Fermion Sign Factors for the Overlap of States in the Occupation Number Basis: Page 1

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1. So long as the fermionic states are defined consistently, the sign factors will give the same energy spectrum.

First I will define the states in a particular way.

Using the notation $|n_0\, n_1 \, \ldots \, n_{N-1}\rangle$ to denote the states, with $n_0, n_1 \ldots n_{N-1} \in \{0, 1\}$ giving the occupation number for each of the sites, I have for the $N=4$ case:

$|0000\rangle = |0\rangle$, $|1000\rangle = c_0^\dagger|0\rangle$, $|0100\rangle = c_1^\dagger|0\rangle$,

$|0010\rangle = c_2^\dagger|0\rangle$, $|0001\rangle = c_3^\dagger|0\rangle$, $|1100\rangle = c_1^\dagger c_0^\dagger|0\rangle$,

$|1010\rangle = c_2^\dagger c_0^\dagger|0\rangle$, $|1001\rangle = c_3^\dagger c_0^\dagger|0\rangle$, $|0110\rangle = c_2^\dagger c_1^\dagger|0\rangle$,

$|0101\rangle = c_3^\dagger c_1^\dagger|0\rangle$, $|0011\rangle = c_3^\dagger c_2^\dagger|0\rangle$, $|1110\rangle = c_2^\dagger c_1^\dagger c_0^\dagger|0\rangle$,

$|1101\rangle = c_3^\dagger c_1^\dagger c_0^\dagger|0\rangle$, $|1011\rangle = c_3^\dagger c_2^\dagger c_0^\dagger|0\rangle$,

$|0111\rangle = c_3^\dagger c_2^\dagger c_1^\dagger|0\rangle$, $|1111\rangle = c_3^\dagger c_2^\dagger c_1^\dagger c_0^\dagger|0\rangle$.

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In general then, a state $|n_0 n_1 \ldots n_{N-1}\rangle$ is given by $(c^\dagger_{N-1})^{n_{N-1}} \ldots (c^\dagger_1)^{n_1} (c^\dagger_0)^{n_0} |0\rangle$.

2. So how do $c^\dagger_i$ or c_i affect the sign prefactor of state $|n_0 n_1 \ldots n_{N-1}\rangle$, defined above? We will assume that these operators do not annihilate the state.

- for $c^\dagger_i$, we will have to move it past all operators $c^\dagger_j$ where $j > i$ to put it in its correct position. Each movement (commutation) carries a factor of (-1). The sign factor will thus be

$$\boxed{\prod_{j=i+1}^{N-1} (-1)^{n_j}}$$

because if an operator $c^\dagger_j$ isn't there, $n_j = 0$ and thus there will be no sign contribution. The new state is

$$\prod_{j=i+1}^{N-1} (-1)^{n_j} |n_0 n_1 \ldots n_i + 1 \ldots n_{N-1}\rangle .$$

- For c_i, again we have to commute the operator across all $c_j^\dagger$ operators where $j > i$. Then we will have $c_i c_i^\dagger$, the "hole" operator which is equal to one since the particle at site i is now annihilated. The operator $c_i c_i^\dagger$ commutes with all the state operators, so no additional sign factors are introduced. Thus the sign factors follow the same formula as for $c_i^\dagger$ and the new state is

$$\prod_{j=i+1}^{N-1} (-1)^{n_j} |n_0 \, n_1 \ldots n_i - 1 \ldots n_{N-1}\rangle$$

3. <u>Compositions</u> Now we can get sign factors for any combination of fermionic operators on $|n_0 \, n_1 \ldots n_{N-1}\rangle$. But – it's important to update the $n_0 \ldots n_{N-1}$ values after the application of each operator. For example, for $c_{i_1}^\dagger c_{i_2}$, get the sign factor for c_{i_2}, update n_{i_2}, and then get the sign factor for $c_{i_1}^\dagger$.