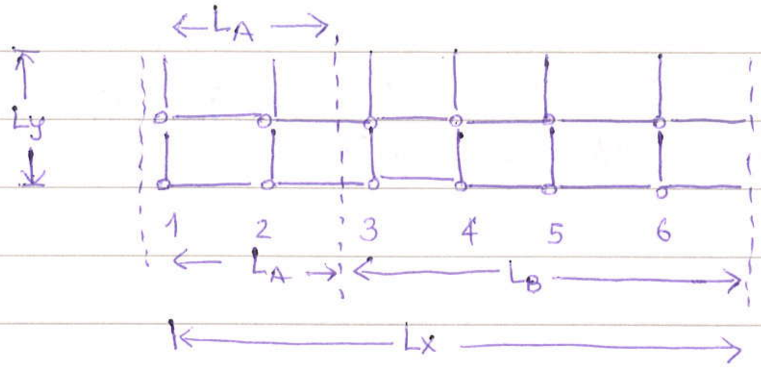
1. In this document the numerical implementation of Entanglement Entropy for a $U(1)$ quantum link model. In the discussion below, we address the computation in a ladder system with the cut described as shown in fig below. The case for any other cut can be implemented the same manner.



The idea is the change L_A, and calculate the entanglement entropy as a function of L_A.

2. Let us denote by, nD = the total number of gauge invariant basis states in a given sector defined by winding numbers (W_x, W_y). Our discussion will never involve the winding numbers changing in real-time dynamics.

The gauge invariant basis states can then be represented as

$$\{|e\rangle\} = \{|e_K\rangle, K = 1, 2, \ldots, nD\} \quad \text{— (1)}$$

A wavefn in this sector is then

$$|\psi\rangle = \sum_{K=1}^{nD} \psi_K |e_K\rangle \quad \text{— (2)}$$

Now, we have two partitions, $\{|e^{(A)}\rangle\}$ and $\{|e^{(B)}\rangle\}$ are the basis states that span these sections. Note that $\{|e^{(A)}\rangle\}$ and $\{|e^{(B)}\rangle\}$ are gauge invariant at all points in the bulk of the regions A and B respectively.

Only at the boundaries, they represent surface charges.

3. The calculation of $\{|e^A\rangle\}$ and $\{|e^B\rangle\}$ is also straightforward. Take each k in $\{|e_k\rangle\}$, and physically split up the systems into the links and sort them into $\{|e^A\rangle\}$ and $\{|e^B\rangle\}$ depending on whether they belong into the region A or B. Then remove the duplicates and obtain:

$$\left.\begin{aligned}\{|e^A\rangle\} &= \{|e^A_{iA}\rangle,\ iA = 1, \ldots, DA\} \\ \{|e^B\rangle\} &= \{|e^B_{iB}\rangle,\ iB = 1, \ldots, DB\}\end{aligned}\right\} \quad (3)$$

4. Given an eigenvector $|\psi\rangle$:

$$|\psi\rangle = \sum_k^{nD} \psi_k |e_k\rangle \qquad -4.1$$

$$= \sum_{iA}^{DA} \sum_{iB}^{DB} \chi_{i_A i_B} |e^A_{iA}\rangle \otimes |e^B_{iB}\rangle \qquad -4.2$$

$$= \sum_{l}^{n_\chi} \chi_l |\tilde{e}_l^{(A)}\rangle \otimes |\tilde{e}_l^{(B)}\rangle \qquad -4.3$$

First, patch individual basis vectors, say the i_A from $\{|e^A\rangle\}$ and i_B from the list $\{|e^B\rangle\}$ and obtain the corresponding ~~elemen~~ matrix element $\chi(iA, iB)$. Clearly χ is a rectangular matrix of dimensions DA × DB.

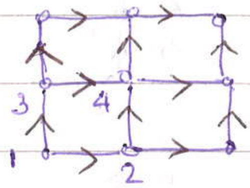
From step (4.2) to (4.3), one can implement the Schmidt decomposition, which gives the ~~positive~~ real and non-negative Schmidt values, χ_l, $l = 1, \ldots, n_\chi$, $\chi = \min\{DA, DB\}$.

The von-Neumann entropy for this subpartion of the state $|\psi\rangle$ is

$$S = -\sum_{i=1}^{\chi} |\chi_i|^2 \ln(|\chi_i|^2) \qquad -4.4$$

5. As a simple example, consider the case of the 2X2 lattice, which has 18 gauge invariant states.

These are the 18 states:

(1)


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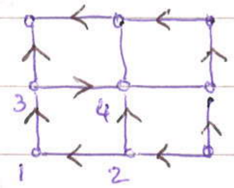

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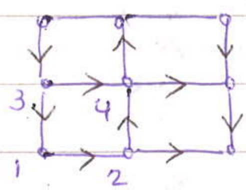

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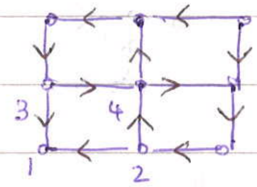

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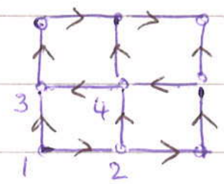

