$$E\left[\left(\frac{1}{N}\sum(x_i-\mu)\right)\left(\frac{1}{N}\sum(x_j-\mu)\right)\right] = var(\bar{X})$$

$$= \frac{1}{N^2} E\left[\left(\sum x_i - N\mu\right)\left(\sum x_j - N\mu\right)\right]$$

$$= \frac{1}{N^2} E\left[\sum x_i \sum x_j - N\mu\left(\sum(x_i+x_j)\right)\right] + E[\mu^2]$$

$$= \frac{1}{N^2} E\left[\sum x_i \sum x_j\right] - \frac{2}{N}\mu\, E\left[\sum x_i\right] + \mu^2 + \overset{=0}{var[\mu]}$$

$$= \frac{1}{N^2} E\left[\sum x_i \sum x_j\right] - 2\mu^2 + \mu^2$$

$$= \frac{1}{N^2} E\left[\sum\left(x_i \sum x_j\right)\right] - \mu^2 = \frac{1}{N^2}\sum E\left[x_i \sum x_j\right] - \mu^2$$

$$= \frac{1}{N^2}\left(\sum\left(\mu E\left[\sum x_j\right] + cov\left(x_i, \sum x_j\right)\right)\right) - \mu^2$$

$$= \frac{\mu}{N^2}\sum_{i=1}^{N} E\left[\sum_{j=1}^{N} x_j\right] + \frac{1}{N^2}\sum_{i=1}^{N} cov\left(x_i, \sum_j x_j\right) - \mu^2$$

$$= \frac{\mu}{N^2}\sum_i\sum_j \mu - \mu^2 + \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} cov(x_i, x_j)$$

$$= \cancel{\frac{\mu\cdot\mu\cdot N^2}{N^2}} - \cancel{\mu^2} + \frac{1}{N^2}\sum_i\sum_j cov(x_i, x_j)$$

so $\mathrm{var}(\bar{X}) = \frac{1}{N^2} \sum_i \sum_j \mathrm{cov}(x_i, x_j)$

$c(\tau) = \mathrm{cov}(X_i, X_{i+\tau}) = \mathrm{cov}(X_{j+\tau}, X_j)$

$$\mathrm{var}(\bar{X}) = \frac{1}{N^2}\begin{pmatrix} \mathrm{cov}(x_1, x_1) & \cdots & \mathrm{cov}(x_N, x_1) \\ \vdots & \ddots & \\ \mathrm{cov}(x_1, x_N) & & \mathrm{cov}(x_N, x_N) \end{pmatrix}$$

$$= \frac{1}{N^2}\begin{pmatrix} c(0) + & \cdots & + c(N-1) \\ \vdots & \ddots & \vdots \\ + c(N-1) & \cdots & + c(0) \end{pmatrix}$$

$$= \frac{N c(0)}{N^2} + \frac{2}{N^2} \sum_{i=1}^{N-1} c(i)(N-i)$$

twice sum each diagonal

$$= \frac{\sigma^2}{N} + \frac{2}{N}\sigma^2 \sum_{i=1}^{N-1} \frac{c(i)}{c(0)} \frac{(N-i)}{N}$$

$$\boxed{\mathrm{var}(\bar{X}) = \frac{\sigma^2}{N}\left(1 + 2\sum_{\tau=1}^{N-1}\left(1 - \frac{\tau}{N}\right)\rho(\tau)\right)}$$