UPPSALA
UNIVERSITET

Welcome to Sequential Monte Carlo methods!!

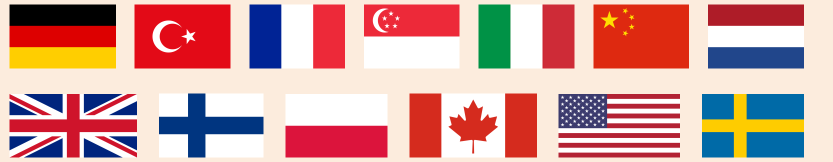
Lecture 1 – Introduction and probabilistic modelling

Thomas Schön, Uppsala University

2017-08-24

Who are we? Students

80 participants registered for the course, representing the following 13 different countries:



35 different universities/companies.

Use this week not only to lean about SMC methods, but also to get to know new friends. Hopefully new research ideas will be initiated.

Who are we? Teachers

Andreas Svensson

Thomas Schön

Christian A. Naesseth

Johan Wågberg

Lawrence Murray

Fredrik Lindsten

Aim of this course

Aim: To provide an introduction to the theory and application of sequential Monte Carlo (SMC) methods.

After the course you should be able to derive your own SMC-based algorithms allowing you to do inference in nonlinear models.

Day 1-3: Focus on state space models (SSMs). How to learning them from data and how to estimate their hidden states.

Day 4: Using SMC for inference in general probabilistic models.

Vision for the joint course and workshop

Our **vision** for the joint course and workshop is to open up the field of SMC to new researchers.

The idea is for you to have the chance to learn about SMC and then get to meet some of the people who have developed the methods and some of those who continue to develop it further as we speak. It is a field that evolves fast for good reasons.

Course: August 24 – August 29.
www.it.uu.se/research/systems_and_control/education/2017/smc

SMC workshop: August 31 – September 1.
www.it.uu.se/conferences/smc2017

Ex) Indoor positioning (engineering)

Aim: Compute the position of a person moving around indoors using variations in the ambient magnetic field and the motion of the person (acceleration and angular velocities). All of this observed using sensors in a standard smartphone.

Ex) Indoor positioning (engineering)

Key ingredients of the solution:

1. The particle filter for computing the position
2. The Gaussian process for building and representing the map of the ambient magnetic field
3. Inertial sensor signal processing

Movie – map making: www.youtube.com/watch?v=enlMiUqPVJo

Movie – indoor positioning result

You can meet and talk to Arno Solin during the workshop.

Arno Solin, Simo Särkkä, Juho Kannala and Esa Rahtu. **Terrain navigation in the magnetic landscape: Particle filtering for indoor positioning.** *Proc. of the European Navigation Conf. (ENC)*, Helsinki, Finland, June, 2016.

Arno Solin, Manon Kok, Niklas Wahlström, Thomas B. Schön and Simo Särkkä. **Modeling and interpolation of the ambient magnetic field by Gaussian processes.** *IEEE Trans. on Robotics*, 2017 (Conditionally accepted).

Ex) Epidemiological modelling (statistics)

Aim: To learn a model explaining the seasonal influenza epidemics and then make use of this model to compute predictions.

Susceptible/infected/recovered (SIR) model:

$$
\begin{aligned}
S_{t+dt} &= S_t + \mu \mathcal{P} dt - \mu S_t dt - (1 + F v_t)\beta_t S_t \mathcal{P}^{-1} I_t dt, \\
I_{t+dt} &= I_t - (\gamma + \mu) I_t dt + (1 + F v_t)\beta_t S_t \mathcal{P}^{-1} I_t dt, \\
R_{t+dt} &= R_t + \gamma I_t dt - \mu R_t dt, \\
\beta_t &= R_0(\gamma + \mu)(1 + \alpha \sin(2\pi t/12)),
\end{aligned}
$$

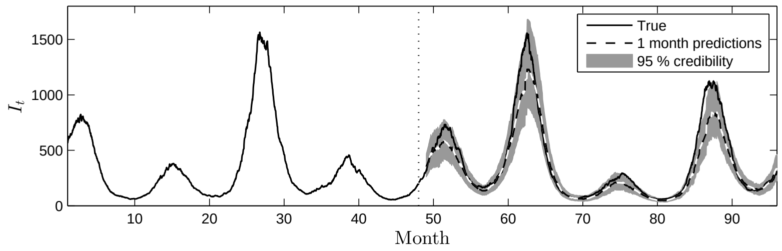
Measurements:

$$
y_k = \rho \operatorname{logit}(\bar{I}_k / \mathcal{P}) + e_k, \qquad e_k \sim \mathcal{N}(0, \sigma^2).
$$

Information about the unknown parameters $\theta = (\gamma, R_0, \alpha, F, \rho, \sigma)$ and states $x_t = (S_t, I_t, R_t)$ has to be learned from measurements.

