- Quantum Link Models

- Consider a usual "quantum spin" model.

$$\mathcal{H} = J\sum_{x,\mu} \left(S_x^1 \cdot S_{x+\mu}^1 + S_x^2 \cdot S_{x+\mu}^2 + S_x^3 \cdot S_{x+\mu}^3 \right)$$

Define: $S^1 = \sum_x S_x^1$; etc.

Then, $[S^1, H] = \left[\sum_y S_y, \sum_{x,\mu} J\left(S_x^1 \cdot S_{x+\mu}^1 + S_x^2 \cdot S_{x+\mu}^2 + S_x^3 \cdot S_{x+\mu}^3\right)\right]$

$$= \sum_y \sum_{x,\mu} J\left[S_y^1, S_x^1 \cdot S_{x+\mu}^1 + S_x^2 S_{x+\mu}^2 + S_x^3 S_{x+\mu}^3\right]$$

Next: $[S_y^1, S_x^1 \cdot S_{x+\mu}^1] = \underbrace{[S_y^1, S_x^1]}_{0} S_{x+\mu}^1 + S_x^1 \underbrace{[S_y^1, S_{x+\mu}^1]}_{0}$

$$[S_y^1, S_x^2 S_{x+\mu}^2] = [S_y^1, S_x^2]\, S_{x+\mu}^2 + S_x^2 \,[S_y^1, S_{x+\mu}^2]$$
$$= i\varepsilon^{123} S_x^3 S_{x+\mu}^2 + i\varepsilon^{123} S_x^2 S_{x+\mu}^3$$

and $[S_y^1, S_x^3 S_{x+\mu}^3] = i\varepsilon^{132} S_x^2 S_{x+\mu}^3 + i\varepsilon^{132} S_x^3 S_{x+\mu}^2 \longrightarrow$ which cancels the previous commutator

Thus, $[S^1, H] = 0$ & similarly $[S^2, H] = [S^3, H] = 0$

which means $[\vec{S}, H] = 0$ for $O(3)$ model.

- As a quantum $U(1)$ link model,

replace $u_{x,\mu} = \exp(i\varphi_{x,\mu}) \longrightarrow U_{x,\mu} = C_{x,\mu} + iS_{x,\mu}$

$$U_{x,\mu}^\dagger = C_{x,\mu} - iS_{x,\mu}$$

Clearly, $C_{x,\mu}^\dagger = C_{x,\mu}$ & $S_{x,\mu}^\dagger = S_{x,\mu}$.

- Action must remain invariant under local $U(1)$ transf.

Transf of U:

$$U'_{x,\mu} = \exp(-i\alpha_x G_x)\, U_{x,\mu} \exp(i\alpha_x G_x) \qquad \text{left G.T.}$$
$$\equiv \exp(i\alpha_x) U_{x,\mu}$$

and $U'_{x,\mu} = \exp(-i\alpha_{x+\mu} G_{x+\mu})\, U_{x,\mu} \exp(i\alpha_{x+\mu} G_{x+\mu})$ right G.T.

$$\equiv U_{x,\mu} \exp(-i\alpha_{x+\mu})$$

- Under an arbitrary G.T. acting on different sites :- $\prod_x \exp(i\alpha_x G_x)$,

$$\mathcal{U}'_{x,\mu} = \prod_m \exp(-i\alpha_m G_m)\, U_{x,\mu} \prod_n \exp(i\alpha_n G_n)$$

Generators of G.T.'s are hermitian ops. $G_L^\dagger = G_L$

Then $M = \exp(-i\alpha_l G_l)$; $M^\dagger = M^{-1}$: unitary transf. matrices

- Commutation relations. Consider the infinitesimal transf :-

$$\exp(-i\alpha_x G_x)\, U_{x,\mu} \exp(i\alpha_x G_x) = \exp(i\alpha_x)\, U_{x,\mu}$$

$\downarrow$ expanding out

$$(1 - i\alpha_x G_x)\underbrace{[C_{x,\mu} + iS_{x,\mu}]}_{U_{x,\mu}}(1 + i\alpha_x G_x) = (1 + i\alpha_x)[C_{x,\mu} + iS_{x,\mu}]$$

Simplify to get :- $[U_{x,\mu}, G_x] = U_{x,\mu}$ —— 1

Using $\exp(-i\alpha_{x+\mu} G_{x+\mu})\, U_{x,\mu} \exp(i\alpha_{x+\mu} G_{x+\mu}) = U_{x,\mu} \exp(-i\alpha_{x+\mu})$

$\Rightarrow$ $[U_{x,\mu}, G_{x+\mu}] = -U_{x,\mu}$ —— 2

Thus : $[G_x, U_{y,\mu}] = -\delta_{x,y} U_{y,\mu}$; $[G_{x+\mu}, U_{x,\mu}] = U_{x,\mu}$

$\hookrightarrow [G_x, U_{y,\mu}] = \delta_{x,y+\mu} U_{y,\mu}$

Giving: $\boxed{[G_x, U_{y,\mu}] = (\delta_{x,y+\mu} - \delta_{x,y})\, U_{y,\mu}}$

and $\boxed{[G_x, U^\dagger_{y,\mu}] = (\delta_{x,y} - \delta_{x,y+\mu})\, U^\dagger_{y,\mu}}$

$$[G_x, U_{y,\mu}] = [G_x, C_{x,\mu}] + i[G_x, S_{x,\mu}]$$

$$\|$$

$$(\delta_{x,y+\mu} - \delta_{x,y})[C_{x,\mu} + iS_{x,\mu}] = (\delta_{x,y+\mu} - \delta_{x,y})C_{x,\mu} + i(\delta_{x,y+\mu} - \delta_{x,y})S_{x,\mu}$$

