Problem Set 6

Parvesh Adi Lachman

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1 Problem 1

Claim: For any integer $n \geq 1$,

$$\sum_{i=1}^{n} i(i+3) = \frac{n(n+1)(n+5)}{3}. \tag{1}$$

Step 0: For all $n \geq 1$, we want to show $\sum_{i=1}^{n} i(i+3) = \frac{n(n+1)(n+5)}{3}$.

Step 1: For all $n \geq 1$, let $P(n)$ be the property that

$$4 + 10 + 18 + 28 + \cdots + (n-1)((n-1)+3) + n(n+3) = \frac{n(n+1)(n+5)}{3}.$$

We want to show that $P(n)$ is true for all $n \geq 1$.

Step 2: As a base case, consider $n = 1$. We will show that $P(1)$ is true: that is, that $1(1+3) = \frac{1(1+1)(1+5)}{3}$. Fortunately, this is true since LHS $= 1(1+3) = 4$ and RHS $= \frac{1(1+1)(1+5)}{3} = 4$ thus LHS = RHS.

Step 3: For the induction hypothesis, suppose (hypothetically) that $P(k)$ were true for some fixed $k \geq 1$. That is suppose that $\sum_{i=1}^{k} i(i+3) = \frac{k(k+1)(k+5)}{3}$, which is $[4 + 10 + 18 + 28 + \cdots + (k-1)((k-1)+3) + k(k+3)] = \frac{k(k+1)(k+5)}{3}$.

Step 4: Now we prove that $P(k+1)$ is true, using the (hypothetical) induction assumption that $P(k)$ is true. That is, we prove that $\sum_{i=1}^{k+1} i(i+3) = \frac{(k+1)(k+2)(k+6)}{3}$ which is $[4 + 10 + 18 + 28 + \cdots + k(k+3) + (k+1)((k+1)+3)]$.

Step 5: The proof that $P(k+1)$ is true (given that $P(k)$ is true) is a follows:

$$\begin{array}{rcll}
\text{Left hand side of } P(k+1) & = & 4+10+18+28+\cdots+k(k+3) & \\
& + & (k+1)((k+1)+3) & \\
& = & (4+10+18+28+\cdots+k(k+3)) & \\
& + & (k+1)((k+1)+3) & \\
& = & \frac{k(k+1)(k+5)}{3}+(k+1)((k+1)+3) & \text{By IH} \\
& = & \frac{k(k+1)(k+5)}{3}+\frac{3(k+1)(k+4)}{3} & \text{By algebra} \\
& = & \frac{k(k+1)(k+5)+3(k+1)(k+4)}{3} & \text{By algebra} \\
& = & \frac{(k+1)(k^2+5k)+(k+1)(3k+12)}{3} & \text{By algebra} \\
& = & \frac{(k+1)(k^2+8k+12)}{3} & \text{By factoring } (k+1) \\
& & & \text{and algebra} \\
& = & \frac{(k+1)(k+2)(k+6)}{3} & \text{By factoring} \\
& = & \text{Right hand side of } P(k+1) &
\end{array}$$

Therefore we have shown that *if* $P(k)$ is true, then $P(k+1)$ is also true for any $k \geq 1$.

Step 6: The steps above have shown that for any $k \geq 1$, if $P(k)$ is true, then $P(k+1)$ is also true. Combined with the base case, which shows that $P(1)$ is true, we have shown that for all $n \geq 1$, $P(n)$ is true, as desired.

2 Problem 2

Claim: Let $n \geq 2$, and let $A_1, A_2, \ldots, A_n$ be sets from some universal set U. For all $n \geq 2$,

$$\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{j=1}^{n} \overline{A_j} \tag{2}$$

Step 0: For all $n \geq 2$, we want to show that $\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{j=1}^{n} \overline{A_j}$

Step 1: For any $n \geq 2$, let $P(n)$ be the property that,

$$\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{j=1}^{n} \overline{A_j}$$

We want to show that $P(n)$ is true for all $n \geq 2$.

