Numeric Optimization Assignment 4

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1.

(a) KKT conditions for (1):

- $\nabla \mathcal{L}(x^*,\, \lambda^*) = \begin{bmatrix} 0.2(x_1-3) \\ \\ 2x_2 \end{bmatrix} - \begin{bmatrix} -2\lambda x_1 \\ \\ -2\lambda x_2 \end{bmatrix} = \vec{0}$
- $-(x_1^*)^2 - (x_2^*)^2 + 1 \geq 0$
- $\lambda \geq 0$
- $\lambda[-(x_1^*)^2 - (x_2^*)^2 + 1] = 0$

(b) Since $x_2 + \lambda x_2 = 0$ and $\lambda \geq 0$, we have $x_2 = 0$, consider the following cases:

i. $\lambda = 0$

We then have: $\begin{cases} x_1 = 3 \\ x_2 = 0 \end{cases}$, however, this is contradict to $-x_1^2 - x_2^2 + 1 = -9 \ngeq 0$.

ii. $-x_1^2 - x_2^2 + 1 = 0$

By this, we have $\begin{cases} x_1 = \pm 1 \\ x_2 = 0 \end{cases}$, and $\begin{cases} \lambda = 0.2 & x_1 = 1 \\ \lambda = -0.4 & x_1 = -1 \end{cases}$.

To sum up, we have $\begin{cases} x_1^* = 1 \\ x_2^* = 0 \\ \lambda = 0.2 \end{cases}$.

(c) The reduced-Hessian is

$\nabla_{xx}^2 \mathcal{L}(x,\, \lambda) = \begin{bmatrix} 0.2 + 2\lambda & 0 \\ \\ 0 & 2 + 2\lambda \end{bmatrix}$

The constraint is active at point $(x_1,\, x_2) = (1,\, 0)$, we only need to check if $w^\top \nabla_{xx}^2 \mathcal{L}(x^*,\, \lambda^*) w \geq 0$ for all w in the critical directions.

Since the reduced-Hessian matrix is PD for $\lambda^* = 0.2$, the second order condition holds for point $(x_1,\, x_2) = (1,\, 0)$.

2. By KKT conditions, we have:

(a) $\nabla \mathcal{L} = Ap^* + g - 2\mu p = 0$

(b) $-p^{*\top} p^* + \Delta^2 \geq 0$

(c) $\mu \geq 0$

(d) $\mu(-p^{*\top} p^* + \Delta^2) = 0$

Setting $\lambda = -2\mu$.

By (d) we have $\mu(\|p^*\| - \Delta)(\|p^*\| + \Delta) = 0$, and since $\Delta \geq 0$, we then have $\lambda(\|p^*\| - \Delta) = 0$.

And by (a):

$$\begin{aligned} Ap^* + g - \lambda p^* &= 0 \\ Ap^* - \lambda p^* &= -g \\ (A - \lambda I)p^* &= -g \end{aligned}$$

But I don't know how to derive the thrid property.

3. The source codes are in the `coding` folder.

   The following plot shows the trace of the interior point method (t_0 is the initial point) applied on solving the problem in the last assignment: