Test 2

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1 Problem 1

Claim: Let $G = (W, E)$ be a (simple, undirected) graph with $|W| \geq 5$. Then there are two distinct subsets of nodes $U, V \subseteq W$ with $|U| = |V| = 2$ such that $\sum_{x \in U} deg(x) = \sum_{x \in V} deg(x)$.

Proof: We will prove this claim using the Pigeonhole Principle.

- Let the pigeons (A) be the set of all subsets of W with exactly two elements.
- Let the pigeonholes (B) be the set of all possible sums of degrees of nodes in subsets of W with exactly two elements. Thus, $B := \{0, 1, 2, ..., 2|W| - 1\}$
- Let $f : A \to B$ be defined by $f(T) := \sum_{x \in T} deg(x)$. Note that f is a well defined function:
  (1) for any pigeon $T \in A$, $f(T) = \sum_{x \in T} deg(x)$ is computable because T is a finite set and $deg(x)$ is defined for all $x \in W$.
  (2) for any pigeon $T \in A$, if $f(T_1) = b$ and $f(T_2) = c$ then $b = c$ because $f(T_1) = \sum_{x \in T_1} deg(x)$ and $f(T_2) = \sum_{x \in T_2} deg(x)$, and since T_1 and T_2 are subsets of W, $\sum_{x \in T_1} deg(x) = \sum_{x \in T_2} deg(x)$.
  (3) for any pigeon $T \in A$, $f(T) = \sum_{x \in T} deg(x)$ is within the codomain B because T is a subset of W and $deg(x)$ is defined for all $x \in W$.

	Mathematical Reasoning	**Reason this Statement is True (From the Approved List)**
$\Longrightarrow$	$\lvert A\rvert = \binom{\lvert W\rvert(\lvert W\rvert-1)}{2}$, where $\lvert W\rvert \geq 5$	Because $A = \binom{\lvert W\rvert}{2}$
$\Longrightarrow$	$\lvert B\rvert = 2\lvert W\rvert - 1$, where $\lvert W\rvert \geq 5$	Because $B = \{0, 1, 2, ..., 2\lvert W\rvert - 1\}$
$\Longrightarrow$	$\lvert A\rvert > \lvert B\rvert$	Since $\binom{\lvert W\rvert(\lvert W\rvert-1)}{2} > 2\lvert W\rvert - 1$
$\Longrightarrow$	$\exists a_1 a_2 \in A : [(a_1 \neq a_2) \wedge (f(a_1) = f(a_2))]$	By the Pigeonhole Principle
$\Longrightarrow$	$\exists U, V \subseteq W : [(U \neq V) \wedge (\lvert U\rvert = \lvert V\rvert = 2) \wedge (\sum_{x \in U} deg(x) = \sum_{x \in V} deg(x))]$	By def of A and f

$\square$

2 Problem 2

Recall the following definitons from lecture about a function $g : A \to B$:

one to one: $\forall n, m \in A : (n \neq m) \implies (g(n) \neq g(m))$

onto: $\forall b \in B : \exists a \in A : g(a) = b$

Claim: Let $f : \mathbb{N} \to \mathbb{Z}$ be defined by $f(n) := \sum_{v \in K_n} deg(v)$, where K_n is the complete graph on n nodes.

(a) For all $x \in S$, we want to show that if $P(x)$ is true then $Q(x)$ is also true.

(b) $\neg Q(x) \implies \neg P(x)$.

(c) To prove $P(x) \implies Q(x)$ by contrapositive we assume $\neg Q(x)$ and use that to show $\neg P(x)$.

(d) $\forall n, m \in \mathbb{N} : (n \neq m) \implies (f(n) \neq f(m))$

(e) **Proof:** We want to show $\forall n, m \in \mathbb{N} : (n \neq x) \implies (f(n) \neq f(m))$. To prove this statement, we want to show that for any $n, m \in \mathbb{N}$, if $n \neq m$ then $f(n) \neq f(m)$. We will prove the equivalent contrapositive, that is, $\exists n, m \in \mathbb{N} : f(n) = f(m) \implies n = m$. To do this we will assume $f(n) = f(m)$ and use that to show $n = m$.