Giving
$$\boxed{\begin{aligned}[G_x, C_{x,\mu}] &= i(\delta_{x,y+\mu} - \delta_{x,y})S_{x,\mu} \\ [G_x, S_{x,\mu}] &= i(\delta_{x,y} - \delta_{x,y+\mu})C_{x,\mu}\end{aligned}}$$

- A representation for these ~~...~~ operators are

$$C_{x,\mu} = S^1_{x,\mu} \, ; \quad S_{x,\mu} = S^2_{x,\mu} \quad \Rightarrow \quad U_{x,\mu} = S^+_{x,\mu}$$
$$U^+_{x,\mu} = S^-_{x,\mu}$$

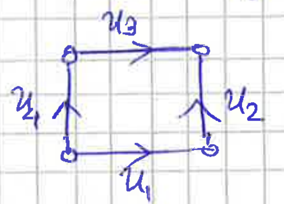
- Gauss' Law $\vec{\nabla}\cdot\vec{E} = \rho = \partial_x E_x + \partial_y E_y + \partial_z E_z$

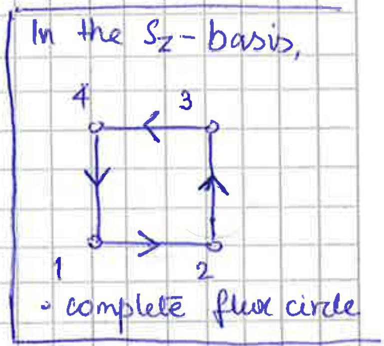
In discrete units,

$$\rho = (E_{n+\hat{x}} - E_n) + (E_{n+\hat{y}} - E_n) + (E_{n+z} - E_n)$$

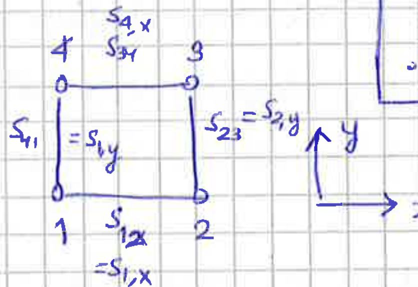
- In the S^z basis, the Hamiltonian of a single plaquette is

$$U_p = U_1 U_2 U_3^+ U_4^+$$
$$= \sigma_1^+ \sigma_2^+ \sigma_3^- \sigma_4^-$$
$$H = \frac{J}{2}\left(U_p + U_p^+\right)$$





- For the single plaquette:



$G|\psi\rangle = 0 \Rightarrow$ Gauss law.

$$G_1 = -S_{1,x} - S_{1,y}$$
$$\left(-S^z_{1,x} - S^z_{1y}\right)|\psi\rangle = 0 = \left(-S^3_{1,x} - S^3_{1,y}\right)|S_{12} S_{23} S_{34} S_{41}\rangle$$

$\Rightarrow$ if $S_{12} = \begin{pmatrix}1\\0\end{pmatrix}$, Then $S_{41} = \begin{pmatrix}0\\1\end{pmatrix}$

Similar application on the other vertices give: $S_{23} = \begin{pmatrix}1\\0\end{pmatrix}$; $S_{34} = \begin{pmatrix}0\\1\end{pmatrix}$

and hence, $|1\rangle = \left|\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right\rangle$

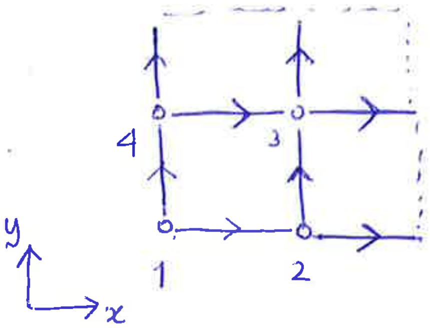
The other gauge ineq. state is $|2\rangle = \left|\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right\rangle$

$$H = J/2\left[\sigma^+_{1,x}\sigma^+_{2y}\sigma^-_{4,x}\sigma^-_{1,y} + \sigma^+_{4,y}\sigma^+_{4,x}\sigma^-_{2y}\sigma^-_{1,x}\right]$$

giving $\langle 1|H|1\rangle = \langle 2|H|2\rangle = 0$ & $\langle 1|H|2\rangle = \langle 2|H|1\rangle = J/2$

$$H = \begin{pmatrix}0 & J/2\\ J/2 & 0\end{pmatrix} \; ; \quad \text{evals: } \lambda = \pm J/2$$

o. Next, let us consider the 2x2 lattice in some detail (with PBC)



$$\mathcal{H} = \frac{J}{2} \sum_{x;\,\mu\nu} \left[U_{x,\mu} U_{x+\mu,\nu} U^{\dagger}_{x+\nu,\mu} U^{\dagger}_{x,\nu} + h.c. \right]$$

$$\mathcal{H} = \frac{J}{2} \Big[U_{1,x} U_{2,y} U^{\dagger}_{4,x} U^{\dagger}_{1,y} + h.c.$$

$$+ U_{2,x} U_{1,y} U^{\dagger}_{3,x} U^{\dagger}_{2,y} + h.c.$$

$$+ U_{3,x} U_{4,y} U^{\dagger}_{2,x} U^{\dagger}_{3,y} + h.c. + U_{4,x} U_{3,y} U^{\dagger}_{1,x} U^{\dagger}_{4,y} + h.c. \Big]$$

o. Using Gauss Law, 18 Gauge Inequiv. states

o. Let us list them; and try to act the Hamiltonian on each state. Note that if each plaquette can be represented by units of ~~flux~~ flux (± 1), then the effect of the Hamiltonian is to reverse the direction of flux.

