A-Level Mathematics

Core 3 and Core 4

D I F F E R E N T I A T I O N

Core 3

The Chain Rule
The Product Rule
The Quotient Rule

Core 4

Parametric Differentiation
Implicit Differentiation

$$\frac{dy}{dx}$$

$$f'(x)$$

Chapter 1.

1.1 The Chain Rule

To begin with, the only standard derivative used is that

$$\text{If } y = x^n \text{ then } \frac{dy}{dx} = n x^{n-1}$$

where x is a variable and n is a constant

The chain rule gives us a rapid method of differentiating expressions such as;

$$y = (4x + 5)^2$$

This example can be differentiated without the chain rule as follows;

$$y = (4x + 5)^2$$
$$y = (4x + 5)(4x + 5)$$
$$y = 16x^2 + 40x + 25$$
$$\frac{dy}{dx} = 32x + 40$$
$$\frac{dy}{dx} = 8(4x + 5)$$

But is more briskly done using the chain rule;

$$y = (4x + 5)^2$$
$$\frac{dy}{dx} = 2(4x + 5)^1 \times 4$$
$$\frac{dy}{dx} = 8(4x + 5)$$

Over the page, try the three further examples.
The answers are on the page after that, but try them yourself first.

Try 1. $y = (3x + 7)^3$

Try 2. $y = (8x + 1)^5$

Try 3. $y = (4x^3 + 8)^4$

Answer 1. $y = (3x + 7)^3$

$$y = (3x + 7)^3$$
$$\frac{dy}{dx} = 3\,(3x + 7)^2 \times 3$$
$$\frac{dy}{dx} = 9\,(3x + 7)^2$$

Answer 2. $y = (8x + 1)^5$

$$y = (8x + 1)^5$$
$$\frac{dy}{dx} = 5\,(8x + 1)^4 \times 8$$
$$\frac{dy}{dx} = 40\,(8x + 1)^4$$

Answer 3. $y = (4x^3 + 8)^4$

$$y = (4x^3 + 8)^4$$
$$\frac{dy}{dx} = 4\,(4x^3 + 8)^3 \times (12x^2)$$
$$\frac{dy}{dx} = 48x^2\,(4x^3 + 8)^3$$

1.2 Exercise

Question 1.

Differentiate $y = (5x^2 + 7)^3$

Question 2.
Differentiate $y = 5\,(11 - 6x^2)^5$

Question 3.
Differentiate $y = \sqrt{9 - 5x}$

Question 4.
Differentiate the following function,

$$f(x) = 4\,(9 + 14x)^{\frac{3}{2}}$$

Question 5.
Differentiate the following function,

$$f(x) = \frac{5}{(7x + 8)^3}$$

Question 6.
Find $\frac{dy}{dx}$ when,

$$y = \frac{5}{3(7 - 2x)^5}$$

Question 7.
Differentiate the following function;

$$y = 6 + \frac{1}{(3x^2 + 2)}$$

Question 8.
Consider the curve,

$$y = (2x - 3)^4$$

Find the equation of the tangent to the curve at the point (2, 1).
Give your answer in the form $y = mx + c$.

Question 9.

(a) A curve has the following equation,

$$y = (x + 3)^3 - 4(x + 3)$$

Find the coordinates of points on the curve with gradient 8.

(b) Repeat part **(a)** for the curve with the following equation,

$$y = x^3 - 4x$$

(c) How are your part **(a)** and part **(b)** answers related ?

Question 10.

(a) Find the equation of the tangent to the curve,

$$y = \frac{2}{x^2 - 3}$$

at the point (3, $\frac{1}{3}$).

Give your answer in the form $ax + by + c = 0$.

(b) Find the normal to the curve at the same point.

Again, give your answer in the form $ax + by + c = 0$.

1.3 Answers.

1.3.1 Solutions (Exercise)

Answer 1.

$$\frac{dy}{dx} = 30\,x\,(5x^2 + 7)^2$$

Answer 2.

$$\frac{dy}{dx} = -300\,x\,(11 - 6\,x^2)^4$$

Answer 3.

$$\frac{dy}{dx} = -\frac{5}{2}\,(9 - 5x)^{-\frac{1}{2}}$$

or

$$\frac{dy}{dx} = -\frac{5}{2\sqrt{9 - 5x}}$$

Answer 4.

$$f'(x) = 84\,(9 + 14x)^{-\frac{1}{2}}$$

or

$$f'(x) = 84\sqrt{9 + 14\,x}$$

Answer 5.

$$f'(x) = -105\,(7x + 8)^{-4}$$

or

$$f'(x) = -\frac{105}{(7x + 8)^4}$$

Answer 6.

$$\frac{dy}{dx} = \frac{50}{3}\,(7 - 2x)^{-6}$$

or

$$\frac{dy}{dx} = \frac{50}{3\,(7 - 2x)^6}$$

Answer 7.

$$\frac{dy}{dx} = -\frac{6x}{(3x^2 + 2)^2}$$

Answer 8.

$$\frac{dy}{dx} = 8(2x - 3)^3$$

When $x = 2$, this gives,

$$\frac{dy}{dx} = 8$$

Thus,

$$y = 8x - 15$$

Answer 9.
(a)

$$\frac{dy}{dx} = 3(x + 3)^2 - 4$$

So, need to solve,

$$\frac{dy}{dx} = 8$$

Which gives two solutions,

$$(-5,\ 0)\ and\ (-1,\ 0)$$

(b)

$$(-2,\ 0)\ and\ (2,\ 0)$$

(c)

$$\textit{First pair of answers is second pair translated by } \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

Answer 10.
(a)

$$x + 3y - 4 = 0$$

(b)

$$9x - 3y - 26 = 0$$

2.1 Preparing for The Product Rule

In mathematics a product is a multiplication. e.g. The product of 5 and 8 is 40.
The Product Rule is a rule for differentiating expressions that multiply each other.
It says;

$$If \quad f = uv \quad then \quad f' = u\,v' + u'\,v$$

Suppose we were asked to differentiate the product of x^5 and x^3

$$y = x^5\ x^3$$
$$\frac{dy}{dx} = x^5\ 3x^2 + 5x^4\ x^3$$
$$\frac{dy}{dx} = 3x^7 + 5x^7$$
$$\frac{dy}{dx} = 8x^7$$

A more sensible way to do this problem would be to simplify first then differentiate;

$$y = x^5 x^3$$
$$y = x^8$$
$$\frac{dy}{dx} = 8x^7$$

The first method let's us practice the product rule before having to use it
on questions that can't be done using the second method.

2.2 Exercise

Differentiate each of the following products using the method specified.

(i) Use the Product Rule;

$$y = x^9\ x^{11}$$

(ii) Use algebra to simplify first;

$$y = x^9 \; x^{11}$$

(iii) Use the product rule;

$$y = 3x^5 \quad 4x^7$$

(iv) Use algebra to simplify first;

$$y = 3x^5 \quad 4x^7$$

(v) Use the product rule;

$$y = x \; x$$

(vi) Use algebra to simplify first;

$$y = x \; x$$

(vii) Use the product rule;

$$y = 8x^{\frac{3}{2}} \; 6x^{\frac{5}{2}}$$

(viii) Use algebra to simplify first;

$$y = 8x^{\frac{3}{2}} \; 6x^{\frac{5}{2}}$$

(ix) Use the product rule;

$$y = x^{-3} x^{8}$$

(x) Use algebra to simplify first;

$$y = x^{-3} x^{8}$$

(xi) Use the product rule;

$$y = (x^2 - 1)(x^2 + 1)$$

(xii) Use algebra to simplify first;

$$y = (x^2 - 1)(x^2 + 1)$$

(xiii) Use the product rule to show that;

$$y = x^4 \; (3x^2 + 1)$$

has a derivative given by;

$$\frac{dy}{dx} = 2x^3 \; (9x^2 + 2)$$

and a second derivative given by;

$$\frac{d^2y}{dx^2} = 6x^2 \; (15x^2 + 2)$$

before determining the third derivative.

