Lattice Gauge Theory using Fermions

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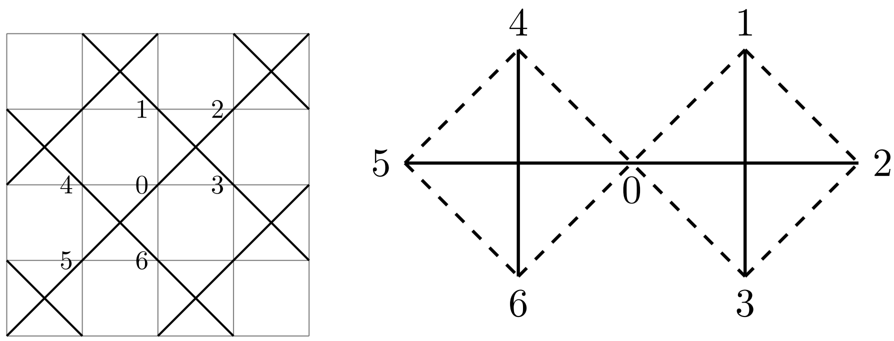
In this set of notes, we first review the connection of the Hamiltonian of a $U(1)$ lattice gauge theory constructed with quantum links on a square lattice with that of a pyrochlore spin-ice model projected on a 2-d checkerboard lattice.

1 Antiferromagnets and Gauge Theories

Consider the anti-ferromagnetic Heisenberg model on a checkerboard lattice. The checkerboard lattice can be realized by putting diagonal bonds on every alternate plaquette on the square lattice as shown in Fig 1. This connectivity mimics the local environment as in a pyrochlore lattice [1].



The antiferromagnet is described by the Hamiltonian:

$$H = J_z \sum_{<ij>} S_i^z S_j^z + J_\perp \sum_{<ij>} (S_i^+ S_j^- + S_i^- S_j^+) \tag{1}$$

As usual the $< ij >$ run over nearest neighbor bonds. For the checkerboard lattice, this indicates the 6 bonds as illustrated in Fig 1. The spin at site 0 is connected to the spins at sites 1,2,3,4,5,6. For $J_\perp = 0$, the nearest neighbor interaction between the spins give the following contributions for H_{S_0} (also see Fig 1 for a simplified depiction of the connectivity):

$$H_{S_0} = J_z (S_0^z S_1^z + S_0^z S_2^z + S_0^z S_3^z + S_0^z S_4^z + S_0^z S_5^z + S_0^z S_6^z) \tag{2}$$

In this limit, the minimal energy for a given S_0^z (say $S_0^z = \frac{1}{2}$), is obtained when four of the six spins have the same orientation as S_0^z, and two have opposite orientations, each situated on different sides of S_0.

Add descriptions about the ground state, with some cartoons.
show how the ground states are related to each other in perturbation theory.

2 Quantum link models with quantum spins

Quantum link models are extensions of Wilson-type lattice gauge theories, and show rich physics beyond the phenomena of conventional Wilson gauge theories. Let's review the simplest $U(1)$ link model realized with quantum spins $S = \frac{1}{2}$.

The Hamiltonian of the link model is

$$H = \frac{g^2}{2} \sum_{x,\mu} E_{x,\mu}^2 - J \sum_{\square} (U_\square + U_\square^\dagger) + \lambda \sum_{\square} (U_\square + U_\square^\dagger)^2. \tag{3}$$

where $E_{x,\mu}$ is the electric field operator defined on the link joining the sites x and $x + \hat{\mu}$. The plaquette operator is defined via the parallel transport operator $U_{x,\mu}$ as:

$$U_\square = U_{x,\mu} U_{x+\hat{\mu},\nu} U_{x+\hat{\nu},\mu}^\dagger U_{x,\nu}^\dagger \tag{4}$$

Each link joining has 3 operators $U_{x,\mu}$, $U_{x,\mu}^\dagger$ and $E_{x,\mu}$ which can be realized by the generators of a $SU(2)$ algebra. The operators satisfy the following commutation relations:

$$[E_{x,\mu}, U_{y,\nu}] = U_{x,\mu}\delta_{x,y}\delta_{\mu,\nu}; \quad [E_{x,\mu}, U_{y,\nu}^\dagger] = -U_{x,\mu}^\dagger \delta_{x,y}\delta_{\mu,\nu}; \quad [U_{x,\mu}, U_{y,\nu}^\dagger] = 2E_{x,\mu}\delta_{x,y}\delta_{\mu,\nu}. \tag{5}$$

The Hamiltonian has a local $U(1)$ invariance generated by the operator

$$G_x = \sum_\mu (E_{x,\mu} - E_{x-\hat{\mu},\mu}); \quad [G_x, H] = 0, \text{for all } \ x. \tag{6}$$

Because of the local commutation relations, we need to additional conditions to specify the Hilbert space. In the context of particle physics it is usual to choose a vacuum which does not have any charges. Mathematically, this is expressed as: $G_x|\psi\rangle = 0$. $|\psi\rangle$ is a physical state of the theory.

