QF4102 Assignment 3
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Question 1

(ii) Result of Monte Carlo Estimation (without control variate):

P	Mean Estimate	Standard Error
300	2.3153	0.044124
3000	2.2908	0.014465
30000	2.2865	0.0035253

Comment: As the number of price-path bundle increase, the standard error **decreases significantly** and the mean estimate of option value also become **more consistent**.

P	Mean Estimate	Standard Error
300000	2.2918	0.0011188

(iii) The suggested control variate is a combination of a European put on S_1 with weight of 0.75 and a European call on S_2 with weight of 0.25.

The payoff function of this control variate is defined to be:

$$0.75 \times \ max(X - S_{1T}, 0) + 0.25 \ \times \ \ max(S_{2T} - X, 0)$$

Rationale: Exact values of European put and call option can be calculated using BS_call and BS_put which makes the control variate useful since we can leverage on **known analytic solutions** for variance reduction. Also, we can treat this option as $0.5 \times \ max(X - S_{1T}, 0) + 0.5 \ \times \ \ max(S_{2T} - S_{1T}, 0)$ and $max(S_{2T} - S_{1T}, 0)$ can be treated as $max(S_{2T} - X, 0)$ & $max(X - S_{1T}, 0)$

(iv) Result of Monte Carlo Estimation (with control variate suggested in (iii))

P	Mean Estimate	Standard Error
300	2.267	0.019196
3000	2.2821	0.0054302
30000	2.2916	0.0015756

Comment: An additional test case with P=300000 is tabulated in (ii) **without control variate**, suppose this is the true value of the estimate of the two-asset option. We can observe that the convergence to true value **with control variate** is computationally faster. The standard errors are also **significantly reduced** for small sample sizes which is a direct result of such a <u>variance reduction technique</u>.

Question 2

(i)

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i) $\mathcal{L}_{BS} = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0$

For American,

$\mathcal{L}_V = 0 \Leftrightarrow -\mathcal{L}_V = 0$, if $V > p(S)$, continue holding

$\mathcal{L}_V \le 0 \Leftrightarrow -\mathcal{L}_V \ge 0$, if $V = p(S)$, early ex.

Transforming, $\tau = T - t \in [0, T]$

$x = \ln(S/X) \in (-\infty, \infty)$, $V(S_T, T) = (S_T - X)^+$

initial condition

$V(S,t) = X u(x, \tau)$ $\quad u(x = \ln(S/X), 0) = (S/X - 1)^+ = (e^x - 1)^+$

$V(S,T) = X u(x, 0)$,

$\frac{\partial x}{\partial S} = \frac{1}{S/X}(1/X) = 1/S$, $\quad \frac{\partial \tau}{\partial t} = -1$, $\quad \frac{\partial V}{\partial t} = X \frac{\partial u}{\partial \tau}\frac{\partial \tau}{\partial t} = -X\frac{\partial u}{\partial \tau}$

$\frac{\partial V}{\partial S} = X \frac{\partial u}{\partial x}\frac{\partial x}{\partial S}$ $\qquad \frac{\partial^2 V}{\partial S^2} = \frac{X}{S}\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial S} + \frac{\partial u}{\partial x}(-X/S^2)$

$= X/S \frac{\partial u}{\partial x}$ $\qquad = \frac{X}{S^2}\frac{\partial^2 u}{\partial x^2} - \frac{X}{S^2}\frac{\partial u}{\partial x}$

$\mathcal{L}_V = -X\frac{\partial u}{\partial \tau} + \frac{1}{2}\sigma^2 \cancel{S^2}\left(\frac{X}{\cancel{S^2}}\frac{\partial^2 u}{\partial x^2} - \frac{X}{\cancel{S^2}}\frac{\partial u}{\partial x}\right) + (r-q)\cancel{S}\frac{X}{\cancel{S}}\frac{\partial u}{\partial x} - rXu$

$= -X\left(\frac{\partial u}{\partial \tau} - \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2} - (r - q + \frac{1}{2}\sigma^2)\frac{\partial u}{\partial x} + ru\right)$

$\mathcal{L}_V = 0 = \frac{\partial u}{\partial \tau} - \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2} - (r - q + \frac{1}{2}\sigma^2)\frac{\partial u}{\partial x} + ru$, continue

