Intermediate Macroeconomics Recitation 1

Topics: partial derivatives, Taylor series approximations, and optimization.

1 Partial derivatives

Partial derivatives look at the variation of a function $f(x_1, \ldots, x_n)$ brought about by the change in only one variable, say x_i. This can be referred to as $\partial f/\partial x_i$, f_{x_i}, f_i, or $D_i f$.

Definition 1. *Let $f : \mathbb{R}^n \to \mathbb{R}$. Then for each variable x_i at each point $\mathbf{x}^0 = (x_1^0, \ldots, x_n^0)$ in the domain of f,*

$$\frac{\partial f}{\partial x_i}(x_1^0, \ldots, x_n^0) = \lim_{h \to 0} \frac{f(x_1^0, \ldots, x_i^0 + h, \ldots, x_n^0) - f(x_1^0, \ldots, x_i^0, \ldots, x_n^0)}{h}$$

if the limit exists. Only the ith variable changes; the others are treated as constants.

For example, in the case of two arguments, we can write:

$$z = F(x, y)$$

A partial derivatives of z with respect to x, denoted by $\partial z/\partial x = \partial F(x,y)/\partial x$ measures the *marginal change* in z in response to a *marginal change* in x while y is kept *constant*. When computing partial derivatives, the usual rules from calculus apply and you should know how to deal with the most common of them.

Example 1. *Consider the function $f(x,y) = 3x^2y^2 + 4xy^3 + 7y$. Compute the partial derivatives with respect to x and y.*

In this course, the two most common types of functions we are going to see are production functions and utility functions.

1.1 Production functions

A production function measures how much output Y_t we get in period t from combining some amount of capital K_t and labor L_t with total factor productivity (TFP) A_t. A general way to write this production function is:

$$Y_t = F(A_t, K_t, L_t),$$

Commonly used production functions are the Cobb Douglas production function:

$$Y_t = AK_t^{\alpha} L_t^{1-\alpha},$$

and the Constant Elasticity of Substitution (CES) production function:

$$Y_t = A\left[\pi K_t^{\frac{\sigma-1}{\sigma}} + (1-\pi) L_t^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}.$$

For the production function, the partial derivative with respect to K is called the **marginal product of capital** (MPK). This is the rate at which output changes with respect to capital, with labor held fixed. Similarly, the **marginal product of labor** (MPL) measures the rate at which output changes with respect to labor, with capital held fixed.

We can use derivatives to learn some properties about production functions.

Example 2. *Compute the partial derivatives of the Cobb-Douglas production function.*

The partial derivatives of the Cobb-Douglas production function are both positive. This property is called **positive marginal returns**. It means that as either input (capital or labor) increases, holding the other one constant, output increases. Mathematically,

$$MPL = \frac{\partial F}{\partial L} \geq 0 \text{ and } MPK = \frac{\partial F}{\partial K} \geq 0$$

Example 3. *Compute the second derivatives of the Cobb-Douglas production function.*

The second derivatives of the Cobb-Douglas production function are both negative. This property is called **diminishing marginal returns**. It means that if we keep on increasing either input (capital or labor), holding the other one constant, output will continue to increase but by decreasing amounts. Mathematically,

$$\frac{\partial^2 F}{\partial K^2} < 0 \text{ and } \frac{\partial^2 F}{\partial L^2} < 0$$

A third property of many production functions is that they are **constant returns to scale**. This means that if you increase all the inputs *proportionally*, output increases by the same proportion. Mathematically, this means

$$Y = F(K, L) \Rightarrow F(2K, 2L) = 2Y$$

We can triple the inputs K and L, we get triple the amount of output. In general, we have

$$F(\lambda K, \lambda L, A) = \lambda F(K, L, A) \text{ for any } \lambda > 0$$

What is *Increasing returns to scale (IRS)*? If we double the physical and labor inputs, and use the same formula, we will have more than twice the amount of cookies, i.e.

$$Y = F(K, L, A) \Rightarrow F(2K, 2L, A) > 2Y$$

Similarly, *Decreasing returns to scale (DRS)*? If we double the physical and labor inputs, and use the same formula, we will have less than twice the amount of cookies, i.e.

$$Y = F(K, L, A) \Rightarrow F(2K, 2L, A) < 2Y$$

1.2 Utility function

A utility function links a level of value an individual gets from consuming different types of goods. For example: your utility from consumption and leisure is $U(c, l)$. Then $\partial U/\partial c$ is the marginal utility of consumption and $\partial U/\partial l$ is the marginal utility of leisure.

Example 4. *Compute the partial derivatives of the utility function* $U = a \ln C + b \ln(L)$.

2 Taylor series approximations

Taylor series approximations are approximation of functions into a polynomial. Suppose we would like to understand the properties of a function near x_0. A Taylor series approximation gives:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + ...$$

Given that we have sufficient knowledge of the function at x_0 $\{f(x_0), f'(x_0), f''(x_0), ...\}$, we are also able to study the movement of the function near x_0.

In general, for a polynomial of degree n:

$$P(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

Let $f(x) = log(1+x)$. We know that $f(0) = log(1) = 0$ and we want to study the properties of the function near 0. Recall that,

$$\frac{d}{dx} log(g(x)) = \frac{g'(x)}{g(x)}.$$

Hence, $f'(x) = \frac{1}{1+x}$ so that $f'(0) = 1$. Now, instead of dealing with the exact form of $f(x)$ away from $x = 0$, the Taylor expansion gives the following:

$$\begin{aligned} f(x) &\approx f(0) + f'(0)(x - 0) \\ &= x \end{aligned}$$

Therefore, we approximate $log(1 + x) = x$ when x is near 0. How precise is this approximation?

$$\begin{aligned} log(1.02) &= 0.0198 \\ log(1.1) &= 0.0953 \\ log(1.2) &= 0.1823 \\ log(1.5) &= 0.4055 \end{aligned}$$

The approximation gets poor when x is no longer 'close' to 0. However, it does give a relatively good approximation near 0. Now, why are we particularly interested in this function?