2.3 Answers.

2.3.1 Solutions (2.2 Exercise)

(i) Using the Product Rule;

$$y = x^{9}\ x^{11}$$

$$\frac{dy}{dx} = x^{9}\ 11x^{10} + 9x^{8}\ x^{11}$$

$$\frac{dy}{dx} = 11x^{19} + 9x^{19}$$

$$\frac{dy}{dx} = 20x^{19}$$

(ii) Using algebra to simplify first;

$$y = x^{9}\ x^{11}$$

$$y = x^{20}$$

$$\frac{dy}{dx} = 20x^{19}$$

(iii) Using the product rule;

$$y = 3x^{5}\quad 4x^{7}$$

$$\frac{dy}{dx} = 3x^{5}\ 28x^{6} + 15x^{4}\,4x^{7}$$

$$\frac{dy}{dx} = 84x^{11} + 60x^{11}$$

$$\frac{dy}{dx} = 144x^{11}$$

(iv) Using algebra to simplify first;

$$y = 3x^{5}\quad 4x^{7}$$

$$y = 12x^{12}$$

$$\frac{dy}{dx} = 144x^{11}$$

(v) Using the product rule;

$$y = x\ \ x$$

$$\frac{dy}{dx} = x\,1 + 1\,x$$

$$\frac{dy}{dx} = 2\,x$$

(vi) Using algebra to simplify first;

$$y = x \; x$$
$$y = x^2$$
$$\frac{dy}{dx} = 2x$$

(vii) Using the product rule;

$$y = 8x^{\frac{3}{2}} \; 6x^{\frac{5}{2}}$$
$$\frac{dy}{dx} = 8x^{\frac{3}{2}} \; 15x^{\frac{3}{2}} + 12x^{\frac{1}{2}} \; 6x^{\frac{5}{2}}$$
$$\frac{dy}{dx} = 120x^3 + 72x^3$$
$$\frac{dy}{dx} = 192x^3$$

(viii) Using algebra to simplify first;

$$y = 8x^{\frac{3}{2}} \; 6x^{\frac{5}{2}}$$
$$y = 48x^4$$
$$\frac{dy}{dx} = 192x^3$$

(ix) Using the product rule;

$$y = x^{-3} \; x^8$$
$$\frac{dy}{dx} = x^{-3} \; 8x^7 - 3x^{-4} \; x^8$$
$$\frac{dy}{dx} = 8x^4 - 3x^4$$
$$\frac{dy}{dx} = 5x^4$$

(x) Using algebra to simplify first;

$$y = x^{-3} \; x^8$$
$$y = x^5$$
$$\frac{dy}{dx} = 5x^4$$

(xi) Using the product rule;

$$y = (x^2 - 1)(x^2 + 1)$$

$$\frac{dy}{dx} = (x^2 - 1)\,2x + 2x\,(x^2 + 1)$$

$$\frac{dy}{dx} = 2x^3 - 2x + 2x^3 + 2x$$

$$\frac{dy}{dx} = 4x^3$$

(xii) Using algebra to simplify first;

$$y = (x^2 - 1)(x^2 + 1)$$

$$y = x^4 - 1 \qquad \textit{difference of two squares}$$

$$\frac{dy}{dx} = 4x^3$$

(xiii) Using the product rule to show that;

$$y = x^4\,(3x^2 + 1)$$

has a derivative given by;

$$\frac{dy}{dx} = 2x^3\,(9x^2 + 2)$$

and a second derivative given by;

$$\frac{d^2y}{dx^2} = 6x^2\,(45x^2 + 2)$$

before determining the third derivative.

$$\frac{d^3y}{dx^3} = 24x\,(15x^2 + 1) \qquad or \qquad \frac{dy}{dx} = 360x^3 + 24x$$

Chapter 3.

3.1 The Product Rule

Suppose that we have a function, f , that is the product of two other functions, u and v.
i.e.

$$f(x) = u(x) \times v(x)$$

The derivative of f, is given by,

$$f'(x) = u(x) \times v'(x) + u'(x) \times v(x)$$

More succinctly,

$$\textit{If} \quad f = uv \quad \textit{then} \quad f' = u\,v' + u'v$$

We'll look at *why* this is the rule later.
First, let's look at how to apply it to an example.

3.2 An example

Differentiate,

$$y = x^3\,(2x + 5)^3$$

by using,

(i) The product rule.

(ii) Converting the given expression into a form to which the chain rule can be applied.

You should, of course, get the same solution each time.

Typically, the problem will not be easily convertable into something to which the chain rule can be applied, so the product rule is often the only viable method.

Try to solve the next problem yourself before checking your solution with mine, which is on the following page.

Try 1. Use the product rule to show that the derivative of,

$$y = (2x + 3)^2 (5x - 1)^3$$

is,

$$\frac{dy}{dx} = (2x + 3)(5x - 1)^2 (50x + 41)$$

Answer 1.

$$If \;\; f = uv \quad then \quad f' = u\,v' + u'v$$

$$y = (2x + 3)^2 \, (5x - 1)^3$$

$$\frac{dy}{dx} = (2x + 3)^2 \; 3\,(5x - 1)^2 \; 5 + 2\,(2x + 3)^1 \; 2\,(5x - 1)^3$$

$$\frac{dy}{dx} = (2x + 3)\;(5x - 1)^2 \;\{ 15\,(2x + 3) \; + \; 4\,(5x - 1) \}$$

$$\frac{dy}{dx} = (2x + 3)\;(5x - 1)^2 \;\{ 30x + 45 + 20x - 4 \}$$

$$\frac{dy}{dx} = (2x + 3)\;(5x - 1)^2 \;\{ 50x + 41 \}$$

3.3 Exercise

Question 1.

Use the product rule to show that the derivative of,

$$y = x^5\,(x - 1)^2$$

is,

$$\frac{dy}{dx} = x^4\,(x - 1)(7x - 5)$$

Question 2.

Use the product rule to show that the derivative of,

$$y = x^7 (6x + 5)^3$$

is,

$$\frac{dy}{dx} = 5x^6 (6x + 5)^2 (12x + 7)$$

Question 3.

Use the product rule to show that the derivative of,

$$y = (x^2 - 3)(x + 1)^2$$

is,

$$\frac{dy}{dx} = 2(x^2 - 1)(2x + 3)$$

Question 4.

Use the product rule to show that the derivative of,

$$y = (4x + 1)^{\frac{3}{2}}(x^2 + 5)$$

is,

$$\frac{dy}{dx} = \sqrt{4x + 1}\,(14x^2 + 2x + 30)$$

Question 5.

Use the product rule to show that the derivative of,

$$y = (2x - 3)^3(x^2 + 1)^2$$

is,

$$\frac{dy}{dx} = 2(2x - 3)^2(x^2 + 1)(7x^2 - 6x + 3)$$

Question 6.
Use the product rule to find the derivative of,

$$y = x^3 (5x + 1)^2$$

Question 7.
Find the x component of the coordinates of the stationary points on the curve

$$y = (x^2 - 1) \sqrt{1 + x}$$

3.4 Answers.

3.4.1 Solutions (3.2 Example)

(i) Using the product rule...

$$\textit{If} \quad f = uv \quad \textit{then} \quad f' = u\,v' + u'\,v$$

$$y = x^3\,(2x + 5)^3$$

$$\frac{dy}{dx} = x^3\,3(2x + 5)^2\,2 + 3x^2\,(2x + 5)^3$$

$$\frac{dy}{dx} = 6x^3(2x + 5)^2 + 3x^2\,(2x + 5)^3$$

$$\frac{dy}{dx} = 3x^2\,(2x + 5)^2\,\{2x + (2x + 5)\}$$

$$\frac{dy}{dx} = 3x^2\,(2x + 5)^2\,\{4x + 5\}$$

(ii) Converting the given expression into a form to which the chain rule can be applied.

$$y = x^3\,(2x + 5)^3$$

$$\frac{dy}{dx} = \{x\,(2x + 5)\}^3$$

$$\frac{dy}{dx} = \{2x^2 + 5x\}^3$$

$$\frac{dy}{dx} = 3\,\{2x^2 + 5x\}^2\,(4x + 5)$$

$$\frac{dy}{dx} = 3\,\{x\,(2x + 5)\}^2\,(4x + 5)$$

$$\frac{dy}{dx} = 3x^2\,(2x + 5)^2\,(4x + 5)$$

Chapter 4.

4.1 The Quotient Rule

$$If \quad f = \frac{u}{v} \quad then \quad f' = \frac{v\,u' - v'\,u}{v^2}$$

4.2 Example

Differentiate,

$$y = \frac{x + 4}{x + 5}$$

by using,

(i) The quotient rule.

(ii) Algebraic 'trickiness' and simple differentiation.