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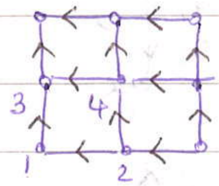

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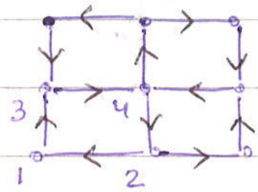

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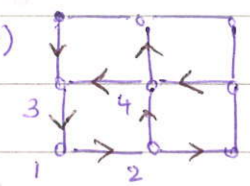

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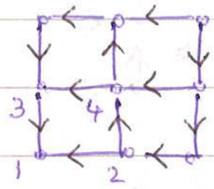

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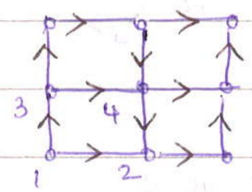

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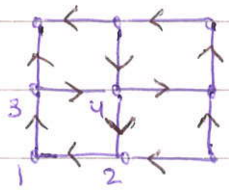

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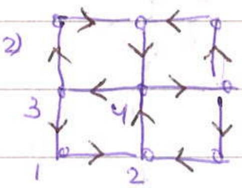

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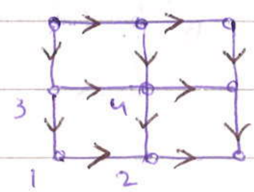

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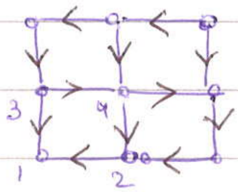

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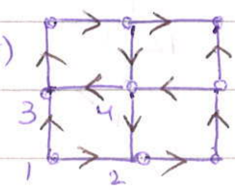

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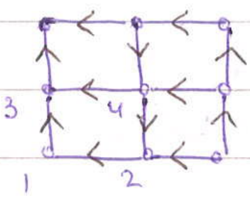

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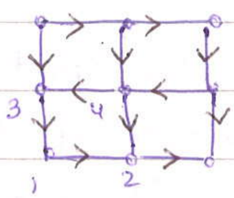

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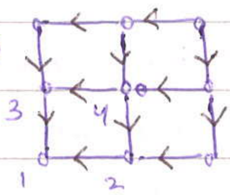
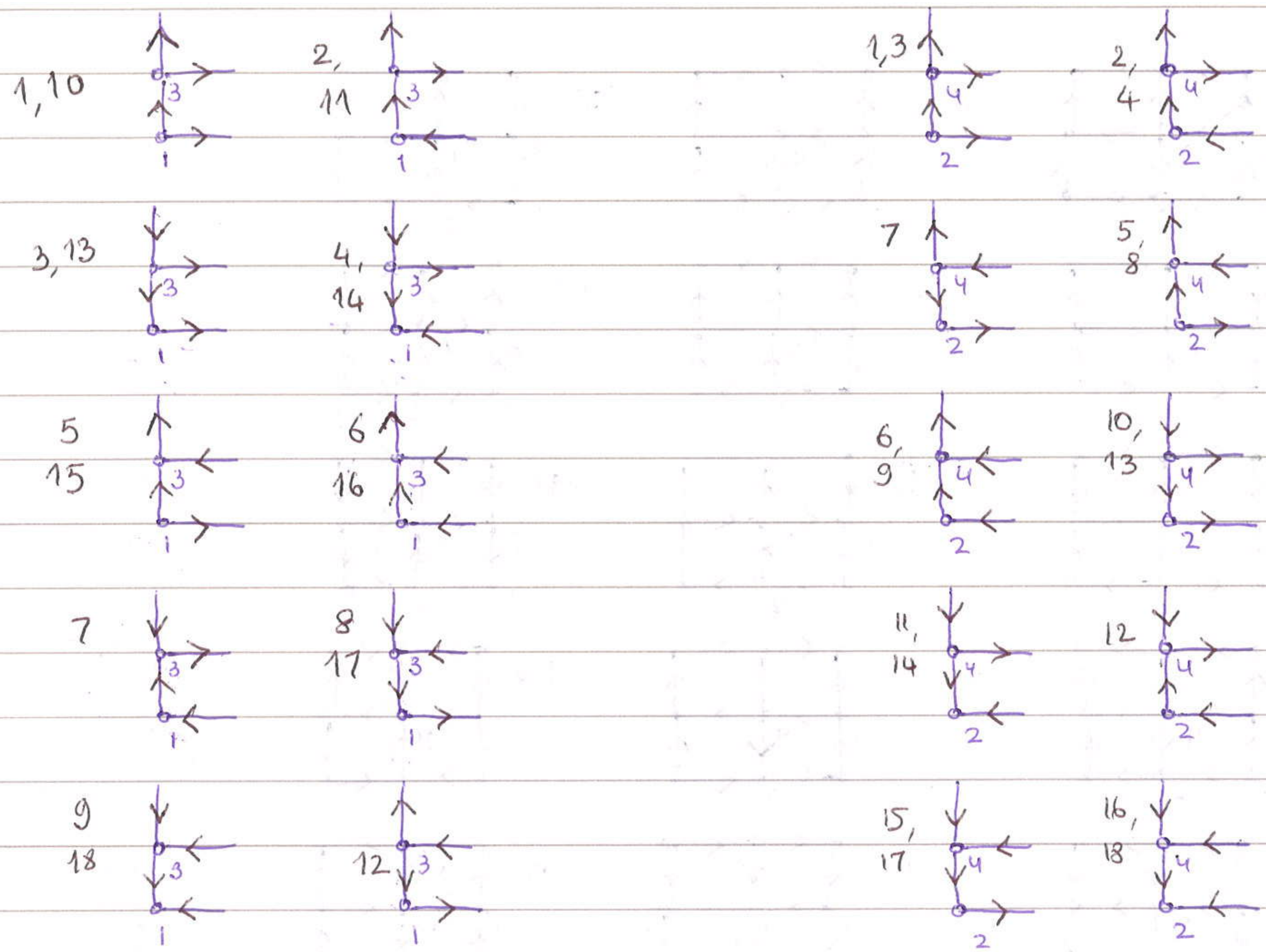


On cutting the system ~~xxxx~~ through $L_A = 1$, we get the following basis states in A & B:

A : B :



The example here shows how the splitting happens in practise. The numbers show which of the ~~b~~ full basis states these set in A & B correspond to.

As is evident, the basis in each partition is much lesser than $2^4 = 16$, since ~~*~~ Gauss law partially rules out many states (the info about Gauss Law is inherited since the Gauge inv. states of the full model are cut open to get these states).

6. Let us now discuss the calculation of entanglement entropy as a function of real-time when the state is evolving under unitary dynamics

given a quantum state, $|\Phi\rangle$, we do the eigenvector decomposition

$$|\Phi\rangle = \sum_{K=1}^{n_D} \alpha_K |\Psi_K\rangle ; \qquad |\Psi_K\rangle \to K^{th} \text{ eigenvector}$$
$$n_D \to \text{total \# of eigenvectors}$$

Now: $$e^{iHt}|\Phi\rangle = \sum_{K=1}^{n_D} \alpha_K e^{iE_K t} |\Psi_K\rangle = \sum_{K=1}^{n_D} \alpha_K \cos(E_K t) |\Psi_K\rangle$$

(the second equality ONLY follows when the spectrum is symmetric about zero, ~~...~~ E=0)

Recall that after a Schmidt decomposition, an eigenstate can be expressed as:

$$|\Psi_K\rangle = \sum_{i}^{n_\chi} \chi_i^{(K)} |\tilde{e}_i^{(A)}\rangle \otimes |\tilde{e}_i^{(B)}\rangle$$

$$e^{iHt}|\Psi_K\rangle = \sum_{i}^{n_\chi} \chi_i^{(K)} e^{iE_K t} |\tilde{e}_i^{(A)}\rangle \otimes |\tilde{e}_i^{(B)}\rangle = \sum_{i}^{n_\chi} \underbrace{\chi_i^{(K)}(t)}_{\chi_i^{(K)} e^{iE_K t}} |\tilde{e}_i^{(A)}\rangle \otimes |\tilde{e}_i^{(B)}\rangle$$

clearly then, the entanglement entropy of the K^{th} eigenstate

$$S^{(K)}(t) = -\sum_{i=1}^{n_\chi} |\chi_i^{(K)}(t)|^2 \ln\left(|\chi_i^{(K)}(t)|^2\right)$$

does NOT change as a function of time

For the time evolution of $|\Phi\rangle = |\Phi(t)\rangle$ however,

$$|\Phi(t)\rangle = \sum_{K=0}^{nD} \left(\alpha_K e^{iE_K t}\right) |\Psi_K\rangle = \sum_{K=0}^{n_D} \left(\alpha_K e^{iE_K t}\right) \sum_{l=1}^{n_\chi} \chi_l^{(K)} \, |\tilde{e}_l^{(A)}\rangle \otimes |\tilde{e}_l^{(B)}\rangle$$

Note that n_χ = total number of Schmidt values is a geometric property and does not change ~~under~~ for different eigenvalues.

Thus, interchanging the order of summation

$$|\Phi(t)\rangle = \sum_{l=1}^{n_\chi} \underbrace{\left(\sum_{k=0}^{n_D} \alpha_K e^{iE_K t} \chi_l^{(K)}\right)}_{\chi_l} \quad |\tilde{e}_l^{(A)}\rangle \otimes |\tilde{e}_l^{(B)}\rangle$$

$$\chi_l(t) = \sum_{K=0}^{n_D} \alpha_K e^{iE_K t} \chi_l^{(K)}$$

The entanglement entropy is then obtained as usual by:

$$|\Phi(t)\rangle = \sum_{l=1}^{n_\chi} \chi_l(t) \; |\tilde{e}_l^{(A)}\rangle \otimes |\tilde{e}_l^{(B)}\rangle$$

$$S(t) = -\sum_{l}^{n_\chi} |\chi_l(t)|^2 \ln\left(|\chi_l(t)|^2\right).$$