It turns out that in many macroeconomic series, a linear time trend seems to be the case. This can be verified if we plot the log of the series over time. In particular, suppose that we plot log real GDP over time. That is, we plot the y axis for log of real GDP and the x axis for time. We have seen in class that this log scale (or ratio scale as in Jones) produces a graph that has a strong linear time trend. It turns out that the slope of this time trend is actually the growth rate of real GDP. This again comes from the above first order TSE.

$$
\begin{aligned}
\text{slope} &= \frac{log(rGDP_t) - log(rGDP_{t-1})}{t-(t-1)} \\
&= log\left(\frac{rGDP_t}{rGDP_{t-1}}\right) \\
&= log\left(\frac{rGDP_{t-1} + (rGDP_t - rGDP_{t-1})}{rGDP_{t-1}}\right) \\
&= log\left(1 + \text{GrowthRate}_t\right) \\
&= \text{GrowthRate}_t
\end{aligned}
$$

3 Optimization

The standard modelling approach in macroeconomics starts from considering how, for example, individuals and firms make decisions and then aggregating these decisions, imposing some equilibrium conditions, to understand how macro variables behave. We generally model the behavior of these single entities through some optimization problem: consumer maximize their happiness, firms maximize their profits, firms minimize their cost, etc. There are some important assumptions when we write an optimization problem and Jon has discussed some of them (example: firms maximization problem). But optimization, both constrained and uncostrained, is an important toll that you must be comfortable with. We are going to review the basic steps using 2 examples: one unconstrained and one constrained.

3.1 Unconstrained maximization

$$Max \quad or \quad Min \quad f(x,y)$$

1. Take first order conditions (FOC).
2. Solve optimal choice of x^*, y^*.
3. Plug x^*, y^* back to the objective function, get maximum or minimum value.

Example 5. *Consider the profit maximization problem*

$$\Pi = AK^{\alpha}L^{\beta} - wL - rK$$

where $\alpha, \beta < 1$

Solution method: Take derivatives and equalize them to 0, we called this first order conditions (FOCs)

$$\begin{aligned}
\frac{\partial \Pi}{\partial K} &= A\alpha K^{\alpha-1}L^{\beta} - r = 0 \\
\frac{\partial \Pi}{\partial L} &= A\beta K^{\alpha}L^{\beta-1} - w = 0
\end{aligned}$$

which means

$$MRTS = \frac{MPK}{MPL} = \frac{r}{w} = \frac{\alpha}{\beta}\frac{L}{K}$$

is the marginal rate of transformation. Given r *and* w*, we can find* K *and* L *(when* $\alpha+\beta < 1$*)*

$$\begin{aligned}
r &= A\alpha K^{\alpha-1}\left(\frac{\beta}{\alpha}\frac{r}{w}K\right)^{\beta} \\
&\Rightarrow K = \left(A\alpha^{1-\beta}\alpha^{\beta}r^{\beta-1}w^{-\beta}\right)^{\frac{1}{1-\alpha-\beta}} \\
&\Rightarrow L = \left(A\beta^{1-\alpha}\alpha^{\alpha}w^{\alpha-1}r^{-\alpha}\right)^{\frac{1}{1-\alpha-\beta}}
\end{aligned}$$

with K *and* L*, we can plug back to profit function above to find the maximized profit.*

If $\alpha + \beta = 1$*, i.e. we have Cobb-Douglas Production function, then we get*

$$\begin{aligned}
\alpha Y &= rK \\
(1-\alpha)Y &= wL
\end{aligned}$$

that means we have the share of labor (and capital) in total GDP is constant over time. The share of capital is α *and share of labor is* $1-\alpha$*. We can use data to find* α*.*

3.2 Constrained maximization

$$Max \quad or \quad Min \quad f(x,y)$$
$$s.t. \quad g(x,y) = 0$$

1. Plug the constraint equation into the objective function.
2. Solve unconstrained optimization problem.

Example 6. *Consider the utility function:*

$$\begin{aligned}
\max_{c,l} U(c,l) &= a\ln c + b\ln l \\
s.t. &: pc = wH \\
&: l = 1 - H \text{(We normalize the total amount of time to be 1)}
\end{aligned}$$

where c *is consumption,* l *is leisure and* H *is hours worked,* p *is consumption goods prices.*

For simplicity, assume $p = 1$. We can then substitute the constraints into the objective function as follows:

$$\max_{c,l} a \ln c + b \ln l = \max_{H} a \ln(wH) + b \ln(1 - H)$$

As before, we take FOC w.r.t. H:

$$\begin{aligned}
\frac{aw}{wH} &= \frac{b}{1-H} \Rightarrow \frac{\frac{b}{1-H}}{\frac{a}{c}} = w \\
\frac{a}{H} &= \frac{b}{1-H} \Rightarrow H = \frac{a}{a+b} \\
&\Rightarrow c^* = \frac{wa}{a+b}
\end{aligned}$$

The condition that $\frac{\frac{b}{1-H}}{\frac{a}{c}} = w$ means the Marginal rate of substitution between working and consumption is the wage.