Test 2

Parvesh Adi Lachman

November 2023

1 Problem 1

2 Problem 2

Recall the following definitons from lecture about a function $g : A \to B$:

one to one: $\forall n, m \in A : (n \neq m) \implies (g(n) \neq g(m))$

onto: $\forall b \in B : \exists a \in A : g(a) = b$

Let $f : \mathbb{N} \to \mathbb{Z}$ be defined by $f(n) := \sum_{v \in K_n} deg(v)$, where K_n is the complete graph on n nodes.

(a) Suppose you are trying to prove a statement of the form $\forall x \in S : [P(x) \implies Q(x)]$. What is the first line of this "for all" proof, as we've seen in this courses?

(b) Suppose you are trying to prove a statement of the form $P(x) \implies Q(x)$. What is the contrapositive of this claim?

$\neg Q(x) \implies \neg P(x)$.

3 Problem 3

Consider the following sequence of numbers similar to (But not the same as) the Sharp numbers.

$$
\begin{aligned}
&d_1 = 2 \\
&d_2 = 4 \\
&d_n = d_{n-1} + 2 \cdot d_{n-2}, \text{ for } n \geq 3
\end{aligned}
$$

Claim: For all $n \geq 1$, $d_n = 2^n$

Step 0: For all $n \geq 1$, we want to show that $d_n = 2^n$.

Step 1: For any $n \geq 1$, let $P(n)$ be the property that $d_n = 2^n$. We want to show $\forall n \geq 1 : P(n)$.

Step 2: As base cases consider when

$n = 1$. We will show that $P(1)$ is true: that is, that $d_1 = 2^1$. Fortunately,

$$\text{left hand side} = d_1 = 2 = 2^1 = \text{right hand side}$$

$n = 2$. We will show that $P(2)$ is true: that is, that $d_2 = 2^2$. Fortunately,

$$\text{left hand side} = d_2 = 4 = 2^2 = \text{right hand side}$$

Step 3: Let $k \geq 2$. For the induction hypothesis, suppose that $P(1), .., P(k)$ are true, or equivalently, that for all $1 \leq k' \leq k : P(k')$. That is, suppose that

$$\forall 1 \leq k' \leq k : d_{k'} = 2^{k'}$$

Step 4: Now we prove that $P(k+1)$ is true, using our induction assumptions that $P(1), .., P(k+1)$ are true. That is, we prove that

$$d_{k+1} = 2^{k+1}$$

Step 5: The proof that $P(k+1)$ is true (given that $P(1), ..., P(k)$ are true) is as follows:

$$
\begin{array}{rcll}
\text{Left hand side of } P(k) & = & d_{k+1} & \\
& = & d_k + 2 \cdot d_{k-1} & \text{By def of sequence} \\
& = & 2^k + 2 \cdot 2^{k-1} & \text{By IH} \\
& = & 2^k + 2^k & \text{By algebra} \\
& = & 2 \cdot 2^k & \text{By algebra} \\
& = & 2^{k+1} & \text{By algebra} \\
& = & \text{Right hand side of } P(k+1) &
\end{array}
$$

Step 6: The steps above have shown that for any $k \geq 2$, if $P(1), ..., P(k)$ are true, then $P(k+1)$ is also true. Combined with the base cases which show that $P(1)$ and $P(2)$ are true, we have shown that for all $n \geq 1$, $P(n)$ is true, as desired. $\square$