(i)

(ii)

4.3 Exercise

Question 1.

Given that

$$y = \frac{4x}{x + 3}$$

use the quotient rule to show that the derivative is given by;

$$\frac{dy}{dx} = \frac{12}{(x + 3)^2}$$

Question 2.

Given that

$$y = \frac{x^2}{(x + 5)}$$

use the quotient rule to show that the derivative is given by;

$$\frac{dy}{dx} = \frac{x(x + 10)}{(x + 5)^2}$$

Question 3.
Given that

$$y = \frac{5x - 2}{3x + 1}$$

use the quotient rule to show that the derivative is given by;

$$\frac{dy}{dx} = \frac{11}{(3x + 1)^2}$$

Question 4.
Given that

$$y = \frac{x^2 + 1}{x^2 + 4}$$

use the quotient rule to show that the derivative is given by;

$$\frac{dy}{dx} = \frac{6x}{(x^2 + 4)^2}$$

Question 5.
Given that

$$y = \frac{x^5}{(2x + 1)^3}$$

use the quotient rule to show that the derivative is given by;

$$\frac{dy}{dx} = \frac{x^4 (4x + 5)}{(2x + 1)^4}$$

Question 6.
Given that

$$y = \frac{x^7}{(3x + 2)^5}$$

use the quotient rule to show that the derivative is given by;

$$\frac{dy}{dx} = \frac{2x^6 (3x + 7)}{(3x + 2)^6}$$

Question 7.
Given that

$$y = \frac{2(x+3)^3}{\sqrt{x}}$$

use the quotient rule to show that the derivative is given by;

$$\frac{dy}{dx} = \frac{(x+3)^2(5x-3)}{x^{\frac{3}{2}}}$$

Question 8.
Given that

$$y = x^2\sqrt{x+5}$$

use the PRODUCT rule to show that the derivative is given by;

$$\frac{dy}{dx} = \frac{5x(x+4)}{2\sqrt{x+5}}$$

Chapter 5.

Differentiation : Core 3

5.1 Exponential & Logarithmic Functions

The exponential function has the remarkable property of being its own derivative. In other words,

If

$$y = e^x$$

Then

$$\frac{dy}{dx} = e^x$$

The inverse of the exponential function is the natural logarithm function. This has a rather surprising derivative.

If

$$y = ln\, x$$

Then

$$\frac{dy}{dx} = \frac{1}{x}$$

Proof :

$$y = ln\, x$$

$$\textit{exponentiate both sides}$$

$$e^y = e^{ln\, x}$$

$$e^y = x$$

$$x = e^y$$

$$\frac{dx}{dy} = e^y$$

$$\frac{dx}{dy} = x$$

$$\frac{dy}{dx} = \frac{1}{x} \qquad \square$$

Armed with these two results, one for exponentials and one for logarithms, we can now tackle many more differentiation questions.

5.2 Did you know ?

e is the number 2.71828 18284 59045 23536 02875 ...

Like π this is an irrational number and like π it crops up in many surprising places throughout mathematics.

5.3 Reminders

$$If \quad f = uv \quad then \quad f' = u\,v' + u'\,v$$

$$If \quad f = \frac{u}{v} \quad then \quad f' = \frac{v\,u' - v'\,u}{v^2}$$

5.4 Examples

Differentiate the following with respect to *x*;

Chain rule examples

(i)

$$y = e^{8x^3 + 5x^2}$$

(ii)

$$y = ln\,(x^2 + e^x)$$

Product rule examples

(iii)

$$y = x\,e^{3x}$$

(iv)

$$y = x^3\ ln\,x$$

Quotient rule examples

(v)

$$y = \frac{ln\,x}{x}$$

(vi)

$$y = \frac{x}{e^x}$$

5.5 Exercise

Question 1.

Problems involving only the chain rule and the exponential function

Differentiate each of the following with respect to *x*;

(i)

$$y = e^{5x}$$

(ii)

$$y = e^{x^2}$$

(iii)

$$y = 8e^{6x} + 6e^{5x}$$

(iv)

$$y = 14e^{\sqrt{x}}$$

(v)

$$y = e^{-x}$$

(vi)

$$y = \frac{4}{e^{5x}}$$

(vii) HINT : *When you divide the indices subtract*

$$y = \frac{e^{7x}}{e^{3x}}$$

(viii) HINT : *First expand the brackets*

$$y = e^{3x}\left(e^{4x} + 7e^{-9x}\right)$$

Question 2.

Problems involving only the chain rule and the logarithm function

Differentiate each of the following with respect to *x*;

(i)

$$y = 3\ln x$$

(ii)

$$y = \ln x^3$$

(iii)

$$y = \ln(5x)$$

(iv)

$$y = \ln(7x)$$

(v)

$$y = \ln(4x^3 + 3)$$

(vi)

$$y = \ln(1 - e^x)$$

(vii)

$$y = \frac{7\ln x}{3}$$

(viii)

$$y = (\ln x)^3$$

(ix)

$$y = (\ln x)^{-4}$$

(x)

$$y = \frac{4}{\ln x}$$

Question 3.

Problems involving the product rule and the exponential function

Differentiate each of the following with respect to *x*;

(i)

$$y = x^3 e^x$$

(ii)

$$y = x e^{-x}$$

(iii)

$$y = \sqrt{x}\, e^{6x}$$

(iv)

$$y = x^2 (1 - e^{-3x})$$

Question 4.

Problems involving the product rule and the logarithm function

Differentiate each of the following with respect to *x*;

(i)

$$y = x^3 \ln x$$

(ii)

$$y = x \ln (x^2 + 1)$$

(iii)

$$y = (x^2 + 1)\ \ln (7x)$$

(iv)

$$y = \ln x^2\ \ln x^3$$

Question 5.

Problems involving the quotient rule and the exponential function

Differentiate each of the following with respect to *x*;

(i)

$$y = \frac{e^{x}}{x}$$

(ii)

$$y = \frac{e^{3x}}{(x^2 - 1)}$$

(iii)

$$y = \frac{4e^{7x}}{\sqrt{x}}$$

(iv)

$$y = \frac{x^2 + 5}{e^{3x}}$$

Question 6.

Problems involving the quotient rule and the logarithm function

Differentiate each of the following with respect to *x*;

(i)

$$y = \frac{x}{\ln x}$$

(ii)

$$y = \frac{\ln x}{x^2}$$

(iii)

$$y = \frac{x^3}{\ln x}$$

(iv)

$$y = \frac{12 \ln (2x)}{x}$$

Question 7.

Differentiate each of the following with respect to *x*;

$$y = e^{x} \ln x$$

Question 8.

Differentiate each of the following with respect to *x*;

$$y = e^{5x} \ln x$$

Question 9.

A curve has equation

$$y = \frac{3x + 2}{2x + 3}$$

(i) Find $\frac{dy}{dx}$

(ii) Find the equation of the tangent to the curve when $x = 1$.

Give your answer in the form $a y + b x + c = 0$ for integer a, b and c.

Question 10.

A curve has equation

$$y = x e^{x}$$

(i) Find $\frac{dy}{dx}$

(ii) Find the equation of the tangent to the curve when $x = 4$.
Give an exact answer.

Question 11.

A curve has equation

$$y = \frac{ln\,x}{x^2}$$

(i) Find an expression for $\frac{dy}{dx}$

(ii) Find the equation of the tangent to the curve when $x = 1$.
Give your answer in the form $a\,y + b\,x + c = 0$ for integer a, b and c.

Question 12.

A curve has equation

$$y = \frac{x^2}{(4x - 1)}$$

(i) Find an expression for $\frac{dy}{dx}$

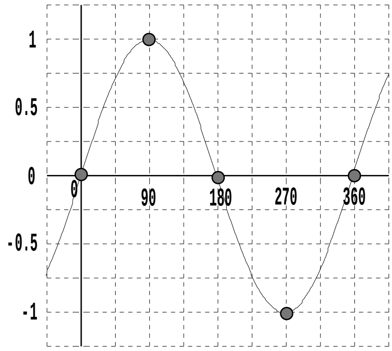
(ii) Find the equation of the tangent to the curve when $x = 1$.
Give your answer in the form $a\,y + b\,x + c = 0$ for integer a, b and c.

Chapter 6.