An equivalent of expressing this is as follows. If $|\psi\rangle$ is an eigenstate of the Hamiltonian, then we have

$$H|\psi\rangle = E|\psi\rangle \tag{7}$$

$$\exp(i\theta_x G_x) H \exp(-i\theta_x G_x) \exp(i\theta_x G_x)|\psi\rangle = E|\psi\rangle \tag{8}$$

We note that any representation of the operators $E_{x,\mu}$, $U_{x,\mu}$ and $U_{x,\mu}^\dagger$ is permissible as long as the commutation relations in equation 5 are satisfied. The usual, and well-known case of Wilson-type lattice gauge theory uses an infinite dimensional representation, by defining a quantum $U(1)$ rotor on each of the links. In this case, the $U_{x,\mu}$ is a unitary operator, and the commutation relation between $U_{x,\mu}$ and $U_{x,\mu}^\dagger$ vanishes.

As was realized by different authors [2], the operators can also be represented via a quantum spin operator $\vec{S}_{x,\mu}$ living on each link, giving rise to a finite dimensional representation, in constrast to the infinite dimensional case as in the Wilson theory. The raising and the lowering spin operators can be identified with the quantum link gauge fields, and the z-component with the electric field:

$$U_{x,\mu} = S_{x,\mu}^+; \quad U_{x,\mu}^\dagger = S_{x,\mu}^-; \quad E_{x,\mu} = S_{x,\mu}^z. \tag{9}$$

$U(1)$ quantum link models are known to have very interesting ground state properties, which have been discussed elsewhere [3]. In the context of condensed matter physics, they can be connected with the quantum dimer model on a square lattice [4]. Coupled to fermions they are also under investigation [5] using the ideas of fermion bags [6].

3 Quantum link models with fermions

We discuss the possibility of constructing even more generalised gauge theories with fermions, which can in principle very different features as the ones shown by the link model constructed with quantum spins [3]. As in the case of the spins, we have to construct as set of fermionic operators which satisfy the relevant commutation relations. We do this by choosing fermionic creation and annhiliation operators on every link $c_{x,\mu}, c^\dagger_{x,\mu}$ respectively, which can change the occupation number of fermions on a link, $n_{x,\mu}$. A similar model has been investigated in [7].

Within our construction, we note that the same commutation relations (equation 5) required for $U(1)$ gauge invariance of the Hamiltonian in equation 3 can be realized with the fermionic operators:

$$U_{x,\mu} = c^\dagger_{x,\mu}; \quad U^\dagger_{x,\mu} = c_{x,\mu}; \quad E_{x,\mu} = n_{x,\mu} - \frac{1}{2}. \tag{10}$$

As is usual, operators defined on different links commute. This is true of the Wilson theory as well as the quantum link models realized with quantum spins (with a spin-representation S). However, for the construction with fermions, the operators satisfy anti-commutation relations for operators on different links:

$$\{c_{x,\mu}, c_{y,\nu}\} = \{c^\dagger_{x,\mu}, c^\dagger_{y,\nu}\} = \{c^\dagger_{x,\mu}, c_{y,\nu}\} = 0, \quad \text{for} \quad x \neq y, \quad \text{and } \mu \neq \nu. \tag{11}$$

The Hamiltonian is still given by equation 3, but the plaquette terms are now defined as:

$$U_\Box = c_{x,\mu} c_{x+\hat{\mu},\nu} c^\dagger_{x+\hat{\nu},\mu} c^\dagger_{x,\nu}. \tag{12}$$

The Gauss Law is simply the difference of the occupation numbers of the links touching a site (since the constant factors of $\frac{1}{2}$ cancel):

$$G_x = \sum_\mu \left(n_{x,\mu} - n_{x-\mu,\mu}\right). \tag{13}$$

4 The Spin and the Fermion Link models compared

In order to understand the differences and the similarities to between the two different representations of the lattice gauge theory, we consider the simplest possible setting: the case of 2×2 lattice with periodic boundary conditions. For this system, there are 4 sites, and 8 links. Implemented without any further constraints, this would have a $2^8 = 256$ states, but due to Gauss' Law with $Q_x = G_x = 0$ for every site gives rise to only 18 total states of the system. The corresponding states in the spin and the fermion representation is shown in Fig 3 and Fig 4 respectively.

Before constructing the Hamiltonian explicitly for the 2×2 system, we divide the basis states into different winding number sectors in x- and y- directions (W_x, W_y). The winding number commutes with the Hamiltonian and in the basis we have chosen the Hamiltonian is block diagonal.

