Numeric Optimization Assignment 3

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1.

(a) We can have A in the following form:

$$A = \begin{bmatrix} \frac{1}{2} & cos(x_0) & \cdots & cos(mx_0) & sin(x_0) & \cdots & sin(mx_0) \\ \frac{1}{2} & cos(x_1) & \cdots & cos(mx_1) & sin(x_1) & \cdots & sin(mx_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2} & cos(x_{n-1}) & \cdots & cos(mx_{n-1}) & sin(x_{n-1}) & \cdots & sin(mx_{n-1}) \end{bmatrix}$$

then we can transform the problem $f(x_k) - y_k$ to $Ax - b$.

(b) To prove that each column vector of A is orthogonal to each other, we need to consider the following cases:

i. $[cos(\alpha x_0),\ cos(\alpha x_1),\ \cdots,\ cos(\alpha x_{n-1})]$ and $[cos(\beta x_0),\ cos(\beta x_1),\ \cdots,\ cos(\beta x_{n-1})]$.

ii. $[sin(\alpha x_0),\ sin(\alpha x_1),\ \cdots,\ sin(\alpha x_{n-1})]$ and $[sin(\beta x_0),\ sin(\beta x_1),\ \cdots,\ sin(\beta x_{n-1})]$.

iii. $\left[\frac{1}{2},\ \frac{1}{2},\ \cdots,\ \frac{1}{2}\right]$ and $[cos(\alpha x_0),\ cos(\alpha x_1),\ \cdots,\ cos(\alpha x_{n-1})]$.

iv. $\left[\frac{1}{2},\ \frac{1}{2},\ \cdots,\ \frac{1}{2}\right]$ and $[sin(\alpha x_0),\ sin(\alpha x_1),\ \cdots,\ sin(\alpha x_{n-1})]$.

v. $[cos(\alpha x_0),\ cos(\alpha x_1),\ \cdots,\ cos(\alpha x_{n-1})]$ and $[sin(\beta x_0),\ sin(\beta x_1),\ \cdots,\ sin(\beta x_{n-1})]$.

To continue to proof, we need the following equations ($r \in N$):

$$\begin{cases} \displaystyle\sum_{m=1}^{r} cos(m\theta) = \tfrac{1}{2sin\frac{\theta}{2}}\left[sin\tfrac{2r+1}{2}\theta - sin\tfrac{\theta}{2}\right] = \tfrac{1}{sin\frac{\theta}{2}} cos((r+1)\theta)sin(r\theta) \\ \displaystyle\sum_{m=1}^{r} sin(m\theta) = \tfrac{1}{2sin\frac{\theta}{2}}\left[cos\tfrac{\theta}{2} - cos\tfrac{2r+1}{2}\theta\right] = \tfrac{1}{sin\frac{\theta}{2}} sin((r+1)\theta)sin(r\theta) \end{cases}$$

Furthermore, since $x_0 = x_k$, we can replace x_0 with x_k. Also, by the product-to-sum identities, the inner products of above cases can be reduced to:

i.

$$\begin{aligned} \sum_{k=1}^{n} cos(\alpha x_k)cos(\beta x_k) &= \frac{1}{2}\sum_{k=1}^{n}[cos((\alpha+\beta)x_k) + cos((\alpha-\beta)x_k)] \\ &= \frac{1}{2}\sum_{k=1}^{n} cos((\alpha+\beta)\frac{2\pi k}{n}) + \frac{1}{2}\sum_{k=1}^{n} cos((\alpha-\beta)\frac{2\pi k}{n}) \\ &= \frac{1}{2sin\frac{(\alpha+\beta)\pi}{n}} cos((n+1)\theta)sin((\alpha+\beta)\pi) + \frac{1}{2}\sum_{k=1}^{n} cos((\alpha-\beta)\frac{2\pi k}{n}) \\ &= 0 + \tfrac{1}{2sin\frac{(\alpha-\beta)\pi}{n}} cos((n+1)\theta)sin((\alpha-\beta)\pi) \\ &= 0 \end{aligned}$$

ii.

$$\begin{aligned}
\sum_{k=1}^{n} sin(\alpha x_k) sin(\beta x_k) &= \frac{1}{2}\sum_{k=1}^{n}[cos((\alpha+\beta)x_k) - cos((\alpha-\beta)x_k)] \\
&= \frac{1}{2}\sum_{k=1}^{n} cos((\alpha+\beta)\frac{2\pi k}{n}) - \frac{1}{2}\sum_{k=1}^{n} cos((\alpha-\beta)\frac{2\pi k}{n}) \\
&= \frac{1}{2sin\frac{(\alpha+\beta)\pi}{n}} cos((n+1)\theta) sin((\alpha+\beta)\pi) - \frac{1}{2}\sum_{k=1}^{n} cos((\alpha-\beta)\frac{2\pi k}{n}) \\
&= 0 - \tfrac{1}{2sin\frac{(\alpha-\beta)\pi}{n}} cos((n+1)\theta) sin((\alpha-\beta)\pi) \\
&= 0
\end{aligned}$$

iii.