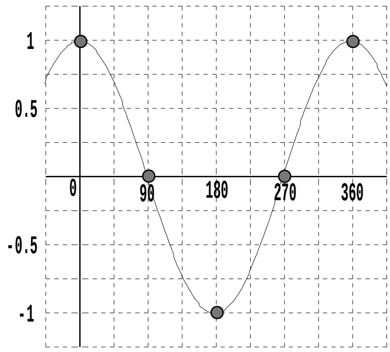
Differentiation : Core 3

6.1 Sine & Cosine Functions

From looking at their graphs, the *sine* and *cosine* functions are obviously related. The *cosine* curve is simply the *sine* curve shifted 90° to the left.



$$y = \sin x$$



$$y = \cos x$$

From staring at the graphs, it seems plausable that...

If $y = \sin x$ then $\frac{dy}{dx} = \cos x$

... and this is indeed the case, provided that x is measured in RADIANS.

In consequence:

$$y = \cos x$$

$$y = (1 - \sin^2 x)^{\frac{1}{2}} \qquad \text{Using} \quad \cos^2 x + \sin^2 x = 1$$

$$\frac{dy}{dx} = \frac{1}{2}(1 - \sin^2 x)^{-\frac{1}{2}}(-2\sin x)(\cos x)$$

$$\frac{dy}{dx} = -\frac{\sin x \cos x}{\sqrt{1 - \sin^2 x}}$$

$$\frac{dy}{dx} = -\frac{\sin x \cos x}{\cos x}$$

$$\frac{dy}{dx} = -\sin x$$

6.2 Exercise.

Question 1.

Differentiate with respect to x;

(i) $y = sin\ (\ 5x\ )$

(ii) $y = cos\ (\ 8\ x^2\ )$

(iii) $y = 6\ sin\ (\ 3\ x^3\ )$

(iv) $y = sin^3\ x$

(v) $y = cos^4\ (\ 7x\ )$

(vi) $y = \sqrt{sin\ x}$

(vii) $y = cos\ (3\ x^5 + 2\ x^2)$

(viii) Show that if

$y = (sin\ x)^{-1}$

then

$$\frac{dy}{dx} = -\ csc\ x\ \ cot\ x$$

(ix) $y = csc\ x$

(x) $y = csc\ (9x)$

Part (ix) is an established, quotable result also on the examination formula sheet.

Question 2.
Find the derivative with respect to x of the following;

(i) $y = e^{4x}$

(ii) $y = 7e^{7x + 2}$

(iii) $y = e^{\sin x}$

(iv) $y = 3e^{2 \sin(5x)}$

(v) $y = \ln(5x)$

(vi) $y = \ln(\sin x)$

(vii) $y = cos \left(\frac{1}{x} \right)$

(viii) Show that if

$y = (cos\, x)^{-1}$

then

$\frac{dy}{dx} = sec\, x \; tan\, x$

(ix) $y = sec\, x$

(x) $y = sec (8 x^2)$

Part (ix) is an established, quotable result also on the examination formula sheet.

Question 3.

(i) Use the product rule to differentiate $f(x)$.

$$f(x) = x \sin x$$

(ii) Show that

$$f'\left(\frac{\pi}{6}\right) = \frac{6 + \sqrt{3}\,\pi}{12}$$

Question 4.

$$g(x) = e^{5x} + \ln 4x$$

Write down an expression for $g'(x)$

Question 5.

$$y = x^4 \; e^{2x}$$

(i) Use the product rule to show that

$$\frac{dx}{dx} = 2x^3 e^{2x}(x + 2)$$

(ii) Find the exact value of the derivative when $x = 3$.

Question 6.
Find the derivative with respect to x of the following;

(i) $y = sin^3 (3x)$

(ii) $y = cos (0.75 x^2)$

(iii) $y = 3 \cos (5 - x^2)$

(iv) $y = e^{-x}$

(v) $y = \frac{4}{x^3}$

(vi) $y = ln (8 x^2)$

(vii) $y = \frac{1}{e^{3x}} + x$

(viii) $y = sin \left(e^{2x} \right)$

(ix) $y = csc^2 x$

(x) $y = ln (\cos x)$

Question 7.

(i) Use the quotient rule to differentiate $f(x)$.

$$h(x) = \frac{\sin x}{x}$$

(ii) Show that

$$h'(\pi) = -\frac{1}{\pi}$$

Question 8.

$$g(x) = \sin(x^2) + \cos^2 x$$

Write down an expression for $g'(x)$

Question 9.

$$y = \frac{\sin x}{\cos x}$$

(i) Use the quotient rule to show that

$$\frac{dx}{dx} = \sec^2 x$$

(ii) Hence differentiate $y = \tan(5x)$

Question 10.
With question 9 in mind, find the derivative of $y = cot\ x$

Question 11.
Look over this chapter, and complete the following table.

Table of Derivatives of Trigonometric Functions

$f\,(x)$	$f\,'\,(x)$
$sin\ x$	
$cos\ x$	
$tan\ x$	
$csc\ x$	
$sec\ x$	
$cot\ x$	

Chapter 7.

Differentiation : Core 3

7.1 Table of Derivatives of Trigonometric Functions

The following table summarises some important results from exercise 6.2.
Most of these results are contained in the Formula Book provided in examinations.

$f(x)$	$f'(x)$	*In FB ?*
$\sin x$	$\cos x$	*No*
$\cos x$	$-\sin x$	*No*
$\tan x$	$\sec^2 x$	*Yes*
$\csc x$	$-\csc x \cot x$	*Yes*
$\sec x$	$\sec x \tan x$	*Yes*
$\cot x$	$-\csc^2 x$	*Yes*

7.2 Example

Write down the derivative of

(i)

$$y = \sec(7x)$$

(ii)

$$y = \sec^5 x$$

(iii)

$$y = \ln(\sec x)$$

(iv)

$$y = e^{\sec x}$$

7.3 Past Examination Questions

See separate sheets

Chapter 8.

8.1 Revision for Test.

Differentiation : Core 3

Table of standard derivatives.

As given in the test.

y	$\frac{dy}{dx}$	y	$\frac{dy}{dx}$
x^n	$n x^{n-1}$	$\sin x$	$\cos x$
e^x	e^x	$\cos x$	$-\sin x$
$\ln x$	$\frac{1}{x}$	$\tan x$	$\sec^2 x$
		$\sec x$	$\sec x \tan x$
		$\csc x$	$-\csc x \cot x$
		$\cot x$	$-\csc^2 x$

Question 1.

Differentiate each of the following;

(i) $y = 7x^4$ **(ii)** $y = 11\sqrt{x}$

[2 marks]

Question 2.

By first expanding the brackets, find the derivative of each of the following;

(i) $y = (x + 6)(2x - 5)$ **(ii)** $y = \sqrt{x}\left(\frac{1}{\sqrt{x}} + 3\sqrt{x}\right)$

[4 marks]

Question 3.

Use the *chain rule* to find the derivative of each of the following;

(i)

$y = 4(x^3 + 3)^5$

(ii)

$y = 4 \cos(2x)$

(iii)

$y = \sec^3 x$

(iv)

$y = e^{\sin x} + e^{\cos x}$

[8 marks]

Question 4.

The rule for the differentiation of products tells us that;

$$\text{If} \qquad f = u\,v$$

$$\text{Then} \qquad f' = u\,v' + u'\,v$$

(i) Using this rule, find $\frac{dy}{dx}$ if $y = x \ln x$

[2 marks]

(ii) Find also $\frac{d^2 y}{dx^2}$

[2 marks]

Question 5.
Consider the function

$$f(x) = tan(3x)$$

Determine the value of

$$f'\left(\frac{\pi}{18}\right)$$

[4 marks]

Question 6.
The rule for the differentiation of quotients tells us that;

$$If \qquad f = \frac{u}{v}$$

$$Then \qquad f' = \frac{v\,u' - v'\,u}{v^2}$$

Using this rule, show that,

$$If \qquad y = \frac{x^3 - 1}{x^3 + 1} \qquad then \qquad \frac{dy}{dx} = \frac{6x^2}{(x^3 + 1)^2}$$

[4 marks]

Question 7.
Find the equation of the tangent to the curve,

$$y = \frac{11}{x^2 - 3}$$

when $x = 5$.

Give the answer in the form $ax + by + c = 0$
where a, b and c are <u>INTEGERS</u> to be found.

[6 marks]

Question 8.