~~If these~~ If any plaquette does not have a well formed flux circle, that state is eliminated.

Naively counting, this gives us 18 states.

2 × $\begin{matrix} + & + \\ + & + \end{matrix}$

2×2 × $\begin{matrix} + & - \\ + & - \end{matrix}$ $\quad \begin{matrix} + & + \\ + & - \end{matrix}$ ×2×3

$\begin{matrix} - & - \\ - & + \end{matrix}$ ×2×3

18 states ?

$\begin{pmatrix} + & + \\ + & + \end{pmatrix} \times 2 \; (+ \leftrightarrow -)$

placing of −

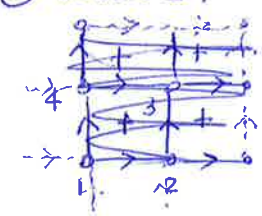
$\begin{pmatrix} + & + \\ + & - \end{pmatrix} \times 3 \times 2 \; (+ \leftrightarrow -)$ $\quad \underbrace{\begin{matrix} + & + \\ - & - \end{matrix}}_{\times 2} \quad \underbrace{\begin{matrix} + & - \\ + & - \end{matrix}}_{\times 2}$

$\begin{pmatrix} - & - \\ - & + \end{pmatrix} \times 3 \times 2$

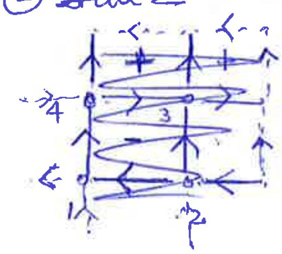
$2 + 6 + 6 + 4 = 18$

o. ~~Now, let's print out the states from the computer program to cross-check:~~

① ~~State 1~~:

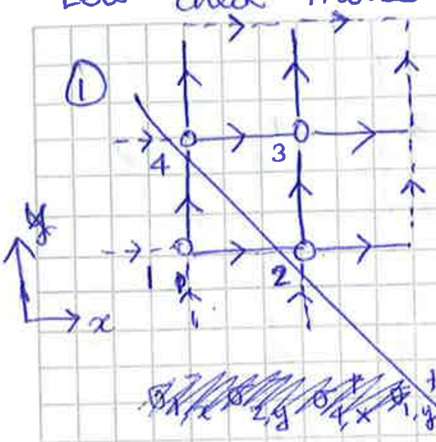


② ~~State 2~~



$U_p \equiv \sigma^+\sigma^+\sigma^-\sigma^- \qquad U_p^\dagger = \sigma^-\sigma^-\sigma^+\sigma^+$

- Lets check more carefully about the gauge inequivalent states



Applying Gauss Law at ~~diff~~ site 1:

$$G_1|\psi_1\rangle = 0 = (-G_{1,x} - G_{1,y} + G_{2,x} + G_{4,y})|\psi_1\rangle$$

If, $S_{1,x} = \binom{1}{0}$, $S_{2,x} = \binom{1}{0}$, $S_{1,y} = \binom{1}{0}$, $G_{4,y} = \binom{1}{0}$

At site 4 $(-G_{4,x} - G_{4,y} + G_{3,x} + G_{1,y})|\psi\rangle = 0$

$S_{1,y} = S_{4,y} = \binom{1}{0}$; $\quad G_{4,x} = G_{3,x} = \binom{1}{0}$

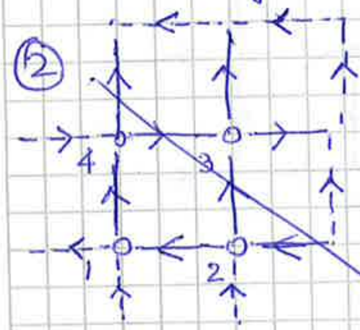
Then, the entire state is :-

$$|\psi_1\rangle = \left|\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{1}{0}\binom{1}{0}\right\rangle$$

$$= |S_{1,x} S_{1,y} S_{2,x} S_{2,y} S_{3,x} S_{3,y} S_{4,x} S_{4,y}\rangle$$

$\mathcal{H}|\psi_1\rangle = 0 \rightarrow$ no flux configuration is completed.

- So clearly our "naive" counting was wrong!



$$G_1|\psi_2\rangle = 0 = (-\underbrace{G_{1,x}}_{+1 = -(-1)} - \underbrace{G_{1,y}}_{-1} + \underbrace{G_{2,x}}_{+1} + \underbrace{G_{4,y}}_{-1})|\psi_2\rangle$$

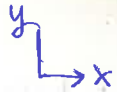
$S_{1,x} = \binom{0}{1}$; $S_{1,y} = \binom{1}{0}$; $S_{2,x} = \binom{1}{0}$; $S_{4,y} = \binom{0}{1}$

