Problem Set 7

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1 Problem 1

Consider the algorithm below, which takes an $n \geq 0$ and finds it remainder when divided by $c \geq 1$

function $\textsc{Remainder}(n)$:
  if $n \leq c - 1$ **then**
    return n
  else
    return Remainder$(n - c)$
  end if
end function

Claim: Let $c \geq 1$. For any $n \geq 0$, **remainder**$(n) = n \mod c$.

Step 0: For $n \geq 0$, we want to show that **remainder**$(n) = n \mod c$.

Step 1: For any $n \geq 0$, Let $P(n)$ be the property that **remainder**$(n) = n \mod c$.

We want to show that $P(n)$ is true for all $n \geq 0$.

Step 2: As base cases, consider when

$n = 0$. We will show that $P(0)$ is true: that is **remainder**$(0) = 0 \mod c$. Fortunately, this is true since $c \geq 1$ and in the algorithm, if $n \leq c - 1$, and in this case $n = 0$, $c \geq 1$. This implies $0 \leq 1 - 1 = 0 \leq 0$ which is true, then **remainder**$(n) = n$. Thus, **remainder**$(0) = 0$, so RHS $= 0$. Also, $0 \mod c = 0$, so LHS $= 0$. Thus, LHS $=$ RHS, so $P(0)$ is true.

Additionally, for any $n \leq c - 1$, **remainder**$(n) = n = (n \mod c)$, this is true since due to the definition of the algorithm.

Step 3: Let $k \geq 1$. For the induction hypothesis, suppose that $P(0), \ldots, P(k)$ are true, or equivalently, that for all $0 \leq k' \leq k : P(k')$. That is, suppose that $\mathbf{remainder}(k') = k' \mod c$.

Step 4: Now we prove that $P(k+1)$ is true, using our induction assumptions that $P(0), \ldots, P(k)$ are true. That is, we prove that $\mathbf{remainder}(k+1) = (k+1) \mod c$.

Step 5: If $k+1 \leq c-1$, then $\mathbf{remainder}(k+1) = k+1 = (k+1 \mod c)$ by definition of algorithm. Otherwise the proof that $P(k+1)$ is true (given that $P(0), \ldots, P(k)$ are true) is as follows:

$$
\begin{array}{rcll}
\text{Left hand side of } P(k+1) & = & \mathbf{remainder}(k+1) & \\
 & = & \mathbf{remainder}((k+1)-c) & \text{By def of algorithm, since } k+1 \geq c-1 \\
 & = & ((k+1)-c) \mod c & \text{By IH, since } 0 \leq (k+1)-c \leq k \\
 & = & (k+1) \mod c - c \mod c & \text{By def of mod} \\
 & = & (k+1) \mod c & \text{Since } c \geq 1,\ c \mod c = 0 \\
 & = & \text{Right hand side of } P(k+1) &
\end{array}
$$

Step 6: The steps above have shown that for any $k \geq 1$, if $P(0), \ldots, P(k)$ are true, then $P(k+1)$ is also true. Combined with the base cases, which show that $P(0)$ are true, we have shown that for all $n \geq 0$, $P(n)$ is true, as desired.

2 Problem 2

Claim: Let $n, c \geq 1$ and $c \leq n$. The number of simple paths of length c in the complete graph on n nodes is $\frac{n!}{(n-c-1)!}$ which is equal to $n(n-1)\cdots(n-c)$.

complete graph K_n: an undirected graph on n nodes with an edge between every pair of nodes.

simple path: a sequence of distinct nodes with edges between consecutive nodes in the sequence.

length of a path: the number of *edges* in the path (**not** number of nodes).

Step 0: For all $c \geq 1$, we want to show that the number of simple paths of length c in the complete graph on n nodes is $\frac{n!}{(n-c-1)!}$ which is equal to $n(n-1)\cdots(n-c)$.

Step 1: For any $c \geq 1$, Let $P(c)$ be the property that the number of simple paths of length c in the complete graph on n nodes is $\frac{n!}{(n-c-1)!}$ which is equal to $n(n-1)\cdots(n-c)$.

Step 2: As base cases, consider when $c = 1$. We will show that $P(1)$ is true: that is the number of simple paths of length 1 in the complete graph on n nodes is $\frac{n!}{(n-1-1)!}$ which is equal to $n(n-1)$. Fortunately, this is true since the number of simple paths of length 1 in the complete graph on n nodes is $n(n-1)$, and $\frac{n!}{(n-1-1)!} = n!$, and $n! = n(n-1)$. Thus, the number of simple paths of length 1 in the complete graph on n nodes is $\frac{n!}{(n-1-1)!}$ which is equal to $n(n-1)$, so LHS = RHS. Thus, $P(1)$ is true.

Step 3: For the induction hypothesis, suppose (hypothetically) that $P(k)$ is true for some fixed $k \geq 1$. That is, suppose that the number of simple paths of length k in the complete graph on n nodes is $\frac{n!}{(n-k-1)!}$ which is equal to $n(n-1)\cdots(n-k)$.

Step 4: Now we prove that $P(k+1)$ is true, using our induction assumptions that $P(k)$ is true. That is, we prove that the number of simple paths of length $k+1$ in the complete graph on n nodes is $\frac{n!}{(n-(k+1)-1)!}$ which is equal to $n(n-1)\cdots(n-(k+1))$.