The function $f(x)$ is given below

$$f(x) = \frac{x^2 \sin(2x)}{9\pi}$$

(i) Find $f'(x)$

(ii) Show that

$$f'\left(\frac{\pi}{4}\right) = \frac{1}{18}$$

[4 marks]

Question 9.

If

$$\frac{d}{dx}\left(\ln \sqrt{ax + b} \right) = \frac{4}{ax + 1}$$

Find the values of *a* and *b*.

[4 marks]

Chapter 9.

9.1 Test.

Differentiation : Core 3

Table of standard derivatives.

y	$\frac{dy}{dx}$	y	$\frac{dy}{dx}$
x^n	$n x^{n-1}$	$\sin x$	$\cos x$
e^x	e^x	$\cos x$	$-\sin x$
$\ln x$	$\frac{1}{x}$	$\tan x$	$\sec^2 x$
		$\sec x$	$\sec x \tan x$
		$\csc x$	$-\csc x \cot x$
		$\cot x$	$-\csc^2 x$

Question 1.

Differentiate each of the following;

(i) $y = 13x^5$ **(ii)** $y = \frac{24}{x}$

[2 marks]

Question 2.

By first expanding the brackets, find the derivative of each of the following;

(i) $y = (x + 7)(3x - 2)$ **(ii)** $y = \frac{1}{\sqrt{x}}(3\sqrt{x} + x)$

[4 marks]

Question 3.

Use the *chain rule* to find the derivative of each of the following;

(i)

$y = (5x^3 + 1)^4$

(ii)

$y = tan\ (5x)$

(iii)

$y = csc^4 x$

(iv)

$y = ln\,2x + e^{-3x}$

[8 marks]

Question 4.

The rule for the differentiation of products tells us that;

$$If \quad f = u\,v$$

$$Then \quad f' = u\,v' + u'\,v$$

(i) Using this rule, find $\frac{dy}{dx}$ if $y = x\,\mathrm{e}^{x}$

[2 marks]

(ii) Find also $\frac{d^2y}{dx^2}$

[2 marks]

Question 5.
Consider the function

$$f(x) = cot\ (6x)$$

Determine the value of

$$f'\left(\frac{\pi}{18}\right)$$

[4 marks]

Question 6.
The rule for the differentiation of quotients tells us that;

$$If \qquad f = \frac{u}{v}$$

$$Then \qquad f' = \frac{v u' - v' u}{v^2}$$

Using this rule, show that,

$$If \qquad y = \frac{x^5 - 2}{x^5 + 2} \qquad then \qquad \frac{dy}{dx} = \frac{20x^4}{(x^5 + 2)^2}$$

[4 marks]

Question 7.
Find the equation of the tangent to the curve,

$$y = \frac{28}{x^2 - 2}$$

when $x = 3$.

Give the answer in the form $ax + by + c = 0$
where a, b and c are INTEGERS to be found.

[6 marks]

Question 8.

The function $f(x)$ is given below

$$f(x) = \frac{x^2 \sin(3x)}{9\pi}$$

(i) Find $f'(x)$

(ii) Show that

$$f'\left(\frac{\pi}{3}\right) = -\frac{\pi}{27}$$

[4 marks]

Question 9.

$$f(x) = ln(2x - 1) - x$$

Show that the maximum value of $f(x)$ is $ln\ 2 - 1.5$

[4 marks]

Chapter 10.

10.1 Further 'Tricks' & Practice.

Differentiation : Core 3

Table of standard derivatives.

y	$\frac{dy}{dx}$	y	$\frac{dy}{dx}$
x^n	$n x^{n-1}$	$\sin x$	$\cos x$
e^x	e^x	$\cos x$	$-\sin x$
$\ln x$	$\frac{1}{x}$	$\tan x$	$\sec^2 x$
		$\sec x$	$\sec x \tan x$
		$\csc x$	$-\csc x \cot x$
		$\cot x$	$-\csc^2 x$

Example

Find the derivative of the inverse sine function

$$y = \sin^{-1} x$$

Note the following mathematical 'glitch' :

$$\sin^{-1} x \neq \frac{1}{\sin x} \qquad \sin^{-1} x \neq (\sin x)^{-1} \qquad \sin^{-1} x \neq \csc x$$

$$y = \sin^{-1} x$$

$$x = \sin y$$

$$\frac{dx}{dy} = \cos y$$

$$\text{But } \cos^2 y + \sin^2 y = 1 \text{ so...}$$

$$\frac{dx}{dy} = \sqrt{1 - \sin^2 y}$$

$$\frac{dx}{dy} = \sqrt{1 - x^2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

There is an issue with why we took the positive square root and not the negative one. Sketch the graph of $y = \sin^{-1} x$ to see why it's the positive square root that is taken.

10.2 Exercise

Question 1.
Show that the derivative of the inverse tangent function

$$y = tan^{-1} x$$

is

$$\frac{dy}{dx} = \frac{1}{1 + x^2}$$

Do this by trying to construct working that is similar to the example.

The following trigonometry formula will be useful;

$$1 + tan^2 y = sec^2 y$$

This proof was asked for in the January 2010 C3 examination and was worth 5 marks.

Question 2.
Show that the derivative of the inverse cotangent function

$$y = cot^{-1} x$$

is

$$\frac{dy}{dx} = -\frac{1}{1 + x^2}$$

Do this by trying to construct working that is similar to the example.

The following trigonometry formula will be useful;

$$cot^2 y + 1 = csc^2 y$$

Question 3.
Show that the derivative of;

$$y = sec\,x\,tan\,x$$

is

$$\frac{dy}{dx} = sec\,x\,(\,2\,sec^2 x - 1\,)$$

Question 4.
Show that the derivative of;

$$y = csc\,x\,cot\,x$$

is

$$\frac{dy}{dx} = csc\,x\,(\,1 - 2\,csc^2 x\,)$$

Question 5.
Consider the function;

$$f(x) = \frac{8}{(1 - 3x)^3}$$

Show that;

$$f'(1) = \frac{9}{2}$$

Question 6.
C3 examination question,January 2009
Find the equation of the tangent to the curve

$$x = cos\ (2y + \pi) \quad at \quad \left(0,\ \frac{\pi}{4}\right)$$

Give your answer in the form $y = ax + b$, where a and b are constants to be found.

[6 marks]

Question 7.
The curve

$$y = ln\,(x^2 - 3)$$

crosses the x-axis at A and B.

(i) Find the coordinates of A and B.

(ii) The normals at A and B meet at P.
Find the coordinates of P.

Question 8.

The curve

$$y = \frac{2x + 1}{2x - 1}$$

crosses the x-axis at A and the y-axis at B.

Find the point of intersection of the tangents to the curve at A and B.

Chapter 11.

Differentiation : Core 3

11.1 Exercise.

Question 1.

Differentiate each of the following with respect to *x*;

(i)	**(ii)**	**(iii)**
$y = e^x$	$y = x^e$	$y = e^e$

[3 marks]

Question 2.

Write down the derivative of each of the following with respect to *x*;

(i)	**(ii)**	**(iii)**
$y = \ln x$	$y = \ln x^2$	$y = \ln \frac{1}{x}$

[3 marks]

Question 3.

Find the equation of the tangent to the curve

$$y = e^{\frac{1}{2}x}$$

at the point where it intercepts the *y*-axis.

Write your answer in the form $ay + bx + c = 0$, where $a, b, c \in \mathbb{Z}$.

[10 marks]

Question 4.

Find the points on the following curve where the gradient is 3.

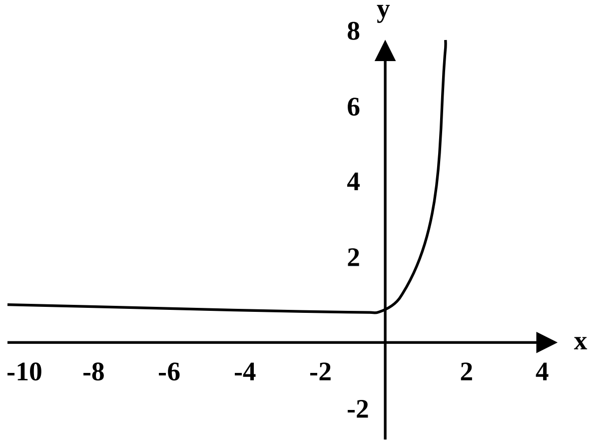
$$y = 5\sqrt{x} - \frac{1}{2} ln\ x \qquad x \in \mathbb{R},\ x > 0.$$

Give your answers correct to three significant figures.