Ex) Epidemiological modelling (statistics)

Compute $p(\theta, x_{1:T} \mid y_{1:T})$, where $y_{1:T} = (y_1, y_2, \ldots, y_T)$ and use it to compute the predictive distribution.



Disease activity (number of infected individuals I_t) over an eight year period.

Fredrik Lindsten, Michael I. Jordan and Thomas B. Schön. **Particle Gibbs with ancestor sampling.** *Journal of Machine Learning Research (JMLR)*, 15:2145-2184, June 2014.

Ex) Probabilistic programming (machine learning)

A **probabilistic program** encodes a **probabilistic model** according to the semantics of a particular probabilistic programming language.

The memory state of a running probabilistic program evolves dynamically and stochastically in time and so is a **stochastic process**.

SMC is a common inference method for probabilistic programs.

Creates a clear separation between the model and the inference methods. Opens up for the automation of inference!

```
x ~ Bernoulli(p);    assume(x)
if (x) {             value(x)
  y ~ N(0,1);        assume(y)
} else {
  y <- 0;
```

More during lecture 17 and the workshop (3 talks + posters).

Course structure – overview

- 18 lectures (45 min. each)
- Credits offered: 6ECTS (Swedish system)
- Practicals (solve exercises and hand-in assignments, **discuss and ask questions**)
- Hand-in assignments. You can collaborate, but the reports with the solutions are individual.
- Complete course information (including lecture slides) is available from the course website: `www.it.uu.se/research/systems_and_control/education/2017/smc`
- **Feel free to ask questions at any time!**

Course practicalities

- Lecture rooms (different on Monday and Tuesday). See course website for map
- Course photo on Monday at 10.00
- Social event on Saturday coordinated by Anna Wigren and Andreas Svensson (there is a separate e-mail list for this)
- Course e-mail list (if you are not on it, let us know)
- Lunch is **available for registered participants** (who have already registered using the Google form)

The only way to really learn something is by **implementing it on your own**.

Outline – 4 days

Day 1 – Probabilistic modelling and particle filtering basics

a) Probabilistic modelling of dynamical systems and filtering

b) Introduce Monte Carlo and derive the bootstrap particle filter

Day 2 – Particle filtering and parameter learning

a) Auxiliary particle filter, full adaptation and practicalities

b) Maximum likelihood parameter learning, convergence

Day 3 – Bayesian parameter learning

a) Particle Metropolis Hastings

b) Particle Gibbs

Day 4 – Beyond state space models (outlooks)

a) General target sequences and SMC samplers

b) High-dimensional SSM and probabilistic programming

Outline – Lecture 1

Aim: Introduce the course and provide background on probabilistic modelling.

Outline:

1. Course introduction and practicalities
2. Probabilistic modelling
3. Key probabilistic objects
4. Ex. probabilistic autoregressive modelling

Probabilistic modelling

Modelling

Mathematical model: A compact representation—set of assumptions—of some phenomenon of interest.

Most of the course (day 1-3) is concerned with dynamical phenomena. The methods are more general than that and during the last day we will broaden the scope significantly.

Dynamical phenomena produce temporal measurements (data) arriving as a **sequence**

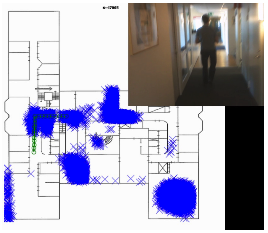
$$y_{1:t} = (y_1, y_2, \ldots, y_t).$$

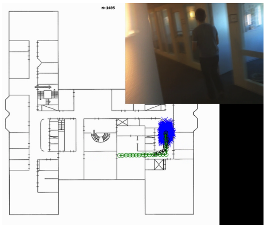
Nice introduction to probabilistic modelling in Machine Learning

Ghahramani, Z. **Probabilistic machine learning and artificial intelligence.** *Nature* 521:452-459, 2015.

Representing and modifying uncertainty

It is important to maintain a solid representation of uncertainty in all mathematical objects and throughout all calculations.





The two basic rules from probability theory

Let X and Y be continuous random variables[1]. Let $p(\cdot)$ denote a general probability density function.