Clearly, the states 1,6,13, and 18 are eigenstates of the Hamiltonian with eigenvalue $= 0$. These are the states that carry maximal winding. The ground state lies in the zero winding sector, and hence is characterized by a six-dimensional vector. The action of the Hamiltonian on the different states is given

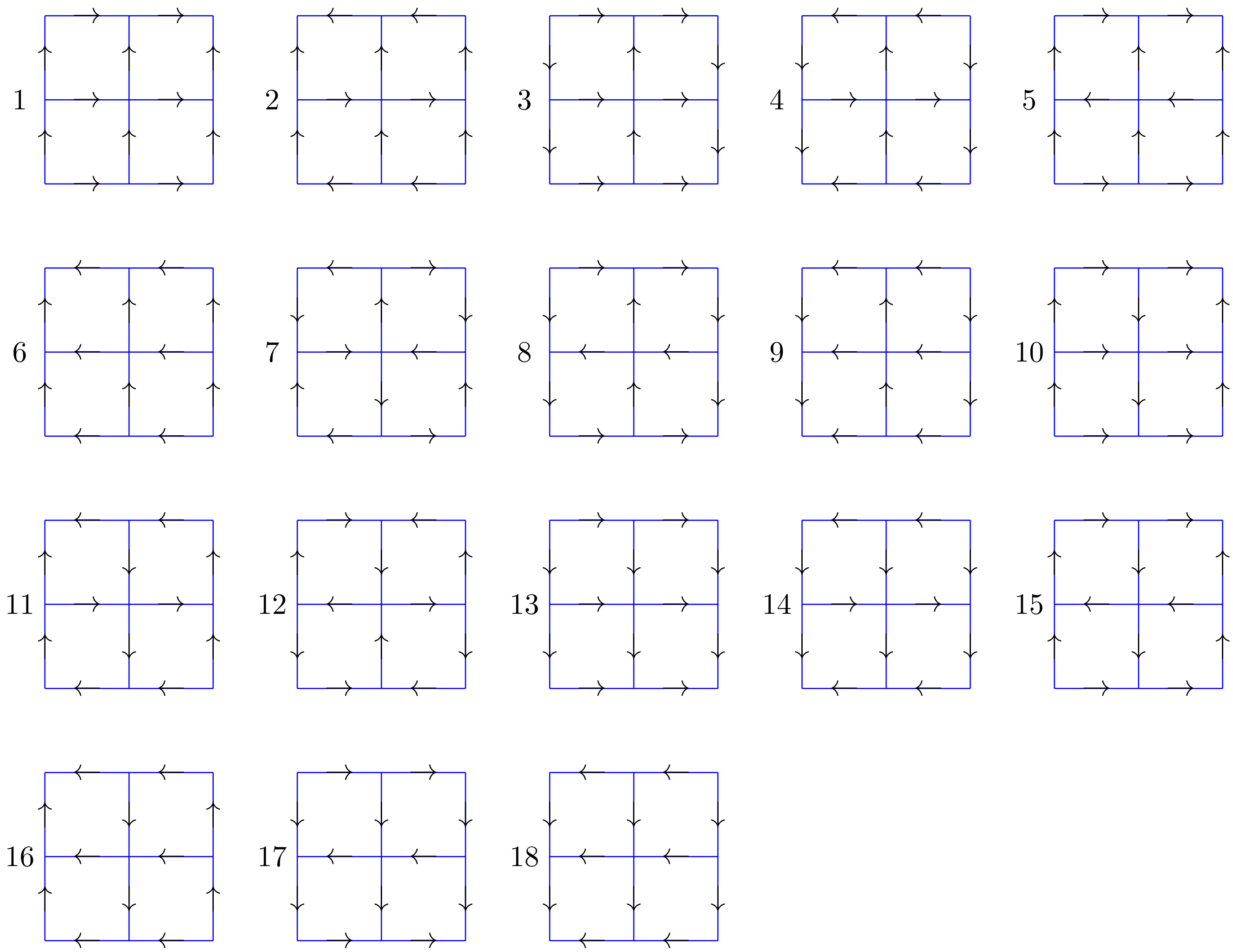


Figure 1: Cartoon states of the quantum link model realized with quantum spin $S = \frac{1}{2}$. The arrows represent the S^z component of the spins. Vertical arrows pointing up represent $S^z = \frac{1}{2}$, those pointing down represent $S^z = -\frac{1}{2}$. Horizontal left pointing arrows indicate $S^z = -\frac{1}{2}$, and right pointing arrows indicate $S^z = \frac{1}{2}$.