[10 marks]

Question 5.

A sketch graph of the function $f(x) = e^{2x} - e^{x} + 1$ is shown below.



(i) Find $f'(x)$

(ii) Explain, briefly, why the equation $e^{x} = 0$ has no solution for $x \in \mathbb{R}$.

(iii) Find the value of x such that $f'(x) = 0$.

(iv) The graph of the $f(x)$ has a turning point of the form ($\ln a$, b).
Determine the value of a and of b.

(v) Find $f''(x)$ at the turning point.

(vi) Explain what your part **(v)** answer tells you about the turning point.

[14 marks]

Chapter 12.

12.1 Graphing Parametric Equations

Investigation

A curve is described by the parametric equations

$$x = 20 \cos \theta°$$

$$y = 15 \sin \theta°$$

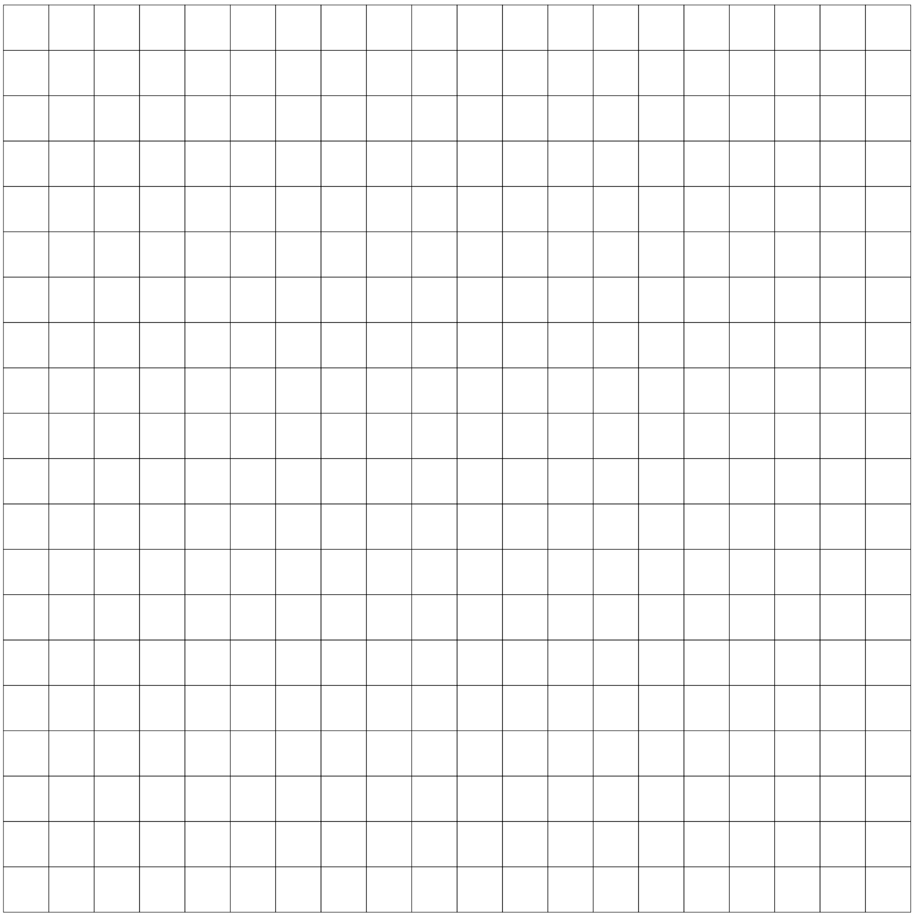
Complete the following tables and graph the resulting curve.
Work to 1 decimal place.

	0	10	20	30	40	50	60	70	80	90
$x = 20 \cos \theta°$										
$y = 15 \sin \theta°$										

	100	110	120	130	140	150	160	170	180
$x = 20 \cos \theta°$									
$y = 15 \sin \theta°$									

	190	200	210	220	230	240	250	260	270
$x = 20 \cos \theta°$									
$y = 15 \sin \theta°$									

	280	290	300	310	320	330	340	350	360
$x = 20 \cos \theta°$									
$y = 15 \sin \theta°$									



12.2 How to obtain the curve's Cartesian Equation

12.3 Exercise

Question 1.

Find the Cartesian equations of each of these curves in the form $y = f(x)$.

(i)

$x = 8t$

$y = \dfrac{8}{t}$

(ii)

$x = 14t$

$y = 7t^2$

(iii)

$x = \dfrac{20}{t}$

$y = t^2$

(iv)

$x = t - 2$

$y = t^2 + 3$

Question 2.

Find the Cartesian equations of each of these curves, by making use of the formula

$$\cos^2 x + \sin^2 x = 1$$

(i)

$x = 2\cos\theta$

$y = 3\sin\theta$

(ii)

$x = 3\sec\theta$

$y = 5\tan\theta$

Chapter 13.

13.1 Differentiation Of Parametric Equations

Having graphed the parametric equations

$$x = 20 \cos \theta°$$

$$y = 15 \sin \theta°$$

we noted that the resulting curve is smooth and continuous.
At every point a tangent could be drawn, so the curve had a gradient at every point.

The gradient can be found, in terms of θ, by differentiating the two parametric equations separately

$$\frac{dx}{d\theta} = -20 \sin \theta \qquad \frac{dy}{d\theta} = 15 \cos \theta$$

then combining them using the facts that

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} \qquad \textit{and} \qquad \frac{d\theta}{dx} = \frac{1}{\left(\frac{dx}{d\theta}\right)}$$

like this

$$\frac{dy}{dx} = 15 \cos \theta \times \frac{1}{-20 \sin \theta}$$

$$= -\frac{3}{4} \cot \theta$$

13.2 Exercise

Question 1.

Find $\frac{dy}{dx}$ in terms of θ.

$$x = 3 \sin \theta$$

$$y = 5 \cos \theta$$

Question 2.

$$x = 2\,cot\,\theta$$

$$y = 3\,tan\,\theta$$

Show that

$$\frac{dy}{dx} = -\frac{3}{2}\,tan^2\,\theta$$

Question 3.

The rule for the differentiation of products tells us that;

$$If \qquad f = u\,v$$

$$Then \qquad f' = u\,v' + u'\,v$$

Consider the pareametric equations

$$x = -\,cos\,\theta$$

$$y = \theta\,sin\,\theta$$

Show that

$$\frac{dy}{dx} = \theta\,cot\,\theta + tan\,\theta$$

Question 4.

$$x = 5t^4$$

$$y = 3\ t^5$$

Show that

$$\frac{dy}{dx} = \frac{3}{4}t$$

Question 5.

$$x = t^2 + t$$

$$y = 2t - t^2$$

Find $\frac{dy}{dx}$ in terms of t

Question 6.

$$x = (4t + 2)^3$$

$$y = (3t + 4)^2$$

Find $\frac{dy}{dx}$ in terms of t

Question 7.

$$x = 5t^2$$

$$y = 4t^3$$

(i) Determine the value of x and the value of y when $t = 2$.
This is a point on the curve.

(ii) Find $\frac{dy}{dx}$ in terms of t

(iii) Use your part **(ii)** answer to show that when $t = 2$, $\frac{dy}{dx} = \frac{12}{5}$

(iv) Use your part **(i)** point, and your part **(iii)** gradient, to find the equation of the tangent to the curve when $t = 2$.
Give the answer in the form $ax + by + c = 0$
where a, b and c are <u>INTEGERS</u> to be found.

Question 8.

$$x = \cos t$$

$$y = \sin 2t$$

(i) Determine the value of x and the value of y when $t = \frac{\pi}{6}$
This is a point on the curve.

(ii) Find $\frac{dy}{dx}$ in terms of t

(iii) Use your part **(ii)** answer to show that when $t = \frac{\pi}{6}$, $\frac{dy}{dx} = -2$

(iv) Use your part **(i)** point, and your part **(iii)** gradient, to find the equation of the tangent to the curve when $t = \frac{\pi}{6}$
Give an exact answer in the form $y = mx + c$ where m & c are constants.

Chapter 14.

Differentiation : Core 4

14.1 Past Paper Work

Question 1.

$$x = \sin^2\theta$$

$$y = \cos 2\theta$$

(i) Show that

$$\frac{dx}{d\theta} = \sin 2\theta$$

(ii) Show that

$$\frac{dy}{dx} = -2$$

(iii) What does your part **(ii)** result tell you about the graph of these parametric equations ?

