau} X_{t+\tau-1} + W_{t+\tau}$

$$= \phi_{t+\tau}\left(\phi_{t+\tau-1} X_{t+\tau-2} + W_{t+\tau-1}\right) + W_{t+\tau}$$

$$\vdots$$

$$\cancel{X_{t+\tau} = \prod_{i=0}^{\tau-1} \phi_{t+\tau-i} \cdot X_t + \sum_{i=0}^{\tau-1}\left(\prod_{j=0}^{i} \phi_{t+\tau-j}\right) \cdot W_2}$$

$$\boxed{X_{t+\tau} = \prod_{i=0}^{\tau-1} \phi_{t+\tau-i} \cdot X_t + \sum_{i=0}^{\tau-1}\left(\prod_{j=0}^{i} \phi_{t+\tau-j}\right) W_{t+\tau-i}}$$

E.

$$cor(X_t, X_{t+\tau}) = E\left[\frac{X_t \cdot \underline{\qquad\qquad}}{\sqrt{var[v_t]\, var[v_{t+\tau}]}}\right]$$

I have no idea how one would go about plotting this...

2. $X_{t+1} = \phi(t+1) X_t + W_t$

Note: ϕ_{t+1} here is same as $\phi(t+1)$

A. $var(X_{t+1}) = var(\phi(t+1) X_t) + \sigma_w^2$

$$var(X_{t+1}) = \phi_{t+1}^2 \, var X_t + \underbrace{\sigma_w^2}_{\text{constant ind. of } t}$$

$$var(X_{t+2}) = \phi_{t+2}^2 \, var X_{t+1} + \sigma_w^2$$

$$var(X_{t+\tau}) - \phi_{t+\tau}^2 \, var(X_{t+\tau-1}) = \sigma_w^2$$

$$\begin{bmatrix} -\phi_{t+1}^2 & 1 & 0 & \cdots & & 0 \\ 0 & -\phi_{t+2}^2 & 1 & 0 & \cdots & \\ 0 & 0 & \ddots & \ddots & & \\ \vdots & & & \ddots & \ddots & \\ 0 & \cdots & & & -\phi_{t+11}^2 & 1 \\ 1 & & & & & -\phi_t \end{bmatrix} \begin{bmatrix} var(X_t) \\ \vdots \\ \vdots \\ var(X_{t+11}) \end{bmatrix} = \begin{bmatrix} \sigma_w^2 \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}$$

B. Multiply bottom row by $\frac{1}{\phi_t^2}$ and add to one above, repeat:

$$\begin{bmatrix} \frac{1}{\phi_t \phi_{t+2} \cdots} & -\phi_{t+1}^2 & 0 & \cdots & & 0 \\ 1/\phi_t^2 \phi_{t+11}^2 \cdots & & \ddots & & & \\ \vdots & & & \ddots & & \\ 1/\phi_t^2 & & & & \ddots & \\ 1 & 0 & \cdots & & 0 & -\phi_t^2 \end{bmatrix} \quad \vec{V}_t = \begin{bmatrix} \sigma_w^2 \left(\sum_0^{11} \prod_0^{i} \frac{1}{\phi_{t+12-j}^2} + 1 \right) \\ \vdots \\ \sigma_w^2 \end{bmatrix}$$