Step 5: The proof that $P(k+1)$ is true (given that $P(k)$ is true) is as follows:

Left hand side of $P(k+1)$	$=$	The number of simple paths of length $k+1$ in K_n	
	$=$	The number of simple paths of length k in K_n * $(n-(k+1))$	Since there are $k+1$ nodes in length k path
	$=$	$\frac{n!}{(n-k-1)!} \cdot (n-(k+1))$	By IH
	$=$	$\frac{n!}{(n-k-1)!} \cdot (n-k-1)$	By algebra
	$=$	$\frac{n!}{(n-k-2)!}$	By def of factorial
	$=$	$\frac{n!}{(n-(k+1)-1)!}$	By algebra
	$=$	Right hand side of $P(k+1)$	

Step 6: The steps above have shown that if $P(k)$ is true, then $P(k+1)$ is also true. Combined with the base case, which shows that $P(1)$ is true, we have shown that for all $c \geq 1$, $P(c)$ is true, as desired.

3 Problem 3

Recall the Fibonacci numbers, as defined by:

$$
\begin{aligned}
&f_1 = 1 \\
&f_2 = 1 \\
&f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 3
\end{aligned}
$$

Recall the Sharp numbers from PS6, as defined by:

$$
\begin{aligned}
&s_1 = 2 \\
&s_2 = 4 \\
&s_n = s_{n-1} + s_{n-2} \text{ for } n \geq 3
\end{aligned}
$$

Claim: For all $n \geq 3$, $s_n = 4 \cdot f_{n-1} + 2 \cdot f_{n-2}$.

Step 0: For $n \geq 3$, we want to show that $s_n = 4 \cdot f_{n-1} + 2 \cdot f_{n-2}$.

Step 1: For any $n \geq 3$, Let $P(n)$ be the property that $s_n = 4 \cdot f_{n-1} + 2 \cdot f_{n-2}$. We want to show that $P(n)$ is true for all $n \geq 3$.

Step 2: As base cases, consider when

$n = 3$. We will show that $P(3)$ is true: that is $s_3 = 4 \cdot f_{3-1} + 2 \cdot f_{3-2}$. Fortunately, this is true since $s_3 = s_2 + s_1 = 4 + 2 = 6$ and $4 \cdot f_{3-1} + 2 \cdot f_{3-2} = 4 \cdot f_2 + 2 \cdot f_1 = 4 \cdot 1 + 2 \cdot 1 = 6$. Thus, $s_3 = 4 \cdot f_{3-1} + 2 \cdot f_{3-2}$, so LHS = RHS. Thus, $P(3)$ is true.

$n = 4$. We will show that $P(4)$ is true: that is $s_4 = 4 \cdot f_{4-1} + 2 \cdot f_{4-2}$. Fortunately, this is true since $s_4 = s_3 + s_2 = 6 + 4 = 10$ and $4 \cdot f_{4-1} + 2 \cdot f_{4-2} = 4 \cdot f_3 + 2 \cdot f_2 = 4 \cdot 2 + 2 \cdot 1 = 10$. Thus, $s_4 = 4 \cdot f_{4-1} + 2 \cdot f_{4-2}$, so LHS = RHS. Thus, $P(4)$ is true.

Step 3: Let $k \geq 4$. For the induction hypothesis, suppose that $P(3), P(4), \ldots, P(k)$ are true, or equivalently, that for all $3 \leq k' \leq k : P(k')$. That is, suppose that $s_{k'} = 4 \cdot f_{k'-1} + 2 \cdot f_{k'-2}$.

Step 4: Now we prove that $P(k+1)$ is true, using our induction assumptions that $P(3), P(4), \ldots, P(k)$ are true. That is, we prove that $s_{k+1} = 4 \cdot f_{k+1-1} + 2 \cdot f_{k+1-2}$.

Step 5: The proof that $P(k+1)$ is true (given that $P(3), P(4) \ldots, P(k)$ are true) is as follows:

$$
\begin{array}{rcll}
\text{Left hand side of } P(k+1) & = & s_{k+1} & \\
& = & s_k + s_{k-1} & \text{By def of sequence} \\
& & & \text{(Sharp numbers)} \\
& = & (4 \cdot f_{k-1} + 2 \cdot f_{k-2}) + (4 \cdot f_{k-2} + 2 \cdot f_{k-3}) & \text{By IH} \\
& = & 4 \cdot f_{k-1} + 2 \cdot f_{k-2} + 4 \cdot f_{k-2} + 2 \cdot f_{k-3} & \text{By algebra} \\
& = & 4 \cdot f_{k-1} + 4 \cdot f_{k-2} + 2 \cdot f_{k-2} + 2 \cdot f_{k-3} & \text{By rewriting} \\
& = & 4(f_{k-1} + f_{k-2}) + 2(f_{k-2} + f_{k-3}) & \text{By factoring and algebra} \\
& = & 4 \cdot f_k + 2 \cdot f_{k-1} & \text{By def of sequence} \\
& & & \text{(Fibonacci numbers)} \\
& = & \text{Right hand side of } P(k+1) &
\end{array}
$$

Step 6: The steps above have shown that for any $k \geq 4$, if $P(3), P(4), \ldots, P(k)$ are true, then $P(k+1)$ is also true. Combined with the base cases, which show that $P(3), P(4)$ are true, we have shown that for all $n \geq 3$, $P(n)$ is true, as desired.