Question 2.

$$x = t^2$$

$$y = \frac{10}{t}$$

(i) Determine the value of x and the value of y when $t = 5$.
This is a point on the curve.

(ii) Find $\frac{dy}{dx}$ in terms of t

(iii) Use your part **(ii)** answer to show that when $t = 5$, $\frac{dy}{dx} = -\frac{1}{25}$

(iv) Use your part **(i)** point, and your part **(iii)** gradient, to find the equation of the tangent to the curve when $t = 5$.
Give the answer in the form $ax + by + c = 0$
where a, b and c are <u>INTEGERS</u> to be found.

(Questions 3, 4, 5, 6 and 7 available separately)

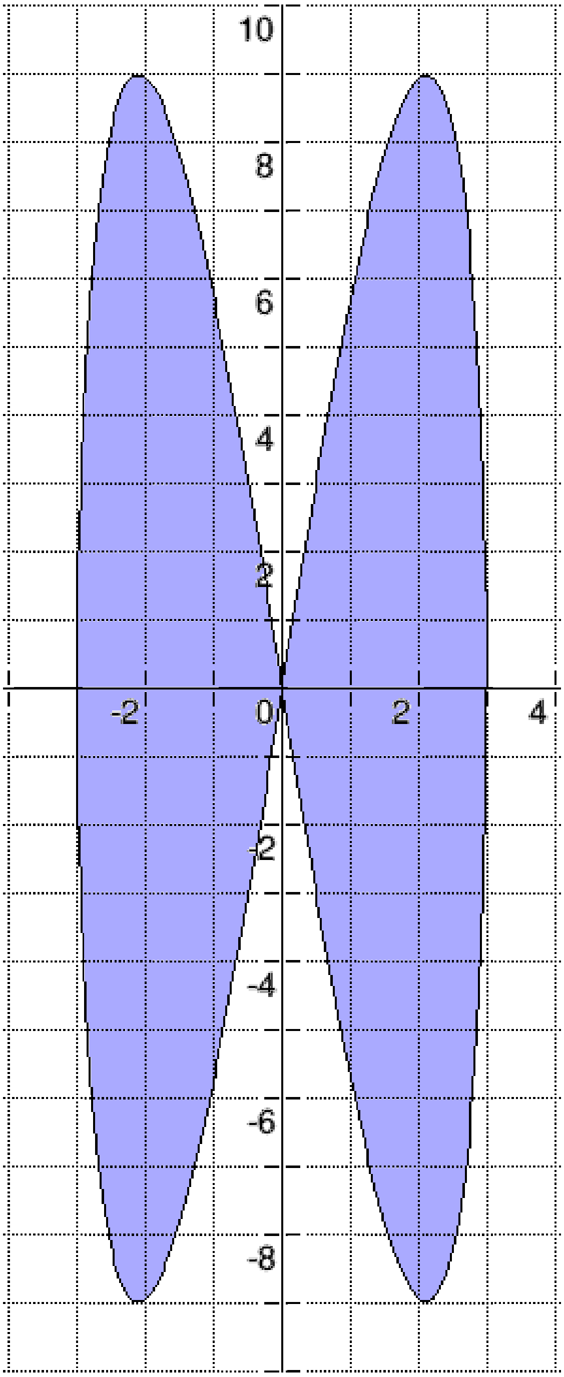
Chapter 15.

15.1 Past Paper Work

Example

$$x = 3\cos t \qquad y = 9\sin 2t \qquad 0 \leqslant t < 2\pi$$

Find the cartesian equation of the curve in the form $y^2 = f(x)$



Chapter 16.

16.1 Implicit Differentiation

Together Questions

Differentiate with respect to *x*.

Present your answer in the form $\frac{dy}{dx} = f(x,y)$

(i)

$x = 4y^3$

(ii)

$3x^2 = 5y^3$

(iii)

$x^3 y^2 = 12$

(iv)

$7x^4 = e^{5y}$

(v)

$5e^{2x} = ln\, y$

(vi)

$x^2 + y^2 = 64$

16.2 Exercise

Question 1.

Differentiate with respect to *x*.

Present your answer in the form $\frac{dy}{dx} = f(x,y)$

(i)

$$x^8 = 4y^5$$

(ii)

$$x^2 = 5y^3 + 4y + 7$$

(iii)

$$x^4 y^5 = 72$$

(iv)

$$5y^4 = e^{5x}$$

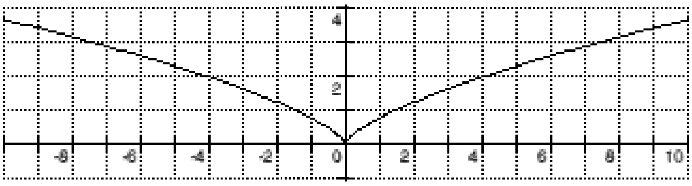
(v)

$$5e^{3y} = \ln x$$

(vi)

$$x^4 + y^4 = 256$$

Question 2.

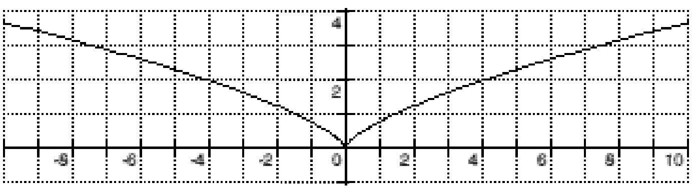


$$x^2 = 2y^3$$

(i) Find the gradient at the point (4, 2)

(ii) Hence determine the equation of the tangent to the curve at (4, 2)

(iii) Draw your tangent on the graph below.



Question 3.

C4 examination question from January 2009, Q1.

A curve C has the equation

$$y^2 - 3y = x^3 + 8$$

(a) Find $\frac{dy}{dx}$ in terms of x and y.

[4 marks]

(b) Hence find the gradient of C at the point where $y = 3$.

[3 marks]

Question 4.

C4 examination question from June 2006, Q1.

A curve C is described by the equation

$$3x^2 - 2y^2 + 2x - 3y + 5 = 0$$

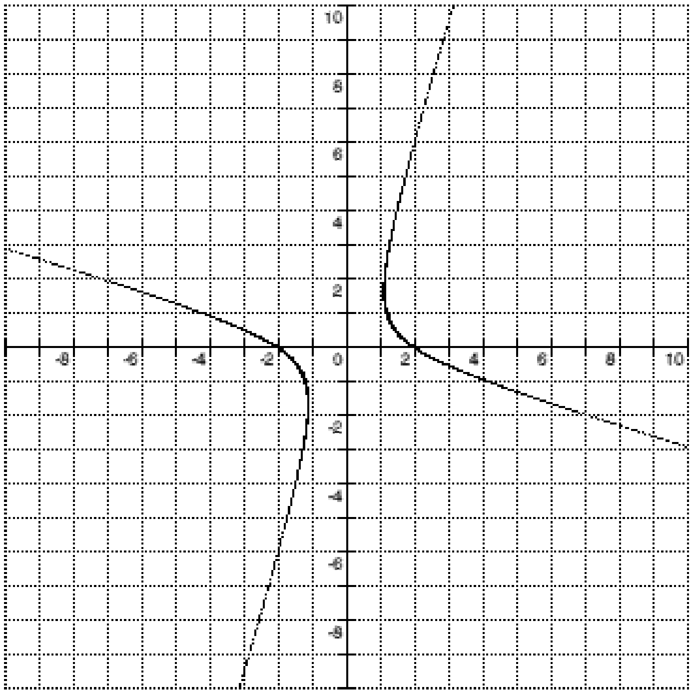
Find an equation of the normal to C at the point (0, 1), giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

[7 marks]

Chapter 17.

17.1 Tangent and Normal Implicitly

Example



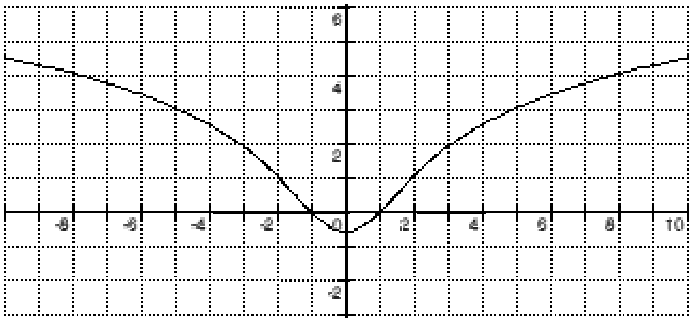
$$x^2 + 3xy - y^2 = 4$$

(i) Find the equation of the normal to the curve at (2, 6) in the form $y = mx + c$

(ii) Draw the normal on the graph above.