$-\mathcal{L}_V \ge 0 \Leftrightarrow \frac{\partial u}{\partial \tau} - \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2} - (r - q + \frac{1}{2}\sigma^2)\frac{\partial u}{\partial x} + ru \ge 0$, early ex.

set $x_{min} < 0$, where $S_{min} \in (0, X)$ $\qquad$ sol domain $\Rightarrow x \in [x_{min}, x_{max}]$

$x_{max} > 0$, where $S_{max} \in (X, \infty)$

$\Delta x = \frac{x_{max} - x_{min}}{I}$, $\Delta t = \frac{T}{N}$

$u_n^i = u(x_n^i, t_n)$, $x_n^i = x_{min} + i\Delta x$

if $S << X$, $u = 0$

$S >> X$, $u = e^x - 1$

if

$V_n^i = X u(x, \tau) > (S - X) = p(S)$

$u_n^i(x, \tau) > S/X - 1 = e^x - 1$ then continue.

otherwise early ex. and $V_n^i = e^x - 1$.

(ii)

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$$\frac{\partial u}{\partial \tau} - \frac{1}{2}\theta^2 \frac{\partial^2 u}{\partial x^2} - \left(r - q + \frac{1}{2}\theta^2\right)\frac{\partial u}{\partial x} + ru = 0$$

backward time : $\frac{\partial u}{\partial \tau} = \frac{u_n^i - u_{n-1}^i}{\Delta t} + O(\Delta t)$

forward space : $\frac{\partial u}{\partial x} = \frac{u_n^{i+1} - u_n^i}{\Delta x} + O(\Delta x)$

centred space : $\frac{\partial^2 u}{\partial x^2} = \frac{u_n^{i-1} - 2u_n^i + u_n^{i+1}}{\Delta x^2} + O(\Delta x^2)$

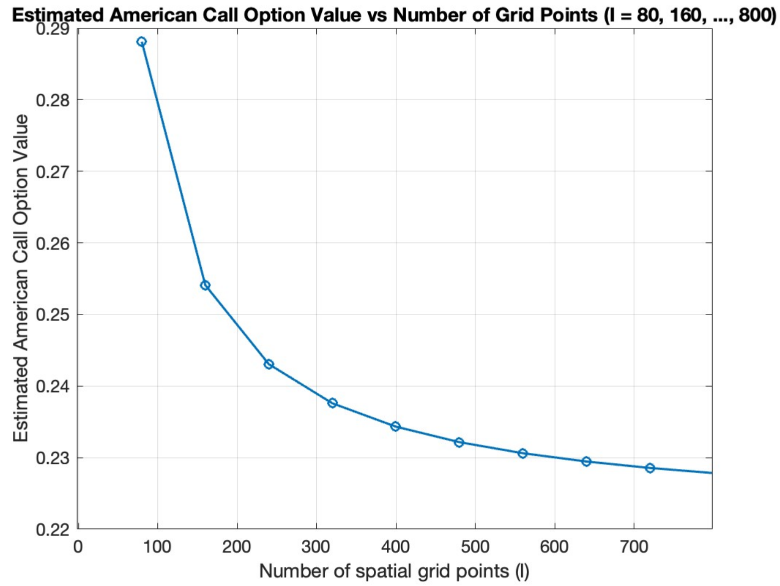
for $i = 1, 2, \ldots I-1$

$$u_{n-1}^i = \left(1 - r\Delta\tau - \frac{\Delta\tau}{\Delta x}\left(r - q + \frac{1}{2}\theta^2\right) - \theta^2\frac{\Delta\tau}{\Delta x^2}\right)u_n^i$$
$$+ \frac{\Delta\tau}{\Delta x}\left(\frac{1}{2\Delta x}\theta^2 + r - q + \frac{1}{2}\theta^2\right)u_n^{i+1} + \frac{1}{2}\theta^2\frac{\Delta\tau}{\Delta x^2}u_n^{i-1}$$

terminal condition / payoff.

$$(S - X)^+ = X(S/X - 1)^+$$
$$= X(e^x - 1)^+$$

(v)



Comment: As the number of spatial grid point increases, the estimated American call option value **converges** to a stable value, this implies that the numerical solution is approaching the true solution as the grid is refined. The option value is higher than the stable value when the number of spatial grid point is small as $xmax = 4 \Rightarrow S_{max} = 54.5$, and S_{max} accounts more of the option value, as the number of spatial grid points increase, the significance of S_{max} decreases.