1. Marginalization (integrate out a variable): $p(x) = \int p(x, y)\mathrm{d}y$.
2. Conditional probability: $p(x, y) = p(x \mid y)p(y)$.

Combine them into Bayes' rule:

$$p(x \mid y) = \frac{p(y \mid x)p(x)}{p(y)} = \frac{p(y \mid x)p(x)}{\int p(y \mid x)p(x)\mathrm{d}x}.$$

[1]Notation: Upper-case letters for random variables (r.v.) X when we talk about models. Lower-case letters for realizations of the r.v., $X = x$ and in algorithms. We will not use bold face for vectors.

Basic variables classes

Measurements $y_{1:T} = (y_1, y_2, \dots, y_T)$: The measured data somehow obtained from the phenomenon we are interested in.

Unknown (static) model parameters θ: Describes the model, but unknown (or not known well enough) to the user.

Unknown model variables x_t (changing over time): Describes the state of the phenomenon at time t (in the indoor positioning example above x_t includes the unknown position).

Explanatory variables u: Known variables that we do not bother to model as stochastic.

A key task is often to learn θ and/or x_t based on the available measurements $y_{1:T}$.

Computational problems

The problem of learning a model based on data leads to computational challenges, both

- **Integration:** e.g. the high-dimensional integrals arising during marg. (averaging over all possible parameter values θ):

$$p(y_{1:T}) = \int p(y_{1:T} \mid \theta)p(\theta)\mathrm{d}\theta.$$

- **Optimization:** e.g. when extracting point estimates, for example by maximizing the posterior or the likelihood

$$\widehat{\theta} = \arg\max_{\theta} \, p(y_{1:T} \mid \theta)$$

Typically impossible to compute exactly, use approximate methods

- Monte Carlo (MC), Markov chain MC (MCMC), and sequential MC (SMC).
- Variational inference (VI).

Probabilistic autoregressive model

Probabilistic autoregressive model

An autoregressive model of order n is given by

$$Y_t = A_1 Y_{t-1} + A_2 Y_{t-2} + \cdots + A_n Y_{t-n} + E_t, \quad E_t \sim \mathcal{N}(\mu, \tau^{-1})$$

where μ and τ are known explanatory variables ($\mu = 0, \tau \neq 0$).

The unknown model variables are collected as

$$\theta = (A_1, A_2, \ldots, A_n)^\mathsf{T}$$

with the prior

$$\theta \sim \mathcal{N}(0, \rho^{-1} I_n), \qquad \text{where } \rho \text{ assumed to be known.}$$

Task: Compute the posterior $p(\theta \mid y_{1:T})$.

Probabilistic autoregressive model

Full probabilistic model $p(\theta, y_{1:T}) = p(y_{1:T} \mid \theta) p(\theta)$, where the data distribution is given by

$$p(y_{1:T} \mid \theta) = p(y_T \mid y_{1:T-1}, \theta) p(y_{1:T-1} \mid \theta) = \cdots = \prod_{t=1}^{T} p(y_t \mid y_{1:t-1}, \theta).$$

From the model we have that

$$p(y_t \mid y_{1:t-1}, \theta) = \mathcal{N}(y_t \mid \theta^\mathsf{T} z_t, \tau^{-1}),$$

where $Z_t = (Y_{t-1}, Y_{t-2}, \ldots, Y_{t-n})^\mathsf{T}$. Hence,

$$p(y_{1:T} \mid \theta) = \prod_{t=1}^{T} \mathcal{N}\left(y_t \,\middle|\, \theta^\mathsf{T} z_t, \tau^{-1}\right) = \mathcal{N}(\boldsymbol{y} \mid \boldsymbol{z}\theta, \tau^{-1} I_T),$$

where we have made use of $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_T)^\mathsf{T}$ and $\boldsymbol{Z} = (Z_1, Z_2, \ldots, Z_T)^\mathsf{T}$.