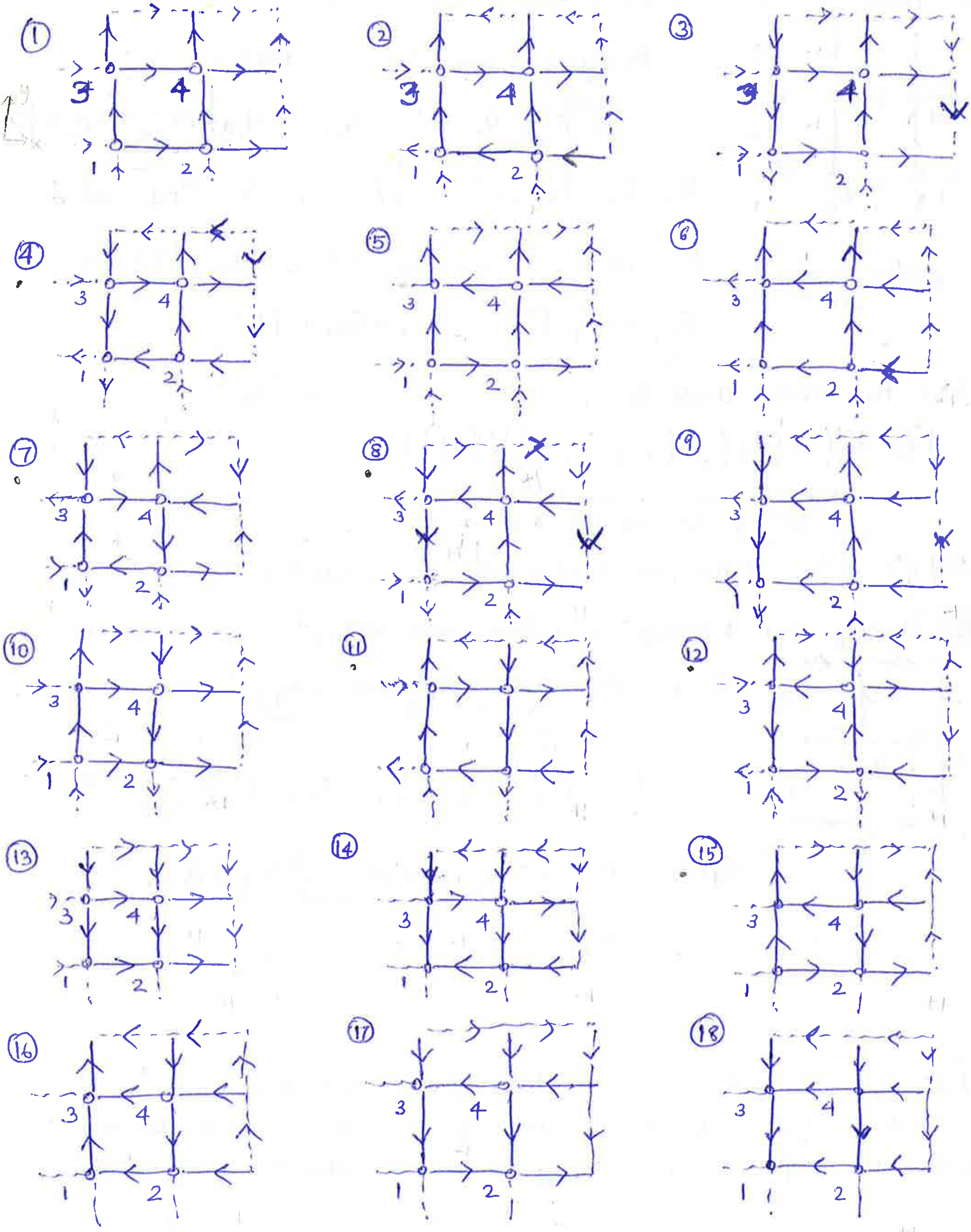
$$G_4|\psi_2\rangle = 0 = (-\underbrace{G_{4,x}}_{-1} - \underbrace{G_{4,y}}_{+1} + \underbrace{G_{3,x}}_{-1} + \underbrace{G_{1,y}}_{+1})|\psi_2\rangle$$

$S_{4,x} = \binom{0}{1}$; $S_{4,y} = \binom{1}{0}$; $S_{3,x} = \binom{0}{1}$; $S_{1,y} = \binom{1}{0}$

Lets just write out the states first, and then we'll check how they are related. In drawing the states, we'll assume that $+1$ refers to $\binom{1}{0}$ or $\rightarrow$ & -1 refers to $\binom{0}{1}$ or $\leftarrow$

All states on 2X2 PBC lattice.





The full Hamiltonian is:

$$\mathcal{H} = \frac{J}{2}\Big[\sigma^+_{1,x}\sigma^+_{2,y}\sigma^-_{3,x}\sigma^-_{1,y} + \sigma^+_{1,y}\sigma^+_{3,x}\sigma^-_{2,y}\sigma^-_{1,x}$$
$$+ \sigma^+_{2,x}\sigma^+_{1,y}\sigma^-_{4,x}\sigma^-_{2,y} + \sigma^+_{2,y}\sigma^+_{4,x}\sigma^-_{1,y}\sigma^-_{2,x}$$
$$+ \sigma^+_{3,x}\sigma^+_{4,y}\sigma^-_{1,x}\sigma^-_{3,y} + \sigma^+_{3,y}\sigma^+_{1,x}\sigma^-_{4,y}\sigma^-_{3,x}$$
$$+ \sigma^+_{4,x}\sigma^+_{3,y}\sigma^-_{2,x}\sigma^-_{4,y} + \sigma^+_{4,y}\sigma^+_{2,x}\sigma^-_{3,y}\sigma^-_{4,x} \Big]$$

Lets use tabulate these states explicity in the S_z basis:

$(\nabla\cdot E)_x|\psi\rangle = 0 \Rightarrow \left(\sum_\mu E_{x+\mu} - E_x\right)|\psi\rangle$

~~In 2-D; for our case writing $E_x = \sigma^3_x$~~

$$G_x = \sum_\mu \left(S^3_{x-\mu,\mu} - S^3_{x,\mu}\right) \Rightarrow$$

$G_1 = -S^3_{1,x} - S^3_{1,y} + S^3_{2,x} + S^3_{3,y}$ — ①

$G_2 = -S^3_{2,x} - S^3_{2,y} + S^3_{1,x} + S^3_{4,y}$ — ②

$G_3 = -S^3_{3,x} - S^3_{3,y} + S^3_{4,x} + S^3_{1,y}$ — ③

$G_4 = -S^3_{4,x} - S^3_{4,y} + S^3_{3,x} + S^3_{2,y}$ — ④

- State 1: $|\psi_1\rangle$

Choose $S^3_{1,x} = \begin{pmatrix}1\\0\end{pmatrix}$; $S^3_{2,x} = \begin{pmatrix}1\\0\end{pmatrix}$; $S^3_{1,y} = \begin{pmatrix}1\\0\end{pmatrix}$; $S^3_{3,y} = \begin{pmatrix}1\\0\end{pmatrix}$ in eq ①

$S^3_{2,y} = \begin{pmatrix}1\\0\end{pmatrix}$; $S^3_{4,y} = \begin{pmatrix}1\\0\end{pmatrix}$ in eq ②

$S^3_{4,x} = \begin{pmatrix}1\\0\end{pmatrix}$; $S^3_{3,x} = \begin{pmatrix}1\\0\end{pmatrix}$ in eq ③. $S^3_{4,x} = \begin{pmatrix}1\\0\end{pmatrix}$

Thus: $|\psi_1\rangle = \left|\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right\rangle$

$= |S_{1,x}\ S_{1,y}\ S_{2,x}\ S_{2,y}\ S_{3,x}\ S_{3,y}\ S_{4,x}\ S_{4,y}\rangle$

- Clearly $\mathcal{H}$ acting on this will give $\underline{0}$

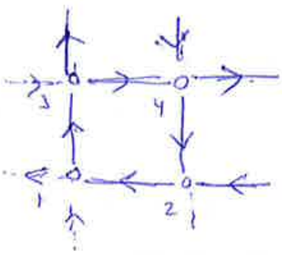
$\mathcal{H}|\psi_1\rangle = 0$

Now, suppose we make the choice:

$S_{1,x} = \begin{pmatrix}0\\1\end{pmatrix}$, $S_{1,y} = \begin{pmatrix}1\\0\end{pmatrix}$, $S_{2,x} = \begin{pmatrix}0\\1\end{pmatrix}$

$S_{3,y} = \begin{pmatrix}1\\0\end{pmatrix}$; $S_{2,y} = \begin{pmatrix}0\\1\end{pmatrix}$, $S_{4,y} = \begin{pmatrix}0\\1\end{pmatrix}$

$S_{3,x} = \begin{pmatrix}1\\0\end{pmatrix}$ $S_{4,x} = \begin{pmatrix}1\\0\end{pmatrix}$



$\equiv |\psi_{11}\rangle$

Then:

$$(\underbrace{-S^3_{1,x}}_{+1} \underbrace{- S_{1,y}}_{-1} + \underbrace{S_{2,x}}_{-1} + \underbrace{S_{3,y}}_{+})|\psi\rangle = 0$$

$$(\underbrace{-S_{2,x}}_{+1} \underbrace{- S_{2,y}}_{+1} + \underbrace{S_{1,x}}_{-1} + \underbrace{S_{4,y}}_{-1})|\psi\rangle = 0$$

$$(\underbrace{-S_{3,x}}_{-1} \underbrace{- S_{3,y}}_{-1} + \underbrace{S_{4,x}}_{+1} + \underbrace{S_{1,y}}_{+1})|\psi\rangle = 0$$

$$(\underbrace{-S_{4,x}}_{-1} \underbrace{- S_{4,y}}_{+1} + \underbrace{S_{3,x}}_{+1} + \underbrace{S_{2,y}}_{-1})|\psi\rangle$$

Thus, as is expected, choosing differently the conventions, give rise to different states. Henceforth we'll use the following conventions:

$\rightarrow\circ \begin{pmatrix}1\\0\end{pmatrix}$ $\circ\leftarrow \begin{pmatrix}0\\1\end{pmatrix}$ $\uparrow \begin{pmatrix}1\\0\end{pmatrix}$ $\downarrow \begin{pmatrix}0\\1\end{pmatrix}$

Moreover, note that there are many states that get annihilated by the action of H, so, our "naive" state counting in the flux basis was not correct.

Lets list all the 18-states; and then we'll ~~check~~ check what happens on H acts on it.

$\sigma_3 = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$

$|\psi\rangle = |S_{1,x} S_{1,y} S_{2,x} S_{2,y} S_{3,x} S_{3,y} S_{4,x} S_{4,y}\rangle$ denotes the generic state.