	Mathematical Reasoning	**Reason this Statement is True (From the Approved List)**
$\implies$	$f(n) = f(m)$	Given
$\implies$	$\sum_{v \in K_n} \deg(v) = \sum_{v \in K_m} \deg(v)$	By the definition of f
$\implies$	$n(n-1) = m(m-1)$	By the definition of K_n and K_m, WLOG $\sum_{v \in K_n} deg(v) = 2\lvert E\rvert$, where $\lvert E\rvert = \frac{n(n-1)}{2}$, thus by algebra $\sum_{v \in K_n} deg(v) = \lvert E\rvert$
$\implies$	$n^2 - n = m^2 - m$	By algebra
$\implies$	$n^2 - n - m^2 + m = 0$	By algebra
$\implies$	$(n-m)(n+m-1) = 0$	Since $n^2 - n - m^2 + m = (n-m)(n+m-1)$ (By algebra)
$\implies$	$n - m = 0$	Since $n, m \in \mathbb{N}$, $n + m - 1 \neq 0$
$\implies$	$n = m$	By algebra

Thus, we have shown that for any $n, m \in \mathbb{N}$, if $f(n) = f(m)$ then $n = m$. $\square$

(f) **Counter Example:** To prove f is not onto, we want to show the negation of the definition of onto. That is, we want to show that $\exists b \in \mathbb{Z} : \forall a \in \mathbb{N} : f(a) \neq b$. Consider $b = 0, b \in \mathbb{Z}$. In a complete graph K_n, there exists no $f(a) = \sum_{v \in K_a} deg(v)$. Meaning there is no $a \in \mathbb{N}$ that a maps to $b \in \mathbb{Z}$. Thus f is not onto.

3 Problem 3

Consider the following sequence of numbers similar to (But not the same as) the Sharp numbers.

$$d_1 = 2$$
$$d_2 = 4$$
$$d_n = d_{n-1} + 2 \cdot d_{n-2}, \text{ for } n \geq 3$$

Claim: For all $n \geq 1$, $d_n = 2^n$

Step 0: For all $n \geq 1$, we want to show that $d_n = 2^n$.

Step 1: For any $n \geq 1$, let $P(n)$ be the property that $d_n = 2^n$. We want to show $\forall n \geq 1 : P(n)$.

Step 2: As base cases consider when

$n = 1$. We will show that $P(1)$ is true: that is, that $d_1 = 2^1$. Fortunately,

$$\text{left hand side} = d_1 = 2 = 2^1 = \text{right hand side}$$

$n = 2$. We will show that $P(2)$ is true: that is, that $d_2 = 2^2$. Fortunately,

$$\text{left hand side} = d_2 = 4 = 2^2 = \text{right hand side}$$

Step 3: Let $k \geq 2$. For the induction hypothesis, suppose that $P(1), .., P(k)$ are true, or equivalently, that for all $1 \leq k' \leq k : P(k')$. That is, suppose that

$$\forall 1 \leq k' \leq k : d_{k'} = 2^{k'}$$

Step 4: Now we prove that $P(k+1)$ is true, using our induction assumptions that $P(1), .., P(k+1)$ are true. That is, we prove that

$$d_{k+1} = 2^{k+1}$$

Step 5: The proof that $P(k+1)$ is true (given that $P(1), ..., P(k)$ are true) is as follows:

$$\begin{array}{lcll}
\text{Left hand side of } P(k) & = & d_{k+1} & \\
 & = & d_k + 2 \cdot d_{k-1} & \text{By def of sequence} \\
 & = & 2^k + 2 \cdot 2^{k-1} & \text{By IH since } 1 \leq k-1 \leq k \\
 & = & 2^k + 2^k & \text{By algebra} \\
 & = & 2 \cdot 2^k & \text{By algebra} \\
 & = & 2^{k+1} & \text{By algebra} \\
 & = & \text{Right hand side of } P(k+1) &
\end{array}$$

Step 6: The steps above have shown that for any $k \geq 2$, if $P(1), ..., P(k)$ are true, then $P(k+1)$ is also true. Combined with the base cases which show that $P(1)$ and $P(2)$ are true, we have shown that for all $n \geq 1$, $P(n)$ is true, as desired. $\square$