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{2} cos(\alpha x_k) &= \frac{1}{2}\sum_{k=1}^{n} cos(\frac{2\alpha\pi k}{n}) \\
&= \frac{1}{2sin\frac{\alpha\pi}{n}} cos((n+1)\theta) sin(\alpha\pi) \\
&= 0
\end{aligned}$$

iv.

$$\begin{aligned}
\sum_{k=1}^{n} \frac{1}{2} sin(\alpha x_k) &= \frac{1}{2}\sum_{k=1}^{n} sin(\frac{2\alpha\pi k}{n}) \\
&= \frac{1}{2sin\frac{\alpha\pi}{n}} sin((n+1)\theta) sin(\alpha\pi) \\
&= 0
\end{aligned}$$

v.

$$\begin{aligned}
\sum_{k=1}^{n} cos(\alpha x_k) sin(\beta x_k) &= \frac{1}{2}\sum_{k=1}^{n}[sin((\alpha+\beta)x_k) - sin((\alpha-\beta)x_k)] \\
&= \frac{1}{2}\sum_{k=1}^{n} sin((\alpha+\beta)\frac{2\pi k}{n}) - \frac{1}{2}\sum_{k=1}^{n} sin((\alpha-\beta)\frac{2\pi k}{n}) \\
&= \frac{1}{2sin\frac{(\alpha+\beta)\pi}{n}} sin((n+1)\theta) sin((\alpha+\beta)\pi) - \frac{1}{2}\sum_{k=1}^{n} sin((\alpha-\beta)\frac{2\pi k}{n}) \\
&= 0 - \tfrac{1}{2sin\frac{(\alpha-\beta)\pi}{n}} sin((n+1)\theta) sin((\alpha-\beta)\pi) \\
&= 0
\end{aligned}$$

Thus each column vector of A is orthogonal to each other.

(c) By the normal equation, $x^* = (A^\top A)^{-1} A^\top b$.
Since each column vector of A is orthogonal to each other, $A^\top A$ and $(A^\top A)^{-1}$ are diagonal matrices as shown below:

$$A^\top A = \begin{bmatrix} \frac{n}{4} & 0 & \cdots & 0 \\ 0 & \frac{n}{2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{n}{2} \end{bmatrix}, \quad (A^\top A)^{-1} = \begin{bmatrix} \frac{4}{n} & 0 & \cdots & 0 \\ 0 & \frac{2}{n} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{2}{n} \end{bmatrix}$$

A little bit calculation bring us:

$$\begin{aligned}
x^* &= \begin{bmatrix} \frac{4}{n} & 0 & \cdots & 0 \\ 0 & \frac{2}{n} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{2}{n} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} \\ cos(x_0) & cos(x_1) & \cdots & cos(x_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ cos(mx_0) & cos(mx_1) & \cdots & cos(mx_{n-1}) \\ sin(x_0) & sin(x_1) & \cdots & sin(x_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ sin(mx_0) & sin(x_{n-1}) & \cdots & sin(mx_{n-1}) \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{n} & \frac{2}{n} & \cdots & \frac{2}{n} \\ \frac{2}{n}cos(x_0) & \frac{2}{n}cos(x_1) & \cdots & \frac{2}{n}cos(x_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{n}cos(mx_0) & \frac{2}{n}cos(mx_1) & \cdots & \frac{2}{n}cos(mx_{n-1}) \\ \frac{2}{n}sin(x_0) & \frac{2}{n}sin(x_1) & \cdots & \frac{2}{n}sin(mx_{n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{n}sin(mx_0) & \frac{2}{n}sin(mx_1) & \cdots & \frac{2}{n}sin(mx_{n-1}) \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{n}\sum y_k \\ \frac{2}{n}\sum y_k cos(x_k) \\ \vdots \\ \frac{2}{n}\sum y_k cos(mx_k) \\ \frac{2}{n}\sum y_k sin(x_k) \\ \vdots \\ \frac{2}{n}\sum y_k sin(mx_k) \end{bmatrix}
\end{aligned}$$

(d) Since $A^\top A$ is a diagonal matrix, the computation cost of calulating the inverse of the matrix will be much lower than the one of polynomial basis. Also, this kind of approximation performs well on periodic functions. However, when n is not sufficiently large, this approximation may not be able to approximate the original function well.

2. The source codes are in the `coding` folder.

   The optimal solution is at point $(320,\ 360)$, and the following plot shows the trace of the simplex method (t_0 is the initial point and t_3 is the optimal solution):