Moreover to simplify notation, we'll denote

$\begin{pmatrix}1\\0\end{pmatrix} \longleftrightarrow +1$ & $\begin{pmatrix}0\\1\end{pmatrix} \longleftrightarrow -1$

- with the understanding that

$s^+|+1\rangle = 0 = s^-|-1\rangle$

$s^+|-1\rangle = |+1\rangle$; $s^-|+1\rangle = |-1\rangle$

States -

$$|\psi_1\rangle = |+1, +1, +1, +1, +1, +1, +1, +1\rangle$$

$$|\psi_2\rangle = |-1, +1, -1, +1, +1, +1, +1, +1\rangle$$

$$|\psi_3\rangle = |+1, -1, +1, +1, +1, -1, +1, +1\rangle$$

$$|\psi_4\rangle = |-1, -1, -1, +1, +1, -1, +1, +1\rangle$$

$$|\psi_5\rangle = |+1, +1, +1, +1, -1, +1, -1, +1\rangle$$

$$|\psi_6\rangle = |-1, +1, -1, +1, -1, +1, -1, +1\rangle$$

$$|\psi_7\rangle = |-1, +1, +1, -1, +1, -1, -1, +1\rangle$$

$$|\psi_8\rangle = |+1, -1, +1, +1, -1, -1, -1, +1\rangle$$

$$|\psi_9\rangle = |-1, -1, -1, +1, -1, -1, -1, +1\rangle$$

$$|\psi_{10}\rangle = |+1, +1, +1, -1, +1, +1, +1, -1\rangle$$

$$|\psi_{11}\rangle = |-1, +1, -1, -1, +1, +1, +1, -1\rangle$$

$$|\psi_{12}\rangle = |+1, -1, -1, +1, -1, +1, +1, -1\rangle$$

$$|\psi_{13}\rangle = |+1, -1, +1, -1, +1, -1, +1, -1\rangle$$

$$|\psi_{14}\rangle = |-1, -1, -1, -1, +1, -1, +1, -1\rangle$$

$$|\psi_{15}\rangle = |+1, +1, +1, -1, -1, +1, -1, -1\rangle$$

$$|\psi_{16}\rangle = |-1, +1, -1, -1, -1, +1, -1, -1\rangle$$

$$|\psi_{17}\rangle = |+1, -1, +1, -1, -1, -1, -1, -1\rangle$$

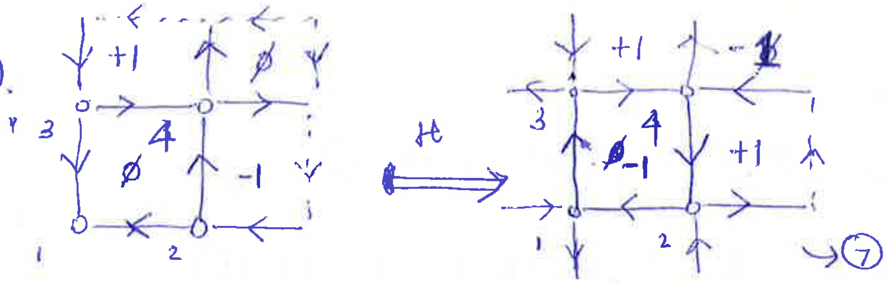
$$|\psi_{18}\rangle = |-1, -1, -1, -1, -1, -1, -1, -1\rangle$$

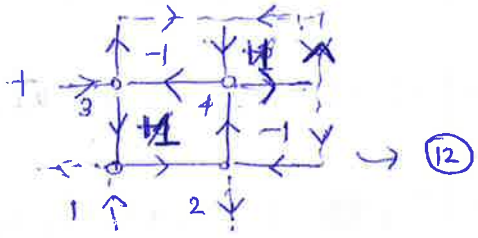
① $\mathcal{H}|\psi_1\rangle = 0$

② $\mathcal{H}|\psi_2\rangle = 0$

③ $\mathcal{H}|\psi_3\rangle = 0$

④.





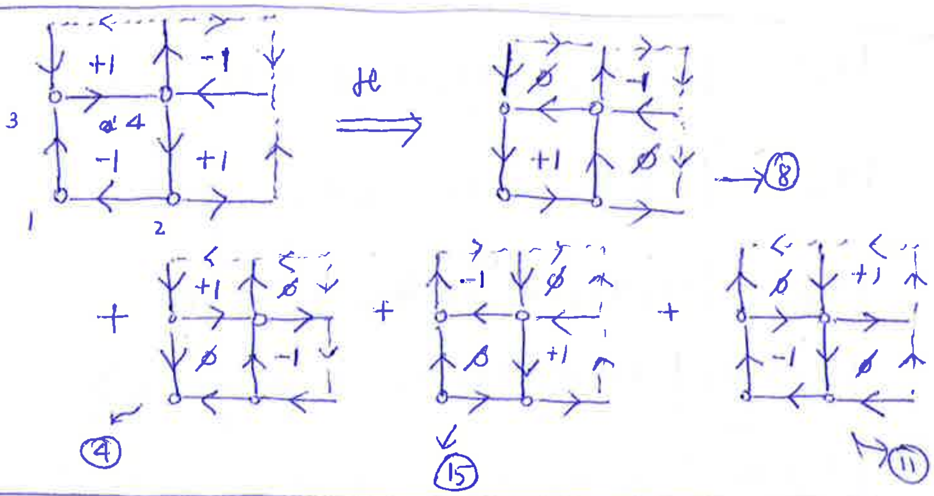
$$\mathcal{H}|\psi_4\rangle = \frac{J}{2}\left(|\psi_7\rangle + |\psi_{12}\rangle\right)$$

$$\boxed{\begin{aligned}\langle\psi_7|\mathcal{H}|\psi_4\rangle &= J/2\\ \langle\psi_{12}|\mathcal{H}|\psi_4\rangle &= J/2\end{aligned}}$$

⑤ $\mathcal{H}|\psi_5\rangle = 0$;

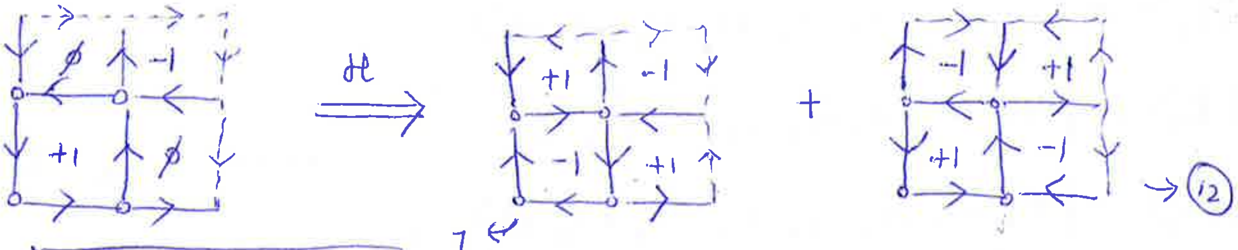
⑥ $\mathcal{H}|\psi_6\rangle = 0$

⑦



$$\boxed{\langle\psi_8|\mathcal{H}|\psi_7\rangle = J/2 = \langle\psi_{15}|\mathcal{H}|\psi_7\rangle = \langle\psi_{11}|\mathcal{H}|\psi_7\rangle}$$

⑧

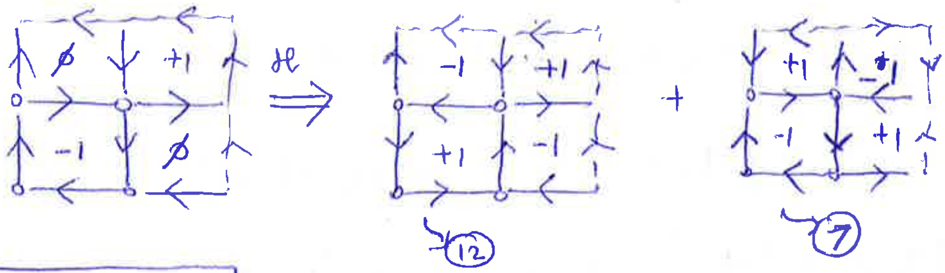


$$\boxed{\langle\psi_{12}|\mathcal{H}|\psi_8\rangle = J/2}$$

⑨ $\mathcal{H}|\psi_9\rangle = 0$

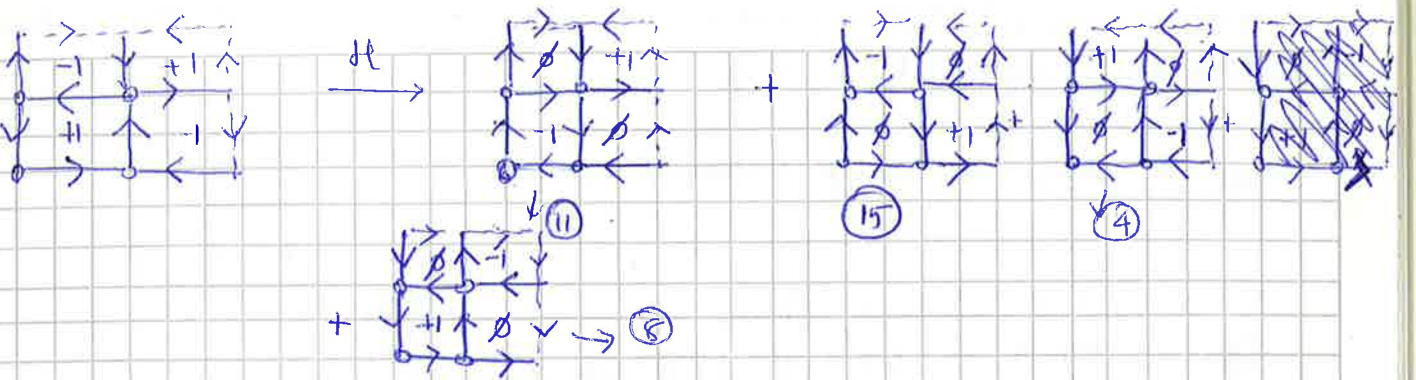
⑩ $\mathcal{H}|\psi_{10}\rangle = 0$

⑪



$$\boxed{\langle\psi_{12}|\mathcal{H}|\psi_{11}\rangle = J/2}$$

(12)

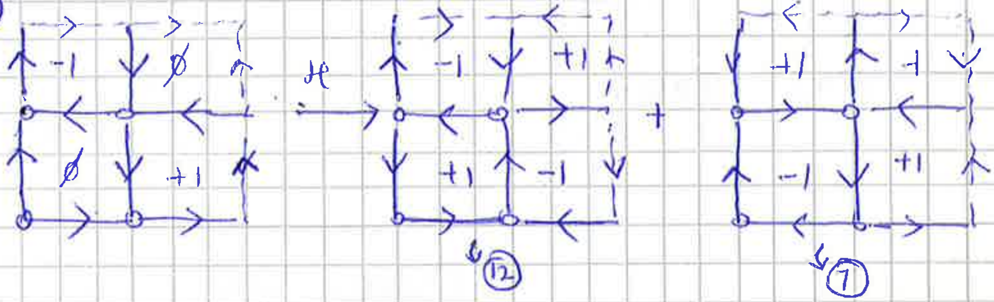


$$\boxed{\langle \psi_{15} | \mathcal{H} | \psi_{12} \rangle = J/2}$$

(13) $\mathcal{H}|\psi_{13}\rangle = 0$

(14) $\mathcal{H}|\psi_{14}\rangle = 0$

(15)

