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Q1、 a. $E[y_i^2] = E[E[y_i^2|X]]$

$$E[y_i^2|X] = E[(x_i'\beta+e_i)'(x_i'\beta+e_i)|x_i]$$
$$= E[(\beta'x_i+e_i)(x_i'\beta+e_i)|x_i]$$
$$= E[\beta'x_ix_i'\beta + e_ix_i'\beta + \beta'x_ie_i + e_i^2|x_i]$$
$$= \beta'E[x_ix_i'|x_i]\beta + E[e_i^2|x_i] \Rightarrow E[y_i^2] = \beta'E[x_ix_i']\beta + E[e_i^2] \ \#$$

b. $Var[y_i] = Var[(x_i'\beta+e_i)]$

$(\because E[e_i|x_i]=0)$ $\quad = \underline{Var[x_i'\beta]} + Var[e_i]$

$$\parallel$$

$$E[(x_i'\beta - \underline{E[x_i'\beta]})^2]$$
$$= E[(x_i'\beta - E[x_i']\beta)^2]$$
$$= E[((x_i'-E[x_i'])\beta)^2]$$

$$((x_i'-E[x_i'])\beta)^2$$
$$= ((x_i'-E[x_i'])\beta)'((x_i'-E[x_i'])\beta)$$
$$= \beta'(x_i-E[x_i])(x_i-E[x_i])'\beta$$
$$= \beta'(x_i-E[x_i])^2\beta$$

$$= \beta' var[x_i]\beta$$

$$\therefore Var[y_i] = \beta' var[x_i]\beta + var[e_i] \ \#$$

Q2、 Known: $E[e_i|x_i]=0$ 、 $E[e_i^2|x_i]=\sigma^2$ 、 $E[y_i]^2 = \beta'E[x_ix_i']\beta + E[e_i^2]$

$E[\tilde{\beta}|X] = E[\hat{\beta}|X] = \beta$ (unbiasedness) 、 $var[y_i] = \beta' var[x_i] + var[e_i]$

For $\tilde{\beta} = A'Y$, $var[\tilde{\beta}] = A'var[Y]A = \sigma^2 A'A$

$\hat{\beta} = (X'X)^{-1}X'Y$, $var[\hat{\beta}] = (X'X)^{-1}X'\,var[Y]\,X(X'X)^{-1} = \sigma^2(X'X)^{-1}$

$\therefore \underline{Var[\tilde{\beta}] - Var[\hat{\beta}]} = (A'A - (X'X)^{-1})\,\sigma^2$

To prove positive semi-definite, for $c \neq 0$, $c'(A'A-(X'X)^{-1})c \geq 0$

suppose $A = X(X'X)^{-1} + E$, then $A'A = ((X'X)^{-1}X' + E')(X(X'X)^{-1} + E)$

$A'A - (X'X)^{-1} = E'X(X'X)^{-1} + (X'X)^{-1}X'E + E'E = E'E \geq 0$

$\Rightarrow$ It's proved that $Var[\tilde{\beta}] - Var[\hat{\beta}]$ is positive semi-definite $\#$

Q3. $\hat{\beta}_{GLS}$ reduces to the LS estimator $\hat{\beta}$ in the case $\Omega = \sigma^2 I_n$

a. $\because E[e_i^2|X_i] = \sigma^2 X_i \Rightarrow E[ee'|X] = \Omega$, $\Omega^{-1/2} Y = \Omega^{-1/2} X\beta + \Omega^{-1/2} e$
(Heteroscedasticity) ($\Omega^{1/2}$ is symmetric & positive definite & $\Omega = \Omega^{1/2}\Omega^{1/2}$)

$\hat{\beta}_{GLS} = (X^{*\prime}X^*)^{-1} X^{*\prime} Y^*$; $X^* = \Omega^{-1/2} X$; $Y^* = \Omega^{-1/2} Y$

$= (X'\Omega^{-1/2}\Omega^{-1/2}X)^{-1}(X'\Omega^{-1/2}\Omega^{-1/2}Y)$

$= (X'\Omega^{-1}X)^{-1}(X'\Omega^{-1}Y)$ ※ Formula of $\hat{\beta}_{GLS}$

b. $var[\hat{\beta}] = E[(\hat{\beta}-\beta)'(\hat{\beta}-\beta)] = (X'X)^{-1}X' \, var[Y] X (X'X)^{-1} = \sigma^2 (X'X)^{-1}$

$var[\hat{\beta}_{GLS}] = E[(\hat{\beta}_{GLS}-\beta)'(\hat{\beta}_{GLS}-\beta)]$; $\hat{\beta}_{GLS} - \beta = (X'\Omega^{-1}X)^{-1}(X'\Omega^{-1}e)$

$= E[(\hat{\beta}_{GLS}-\beta)(\hat{\beta}_{GLS}-\beta)']$

$= E[(X'\Omega^{-1}X)^{-1}(X'\underbrace{\Omega^{-1}e)(e'}_{I}\Omega^{-1}X)(X'\Omega^{-1}X)^{-1}]$

$= E[(X'\Omega^{-1}X)^{-1}(X'\Omega^{-1}X)(X'\Omega^{-1}X)^{-1}]$

$= (X'\Omega^{-1}X)^{-1}$

$\because \hat{\beta}$ is the estimator of β & $\hat{\beta}_{GLS}$ is the generalized estimator of β
By Gauss-Markov Theorom, $\hat{\beta}_{GLS}$ is not less efficient than $\hat{\beta}$, so $Var[\hat{\beta}] \geq Var[\hat{\beta}_{GLS}]$,
That is, $Var[\hat{\beta}] - Var[\hat{\beta}_{GLS}]$ is positive semi-definite.

Q4. a. when $\beta_2 = 0$, $Y_i = X_{1i}'\beta_1 + e_i = X_{1i}'\gamma + u_i$

$\because E[e_i|X_i] = 0$, $\therefore E[\hat{\gamma}|X] = \beta_1$ ※

$\because E[e_i^2|X_i] = \sigma^2$, $E[e'e|X] = \sigma^2 I_n$, $\therefore Var[\hat{\gamma}|X] = Var[(X_1'X_1)^{-1}X_1'e] = \sigma^2 (X_1'X_1)^{-1}$ ※

b. when $\beta_2 \neq 0$, $Y_i = X_{1i}'\beta_1 + X_{2i}'\beta_2 + e_i = X_{1i}'\gamma + u_i$

$E[\hat{\gamma}|X] = E[(X_1'X_1)^{-1}X_1 Y|X]$

$= E[(X_1X_1')^{-1}X_1(X_1'\beta_1 + X_2'\beta_2 + e)|X]$

$= E[\beta_1|X] + E[(X_1X_1')^{-1}X_1X_2'\beta_2|X] + 0$ $\nearrow \because E[e|X]=0$

$= \beta_1 + (X_1X_1')^{-1}X_1X_2'\beta_2$ ※

$Var[\hat{\gamma}|X] = Var[(X_1'X_1)^{-1}X_1'e] = \sigma^2 (X_1'X_1)^{-1}$ ※