Problem Set 8

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1 Problem 1

Claim: Let $G = (V, E)$ be any undirected graph, and let $f : V \to \{1, 2, ..., k\}$ be a k-colouring of G, where $k = \chi(G)$. The binary relation $\equiv_f$ on V defined by

$$u \equiv_f v \text{ if and only if } f(u) = f(v)$$

(a)

	Mathematical Reasoning	**Reason this Statement is True (From the Approved List)**
WTS	$\forall v \in V : v \equiv_f v$	By def of reflexive
Consider any $v \in V$. WTS	$v \equiv_f v$	By rephrasing
	$f(v) = f(v)$	Since = is reflexive
$\Rightarrow$	$v \equiv_f v$	By definition $\equiv_f$

(b)

	Mathematical Reasoning	**Reason this Statement is True (From the Approved List)**
WTS	$\forall v, u \in V$, *if* $u \equiv_f v$ then $v \equiv_f u$	By def of symmetric
Consider any $u, v \in V$. Suppose	$u \equiv_f v$	By this is the antecedent
WTS	$v \equiv_f u$	This is the consequent
$\Rightarrow$	$f(u) = f(v)$	By definition of $u \equiv_f v$
$\Rightarrow$	$f(v) = f(u)$	Since = is symmetric
$\Rightarrow$	$v \equiv_f u$	By def of $\equiv_f$

(c)

	Mathematical Reasoning	**Reason this Statement is True (From the Approved List)**
WTS	$\forall v, u, w \in V$, *if* $u \equiv_f v \wedge v \equiv_f w \Rightarrow u \equiv_f w$	By def of transitive
Consider any $u, v, w \in V$. Suppose	$u \equiv_f v \wedge v \equiv_f w$	By this is the antecedent
WTS	$u \equiv_f w$	This is the consequent
$\Rightarrow$	$f(u) = f(v) \wedge f(v) = f(w)$	By def of $u \equiv_f v \wedge v \equiv_f w$
$\Rightarrow$	$f(u) = f(w)$	By transivity of $=$
$\Rightarrow$	$u \equiv_f w$	By definition of $\equiv_f$

(d)

There are c equivalence classes of $\equiv_f$.

Informally, the equivalence classes are: the sets of vertices that have the same colour.

In precise mathematical language, the equivalence classes of $\equiv_f$ are:

$$\{v \in V : f(v) = 1\}, \{v \in V : f(v) = 2\}, ..., \{v \in V : f(v) = c\}$$

The proof that these are the equivalence classes is as follows:

	Mathematical Reasoning	**Reason this Statement is True (From the Approved List)**
		By def of equivalence class
$\Rightarrow$		
$\Rightarrow$		

2 Problem 2

Claim: Suppose a graph $G = (V, E)$ is 2-colourable using the colouring $f : V \to \{0, 1\}$. Then for any path Q, the length of Q has a parity $|f(u) - f(v)|$, where u and v are the endpoints of Q.

Step 0: For all $\text{len}(Q) \geq 0$, we want to show that $\text{len}(Q)$ has a parity of $|f(u) - f(v)|$, where u and v are the endpoints of Q.

Step 1: For any $n \geq 0$, let $P(n)$ be the property that for all paths of length n, $\text{parity}(n) = |f(u) - f(v)|$, where u and v are the path's endpoints.

Step 2: As a base case, consider when $n = 0$. We will show that $P(0)$ is true: that is, that $\text{parity}(0) = |f(u) - f(v)|$. Consider any path of length 0. We want to show $\text{parity}(0) = |f(u) - f(v)|$. Fortunately, since this path has no edges, the endpoints are the same node. Therefore, $\text{parity}(0) = |f(u) - f(v)|$ is true.

Step 3: Let $k \geq 0$. For the induction hypothesis, suppose $P(k)$ is true. That is, suppose that for all paths of length k, $\text{parity}(k) = |f(u) - f(v)|$, where u and v are the path's endpoints.

Step 4: Now we prove that $P(k+1)$ is true, using the (hypothetical) induction assumption that $P(k)$ is true. That is, we prove for all paths of length $k+1$, $\text{parity}(k+1) = |f(u) - f(v)|$, where u and v are the path's endpoints.

Step 5: The proof that $P(k+1)$ is true (given that $P(k)$ is true) is as follows:

Consider any path of length $k+1$. We want to show that $\text{parity}(k+1) = |f(u) - f(v)|$. This path can be split into two paths: one of length k and one of length 1.

There are two cases:
$|f(u) - f(v)| = 1$:

$\Rightarrow$	$\text{parity}(k+1)$	LHS of $P(k+1)$
$\Rightarrow$	$\text{parity}(k) + 1$	By def of parity
$\Rightarrow$	$\lvert f(u) - f(v)\rvert + 1$	By IH
$\Rightarrow$	$1 + 1$	By def of $\lvert f(u) - f(v)\rvert$
$=$	2	By algebra
$\Rightarrow$	even	By def of even

$|f(u) - f(v)| = 0$

$\Rightarrow$	$\text{parity}(k+1)$	LHS of $P(k+1)$
$\Rightarrow$	$\text{parity}(k) + 1$	By def of parity
$\Rightarrow$	$\lvert f(u) - f(v)\rvert + 1$	By IH
$\Rightarrow$	$0 + 1$	By def of $\lvert f(u) - f(v)\rvert$
$=$	1	By algebra
$\Rightarrow$	odd	By def of odd

Therefore we have shown that *if* $P(k)$ is true, *then* $P(k+1)$ is true, for all $k \geq 0$.

Step 6: The steps above have shown that for any $k \geq 0$, if $P(k)$ is true, then $P(k+1)$ is also true. Combined with the base case, which shows that $P(0)$ is true, we have shown that for all $n \geq 0$, $P(n)$ is true, as desired.

3 Problem 3

Consider the algorithm below, which takes as input a connected graph $G = (V, E)$ and tries to colour it with two colours T, F (for "true" and "false").

Algorithm 1: two-colour($G = (V, E)$):

`// Initialization`

Pick an arbitrary element $u_0 \in V$, label it T (i.e. set $f(u_0) = T$);

Set $i \leftarrow 1$;

`// Colouring Process`

while *there is an unlabeled* $v \in V$ *with a labeled neighbor* w **do**

    Label v with $\neg f(w)$ (i.e. set $f(u) = \neg f(w)$) and set $u_i = v$;

    Update $i \leftarrow i + 1$;

`// Output`

return f;

Claim: For any connected graph $G = (V, E)$, if G has no odd-length cycles, then G is bipartite.

Contrapositive: For any connected graph $G = (V, E)$, if G is not bipartite, then G has an odd-length cycle.

	Mathematical Reasoning	**Reason this Statement is True (From the Approved List)**
We want to show	For any connected graph, if G is not bipartite, the G has an odd-length cycle	Since this is the contrapositive of our claim
Suppose	G is a connected graph that is not bipartite	By assumption
$\Rightarrow$		
$\Rightarrow$		
$\Rightarrow$		
$\Rightarrow$		
$\Rightarrow$		