Step 2: As a base case, consider when $n = 2$. We will show that $P(2)$ is true: that is, that $\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$. Fortunately, this is true by definition of De Morgan's law, since De Morgan's law states that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Step 3: For the induction hypothesis, suppose (hypothetically) that $P(k)$ is true for some fixed $k \geq 2$. That is, suppose that $\overline{\bigcup_{i=1}^{k} A_i} = \bigcap_{j=1}^{k} \overline{A_j}$.

Step 4: Now we prove that $P(k+1)$ is true using the (hypothetical) induction assumption that $P(k)$ is true. That is, we prove that

$$\overline{\bigcup_{i=1}^{k+1} A_i} = \bigcap_{j=1}^{k+1} \overline{A_j}$$

Step 5: The proof that $P(k+1)$ is true (given that $P(k)$ is true) is as follows:

$$\begin{array}{rcll}
\text{Left hand side of } P(k+1) & = & \overline{\bigcup_{i=1}^{k+1} A_i} & \\
& = & \overline{\left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}} & \text{By pulling the (k+1)st term out of a union} \\
& = & \overline{\bigcup_{i=1}^{k} A_i} \cap \overline{A_{k+1}} & \text{By De Morgan's Law} \\
& = & \bigcup_{j=1}^{k} \overline{A_j} \cap \overline{A_{k+1}} & \text{By IH} \\
& = & \bigcup_{j=1}^{k+1} \overline{A_j} & \text{By pushing in the (k+1)st into the intersection} \\
& = & \text{Right hand side of } P(k+1) &
\end{array}$$

Therefore we have shown that *if* $P(k)$ is true, then $P(k+1)$ is also true for any $k \geq 2$.

Step 6: The steps above have shown that for any $k \geq 2$, if $P(k)$ is true, then $P(k+1)$ is also true. Combined with the base case, which shows that $P(2)$ is true, we have shown that for all $n \geq 2$, $P(n)$ is true, as desired.

3 Problem 3

Claim: For all $n \geq 1$,

$$\sum_{i=1}^{n} s_i = s_{n+2} - 4 \tag{3}$$

Step 0: For all $n \geq 1$, we want to show that $\sum_{i=1}^{n} s_i = s_{n+2} - 4$.

Step 1: For any $n \geq 1$, let $P(n)$ be the property that

$$\sum_{i=1}^{n} s_i = s_{n+2} - 4$$

We want to show that $P(n)$ is true for all $n \geq 1$.

Step 2: As a base case, consider when $n = 1$. We will show that $P(1)$ is true: that is, that $\sum_{i=1}^{1} s_i = s_{1+2} - 4$. Fortunately, this is true, using the sharp number sequence, $s_1 = 2$ and $s_{1+2} - 4 = s_3 - 4 = 6 - 4 = 2$, thus LHS = RHS.

Step 3: For the induction hypothesis, suppose (hypothetically) that $P(k)$ were true for some fixed $k \geq 1$. That is suppose that

$$\sum_{i=1}^{k} s_i = s_{k+2} - 4$$

Step 4: Now we prove $P(k+1)$ is true, using the (hypothetical) induction assumption that $P(k)$ is true. That is, we prove that

$$\sum_{i=1}^{k+1} s_i = s_{(k+1)+2} - 4 = \sum_{i=1}^{k+1} s_i = s_{k+3} - 4$$

Step 5: The proof that $P(k+1)$ is true (given that $P(k)$ is true) is as follows:

Left hand side of $P(k+1)$	$=$	$\sum_{i=1}^{k+1} s_i$	
	$=$	$\sum_{i=1}^{k} s_i + s_{k+1}$	By pulling out The (k+1)st term out the summation
	$=$	$s_{k+2} - 4 + s_{k+1}$	By IH
	$=$	$s_{k+2} + s_{k+1} - 4$	By rewriting
	$=$	$s_{k+3} - 4$	By def of our sequence
	$=$	Right hand side of $P(k+1)$	

Therefore we have shown that *if* $P(k)$ is true, then $P(k+1)$ is also true for any $k \geq 1$.

Step 6: The steps above have shown that for any $k \geq 1$, if $P(k)$ is true, then $P(k+1)$ is also true. Combined with the base case, which shows that $P(1)$ is true, we have shown that for all $n \geq 1$, $P(n)$ is true, as desired.