Probabilistic autoregressive model

$$p(\theta, \boldsymbol{y}) = \underbrace{\mathcal{N}(\boldsymbol{y} \mid \boldsymbol{z}\theta, \tau^{-1} I_T)}_{p(\boldsymbol{y} \mid \theta)} \underbrace{\mathcal{N}(\theta \mid 0, \rho^{-1} I_n)}_{p(\theta)}$$
$$= \mathcal{N}\left(\begin{pmatrix} \theta \\ \boldsymbol{y} \end{pmatrix} \,\middle|\, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \rho^{-1} I_2 & \rho^{-1} \boldsymbol{z}^\mathsf{T} \\ \rho^{-1} \boldsymbol{z} & \tau^{-1} I_T + \rho^{-1} \boldsymbol{z}\boldsymbol{z}^\mathsf{T} \end{pmatrix}\right).$$

The posterior is given by

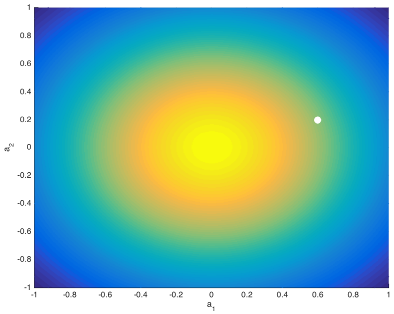
$$p(\theta \mid \boldsymbol{y}) = \mathcal{N}(\theta \mid m_T, S_T),$$

where

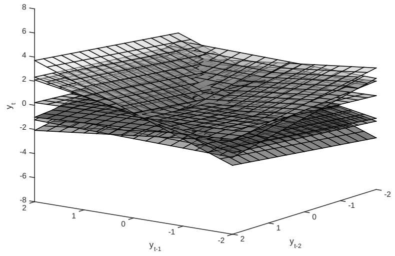
$$m_T = \tau S_T \boldsymbol{z}^\mathsf{T} \boldsymbol{y},$$
$$S_T = \left(\rho^{-1} I_2 + \sigma \boldsymbol{z}^\mathsf{T} \boldsymbol{z}\right)^\mathsf{T}.$$

Ex) Situation before any data is used

$$Y_t = {\color{red}A_1} Y_{t-1} + {\color{red}A_2} Y_{t-2} + E_t, \qquad E_t \sim \mathcal{N}(0, 0.2).$$



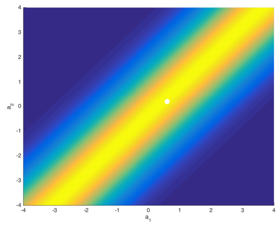
Prior



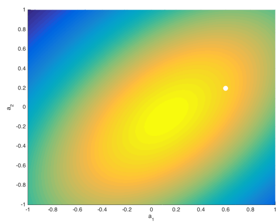
7 samples from the prior

White dot – true value for ${\color{red}\theta} = (0.6, 0.2)$.

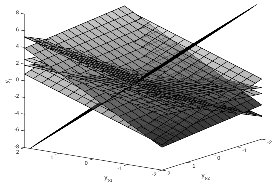
Ex) Situation after y_1 is obtained



Likelihood

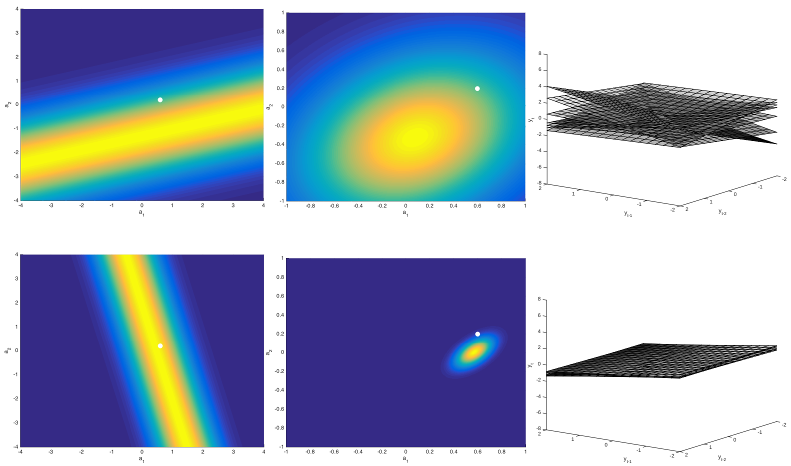


Posterior



7 samples from the posterior

Ex) Situation after $y_{1:2}$ and $y_{1:20}$



Likelihood

Posterior

7 samples from the posterior

A few concepts to summarize lecture 1

Mathematical model: A compact representation—set of assumptions—of some phenomenon of interest.

Probabilistic modelling: Provides the capability to represent and manipulate **uncertainty** in data, models, decisions and predictions.

Full probabilistic model: The joint distribution of all observed (here $y_{1:T}$) and unobserved (here ${\color{red}\theta}$) variables.

Data distribution/likelihood: Distribution describing the observed data conditioned on unobserved variables.

Prior distribution: Encodes initial assumptions on the unobserved variables.

Posterior distribution: Conditional distribution of the unobserved variables given the observed variables.