Problem Set 4

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1 Problem 1

Claim: Let $p, q \geq 1$ be integers. If $d \nmid n^p$ then $d \nmid n$.

a)

n	p	n^p	d	$d \mid n$	$d \mid n^p$	$d \nmid n$	$d \nmid n^p$
4	2	16	3	F	F	T	T
3	3	9	4	F	F	T	T
9	2	81	7	F	F	T	T

b) Contrapositive of the claim: Let $p, q \geq 1$ be integers. *if* $d \mid n$ then $d \mid n^p$.

c) Proof:

	Mathematical Reasoning	**Reason this Statement is True (From the Approved List)**
WTS	*If* $d \mid n$ then $d \mid n^p$	Since it is the contrapositive of the claim
Suppose	$d \mid n$	By assumption
$\Rightarrow$	$n = df, f \in \mathbb{Z}$	By def of divisible by d
$\Rightarrow$	$n^p = (df)^p, p \in \mathbb{Z}$	Putting both sides to the power of p
$\Rightarrow$	$n^p = d^p f^p$	By algebra
$\Rightarrow$	$n^p = d^p k, k \in \mathbb{Z}$	Since prod of ints is int
$\Rightarrow$	$n^p = d(d^{p-1})k$	By factoring d
$\Rightarrow$	$n^p = dck, c, k \in \mathbb{Z}$	Since prod of ints is int
$\Rightarrow$	$n^p = dv, v \in \mathbb{Z}$	Since prod of ints is int
$\Rightarrow$	n^p is divisible by d	by def of divisible by d
$\Rightarrow$	$d \mid n^p$	By def of divisible by d

d) The converse of the claim is: Let $p, q \geq 1$ be integers. If $d \nmid n$ then $d \nmid n^p$

e) The converse is false.

Suppose $d = 4, p = 2, n = 6$, Then $d \nmid n = 4 \nmid 6$, we know that $n^p = 6^2 = 36$. However $d \mid n^p = 4 \mid 36$, this is true since $36 \div 4 = 9$. This shows that the converse is false since an implication is only false when T $\Rightarrow$ F, which is what happens in the converse. Thus we can conclude that the converse is false.

2 Problem 2

Claim: Let $f : 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}$ be defined by $f(X) = \{x : x \in X \text{ and } x \text{ is even}\}$. Then f is a function. ($2^{\mathbb{Z}}$ is just alternate notation for $P(\mathbb{Z})$, the powerset of $\mathbb{Z}$).

a) $f(X) = X \cap 2\mathbb{Z}$

Reasoning: Since $x \in X$ and x is even, along with that we know that $2^{\mathbb{Z}}$ is the set of even integers. Thus the function, $f(X)$ should be the intersection between X and $2^{\mathbb{Z}}$.

b) Proof:

	Mathematical Reasoning	**Reason this Statement is True (From the Approved List)**
WTS	For each $X \in 2^{\mathbb{Z}}$ $f(X)$ is defined/computable	By def of property 1
$\Rightarrow$	$X \in 2^{\mathbb{Z}}$	By assumption
$\Rightarrow$	$X \cap 2\mathbb{Z} \in 2^{\mathbb{Z}}$	Since $(Y \cap Z) \subseteq Y = (Y \cap Z) \in 2^{Y}$
$\Rightarrow$	$f(X) \in 2^{\mathbb{Z}}$	By def of f

c) Proof:

	Mathematical Reasoning	**Reason this Statement is True (From the Approved List)**
Suppose	We have two sets A, B s.t. $A = B$, WTS $f(A) = f(B)$	By def of Property 2
$\Rightarrow$	$A = B$	By assumption
$\Rightarrow$	$f(A) = A \cap 2\mathbb{Z}$ and $f(B) = B \cap 2\mathbb{Z}$	By def of f
$\Rightarrow$	$f(A) = f(B)$	By substitution since $A = B$

d) Proof:

	Mathematical Reasoning	**Reason this Statement is True (From the Approved List)**
Suppose	$X \in 2^{\mathbb{Z}}$	By assumption
$\Rightarrow$	$X \subseteq \mathbb{Z}$	Since $\mathbb{Z} \in \mathcal{P}(\mathbb{Z})$
$\Rightarrow$	$X \cap 2\mathbb{Z} \subseteq \mathbb{Z}$	Since $2\mathbb{Z} \subseteq \mathbb{Z}$ and since $(Y \cap Z) \subseteq Y$ for any sets Y, Z
$\Rightarrow$	$f(X) \subseteq \mathbb{Z}$	By substitution
$\Rightarrow$	$f(X) \in 2^{\mathbb{Z}}$	Since $\mathbb{Z} \in \mathcal{P}(\mathbb{Z})$

e) Proof:

	Mathematical Reasoning	**Reason this Statement is True (From the Approved List)**
Suppose	$X \in 2^{\mathbb{Z}}$	By assumption
$\Rightarrow$	$X \subseteq \mathbb{Z}$	Since $\mathbb{Z} \in \mathcal{P}(\mathbb{Z})$
$\Rightarrow$	$X \subseteq 2\mathbb{Z}$	By def of f
$\Rightarrow$	$X \cap 2\mathbb{Z} \subseteq 2\mathbb{Z}$	Since $(Y \cap Z) \subseteq Y$ for any sets Y, Z
$\Rightarrow$	$f(X) \subseteq 2\mathbb{Z}$	By substitution
$\Rightarrow$	$f(X) \in 2^{2\mathbb{Z}}$	Since $\mathbb{Z} \in \mathcal{P}(\mathbb{Z})$

3 Problem 3

Claim: Let A, B, C be sets. If $f : A \to C$ is not a function then $B \not\subseteq C$ or $f : A \to B$ is not a function.

a) Contrapositive of the claim: Let A, B, C be sets. If $B \subseteq C$ and $f : A \to B$ is a function then $f : A \to C$ is a function.

b) **Properties of the antecedent**:

- For each $a \in A, f(a)$ is computable/defined because $\forall a \in A, \exists b \in B : f(a) = b$
- For each $a \in A, f(a)$ does not produce two different outputs because $\forall a \in A, \forall b_1, b_2 \in B$ if $f(a) = b_1$ and $f(a) = b_2$ then $b_1 = b_2$
- For each $a \in A, f(a) \in B$

c) **Properties of the consequent**:

- For each $a \in A, f(a)$ is computable/defined because $\forall a \in A, \exists c \in C : f(a) = c$
- For each $a \in A, f(a)$ does not produce two different outputs because $\forall a \in A, \forall c_1, c_2 \in C$ if $f(a) = c_1$ and $f(a) = c_2$ then $c_1 = c_2$
- For each $a \in A, f(a) \in C$

d)

	Mathematical Reasoning	**Reason this Statement is True**
WTS that	If $B \subseteq C$ and $f : A \to B$ is a function, then $f : A \to C$ is a function.	Since this is the contrapositive of our claim.
Suppose	If $B \subseteq C$ and $f : A \to B$ is a function.	By assumption.
i.e., Suppose	• For each $a \in A$, $f(a)$ is computable/defined because $\forall a \in A, \exists b \in B$: $f(a) = b$ • For each $a \in A$, $f(a)$ does not produce two different outputs because $\forall a \in A, \forall b_1, b_2 \in B$ if $f(a) = b_1$ and $f(a) = b_2$ then $b_1 = b_2$ • For each $a \in A, f(a) \in B$	By the 3 properties of functions.
WTS	$f : A \to C$ is a function	because this is the consequent
i.e., WTS	• For each $a \in A, f(A)$ is computable/defined because $\forall a \in A, \exists c \in C : f(a) = c$ • For each $a \in A, f(A)$ does not produce two different outputs because $\forall a \in A, \forall c_1, c_2 \in C$ if $f(a) = c_1$ and $f(a) = c_2$ then $c_1 = c_2$ • For each $a \in A, f(a) \in C$	By the 3 properties of functions

Property 1:

	Mathematical Reasoning	**Reason this Statement is True**
$\Rightarrow$	$\forall a \in A, \exists b \in B : f(a) = b$	By assumption
$\Rightarrow$	$\forall a \in A, \exists c \in C : f(a) = c$	Since $B \subseteq C$ and $\exists b \in B = \exists b \in C = \exists c \in C$

Property 2:

	Mathematical Reasoning	**Reason this Statement is True**
$\Rightarrow$	$\forall a \in A, \forall b_1, b_2 \in B$ if $f(a) = b_1$ and $f(a) = b_2$ then $b_1 = b_2$	By assumption
$\Rightarrow$	$\forall a \in A, \forall c_1, c_2 \in C$ if $f(a) = c_1$ and $f(a) = c_2$ then $c_1 = c_2$	Since $B \subseteq C$ and $b_1, b_2 \in C$ thus can be written as c_1, c_2

Property 3:

	Mathematical Reasoning	**Reason this Statement is True**
$\Rightarrow$	$a \in A, f(a) \in B$	By assumption
$\Rightarrow$	$a \in A, f(a) \in C$	Since $B \subseteq C$