17.2 Exercise.

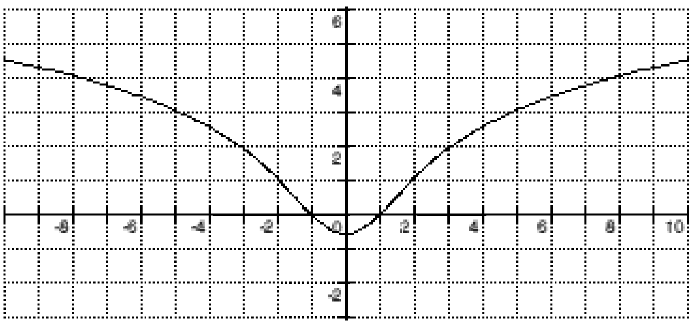
Question 1.



$$e^y + y = x^2$$

(i) Find the equation of the targent at (1, 0) in the form $y = mx + c$

(ii) Draw the tangent at (1, 0) on the graph below.



Question 2.

C4 examination question from January 2012, Q1.

The curve *C* has the equation

$$2x + 3y^2 + 3x^2 y = 4x^2$$

The point *P* on the curve has coordinates (-1, 1).

(a) Find the gradient of the curve at *P*.

[5 marks]

(b) Hence find the equation of the normal to *C* at *P*, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

[3 marks]

Question 3.

C4 examination question from January 2006, Q1.

The curve C is described by the equation

$$3x^2 + 4y^2 - 2x + 6xy - 5 = 0$$

Find an equation of the tangent to C at the point (1. -2), giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

[7 marks]

Question 4.

C4 examination question from June 2005, Q2.

A curve C has equation

$$x^2 + 2xy - 3y^2 + 16 = 0$$

Find the coordinates of the points on the curve where $\frac{dy}{dx} = 0$.

[7 marks]

Question 5.

C4 examination question from January 2008, Q5.

A curve C is described by the equation

$$x^3 - 4y^2 = 12xy$$

(a) Find the coordinates of the two points on the curve where $x = -8$

[3 marks]

(b) Find the gradient of the curve at each of these points.

[6 marks]

Question 6.

C4 examination question from June 2008, Q4.

A curve has equation

$$3x^2 - y^2 + xy = 4$$

The points P and Q lie on the curve.

The gradient of the tangent to the curve is $\frac{8}{3}$ at P and at Q

(a) Use implicit differentiation to show that $y - 2x = 0$ at P and at Q.

[6 marks]

(b) Find the coordinates of P and Q.

[3 marks]

Question 7.

C4 examination question from June 2011, Q5.

Find the gradient of the curve with equation

$$\ln y = 2x \ln x, \qquad x > 0, \qquad y > 0$$

at the point on the curve where $x = 2$.

Give your answer as an exact value.

[7 marks]

Question 8.

C4 examination question from June 2009, Q4.

The curve *C* has the equation

$$y e^{-2x} = 2x + y^2$$

(a) Find $\frac{dy}{dx}$ in terms of x and y.

[5 marks]

The point *P* on *C* has coordinates (0, 1).

(b) Find the equation of the normal to *C* at *P*, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

[4 marks]

Question 9.

C4 examination question from January 2010, Q3.

The curve C has the equation

$$\cos 2x + \cos 3y = 1, \qquad -\frac{\pi}{4} < x < \frac{\pi}{4}, \qquad 0 \leqslant y \leqslant \frac{\pi}{6}$$

(a) Find $\frac{dy}{dx}$ in terms of x and y.

[3 marks]

The point P lies on C where $x = \frac{\pi}{6}$

(b) Find the value of y at P.

[3 marks]

(c) Find the equation of the tangent to C at P, giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

[3 marks]

Question 10.

C4 examination question from January 2007, Q5.

A set of curves is given by the equation

$$\sin x + \cos y = 0.5$$

(a) Use implicit differentiation to find an expression for $\frac{dy}{dx}$

[2 marks]

For $-\pi < x < \pi$ and $-\pi < y < \pi$

(b) Find the coordinates of the points where $\frac{dy}{dx} = 0$

[5 marks]

For this final question you need to know that

$$If \qquad y = a^x \qquad Then \qquad \frac{dy}{dx} = (\ln a) \; a^x$$

Question 11.

C4 examination question from June 2010, Q3.

The curve C has equation

$$2^x + y^2 = 2xy$$

Find the exact value of $\frac{dy}{dx}$ at the point on C with coordinates (3. 2).

[7 marks]

Chapter 18.

Differentiation : Core 4

18.1 TEST

Question 1.

Differentiate each of the following;

(i) $y = \frac{2x^5}{15}$ **(ii)** $y = \ln(2x^3 + 7)$

(iii) $y = \frac{5}{(4x^2 - 3)}$ **(iv)** $y = e^{\sqrt{x}}$

[8 marks]

Question 2.

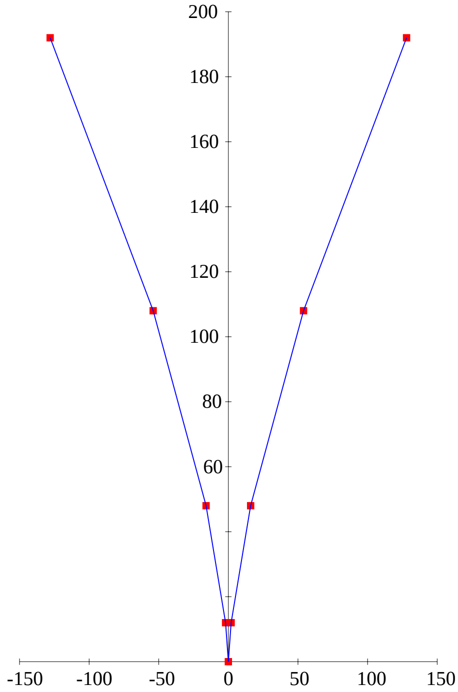
The following is an examination question from the January 2002 paper.

Use the derivatives of *sin x* and *cos x* to prove that the derivative of *tan x* is $sec^2 x$.

[4 marks]

Question 3.

A graph of a semi-cubical parabola is shown below.
It has parametric equations; $x = 2t^3$, $y = 12t^2$.



(i) Find, in terms of t,

(a) $\dfrac{dx}{dt}$ **(b)** $\dfrac{dy}{dt}$ **(c)** $\dfrac{dy}{dx}$

(ii) Write down the coordinates of the point on the curve that corresponds to the parameter t having the value 2.

(iii) What is the gradient of the curve at your part **(ii)** point ?

(iv) By making use of your part **(ii)** and **(iii)** answers, determine the equation of the tangent to the curve from the point at which $t = 2$.

[10 marks]

Question 4.

The famous curve known as the Folium of Descartes has equation;

$$y^3 + x^3 = 3xy$$

Differentiate Descartes' Folium by means of implicit differentiation.

Write your answer in the form $\frac{dy}{dx} = f(x, y)$.

[8 marks]

Question 5.

The parametric equations of a curve are;

$$x = t + \frac{1}{t}, \qquad y = t - \frac{1}{t}.$$

(i) Complete the following table by way of working out some points on the graph of this curve.

t	-5	-4	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{3}$	$-\frac{1}{4}$	$-\frac{1}{5}$
x									
y									

t	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3	4	5
x									
y									

(ii) On the graph paper provided opposite plot a graph of the curve.

(iii) What is the name of this type of curve ?

(iv) Find, in terms of *t*, an expression for the derivative of this curve.

[10 marks]

Chapter 19.

19.1 Connected Rates Of Change

Example

The radius of a circular coffee stain on a carpet is increasing at 0.4 mm.s^{-1}

(a) Write down the well known formulae for

(i) The circumference of a circle.

(ii) The area of a circle.

(b) Find the rate at which the circumference is increasing when the stain is 24mm.

(c) Find the rate at which the area is increasing when the circumference is 44mm.

19.2 Exercise

See separate sheets