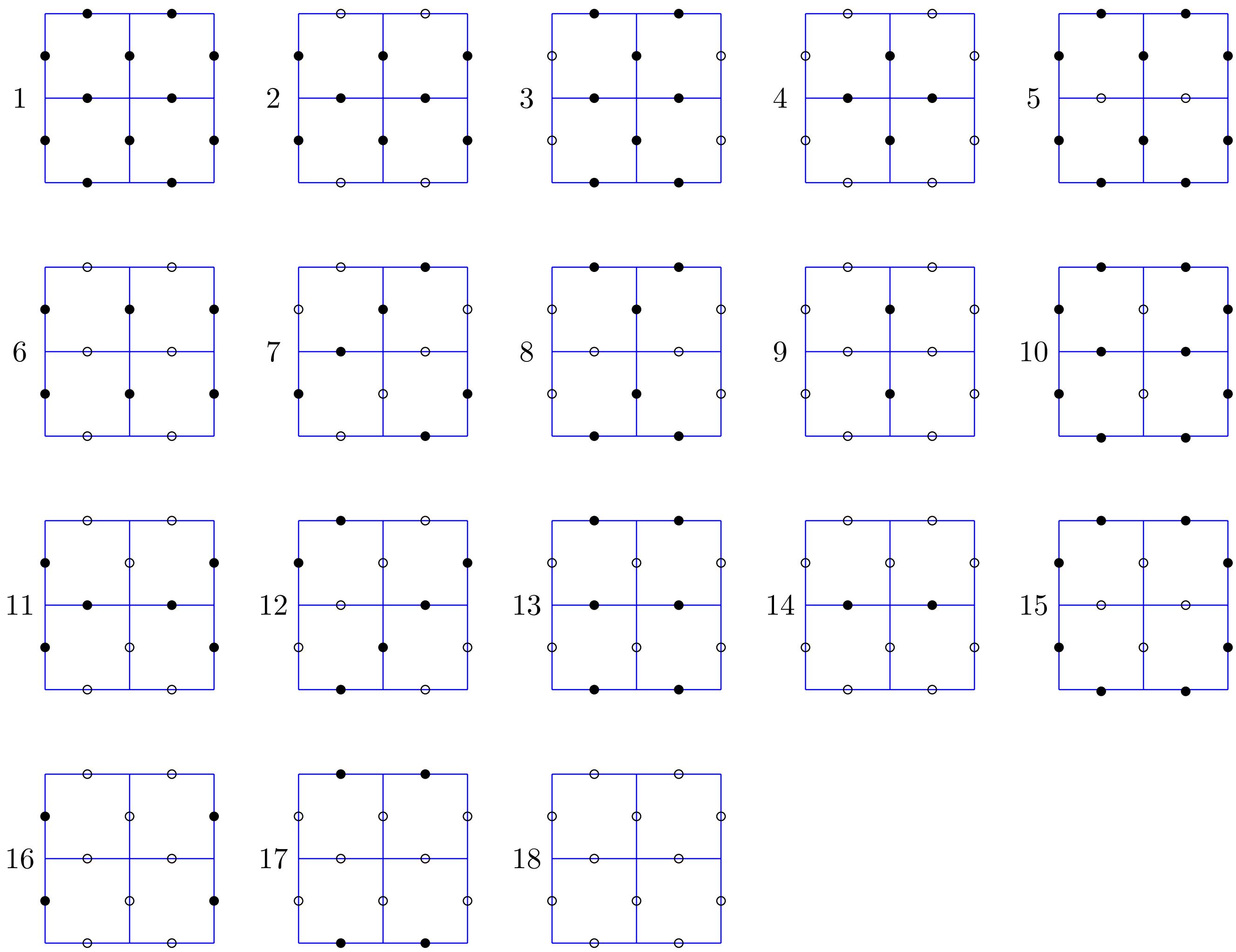


Figure 2: Cartoon states of the quantum link model realized with fermions. Open and filled circles represent the absence and presence of the fermions on the links respectively. There is an one to one map between the cartoon states in the spin $S = \frac{1}{2}$ basis and the fermion number basis.

(W_x, Wy)	list of states
(0,0)	4, 7, 8, 11, 12, 15
(1,0)	2, 5
(0,1)	3, 10
(-1,0)	14, 17
(0,-1)	9, 16
(1,1)	1
(1,-1)	6
(-1,1)	13
(-1,-1)	18

by:

$$H|4\rangle = -J(|7\rangle + |15\rangle) \tag{14}$$

$$H|7\rangle = -J(|4\rangle + |8\rangle + |11\rangle + |15\rangle) \tag{15}$$

$$H|8\rangle = -J(|7\rangle + |12\rangle) \tag{16}$$

$$H|12\rangle = -J(|4\rangle + |8\rangle + |11\rangle + |15\rangle) \tag{17}$$

$$H|15\rangle = -J(|7\rangle + |12\rangle) \tag{18}$$

Thus the eigenvalues are obtained by diagonalizing the matrix:

$$H_{(0,0)} = -J \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{19}$$

For J=1, the eigenvalues are -2.82843, 0, 0, 0, 0, 2.82843.

References

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