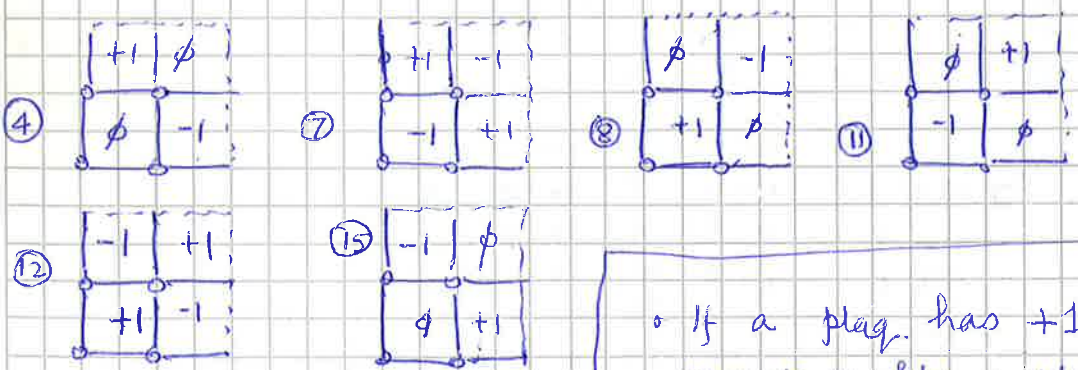


(16) $\mathcal{H}|\psi_{16}\rangle = 0$

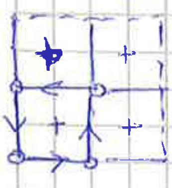
(17) $\mathcal{H}|\psi_{17}\rangle = 0$

(18) $\mathcal{H}|\psi_{18}\rangle = 0$

- Lets see what was going wrong in our naive conting of flux states. For example, note that the only states ~~__~~ possible are:



Note that for a config like



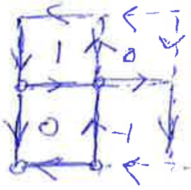
$\rightarrow$ will clearly violate Gauss Law

- If a plaq. has +1 flux, then nearest neighbours plaquette cannot have +1, they must have either -1 or ϕ. Moreover, since the net flux has to vanish, ~~__~~ a +1 ~~_~~ plaquette is always accompanied by a -1 plaquette.

- NN-array

DIM +1 → +x	DIM −1 → −x
DIM +2 → +y	DIM −2 → −y

- Need to store the gauge invariant states.



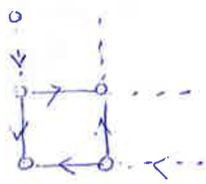
Eg:- $\underbrace{-1\ -1}_{0}\ \underbrace{-1\ \ 1}_{1}\ \underbrace{1\ -1}_{2}\ \underbrace{1\ \ 1}_{3}$ 4 spinstate

0 −1 1 0 4 flux

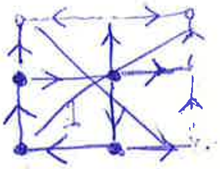
f(x)
−1 → 1
1 → −1

f(x): 1 → −1
−1 → 1

−1+2 = 1
1+2 = 3

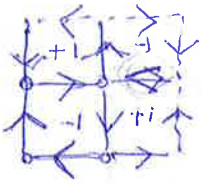


−1 (−1) (−1) (1) 1 −1 (1) 1



state 4 $\xrightarrow{H}$ state 7 + state 12

H ⟶ ① ~~−1 1 1 −1 1 1 1 1~~
② −1 −1 −1 1 −1 1 1 −1



state 4 $\xrightarrow{H}$ ① −1 1 1 −1 1 −1 −1 1
② −1 −1 −1 1 −1 1 1 −1

0 — 4 ⟹
1 — 7
2 — 8
3 — 11
4 — 12
5 — 15

1 + 4	← 0	4	→	7 + 12
0 + 2 + 3 + 5	← 1	7	→	4 + 8 + 11 + 15
1 + 4	← 2	8	→	7 + 12
1 + 4	← 3	11	→	7 + 12
0 + 2 + 3 + 5	← 4	12	→	4 + 8 + 11 + 15
1 + 4	← 5	15	→	7 + 12

	0	1	2	3	4	5
0	0	J/2			J/2	
1	J/2	0	J/2	J/2		J/2
2		J/2	0		J/2	
3		J/2		0	J/2	
4	J/2		J/2	J/2	0	J/2
5		J/2			J/2	0

Check with the full 18 state Hamiltonian matrix :-

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1																		
2																		
3							0					0						
4							$J/2$					$J/2$						
5																		
6																		
7				$J/2$				$J/2$			$J/2$				$J/2$			
8							$J/2$					$J/2$						
9																		
10																		
11							$J/2$					$J/2$						
12				$J/2$				$J/2$			$J/2$				$J/2$			
13																		
14																		
15							$J/2$				$J/2$							
16																		
17																		
18																		

all other entries zero

Eigenvalues of the 6X6 matrix (2X2 lattice) : $(0, 0, 0, 0, -\sqrt{2}J, +\sqrt{2}J)$

Eigenvalues of the single plaq. :- $(-J/2, J/2)$